Weighted Positivity Of Second Order Elliptic Systems

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Abstract. Integral inequalities that concern the weighted positivity of a differential operator have important applications in qualitative theory of elliptic boundary value problems. Despite the power of these inequalities, however, it is far from clear which operators have this property. In this paper, we study weighted integral inequalities for general second order elliptic systems in $I\!R^n$ $(n \geq 3)$ and prove that, with a weight, smooth and positive homogeneous of order $2-n$, the system is weighted positive only if the weight is the fundamental matrix of the system, possibly multiplied by a semi-positive definite constant matrix.

Keywords: Weighted positivity, elliptic system, fundamental matrix

1. Introduction

The goal of this paper is to study a subclass of weighted integral inequalities of the type

$$\int_\Omega Lu \cdot \Psi u \, dx \geq 0, \tag{1}$$

where Ω is a domain in $I\!R^n$ $(n \geq 3)$, $L(x, D_x)$ is an elliptic operator of order $2m$ and Ψ is positive homogeneous of order $2m-n$. This inequality, or in other words the weighted positivity of the operator L, has a number of applications in qualitative theory of elliptic boundary value problems [1–6]. For the time being, the weighted positivity has been established for certain scalar operators, and it remains an interesting question whether a similar property holds for systems. In this paper, we study the weighted positivity of general second order elliptic systems and prove the following necessary condition for (1): if the weight Ψ is smooth in $I\!R^n \backslash \{0\}$ and is positive homogeneous of order $2-n$, then Ψ must be the fundamental matrix of $L^T(0, D_x)$ multiplied by a semi-positive definite constant matrix. It is worth noting that this result is new even for the Laplacian, in which case Ψ is the fundamental solution of $-\Delta$.

This paper is organized as follows. In Section 2 we formulate the main result along with several remarks, and in Section 3 we give its proof.

2. Formulation of the Theorem

2.1. General Second-Order Elliptic Systems

Let Ω be a domain in $\mathbb{R}^n$ $(n \geq 3)$ with smooth boundary and assume $0 \in \Omega$. Consider the second order elliptic system on Ω defined by

$$\begin{aligned} L_i(x, D_x)u &:= \sum_{j=1}^{N} \sum_{\alpha,\beta=1}^{n} -A_{ij}^{\alpha\beta}(x) \frac{\partial^2 u_j}{\partial x_\alpha \partial x_\beta} \\ &=: -A_{ij}^{\alpha\beta}(x) D_{\alpha\beta} u_j \qquad\qquad (i = 1, 2, \ldots, N), \end{aligned} \tag{2}$$

where as usual repeated indices indicate summation. We assume throughout this paper that $A_{ij}^{\alpha\beta}(x)$ are real-valued, continuous functions on Ω and there exists $\lambda > 0$ such that the strong Legendre condition

$$A_{ij}^{\alpha\beta}(x) \xi_\alpha^i \xi_\beta^j \geq \lambda |\xi|^2, \qquad \forall \xi \in \mathbb{R}^{nN}$$

holds uniformly on Ω. Without loss of generality, we may also assume that

$$A_{ij}^{\alpha\beta}(x) = A_{ij}^{\beta\alpha}(x) \qquad (i, j = 1, 2, \ldots, N,\ \alpha, \beta = 1, 2, \ldots, n).$$

DEFINITION 2.1. *The operator L is said to be positive with weight $\Psi(x) = (\Psi_{ij}(x))_{i,j=1}^N$ if*

$$\int_\Omega Lu \cdot \Psi u \, dx = -\int_\Omega A_{ik}^{\alpha\beta}(x) D_{\alpha\beta} u_k(x) \cdot u_j(x) \Psi_{ij}(x) \, dx \geq 0 \tag{3}$$

for all real valued, smooth vector functions $u = (u_i)_{i=1}^N$, $u_i \in C_0^\infty(\Omega)$.

Remark. The positivity of $L(x, D_x)$ actually reduces to the positivity of $L(0, D_x)$ (with the same weight). Indeed, if $u = (u_i)_{i=1}^N$ is a smooth vector function that is supported near the origin (say, in a δ-ball B_δ) and $u_\epsilon(x) = u(\epsilon^{-1}x)$, then

$$\begin{aligned} \int_\Omega Lu_\epsilon \cdot \Psi u_\epsilon \, dx &= -\int_{B_{\epsilon\delta}} A_{ik}^{\alpha\beta}(x) D_{\alpha\beta} u_{\epsilon k}(x) \cdot u_{\epsilon j}(x) \Psi_{ij}(x) \, dx \\ &= -\epsilon^{-n} \int_{B_{\epsilon\delta}} A_{ik}^{\alpha\beta}(x) (D_{\alpha\beta} u_k)(\epsilon^{-1}x) \cdot u_j(\epsilon^{-1}x) \Psi_{ij}(\epsilon^{-1}x) \, dx \\ &= -\int_{B_\delta} A_{ik}^{\alpha\beta}(\epsilon y) D_{\alpha\beta} u_k(y) \cdot u_j(y) \Psi_{ij}(y) \, dy \qquad (x = \epsilon y). \end{aligned}$$

Since the integrand in the last integral is bounded by

$$r^{2-n} \|A_{ik}^{\alpha\beta}\|_{L^\infty(B_\delta)} \|u\|_{C^2}^2 \|\Psi_{ij}\|_{L^\infty(S^{n-1})},$$

which is clearly in $L^1(B_\delta)$, the dominated convergence theorem and the continuity of $A_{ij}^{\alpha\beta}$ implies that

$$\begin{aligned}\lim_{\epsilon\to 0^+}\int_\Omega Lu_\epsilon \cdot \Psi u_\epsilon\,dx &= -\int_{B_\delta} A_{ik}^{\alpha\beta}(0)D_{\alpha\beta}u_k(y)\cdot u_j(y)\Psi_{ij}(y)\,dy\\ &= \int_\Omega L(0,D_x)u\cdot\Psi u\,dx.\end{aligned}$$

Hence the positivity of L is in effect a local property at the origin.

In the sequel, we shall establish a necessary condition for (3) under the assumptions that

$$\begin{aligned}&\Psi_{ij}\in C^\infty(I\!\!R^n\backslash\{0\}) \qquad (i,j=1,2,\ldots,N)\\ &\Psi(x)=|x|^{2-n}\Psi\left(\frac{x}{|x|}\right)=:r^{2-n}\Psi(\omega),\end{aligned}\tag{4}$$

where $r=|x|$ and $\omega=x/|x|$. The main result will be formulated below.

2.2. Other Notations

We denote by $B_R(x_0)$ the n-dimensional ball centered at x_0 with radius R and by S^{n-1} the n-dimensional sphere. If $x_0=0$, we write B_R instead of $B_R(0)$. Generally we use $\nu=(\nu_j)_{j=1}^n$ to denote a surface outward normal, and to simplify writings, we write $\int v\,dx$ instead of $\int_{I\!\!R^n} v\,dx$ if $\Omega=I\!\!R^n$.

We usually use $u=(u_i)_{i=1}^N$ to denote a vector valued function and use v,w to denote its scalar components. The Euclidean norm of a vector is always denoted by $|\cdot|$, and in the following,

$$|u|^2=\sum_i u_i^2,\qquad |Du|^2=\sum_{i,j}(D_iu_j)^2.$$

As usual, the Fourier transform of u is denoted by $\widehat{u}$.

We also identify the elements in $C^\infty(S^{n-1})$ with those in $C^\infty(I\!\!R^n\backslash\{0\})$ that are homogeneous of degree 0. This is to say, to each $v\in C^\infty(S^{n-1})$ we associate a $\widetilde{v}\in C^\infty(I\!\!R^n\backslash\{0\})$ with

$$\widetilde{v}(x)=v\left(\frac{x}{|x|}\right).$$

Similarly, to each $\widetilde{v}\in C^\infty(I\!\!R^n\backslash\{0\})$ we associate a $v\in C^\infty(S^{n-1})$ with

$$v(\omega)=\widetilde{v}(\omega).$$

In this convention, we understand that

$$D_\alpha v(\omega) = D_\alpha \left[\widetilde{v}\left(\frac{x}{|x|}\right)\right],$$

and

$$(D_\alpha v)(\omega) = D_\alpha \widetilde{v}(x)\big|_{x=\omega} \qquad (\alpha = 1, 2, \ldots, n).$$

2.3. The Main Theorem

The main result we shall establish in this paper is the following theorem.

THEOREM 2.2. *Suppose L is an elliptic operator as defined in* (2) *and Ψ satisfies* (4)*. If L is positive with weight Ψ (and so is $L(0, D_x)$), then $L^T(0, D_x)\Psi = \delta M$ where δ is the Dirac delta function, $L^T(0, D_x)$ is the formal adjoint of $L(0, D_x)$,*

$$L_i^T(0, D_x)u := -A_{ji}^{\alpha\beta}(0) D_{\alpha\beta} u_j \qquad (i = 1, 2, \ldots, N),$$

and $M \in I\!R^{N\times N}$ is a symmetric, semi-positive definite matrix. Furthermore,

$$\sum_{i,\alpha,\beta} A_{ip}^{\alpha\beta}(r\omega)\xi_\alpha\xi_\beta\Psi_{ip}(\omega) \geq 0, \qquad \forall \xi \in I\!R^n \qquad (p = 1, 2, \ldots, N)$$

for all $r > 0$, $\omega \in S^{n-1}$ such that $r\omega \in \Omega$. That is to say, the $n \times n$ matrix $(\sum_i A_{ip}^{\alpha\beta}(r\omega)\Psi_{ip}(\omega))_{\alpha,\beta=1}^n$ is pointwise semi-positive definite.

Remark. Several extensions of the above result are possible. First, in this theorem we considered only real coefficient elliptic operators and real valued test functions. It is then natural to ask whether the same result holds for complex cases. Second, it is interesting to ask whether the set of operators that are positive in the sense of (1) is "open" in some suitable topology. In other words, we wonder whether a "small" perturbation of a positive operator still leaves the operator positive. Finally, it would be interesting to apply the above theorem to concrete problems, say the Lamé system. We actually proved (not shown here) that the Lamé system on $I\!R^3$:

$$Lu := -\mu\Delta u - (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,u, \qquad u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

is positive with weight Φ, its fundamental matrix, if $(\lambda/\mu) + 1$ is small and fails to be so if λ/μ is large. It would be interesting to find the "critical value" λ_0/μ_0 for which the system changes its behavior.

3. Proof of Theorem 2.2

In this section we give the proof of Theorem 2.2. Without loss of generality, for the first part of the theorem we may assume $\Omega = \mathbb{R}^n$ and L is a constant coefficient elliptic operator.

First some preliminaries.

3.1. Spherical Harmonics

Let $\mathscr{H}_k$ denote the linear space of homogeneous polynomials of degree k that are harmonic; they are the so-called *solid spherical harmonics of degree k*. The space of restrictions of $\mathscr{H}_k$ to the unit sphere, H_k, are the so-called *surface spherical harmonics of degree k*. It is well known that each $f \in L^2(S^{n-1})$ admits the decomposition

$$f(\omega) = \sum_{k=0}^{\infty} Y_k(\omega), \qquad Y_k \in H_k,$$

where the series converges in the L^2 sense. Since H_k can be shown to be mutually orthogonal (see, for example, [7]), Parseval's identity

$$\int_{S^{n-1}} f(\omega) g(\omega)\, d\sigma = \sum_{k=0}^{\infty} \int_{S^{n-1}} Y_k(\omega) Z_k(\omega)\, d\sigma$$

holds for all $f, g \in L^2(S^{n-1})$ where

$$f(\omega) = \sum_{k=0}^{\infty} Y_k(\omega), \qquad g(\omega) = \sum_{k=0}^{\infty} Z_k(\omega).$$

3.2. Support at the Origin

The first observation we make is that, in order for L to be positive with weight Ψ, $L^T\Psi$ has to be supported at the origin.

PROPOSITION 3.1. *Suppose L is a constant coefficient elliptic operator as defined in* (2) *and Ψ satisfies* (4)*. If L is positive with weight Ψ, then $L^T\Psi$ is supported at the origin.*

We start the proof of this proposition by observing some elementary properties of the matrix Ψ.

LEMMA 3.2. *Suppose $\Psi = (\Psi_{ij})_{i,j=1}^{N}$ satisfies* (4)*. Then*

$$D_\alpha \Psi_{ij}(x) = r^{1-n} \Psi_{ij}^{\alpha}(\omega) \qquad (i, j = 1, 2, \ldots, N),$$
$$D_{\alpha\beta} \Psi_{ij}(x) = r^{-n} \Psi_{ij}^{\alpha\beta}(\omega) \qquad (\alpha, \beta = 1, 2, \ldots, n),$$

where $\Psi_{ij}^{\alpha}, \Psi_{ij}^{\alpha\beta} \in C^\infty(S^{n-1})$ *and*

$$\int_{S^{n-1}} \Psi_{ij}^{\alpha\beta}(\omega)\, d\sigma = 0.$$

Proof. According to (4),

$$\begin{aligned}
D_\alpha \Psi_{ij}(x) &= D_\alpha\Big(r^{2-n}\Psi_{ij}(\omega)\Big) \\
&= (2-n)r^{1-n}\Psi_{ij}(\omega)\cdot\frac{x_\alpha}{r} + r^{1-n}(D_\beta\Psi_{ij})(\omega)\Big(\delta_{\alpha\beta} - \frac{x_\alpha}{r}\cdot\frac{x_\beta}{r}\Big) \\
&= r^{1-n}\Big[(2-n)\omega_\alpha\Psi_{ij}(\omega) + (D_\alpha\Psi_{ij})(\omega) - \omega_\alpha\omega_\beta(D_\beta\Psi_{ij})(\omega)\Big] \\
&=: r^{1-n}\Psi_{ij}^{\alpha}(\omega),
\end{aligned}$$

where

$$\Psi_{ij}^{\alpha}(\omega) = (2-n)\omega_\alpha\Psi_{ij}(\omega) + (D_\alpha\Psi_{ij})(\omega) - \omega_\alpha\omega_\beta(D_\beta\Psi_{ij})(\omega).$$

Similarly one can show that

$$D_{\alpha\beta}\Psi_{ij}(x) = D_\alpha\left(r^{1-n}\Psi_{ij}^{\beta}(\omega)\right) = r^{-n}\Psi_{ij}^{\alpha\beta}(\omega).$$

To prove the last statement, we integrate the above identity on $B_2\backslash B_1$ and obtain

$$\int_{B_2\backslash B_1} D_\alpha\left(r^{1-n}\Psi_{ij}^{\beta}(\omega)\right)\, dx = \int_{B_2\backslash B_1} r^{-n}\Psi_{ij}^{\alpha\beta}(\omega)\, dx.$$

Note that

$$\begin{aligned}
\int_{B_2\backslash B_1} D_\alpha\left(r^{1-n}\Psi_{ij}^{\beta}(\omega)\right)\, dx &= \int_{\partial(B_2\backslash B_1)} r^{1-n}\Psi_{ij}^{\beta}(\omega)\nu_\alpha\, d\sigma \\
&= \int_{\partial B_2} r^{1-n}\Psi_{ij}^{\beta}(\omega)\omega_\alpha\, d\sigma - \int_{\partial B_1} r^{1-n}\Psi_{ij}^{\beta}(\omega)\omega_\alpha\, d\sigma \\
&= \int_{S^{n-1}} \Psi_{ij}^{\beta}(\omega)\omega_\alpha\, d\sigma - \int_{S^{n-1}} \Psi_{ij}^{\beta}(\omega)\omega_\alpha\, d\sigma \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\int_{B_2\backslash B_1} r^{-n}\Psi_{ij}^{\alpha\beta}(\omega)\, dx &= \int_1^2 r^{-1}\, dr \int_{S^{n-1}} \Psi_{ij}^{\alpha\beta}(\omega)\, d\sigma \\
&= \log 2 \int_{S^{n-1}} \Psi_{ij}^{\alpha\beta}(\omega)\, d\sigma.
\end{aligned}$$

So the result follows.

Since the proof of Proposition 3.1 is long, we break it up into two lemmas.

LEMMA 3.3. *Under the assumptions of Proposition 3.1, if L is positive with weight Ψ, then $(L^T\Psi)_{pp}$ $(p = 1, 2, \ldots, N)$ is supported at the origin.*

Proof. Step 1. By definition, we wish to show that

$$\sum_{i,\alpha,\beta} A_{ip}^{\alpha\beta} D_{\alpha\beta}\Psi_{ip} = 0 \quad \text{on} \quad I\!\!R^n\backslash\{0\} \qquad (p = 1, 2, \ldots, N).$$

Taking $u = (u_i)_{i=1}^N$ where

$$u_i = \begin{cases} 0, & i \neq p \\ v, & i = p \end{cases}, \qquad v \in C_0^\infty(I\!\!R^n\backslash\{0\}),$$

we have

$$\begin{aligned}
\int Lu \cdot \Psi u \, dx &= -\int A_{ik}^{\alpha\beta} D_{\alpha\beta}u_k \cdot u_j \Psi_{ij} \, dx \\
&= -\int \sum_{i,\alpha,\beta} A_{ip}^{\alpha\beta} D_{\alpha\beta}v \cdot v\Psi_{ip} \, dx \\
&= \int \sum_{i,\alpha,\beta} A_{ip}^{\alpha\beta} D_\alpha v D_\beta v \cdot \Psi_{ip} \, dx + \int \sum_{i,\alpha,\beta} A_{ip}^{\alpha\beta} D_\alpha v \cdot v D_\beta \Psi_{ip} \, dx \\
&=: I_1 + I_2.
\end{aligned}$$

Step 2. By assumption (4), it is easy to see that

$$|I_1| \leq C \int r^{2-n}|Dv|^2 \, dx. \tag{5}$$

As for I_2, we observe $D_\alpha v \cdot v = \frac{1}{2}D_\alpha(v^2)$, so integrating by part once more gives

$$I_2 = -\frac{1}{2} \int \sum_{i,\alpha,\beta} A_{ip}^{\alpha\beta} v^2 D_{\alpha\beta}\Psi_{ip} \, dx. \tag{6}$$

Now assume

$$\sum_{i,\alpha,\beta} A_{ip}^{\alpha\beta} D_{\alpha\beta}\Psi_{ip} \not\equiv 0 \quad \text{on} \quad I\!\!R^n\backslash\{0\}.$$

By Lemma 3.2,

$$D_{\alpha\beta}\Psi_{ip} = r^{-n}\Psi_{ip}^{\alpha\beta}(\omega),$$

so we may write

$$\sum_{i,\alpha,\beta} A_{ip}^{\alpha\beta} D_{\alpha\beta}\Psi_{ip} = \sum_{i,\alpha,\beta} A_{ip}^{\alpha\beta} r^{-n}\Psi_{ip}^{\alpha\beta}(\omega) =: r^{-n}\Psi''_{pp}(\omega),$$

where

$$\Psi''_{pp}(\omega) = \sum_{i,\alpha,\beta} A_{ip}^{\alpha\beta} \Psi_{ip}^{\alpha\beta}(\omega) \not\equiv 0,$$

$$\int_{S^{n-1}} \Psi''_{pp}(\omega)\, d\sigma = 0.$$

Substituting this into (6) and switching to spherical coordinates, we have

$$I_2 = -\frac{1}{2} \int_0^\infty r^{-1}\, dr \int_{S^{n-1}} v^2 \Psi''_{pp}(\omega)\, d\sigma. \tag{7}$$

Step 3. Let

$$\Psi''_{pp}(\omega) = \sum_{k=m}^{\infty} Y_k(\omega), \qquad Y_k \in H_k$$

where $Y_m \not\equiv 0$. Note that $m \geq 1$ since

$$\int_{S^{n-1}} \Psi''_{pp}(\omega)\, d\sigma = 0.$$

Now take $v(x) = \zeta(r)\Omega(\omega)$ where

$$\zeta \in C_0^\infty(0,\infty) \quad \text{is to be determined later,}$$
$$\Omega(\omega) = \epsilon^{-1} + Y_m(\omega), \quad \epsilon > 0.$$

Substituting this into (7), applying Parseval's identity and recalling that $m \geq 1$, we have

$$\begin{aligned} I_2 &= -\frac{1}{2} \int_0^\infty r^{-1}\zeta^2(r)\, dr \int_{S^{n-1}} \left(\epsilon^{-1} + Y_m(\omega)\right)^2 \sum_{k=m}^{\infty} Y_k(\omega)\, d\sigma \\ &\leq -\frac{1}{2} \int_0^\infty r^{-1}\zeta^2(r)\, dr \left(2\epsilon^{-1} \int_{S^{n-1}} Y_m^2(\omega)\, d\sigma + C\right). \end{aligned}$$

This implies, for small ϵ, that

$$I_2 \leq -C_0 \epsilon^{-1} \int_0^\infty r^{-1}\zeta^2(r)\, dr.$$

On the other hand, we note that (5) implies that

$$\begin{aligned} |I_1| &\leq C \int r^{2-n} \left[(\zeta'(r))^2 \Omega^2(\omega) + r^{-2}\zeta^2(r) |\nabla_\sigma \Omega(\omega)|^2\right] dx \\ &= C \int_0^\infty r(\zeta'(r))^2\, dr \int_{S^{n-1}} \Omega^2(\omega)\, d\sigma \\ &\quad + C \int_0^\infty r^{-1}\zeta^2(r)\, dr \int_{S^{n-1}} |\nabla_\sigma Y_m(\omega)|^2\, d\sigma \\ &=: I_{11} + I_{12}, \end{aligned}$$

where ∇_σ is the spherical part of the gradient D.

Step 4. We first choose ϵ small enough so that

$$C\int_{S^{n-1}} |\nabla_\sigma Y_m(\omega)|^2\, d\sigma < C_0(2\epsilon)^{-1},$$

where C is the constant appearing in (5). For this fixed ϵ, we have

$$I_{12} < C_0(2\epsilon)^{-1}\int_0^\infty r^{-1}\zeta^2(r)\, dr, \qquad \forall \zeta \in C_0^\infty(0,\infty).$$

Next, we appeal to Lemma 3.4 below and choose ζ so that

$$I_{11} < C_0(2\epsilon)^{-1}\int_0^\infty r^{-1}\zeta^2(r)\, dr.$$

This shows that

$$I_1 + I_2 < 0$$

and gives us the desired contradiction.

LEMMA 3.4. *For any given $C > 0$, there exists $\zeta \in C_0^\infty(0,\infty)$ so that*

$$\int_0^\infty r^{-1}\zeta^2(r)\, dr \geq C\int_0^\infty r(\zeta'(r))^2\, dr.$$

Proof. Take $\varphi \in C^\infty(I\!R)$ such that

$$\varphi(r) = \begin{cases} 0, & r \leq 0 \\ 1, & r \geq 1 \end{cases}, \qquad 0 \leq \varphi \leq 1.$$

For $0 < \delta < \frac{1}{4}$, define

$$\zeta_\delta(r) = \begin{cases} \varphi(\delta^{-1}r - 1), & 0 \leq r < 1 \\ \varphi(-r+2), & r \geq 1 \end{cases}.$$

Clearly $\zeta_\delta \in C_0^\infty(0,\infty)$ (Fig.1). Now

$$\begin{aligned}
\int_0^\infty r^{-1}\zeta_\delta^2(r)\, dr &\geq \int_{2\delta}^1 r^{-1}\, dr = \log\frac{1}{2\delta}, \\
\int_0^\infty r(\zeta_\delta'(r))^2\, dr &= \int_\delta^{2\delta} r(\zeta_\delta'(r))^2\, dr + \int_1^2 r(\zeta_\delta'(r))^2\, dr \\
&\leq \delta^{-2}\|\varphi'\|_\infty^2 \int_\delta^{2\delta} r\, dr + \|\varphi'\|_\infty^2 \int_1^2 r\, dr \\
&\leq C\|\varphi'\|_\infty^2.
\end{aligned}$$

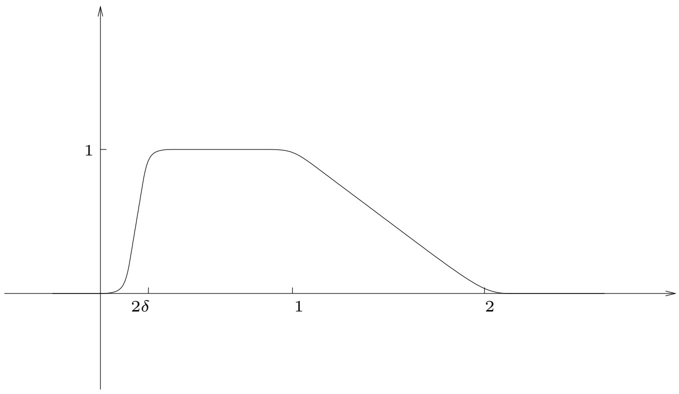

Figure 1. The function ζ_δ used in Lemma 3.4.

So the result follows by choosing δ sufficiently small.

While Lemma 3.3 proves the statement of Proposition 3.1 for diagonal elements of $L^T\Psi$, the next one takes care of the off-diagonal elements.

LEMMA 3.5. *Under the assumptions of Proposition 3.1, if L is positive with weight Ψ, then $(L^T\Psi)_{pq}$ $(p, q = 1, 2, \ldots, N,\ p \neq q)$ is supported at the origin.*

Proof. Step 1. By definition, we wish to show that

$$A_{ip}^{\alpha\beta} D_{\alpha\beta}\Psi_{iq} = 0 \quad \text{on} \quad I\!R^n\backslash\{0\} \qquad (p, q = 1, 2, \ldots, N,\ p \neq q).$$

Taking $u = (u_i)_{i=1}^N$ where

$$u_i = \begin{cases} 0, & i \neq p, q \\ v, & i = p \\ w, & i = q \end{cases}, \qquad v, w \in C_0^\infty(I\!R^n\backslash\{0\}),$$

we have

$$\begin{aligned} \int Lu \cdot \Psi u\, dx &= -\int A_{ik}^{\alpha\beta} D_{\alpha\beta}u_k \cdot u_j \Psi_{ij}\, dx \\ &= \int A_{ik}^{\alpha\beta} D_\alpha u_k D_\beta u_j \cdot \Psi_{ij}\, dx + \int A_{ik}^{\alpha\beta} D_\alpha u_k \cdot u_j D_\beta \Psi_{ij}\, dx \\ &=: I_1 + I_2. \end{aligned}$$

Step 2. By assumption (4) and Cauchy's inequality, it is easy to see that

$$|I_1| \leq C \int r^{2-n}|Du|^2\, dx \leq C \int r^{2-n}\Big(|Dv|^2 + |Dw|^2\Big)\, dx. \tag{8}$$

As for I_2, it follows from Lemma 3.3 that

$$\int A_{ip}^{\alpha\beta} D_\alpha v \cdot v D_\beta \Psi_{ip}\, dx = \int A_{iq}^{\alpha\beta} D_\alpha w \cdot w D_\beta \Psi_{iq}\, dx = 0.$$

So

$$\begin{aligned} I_2 &= \int A_{ip}^{\alpha\beta} D_\alpha v \cdot w D_\beta \Psi_{iq}\, dx + \int A_{iq}^{\alpha\beta} D_\alpha w \cdot v D_\beta \Psi_{ip}\, dx \\ &= -\int A_{ip}^{\alpha\beta} v \Big(D_\alpha w D_\beta \Psi_{iq} + w D_{\alpha\beta} \Psi_{iq} \Big)\, dx + \int A_{iq}^{\alpha\beta} D_\alpha w \cdot v D_\beta \Psi_{ip}\, dx \\ &= -\int A_{ip}^{\alpha\beta} v w D_{\alpha\beta} \Psi_{iq}\, dx + \int v D_\alpha w \Big(A_{iq}^{\alpha\beta} D_\beta \Psi_{ip} - A_{ip}^{\alpha\beta} D_\beta \Psi_{iq} \Big)\, dx \\ &= -\int r^{-n} v w \Psi''_{pq}(\omega)\, dx + \int r^{1-n} v D_\alpha w \Big(\Psi'_{\alpha qp}(\omega) - \Psi'_{\alpha pq}(\omega) \Big)\, dx \\ &=: I_{21} + I_{22}. \end{aligned} \tag{9}$$

In the derivation of (9) we have used Lemma 3.2 again, where

$$\begin{aligned} \Psi'_{\alpha pq}(\omega) &= A_{ip}^{\alpha\beta} \Psi_{iq}^{\beta}(\omega), \\ \Psi''_{pq}(\omega) &= A_{ip}^{\alpha\beta} \Psi_{iq}^{\alpha\beta}(\omega), \qquad \int_{S^{n-1}} \Psi''_{pq}(\omega)\, d\sigma = 0. \end{aligned}$$

Now assume

$$A_{ip}^{\alpha\beta} D_{\alpha\beta} \Psi_{iq} \not\equiv 0 \quad \text{on} \quad \mathbb{R}^n \backslash \{0\}.$$

Then we have

$$\Psi''_{pq}(\omega) \not\equiv 0.$$

Step 3. Let

$$\Psi''_{pq}(\omega) = \sum_{k=m}^{\infty} Y_k(\omega), \qquad Y_k \in H_k$$

where $Y_m \not\equiv 0$. Note that $m \geq 1$ since

$$\int_{S^{n-1}} \Psi''_{pq}(\omega)\, d\sigma = 0.$$

Now take

$$\begin{aligned} v(x) &= \zeta(r) Y_m(\omega), \\ w(x) &= \epsilon^{-1} \zeta(r), \qquad \epsilon > 0, \end{aligned}$$

where $\zeta \in C_0^\infty(0,\infty)$ is to be determined later. Substituting this into (9), applying Parseval's identity and recalling that $m \geq 1$, we have

$$\begin{aligned}
I_{21} &= -\int_0^\infty r^{-1}\zeta^2(r)\,dr \int_{S^{n-1}} \epsilon^{-1} Y_m(\omega) \sum_{k=m}^\infty Y_k(\omega)\,d\sigma \\
&= -\epsilon^{-1}\int_0^\infty r^{-1}\zeta^2(r)\,dr \int_{S^{n-1}} Y_m^2(\omega)\,d\sigma \\
&= -C_0\epsilon^{-1}\int_0^\infty r^{-1}\zeta^2(r)\,dr, \\
I_{22} &= \epsilon^{-1}\int_0^\infty \zeta(r)\zeta'(r)\,dr \int_{S^{n-1}} \omega_\alpha Y_m(\omega)\Big(\Psi'_{\alpha qp}(\omega) - \Psi'_{\alpha pq}(\omega)\Big)\,d\sigma \\
&= 0.
\end{aligned}$$

So

$$I_2 = -C_0\epsilon^{-1}\int_0^\infty r^{-1}\zeta^2(r)\,dr.$$

On the other hand, (8) implies that

$$\begin{aligned}
|I_1| &\leq C\int r^{2-n}\Big[(\zeta'(r))^2 Y_m^2(\omega) + r^{-2}\zeta^2(r)|\nabla_\sigma Y_m(\omega)|^2 + \epsilon^{-2}(\zeta'(r))^2\Big]\,dx \\
&= C\int_0^\infty r(\zeta'(r))^2\,dr \int_{S^{n-1}} \Big(Y_m^2(\omega) + \epsilon^{-2}\Big)\,d\sigma \\
&\quad + C\int_0^\infty r^{-1}\zeta^2(r)\,dr \int_{S^{n-1}} |\nabla_\sigma Y_m(\omega)|^2\,d\sigma.
\end{aligned}$$

Now we may proceed as in Lemma 3.3 and choose ϵ, ζ appropriately to derive the desired contradiction.

Now Proposition 3.1 is a direct consequence of Lemma 3.3 and Lemma 3.5.

3.3. Positive Definiteness of $L^T\Psi$

By Proposition 3.1, we can write $L^T\Psi$ as

$$L^T\Psi = \delta M,$$

where δ is the Dirac delta function and M is a real $N \times N$ matrix. Now we show M is symmetric and semi-positive definite.

PROPOSITION 3.6. *Suppose L is a constant coefficient elliptic operator as defined in* (2) *and Ψ satisfies* (4)*. If L is positive with weight Ψ, then $L^T\Psi = \delta M$ where δ is the Dirac delta function and $M \in I\!R^{N\times N}$ is a symmetric, semi-positive definite matrix.*

We start the proof of this proposition by writing M explicitly in terms of $A_{ij}^{\alpha\beta}$ and Ψ_{ij}.

LEMMA 3.7. *Under the assumptions of Proposition 3.6, if L is positive with weight* Ψ*, then* $L^T\Psi = \delta M$ *where* $M \in I\!R^{N\times N}$*,*

$$\begin{aligned} M_{pq} &= -\int_{S^{n-1}} A_{ip}^{\alpha\beta}\omega_\alpha \Psi_{iq}^{\beta}(\omega)\, d\sigma \\ &= -\int_{S^{n-1}} \omega_\alpha \Psi'_{\alpha pq}(\omega)\, d\sigma \qquad\qquad (p,q = 1,2,\dots,N). \end{aligned}$$

Here $\Psi_{iq}^{\beta}(\omega), \Psi'_{\alpha pq}(\omega)$ *are as defined in Lemma 3.2 and Lemma 3.5.*

Proof. By Lemma 3.2 and Proposition 3.1, for any $u \in C_0^\infty(I\!R^n)$,

$$\begin{aligned} M_{pq}u(0) &= \Big\langle (L^T\Psi)_{pq}, u\Big\rangle = \Big\langle -A_{ip}^{\alpha\beta} D_{\alpha\beta}\Psi_{iq}, u\Big\rangle \\ &= -\int A_{ip}^{\alpha\beta}\Psi_{iq} D_{\alpha\beta}u\, dx \\ &= \int A_{ip}^{\alpha\beta} D_\beta\Psi_{iq} D_\alpha u\, dx \\ &= \lim_{\epsilon\to 0}\int_{I\!R^n\setminus B_\epsilon} A_{ip}^{\alpha\beta} D_\beta\Psi_{iq} D_\alpha u\, dx \\ &= \lim_{\epsilon\to 0}\left(\int_{\partial B_\epsilon} A_{ip}^{\alpha\beta} D_\beta\Psi_{iq}\cdot u\nu_\alpha\, d\sigma - \int_{I\!R^n\setminus B_\epsilon} A_{ip}^{\alpha\beta} D_{\alpha\beta}\Psi_{iq}\cdot u\, dx\right) \\ &= -\lim_{\epsilon\to 0}\int_{\partial B_\epsilon} A_{ip}^{\alpha\beta} r^{1-n}\Psi_{iq}^{\beta}(\omega)\cdot u\omega_\alpha\, d\sigma \\ &= -\lim_{\epsilon\to 0}\int_{S^{n-1}} A_{ip}^{\alpha\beta}\Psi_{iq}^{\beta}(\omega) u(\epsilon\omega)\omega_\alpha\, d\sigma \\ &= -u(0)\int_{S^{n-1}} A_{ip}^{\alpha\beta}\omega_\alpha\Psi_{iq}^{\beta}(\omega)\, d\sigma. \end{aligned}$$

So the result follows.

As before, we break up the proof of Proposition 3.6 into two Lemmas.

LEMMA 3.8. *Under the assumptions of Proposition 3.6, if L is positive with weight* Ψ*, then* $L^T\Psi = \delta M$ *where* $M \in I\!R^{N\times N}$ *is symmetric.*

Proof. Step 1. By definition, we wish to show that

$$M_{pq} = M_{qp} \qquad (p,q = 1,2,\dots,N,\ p \neq q).$$

As in the proof of Lemma 3.5, we take $u = (u_i)_{i=1}^N$, where

$$u_i = \begin{cases} 0, & i \neq p,q \\ v, & i = p \\ w, & i = q \end{cases}, \qquad v,w \in C_0^\infty(I\!R^n\setminus\{0\}),$$

and obtain

$$\int Lu \cdot \Psi u \, dx = \int A_{ik}^{\alpha\beta} D_\alpha u_k D_\beta u_j \cdot \Psi_{ij} \, dx + \int A_{ik}^{\alpha\beta} D_\alpha u_k \cdot u_j D_\beta \Psi_{ij} \, dx$$
$$=: I_1 + I_2.$$

Step 2. As before, we have

$$|I_1| \leq C \int r^{2-n} \Big(|Dv|^2 + |Dw|^2 \Big) \, dx \tag{10}$$

and

$$I_2 = -\int r^{-n} v w \Psi''_{pq}(\omega) \, dx + \int r^{1-n} v D_\alpha w \Big(\Psi'_{\alpha qp}(\omega) - \Psi'_{\alpha pq}(\omega) \Big) \, dx.$$

Note that

$$\Psi''_{pq}(\omega) \equiv 0$$

by Proposition 3.1, so

$$I_2 = \int r^{1-n} v D_\alpha w \Big(\Psi'_{\alpha qp}(\omega) - \Psi'_{\alpha pq}(\omega) \Big) \, dx. \tag{11}$$

Step 3. Now take

$$v(x) = \zeta(r),$$
$$w(x) = \eta(r) := -\epsilon \operatorname{sgn}(M_{pq} - M_{qp}) \int_0^r \rho^{-1} \zeta(\rho) \, d\rho, \qquad \epsilon > 0,$$

where $\zeta \in C_0^\infty(0, \infty)$ is to be determined later. Substituting this into (11), switching to spherical coordinates and applying Lemma 3.7, we have

$$\begin{aligned} I_2 &= \int_0^\infty \zeta(r) \eta'(r) \, dr \int_{S^{n-1}} \omega_\alpha \Big(\Psi'_{\alpha qp}(\omega) - \Psi'_{\alpha pq}(\omega) \Big) \, d\sigma \\ &= -\epsilon \operatorname{sgn}(M_{pq} - M_{qp}) \cdot (M_{pq} - M_{qp}) \int_0^\infty r^{-1} \zeta^2(r) \, dr \\ &= -\epsilon |M_{pq} - M_{qp}| \int_0^\infty r^{-1} \zeta^2(r) \, dr. \end{aligned}$$

On the other hand, (10) implies that

$$\begin{aligned} |I_1| &\leq C \int r^{2-n} \Big[(\zeta'(r))^2 + (\eta'(r))^2 \Big] \, dx \\ &= C \left[\int_0^\infty r (\zeta'(r))^2 \, dr + \epsilon^2 \int_0^\infty r^{-1} \zeta^2(r) \, dr \right] \\ &=: I_{11} + I_{12}. \end{aligned}$$

Step 4. Assume $M_{pq} - M_{qp} \neq 0$. We first choose ϵ small enough so that

$$C\epsilon < \frac{1}{2}|M_{pq} - M_{qp}|,$$

where C is the constant appearing in (10). For this fixed ϵ, we have

$$I_{12} < \frac{\epsilon}{2}|M_{pq} - M_{qp}| \int_0^\infty r^{-1}\zeta^2(r)\,dr, \qquad \forall \zeta \in C_0^\infty(0,\infty).$$

Next, we appeal to Lemma 3.9 below and choose ζ so that

$$I_{11} < \frac{\epsilon}{2}|M_{pq} - M_{qp}| \int_0^\infty r^{-1}\zeta^2(r)\,dr.$$

This shows that

$$I_1 + I_2 < 0$$

and gives us the desired contradiction.

LEMMA 3.9. *For any given $C > 0$, there exists $\zeta \in C_0^\infty(0,\infty)$ so that*

$$\int_0^\infty r^{-1}\zeta^2(r)\,dr \geq C \int_0^\infty r(\zeta'(r))^2\,dr$$

and

$$r^{-1}\zeta(r) = \eta'(r) \quad \textit{for some } \eta \in C_0^\infty(0,\infty).$$

Proof. We first note that for any $\zeta \in C_0^\infty(0,\infty)$,

$$r^{-1}\zeta(r) = \eta'(r) \quad \text{for some } \eta \in C_0^\infty(0,\infty)$$

if and only if

$$\int_0^\infty r^{-1}\zeta(r)\,dr = 0.$$

Let $\varphi \in C^\infty(I\!R)$ be as given in Lemma 3.4. For $0 < \delta < \frac{1}{4}$ and $R > \frac{5}{4}$, define

$$\zeta_{\delta,R}(r) = \begin{cases} \varphi(\delta^{-1}r - 1), & 0 \leq r < \frac{3}{4} \\ 2\varphi(-2r + \frac{5}{2}) - 1, & \frac{3}{4} \leq r < R \\ \varphi(R^{-1}r - 1) - 1, & r \geq R \end{cases}.$$

Clearly $\zeta_{\delta,R} \in C_0^\infty(0,\infty)$ (Fig.2). For each δ small, we may choose $R = R_\delta$ so that

$$\int_0^\infty \zeta_{\delta,R_\delta}(r)\,dr = 0.$$

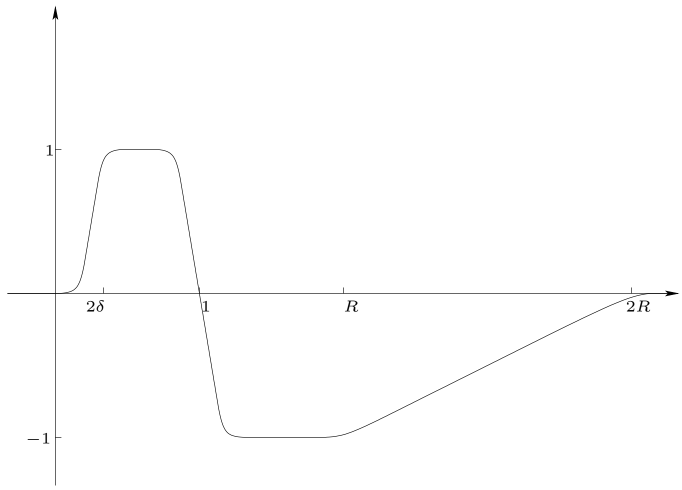

Figure 2. The function $\zeta_{\delta,R}$ used in Lemma 3.9.

This is always possible since the integral above changes continuously with R and

$$\int_0^\infty r^{-1}\zeta_{\delta,5/4}(r)\,dr > 0 \quad \text{if } \delta \text{ is sufficiently small,}$$
$$\int_0^\infty r^{-1}\zeta_{\delta,R}(r)\,dr \to -\infty \quad \text{as } R\to\infty.$$

Now

$$\begin{aligned}
&\int_0^\infty r^{-1}\zeta^2_{\delta,R_\delta}(r)\,dr \geq \int_{2\delta}^{3/4} r^{-1}\,dr = \log\frac{3}{8\delta},\\
&\int_0^\infty r(\zeta'_{\delta,R_\delta}(r))^2\,dr = \left[\int_\delta^{2\delta} + \int_{3/4}^{5/4} + \int_{R_\delta}^{2R_\delta}\right] r(\zeta'_{\delta,R_\delta}(r))^2\,dr\\
&\qquad \leq \delta^{-2}\|\varphi'\|^2_\infty \int_\delta^{2\delta} r\,dr + 16\|\varphi'\|^2_\infty \int_{3/4}^{5/4} r\,dr + R_\delta^{-2}\|\varphi'\|^2_\infty \int_{R_\delta}^{2R_\delta} r\,dr\\
&\qquad \leq C\|\varphi'\|^2_\infty.
\end{aligned}$$

So the result follows by choosing δ sufficiently small.

Now we show M is semi-positive definite.

LEMMA 3.10. *Under the assumptions of Proposition 3.6, if* L *is positive with weight* Ψ*, then* $L^T\Psi = \delta M$ *where* $M \in \mathbb{R}^{N\times N}$ *is semi-positive definite.*

Proof. Step 1. Take $u=(u_i)_{i=1}^N$ where $u_i \in C_0^\infty(I\!R^n)$ (note that u_i does not necessarily vanish near the origin). As before we have

$$\begin{aligned}\int Lu\cdot\Psi u\,dx &= \int A_{ik}^{\alpha\beta}D_\alpha u_k D_\beta u_j\cdot\Psi_{ij}\,dx+\int A_{ik}^{\alpha\beta}D_\alpha u_k\cdot u_j D_\beta\Psi_{ij}\,dx\\ &=: I_1+I_2.\end{aligned}$$

Step 2. Clearly

$$|I_1|\le C\int r^{2-n}|Du|^2\,dx. \tag{12}$$

As for I_2, we write

$$\begin{aligned}I_2 &= \int\sum_{k<j}A_{ik}^{\alpha\beta}D_\alpha u_k\cdot u_j D_\beta\Psi_{ij}\,dx+\int\sum_{k>j}A_{ik}^{\alpha\beta}D_\alpha u_k\cdot u_j D_\beta\Psi_{ij}\,dx\\ &\quad+\int\sum_{k=j}A_{ik}^{\alpha\beta}D_\alpha u_k\cdot u_j D_\beta\Psi_{ij}\,dx\\ &=: I_{21}+I_{22}+I_{23}.\end{aligned}$$

Similar to the calculations in Lemma 3.7, we have

$$\begin{aligned}I_{21} &= \sum_{k<j}\int A_{ik}^{\alpha\beta}D_\alpha u_k\cdot u_j D_\beta\Psi_{ij}\,dx\\ &= -\sum_{k<j}\left(u_k(0)u_j(0)\int_{S^{n-1}}\omega_\alpha\Psi'_{\alpha kj}(\omega)\,d\sigma+\int A_{ik}^{\alpha\beta}u_k D_\alpha u_j D_\beta\Psi_{ij}\,dx\right)\\ &= \sum_{k<j}u_k(0)u_j(0)M_{kj}-\sum_{j<k}\int A_{ij}^{\alpha\beta}u_j D_\alpha u_k D_\beta\Psi_{ik}\,dx,\\ I_{23} &= \sum_{k=j}\int A_{ik}^{\alpha\beta}D_\alpha u_k\cdot u_j D_\beta\Psi_{ij}\,dx\\ &= \frac{1}{2}\sum_{k=j}u_k(0)u_j(0)M_{kj}.\end{aligned}$$

Since M is symmetric, this implies that

$$\begin{aligned}I_2 &= \frac{1}{2}\sum_{k,j}u_k(0)u_j(0)M_{kj}+\sum_{k>j}\int u_j D_\alpha u_k\left(A_{ik}^{\alpha\beta}D_\beta\Psi_{ij}-A_{ij}^{\alpha\beta}D_\beta\Psi_{ik}\right)\,dx\\ &= \frac{1}{2}u^T(0)Mu(0)+\sum_{k>j}\int u_j D_\alpha u_k\left(A_{ik}^{\alpha\beta}D_\beta\Psi_{ij}-A_{ij}^{\alpha\beta}D_\beta\Psi_{ik}\right)\,dx.\end{aligned} \tag{13}$$

Step 3. Assume M is not semi-positive definite, then there exists $\xi\in I\!R^N$ such that

$$\xi^T M\xi<0.$$

Take

$$u_j(x) = \xi_j \varphi\left(\frac{\log r}{\log \epsilon} - 1\right), \qquad 0 < \epsilon < 1 \qquad (j = 1, 2, \ldots, N),$$

where $\varphi \in C^\infty(I\!R)$ is as given in Lemma 3.4. Substituting this into (13), switching to spherical coordinates and applying Lemma 3.8, we have

$$\begin{aligned} I_2 &= \frac{1}{2}\xi^T M \xi + \sum_{k>j} \xi_j \xi_k (M_{jk} - M_{kj}) \int_0^\infty \varphi\left(\frac{\log r}{\log \epsilon} - 1\right)\left[\varphi\left(\frac{\log r}{\log \epsilon} - 1\right)\right]' dr \\ &= \frac{1}{2}\xi^T M \xi. \end{aligned}$$

On the other hand

$$\begin{aligned} |Du|^2 = \sum_i |Du_i|^2 &= \sum_i \left| \frac{\xi_i}{\log \epsilon} \varphi'\left(\frac{\log r}{\log \epsilon} - 1\right) \frac{\omega}{r} \right|^2 \\ &= \frac{|\xi|^2}{r^2 \log^2 \epsilon} \left[\varphi'\left(\frac{\log r}{\log \epsilon} - 1\right)\right]^2, \end{aligned}$$

so (12) implies that

$$\begin{aligned} |I_1| &\le C \int \frac{|\xi|^2}{r^n \log^2 \epsilon} \left[\varphi'\left(\frac{\log r}{\log \epsilon} - 1\right)\right]^2 dx \\ &\le \frac{C|\xi|^2}{\log^2 \epsilon} \|\varphi'\|_\infty^2 \int_{\epsilon^2}^{\epsilon} \frac{1}{r}\, dr \\ &= \frac{C|\xi|^2}{|\log \epsilon|} \|\varphi'\|_\infty^2. \end{aligned}$$

This shows that

$$I_1 + I_2 < 0 \qquad \text{if } \epsilon \text{ is sufficiently small}$$

and gives us the desired contradiction.

Now Proposition 3.6 is a direct consequence of Lemma 3.8 and Lemma 3.10.

It is natural to ask whether one can improve the results of Proposition 3.6 by showing that actually $M = I$, the $N \times N$ identity matrix. The following example shows that this is not the case.

Example. Assume $n \ge 3$ and consider $L = -\Delta \cdot I$, where Δ is the Laplacian:

$$\Delta u = D_{\alpha\alpha} u, \qquad \forall u \in C^2(I\!R^n),$$

and I is the $N \times N$ identity matrix. It is not hard to see that the fundamental matrix of $L^T = L$ is given by $\Phi = \gamma I$, where

$$\gamma(x) = \frac{1}{\omega_n(n-2)} \cdot r^{2-n}, \qquad \omega_n = \int_{S^{n-1}} dx$$

is the fundamental solution of $-\Delta$. For any $M \in I\!R^{N\times N}$ with M symmetric and semi-positive definite, we have

$$M = P^T \Lambda P$$

where P is orthogonal and Λ is diagonal with non-negative diagonal elements $\lambda_1, \ldots, \lambda_N$. Now for any $u = (u_i)_{i=1}^N$, $u_i \in C_0^\infty(I\!R^n)$ $(i = 1, 2, \ldots, N)$,

$$\begin{aligned}\int Lu \cdot (\Phi M) u \, dx &= -\int \gamma (\Delta u)^T M u \, dx \\ &= -\int \gamma (\Delta u)^T P^T \Lambda P u \, dx \\ &= -\int \gamma (\Delta (Pu))^T \Lambda (Pu) \, dx.\end{aligned}$$

Setting $v = Pu$, we have

$$\begin{aligned}-\int \gamma (\Delta v)^T \Lambda v \, dx &= \frac{1}{2} \lambda_i v_i^2(0) + \int \lambda_i |Dv_i|^2 \gamma \, dx \\ &\geq \min_{i=1,2,\ldots,N} \{\lambda_i\} \left(\frac{1}{2} |v(0)|^2 + \int |Dv|^2 \gamma \, dx \right) \\ &\geq 0.\end{aligned}$$

3.4. Pointwise Positive Definiteness

With judicious choices of the test function u, we now proceed to show the pointwise "positive definiteness" of Ψ.

PROPOSITION 3.11. *Suppose L is an elliptic operator as defined in* (2) *and Ψ satisfies* (4). *If L is positive with weight Ψ, then*

$$\sum_{i,\alpha,\beta} A_{ip}^{\alpha\beta}(r\omega) \xi_\alpha \xi_\beta \Psi_{ip}(\omega) \geq 0, \qquad \forall \xi \in I\!R^n \qquad (p = 1, 2, \ldots, N)$$

for all $r > 0$, $\omega \in S^{n-1}$ such that $r\omega \in \Omega$. That is to say, the $n \times n$ matrix $(\sum_i A_{ip}^{\alpha\beta}(r\omega) \Psi_{ip}(\omega))_{\alpha,\beta=1}^n$ is pointwise semi-positive definite.

Proof. Let $r > 0,\ \omega \in S^{n-1}$ be fixed and $r\omega \in \Omega$. We follow the idea in [5] and take $u = (u_j)_{j=1}^N$, where

$$u_j(x) = \begin{cases} 0, & j \neq p \\ \epsilon^{-n/2}|\xi|^{-1}\eta(\epsilon^{-1}(x - r\omega))e^{ix\cdot\xi}, & j = p \end{cases},$$
$$\epsilon > 0,\ \xi \in I\!R^n,\ 0 \not\equiv \eta \in C_0^\infty(I\!R^n).$$

By definition (with $y = \epsilon^{-1}(x - r\omega)$),

$$\begin{aligned}
&\mathrm{Re}\int_\Omega Lu \cdot \Psi\overline{u}\,dx \\
&\quad = \mathrm{Re}\Big\{ -\epsilon^{-n}|\xi|^{-2}\int_\Omega \sum_{j,\alpha,\beta} A_{jp}^{\alpha\beta} D_{\alpha\beta}\Big[\eta(y)e^{ix\cdot\xi}\Big]\cdot \eta(y)e^{-ix\cdot\xi}\Psi_{jp}\,dx\Big\} \\
&\quad = -\epsilon^{-n}|\xi|^{-2}\int_\Omega \sum_{j,\alpha,\beta} A_{jp}^{\alpha\beta}\Big[\epsilon^{-2}\eta''(y) - \xi_\alpha\xi_\beta\eta(y)\Big]\eta(y)\Psi_{jp}\,dx \\
&\quad \geq 0.
\end{aligned}$$

We first let $|\xi| \to \infty$ along a fixed direction and obtain

$$\epsilon^{-n}\frac{\xi_\alpha}{|\xi|}\cdot\frac{\xi_\beta}{|\xi|}\int_\Omega \sum_{j,\alpha,\beta} A_{jp}^{\alpha\beta}\eta^2(y)\Psi_{jp}\,dx \geq 0.$$

By substituting $y = \epsilon^{-1}(x - r\omega)$ for x and letting $\epsilon \to 0$, we then conclude that

$$\sum_{j,\alpha,\beta} A_{jp}^{\alpha\beta}(r\omega)\frac{\xi_\alpha}{|\xi|}\cdot\frac{\xi_\beta}{|\xi|}r^{2-n}\Psi_{jp}(\omega)\int \eta^2\,dx \geq 0,$$

which is what to be shown.

This completes the proof of Theorem 2.2.

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