Boundedness of the gradient and Wiener test for the solutions of the biharmonic equation

Svitlana Mayboroda and Vladimir Maz'ya*

1 Introduction

The maximum principle for harmonic functions is one of the fundamental results in the theory of elliptic equations. It holds in arbitrary domains and guarantees that every solution to the Dirichlet problem for the Laplace equation, with bounded data, is bounded. An analogue of the maximum principle for the higher order elliptic operators is unknown, even for the model case of the bilaplacian (see Problem 4.3, p.275, in J. Nečas's book [19]).

To be more specific, let $\Omega\subset\mathbb{R}^n$ be a bounded domain and consider the boundary value problem

$$\Delta^2 u=f \text{ in } \Omega,\quad u\in \mathring{W}_2^2(\Omega), \tag{1.1}$$

where the Sobolev space $\mathring{W}_2^2(\Omega)$ is a completion of $C_0^\infty(\Omega)$ in the norm $\|u\|_{\mathring{W}_2^2(\Omega)}=\|\Delta u\|_{L^2(\Omega)}$ and f is a reasonably nice function. According to the counterexamples built in [17] and [20], the gradient of the solution to the Dirichlet problem (1.1) is not necessarily bounded when $n\geq 4$ (thus, the maximum principle fails). In dimension three this problem has been open.

The absence of any information about the geometry of the domain puts this question beyond the scope of applicability of the previously devised methods, which typically rely on specific assumptions on Ω: positive results have been available only when $\Omega\subset\mathbb{R}^3$ is sufficiently smooth ([3]), Lipschitz ([21]), or diffeomorphic to a polyhedron ([8], [16]). The techniques developed in the present paper allow us to establish the boundedness of the gradient of a biharmonic function under no restrictions on the underlying domain. We prove the following:

Theorem 1.1 *Let Ω be an arbitrary bounded domain in $\mathbb{R}^3$ and let u be the solution to the Dirichlet problem* (1.1). *Then*

$$|\nabla u|\in L^\infty(\Omega). \tag{1.2}$$

Moreover, as a by-product of this result we obtain the estimate

$$|\nabla_x\nabla_y\, G(x,y)|\leq C|x-y|^{-1},\qquad x,y\in\Omega, \tag{1.3}$$

*2000 *Math Subject Classification:* 35J40, 35J30, 35B65.
Key words: Biharmonic equation, Dirichlet problem, general domains.
The second author is partially supported by NSF grant DMS 0500029.

where G is Green's function for the biharmonic equation and C is a numerical constant independent of Ω.

Theorem 1.1 is sharp in the sense that the solution generally does not exhibit more regularity than implied by (1.2). Indeed, let Ω be the three-dimensional punctured unit ball $B_1 \setminus \{O\}$, where $B_r = \{x \in \mathbb{R}^3 : |x| < r\}$, and consider a function $\eta \in C_0^\infty(B_{1/2})$ such that $\eta = 1$ on $B_{1/4}$. Let

$$u(x) := \eta(x)|x|, \qquad x \in B_1 \setminus \{O\}. \tag{1.4}$$

A straightforward computation shows that $u \in \mathring{W}_2^2(\Omega)$ and $\Delta^2 u \in C_0^\infty(\Omega)$. While ∇u satisfies (1.2), it is not continuous at the origin. Therefore, the improvement of (1.2) is not possible in general and must depend on some delicate properties of the domain.

Even in the case of the Laplacian an analogous issue, the continuity of the solution to the Dirichlet problem, is very subtle. It has been resolved in 1924, when Wiener gave his famous criterion for the regularity of a boundary point [23]. Wiener's result set a new framework for the subject of potential theory and influenced the development of partial differential equations, function spaces and probability. Over the years, it has been extended to a variety of second order elliptic and parabolic equations ([10], [7], [6], [4], [11], [2], [24], [9], [5]; see also the review papers [14], [1]). However, the case of higher order operators is largely underdeveloped. While substantial progress has been made in the study of the continuity of solutions for a certain family of higher order elliptic equations in [15] (see also [12], [13]), there have been no general results providing necessary or sufficient geometrical conditions for continuity of the derivatives of solutions, in particular, describing the phenomenon addressed in (1.4).

In what follows, we establish an analogue of the Wiener's test governing the gradient of a solution to problem (1.1). To set the stage, let us recall the original Wiener's criterion. Roughly speaking, it states that a point $O \in \partial\Omega$ is regular (i.e. every solution to the Dirichlet problem for the Laplacian, with continuous data, is continuous at O) if and only if the complement of the domain near the point O, measured in terms of the Wiener (harmonic) capacity, is sufficiently large. More specifically, the harmonic capacity of a compactum $K \subset \mathbb{R}^n$ can be defined as

$$\operatorname{cap}(K) := \inf\Big\{ \|\nabla u\|^2_{L^2(\mathbb{R}^n)} : \ u \in C_0^\infty(\mathbb{R}^n), \ u = 1 \text{ in a neighborhood of } K \Big\}, \tag{1.5}$$

where $n \geq 3$. Then the regularity of the point O is equivalent to the condition

$$\int_0^1 \operatorname{cap}(\overline{B_s} \setminus \Omega) s^{1-n}\, ds = +\infty. \tag{1.6}$$

The aforementioned developments in [15] extend (1.5)–(1.6) to the context of the biharmonic equation in dimensions 4, 5, 6 and 7, using potential-theoretic Bessel capacity of order four in place of (1.5). In dimension three, on the other hand, it is natural to expect a higher order of smoothness for solutions of the biharmonic equation or, more generally, higher order elliptic equations. However, being confined to the issue of continuity of a solution, the results in [15] can not be extended to treat its derivatives.

Turning to this problem, we start with a suitable notion of capacity. Let Π denote the space of functions

$$P(x) = b_0 + b_1 \tfrac{x_1}{|x|} + b_2 \tfrac{x_2}{|x|} + b_3 \tfrac{x_3}{|x|}, \qquad x \in \mathbb{R}^3 \setminus \{O\}, \qquad b_i \in \mathbb{R}, \qquad i = 0, 1, 2, 3, \tag{1.7}$$

and $\Pi_1 := \{P \in \Pi : \|P\|_\Pi = 1\}$. Then, given a compactum $K \subset \mathbb{R}^3 \setminus \{0\}$ and $P \in \Pi_1$, let

$$\mathrm{Cap}_P(K) := \inf\Big\{ \|\Delta u\|^2_{L^2(\mathbb{R}^3)} : \ u \in \mathring{W}^2_2(\mathbb{R}^3 \setminus \{0\}), \ u = P \text{ in a neighborhood of } K \Big\}. \tag{1.8}$$

Such biharmonic capacity first appeared in [18], in conjunction with the upper estimates on $\sup_r (\frac{1}{r^3} \int_{B_r} |\nabla u(x)|^6 \, dx)^{1/6}$ for a solution of (1.1).

We say that a point $O \in \partial\Omega$ is 1-regular if for every $f \in C_0^\infty(\Omega)$ the solution to (1.1) is continuously differentiable at O, i.e. $\nabla u(x) \to 0$ as $x \to O$; and O is 1-irregular otherwise. Our main result in this direction is the following.

Theorem 1.2 *Let Ω be an open set in $\mathbb{R}^n$. If for some $a \geq 4$ and some $c > 0$*

$$\int_0^c \inf_{P \in \Pi_1} \mathrm{Cap}_P(\overline{C_{s,as}} \setminus \Omega) \, ds = +\infty, \tag{1.9}$$

then the point O is 1-regular.

Conversely, if the point $O \in \partial\Omega$ is 1-regular then for every $c > 0$ and every $a \geq 8$

$$\inf_{P \in \Pi_1} \int_0^c \mathrm{Cap}_P(\overline{C_{s,as}} \setminus \Omega) \, ds = +\infty. \tag{1.10}$$

Here $C_{s,as}$ is the annulus $\{x \in \mathbb{R}^3 : s < |x| < as\}$.

In [18] the authors defined the biharmonic capacity as the infimum over $P \in \Pi_1$ of $\mathrm{Cap}_P(\cdot)$. However, the proof of the "necessity" in Theorem 1.2 required that we place the infimum outside the integral in (1.10). Later, in §9, we show that this is a natural effect as (1.9) cannot play a role of the necessary condition for 1-regularity.

Our results are accompanied by the corresponding estimates, in particular, we prove the following refinement of (1.3):

$$|\nabla_x \nabla_y G(x,y)| \leq C \begin{cases} |x-y|^{-1} \exp\Big(-c \int_{c_1|y|}^{c_2|x|} \inf\limits_{P \in \Pi_1} \mathrm{Cap}_P(\overline{C_{s,as}} \setminus \Omega) \, ds\Big), & \text{if } \ |y| \leq c_0|x|, \\ |x-y|^{-1} \exp\Big(-c \int_{c_1|x|}^{c_2|y|} \inf\limits_{P \in \Pi_1} \mathrm{Cap}_P(\overline{C_{s,as}} \setminus \Omega) \, ds\Big), & \text{if } \ |x| \leq c_0|y|, \\ |x-y|^{-1}, & \text{if } \ c_0|y| \leq |x| \leq c_0^{-1}|y|, \end{cases}$$

where $a \geq 4$ and c_0, c_1, c_2 are some constants depending on a.

It has to be noted that Theorem 1.2 brings up a peculiar role of circular cones for 1-regularity of a boundary point. For example, if the complement of Ω is a compactum located on the circular cone $\{x \in \mathbb{R}^3 \setminus \{0\} : b_0|x| + b_1x_1 + b_2x_2 + b_3x_3 = 0\}$ such that the harmonic capacity $\operatorname{cap}(\mathbb{R}^3 \setminus \Omega) = 0$, then $\operatorname{Cap}_P(\mathbb{R}^3 \setminus \Omega) = 0$ for P associated to the same b_i's. Hence, by Theorem 1.2, the point O is not 1-regular.

Another surprising effect, strikingly different from the classical theory, is that 1-irregularity turns out to be unstable under affine transformations of coordinates.

In conclusion, we provide some examples further illustrating the geometric nature of conditions (1.9)–(1.10). Among them is the model case when Ω has an inner cusp, i.e. in a neighborhood of the origin $\Omega = \{(r,\theta,\phi) : 0 < r < c,\ h(r) < \theta \leq \pi,\ 0 \leq \phi < 2\pi\}$, where h is a non-decreasing function such that $h(br) \leq h(r)$ for some $b > 1$. For such domain Theorem 1.2 yields the following criterion:

$$\text{the point } O \text{ is 1-regular} \quad \text{if and only if} \quad \int_0^1 s^{-1} h(s)^2\, ds = +\infty. \tag{1.11}$$

Some other examples can be found in the body of the paper.

2 The global estimates

Let us start with a few remarks about the notation.

Let (r,ω) be the spherical coordinates in $\mathbb{R}^3$, i.e. $r = |x| \in (0,\infty)$ and $\omega = x/|x| \in S^2$, the unit sphere. Occasionally we will write the spherical coordinates as (r,θ,ϕ), where $\theta \in [0,\pi]$ stands for the colatitude and $\phi \in [0,2\pi)$ is the longitudinal coordinate, i.e.

$$\omega = x/|x| = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta). \tag{2.1}$$

Now let $t = \log r^{-1}$. Then by κ and $\varkappa$ we denote the mappings

$$\mathbb{R}^3 \ni x \xrightarrow{\kappa} (r,\phi,\theta) \in [0,\infty) \times [0,2\pi) \times [0,\pi]; \qquad \mathbb{R}^3 \ni x \xrightarrow{\varkappa} (t,\omega) \in \mathbb{R} \times S^2. \tag{2.2}$$

The symbols δ_ω and ∇_ω refer, respectively, to the Laplace-Beltrami operator and the gradient on S^2.

For any domain $\Omega \subset \mathbb{R}^3$ a function $u \in C_0^\infty(\Omega)$ can be extended by zero to $\mathbb{R}^3$ and we will write $u \in C_0^\infty(\mathbb{R}^3)$ whenever convenient. Similarly, the functions in $\mathring{W}_2^2(\Omega)$ will be extended by zero and treated as functions on $\mathbb{R}^3$ without further comments.

We open with the following generic identity.

Lemma 2.1 *Let Ω be an open set in $\mathbb{R}^3$, $u \in C_0^\infty(\Omega)$ and $v = e^t(u \circ \varkappa^{-1})$. Then*

$$\begin{aligned}
&\int_{\mathbb{R}^3} \Delta u(x) \Delta\Big(u(x)|x|^{-1} G(\log|x|^{-1})\Big)\, dx \\
&\quad = \int_{\mathbb{R}} \int_{S^2} \Big[(\delta_\omega v)^2 G + 2(\partial_t \nabla_\omega v)^2 G + (\partial_t^2 v)^2 G - (\nabla_\omega v)^2 \Big(\partial_t^2 G + \partial_t G + 2G\Big) \\
&\quad -(\partial_t v)^2 \Big(2\partial_t^2 G + 3\partial_t G - G\Big) + \frac{1}{2} v^2 \Big(\partial_t^4 G + 2\partial_t^3 G - \partial_t^2 G - 2\partial_t G\Big)\Big]\, d\omega dt,
\end{aligned} \tag{2.3}$$

for every function G on $\mathbb{R}$ such that both sides of (2.3) are well-defined.

Proof. In the system of coordinates (t,ω) the 3-dimensional Laplacian can be written as

$$\Delta = e^{2t}\Lambda(\partial_t,\delta_\omega), \quad \text{where} \quad \Lambda(\partial_t,\delta_\omega) = \partial_t^2 - \partial_t + \delta_\omega. \tag{2.4}$$

Then passing to the coordinates (t,ω), we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} \Delta u(x)\Delta\Big(u(x)|x|^{-1}G(\log|x|^{-1})\Big)\,dx = \int_{\mathbb{R}}\int_{S^2} \Lambda(\partial_t-1,\delta_\omega)v\,\Lambda(\partial_t,\delta_\omega)(vG)\,d\omega dt \\
&= \int_{\mathbb{R}}\int_{S^2} \left(\partial_t^2 v - 3\partial_t v + 2v + \delta_\omega v\right)\,\left(\partial_t^2(vG) - \partial_t(vG) + G\,\delta_\omega v\right)\,d\omega dt \\
&= \int_{\mathbb{R}}\int_{S^2} \left(\partial_t^2 v - 3\partial_t v + 2v + \delta_\omega v\right) \\
&\qquad\qquad \times \left(G\,\delta_\omega v + G\,\partial_t^2 v + (2\partial_t G - G)\,\partial_t v + (\partial_t^2 G - \partial_t G)\,v\right)\,d\omega dt \\
&= \int_{\mathbb{R}}\int_{S^2} \Big(\left((\delta_\omega v)^2 + 2\,\delta_\omega v \partial_t^2 v + \,(\partial_t^2 v)^2\right) G \\
&\quad + \left(v\delta_\omega v + v\partial_t^2 v\right)\,\left(\partial_t^2 G - \partial_t G + 2G\right) + \left(\delta_\omega v\partial_t v + \partial_t^2 v\partial_t v\right)(2\partial_t G - 4G) \\
&\quad + (\partial_t v)^2\,(-6\partial_t G + 3G) + v\partial_t v\left(-3\partial_t^2 G + 7\partial_t G - 2G\right) + v^2\left(2\partial_t^2 G - 2\partial_t G\right)\Big)\,d\omega dt.
\end{aligned} \tag{2.5}$$

This, in turn, is equal to

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{S^2} \Big(G\,(\delta_\omega v)^2 - 2G\,\delta_\omega\partial_t v\partial_t v + G\,(\partial_t^2 v)^2 \\
&\quad + (\nabla_\omega v)^2\,\left(-\partial_t^2 G - (\partial_t^2 G - \partial_t G + 2G) + (\partial_t^2 G - 2\partial_t G)\right) \\
&\quad + (\partial_t v)^2\left(-(\partial_t^2 G - \partial_t G + 2G) + (-\partial_t^2 G + 2\partial_t G) + (-6\partial_t G + 3G)\right) \\
&\quad + v\partial_t v\left(-(\partial_t^3 G - \partial_t^2 G + 2\partial_t G) + (-3\partial_t^2 G + 7\partial_t G - 2G)\right) \\
&\quad + v^2\left(2\partial_t^2 G - 2\partial_t G\right)\Big)\,d\omega dt,
\end{aligned} \tag{2.6}$$

and integrating by parts once again we obtain (2.3). $\square$

In order to single out the term with v^2 in (2.3) we shall need the following auxiliary result.

Lemma 2.2 *Consider the equation*

$$\frac{d^4 g}{dt^4} + 2\frac{d^3 g}{dt^3} - \frac{d^2 g}{dt^2} - 2\frac{dg}{dt} = \delta, \tag{2.7}$$

where δ stands for the Dirac delta function. A unique solution to (2.7) which is bounded and vanishes at $+\infty$ is given by

$$g(t) = -\frac{1}{6}\begin{cases} e^t - 3, & t < 0, \\ e^{-2t} - 3\,e^{-t}, & t > 0. \end{cases} \tag{2.8}$$

Proof. Since the equation (2.7) is equivalent to

$$\frac{d}{dt}\left(\frac{d}{dt}+2\right)\left(\frac{d}{dt}+1\right)\left(\frac{d}{dt}-1\right)g=\delta, \tag{2.9}$$

a bounded solution of (2.7) vanishing at $+\infty$ must has the form

$$g(t)=\begin{cases} a\,e^{t}+b, & t<0,\\ c\,e^{-2t}+d\,e^{-t}, & t>0, \end{cases} \tag{2.10}$$

for some constants a,b,c,d. Once this is established, we find the system of coefficients so that $\partial_t^k g$ is continuous for $k=0,1,2$ and $\lim_{t\to 0^+}\partial_t^3 g(t)-\lim_{t\to 0^-}\partial_t^3 g(t)=1$. $\square$

With Lemma 2.2 at hand, a suitable choice of the function G yields the positivity of the left-hand side of (2.3), one of the cornerstones of this paper. The details are as follows.

Lemma 2.3 *Let Ω be a bounded domain in $\mathbb{R}^3$, $O\in\mathbb{R}^3\setminus\Omega$, $u\in C_0^\infty(\Omega)$ and $v=e^t(u\circ\varkappa^{-1})$. Then for every $\xi\in\Omega$ and $\tau=\log|\xi|^{-1}$ we have*

$$\frac{1}{2}\int_{S^{n-1}}v^2(\tau,\omega)\,d\omega\le\int_{\mathbb{R}^n}\Delta u(x)\Delta\Big(u(x)|x|^{-1}g(\log(|\xi|/|x|))\Big)\,dx, \tag{2.11}$$

where g is given by (2.8).

Proof. Representing v as a series of spherical harmonics and noting that the eigenvalues of the Laplace-Beltrami operator on the unit sphere are $k(k+1)$, $k=0,1,...$, we arrive at the inequality

$$\int_{S^2}|\delta_\omega v|^2\,d\omega\ge 2\int_{S^2}|\nabla_\omega v|^2\,d\omega. \tag{2.12}$$

Now, let us take $G(t)=g(t-\tau)$, $t\in\mathbb{R}$. Since $g\ge 0$, the combination of Lemma 2.2, (2.3) and (2.12) allows to obtain the estimate

$$\begin{aligned}&\int_{\mathbb{R}^n}\Delta u(x)\Delta\Big(u(x)|x|^{-1}g(\log(|\xi|/|x|))\Big)\,dx\\ &\quad\ge\int_{\mathbb{R}}\int_{S^{n-1}}\Big[-(\nabla_\omega v)^2\Big(\partial_t^2 g(t-\tau)+\partial_t g(t-\tau)\Big)\\ &\quad-(\partial_t v)^2\Big(2\partial_t^2 g(t-\tau)+3\partial_t g(t-\tau)-g(t-\tau)\Big)\Big]\,d\omega dt+\frac{1}{2}\int_{S^{n-1}}v^2(\tau,\omega)\,d\omega.\end{aligned} \tag{2.13}$$

Thus, the matters are reduced to showing that

$$\partial_t^2 g+\partial_t g\le 0\qquad\text{and}\qquad 2\partial_t^2 g+3\partial_t g-g\le 0. \tag{2.14}$$

Indeed, we compute

$$\partial_t g(t)=-\frac{1}{6}\begin{cases} e^{t}, & t<0,\\ -2e^{-2t}+3\,e^{-t}, & t>0, \end{cases} \tag{2.15}$$

and

$$\partial_t^2 g(t) = -\frac{1}{6} \begin{cases} e^t, & t < 0, \\ 4e^{-2t} - 3\,e^{-t}, & t > 0, \end{cases} \tag{2.16}$$

which gives

$$\partial_t^2 g(t) + \partial_t g(t) = -\frac{1}{3} \begin{cases} e^t, & t < 0, \\ e^{-2t}, & t > 0, \end{cases} \tag{2.17}$$

and

$$2\partial_t^2 g(t) + 3\partial_t g(t) - g(t) = -\frac{1}{6} \begin{cases} 4e^t + 3, & t < 0, \\ e^{-2t} + 6e^{-t}, & t > 0. \end{cases} \tag{2.18}$$

Clearly, both functions above are non-positive. $\square$

3 The local estimates and the Dirichlet problem

This section is devoted to estimates for a solution of the Dirichlet problem near a boundary point, in particular, the proof of Theorem 1.1. To set the stage, let us first record the well-known result following from the interior estimates for solutions of the elliptic equations.

Lemma 3.1 *Let Ω be an arbitrary domain in $\mathbb{R}^3$, $Q \in \mathbb{R}^3 \setminus \Omega$ and $R > 0$. Suppose*

$$\Delta^2 u = f \text{ in } \Omega, \quad f \in C_0^\infty(\Omega \setminus B_{4R}(Q)), \quad u \in \mathring{W}_2^2(\Omega). \tag{3.1}$$

Then

$$\int_{B_\rho(Q)\cap\Omega} |\nabla^2 u|^2\, dx + \frac{1}{\rho^2} \int_{B_\rho(Q)\cap\Omega} |\nabla u|^2\, dx \le \frac{C}{\rho^4} \int_{C_{\rho,2\rho}(Q)\cap\Omega} |u|^2\, dx \tag{3.2}$$

for every $\rho < 2R$.

Here and throughout the paper $B_r(Q)$ and $S_r(Q)$ denote, respectively, the ball and the sphere with radius r centered at Q and $C_{r,R}(Q) = B_R(Q) \setminus \overline{B_r(Q)}$. When center is at the origin, we will write B_r in place of $B_r(O)$, and similarly $S_r := S_r(O)$ and $C_{r,R} := C_{r,R}(O)$. Also, $\nabla^2 u$ stands for a vector of all second derivatives of u.

We omit the proof of Lemma 3.1 (see, e.g., [3], [22]) and proceed to the estimates for a biharmonic function near the boundary invoking the inequalities derived in §2.

Proposition 3.2 *Let Ω be a bounded domain in $\mathbb{R}^3$, $Q \in \mathbb{R}^3 \setminus \Omega$, and $R > 0$. Suppose*

$$\Delta^2 u = f \text{ in } \Omega, \quad f \in C_0^\infty(\Omega \setminus B_{4R}(Q)), \quad u \in \mathring{W}_2^2(\Omega). \tag{3.3}$$

Then

$$\frac{1}{\rho^4} \int_{S_\rho(Q)\cap\Omega} |u(x)|^2\, d\sigma_x \le \frac{C}{R^5} \int_{C_{R,4R}(Q)\cap\Omega} |u(x)|^2\, dx \quad \text{for every} \quad \rho < R, \tag{3.4}$$

where C is an absolute constant.

Proof. For notational convenience we assume that $Q = O$. Let us start approximating Ω by a sequence of domains with smooth boundaries $\{\Omega_n\}_{n=1}^{\infty}$ with the properties

$$\bigcup_{n=1}^{\infty} \Omega_n = \Omega \quad \text{and} \quad \overline{\Omega}_n \subset \Omega_{n+1} \quad \text{for every} \quad n \in \mathbb{N}. \tag{3.5}$$

Choose $n_0 \in \mathbb{N}$ such that $\operatorname{supp} f \subset \Omega_n$ for every $n \geq n_0$ and denote by u_n the solution of the Dirichlet problem

$$\Delta^2 u_n = f \quad \text{in} \quad \Omega_n, \quad u_n \in \mathring{W}_2^2(\Omega_n), \quad n \geq n_0. \tag{3.6}$$

Since $f \in C_0^{\infty}(\Omega_n)$ for every $n \geq n_0$, the solution to (3.6) exists and is unique. Moreover, $\{u_n\}_{n=n_0}^{\infty}$ converges to u in $\mathring{W}_2^2(\Omega)$ (see, e.g., [19], §6.6).

Next, take some $\eta \in C_0^{\infty}(B_{2R})$ such that

$$0 \leq \eta \leq 1 \text{ in } B_{2R}, \quad \eta = 1 \text{ in } B_R \quad \text{and} \quad |\nabla^k \eta| \leq C/|x|^k, \quad k \leq 4. \tag{3.7}$$

Also, fix $\tau = \log \rho^{-1}$ and let g be the function defined in (2.8).

Consider the difference

$$\int_{\mathbb{R}^3} \Delta\Big(\eta(x) u_n(x)\Big) \Delta\Big(\eta(x) u_n(x) |x|^{-1} g(\log(\rho/|x|))\Big)\, dx$$
$$- \int_{\mathbb{R}^3} \Delta u_n(x) \Delta\Big(u_n(x)|x|^{-1} g(\log(\rho/|x|)) \eta^2(x)\Big)\, dx. \tag{3.8}$$

One can view this expression as

$$\int_{\mathbb{R}^3} \Big([\Delta^2, \eta] u_n(x)\Big) \Big(\eta(x) u_n(x) |x|^{-1} g(\log(\rho/|x|))\Big)\, dx, \tag{3.9}$$

where the integral is understood in the sense of pairing between $\mathring{W}_2^2(\Omega_n)$ and its dual. Evidently, the support of the integrand is a subset of $\operatorname{supp} \nabla\eta \subset C_{R,2R}$, and therefore, the difference in (3.8) is bounded by

$$C \sum_{k=0}^{2} \frac{1}{R^{5-2k}} \int_{C_{R,2R}} |\nabla^k u_n(x)|^2\, dx. \tag{3.10}$$

Since u_n is biharmonic in B_{4R} and η is supported in B_{2R}, the second term in (3.8) is equal to zero. Turning to the first term, we shall employ Lemma 2.3 with $u = \eta\, u_n$. The result of the Lemma holds for such a choice of u. This can be seen directly by inspection of the argument or one can approximate each u_n by a sequence of $C_0^{\infty}(\Omega_n)$ functions in $\mathring{W}_2^2(\Omega_n)$ and then take a limit using that $O \notin \overline{\Omega}_n$. Then (3.8) is bounded from below by

$$\frac{C}{\rho^4} \int_{S_\rho} |\eta(x) u_n(x)|^2\, d\sigma_x. \tag{3.11}$$

Hence, for every $\rho < R$

$$\frac{1}{\rho^4} \int_{S_\rho} |u_n(x)|^2\, d\sigma_x \leq C \sum_{k=0}^{2} \frac{1}{R^{5-2k}} \int_{C_{R,2R}} |\nabla^k u_n(x)|^2\, dx. \tag{3.12}$$

Now the proof can be finished applying Lemma 3.1 and taking the limit as $n \to \infty$. $\square$

By virtue of the interior regularity of biharmonic functions (3.4) yields the pointwise estimates, leading to Theorem 1.1.

Corollary 3.3 *Let Ω be a bounded domain in $\mathbb{R}^3$, $Q \in \mathbb{R}^3 \setminus \Omega$, $R > 0$ and*

$$\Delta^2 u = f \ \text{in}\ \Omega, \quad f \in C_0^\infty(\Omega \setminus B_{4R}(Q)), \quad u \in \mathring{W}_2^2(\Omega). \tag{3.13}$$

Then for every $x \in B_{R/4}(Q) \cap \Omega$

$$|\nabla u(x)|^2 \leq \frac{C}{R^5} \int_{C_{R/4,4R}(Q)\cap\Omega} |u(y)|^2 \, dy, \tag{3.14}$$

and

$$|u(x)|^2 \leq C \frac{|x-Q|^2}{R^5} \int_{C_{R/4,4R}(Q)\cap\Omega} |u(y)|^2 \, dy. \tag{3.15}$$

In particular, for every bounded domain $\Omega \subset \mathbb{R}^3$ the solution to the boundary value problem (3.13) satisfies

$$|\nabla u| \in L^\infty(\Omega). \tag{3.16}$$

Proof. By the interior estimates for solutions of the elliptic equations (see [3])

$$|\nabla u(x)|^2 \leq \frac{C}{d(x)^3} \int_{B_{d(x)/2}(x)} |\nabla u(y)|^2 \, dy, \tag{3.17}$$

where $d(x)$ denotes the distance from x to $\partial\Omega$. Let x_0 be a point on the boundary of Ω such that $d(x) = |x - x_0|$. Since $x \in B_{R/4}(Q) \cap \Omega$ and $Q \in \mathbb{R}^3 \setminus \Omega$, we have $x \in B_{R/4}(x_0)$, and therefore

$$\frac{1}{d(x)^3} \int_{B_{d(x)/2}(x)} |\nabla u(y)|^2 \, dy \leq \frac{C}{d(x)^5} \int_{B_{2d(x)}(x_0)} |u(y)|^2 \, dy \leq \frac{C}{R^5} \int_{C_{3R/4,3R}(x_0)} |u(y)|^2 \, dy, \tag{3.18}$$

using Lemma 3.1 for the first estimate and (3.4) for the second one. Indeed, $d(x) \leq R/4$ and therefore, $2d(x) < 3R/4$. On the other hand, u is biharmonic in $B_{4R}(Q) \cap \Omega$ and

$$|Q - x_0| \leq |Q - x| + |x - x_0| \leq R/2. \tag{3.19}$$

Hence, u is biharmonic in $B_{3R}(x_0) \cap \Omega$ and Proposition 3.2 holds with x_0 in place of Q, $3R/4$ in place of R and $\rho = 2d(x)$. Finally, (3.19) yields

$$C_{3R/4,3R}(x_0) \subset C_{R/4,4R}(Q), \tag{3.20}$$

and that finishes the argument for (3.14).

To prove (3.15), we start with the estimate

$$|u(x)|^2 \leq \frac{C}{d(x)^3} \int_{B_{d(x)/2}(x)} |u(y)|^2 \, dy, \tag{3.21}$$

and then proceed using (3.4), much as in (3.18)–(3.20). $\square$

Using the Kelvin transform, an estimate on a biharmonic function near the origin can be translated into an estimate at infinity. In particular, Proposition 3.2 and Corollary 3.3 lead to the following result.

Proposition 3.4 *Let Ω be a bounded domain in $\mathbb{R}^3$, $Q \in \mathbb{R}^3 \setminus \Omega$, $r > 0$ and assume that*

$$\Delta^2 u = f \ \ in\ \Omega, \quad f \in C_0^\infty(B_{r/4}(Q) \cap \Omega), \quad u \in \mathring{W}_2^2(\Omega). \tag{3.22}$$

Then

$$\frac{1}{\rho^2} \int_{S_\rho(Q)\cap\Omega} |u(x)|^2 \, d\sigma_x \le \frac{C}{r^3} \int_{C_{r/4,r}(Q)\cap\Omega} |u(x)|^2 \, dx, \tag{3.23}$$

for any $\rho > r$.

Furthermore, for any $x \in \Omega \setminus B_{4r}(Q)$

$$|\nabla u(x)|^2 \le \frac{C}{|x-Q|^2 \, r^3} \int_{C_{r/4,4r}(Q)\cap\Omega} |u(y)|^2 \, dy, \tag{3.24}$$

and

$$|u(x)|^2 \le \frac{C}{r^3} \int_{C_{r/4,4r}(Q)\cap\Omega} |u(y)|^2 \, dy. \tag{3.25}$$

Proof. As before, it is enough to consider the case $Q = O$. Retain the approximation of Ω with the sequence of smooth domains Ω_n satisfying (3.5) and define u_n according to (3.6). We denote by $\mathcal{I}$ the inversion $x \mapsto y = x/|x|^2$ and by U_n the Kelvin transform of u_n,

$$U_n(y) := |y| \, u_n(y/|y|^2), \quad y \in \mathcal{I}(\Omega_n). \tag{3.26}$$

Then

$$\Delta^2 U_n(y) = |y|^{-7} (\Delta^2 u_n)(y/|y|^2), \tag{3.27}$$

and therefore, U_n is biharmonic in $\mathcal{I}(\Omega_n) \cap B_{4/r}$. Moreover, (3.27) implies that

$$\int_{\mathcal{I}(\Omega_n)} |\Delta U_n(y)|^2 \, dy = \int_{\Omega_n} |\Delta u_n(x)|^2 \, dx, \tag{3.28}$$

so that

$$U_n \in \mathring{W}_2^2(\mathcal{I}(\Omega_n)) \quad \Longleftrightarrow \quad u_n \in \mathring{W}_2^2(\Omega_n). \tag{3.29}$$

Observe also that Ω_n is a bounded domain with $O \notin \overline{\Omega}_n$, hence, so is $\mathcal{I}(\Omega_n)$ and $O \notin \overline{\mathcal{I}(\Omega_n)}$.

Following Proposition 3.2, we show that

$$\rho^4 \int_{S_{1/\rho}} |U_n(y)|^2 \, d\sigma_y \le C \, r^5 \int_{C_{1/r,4/r}} |U_n(y)|^2 \, dy, \tag{3.30}$$

which after the substitution (3.26) and the change of coordinates yields

$$\frac{1}{\rho^2} \int_{S_\rho} |u_n(x)|^2 \, d\sigma_x \le \frac{C}{r^3} \int_{C_{r/4,r}} |u_n(x)|^2 \, dx. \tag{3.31}$$

Turning to the pointwise estimates (3.24)–(3.25), let us fix some $x \in \Omega \setminus B_{4r}(Q)$. Observe that

$$|\nabla u_n(x)| \le C|x|^{-1} \left| (\nabla U_n)(x/|x|^2) \right| + \left| U_n(x/|x|^2) \right|, \tag{3.32}$$

since $u_n(x) = |x|\, U_n(x/|x|^2)$. Therefore, combining (3.32) and Corollary 3.3 applied to the function U_n, we deduce that

$$|\nabla u_n(x)|^2 \leq C\,\frac{r^5}{|x|^2} \int_{C_{1/(4r),4/r}} |U_n(z)|^2\,dz = \frac{C}{|x|^2\,r^3} \int_{C_{r/4,4r}} |u_n(z)|^2\,dz, \qquad (3.33)$$

and

$$|u_n(x)|^2 \leq Cr^5 \int_{C_{1/(4r),4/r}} |U_n(z)|^2\,dz = \frac{C}{r^3} \int_{C_{r/4,4r}} |u_n(z)|^2\,dz. \qquad (3.34)$$

At this point, we can use the limiting procedure to complete the argument. Indeed, since u_n converges to u in $\mathring{W}_2^2(\Omega)$, the integrals in (3.31), (3.33) and (3.34) converge to the corresponding integrals with u_n replaced by u. Turning to $|\nabla u_n(x)|$, we observe that both u_n and u are biharmonic in a neighborhood of x, in particular, for sufficiently small d

$$|\nabla(u_n(x) - u(x))|^2 \leq \frac{C}{d^5} \int_{B_{d/2}(x)} |u_n(z) - u(z)|^2\,dz. \qquad (3.35)$$

As $n \to \infty$, the integral on the right-hand side of (3.35) vanishes and therefore, $|\nabla u_n(x)| \to |\nabla u(x)|$. The similar considerations apply to $u_n(x)$. □

4 Estimates for Green's function

Let Ω be a bounded three-dimensional domain. As in the introduction, we denote by $G(x,y)$, $x, y \in \Omega$, the Green's function for the biharmonic equation. In other words, for every fixed $y \in \Omega$ the function $G(x,y)$ is the solution of the problem

$$\Delta_x^2 G(x,y) = \delta(x-y), \qquad x \in \Omega, \qquad (4.1)$$

in the space $\mathring{W}_2^2(\Omega)$. Here and throughout the section Δ_x stands for the Laplacian in x variable, and similarly we use the notation Δ_y, ∇_y, ∇_x for the Laplacian and gradient in y, and gradient in x, respectively. As before, $d(x)$ is the distance from $x \in \Omega$ to $\partial\Omega$.

Proposition 4.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Then for every $x, y \in \Omega$*

$$\Big|\nabla_x \nabla_y (G(x,y) - \Gamma(x-y))\Big| \leq \frac{C}{\max\{|x-y|,\, d(x),\, d(y)\}}, \qquad (4.2)$$

where $\Gamma(x-y) = \frac{|x-y|}{8\pi}$ is the fundamental solution for the bilaplacian. In particular,

$$|\nabla_x \nabla_y G(x,y)| \leq C|x-y|^{-1} \qquad \textit{for all} \qquad x, y \in \Omega. \qquad (4.3)$$

Proof. Let us start with some auxiliary calculations. Consider a function η such that

$$\eta \in C_0^\infty(B_{1/2}) \quad \text{and} \quad \eta = 1 \quad \text{in} \quad B_{1/4}, \qquad (4.4)$$

and define a vector-valued function $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3)$ by

$$\mathcal{R}_j(x,y) := \frac{\partial}{\partial y_j} G(x,y) - \eta\left(\frac{x-y}{d(y)}\right) \frac{\partial}{\partial y_j} \Gamma(x-y), \qquad x, y \in \Omega, \tag{4.5}$$

where $j = 1, 2, 3$. Also, let us denote

$$f_j(x,y) := \Delta_x^2 \mathcal{R}_j(x,y) = -\left[\Delta_x^2, \eta\left(\frac{x-y}{d(y)}\right)\right] \frac{\partial}{\partial y_j} \Gamma(x-y), \quad j = 1, 2, 3. \tag{4.6}$$

It is not hard to see that for every j

$$f_j(\cdot, y) \in C_0^\infty(C_{d(y)/4, d(y)/2}(y)) \qquad \text{and} \qquad |f_j(x,y)| \leq C d(y)^{-4}, \quad x, y \in \Omega. \tag{4.7}$$

Then for every fixed $y \in \Omega$ the function $x \mapsto \mathcal{R}_j(x,y)$ is a solution of the boundary value problem

$$\Delta_x^2 \mathcal{R}_j(x,y) = f_j(x,y) \text{ in } \Omega, \quad f_j(\cdot, y) \in C_0^\infty(\Omega), \quad \mathcal{R}_j(\cdot, y) \in \mathring{W}_2^2(\Omega), \tag{4.8}$$

so that

$$\left\|\nabla_x^2 \mathcal{R}_j(\cdot, y)\right\|_{L^2(\Omega)} = \|\mathcal{R}_j(\cdot, y)\|_{W_2^2(\Omega)} \leq C \|f_j(\cdot, y)\|_{W_{-2}^2(\Omega)}, \qquad j = 1, 2, 3. \tag{4.9}$$

Here $W_{-2}^2(\Omega)$ stands for the Banach space dual of $\mathring{W}_2^2(\Omega)$, i.e.

$$\|f_j(\cdot, y)\|_{W_{-2}^2(\Omega)} = \sup_{v:\, \|v\|_{\mathring{W}_2^2(\Omega)} = 1} \int_\Omega f_j(x,y) v(x)\, dx. \tag{4.10}$$

Recall that by Hardy's inequality

$$\left\|\frac{v}{|\cdot - Q|^2}\right\|_{L^2(\Omega)} \leq C \left\|\nabla^2 v\right\|_{L^2(\Omega)} \quad \text{for every} \quad v \in \mathring{W}_2^2(\Omega), \quad Q \in \partial\Omega. \tag{4.11}$$

Then for some $y_0 \in \partial\Omega$ such that $|y - y_0| = d(y)$

$$\begin{aligned}\int_\Omega f_j(x,y) v(x)\, dx &\leq C \left\|\frac{v}{|\cdot - y_0|^2}\right\|_{L^2(\Omega)} \left\|f_j(\cdot, y)|\cdot - y_0|^2\right\|_{L^2(\Omega)} \\ &\leq C d(y)^2 \left\|\nabla^2 v\right\|_{L^2(\Omega)} \|f_j(\cdot, y)\|_{L^2(C_{d(y)/4, d(y)/2}(y))},\end{aligned} \tag{4.12}$$

and therefore, by (4.7)

$$\left\|\nabla_x^2 \mathcal{R}(\cdot, y)\right\|_{L^2(\Omega)} \leq C d(y)^{-1/2}. \tag{4.13}$$

Turning to (4.2), let us first consider the case $|x - y| \geq N d(y)$ for some large N to be specified later. As before, we denote by y_0 some point on the boundary such that $|y - y_0| = d(y)$. Then by (4.7) the function $x \mapsto \mathcal{R}(x,y)$ is biharmonic in $\Omega \setminus B_{3d(y)/2}(y_0)$. Hence, by Proposition 3.4 with $r = 6d(y)$

$$|\nabla_x \mathcal{R}(x,y)|^2 \leq \frac{C}{|x - y_0|^2\, d(y)^3} \int_{C_{3d(y)/2, 24d(y)}(y_0)} |\mathcal{R}(z,y)|^2\, dz, \tag{4.14}$$

provided $|x-y| \geq 4r + d(y)$, i.e $N \geq 25$. The expression above is bounded by

$$\frac{Cd(y)}{|x-y_0|^2} \int_{C_{3d(y)/2,24d(y)}(y_0)} \frac{|\mathcal{R}(z,y)|^2}{|z-y_0|^4}\, dz \leq \frac{C\, d(y)}{|x-y_0|^2} \int_{\Omega} |\nabla_z^2 \mathcal{R}(z,y)|^2\, dz \leq \frac{C}{|x-y|^2}, \quad (4.15)$$

by Hardy's inequality and (4.13).

Now one can directly check that

$$|\nabla_x \nabla_y \Gamma(x,y)| \leq \frac{C}{|x-y|} \qquad \text{for all} \qquad x, y \in \Omega, \quad (4.16)$$

and combine it with (4.14)–(4.15) to deduce that

$$\Big|\nabla_x \nabla_y (G(x,y) - \Gamma(x-y))\Big| \leq \frac{C}{|x-y|} \qquad \text{whenever} \qquad |x-y| \geq N d(y). \quad (4.17)$$

We claim that this settles the case

$$|x-y| \geq N \min\{d(y), d(x)\}. \quad (4.18)$$

Indeed, if $d(y) \leq d(x)$, (4.17) gives the desired result and if $d(y) \geq d(x)$ and $|x-y| \geq N d(x)$, we employ the version of (4.17) with $d(x)$ in place of $d(y)$ which follows from the symmetry of the Green's function and the fundamental solution in x and y variables.

Next, assume that $|x-y| \leq N^{-1} d(y)$. For such x we have $\eta(\frac{x-y}{d(y)}) = 1$ and therefore

$$\frac{\partial}{\partial y_j} (G(x,y) - \Gamma(x-y)) = \mathcal{R}_j(x,y). \quad (4.19)$$

By the interior estimates for solutions of elliptic equations

$$|\nabla_x \mathcal{R}(x,y)|^2 \leq \frac{C}{d(y)^5} \int_{B_{d(y)/8}(x)} |\mathcal{R}(z,y)|^2\, dz, \quad (4.20)$$

since the function $\mathcal{R}$ is biharmonic in $B_{d(y)/8}(x) \subset B_{d(y)/4}(y)$. Now we bound the expression above by

$$\frac{C}{d(y)} \int_{B_{d(y)/4}(y)} \frac{|\mathcal{R}(z,y)|^2}{|z-y_0|^4}\, dz \leq \frac{C}{d(y)} \left\|\nabla_x^2 \mathcal{R}(\cdot, y)\right\|_{L^2(\Omega)}^2 \leq \frac{C}{d(y)^2}. \quad (4.21)$$

When $|x-y| \leq N^{-1} d(y)$, we have

$$(N-1)\, d(y) \leq N d(x) \leq (N+1)\, d(y), \quad (4.22)$$

i.e. $d(y) \approx d(x)$, and therefore (4.20)–(4.21) give the desired result. By symmetry, one can handle the case $|x-y| \leq N^{-1} d(x)$ and hence all $x, y \in \Omega$ such that

$$|x-y| \leq N^{-1} \max\{d(x),\, d(y)\}. \quad (4.23)$$

Finally, it remains to consider the situation when

$$|x-y| \approx d(x) \approx d(y), \quad (4.24)$$

or more precisely, when

$$N^{-1}\,d(x) \le |x-y| \le N d(x) \quad \text{and} \quad N^{-1}\,d(y) \le |x-y| \le N d(y). \tag{4.25}$$

In this case we use the biharmonicity of $x \mapsto G(x,y)$ in $B_{d(x)/(2N)}(x)$. By the interior estimates, with $x_0 \in \partial\Omega$ such that $|x-x_0| = d(x)$, we have

$$\begin{aligned}
|\nabla_x\nabla_y G(x,y)|^2 &\le \frac{C}{d(x)^5}\int_{B_{d(x)/(2N)}(x)} |\nabla_y G(z,y)|^2\,dz \\
&\le \frac{C}{d(x)^5}\int_{B_{d(x)/(2N)}(x)} |\nabla_y \Gamma(z-y)|^2\,dz + \frac{C}{d(x)}\int_{B_{2d(x)}(x_0)} \frac{|\mathcal{R}(z,y)|^2}{|z-x_0|^4}\,dz \\
&\le \frac{C}{d(x)^5}\int_{B_{d(x)/(2N)}(x)} |\nabla_y \Gamma(z-y)|^2\,dz + \frac{C}{d(x)}\int_{\Omega} |\nabla_z^2\mathcal{R}(z,y)|^2\,dz \\
&\le \frac{C}{d(x)^2} + \frac{C}{d(x)d(y)},
\end{aligned} \tag{4.26}$$

invoking Hardy's inequality and (4.13). In view of (4.24) this finishes the argument. □

The Green's function estimates proved in this section allow to investigate the solutions of the Dirichlet problem (1.1) for a wide class of data. For example, consider the boundary value problem

$$\Delta^2 u = \operatorname{div} f, \quad u \in \mathring{W}_2^2(\Omega), \tag{4.27}$$

where $f = (f_1, f_2, f_3)$ is some vector valued function. Then the solution satisfies the estimate

$$|\nabla u(x)| \le C\int_\Omega \frac{|f(y)|}{|x-y|}\,dy, \qquad x \in \Omega, \tag{4.28}$$

provided the integral on the right-hand side of (4.28) is finite.

Indeed, the integral representation formula

$$u(x) = \int_\Omega G(x,y)\operatorname{div} f(y)\,dy, \qquad x \in \Omega, \tag{4.29}$$

follows directly from the definition of the Green's function. It implies that

$$\nabla u(x) = \nabla_x \int_\Omega G(x,y)\operatorname{div} f(y)\,dy = \int_\Omega \nabla_x(\nabla_y G(x,y)\cdot f(y))\,dy, \tag{4.30}$$

and Proposition 4.1 leads to (4.28).

One can further observe that by the mapping properties of the fractional integral operator (4.28) yields the estimate

$$\|\nabla u\|_{L^\infty(\Omega)} \le C\|f\|_{L^{3/2,1}(\Omega)}, \tag{4.31}$$

where $L^{3/2,1}(\Omega)$ is the Lorentz space. In particular,

$$\|\nabla u\|_{L^\infty(\Omega)} \le C\|f\,\|_{L^p(\Omega)}, \quad p > 3/2, \tag{4.32}$$

whenever $f \in L^p(\Omega)$ for some $p > 3/2$.

5 Biharmonic capacity

This section will be devoted to basic properties of the biharmonic capacity of the type (1.8). A part of the results presented here and in §9 have been obtained in [18]. For the convenience of the reader we present a self-contained discussion.

To begin, we introduce the biharmonic capacity of a compactum K relative to some open set $\Omega \subset \mathbb{R}^3 \setminus \{O\}$, $K \subset \Omega$. To this end, recall that Π is the space of functions (1.7) equipped with some norm. For example, we can take

$$\|P\|_\Pi = \sqrt{b_0^2 + b_1^2 + b_2^2 + b_3^2}, \tag{5.1}$$

and $\Pi_1 := \{P \in \Pi : \|P\|_\Pi = 1\}$. A different norm in the space Π would yield an equivalent relative capacity.

Now fix some $P \in \Pi_1$. Then

$$\mathrm{Cap}_P\,(K, \Omega) := \inf \left\{ \int_\Omega (\Delta u(x))^2\, dx : \ u \in \mathring{W}_2^2(\Omega), \ u = P \text{ in a neighborhood of } K \right\}, \tag{5.2}$$

and

$$\mathrm{Cap}\,(K, \Omega) := \inf_{P \in \Pi_1} \mathrm{Cap}_P\,(K, \Omega). \tag{5.3}$$

Observe that in the introduction, for the sake of brevity, we dropped the reference to Ω. There we had $\Omega = \mathbb{R}^3 \setminus \{0\}$.

It follows directly from the definitions that the biharmonic capacity is monotone in the sense that for every $P \in \Pi_1$

$$K_1 \subseteq K_2 \subset \Omega \qquad \Longrightarrow \qquad \mathrm{Cap}_P\,(K_1, \Omega) \le \mathrm{Cap}_P\,(K_2, \Omega), \tag{5.4}$$

$$K \subset \Omega_1 \subseteq \Omega_2 \qquad \Longrightarrow \qquad \mathrm{Cap}_P\,(K, \Omega_1) \ge \mathrm{Cap}_P\,(K, \Omega_2), \tag{5.5}$$

and analogous statements hold for Cap in place of Cap_P.

We shall mostly be concerned with the case when a compactum is contained in some annulus centered at the origin for the reasons that will become apparent in the sequel. In such case, it will be convenient to work with an equivalent definition of capacity by means of the form

$$\Psi[u; \Omega] = \int_{\widetilde{\varkappa}(\Omega)} \left((\partial_r^2 v)^2 + 2r^{-2}(\partial_r v)^2 + 2r^{-2}|\partial_r \nabla_\omega v|^2 + r^{-4}(\delta_\omega v)^2 + 2r^{-4} v \delta_\omega v \right) r^2\, d\omega dr, \tag{5.6}$$

where (r, ω) are the spherical coordinates in the three dimensional space, $\widetilde{\varkappa}$ is the mapping

$$\mathbb{R}^3 \ni x \xrightarrow{\widetilde{\varkappa}} (r, \omega) \in [0, \infty) \times S^2, \tag{5.7}$$

and $v = u \circ \widetilde{\varkappa}^{-1}$.

Lemma 5.1 *For every* $0 < r < R < \infty$ *and every function* $u \in W_2^2(C_{r,R})$

$$\Psi[u; C_{r,R}] = \int_{C_{r,R}} \Bigg[(\Delta u)^2$$

$$- \frac{2}{|x|^4} \Big(x_i \frac{\partial}{\partial x_i} - 1 \Big) \left(\Big(x_j \frac{\partial u}{\partial x_j} + u \Big) \Big(|x|^2 \Delta u - x_i \frac{\partial}{\partial x_i} \Big(x_j \frac{\partial u}{\partial x_j} \Big) - u \Big) + u^2 \Big) \right) \Bigg] \, dx, \quad (5.8)$$

where, as customary, we imply summation on repeated indices. Furthermore, for every open set Ω *in* $\mathbb{R}^3 \setminus \{0\}$ *and every* $u \in \mathring{W}_2^2(\Omega)$

$$\Psi[u; \Omega] = \int_{\Omega} (\Delta u(x))^2 \, dx. \quad (5.9)$$

The formulas (5.8)–(5.9) can be checked directly using the representation of the Laplacian in spherical coordinates

$$\Delta u = r^{-2} \left(\partial_r (r^2 \partial_r) + \delta_\omega \right). \quad (5.10)$$

They give rise to an alternative definition of the biharmonic capacity. Indeed, if K is a compact subset of $\Omega \subset \mathbb{R}^3 \setminus \{0\}$, then for every $P \in \Pi_1$

$$\mathrm{Cap}_P\,(K, \Omega) = \inf\{\Psi[u; \Omega] : \ u \in \mathring{W}_2^2(\Omega), \ u = P \text{ in a neighborhood of } K\} \quad (5.11)$$

and an analogous equality holds for Cap in place of Cap_P.

Lemma 5.2 *Suppose* K *is a compactum in* $\overline{C_{s,as}}$ *for some* $s > 0$, $a > 1$. *Then for every* $P \in \Pi_1$

$$\mathrm{Cap}_P(K, \mathbb{R}^3 \setminus \{O\}) \approx \mathrm{Cap}_P(K, C_{s/2,2as}) \qquad \textit{and} \qquad \mathrm{Cap}_P(K, C_{s/2,2as}) \leq C s^{-1}, \quad (5.12)$$

with the constants are independent of s.

Proof. The inequality

$$\mathrm{Cap}_P(K, \mathbb{R}^3 \setminus \{O\}) \leq \mathrm{Cap}_P(K, C_{s/2,2as}) \quad (5.13)$$

is a consequence of the monotonicity property (5.5). As for the opposite one, we take $u \in \mathring{W}_2^2(\mathbb{R}^3 \setminus \{O\})$ such that $u = P$ in a neighborhood of K and

$$\mathrm{Cap}_P(K, \mathbb{R}^3 \setminus \{O\}) + \varepsilon > \int_{\mathbb{R}^3} |\Delta u(x)|^2 \, dx = \Psi[u; \mathbb{R}^3 \setminus \{O\}]. \quad (5.14)$$

Consider now the cut-off function

$$\zeta \in C_0^\infty(1/2, 2a), \ \zeta = 1 \text{ on } [3/4, 3a/2], \quad (5.15)$$

and let $w(x) := \zeta(|x|/s) u(x)$, $x \in \mathbb{R}^3$. Then

$$w \in \mathring{W}_2^2(C_{s/2,2as}) \quad \text{and} \quad w = P \text{ in a neighborhood of } K. \quad (5.16)$$

Hence,

$$\mathrm{Cap}_P(K, C_{s/2,2as}) \le \Psi[w; C_{s/2,2as}] \tag{5.17}$$

and

$$\Psi[w, C_{s/2,2as}] = \int_{s/2}^{2as} \int_{S^2} \Big((\partial_r^2(\zeta(r/s)v))^2 + 2r^{-2}(\partial_r(\zeta(r/s)v))^2$$

$$+2r^{-2}|\partial_r(\zeta(r/s)\nabla_\omega v)|^2 + r^{-4}\zeta^2(r/s)(\delta_\omega v)^2 + 2r^{-4}\zeta^2(r/s)v\delta_\omega v \Big) r^2\, d\omega dr$$

$$\le \Psi[v, C_{s/2,2as}], \tag{5.18}$$

using the properties of ζ and the one dimensional Hardy's inequality in the r variable. This finishes the proof of the first assertion in (5.12).

As for the second one, observe first that if $v(x) = u(sx)$, $x \in \mathbb{R}^3$, the functions u and v belong to $\mathring{W}_2^2(\mathbb{R}^3 \setminus \{O\})$ simultaneously, and $u = P$ in a neighborhood of K if and only if $v = P$ in a neighborhood of $s^{-1}K := \{x \in \mathbb{R}^3 : \, sx \in K\}$. Also,

$$\int_{\mathbb{R}^3} |\Delta v(x)|^2\, dx = \int_{\mathbb{R}^3} |\Delta_x u(sx)|^2\, dx = s \int_{\mathbb{R}^3} |\Delta_y u(y)|^2\, dy, \tag{5.19}$$

so that

$$s\mathrm{Cap}_P(K, \mathbb{R}^3 \setminus \{O\}) = \mathrm{Cap}_P(s^{-1}K, \mathbb{R}^3 \setminus \{O\}). \tag{5.20}$$

However, $s^{-1}K \subset \overline{C_{1,a}}$ and therefore by (5.12) the right-hand side of (5.20) is controlled by $\mathrm{Cap}_P(\overline{C_{1,a}}, \mathbb{R}^3 \setminus \{O\})$, uniformly in s. □

Lemma 5.3 *Assume that for some $s > 0$, $a > 1$ the function $u \in L^2(C_{s,as})$ is such that $\Psi[u; C_{s,as}] < \infty$. Then there exists $Q = Q(u, s, a) \in \Pi$ with the property*

$$\|u - Q\|^2_{L^2(C_{s,as})} \le Cs^4\Psi[u; C_{s,as}]. \tag{5.21}$$

Proof. Let us start with the expansion of u by means of spherical harmonics:

$$u = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} u_l^m(r) Y_l^m(\omega), \tag{5.22}$$

where Y_l^m are the spherical harmonic functions of the degree l (i.e. corresponding to the l-th eigenvalue of δ_ω) and order m. By Poincaré's inequality, for $l = 0, 1$, and the corresponding m there exist constants $\overline{u_l^m}$ such that

$$\int_s^{as} |u_l^m(r) - \overline{u_l^m}|^2\, dr \le Cs^2 \int_s^{as} |\partial_r u_l^m(r)|^2\, dr. \tag{5.23}$$

Let

$$Q(x) := \overline{u_0^m} + \overline{u_1^1}\,\frac{x_1}{|x|} + \overline{u_1^{-1}}\,\frac{x_2}{|x|} + \overline{u_1^0}\,\frac{x_3}{|x|}, \qquad x \in \mathbb{R}^3 \setminus \{O\}. \tag{5.24}$$

Then (5.23) yields (5.21). □

Proposition 5.4 *Suppose $s>0$, $a\geq 2$ and K is a compact subset of $\overline{C_{s,as}}$. Then for every $u\in L^2(C_{s,as})$ such that $\Psi[u;C_{s,as}]<\infty$ and $u=0$ in a neighborhood of K*

$$\frac{1}{s^3}\int_{C_{s,as}}|u(x)|^2\,dx\leq \frac{C}{\operatorname{Cap}(K,\mathbb{R}^3\setminus\{O\})}\;\Psi[u;C_{s,as}], \tag{5.25}$$

provided $\operatorname{Cap}(K,\mathbb{R}^3\setminus\{O\})>0$.

Proof. For the purposes of this argument let us take $\|P\|_\Pi:=\|P\|_{L^2(C_{1,a})}$ and let $\Pi_1:=\{P\in\Pi:\ \|P\|_\Pi=1\}$ with such a norm. This is an equivalent norm in the space Π and hence it yields the biharmonic capacity equivalent to the one defined in (5.1)–(5.2). We claim that for every $P\in\Pi_1$

$$\operatorname{Cap}_P(K,C_{s/2,2as})\leq Cs^{-4}\|P-u\|^2_{L^2(C_{s,as})}+C\Psi[u;C_{s,as}]. \tag{5.26}$$

To prove this, let us denote by $V_2^2(C_{s,as})$ a collection of functions on $C_{s,as}$ such that

$$\|u\|_{V_2^2(C_{s,as})}:=\left(\frac{1}{s^4}\int_{C_{s,as}}|u(x)|^2\,dx+\Psi[u;C_{s,as}]\right)^{1/2}, \tag{5.27}$$

is finite. One can construct an extension operator

$$\operatorname{Ex}:V_2^2(C_{s,as})\to V_2^2(C_{s/2,2as}) \tag{5.28}$$

with the operator norm independent of s satisfying the properties

$$\operatorname{Ex}u=u\text{ in }C_{s,as},\qquad \operatorname{Ex}P=P\text{ for every }P\in\Pi_1, \tag{5.29}$$

and such that if $u=0$ in some neighborhood of K intersected with $\overline{C_{s,as}}$ then $\operatorname{Ex}u$ vanishes in a neighborhood of K contained in $C_{s/2,2as}$. For example, one can start with the corresponding one-dimensional extension operator and then use the expansion (5.22) to define Ex.

Having this at hand, we define $w(x):=\zeta(|x|/s)(P(x)-\operatorname{Ex}u(x))$, $x\in C_{s/2,2as}$, where ζ is a function introduced in (5.15). Then w satisfies (5.16) and therefore $\operatorname{Cap}_P(K,C_{s/2,2as})$ is controlled by

$$\begin{aligned}\Psi[w;C_{s/2,2as}]&\leq\Psi[P-\operatorname{Ex}u;C_{s/2,2as}]=\Psi[\operatorname{Ex}(P-u);C_{s/2,2as}]\\&\leq Cs^{-4}\|P-u\|^2_{L^2(C_{s,as})}+C\Psi[P-u;C_{s,as}],\end{aligned} \tag{5.30}$$

where the first inequality is proved analogously to (5.18) and the second one follows from the mapping properties of Ex. Using that $\delta_\omega\omega_i=-2\omega_i$, $i=1,2,3$, one can directly check that

$$\Psi[P-u;C_{s,as}]=\Psi[u;C_{s,as}], \tag{5.31}$$

and obtain (5.26).

The next step is to pass from (5.26) to (5.25). Without loss of generality we may assume that $\|u\|_{L^2(C_{s,as})}=s^{3/2}$. Then the desired result reads as

$$\inf_{P\in\Pi_1}\operatorname{Cap}_P(K,C_{s/2,2as})\leq\Psi[u;C_{s,as}]. \tag{5.32}$$

Let $Q = Q(u, s, a)$ be a function in Π satisfying (5.21), and denote by C_0 the constant C in (5.21). First of all, the case

$$\Psi[u; C_{s,as}] \geq 1/(4C_0 s) \tag{5.33}$$

is trivial, since Lemma 5.2 guarantees that the right-hand side of (5.33) is bounded from below by the biharmonic capacity of K, modulo a multiplicative constant.

On the other hand,

$$\Psi[u; C_{s,as}] \leq 1/(4C_0 s) \qquad \Longrightarrow \qquad 2\|u - Q\|_{L^2(C_{s,as})} \leq \|u\|_{L^2(C_{s,as})}, \tag{5.34}$$

by (5.21) and our assumptions on u. This, in turn, implies that

$$\frac{s^{3/2}}{2} \leq \|Q\|_{L^2(C_{s,as})} \leq \frac{3s^{3/2}}{2}. \tag{5.35}$$

Finally, we choose

$$P := \frac{Q}{\|Q\|_{L^2(C_{1,a})}} = s^{3/2} \frac{Q}{\|Q\|_{L^2(C_{s,as})}}. \tag{5.36}$$

Then

$$\|u - P\|^2_{L^2(C_{s,as})} \leq 16\|u - Q\|^2_{L^2(C_{s,as})} \leq 16C_0 s^4 \Psi[u; C_{s,as}], \tag{5.37}$$

by (5.21). Combining (5.37) with (5.26), we complete the argument. □

6 1-regularity of a boundary point

Let Ω be a domain in $\mathbb{R}^3$ and consider the boundary value problem

$$\Delta^2 u = f \text{ in } \Omega, \quad f \in C_0^\infty(\Omega), \quad u \in \mathring{W}_2^2(\Omega). \tag{6.1}$$

We say that the point $Q \in \partial\Omega$ is *1-regular* (with respect to Ω) if for every $f \in C_0^\infty(\Omega)$ the gradient of the solution to (6.1) is continuous, i.e.

$$\nabla u(x) \to 0 \text{ as } x \to Q,\ x \in \Omega. \tag{6.2}$$

Otherwise, $Q \in \partial\Omega$ is called 1-irregular.

Observe that in the case $Q = O$ this definition coincides with the one given in the introduction.

In this section we would like to show that 1-regularity is a local property. In particular, while most of the statements in Sections 1–5 were confined to the case of a bounded domain, the proposition below will allow us to study 1-regularity with respect to any open set in $\mathbb{R}^3$.

Proposition 6.1 *Let Ω be a bounded domain in $\mathbb{R}^3$ and the point $Q \in \partial\Omega$ be 1-regular with respect to Ω. If Ω' is another domain with the property that $B_r(Q) \cap \Omega = B_r(Q) \cap \Omega'$ for some $r > 0$ then Q is 1-regular with respect to Ω'.*

The proof of the proposition rests on the ideas from [15]. It starts with the following result.

Lemma 6.2 *Let Ω be a bounded domain in $\mathbb{R}^3$ and the point $Q \in \partial\Omega$ be 1-regular with respect to Ω. Then*

$$\nabla u(x) \to 0 \text{ as } x \to Q,\ x \in \Omega, \tag{6.3}$$

for every $u \in \mathring{W}_2^2(\Omega)$ satisfying

$$\Delta^2 u = \sum_{\alpha:\,|\alpha|\le 2} \partial^\alpha f_\alpha \text{ in } \Omega, \quad f_\alpha \in L^2(\Omega) \cap C^\infty(\Omega), \quad f_\alpha = 0 \text{ in a neighborhood of } Q. \tag{6.4}$$

Proof. Take some $\eta \in C_0^\infty(\Omega)$ and let v be the solution of the Dirichlet problem

$$\Delta^2 v = \sum_{\alpha:\,|\alpha|\le 2} \partial^\alpha(\eta f_\alpha) \text{ in } \Omega, \quad v \in \mathring{W}_2^2(\Omega), \tag{6.5}$$

and $w := u - v \in W_2^2(\Omega)$. Since the point Q is 1-regular, the function v automatically satisfies (6.3) and it remains to consider w.

Since $f_\alpha = 0$ in a neighborhood of Q, the function w is biharmonic in some neighborhood of Q and, therefore, for some $R > 0$ depending on the supp f_α, we have

$$|\nabla w(x)| \le \frac{C}{d(x)^3} \int_{B_{d(x)/2}(x)} |\nabla w(y)|^2\,dy \le \frac{C}{R^5} \int_{C_{R/4,4R}(Q)} |w(y)|^2\,dy, \quad \forall\, x \in B_{R/4}(Q), \tag{6.6}$$

analogously to (3.17)–(3.18). On the other hand, according to Lemma 2.3 the last expression in (6.6) is controlled by

$$\begin{aligned} &C \sup_{\xi \in C_{R/4,4R}(Q)\cap\Omega} \int_{\mathbb{R}^n} \Delta w(y) \Delta\Big(\frac{w(y)}{|x-Q|}\, g\Big(\log\frac{|\xi - Q|}{|y-Q|}\Big)\Big)\,dy \\ &\le C \sup_{\xi \in C_{R/4,4R}(Q)\cap\Omega} \sum_{\alpha:\,|\alpha|\le 2} \int_{\mathbb{R}^n} (1-\eta(y)) f_\alpha(y) (-\partial_y)^\alpha \Big(\frac{w(y)}{|y-Q|}\, g\Big(\log\frac{|\xi - Q|}{|y-Q|}\Big)\Big)\,dy, \end{aligned} \tag{6.7}$$

where g is given by (2.8). The integral in (6.7) is controlled in terms of the size of supp $(1-\eta)$. The latter can be chosen arbitrarily small as x approaches Q and, thus, $|\nabla w(x)| \to 0$ when $x \to Q$. □

Proof of Proposition 6.1. Consider the solution of the Dirichlet problem

$$\Delta^2 u = f \text{ in } \Omega', \quad f \in C_0^\infty(\Omega'), \quad u \in \mathring{W}_2^2(\Omega'), \tag{6.8}$$

and take some cut-off function $\eta \in C_0^\infty(B_r(Q))$ equal to 1 on $B_{r/2}(Q)$. Then $\eta u \in \mathring{W}_2^2(\Omega)$ and

$$\Delta^2(\eta u) = \eta f + [\Delta^2, \eta] u. \tag{6.9}$$

Since $\eta f \in C_0^\infty(\Omega)$,

$$[\Delta^2, \eta] : \mathring{W}_2^2(\Omega) \longrightarrow (\mathring{W}_2^2(\Omega))^* = W_{-2}^2(\Omega) \quad \text{and} \quad \text{supp}\,([\Delta^2, \eta] u) \subset C_{r/2,r}(Q) \cap \Omega, \tag{6.10}$$

one can write

$$\Delta^2(\eta u) = \sum_{\alpha:\,|\alpha|\leq 2} \partial^\alpha f_\alpha, \quad \text{for some} \quad f_\alpha \in L^2(\Omega)\cap C^\infty(\Omega), \tag{6.11}$$

with $f_\alpha = 0$ in a neighborhood of Q given by the intersection of $B_{r/2}(Q)$ and the complement to supp f. Then, by Lemma 6.2, the gradient of ηu (and therefore, the gradient of u) vanishes as $x \to Q$. □

7 Sufficient condition for 1-regularity

In the following two sections we investigate the necessary and sufficient conditions for the 1-regularity of a point $Q \in \partial\Omega$. As before, by applying a simple shift of coordinates, we can assume that $O \notin \Omega$, and thus we can take $Q = O$. This will be convenient since the origin plays a special role in our definition of the biharmonic capacity.

The following proposition provides the first part of Theorem 1.2, i.e. sufficiency of condition (1.9) for 1-regularity of a boundary point.

Proposition 7.1 *Let Ω be a bounded domain in $\mathbb{R}^3$, $O \in \mathbb{R}^3 \setminus \Omega$, $R > 0$ and*

$$\Delta^2 u = f \text{ in } \Omega, \quad f \in C_0^\infty(\Omega \setminus B_{4R}), \quad u \in \mathring{W}_2^2(\Omega). \tag{7.1}$$

Fix some $a \geq 4$. Then for every $x \in B_{R/a^4} \cap \Omega$

$$\begin{aligned} |\nabla u(x)|^2 + \frac{|u(x)|^2}{|x|^2} \quad &\leq \quad \frac{C}{R^5} \int_{C_{R,4R}\cap\Omega} |u(y)|^2\, dy \\ &\quad \times \exp\left(- c \int_{a^2|x|}^{R/a^2} \text{Cap}\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds \right). \end{aligned} \tag{7.2}$$

In particular, when O is a boundary point of Ω,

$$\text{if} \int_0^{R/a^2} \text{Cap}\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds = +\infty \text{ then } O \text{ is 1-regular.} \tag{7.3}$$

Proof. Fix $s \leq R/a^2$ and let us introduce some extra notation. First,

$$\gamma(s) := \text{Cap}\,(\overline{C_{s,a^2 s}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}). \tag{7.4}$$

Further, let $Q_\tau[u;\Omega]$, $\tau \in \mathbb{R}$, be the quadratic form

$$\begin{aligned} Q_\tau[u;\Omega] \quad := \quad & \int_{\varkappa(\Omega)} \Big[(\delta_\omega v)^2 g(t-\tau) + 2(\partial_t \nabla_\omega v)^2 g(t-\tau) + (\partial_t^2 v)^2 g(t-\tau) \\ & -(\nabla_\omega v)^2 \Big(\partial_t^2 g(t-\tau) + \partial_t g(t-\tau) + 2g(t-\tau) \Big) \\ & -(\partial_t v)^2 \Big(2\partial_t^2 g(t-\tau) + 3\partial_t g(t-\tau) - g(t-\tau) \Big) \Big]\, d\omega dt, \end{aligned} \tag{7.5}$$

where $v = e^t(u \circ \varkappa^{-1})$, g is defined by (2.8) and $\varkappa$ is the change of coordinates (2.2). Throughout this proof $\tau = \log s^{-1}$.

Now take $\eta \in C_0^\infty(B_{2s})$ such that

$$0 \le \eta \le 1 \text{ in } B_{2s}, \quad \eta = 1 \text{ in } B_s \quad \text{and} \quad |\nabla^k \eta| \le C/|x|^k, \quad k \le 4. \tag{7.6}$$

Following the argument in (3.8)–(3.10) and the discussion after (3.10), and then passing to the limit as $n \to \infty$, we have

$$\begin{aligned} Q_\tau[u; B_s] &\le Q_\tau[\eta u; \Omega] \le \int_{\mathbb{R}^3} \Delta\Big(\eta(x)u(x)\Big)\Delta\Big(\eta(x)u(x)|x|^{-1}g(\log(s/|x|))\Big)\,dx \\ &\le C\sum_{k=0}^{2} \frac{1}{s^{5-2k}} \int_{C_{s,2s}} |\nabla^k u(x)|^2\,dx \le \frac{C}{s^5}\int_{C_{s,4s}} |u(x)|^2\,dx. \end{aligned} \tag{7.7}$$

Denote

$$\varphi(s) := \sup_{|x|\le s}\left(|\nabla u(x)|^2 + \frac{|u(x)|^2}{|x|^2}\right) + Q_\tau[u; B_s], \qquad \tau = \log s^{-1}, \qquad s \le \frac{R}{a^2}. \tag{7.8}$$

Then combining (7.7) with Corollary 3.3 and Proposition 3.2,

$$\varphi(s) \le \frac{C}{s^5}\int_{C_{s,16s}} |u(x)|^2\,dx \le \frac{C}{s^5}\int_{C_{s,a^2s}} |u(x)|^2\,dx. \tag{7.9}$$

For $\gamma(s) > 0$ the expression on the right-hand side of (7.9) is further controlled by

$$\frac{C}{s^3}\int_{C_{s,a^2s}} \frac{|u(x)|^2}{|x|^2}\,dx \le \frac{C}{\gamma(s)}\,\Psi\left[\frac{u}{|x|}; C_{s,a^2s}\right] \le \frac{C}{s\gamma(s)}\,Q_\tau[u; C_{s,a^2s}], \tag{7.10}$$

where we used Proposition 5.4 for the first inequality. The second one can be proved directly using that $e^{-\tau} = s$ and calculations from the proof of Lemma 2.3. All in all,

$$\varphi(s) \le \frac{C}{s\gamma(s)}\,Q_\tau[u; C_{s,a^2s}] \le \frac{C}{s\gamma(s)}\left(\varphi(a^2 s) - \varphi(s)\right), \tag{7.11}$$

which, in turn, implies that

$$\varphi(s) \le \frac{1}{1 + C^{-1}\,s\gamma(s)}\,\varphi(a^2 s) \le \exp\left(-cs\gamma(s)\right)\,\varphi(a^2 s), \tag{7.12}$$

since $s\gamma(s)$ is bounded by (5.12). In particular, employing (7.12) for $s = a^{-2j}r$, $r \le R$, $j \in \mathbb{N}$, one can conclude that

$$\varphi(a^{-2l}r) \le \exp\left(-c\sum_{j=1}^{l} a^{-2j}r\,\gamma(a^{-2j}r)\right)\varphi(r), \tag{7.13}$$

for all $l \in \mathbb{N}$.

Let us choose $l \in \mathbb{N}$ so that

$$a^{-2l-4}R \le |x| \le a^{-2l-2}R. \tag{7.14}$$

Next, observe that by (5.4)

$$\begin{aligned}
&\int_{a^2|x|}^{R/a^2} \mathrm{Cap}\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds \\
&\quad \le \sum_{j=1}^{l}\sum_{k=1}^{2} \int_{a^{-2j+k-3}\,R}^{a^{-2j+k-2}\,R} \mathrm{Cap}\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds \\
&\quad \le C \sum_{j=1}^{l}\sum_{k=1}^{2} a^{-2j+k-3}\,R\,\mathrm{Cap}\,(\overline{C_{a^{-2j+k-3}\,R, a^{-2j+k-1}\,R}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}) \\
&\quad \le C \sum_{j=1}^{l}\sum_{k=1}^{2} a^{-2j} r_k\, \gamma(a^{-2j} r_k),
\end{aligned} \tag{7.15}$$

where $r_k = a^{k-3}\,R$, $k = 1, 2$. Using (7.13) with $r = r_k$, we deduce that for every $x \in B_{R/a^4} \cap \Omega$ and l defined by (7.14)

$$|\nabla u(x)|^2 + \frac{|u(x)|^2}{|x|^2} \le \varphi(a^{-2l} r_k) \le \exp\left(-c \sum_{j=1}^{l} a^{-2j} r_k\, \gamma(a^{-2j} r_k)\right) \varphi(R/a), \quad k = 1, 2, \tag{7.16}$$

which implies

$$\begin{aligned}
|\nabla u(x)|^2 + \frac{|u(x)|^2}{|x|^2} &\le \exp\left(-c \sum_{j=1}^{l}\sum_{k=1}^{l} a^{-2j} r_k\, \gamma(a^{-2j} r_k)\right) \varphi(R/a) \\
&\le \exp\left(-\,c \int_{a^2|x|}^{R/a^2} \mathrm{Cap}\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds\right) \varphi(R/a),
\end{aligned} \tag{7.17}$$

by (7.15).

Finally, analogously to (7.7)–(7.9)

$$\begin{aligned}
\varphi(R/a) \;&\le\; \sup_{|x| \le R/a} \left(|\nabla u(x)|^2 + \frac{|u(x)|^2}{|x|^2}\right) + C \sum_{k=0}^{2} \int_{C_{R/a,2R/a}} \frac{|\nabla^k u(x)|^2}{|x|^{5-2k}}\, dy \\
&\le\; \frac{C}{R^5} \int_{C_{R/a,16R/a}} |u(y)|^2\, dy \le \frac{C}{R^5} \int_{C_{R,4R}} |u(y)|^2\, dy,
\end{aligned} \tag{7.18}$$

using Proposition 3.2 for the last inequality. Combining (7.17) and (7.18), we finish the proof of (7.2). The statement (7.3) is a direct consequence of (7.2) and the definition of the 1-regularity. $\square$

Given the result of Proposition 7.1, we can derive the estimates for biharmonic functions at infinity as well as those for the Green's function in terms of the biharmonic capacity of the complement of Ω, in the spirit of (7.2).

Proposition 7.2 *Let Ω be a bounded domain in $\mathbb{R}^3$, $O \in \mathbb{R}^3 \setminus \Omega$, $r > 0$ and assume that*

$$\Delta^2 u = f \ \text{in}\ \Omega, \quad f \in C_0^\infty(B_{r/4} \cap \Omega), \quad u \in \mathring{W}_2^2(\Omega). \tag{7.19}$$

Fix some $a \geq 4$. Then for any $x \in \Omega \setminus B_{a^4 r}$

$$\begin{aligned} |\nabla u(x)|^2 + \frac{|u(x)|^2}{|x|^2} \quad \leq \quad & \frac{C}{|x|^2 r^3} \int_{C_{\frac{r}{4},r} \cap \Omega} |u(y)|^2 \, dy \\ & \times \exp\left(- c \int_{a^2 r}^{|x|/a^2} \operatorname{Cap}(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}) \, ds \right). \end{aligned} \tag{7.20}$$

Proof. Recall the proof of Proposition 3.4. With the notation (3.26) the results (3.27)–(3.29), (3.32) allow to apply Proposition 7.1 to U_n, $R = 1/r$, in order to write

$$\begin{aligned} |\nabla u_n(x)|^2 + \frac{|u_n(x)|^2}{|x|^2} &\leq C \, \frac{\left|(\nabla U_n)(x/|x|^2)\right|^2}{|x|^2} + \left|U_n(x/|x|^2)\right|^2 \\ &\leq C \, \frac{r^5}{|x|^2} \int_{C_{\frac{1}{r},\frac{4}{r}}} |U_n(z)|^2 \, dz \ \times \exp\left(- c \int_{a^2/|x|}^{1/(a^2 r)} \operatorname{Cap}(\overline{C_{s,as}} \setminus \mathcal{I}(\Omega_n), \mathbb{R}^3 \setminus \{O\}) \, ds \right) \\ &\leq \frac{C}{|x|^2 r^3} \int_{C_{\frac{r}{4},r}} |u_n(z)|^2 \, dz \ \times \exp\left(- c \int_{a^2 r}^{|x|/a^2} \operatorname{Cap}(\overline{C_{\frac{1}{s},\frac{a}{s}}} \setminus \mathcal{I}(\Omega_n), \mathbb{R}^3 \setminus \{O\}) \, \frac{ds}{s^2} \right). \end{aligned}$$

We claim that

$$\operatorname{Cap}(\overline{C_{\frac{1}{s},\frac{a}{s}}} \setminus \mathcal{I}(\Omega_n), \mathbb{R}^3 \setminus \{O\}) \approx s^2 \operatorname{Cap}(\overline{C_{s,as}} \setminus \Omega_n, \mathbb{R}^3 \setminus \{O\}), \tag{7.21}$$

where the implicit constants are independent of s.

Indeed,

$$\operatorname{Cap}(\overline{C_{\frac{1}{s},\frac{a}{s}}} \setminus \mathcal{I}(\Omega_n), \mathbb{R}^3 \setminus \{O\}) \approx \operatorname{Cap}(\overline{C_{\frac{1}{s},\frac{a}{s}}} \setminus \mathcal{I}(\Omega_n), C_{\frac{1}{2s},\frac{2a}{s}}), \tag{7.22}$$

and for every $u \in \mathring{W}_2^2(C_{\frac{1}{2s},\frac{2a}{s}})$ the function $y \mapsto |y| \, u(y/|y|^2)$ belongs to $\mathring{W}_2^2(C_{\frac{s}{2a},2s})$ by (3.29) and therefore, if $U(y) := u(y/|y|^2)$ then $U \in \mathring{W}_2^2(C_{\frac{s}{2a},2s})$. In addition, if $u = P$ in a neighborhood of $\overline{C_{\frac{1}{s},\frac{a}{s}}} \setminus \mathcal{I}(\Omega_n)$ then $U(y) = P(y/|y|^2) = P(y)$ for all y in the corresponding neighborhood of $\overline{C_{\frac{s}{a},s}} \setminus \Omega_n$. Finally, by (3.28)

$$\int_{C_{\frac{1}{2s},\frac{2a}{s}}} |\Delta u(x)|^2 \, dx = \int_{C_{\frac{s}{2a},2s}} |\Delta(|y| \, u(y/|y|^2))|^2 \, dy \approx s^2 \int_{C_{\frac{s}{2a},2s}} |\Delta U(y)|^2 \, dy, \tag{7.23}$$

since $u \in \mathring{W}_2^2(C_{\frac{1}{2s},\frac{2a}{s}})$. This proves the "$\geq$" inequality in (7.21). The opposite one can be derived by the same method.

All in all,

$$
\begin{aligned}
&|\nabla u_n(x)|^2 + \frac{|u_n(x)|^2}{|x|^2} \\
&\qquad \leq \frac{C}{|x|^2\, r^3} \int_{C_{\frac{r}{4},r}} |u_n(z)|^2\, dz \ \times \exp\left(-c \int_{a^2 r}^{|x|/a^2} \operatorname{Cap}\,(\overline{C_{s,as}} \setminus \Omega_n, \mathbb{R}^3 \setminus \{O\})\, ds \right) \\
&\qquad \leq \frac{C}{|x|^2\, r^3} \int_{C_{\frac{r}{4},r}} |u_n(z)|^2\, dz \ \times \exp\left(-c \int_{a^2 r}^{|x|/a^2} \operatorname{Cap}\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds \right), \quad (7.24)
\end{aligned}
$$

using the monotonicity property (5.4). Now the argument can be finished using the limiting procedure similar to the one in Proposition 3.4. $\square$

The following Proposition is a more precise version of the estimate on Green's function we announced in the introduction after Theorem 1.2.

Proposition 7.3 *Let Ω be a bounded domain in $\mathbb{R}^3$, $O \in \partial\Omega$. Fix some $a \geq 4$ and let $c_a := 1/(32a^4)$. Then*

$$
\begin{aligned}
&|\nabla_x \nabla_y G(x,y)| \\
&\quad \leq \begin{cases}
\frac{C}{|x-y|} \times \exp\left(-c \int_{32a^2|y|}^{|x|/a^2} \operatorname{Cap}\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds \right), & \text{if} \quad |y| \leq c_a|x|, \\
\frac{C}{|x-y|} \times \exp\left(-c \int_{32a^2|x|}^{|y|/a^2} \operatorname{Cap}\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds \right), & \text{if} \quad |x| \leq c_a|y|, \\
\frac{C}{|x-y|}, & \text{if} \quad c_a|y| \leq |x| \leq c_a^{-1}|y|.
\end{cases}
\end{aligned}
$$

Proof. The estimate for the case $c_a|y| \leq |x| \leq c_a^{-1}|y|$ was proved in Proposition 4.1, and the bound for $|x| \leq c_a|y|$ follows from the one for $|y| \leq c_a|x|$ by the symmetry of the Green's function. Hence, it is enough to consider the case $|y| \leq c_a|x|$ only.

The function $x \mapsto \nabla_y G(x,y)$ is biharmonic in $\Omega \setminus \{y\}$. We use Proposition 7.2 with $r = 32|y|$ to write

$$
\begin{aligned}
|\nabla_x \nabla_y G(x,y)|^2 \ &\leq \ \frac{C}{|x|^2\, |y|^3} \int_{C_{8|y|,32|y|}} |\nabla_y G(z,y)|^2\, dz \\
&\qquad \times \exp\left(-c \int_{32a^2|y|}^{|x|/a^2} \operatorname{Cap}\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds \right), \quad (7.25)
\end{aligned}
$$

for $x \in \Omega \setminus B_{c_a^{-1}|y|}$. Recall now the function $\mathcal{R}$ introduced in the proof of Proposition 4.1. If y_0 is a point on $\partial\Omega$ such that $|y - y_0| = d(y)$, then

$$
C_{8|y|,32|y|} \subset C_{6|y|,34|y|}(y_0), \quad (7.26)
$$

and $\nabla_y G(z,y) = \mathcal{R}(z,y)$ for every $z \in C_{6|y|,34|y|}(y_0)$. Therefore,

$$
\begin{aligned}
\frac{1}{|x|^2\, |y|^3} \int_{C_{8|y|,32|y|}} |\nabla_y G(z,y)|^2\, dz &\leq \frac{1}{|x|^2\, |y|^3} \int_{C_{6|y|,34|y|}(y_0)} |\mathcal{R}(z,y)|^2\, dz \\
&\leq \frac{C}{|x|^2\, d(y)^3} \int_{C_{3d(y)/2,6d(y)}(y_0)} |\mathcal{R}(z,y)|^2\, dz \leq \frac{C}{|x|^2} \leq \frac{C}{|x-y|^2}. \quad (7.27)
\end{aligned}
$$

The second inequality above follows from Proposition 3.4, the third one has been proved in (4.14)–(4.15) and the last one owes to the observation that

$$|x-y| \leq |x| + |y| \leq (1+c_a)|x| \quad \text{whenever} \quad |y| \leq c_a|x|. \tag{7.28}$$

Combining (7.25)–(7.27), we finish the proof. $\square$

8 Necessary condition for 1-regularity

This section will be entirely devoted to the proof of the second part of Theorem 1.2, i.e. the necessary condition for 1-regularity. We recall that $\mathrm{Cap}_P\,(K) = \mathrm{Cap}_P\,(K, \mathbb{R}^3 \setminus \{0\})$ for any compactum K by definition, and begin with

Step I: the set-up. Suppose that for some $P \in \Pi_1$, $c > 0$, $a \geq 8$ the integral in (1.10) is convergent. Then for every $\varepsilon > 0$ there exists a number $N \in \mathbb{N}$ such that

$$\int_0^{2^{-N}} \mathrm{Cap}_P\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds < \varepsilon. \tag{8.1}$$

However, the integral above is bounded from below by

$$\begin{aligned}\sum_{j=N}^{\infty} \int_{2^{-j-1}}^{2^{-j}} \mathrm{Cap}_P\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds &\geq C \sum_{j=N}^{\infty} 2^{-j} \mathrm{Cap}_P\,(\overline{C_{2^{-j},2^{-j-1}a}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}) \\ &\geq C \sum_{j=N}^{\infty} 2^{-j} \mathrm{Cap}_P\,(\overline{C_{2^{-j},2^{-j+2}}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}),\end{aligned} \tag{8.2}$$

using the monotonicity property (5.4). Therefore, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sum_{j=N}^{\infty} 2^{-j}\, \mathrm{Cap}_P\,(\overline{C_{2^{-j},2^{-j+2}}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}) < \varepsilon. \tag{8.3}$$

Now let $K := \overline{B_{2^{-N}}} \setminus \Omega$ and $D := \mathbb{R}^3 \setminus K$. We shall prove that the point O is not 1-regular with respect to D, and therefore with respect to Ω, since D coincides with Ω in a fixed neighborhood of O (see Proposition 6.1).

To this end, fix $P \in \Pi_1$ and let $\mathbb{P}(x) := |x|P(x)$, $x \in \mathbb{R}^3$. Then take some cut-off $\eta \in C_0^\infty(B_2)$ equal to 1 on $B_{3/2}$ and denote $f := -[\Delta^2, \eta]\mathbb{P} \in C_0^\infty(B_2 \setminus B_{3/2})$. Finally, let V be a solution of the boundary value problem

$$\Delta^2 V = f \text{ in } D, \quad V \in \mathring{W}_2^2(D). \tag{8.4}$$

Our goal is to show that $|\nabla V|$ does not vanish as $x \to O$, $x \in D$.

Let us also consider the function $U := V + \eta\, \mathbb{P}$. One can check that

$$\Delta^2 U = 0 \text{ in } D, \quad U = \mathbb{P} \text{ on } K, \quad U \in \mathring{W}_2^2(\mathbb{R}^3). \tag{8.5}$$

Therefore, U can be seen as a version of a biharmonic potential. In fact, it is (8.5) that gave an original idea for the above definition of V.

Step II: the main identity. Let $\mathcal{B}$ denote the bilinear form associated to the quadratic form in (2.3), i.e.

$$\begin{aligned}\mathcal{B}(v,w) \;=\; & \int_{\mathbb{R}}\int_{S^2}\Big[(\delta_\omega v)(\delta_\omega w)\,G + 2(\partial_t\nabla_\omega v)\cdot(\partial_t\nabla_\omega w)\,G + (\partial_t^2 v)(\partial_t^2 w)\,G \\ & -(\nabla_\omega v)\cdot(\nabla_\omega w)\Big(\partial_t^2 G + \partial_t G + 2G\Big) - (\partial_t v)(\partial_t w)\Big(2\partial_t^2 G + 3\partial_t G - G\Big) \\ & +\frac{1}{2}\,v\,w\Big(\partial_t^4 G + 2\partial_t^3 G - \partial_t^2 G - 2\partial_t G\Big)\Big]\,d\omega dt. \qquad (8.6)\end{aligned}$$

As before, we fix some point $\xi \in \mathbb{R}^3$, $\tau := \log|\xi|^{-1}$ and let $G(t) = g(t-\tau)$, $t \in \mathbb{R}$. By $\mathcal{B}_\tau(v,w)$ we shall denote $\mathcal{B}(v,w)$ for this particular choice of G. Then

$$\begin{aligned}&\int_{\mathbb{R}^3}\Delta U(x)\Delta\Big(\mathbb{P}(x)|x|^{-1}\,g(\log(|\xi|/|x|))\Big)\,dx \\ &\qquad + \int_{\mathbb{R}^3}\Delta\mathbb{P}(x)\Delta\Big(U(x)|x|^{-1}\,g(\log(|\xi|/|x|))\Big)\,dx = 2\mathcal{B}_\tau(u,q), \qquad (8.7)\end{aligned}$$

where $u = e^t(U\circ\varkappa^{-1})$ and $q = e^t(\mathbb{P}\circ\varkappa^{-1}) = P\circ\varkappa^{-1}$.

The identity above can be proved along the lines of the argument for Lemma 2.1, as long as the integration by parts and absence of the boundary terms is justified. To this end, we note that for any fixed $\xi \in \mathbb{R}^3$ the function $x \mapsto g(\log(|\xi|/|x|))$ is bounded by a constant as $|x| \to \infty$, while $x \mapsto |x|^{-1}\,g(\log(|\xi|/|x|))$ is bounded by a constant as $x \to O$. If $v_s \in C_0^\infty(D)$, $s \in \mathbb{N}$, is a collection of functions approximating V in the $\mathring{W}_2^2(D)$-norm, we let $u_s := v_s + \eta\mathbb{P}$. Then u_s converges to U in $\mathring{W}_2^2(\mathbb{R}^3)$. This, combined with the above observation about the behavior of g, shows that it suffices to prove (8.7) for u_s in place of U. Finally, since u_s is compactly supported in $\mathbb{R}^3$ and is equal to $\mathbb{P}$ in a neighborhood of 0, it is a matter of direct calculation to establish (8.7).

Since $\frac{|x|}{8\pi}$ is the fundamental solution of the bilaplacian,

$$\Delta^2\mathbb{P}(x) = \Delta^2(b_0|x| + b_1x_1 + b_2x_2 + b_3x_3) = \frac{b_0}{8\pi}\,\delta(x), \qquad (8.8)$$

where δ is the Dirac delta function. Therefore, the second term on the left-hand side of (8.7) is equal (modulo a multiplicative constant) to $U(0) = 0$.

Going further, we show that

$$\int_{\mathbb{R}^3}\Delta U(x)\Delta\Big((U(x) - \mathbb{P}(x))|x|^{-1}\,g(\log(|\xi|/|x|))\Big)\,dx = 0. \qquad (8.9)$$

Indeed, the expression in (8.9) is equal to

$$\begin{aligned}&\int_{\mathbb{R}^3}\Delta U(x)\Delta\Big(V(x)|x|^{-1}\,g(\log(|\xi|/|x|))\Big)\,dx \\ &\qquad + \int_{\mathbb{R}^3}\Delta U(x)\Delta\Big((\eta(x)-1)\mathbb{P}(x)|x|^{-1}\,g(\log(|\xi|/|x|))\Big)\,dx. \qquad (8.10)\end{aligned}$$

Then, using the aforementioned approximation by v_s, $s \in \mathbb{N}$, in the first integral, an observation that $\operatorname{supp}(\eta - 1)\mathbb{P} \subset D$ in the second one, and the biharmonicity of U in D we arrive at (8.9).

Now the combination of (8.7)–(8.10) leads to the identity

$$\int_{\mathbb{R}^3} \Delta U(x) \Delta \Big(U(x)|x|^{-1}\, g(\log(|\xi|/|x|)) \Big)\, dx = 2\mathcal{B}_\tau(u, q). \tag{8.11}$$

Finally, since the identity (2.3) applies to our function U, (8.11) implies that

$$\mathcal{B}_\tau(u, u) = 2\mathcal{B}_\tau(u, q). \tag{8.12}$$

Recall now that g is a fundamental solution of the equation (2.7), and therefore with the notation

$$\begin{aligned} \widetilde{\mathcal{B}}_\tau(v, w) = \int_{\mathbb{R}} \int_{S^2} \Big[&(\delta_\omega v)(\delta_\omega w)\, g(t-\tau) + 2(\partial_t \nabla_\omega v) \cdot (\partial_t \nabla_\omega w)\, g(t-\tau) \\ &+ (\partial_t^2 v)(\partial_t^2 w)\, g(t-\tau) - (\nabla_\omega v) \cdot (\nabla_\omega w) \Big(\partial_t^2 g(t-\tau) + \partial_t g(t-\tau) + 2g(t-\tau) \Big) \\ &- (\partial_t v)(\partial_t w) \Big(2\partial_t^2 g(t-\tau) + 3\partial_t g(t-\tau) - g(t-\tau) \Big) \Big]\, d\omega dt, \end{aligned} \tag{8.13}$$

we have

$$\mathcal{B}_\tau(v, w) = \widetilde{\mathcal{B}}_\tau(v, w) + \frac{1}{2} \int_{S^2} v(\tau, \omega) w(\tau, \omega)\, d\omega. \tag{8.14}$$

Then the equality in (8.12) can be written as

$$\int_{S^2} (u(\tau, \omega) - q(\tau, \omega))^2\, d\omega = \int_{S^2} q^2(\tau, \omega)\, d\omega + 4\widetilde{\mathcal{B}}_\tau(u, q) - 2\widetilde{\mathcal{B}}_\tau(u, u), \tag{8.15}$$

so that if $|\xi| < 3/2$, $\tau = \log|\xi|^{-1}$,

$$\int_{S^2} v^2(\tau, \omega)\, d\omega = \int_{S^2} q^2(\tau, \omega)\, d\omega + 4\widetilde{\mathcal{B}}_\tau(u, q) - 2\widetilde{\mathcal{B}}_\tau(u, u), \tag{8.16}$$

where $v = e^t (V \circ \varkappa^{-1})$.

The identity (8.16) is our major starting point. We shall show that $\widetilde{\mathcal{B}}_\tau(u, q)$ and $\widetilde{\mathcal{B}}_\tau(u, u)$ can be estimated in terms of the series in (8.3) and hence, can be made arbitrarily small by shrinking ε in (8.3). On the other hand,

$$\int_{S^2} q^2(\tau, \omega)\, d\omega = \int_{S^2} \Big(b_0^2 + \sum_{i=1}^3 b_i^2 \omega_i^2 \Big)\, d\omega = 4\pi b_0^2 + \frac{4\pi}{3} \sum_{i=1}^3 b_i^2, \tag{8.17}$$

so that

$$\frac{4\pi}{3} \le \int_{S^2} q^2(\tau, \omega)\, d\omega \le 4\pi. \tag{8.18}$$

Therefore, by (8.16),

$$\int_{S^2} v^2(\tau,\omega)\,d\omega = \frac{C}{|\xi|^4}\int_{S_{|\xi|}} V^2(\xi)\,d\sigma_\xi \tag{8.19}$$

does not vanish as $\xi \to O$, which means that ∇V does not vanish at O either, as desired. It remains to estimate $\widetilde{\mathcal{B}}_\tau(u,q)$ and $\widetilde{\mathcal{B}}_\tau(u,u)$.

Step III: the estimate for $\widetilde{\mathcal{B}}_\tau(u,q)$. Since $q = P\circ\varkappa^{-1}$ is independent of t,

$$\begin{aligned}\widetilde{\mathcal{B}}_\tau(u,q) \;&=\; \int_{\mathbb{R}}\int_{S^2}\Big[(\delta_\omega u)(\delta_\omega q)\,g(t-\tau)\\ &\quad -(\nabla_\omega u)\cdot(\nabla_\omega q)\Big(\partial_t^2 g(t-\tau)+\partial_t g(t-\tau)+2g(t-\tau)\Big)\Big]\,d\omega dt. \end{aligned}\tag{8.20}$$

Next, $\delta_\omega\omega_i = -2\omega_i$ for $i=1,2,3$, and therefore $\delta_\omega q = -2\sum_{i=1}^3 b_i\omega_i$, so that

$$\begin{aligned}\widetilde{\mathcal{B}}_\tau(u,q) \;&=\; \int_{\mathbb{R}}\int_{S^2}\Big[2b_0\delta_\omega u\,g(t-\tau) - (\nabla_\omega u)\cdot(\nabla_\omega q)\Big(\partial_t^2 g(t-\tau)+\partial_t g(t-\tau)\Big)\Big]\,d\omega dt\\ &=\; -\int_{\mathbb{R}}\int_{S^2}\Big[(\nabla_\omega u)\cdot(\nabla_\omega q)\Big(\partial_t^2 g(t-\tau)+\partial_t g(t-\tau)\Big)\Big]\,d\omega dt\\ &\le\; \left(\int_{\mathbb{R}}\int_{S^2}\Big[|\nabla_\omega u|^2\Big(-\partial_t^2 g(t-\tau)-\partial_t g(t-\tau)\Big)\Big]\,d\omega dt\right)^{1/2}\\ &\quad\times\left(\int_{S^2}|\nabla_\omega q|^2\,d\omega\int_{\mathbb{R}}\Big(-\partial_t^2 g(t-\tau)-\partial_t g(t-\tau)\Big)\,dt\right)^{1/2} =: I_1\times I_2, \end{aligned}\tag{8.21}$$

using the Cauchy-Schwarz inequality and the positivity of the weight function (see (2.17)).

Inspecting the argument of Lemma 2.3 one can see that

$$I_1 \le (\widetilde{\mathcal{B}}_\tau(u,u))^{1/2}. \tag{8.22}$$

On the other hand,

$$\begin{aligned} I_2^2 \;&=\; \frac{8\pi}{3}\sum_{i=1}^3 b_i^2\int_{\mathbb{R}}\Big(-\partial_t^2 g(t-\tau)-\partial_t g(t-\tau)\Big)\,dt\\ &=\; \frac{8\pi}{9}\sum_{i=1}^3 b_i^2\left(\int_{-\infty}^{\tau} e^{t-\tau}\,dt + \int_{\tau}^{\infty} e^{-2(t-\tau)}\,dt\right) = \frac{4\pi}{9}\sum_{i=1}^3 b_i^2 \le \frac{4\pi}{9}. \end{aligned}\tag{8.23}$$

Therefore,

$$\widetilde{\mathcal{B}}_\tau(u,q) \le \frac{2\sqrt{\pi}}{3}(\widetilde{\mathcal{B}}_\tau(u,u))^{1/2}. \tag{8.24}$$

Step IV: the estimate for $\widetilde{\mathcal{B}}_\tau(u,u)$, the set-up. Let us now focus on the estimate for $\widetilde{\mathcal{B}}_\tau(u,u)$. To this end, consider the covering of $K = \overline{B_{2^{-N}}}\setminus\Omega$ by the sets $K\cap C_{2^{-j},2^{-j+2}}$, $j\ge N$, and observe that

$$K\cap\overline{C_{2^{-j},2^{-j+2}}} = \overline{C_{2^{-j},2^{-j+2}}}\setminus\Omega, \quad j\ge N+2, \tag{8.25}$$

$$K\cap\overline{C_{2^{-j},2^{-j+2}}} \subseteq \overline{C_{2^{-j},2^{-j+2}}}\setminus\Omega, \quad j = N, N+1. \tag{8.26}$$

Let $\{\eta^j\}_{j=N}^{\infty}$ be the corresponding partition of unity such that

$$\eta^j \in C_0^\infty(C_{2^{-j},2^{-j+2}}), \quad |\nabla^k \eta^j| \le C 2^{kj}, \quad k=0,1,2, \quad \text{and} \quad \sum_{j=N}^{\infty} \eta^j = 1. \tag{8.27}$$

By U^j we denote the capacitary potential of $K \cap \overline{C_{2^{-j},2^{-j+2}}}$ with the boundary data P, i.e. the minimizer for the optimization problem

$$\inf\Big\{ \int_{C_{2^{-j-2},2^{-j+4}}} (\Delta u(x))^2\,dx : \ u \in \mathring{W}_2^2(C_{2^{-j-2},2^{-j+4}}),$$
$$u = P \text{ in a neighborhood of } K \cap \overline{C_{2^{-j},2^{-j+2}}} \Big\}. \tag{8.28}$$

Such U^j always exists and belongs to $\mathring{W}_2^2(C_{2^{-j-2},2^{-j+4}})$ since P is an infinitely differentiable function in a neighborhood of $K \cap \overline{C_{2^{-j},2^{-j+2}}}$. The infimum above is equal to

$$\mathrm{Cap}_P\{K \cap \overline{C_{2^{-j},2^{-j+2}}}, C_{2^{-j-2},2^{-j+4}}\} \approx \mathrm{Cap}_P\{K \cap \overline{C_{2^{-j},2^{-j+2}}}, \mathbb{R}^3 \setminus \{O\}\}. \tag{8.29}$$

Let us now define the function

$$T(x) := \sum_{j=N}^{\infty} |x| \eta^j(x) U^j(x), \quad x \in \mathbb{R}^3, \tag{8.30}$$

and let $\vartheta := e^t (T \circ \varkappa^{-1})$. Then by the Cauchy-Schwarz inequality

$$\widetilde{\mathcal{B}}_\tau(\vartheta,\vartheta) \le C \sum_{k=0}^{2} \sum_{j=N}^{\infty} \int_{C_{2^{-j},2^{-j+2}}} \frac{|\nabla^k (U^j(x))|^2}{|x|^{3-2k}}\,dx. \tag{8.31}$$

Next, since $U^j \in \mathring{W}_2^2(C_{2^{-j-2},2^{-j+4}})$, the Hardy's inequality allows us to write

$$\begin{aligned}
\widetilde{\mathcal{B}}_\tau(\vartheta,\vartheta) &\le C \sum_{j=N}^{\infty} 2^{-j} \int_{C_{2^{-j-2},2^{-j+4}}} |\nabla^2 U^j(x)|^2\,dx \le C \sum_{j=N}^{\infty} 2^{-j} \int_{C_{2^{-j-2},2^{-j+4}}} |\Delta U^j(x)|^2\,dx \\
&\le C \sum_{j=N}^{\infty} 2^{-j}\, \mathrm{Cap}_P\{K \cap \overline{C_{2^{-j},2^{-j+2}}}, \mathbb{R}^3 \setminus \{O\}\} \\
&\le C \sum_{j=N}^{\infty} 2^{-j}\, \mathrm{Cap}_P\{\overline{C_{2^{-j},2^{-j+2}}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}\} < C\varepsilon,
\end{aligned} \tag{8.32}$$

by (8.29), (8.25)–(8.26), the monotonicity property (5.4), and (8.3).

Having at hand (8.32), we need to consider the difference $U - T$ in order to obtain the estimate for $\widetilde{\mathcal{B}}_\tau(u,u)$. Let us denote $W := U - T$, $w := e^t (W \circ \varkappa^{-1})$.

Step V: the estimate for $\mathcal{B}_\tau(w,w)$. First of all, one can show that $W \in \mathring{W}_2^2(D)$. Roughly speaking, it comes from the fact that both U and T belong to $\mathring{W}_2^2(\mathbb{R}^3)$. For U this was pointed out in (8.5), the statement about T can be proved along the lines of (8.31)–(8.32):

$$\begin{aligned}
\|T\|_{\mathring{W}_2^2(\mathbb{R}^3)} &\leq C \sum_{k=0}^{2} \sum_{j=N}^{\infty} 2^{j(4-2k)} \int_{C_{2^{-j},2^{-j+2}}} |\nabla^k(|x|U^j(x))|^2\,dx \\
&\leq C \sum_{j=N}^{\infty} 2^{-2j} \int_{C_{2^{-j-2},2^{-j+4}}} |\Delta U^j(x)|^2\,dx \\
&\leq C \sum_{j=N}^{\infty} 2^{-2j}\,\mathrm{Cap}_P\{\overline{C_{2^{-j},2^{-j+2}}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}\} < C\varepsilon. \qquad (8.33)
\end{aligned}$$

To prove that W actually belongs to $\mathring{W}_2^2(D)$, it remains to show that the boundary data is zero in the sense of $\mathring{W}_2^2(D)$, but that follows fairly directly from the definitions. We leave the details to the interested reader.

Furthermore, $\Delta^2 W = -\Delta^2 T$ in D by (8.5). Then, with the notation $w := e^t(W \circ \varkappa^{-1})$ we have the formula

$$\begin{aligned}
\mathcal{B}_\tau(w,w) &= \int_{\mathbb{R}^3} \Delta W(x)\Delta\Big(W(x)|x|^{-1}\,g(\log(|\xi|/|x|))\Big)\,dx \\
&= -\int_{\mathbb{R}^3} \Delta T(x)\Delta\Big(W(x)|x|^{-1}\,g(\log(|\xi|/|x|))\Big)\,dx. \qquad (8.34)
\end{aligned}$$

In what follows we will show that

$$-\int_{\mathbb{R}^3} \Delta T(x)\Delta\Big(W(x)|x|^{-1}\,g(\log(|\xi|/|x|))\Big)\,dx \leq C\varepsilon^{1/2}(\mathcal{B}_\tau(w,w))^{1/2}. \qquad (8.35)$$

Observe that according to (8.34) and (2.11) the expression on the left-hand side of (8.35) is positive. Next, analogously to (2.5),

$$\begin{aligned}
&-\int_{\mathbb{R}^3} \Delta T(x)\Delta\Big(W(x)|x|^{-1}\,g(\log(|\xi|/|x|))\Big)\,dx \\
&= -\int_{\mathbb{R}}\int_{S^2} \Big(\partial_t^2\vartheta - 3\partial_t\vartheta + 2\vartheta + \delta_\omega\vartheta\Big)\Big(g(t-\tau)\,\delta_\omega w + g(t-\tau)\,\partial_t^2 w \\
&\qquad +(2\partial_t g(t-\tau) - g(t-\tau))\,\partial_t w + (\partial_t^2 g(t-\tau) - \partial_t g(t-\tau))\,w\Big)\,d\omega dt. \qquad (8.36)
\end{aligned}$$

Now recall the formula for $-(2\partial_t^2 g + 3\partial_t g - g)$ from (2.18). It is evident that for any D_t – differential operator in t of the order greater than or equal to 0 we have

$$|D_t g| \leq C(-2\partial_t^2 g - 3\partial_t g + g), \qquad (8.37)$$

where C generally depends on D_t, and hence,

$$\int_{\mathbb{R}}\int_{S^2}(\partial_t w)^2\,|D_t g(t-\tau)|\,d\omega dt$$
$$\le -C\int_{\mathbb{R}}\int_{S^2}(\partial_t w)^2(2\partial_t^2 g(t-\tau)+3\partial_t g(t-\tau)-g(t-\tau))\,d\omega dt \le C\widetilde{\mathcal{B}}_\tau(w,w), \quad (8.38)$$

where the last inequality follows from the calculations in Lemma 2.3. Then for $0\le i+k\le 2$

$$\int_{\mathbb{R}}\int_{S^2}|\partial_t^k\nabla_\omega^i\vartheta|\,|\partial_t w|\,|D_t^{i,k}g(t-\tau)|\,d\omega dt$$
$$\le C\left(\int_{\mathbb{R}}\int_{S^2}|\partial_t^k\nabla_\omega^i\vartheta|^2\,|D_t^{i,k}g(t-\tau)|\,d\omega dt\right)^{1/2}(\widetilde{\mathcal{B}}_\tau(w,w))^{1/2}$$
$$\le C\sum_{j=0}^{2}\left(\int_{\mathbb{R}^3}\frac{|\nabla^j T(x)|^2}{|x|^{5-2j}}\,dx\right)^{1/2}(\widetilde{\mathcal{B}}_\tau(w,w))^{1/2}\le C\varepsilon^{1/2}(\widetilde{\mathcal{B}}_\tau(w,w))^{1/2}, \quad (8.39)$$

where $D_t^{i,k}$ are some differential operators in t and the last inequality follows from (8.31)–(8.32).

For similar reasons,

$$\int_{\mathbb{R}}\int_{S^2}|\partial_t^k\nabla_\omega^i\vartheta|\,|\partial_t^2 w|\,g(t-\tau)\,d\omega dt\le C\varepsilon^{1/2}(\widetilde{\mathcal{B}}_\tau(w,w))^{1/2}, \quad (8.40)$$

and

$$\int_{\mathbb{R}}\int_{S^2}|\partial_t^k\nabla_\omega^i\vartheta|\,|\partial_t\nabla_\omega w|\,g(t-\tau)\,d\omega dt\le C\varepsilon^{1/2}(\widetilde{\mathcal{B}}_\tau(w,w))^{1/2}, \quad (8.41)$$

for $0\le i+k\le 2$.

Invoking (8.39)–(8.41) and integrating by parts, we see that the expression in (8.36) is bounded by

$$\Bigg|\int_{\mathbb{R}}\int_{S^2}\Big(\delta_\omega\vartheta\delta_\omega w\,g(t-\tau)-\nabla_\omega\vartheta\cdot\nabla_\omega w\,(2\partial_t^2 g(t-\tau)+2\partial_t g(t-\tau)+2g(t-\tau))$$
$$+\vartheta w(\partial_t^4 g(t-\tau)+2\partial_t^3 g(t-\tau)-\partial_t^2 g(t-\tau)-2\partial_t g(t-\tau))\Big)\,dtd\omega\Bigg|$$
$$+C\varepsilon^{1/2}(\widetilde{\mathcal{B}}_\tau(w,w))^{1/2}. \quad (8.42)$$

Also,

$$\left|\int_{\mathbb{R}}\int_{S^2}(\delta_\omega\vartheta\cdot\delta_\omega w-2\nabla_\omega\vartheta\cdot\nabla_\omega w)\;g\,dt\,d\omega\right|$$
$$\le\left(\int_{\mathbb{R}}\int_{S^2}\left[(\delta_\omega\vartheta)^2-2(\nabla_\omega\vartheta)^2\right]\;g\,dt\,d\omega\right)^{1/2}\left(\int_{\mathbb{R}}\int_{S^2}\left[(\delta_\omega w)^2-2(\nabla_\omega w)^2\right]\;g\,dt\,d\omega\right)^{1/2}$$
$$\le C\varepsilon^{1/2}(\widetilde{\mathcal{B}}_\tau(w,w))^{1/2}, \quad (8.43)$$

using (2.12) and the Cauchy-Schwarz inequality for the bilinear form on the left-hand side of (8.43). In view of (8.43) and (2.7) the expression in (8.42) is further controlled by

$$\begin{aligned}
&\left|\int_{\mathbb{R}}\int_{S^2} \nabla_\omega \vartheta \cdot \nabla_\omega w \left(-2\partial_t^2 g(t-\tau) - 2\partial_t g(t-\tau)\right) dt d\omega\right| \\
&\qquad + \frac{1}{2}\left|\int_{S^2} \vartheta(\tau,\omega) w(\tau,\omega)\, d\omega\right| + C\varepsilon^{1/2} (\widetilde{\mathcal{B}}_\tau(w,w))^{1/2} \\
&\leq C\varepsilon^{1/2}(\widetilde{\mathcal{B}}_\tau(w,w))^{1/2} + \frac{1}{2}\left(\int_{S^2} \vartheta^2(\tau,\omega)\, d\omega\right)^{1/2} (\mathcal{B}_\tau(w,w))^{1/2}.
\end{aligned} \tag{8.44}$$

For the last inequality we used the positivity of $-2\partial_t^2 g - 2\partial_t g$ (see (2.17)) and the argument similar to (8.38)–(8.39) to estimate the first term, and the Cauchy-Schwarz inequality together with (2.11) for the second one.

Finally, we claim that

$$\int_{S^2} \vartheta^2(\tau,\omega)\, d\omega < C\varepsilon. \tag{8.45}$$

Indeed, by definition (8.45) is equal to

$$\begin{aligned}
&\frac{1}{|\xi|^4}\int_{S_{|\xi|}} T^2(\xi)\, d\sigma_\xi \leq C \sum_{j:\, 2^{-j}\leq|\xi|\leq 2^{-j+2}} \frac{1}{|\xi|^2}\int_{S_{|\xi|}} (U^j(\xi))^2\, d\sigma_\xi \\
&\qquad \leq C \sum_{j:\, 2^{-j}\leq|\xi|\leq 2^{-j+2}} \int_{\mathbb{R}^3} \Delta\Big(|x| U^j(x)\Big) \Delta\Big(U^j(x)\, g(\log(|\xi|/|x|))\Big)\, dx,
\end{aligned} \tag{8.46}$$

using (2.11) for the function $x \mapsto |x|\, U^j(x)$ in $\mathring{W}_2^2(C_{2^{-j-2},2^{-j+4}})$. Finally, the right-hand side of (8.46) is bounded by

$$C \sum_{j:\, 2^{-j}\leq|\xi|\leq 2^{-j+2}} \sum_{k=0}^{2} \int_{C_{2^{-j},2^{-j+2}}} \frac{|\nabla^k(|x| U^j(x))|^2}{|x|^{5-2k}}\, dx < C\varepsilon, \tag{8.47}$$

by the estimate following (8.31). This completes the proof of (8.35), which together with (8.34) yields $\mathcal{B}_\tau(w,w) < \varepsilon$. and therefore,

$$\widetilde{\mathcal{B}}_\tau(w,w) < \mathcal{B}_\tau(w,w) < C\varepsilon. \tag{8.48}$$

The last estimate, in turn, implies that $\mathcal{B}_\tau(u,u) < C\varepsilon$ by the results of Step IV. At last, the combination with (8.24) finishes the argument. $\square$

9 Examples and further properties of the biharmonic capacity

Lemma 9.1 *Consider a domain Ω shaped as an exterior of a cusp in some neighborhood of $O \in \partial\Omega$, i.e.*

$$\Omega \cap B_c = \{(r,\theta,\phi) :\, 0 < r < c,\, \theta > h(r)\}, \quad \textit{for some} \quad c > 0, \tag{9.1}$$

where (r,θ,ϕ), $r \in (0,c)$, $\theta \in [0,\pi]$, $\phi \in [0,2\pi)$, are the spherical coordinates in $\mathbb{R}^3$ and $h(r) : (0,c) \to \mathbb{R}$ is a nondecreasing function satisfying the condition $h(br) \le Ch(r)$ for some $b > 1$ and all $r \in (0,c)$.

Then

$$O \text{ is 1-regular} \quad \text{if and only if} \quad \int_0^c s^{-1} h(s)^2\, ds = +\infty. \tag{9.2}$$

Proof. We claim that for every $P \in \Pi_1$ and every $a \ge 4$

$$\mathrm{Cap}_P\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}) \ge C s^{-1} h(s)^2, \qquad 0 < s < c/a. \tag{9.3}$$

Indeed, recall from Lemma 5.2 that

$$\mathrm{Cap}_P\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}) \approx \mathrm{Cap}_P\,(\overline{C_{s,as}} \setminus \Omega, C_{s/2,2as}). \tag{9.4}$$

By definition of the biharmonic capacity, for every $\varepsilon > 0$ there exist some $u \in \mathring{W}_2^2(C_{s/2,2as})$ such that

$$\mathrm{Cap}_P\,(\overline{C_{s,as}} \setminus \Omega, C_{s/2,2as}) + \varepsilon \ge C \int_{C_{s/2,2as}} (\Delta u(x))^2\, dx, \tag{9.5}$$

and $u = P$ in a neighborhood of $\overline{C_{s,4s}} \setminus \Omega$. Since $u \in \mathring{W}_2^2(C_{s/2,2as})$, by Hardy's inequality

$$\begin{aligned} \int_{C_{s/2,2as}} (\Delta u(x))^2\, dx &= \int_{C_{s/2,2as}} |\nabla^2 u(x)|^2\, dx \\ &\ge C \int_{C_{s/2,2as}} \frac{|u(x)|^2}{|x|^4}\, dx \ge C \int_{C_{s,as} \setminus \Omega} \frac{|P(x)|^2}{|x|^4}\, dx. \end{aligned} \tag{9.6}$$

Using the spherical coordinates the last integral above can be written as

$$\begin{aligned} &\frac{C}{s^4} \int_s^{as} \int_0^{h(r)} \int_0^{2\pi} \left(b_0 + b_1 \sin\theta\cos\phi + b_2 \sin\theta \sin\phi + b_3\cos\theta\right)^2 \; sin\theta\, r^2\, d\phi d\theta dr \\ &\ge \frac{C}{s} \int_0^{h(s)} \left(2b_0^2 + 2b_3^2\cos^2\theta + 4b_0 b_3 \cos\theta + b_1^2 \sin^2\theta + b_2^2 \sin^2\theta\right) \; sin\theta\, d\theta \\ &\ge \frac{C}{s} \left(\cos\theta \Big(b_0^2 + \frac{b_3^2}{3}\cos^2\theta + b_0 b_3 \cos\theta \Big) + (b_1^2 + b_2^2)\Big(\cos\theta - \frac{\cos^3\theta}{3}\Big)\right)\Bigg|_{h(s)}^{0} \\ &\ge \frac{C}{s}\Big(b_0^2 + b_1^2 + b_2^2 + b_3^2\Big)\cos\theta \Bigg|_{h(s)}^{0} \ge \frac{C}{s}\, h(s)^2. \end{aligned} \tag{9.7}$$

Now one can combine (9.5)–(9.7) and let $\varepsilon \to 0$ to obtain (9.3).

Therefore, the divergence of the integral in (9.2) implies that

$$\int_0^{c/a} \inf_{P \in \Pi_1} \mathrm{Cap}_P\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds = +\infty, \tag{9.8}$$

which, in turn, shows that the point O is 1-regular by Theorem 1.2.

Conversely, we claim that there exists $P \in \Pi_1$ such that for every $s \in (0, c/a)$

$$\mathrm{Cap}_P\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}) \leq C s^{-1} h(s)^2. \tag{9.9}$$

Indeed, let us take

$$P(x) := \frac{1}{2}\Big(1 - \frac{x_3}{|x|}\Big), \qquad x \in \mathbb{R}^3. \tag{9.10}$$

Clearly, $P \in \Pi_1$. Next, we choose a function $U \in \mathring{W}_2^2(C_{s/2,2as})$ that is given by P in a neighborhood of $C_{s,as} \setminus \Omega$. To do this, let us introduce two cut-off functions, ζ^θ and ζ^r, such that

$$\zeta^\theta \in C_0^\infty(-1/2, 2),\ \zeta^\theta = 1 \text{ on } [0, 3/2]; \quad \zeta^r \in C_0^\infty(1/2, 2a),\ \zeta^r = 1 \text{ on } [3/4, 3a/2]. \tag{9.11}$$

Then let

$$u(r, \phi, \theta) := \frac{1}{2}(1 - \cos\theta)\, \zeta^\theta\Big(\frac{\theta}{h(as)}\Big)\, \zeta^r\Big(\frac{r}{s}\Big), \tag{9.12}$$

so that

$$u(r, \phi, \theta) = 1 \quad \text{whenever} \quad 0 \leq \theta \leq \frac{3h(as)}{2} \quad \text{and} \quad \frac{3s}{4} \leq r \leq \frac{3as}{2}, \tag{9.13}$$

and

$$u(r, \phi, \theta) = 0 \quad \text{whenever} \quad 2h(as) \leq \theta \leq \pi \quad \text{or} \quad r \notin \Big(\frac{s}{2}, 2as\Big). \tag{9.14}$$

Finally, let $U := u \circ \kappa$, where κ is the change of coordinates in (2.2). Then

$$\int_{C_{s/2,2as}} |\Delta U(x)|^2\, dx = C \int_{s/2}^{2as} \int_0^{2h(as)} \left| \frac{1}{r^2}\, \partial_r(r^2 \partial_r u) + \frac{1}{r^2 \sin\theta}\, \partial_\theta(\sin\theta\, \partial_\theta u) \right|^2 \sin\theta\, d\theta\, r^2\, dr,$$

since u is independent of ϕ. A straightforward calculation shows that for r and θ as above

$$\left| \frac{1}{r^2}\, \partial_r(r^2 \partial_r u) + \frac{1}{r^2 \sin\theta}\, \partial_\theta(\sin\theta\, \partial_\theta u) \right| \leq \frac{C}{s^2}, \tag{9.15}$$

and therefore,

$$\int_{C_{s/2,2as}} |\Delta U(x)|^2\, dx \leq C s^{-1} h(as)^2 \leq C s^{-1} h(s)^2. \tag{9.16}$$

The last inequality follows from the properties of the function h. Indeed, since h is nondecreasing $h(ar) \leq h(br) \leq C h(r)$ when $a \leq b$. If $a > b$ then $h(ar) \leq C^m h\Big(\frac{a}{b^m} r\Big) \leq C^{m+1} h(r)$ for $m \geq \log_b a - 1$.

Finally, if the point O is 1-regular, then by Theorem 1.2 the integral in (1.10) diverges for every $P \in \Pi_1$ and therefore, by (9.9), the integral in (9.2) diverges. $\square$

In order to state the next result, let us recall the definition of the harmonic capacity of a compact set. For an open set $\Omega \subset \mathbb{R}^3 \setminus \{O\}$ and a compactum $e \subset \Omega$

$$\mathrm{cap}\,(e, \Omega) := \inf \left\{ \int_\Omega (\nabla u(x))^2\, dx :\ u \in \mathring{W}_1^2(\Omega),\ u = 1 \text{ in a neighborhood of } e \right\}, \tag{9.17}$$

is a harmonic capacity of the set e relative to Ω. If $\Omega = \mathbb{R}^3 \setminus \{0\}$ then (9.17) coincides with (1.5).

Lemma 9.2 *Let K be a compactum situated on the set*

$$\{x \in \mathbb{R}^3 :\, b_0|x| + b_1x_1 + b_2x_2 + b_3x_3 = 0\}, \quad b_i \in \mathbb{R},\, i = 0,1,2,3, \tag{9.18}$$

such that $O \notin K$. If the harmonic capacity of K equals zero, then

$$\mathrm{Cap}_P(K, \mathbb{R}^3 \setminus \{0\}) = 0 \tag{9.19}$$

for

$$P(x) = \frac{1}{\sqrt{b_0^2 + b_1^2 + b_2^2 + b_3^2}} \Big(b_0 + b_1 \frac{x_1}{|x|} + b_2 \frac{x_2}{|x|} + b_3 \frac{x_3}{|x|} \Big), \qquad x \in \mathbb{R}^3 \setminus \{O\}. \tag{9.20}$$

In particular, $\mathrm{Cap}(K, \mathbb{R}^3 \setminus \{0\}) = 0$.

Proof. By the assumptions of the theorem $O \notin K$. Therefore, there exist $s > 0$, $a > 1$ such that $K \subset \overline{C_{s,as}}$. In the course of proof some constants will depend on s and a. That, however, does not influence the result.

Since

$$\mathrm{cap}(K, C_{s/2,2as}) \approx \mathrm{cap}(K, \mathbb{R}^3 \setminus \{0\}) = 0, \tag{9.21}$$

for every $\varepsilon > 0$ there exists a compactum K_ε with a smooth boundary contained in the set (9.18) and such that

$$K \subset K_\varepsilon \subset C_{s/2,2as} \quad \text{and} \quad \mathrm{cap}(K_\varepsilon, C_{s/2,2as}) < \varepsilon. \tag{9.22}$$

Let u denote the harmonic potential of K_ε, so that

$$u \in \mathring{W}_2^1(C_{s/2,2as}), \quad u = 1 \text{ in } K_\varepsilon, \quad \Delta u = 0 \text{ in } \mathbb{R}^3 \setminus K_\varepsilon, \quad \int_{C_{s/2,2as}} |\nabla u(x)|^2\,dx < \varepsilon. \tag{9.23}$$

Next, given $\alpha < 1$ let

$$v_\alpha(x) = \begin{cases} \alpha^{-4} P(x) u^2(x) (2\alpha - u(x))^2, & \text{if} \quad u(x) \le \alpha, \\ P(x), & \text{if} \quad u(x) > \alpha, \end{cases} \tag{9.24}$$

where $x \in C_{s/2,2as}$ and P is defined by (9.20). Then $v_\alpha \in \mathring{W}_2^2(C_{s/2,2as})$ by (9.23) and $v_\alpha = P$ in a neighborhood of K. Therefore,

$$\begin{aligned} \mathrm{Cap}_P(K, \mathbb{R}^3 \setminus \{0\}) &\approx \mathrm{Cap}_P(K, C_{s/2,2as}) \le \int_{C_{s/2,2as}} |\Delta v_\alpha(x)|^2\,dx \\ &= \alpha^{-8} \int_{x:\, u(x) \le \alpha} \Big| \Delta\Big(P(x) u^2(x)(2\alpha - u(x))^2 \Big) \Big|^2 dx + \int_{x:\, u(x) > \alpha} |\Delta P(x)|^2\,dx. \end{aligned} \tag{9.25}$$

We take $\alpha = \alpha(\varepsilon) < 1$ (close to 1) such that the last term above is less than ε. In addition, on the set $\{x :\, u(x) \le \alpha\}$

$$\Big| \Delta\Big(u^2(x)(2\alpha - u(x))^2 \Big) \Big| \le C|\nabla u|^2, \qquad \Big| \nabla P \cdot \nabla\Big(u^2(x)(2\alpha - u(x))^2 \Big) \Big| \le C|\nabla u|,$$
$$\Big| \Delta P \Big(u^2(x)(2\alpha - u(x))^2 \Big) \Big| \le C|u|, \tag{9.26}$$

so that

$$\int_{x:\,u(x)\le\alpha}\Big|\Delta\Big(P(x)u^2(x)(2\alpha-u(x))^2\Big)\Big|^2\,dx \le C\varepsilon + C\int_{x:\,u(x)\le\alpha}|P(x)|^2|\nabla u|^4\,dx, \tag{9.27}$$

by (9.23).

It remains to estimate the last integral above. To do that, we use the Whitney decomposition of the set $C_{s/2,2as}\setminus K_\varepsilon$. Let us call the corresponding collection of balls $\{B_i\}_{i=1}^\infty$, then

$$\bigcup_{i=1}^\infty B_i = C_{s/2,2as}\setminus K_\varepsilon,\quad \sum_{i=1}^\infty \chi_{B_i}\le C,\quad r(B_i)\approx \operatorname{dist}\Big(B_i,\partial(C_{s/2,2as}\setminus K_\varepsilon)\Big), \tag{9.28}$$

where $r(B_i)$ denotes the radius of B_i. Observe that

$$|u(x)|\le 1,\quad |P(x)|\le Cr_i,\quad \text{if } x\in B_i \text{ such that } \operatorname{dist}(B_i,\partial C_{s/2,2as})\ge \operatorname{dist}(B_i,K_\varepsilon),$$
$$|u(x)|\le Cr_i,\quad |P(x)|\le C,\quad \text{if } x\in B_i \text{ such that } \operatorname{dist}(B_i,\partial C_{s/2,2as})\le \operatorname{dist}(B_i,K_\varepsilon).$$

Since u is harmonic in $C_{s/2,2as}\setminus K_\varepsilon$,

$$|\nabla u|^2\le \frac{C}{r_i^5}\int_{B_i}|u(x)|^2\,dx. \tag{9.29}$$

Therefore, $|P||\nabla u|\le C$ on $C_{s/2,2as}\setminus K_\varepsilon$ and

$$\int_{C_{s/2,2as}}|P(x)|^2|\nabla u|^4\,dx \le \int_{C_{s/2,2as}}|\nabla u|^2\,dx<\varepsilon. \tag{9.30}$$

Letting $\varepsilon\to 0$, we finish the argument. $\square$

Corollary 9.3 *Let Ω be a domain in $\mathbb{R}^3$ such that $O\in\partial\Omega$ and the complement of Ω is a compactum of zero harmonic capacity situated on the set (9.18). Then the point O is not 1-regular.*

Proof. By Lemma 9.2 for the choice of P in (9.20)

$$\mathrm{Cap}_P\,(\overline{C_{s,as}}\setminus\Omega,\mathbb{R}^3\setminus\{O\})\,ds=0, \tag{9.31}$$

for every $s>0$, $a>1$. One can see that such P does not depend on s and a, but only on the initial cone containing the complement of Ω. Therefore,

$$\inf_{P\in\Pi_1}\int_0^c \mathrm{Cap}_P\,(\overline{C_{s,as}}\setminus\Omega,\mathbb{R}^3\setminus\{O\})\,ds=0, \tag{9.32}$$

and hence O is not 1-regular by Theorem 1.2. $\square$

Remark. The set defined by (9.18) is either a circular cone with the vertex at O or a plane containing O. Indeed, the set (9.18) is formed by the rays originating at O and passing through the intersection of the plane $b_0 + b_1x_1 + b_2x_2 + b_3x_3 = 0$ with the unit sphere. If this plane passes through the origin ($b_0 = 0$), it is actually the set (9.18). If it does not, then its intersection with S^2 is a circle giving rise to the corresponding circular cone.

Due to the particular form of elements in the space Π_1 such sets play a special role for our version of the biharmonic capacity and for 1-regularity. This observation is, in particular, supported by Lemma 9.2 and the upcoming example.

We consider a domain whose complement consists of a set of points such that in each dyadic spherical layer three of the points belong to a fixed circular cone, while the fourth one does not. The result below shows that in this case the origin is 1-regular provided the deviation of the fourth point is large enough in a certain sense. The details are as follows.

Lemma 9.4 *Fix some $a \geq 4$. Consider a domain Ω such that in some neighborhood of the origin its complement consists of the set of points*

$$\bigcup_k \left\{A_1^k = (a^{-k}, 0, \alpha),\ A_2^k = (a^{-k}, \pi/2, \alpha),\ A_3^k = (a^{-k}, \pi, \alpha),\ A_4^k = (a^{-k+1/2}, 3\pi/2, \beta_k)\right\}, \tag{9.33}$$

where the points are represented in spherical coordinates (r, ϕ, θ), $r \in (0, c)$ for some $c > 0$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$, $k \in \mathbb{N} \cap (1/2 - \log_a c, \infty)$. Assume, in addition, that

$$0 < \alpha < \pi/2, \quad 0 \leq |\beta_k - \alpha| < \alpha/2, \quad \forall\ k \in \mathbb{N} \cap (1/2 - \log_a c, \infty). \tag{9.34}$$

Then

$$\int_0^{c/a} \inf_{P \in \Pi_1} \mathrm{Cap}_P\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds \geq C \sum_k (\beta_k - \alpha)^2, \tag{9.35}$$

where $C = C(\alpha) > 0$. In particular,

$$\text{if} \quad \sum_k (\beta_k - \alpha)^2 = +\infty \quad \text{then } O \text{ is 1-regular.} \tag{9.36}$$

Proof. To begin, let us observe that

$$\begin{aligned} \int_0^{c/a} \mathrm{Cap}\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds &\geq \sum_k \int_{a^{-k-1/2}}^{a^{-k}} \mathrm{Cap}\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds \\ &\geq \sum_k a^{-k} \min_{s \in (a^{-k-1/2}, a^{-k})} \mathrm{Cap}\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}). \end{aligned} \tag{9.37}$$

For every $s \in (a^{-k-1/2}, a^{-k})$ in the spherical layer $\overline{C_{s,as}}$ there are exactly four points that belong to the complement of Ω, namely, A_i^k, $i = 1, 2, 3, 4$. We aim to show that for each $k \geq 1/2 - \log_a c$

$$\mathrm{Cap}\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}) \geq C a^k (\beta_k - \alpha)^2, \tag{9.38}$$

provided $s \in (a^{-k-1/2}, a^{-k})$.

Take some $P \in \Pi_1$ and consider the distribution

$$T^k(x) := \sum_{i=1}^{4} P(A_i^k)\delta(x - A_i^k). \tag{9.39}$$

Then for every $u \in \mathring{W}_2^2(C_{s/2,2as})$ such that $u = P$ in a neighborhood of $\{A_i^k, i = 1, 2, 3, 4\}$, we have

$$\langle T^k, P\rangle = \sum_{i=1}^{4} P(A_i^k)^2. \tag{9.40}$$

On the other hand, since T^k is supported in the set $\{A_i^k, i = 1, 2, 3, 4\}$,

$$\langle T^k, P\rangle = -\langle \Delta E * T^k, u\rangle = -\langle E * T^k, \Delta u\rangle, \tag{9.41}$$

where $E(x) = 1/(4\pi|x|)$ is the fundamental solution for the Laplacian. By the Cauchy-Schwarz inequality

$$\begin{aligned} |\langle T^k, P\rangle|^2 &\le \|E * T^k\|^2_{L^2(C_{s/2,2as})} \|\Delta u\|^2_{L^2(C_{s/2,2as})} \\ &\le Cs \sum_{i=1}^{4} P(A_i^k)^2 \,\mathrm{Cap}_P\,(\overline{C_{s,as}} \setminus \Omega, C_{s/2,2as}). \end{aligned} \tag{9.42}$$

Therefore, combining (9.40)–(9.42) and taking the infimum in P, we obtain the estimate

$$\mathrm{Cap}\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}) \ge Ca^k \inf_{P\in\Pi_1} \sum_{i=1}^{4} P(A_i^k)^2 = Ca^k \inf_{b\in\mathbb{R}^4:\, \|b\|=1} b\, MM^{\perp}\, b^{\perp}, \tag{9.43}$$

where $b = (b_0, b_1, b_2, b_3)$,

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \sin\alpha & 0 & -\sin\alpha & 0 \\ 0 & \sin\alpha & 0 & -\sin\beta_k \\ \cos\alpha & \cos\alpha & \cos\alpha & \cos\beta_k \end{pmatrix} \tag{9.44}$$

and the superindex $\perp$ denotes matrix transposition. Then the infimum in (9.43) is bounded from below by the smallest eigenvalue of $MM^{\perp}$. The characteristic equation of $MM^{\perp}$ is

$$\begin{aligned} &-\lambda^4 + 8\lambda^3 - \frac{1}{4}\Big(55 - 22\cos(2\alpha) - 3\cos(4\alpha) - 8\cos(\alpha - \beta_k) - \cos(2\alpha - 2\beta_k) - 2\cos(2\beta_k) \\ &-16\cos(\alpha + \beta_k) - 3\cos(2\alpha + 2\beta_k)\Big)\lambda^2 - \frac{1}{2}\sin^2\alpha\Big(-4\cos(2\alpha) + \cos(4\alpha) + 12\cos(\alpha - \beta_k) \\ &-33 + \cos(2\alpha - 2\beta_k) + 20\cos(\alpha + \beta_k) + 3\cos(2\alpha + 2\beta_k)\Big)\lambda = 4\sin^2\alpha(\cos\alpha - \cos\beta_k)^2. \end{aligned}$$

By the Mean Value Theorem for the function arccos and our assumptions on α, β_k there exists $C_0(\alpha)$ independent of β_k such that for all k

$$|\alpha - \beta_k| \le C_0(\alpha)|\cos\alpha - \cos\beta_k|, \tag{9.45}$$

and therefore,

$$4\sin^2\alpha(\cos\alpha-\cos\beta_k)^2 \geq 4\sin^2\alpha(C_0(\alpha))^{-2}|\alpha-\beta_k|^2. \tag{9.46}$$

It follows that

$$\lambda \geq \frac{\sin^2\alpha(C_0(\alpha))^{-2}}{100}\,|\alpha-\beta_k|^2, \tag{9.47}$$

because otherwise the left-hand side of (9.45) is strictly less than its right-hand side. Combined with (9.43), this finishes the proof of (9.35). The statement (9.36) follows from (9.35) and Theorem 1.2. □

Remark. Retain the conditions of Lemma 9.4 and observe that by our construction for every $s \in (0, c/a^{1/3})$ in the spherical layer $\overline{C_{s,a^{1/3}s}}$ there are either exactly three points A_i^k, $i=1,2,3$ for some $k=k(s)$, or exactly one point A_4^k, $k=k(s)$, or no points from the complement of Ω. By Lemma 9.2 it follows that in either case

$$\mathrm{Cap}\,(\overline{C_{s,a^{1/3}s}}\setminus\Omega, \mathbb{R}^3\setminus\{O\}) = 0 \tag{9.48}$$

and hence,

$$\int_0^{c/a^{1/3}} \inf_{P\in\Pi_1} \mathrm{Cap}_P\,(\overline{C_{s,a^{1/3}s}}\setminus\Omega, \mathbb{R}^3\setminus\{O\})\,ds = 0. \tag{9.49}$$

At the same time, if $\sum_k(\alpha-\beta_k)^2$ diverges, then so does the integral in (9.35).

It follows that for the *same* domain Ω the convergence of the integral in (1.9) might depend on the choice of a.

Alternatively, one can say that for the same a the convergence of the integral in (1.9) might depend on the dilation of the domain. In particular, (1.9) *can not* be a necessary condition for the 1-regularity since the concept of 1-regularity is dilation invariant.

Conversely, our proof of the first statement in Theorem 1.2 and Proposition 7.1 relies on Proposition 5.4 which, in turn, follows from the Poincaré-type inequality (5.21). In fact, for every s our choice of P, that allows to estimate the infimum under the integral sign in (1.9), is dictated by the approximating constants in the Poincaré's inequality on (s, as) (see the proof of Lemma 5.3). Therefore, in our argument one can not make a uniform choice of P for all s to substitute (1.9) with (1.10).

Corollary 9.5 *The 1-irregularity is unstable under the affine transformation of coordinates.*

Proof. The proof is based on Corollary 9.3 and Lemma 9.4. Indeed, given the assumptions of Lemma 9.4, if $\beta_k=\alpha$ for all k, then the complement of Ω is entirely contained in the circular cone of aperture α with the vertex at the origin and hence, by virtue of Corollary 9.3, the point O is not 1-regular.

However, if $\beta_k=\alpha+\varepsilon$ for all k, then the series in (9.36) diverges for arbitrary small $\varepsilon>0$, which entails 1-regularity of O. □

References

[1] D. Adams, *Potential and capacity before and after Wiener*, Proceedings of the Norbert Wiener Centenary Congress, 1994 (East Lansing, MI, 1994), 63–83, Proc. Sympos. Appl. Math., 52, Amer. Math. Soc., Providence, RI, 1997.

[2] D. Adams, L. Hedberg, *Function spaces and potential theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 314. Springer-Verlag, Berlin, 1996.

[3] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I*, Comm. Pure Appl. Math., 12 (1959), 623–727.

[4] G. Dal Maso, U. Mosco, *Wiener criteria and energy decay for relaxed Dirichlet problems*, Arch. Rational Mech. Anal. 95 (1986), no. 4, 345–387.

[5] L.C. Evans, R.F. Gariepy, *Wiener's criterion for the heat equation*, Arch. Rational Mech. Anal. 78 (1982), no. 4, 293–314.

[6] E. Fabes, N. Garofalo, E. Lanconelli, *Wiener's criterion for divergence form parabolic operators with C^1-Dini continuous coefficients*, Duke Math. J. 59 (1989), no. 1, 191–232.

[7] E. Fabes, D. Jerison, C. Kenig, *The Wiener test for degenerate elliptic equations*, Ann. Inst. Fourier (Grenoble) 32 (1982), no. 3, vi, 151–182.

[8] V. Kozlov, V. Maz'ya, J. Rossmann, *Spectral problems associated with corner singularities of solutions to elliptic equations*, Mathematical Surveys and Monographs, 85. American Mathematical Society, Providence, RI, 2001.

[9] D. Labutin, *Potential estimates for a class of fully nonlinear elliptic equations*, Duke Math. J. 111 (2002), no. 1, 1–49.

[10] W. Littman, G. Stampacchia, H.F. Weinberger, *Regular points for elliptic equations with discontinuous coefficients*, Ann. Scuola Norm. Sup. Pisa (3), 17 (1963), 43–77.

[11] J. Malý, W.P. Ziemer, *Fine regularity of solutions of elliptic partial differential equations*, Mathematical Surveys and Monographs, 51. American Mathematical Society, Providence, RI, 1997.

[12] V. Maz'ya, *On the behavior near the boundary of solutions to the Dirichlet problem for the biharmonic operator*, Dokl. Akad. Nauk SSSR 18 (1977), 15–19. English transl.: Soviet Math. Dokl. 18 (1977), 1152–1155 (1978).

[13] V. Maz'ya, *Behaviour of solutions to the Dirichlet problem for the biharmonic operator at a boundary point*, Equadiff IV (Proc. Czechoslovak Conf. Differential Equations and their Applications, Prague, 1977), pp. 250–262, Lecture Notes in Math., 703, Springer, Berlin, 1979.

[14] V. Maz'ya, *Unsolved problems connected with the Wiener criterion*, The Legacy of Norbert Wiener: A Centennial Symposium (Cambridge, MA, 1994), 199–208, Proc. Sympos. Pure Math., 60, Amer. Math. Soc., Providence, RI, 1997.

[15] V. Maz'ya, *The Wiener test for higher order elliptic equations.*, Duke Math. J. 115 (2002), no. 3, 479–512.

[16] V. Maz'ya, J. Rossmann, *On the Agmon-Miranda maximum principle for solutions of elliptic equations in polyhedral and polygonal domains*, Ann. Global Anal. Geom. 9 (1991), no. 3, 253–303.

[17] V. Maz'ya, J. Rossmann, *On the Agmon-Miranda maximum principle for solutions of strongly elliptic equations in domains of R^n with conical points*, Ann. Global Anal. Geom. 10 (1992), no. 2, 125–150.

[18] V. Maz'ya, G. M. Tashchiyan, *On the behavior of the gradient of the solution of the Dirichlet problem for the biharmonic equation near a boundary point of a three-dimensional domain*, Sibirsk. Mat. Zh. 31 (1990), no. 6, 113–126; translation in Siberian Math. J. 31 (1990), no. 6, 970–983 (1991).

[19] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Masson et Cie, Éditeurs, Paris; Academia, Éditeurs, Prague 1967.

[20] J. Pipher, G. Verchota, *The Dirichlet problem in L^p for the biharmonic equation on Lipschitz domains*, Amer. J. Math. 114 (1992), no. 5, 923–972.

[21] J. Pipher, G. Verchota, *A maximum principle for biharmonic functions in Lipschitz and C^1 domains*, Comment. Math. Helv. 68 (1993), no. 3, 385–414.

[22] Z. Shen, *On estimates of biharmonic functions on Lipschitz and convex domains*, preprint.

[23] N. Wiener, *The Dirichlet problem*, J. Math. Phys. 3 (1924), pp. 127-146.

[24] N. Trudinger, X.-J. Wang, *On the weak continuity of elliptic operators and applications to potential theory*, (English summary) Amer. J. Math. 124 (2002), no. 2, 369–410.

Svitlana Mayboroda
Department of Mathematics, The Ohio State University,
231 W 18th Av., Columbus, OH, 43210, USA
svitlana@math.ohio-state.edu

Vladimir Maz'ya
Department of Mathematics, The Ohio State University,
231 W 18th Av., Columbus, OH, 43210, USA

Department of Mathematical Sciences, M&O Building,
University of Liverpool, Liverpool L69 3BX, UK

Department of Mathematics, Linköping University,
SE-581 83 Linköping, Sweden
vlmaz@math.ohio-state.edu, vlmaz@mai.liu.se