Uniform asymptotic formulae for Green's tensors in elastic singularly perturbed domains

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Abstract

We present uniform asymptotic approximations of Green's kernels for boundary value problems of elasticity in singularly perturbed domains containing a small hole. We consider the cases of two and three dimensions, for an isotropic Lamé operator and the Dirichlet boundary conditions. The main feature of the asymptotic approximations mentioned is their uniformity with respect to the independent spatial variables. The formal asymptotic formulae are supplied with rigorous derivations and the remainder estimates.

Keywords: Green's tensors, uniform asymptotics, singularly perturbed domains, compound asymptotic expansions.

1 Introduction

Singularly perturbed problems of elasticity occur in models of defects in solids, damage mechanics and models of fracture. The knowledge of Green's kernels enables one to solve boundary value problems and estimate toughness parameters of solids with defects for a general choice of displacement and force terms in the boundary conditions and governing equations.

The papers [6, 7] provide uniform asymptotic approximations for Green's kernels for Laplace's operator in domains with small holes, domains with perturbed external boundaries, and different types of boundary conditions. The asymptotic analysis is based on the method of compound asymptotic expansions developed in [8], and the results include uniform asymptotic approximations for Green's kernels supplied with the rigorous estimates of the remainder terms.

The paper [10] deals with applications of the asymptotic approximations of Green's functions in domains with holes in analysis of eigenvalues of the corresponding spectral problems for the Laplacian. In particular, this is linked to the study of the Lenz shift effect discussed in [3] and [11].

The main feature of the asymptotic approximations of [6, 7] is their uniformity with respect to the independent variables. We extend this theory to the vector case of elasticity equations in two and three dimensions. Our aim is to derive uniform asymptotic approximations for components of Green's tensor in a solid with a small hole, located at a finite distance from the exterior boundary. The motivation for this article came from the asymptotic formulae derived for the Laplacian, in [6].

The structure of the present article is as follows.

Section 2 gives an outline of governing equations and description of the geometry of the singularly perturbed domain. Section 3 includes a result concerning the estimates for the modulus of solutions to the Lamé equation in a domain with a small hole. Section 4 presents one of the main results related to the evaluation of Green's tensor in three dimensions. This section also introduces the notion of the elastic capacity matrix and furthermore includes a detailed discussion of its properties. The case of a planar singularly perturbed domain and construction of the corresponding Green's tensor for the operator of the Lamé system are considered in Section 5. We also give corollaries, in Section 6, showing that under certain constraints on the independent variables, the asymptotic formulae for Green's matrices can be simplified and represented via canonical model fields associated with either unperturbed domain (without any holes) or an unbounded domain containing a hole of finite size (boundary layer domain).

In what follows, Ω_ε is a bounded domain with a small hole, G is the Green's tensor for the unperturbed domain, g is the Green's tensor for the scaled unbounded domain with the finite hole and Γ is the fundamental solution for the Lamé operator in three dimensions. The matrices H and h are the regular parts of Green's matrices G and g, which are given by $H = G - \Gamma$, $h = g - \Gamma$. The matrix P is the capacitary potential for the hole and matrix B represents the elastic capacity of the hole.

We prove the following asymptotic formula for the Green's tensor:

Theorem 1 *Green's tensor $G_\varepsilon(\mathbf{x}, \mathbf{y})$ for the Lamé operator in $\Omega_\varepsilon \subset \mathbb{R}^3$ admits the representation*

$$\begin{aligned} G_\varepsilon(\mathbf{x}, \mathbf{y}) \; = \; & G(\mathbf{x}, \mathbf{y}) + \varepsilon^{-1} g(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) - \Gamma(\mathbf{x}, \mathbf{y}) + P(\varepsilon^{-1}\mathbf{x}) H(\mathbf{O}, \mathbf{y}) \\ & + H(\mathbf{x}, \mathbf{O}) P^T(\varepsilon^{-1}\mathbf{y}) - P(\varepsilon^{-1}\mathbf{x}) H(\mathbf{O}, \mathbf{O}) P^T(\varepsilon^{-1}\mathbf{y}) \\ & - \varepsilon H(\mathbf{x}, \mathbf{O}) B H(\mathbf{O}, \mathbf{y}) + O(\varepsilon^2 (\min\{|\mathbf{x}|, |\mathbf{y}|\})^{-1}) \;, \end{aligned} \tag{1}$$

uniformly with respect to $\mathbf{x}$*,* $\mathbf{y} \in \Omega_\varepsilon$*.*

We also obtain and prove a similar result for the case of a planar domain with a small hole, this formula is given in Section 5.

2 Governing equations and main notations

We now give several notations adopted in the following text. Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2, 3$, with compact closure $\bar{\Omega}$ and smooth boundary $\partial\Omega$. By ω we denote a domain in $\mathbb{R}^n$ with smooth boundary $\partial\omega$ and compact closure $\bar{\omega}$; its complement being $C\bar{\omega} = \mathbb{R}^n \backslash \bar{\omega}$. We shall assume that both Ω and ω contain the origin $\mathbf{O}$ as an interior point. It is also assumed that the minimum distance between $\mathbf{O}$ and the points of $\partial\Omega$ is equal to 1. In addition the maximum distance between $\mathbf{O}$ and the points of $\partial C\bar{\omega}$ will be taken as 1. We introduce the set $\omega_\varepsilon = \{\mathbf{x} : \varepsilon^{-1}\mathbf{x} \in \omega\}$, where ε is a small positive parameter, and the open set $\Omega_\varepsilon = \Omega \backslash \bar{\omega}_\varepsilon$. The notation B_ϱ stands for the open ball centered at $\mathbf{O}$ with radius ϱ.

The main object of our study, Green's tensor for the Lamé operator in $\Omega_\varepsilon \subset \mathbb{R}^n$, $n = 2, 3$, will be denoted by G_ε. The tensor G_ε is a solution of

$$\mu\Delta_{\mathbf{x}} G_\varepsilon(\mathbf{x}, \mathbf{y}) + (\lambda + \mu)\nabla_{\mathbf{x}}(\nabla_{\mathbf{x}} \cdot G_\varepsilon(\mathbf{x}, \mathbf{y})) + \delta(\mathbf{x} - \mathbf{y})I_n = 0I_n \ , \quad \mathbf{x}, \mathbf{y} \in \Omega_\varepsilon \ , \tag{2}$$

$$G_\varepsilon(\mathbf{x}, \mathbf{y}) = 0I_n \ , \quad \mathbf{x} \in \partial\Omega_\varepsilon, \mathbf{y} \in \Omega_\varepsilon \ , \tag{3}$$

where I_n is the $n \times n$ identity matrix. An important property of this tensor is the following symmetry relation

$$G_\varepsilon(\mathbf{x}, \mathbf{y}) = G_\varepsilon^T(\mathbf{y}, \mathbf{x}) \ . \tag{4}$$

In the sequel, along with $\mathbf{x}$ and $\mathbf{y}$, we shall use scaled variables $\boldsymbol{\xi} = \varepsilon^{-1}\mathbf{x}$ and $\boldsymbol{\eta} = \varepsilon^{-1}\mathbf{y}$.

By const we always mean different positive constants depending only on n, λ and μ. The notation $f = O(g)$ is equivalent to the inequality $|f| \leq \text{const}\, g$.

Let $\sigma(u) = [\sigma_{ij}(u)]_{i,j=1}^3$ represent the Cauchy stress tensor, which for an isotropic solid with displacements $u = \{u_k\}_{k=1}^3$ has entries of the form

$$\sigma_{ij}(u) = \lambda\delta_{ij}u_{p,p} + \mu(u_{i,j} + u_{j,i}) \ , \tag{5}$$

here and elsewhere in the text, the repeated are regarded as the indices of summation, and $T_n(u) = \sigma_{ij}(u)n_j$ are the tractions computed for displacements u, where n_j is the j^{th} component of the unit outward normal.

Also $e(u) = [e_{ij}(u)]_{i,j=1}^3$ denotes the strain tensor, whose entries are given by

$$e_{ij}(u) = 2^{-1}(u_{i,j} + u_{j,i}) \ . \tag{6}$$

3 Estimates for the maximum modulus of solutions of elasticity problems in domains with small inclusions

In order to obtain the estimates for the remainders in the representations for G_ε in (1) for three dimensions, and that given in Section 5 for two dimensions, we need an auxiliary result concerning an estimate for the maximum modulus of solutions for Lamé system in domains with small holes. In what follows we shall formulate and prove such a result.

Let $\mathbf{u}$ be the displacement vector which satisfies the Dirichlet boundary value problem in the domain $\Omega_\varepsilon \subset \mathbb{R}^n$,

$$L\,(\partial_{\mathbf{x}})\,\mathbf{u}(\mathbf{x}) := \mu\Delta\mathbf{u}(\mathbf{x}) + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}(\mathbf{x})) = \mathbf{O}\ , \quad \mathbf{x}\in\Omega_\varepsilon\ , \tag{7}$$

$$\mathbf{u}(\mathbf{x}) = \boldsymbol{\varphi}(\varepsilon^{-1}\mathbf{x})\ , \quad \mathbf{x}\in\partial\omega_\varepsilon\ , \tag{8}$$
$$\mathbf{u}(\mathbf{x}) = \boldsymbol{\psi}(\mathbf{x})\ , \quad \mathbf{x}\in\partial\Omega\ , \tag{9}$$

where $\partial_{\mathbf{x}} = \partial/\partial\mathbf{x}$, $\mathbf{O}$ is the zero vector, and we assume that $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are continuous vector functions.

In this section, we prove the following.

Lemma 1 *There exists a unique solution $\mathbf{u}\in C(\bar{\Omega}_\varepsilon)$ of problem (7) − (9) which satisfies the estimate*

$$\max_{\bar{\Omega}_\varepsilon}|\mathbf{u}(\mathbf{x})| \le \text{const}\max\{\|\boldsymbol{\varphi}\|_{C(\partial\omega_\varepsilon)}\ ,\|\boldsymbol{\psi}\|_{C(\partial\Omega)}\}\ . \tag{10}$$

We consider the cases when the dimension n is equal to 3 or 2.

The proof of the theorem involves auxiliary statements related to model domains Ω and $C\bar{\omega} = \mathbb{R}^n\backslash\bar{\omega}$.

3.1 The maximum principle in Ω

Let $\mathbf{u}$ solve the Dirichlet boundary value problem in Ω

$$L\,(\partial_{\mathbf{x}})\,\mathbf{u}(\mathbf{x}) = \mathbf{O}\ , \quad \mathbf{x}\in\Omega\ , \tag{11}$$
$$\mathbf{u}(\mathbf{x}) = \boldsymbol{\psi}(\mathbf{x})\ , \quad \mathbf{x}\in\partial\Omega\ , \tag{12}$$

where $\boldsymbol{\psi}$ is continuous on $\partial\Omega$.

The following assertion is essentially due to Fichera [2], who proved its analogue for then 3-dimensional case. The same argument works for the case of a planar domain and is even simpler.

Lemma 2 (Fichera's maximum principle, see [2]) *There exists a unique solution* $\mathbf{u} \in C(\bar{\Omega})$ *of problem* (11)*,* (12)*. This solution satisfies the estimate*

$$\|\mathbf{u}\|_{C(\bar{\Omega})} \leq A_\Omega \|\boldsymbol{\psi}\|_{C(\partial\Omega)} , \tag{13}$$

where A_Ω *is a constant coefficient.*

3.2 The maximum principle in $C\bar{\omega}$

Let $\mathbf{v}(\boldsymbol{\xi})$ be a solution of the Dirichlet boundary value problem in the unbounded domain $C\bar{\omega}$:

$$L(\partial_{\boldsymbol{\xi}})\,\mathbf{v}(\boldsymbol{\xi}) = \mathbf{O} , \quad \boldsymbol{\xi} \in C\bar{\omega} , \tag{14}$$

$$\mathbf{v}(\boldsymbol{\xi}) = \boldsymbol{\varphi}(\boldsymbol{\xi}) , \quad \boldsymbol{\xi} \in \partial\omega , \tag{15}$$

$$|\mathbf{v}| \to 0 \quad \text{as} \quad |\boldsymbol{\xi}| \to \infty , \tag{16}$$

when $n = 3$

For the two-dimensional case ($n = 2$), the formulation (14)–(16) has to be supplied with the orthogonality conditions for the right-hand side $\boldsymbol{\varphi}$:

$$\int_{\partial\omega} \boldsymbol{\varphi}(\boldsymbol{\xi}) \cdot T_n(\partial_{\boldsymbol{\xi}})\,\zeta^{(j)}(\boldsymbol{\xi})\, ds = 0 , \quad j = 1, 2 . \tag{17}$$

The vector functions $\zeta^{(j)}$ are solutions of the model problem

$$L(\partial_{\boldsymbol{\xi}})\,\zeta^{(j)}(\boldsymbol{\xi}) = \mathbf{O} , \quad \boldsymbol{\xi} \in C\bar{\omega} , \tag{18}$$

$$\zeta^{(j)}(\boldsymbol{\xi}) = \mathbf{O} , \quad \boldsymbol{\xi} \in \partial C\bar{\omega} , \tag{19}$$

$$\zeta^{(j)}(\boldsymbol{\xi}) \sim -\gamma^{(j)}(\boldsymbol{\xi}, \mathbf{O}) + \zeta^{(\infty,j)} \quad \text{as} \quad |\boldsymbol{\xi}| \to \infty , \tag{20}$$

where $\gamma^{(j)}$ are the columns of the fundamental solution for the Lamé operator in an infinite plane, $\zeta^{(\infty,j)}$ is a constant vector and T_n denotes the matrix differential operator of tractions

$$T_n(\partial_{\boldsymbol{\xi}})\,\zeta^{(j)}(\boldsymbol{\xi}) = \begin{pmatrix} \sigma_{11}(\zeta^{(j)})n_1 + \sigma_{12}(\zeta^{(j)})n_2 \\ \sigma_{12}(\zeta^{(j)})n_2 + \sigma_{22}(\zeta^{(j)})n_2 \end{pmatrix}$$

where $\boldsymbol{n} = (n_1, n_2)$ is the unit outward normal on $\partial\omega$. We shall also use the notation $\mathfrak{N}$ for the 2×2 matrix function:

$$\mathfrak{N}(\boldsymbol{\xi}) = \{T_n\zeta^{(1)}(\boldsymbol{\xi}), T_n\zeta^{(2)}(\boldsymbol{\xi})\} . \tag{21}$$

The following assertion results readily from Fichera's maximum modulus principle for a bounded domain [2], combined with the standard asymptotic estimate for the solution at infinity.

Lemma 3 *There exists a unique solution in* $C(C\bar{\omega})$ *of the problem* (14)−(16) ((14)–(17) *for* $n = 2$)*. This solution satisfies the estimate*

$$\sup_{\boldsymbol{\xi} \in C\bar{\omega}} \{|\boldsymbol{\xi}||u(\boldsymbol{\xi})|\} \leq A_{C\bar{\omega}} \|\boldsymbol{\varphi}\|_{C(\partial\omega)} .$$

3.3 The operator notations

We introduce the operators P_Ω and $P_{C\bar{\omega}}$ in such a way that the solutions $\mathbf{u}$, $\mathbf{v}$ of problems (11), (12) and (14)−(16) are represented in the form

$$\mathbf{u} = P_\Omega(\boldsymbol{\psi}) , \quad \mathbf{v} = P_{C\bar{\omega}}(\boldsymbol{\varphi}) . \tag{22}$$

In the case of $n = 2$, we will also use the approximation $\boldsymbol{\pi}_\varepsilon$ of the capacitary potential:

$$\begin{aligned} \boldsymbol{\pi}_\varepsilon \;=\; & D(\log\varepsilon)G(\mathbf{x}, \mathbf{O}) \\ & + P_{C\bar{\omega}_\varepsilon}(I_2 - D(\log\varepsilon)\,\mathrm{Tr}_{\partial\omega_\varepsilon} G(\mathbf{x}, \mathbf{O})) \\ & - P_\Omega(\mathrm{Tr}_{\partial\Omega} P_{C\bar{\omega}_\varepsilon}(I_2 - D(\log\varepsilon)\,\mathrm{Tr}_{\partial\omega_\varepsilon} G(\mathbf{x}, \mathbf{O}))) , \end{aligned}$$

where $D(\log\varepsilon)$ is the 2×2 matrix defined by

$$D = -\frac{1}{K_1} \begin{pmatrix} K_2 \log\varepsilon - \zeta_{22}^\infty + H_{22}(\mathbf{O}, \mathbf{O}) & \zeta_{12}^\infty - H_{12}(\mathbf{O}, \mathbf{O}) \\ \zeta_{21}^\infty - H_{21}(\mathbf{O}, \mathbf{O}) & K_2 \log\varepsilon - \zeta_{11}^\infty + H_{11}(\mathbf{O}, \mathbf{O}) \end{pmatrix} , \tag{23}$$

with

$$\begin{aligned} K_1 \;=\; & (K_2 \log\varepsilon - \zeta_{11}^\infty + H_{11}(\mathbf{O}, \mathbf{O}))\,(K_2 \log\varepsilon - \zeta_{22}^\infty + H_{22}(\mathbf{O}, \mathbf{O})) \\ & - (H_{12}(\mathbf{O}, \mathbf{O}) - \zeta_{12}^\infty)(H_{21}(\mathbf{O}, \mathbf{O}) - \zeta_{21}^\infty) , \end{aligned} \tag{24}$$

$$K_2 = \frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)} , \tag{25}$$

and $H = [H_{ij}]_{i,j=1}^3$ is the regular part of Green's tensor for the domain Ω,

$$\zeta^\infty = [\zeta_{ij}^\infty]_{i,j=1}^3 = \lim_{|\boldsymbol{\xi}|, |\boldsymbol{\eta}| \to \infty} \{\gamma(\boldsymbol{\eta}, \mathbf{O}) + g(\boldsymbol{\xi}, \boldsymbol{\eta})\} , \tag{26}$$

where g is Green's tensor for the unbounded domain $C\bar{\omega}$.

By direct substitution, we can verify that

$$L\,(\partial_\mathbf{x})\,\boldsymbol{\pi}_\varepsilon(\mathbf{x}) = 0 I_2 , \quad \mathbf{x} \in \Omega_\varepsilon , \tag{27}$$

$$\boldsymbol{\pi}_\varepsilon(\mathbf{x}) = 0 I_2 , \quad \mathbf{x} \in \partial\Omega , \tag{28}$$

$$\boldsymbol{\pi}_\varepsilon(\mathbf{x}) = I_2 + O(\varepsilon) , \quad \mathbf{x} \in \partial\omega_\varepsilon . \tag{29}$$

3.4 The proof of Lemma 1 for $n = 2$

First, consider the case when the *homogeneous boundary condition is set on* $\partial\Omega$, so that

$$
\begin{aligned}
L(\partial_{\mathbf{x}})\,\mathbf{u}(\mathbf{x}) &= \mathbf{O}\,, \quad \mathbf{x} \in \Omega_\varepsilon\,, &(30)\\
\mathbf{u}(\mathbf{x}) &= \boldsymbol{\varphi}(\varepsilon^{-1}\mathbf{x})\,, \quad \mathbf{x} \in \partial\omega_\varepsilon\,, &(31)\\
\mathbf{u}(\mathbf{x}) &= \mathbf{O}\,, \quad \mathbf{x} \in \partial\Omega\,. &(32)
\end{aligned}
$$

We are looking for the solution in the form

$$
\begin{aligned}
\mathbf{u} &= P_{C\bar{\omega}_\varepsilon}(\mathbf{g} - A_{\mathbf{g}}) + \boldsymbol{\pi}_\varepsilon A_{\mathbf{g}}\\
&\quad - P_\Omega(\mathrm{Tr}_{\partial\Omega} P_{C\bar{\omega}_\varepsilon}(\mathbf{g} - A_{\mathbf{g}}))\,, \qquad (33)
\end{aligned}
$$

where the constant vector $A_{\mathbf{g}}$ is determined by

$$
A_{\mathbf{g}} = \int_{\partial\omega} \mathfrak{N}^T(\boldsymbol{\xi})\mathbf{g}(\boldsymbol{\xi})\, dS_{\boldsymbol{\xi}}\,, \tag{34}
$$

where the matrix $\mathfrak{N}$ is the same as in (21). We note that

$$
\int_{\partial\omega_\varepsilon} \|\mathfrak{N}\|\, dS_{\mathbf{x}} < C\,, \tag{35}
$$

where C is independent of ε and $\|\mathfrak{N}\|$ is the norm of the matrix $\mathfrak{N}$.

Evaluating the trace of (33) on $\partial\omega_\varepsilon$ we obtain

$$
\boldsymbol{\varphi} = \mathbf{g} + S_\varepsilon \mathbf{g}\,, \tag{36}
$$

where the operator S_ε is defined by

$$
\begin{aligned}
S_\varepsilon \mathbf{g} &= \mathrm{Tr}_{\partial\omega_\varepsilon}(\pi_\varepsilon - I_2)A_{\mathbf{g}}\\
&\quad - \mathrm{Tr}_{\partial\omega_\varepsilon} P_\Omega(T_{\partial\Omega} P_{C\bar{\omega}_\varepsilon}(\mathbf{g} - A_{\mathbf{g}}))\,.
\end{aligned}
$$

By (34), (35) and (29)

$$
\|\mathrm{Tr}_{\partial\omega_\varepsilon}(\boldsymbol{\pi}_\varepsilon - I_2)A_{\mathbf{g}}\|_{C(\partial\omega_\varepsilon)} \le \mathrm{const}\ \varepsilon \|\mathbf{g}\|_{C(\partial\omega_\varepsilon)}\,. \tag{37}
$$

Lemma 3 implies

$$
|\mathbf{x}|\,|P_{C\bar{\omega}_\varepsilon}(\mathbf{g} - A_{\mathbf{g}})(\mathbf{x})| \le \mathrm{const}\ \varepsilon\,, \tag{38}
$$

for all $\mathbf{x} \in \Omega_\varepsilon$.

Combining (37) and (38) we conclude

$$
\|S_\varepsilon\|_{C(\partial\omega_\varepsilon)\to C(\partial\omega_\varepsilon)} \le \mathrm{const}\ \varepsilon\,. \tag{39}
$$

It follows from (36) that $\mathbf{g} = (I + S_\varepsilon)^{-1}\boldsymbol{\varphi}$, and then we deduce

$$\|\mathbf{g}\|_{C(\partial\omega_\varepsilon)} \leq \text{const } \|\boldsymbol{\varphi}\|_{C(\partial\omega_\varepsilon)} \; .$$

Owing to Lemmas 2 and 3 we obtain

$$\max_{\bar{\Omega}_\varepsilon} |\mathbf{u}| \leq \text{const } \|\mathbf{g}\|_{C(\partial\omega_\varepsilon)} \leq \text{const } \|\boldsymbol{\varphi}\|_{C(\partial\omega_\varepsilon)} \; . \tag{40}$$

Second, we consider *the case of the inhomogeneous boundary condition on* $\partial\Omega$

$$L\,(\partial_{\mathbf{x}})\,\mathbf{u}(\mathbf{x}) = \mathbf{O} \, , \quad \mathbf{x} \in \Omega_\varepsilon \, , \tag{41}$$
$$\mathbf{u}(\mathbf{x}) = \boldsymbol{\psi}(\mathbf{x}) \, , \quad \mathbf{x} \in \partial\Omega \, , \tag{42}$$
$$\mathbf{u}(\mathbf{x}) = \mathbf{O} \, , \quad \mathbf{x} \in \partial\omega_\varepsilon \; . \tag{43}$$

The solution is sought in the form

$$\mathbf{u} = P_\Omega \boldsymbol{\psi} + \mathbf{v} \, , \tag{44}$$

where the second term $\mathbf{v}$ is defined as a solution of the problem, which is similar to (30)–(32), with the boundary condition on $\partial\omega_\varepsilon$ being replaced by

$$\mathbf{v}(\mathbf{x}) = -(\text{Tr}_{\partial F_\varepsilon} P_\Omega \boldsymbol{\psi})(\mathbf{x}) \, , \quad \mathbf{x} \in \partial\omega_\varepsilon \; .$$

According to the result of first part of the proof (40), we have

$$\begin{aligned} \max_{\bar{\Omega}_\varepsilon} |\mathbf{v}| &\leq \text{const } \max_{\partial\omega_\varepsilon} |\text{Tr}_{\partial\omega_\varepsilon} P_\Omega \boldsymbol{\psi}| \\ &\leq \text{const } \|\boldsymbol{\psi}\|_{C(\partial\Omega)} \; . \end{aligned} \tag{45}$$

It follows from Lemma 2 that

$$\max_{\bar{\Omega}_\varepsilon} |P_\Omega \boldsymbol{\psi}| \leq \text{const } \|\boldsymbol{\psi}\|_{C(\partial\Omega)} \; . \tag{46}$$

Combining (44), (45) and (46) we deduce

$$\max_{\bar{\Omega}_\varepsilon} |\mathbf{u}| \leq \text{const } \|\boldsymbol{\psi}\|_{C(\partial\Omega)} \; .$$

This completes the proof for the case $n = 2$.

3.5 The proof of Lemma 1 for $n = 3$

First, we address the formulation (30)–(32), where Ω_ε is a domain in $\mathbb{R}^3$, and the inhomogeneous boundary condition is specified on $\partial\omega_\varepsilon$.

The solution is sought in the form

$$\mathbf{u} = P_{C\bar{\omega}_\varepsilon}\mathbf{g} - P_\Omega(\mathrm{Tr}_{\partial\Omega} P_{C\bar{\omega}_\varepsilon}\mathbf{g}) \;, \tag{47}$$

with $\mathbf{g}$ being an unknown function. Evaluating the trace of (47) on $\partial\omega_\varepsilon$ we obtain

$$\boldsymbol{\varphi} = \mathbf{g} + S_\varepsilon \mathbf{g} \;,$$

where $S_\varepsilon \mathbf{g} = -\mathrm{Tr}_{\partial\omega_\varepsilon} P_\Omega(\mathrm{Tr}_{\partial\Omega} P_{C\bar{\omega}_\varepsilon}\mathbf{g})$.

Since $\|\mathrm{Tr}_{\partial\Omega} P_{C\bar{\omega}_\varepsilon}\mathbf{g}\|_{C(\partial\Omega)} \le C\varepsilon$ it follows from Lemma 2 that

$$\|S_\varepsilon\|_{C(\partial\omega_\varepsilon)\to C(\partial\omega_\varepsilon)} \le \text{const } \varepsilon \;.$$

Hence

$$\mathbf{g} = (I + S_\varepsilon)^{-1}\boldsymbol{\varphi} \;,$$

and the following estimate holds

$$\|\mathbf{g}\|_{C(\partial\omega_\varepsilon)} \le \text{const } \|\boldsymbol{\varphi}\|_{C(\partial\omega_\varepsilon)} \;.$$

Applying Lemmas 2 and 3 we conclude

$$\max_{\bar{\Omega}_\varepsilon} |\mathbf{u}| \le \text{const } \|\mathbf{g}\|_{C(\partial\omega_\varepsilon)} \le \text{const } \|\boldsymbol{\varphi}\|_{C(\partial\omega_\varepsilon)} \;.$$

The case when an inhomogeneous boundary condition is set on $\partial\Omega$ is treated similarly to the proof of subsection 3.4.

The proof of the theorem is complete. □

4 Green's tensor for a 3-dimensional domain with a small hole

This part of the paper presents a uniform asymptotic approximation of the Green's tensor $G_\varepsilon(\mathbf{x}, \mathbf{y})$ in a three-dimensional domain with a small hole, as described in Section 2 (see (2) and (3)). Before formulating the asymptotic representation, we list model domains and associated model problems required for the asymptotic algorithm.

4.1 Green's matrices for model domains in three dimensions

Let $G(\mathbf{x},\mathbf{y}) = [G^{(1)}(\mathbf{x},\mathbf{y}), G^{(2)}(\mathbf{x},\mathbf{y}), G^{(3)}(\mathbf{x},\mathbf{y})]$ and $g(\boldsymbol{\xi},\boldsymbol{\eta}) = [g^{(1)}(\boldsymbol{\xi},\boldsymbol{\eta}), g^{(2)}(\boldsymbol{\xi},\boldsymbol{\eta}), g^{(3)}(\boldsymbol{\xi},\boldsymbol{\eta})]$ denote Green's tensors for the Lamé operator

$$L := \mu\Delta + (\lambda+\mu)\nabla(\nabla\cdot\ ) \ , \tag{48}$$

in the sets Ω and $C\bar{\omega} = \mathbb{R}^3\backslash\bar{\omega}$, respectively. The tensor G solves the following problem

$$\mu\Delta_{\mathbf{x}}G(\mathbf{x},\mathbf{y}) + (\lambda+\mu)\nabla_{\mathbf{x}}(\nabla_{\mathbf{x}}\cdot G(\mathbf{x},\mathbf{y})) + \delta(\mathbf{x}-\mathbf{y})I_3 = 0I_3 \ , \quad \mathbf{x},\mathbf{y}\in\Omega \ , \tag{49}$$

$$G(\mathbf{x},\mathbf{y}) = 0I_3 \ , \quad \mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega \ , \tag{50}$$

and the tensor g is solution of

$$\mu\Delta_{\boldsymbol{\xi}}g(\boldsymbol{\xi},\boldsymbol{\eta}) + (\lambda+\mu)\nabla_{\boldsymbol{\xi}}(\nabla_{\boldsymbol{\xi}}\cdot g(\boldsymbol{\xi},\boldsymbol{\eta})) + \delta(\boldsymbol{\xi}-\boldsymbol{\eta})I_3 = 0I_3 \ , \quad \boldsymbol{\xi},\boldsymbol{\eta}\in C\bar{\omega} \ , \tag{51}$$

$$g(\boldsymbol{\xi},\boldsymbol{\eta}) = 0I_3 \ , \quad \boldsymbol{\xi}\in\partial C\bar{\omega}, \boldsymbol{\eta}\in C\bar{\omega} \ , \tag{52}$$

$$g(\boldsymbol{\xi},\boldsymbol{\eta}) \to 0I_3 \quad \text{as} \quad |\boldsymbol{\xi}|\to\infty \ . \tag{53}$$

We represent $G(\mathbf{x},\mathbf{y})$ and $g(\boldsymbol{\xi},\boldsymbol{\eta})$ as

$$G(\mathbf{x},\mathbf{y}) = \Gamma(\mathbf{x},\mathbf{y}) - H(\mathbf{x},\mathbf{y}) \ , \tag{54}$$

and

$$g(\boldsymbol{\xi},\boldsymbol{\eta}) = \Gamma(\boldsymbol{\xi},\boldsymbol{\eta}) - h(\boldsymbol{\xi},\boldsymbol{\eta}) \ , \tag{55}$$

where $\Gamma(\mathbf{x},\mathbf{y}) = [\Gamma_{ij}(\mathbf{x},\mathbf{y})]$, $i,j = 1,2,3$, is the fundamental solution of the Lamé operator whose entries are given by

$$\Gamma_{ij}(\mathbf{x},\mathbf{y}) = (8\pi\mu(\lambda+2\mu)|\mathbf{x}-\mathbf{y}|)^{-1}((\lambda+\mu)(x_i-y_i)(x_j-y_j)|\mathbf{x}-\mathbf{y}|^{-2}+(\lambda+3\mu)\delta_{ij}) \ , \tag{56}$$

and H, h are the regular parts of G, g respectively.

4.2 The elastic capacitary potential matrix

By $P(\boldsymbol{\xi}) = [P^{(1)}(\boldsymbol{\xi}), P^{(2)}(\boldsymbol{\xi}), P^{(3)}(\boldsymbol{\xi})]$ we mean the elastic capacitary potential matrix of the set ω, whose columns satisfy

$$\mu\Delta_{\boldsymbol{\xi}}P^{(j)}(\boldsymbol{\xi}) + (\lambda+\mu)\nabla_{\boldsymbol{\xi}}(\nabla_{\boldsymbol{\xi}}\cdot P^{(j)}(\boldsymbol{\xi})) = \mathbf{O} \quad \text{in } C\bar{\omega} \ , \tag{57}$$

$$P^{(j)}(\boldsymbol{\xi}) = \mathbf{e}^{(j)} \quad \text{on } \partial C\bar{\omega} \ , \tag{58}$$

$$P^{(j)}(\boldsymbol{\xi}) \to \mathbf{O} \quad \text{as} \quad |\boldsymbol{\xi}|\to\infty \ , \tag{59}$$

for $j = 1,2,3$, where $\mathbf{e}^{(j)}$ is a basis vector, whose j^{th} entry is equal to 1, and all other entries are zero.

Lemma 4 *The columns $P^{(j)}$, $j = 1,2,3$, of the elastic capacitary potential satisfy the inequality*

$$\sup_{\boldsymbol{\xi} \in C\bar{\omega}} \{|\boldsymbol{\xi}||P^{(j)}(\boldsymbol{\xi})|\} \leq \text{const} . \tag{60}$$

Proof. The proof follows directly from the maximum principle for unbounded domains (cf. Lemma 3). □

In the sequel, we will need the following lemma, which is a reformulation of that by Kondratiev and Oleinik, in [4] (p. 78).

Lemma 5 *Suppose the columns $u^{(j)}(\boldsymbol{\xi})$ of the matrix $u(\boldsymbol{\xi})$ are solutions of*

$$\mu \Delta u^{(j)}(\boldsymbol{\xi}) + (\lambda + \mu)\nabla(\nabla \cdot u^{(j)}(\boldsymbol{\xi})) = \mathbf{O} , \qquad \text{in } C\bar{\omega} ,$$

and that $|u^{(j)}(\boldsymbol{\xi})| \leq \text{const } (1 + |\boldsymbol{\xi}|)^k$, $k \geq 0$, for $j = 1,2,3$.

Then for $|\boldsymbol{\xi}| > 2$

$$u^{(j)}(\boldsymbol{\xi}) = \mathscr{P}_k^{(j)}(\boldsymbol{\xi}) + \Gamma(\boldsymbol{\xi}, \mathbf{O})C^{(j)} + O(|\boldsymbol{\xi}|^{-2}) , \tag{61}$$

where $\mathscr{P}_k^{(j)}(\boldsymbol{\xi}) = \{\mathscr{P}_i^{(j,k)}(\boldsymbol{\xi})\}_{i=1}^3$, $\mathscr{P}_i^{(j,k)}(\boldsymbol{\xi})$ are polynomials of order not greater than k, $C^{(j)} = \{C_i^{(j)}\}_{i=1}^3$, where $C_i^{(j)}$ are constants.

4.2.1 Properties of the elastic capacity matrix

Let $B = [B_{ij}]$, $i,j = 1,2,3$ be a constant matrix that we shall call the elastic capacity matrix of the set ω. In the present subsection, we will discuss some properties of the elastic capacity matrix. The aim of this subsection is to show that upper and lower elastic capacity (obtained from the maximum and minimum eigenvalues of B, respectively) are equivalent to electrostatic capacity.

Throughout we will need the following Lemma related to the asymptotic behaviour of P.

Lemma 6 *If $|\boldsymbol{\xi}| \geq 2$, then for $P^{(j)}$ the following estimate holds*

$$|P^{(j)}(\boldsymbol{\xi}) - B_{ij}\Gamma^{(i)}(\boldsymbol{\xi}, \mathbf{O})| \leq \text{const } |\boldsymbol{\xi}|^{-2} , \tag{62}$$

for $j = 1,2,3$, where $\Gamma^{(i)}$ are columns of the fundamental solution for the Lamé operator and B_{ij} are entries of the elastic capacity matrix B of the set ω.

Proof. By Lemma 4, it is sufficient to take $P(\boldsymbol{\xi}) = O(1)$, then from Lemma 5, for $|\boldsymbol{\xi}| \geq 2$ the columns $P^{(j)}(\boldsymbol{\xi})$ can be written in the following way

$$P^{(j)}(\boldsymbol{\xi}) = K^{(j)} + \Gamma(\boldsymbol{\xi}, \mathbf{O})C^{(j)} + O(|\boldsymbol{\xi}|^{-2}) \ , \tag{63}$$

where $K^{(j)}$ is vector independent of $\boldsymbol{\xi}$,

Condition (59), implies $K^{(j)} \equiv \mathbf{O}$ and taking $C^{(j)} = B^{(j)}$ we obtain (62). □

We also use the electrostatic potential $\mathcal{P}$ for the unbounded set $C\bar{\omega}$ with electrostatic capacity cap ω, as a solution of the problem

$$\Delta_{\boldsymbol{\xi}}\mathcal{P}(\boldsymbol{\xi}) = 0 \ , \quad \boldsymbol{\xi} \in \Omega_\varepsilon \ , \tag{64}$$

$$\mathcal{P}(\boldsymbol{\xi}) = 1 \ , \quad \boldsymbol{\xi} \in \partial\omega \ , \tag{65}$$

$$\mathcal{P}(\boldsymbol{\xi}) \to 0 \quad \text{as} \quad |\boldsymbol{\xi}| \to \infty \ . \tag{66}$$

The electrostatic energy for a scalar function u in a domain $T \subset \mathbb{R}^n$ is defined as

$$\mathcal{E}(u, T) = \int_T |\nabla u|^2 \, d\mathbf{x} \ . \tag{67}$$

It is well known that for the function $\mathcal{P}$, we have for the energy functional $\mathcal{E}$ in $C\bar{\omega}$

$$\mathcal{E}(\mathcal{P}, C\bar{\omega}) = \int_{C\bar{\omega}} |\nabla \mathcal{P}|^2 \, d\boldsymbol{\xi} = \text{cap } \omega \ . \tag{68}$$

In contrast, the elastic energy functional for a vector $\mathbf{u}$ in the domain T is given by

$$\mathscr{E}(\mathbf{u}, T) = 2^{-1} \int_T e_{ij}(\mathbf{u})\sigma_{ij}(\mathbf{u}) \, d\mathbf{x} \ , \tag{69}$$

also we define the elastic energy matrix $E = [E_{ij}]_{i,j=3}^{3}$ for a matrix $\mathcal{A}$ in the domain T with entries

$$E_{ij}(\mathcal{A}, T) = 2^{-1} \int_T e_{st}(\mathcal{A}^{(i)})\sigma_{st}(\mathcal{A}^{(j)}) \, d\mathbf{x} \ , \tag{70}$$

where $\mathcal{A}^{(i)}$, $i = 1, 2, 3$ are the columns of the matrix $\mathcal{A}$. Clearly, the diagonal entries E_{11}, E_{22} and E_{33} give the elastic energy for the vectors $\mathcal{A}^{(i)}$, $i = 1, 2, 3$ respectively.

We shall show that the elastic energy matrix can be represented in terms of the elastic capacity matrix B of the set ω, by considering the entries of elastic energy matrix for the matrix function P, defined as a solution of (57)–(59).

Lemma 7 *i) For the elastic capacitary potential P, we have*

$$E(P(\boldsymbol{\xi}), C\bar{\omega}) = 2^{-1}B \, , \tag{71}$$

where B is the elastic capacity matrix of the set ω and ii) this matrix is symmetric.

Proof. *i*) We take a ball $B_R = \{\boldsymbol{\xi} : |\boldsymbol{\xi}| < R\}$ with sufficiently large radius R. We consider the component E_{jk} of the elastic energy matrix in the domain $B_R \backslash \bar{\omega}$ as follows

$$\begin{aligned} E_{jk}(P(\boldsymbol{\xi}), B_R\backslash\bar{\omega}) &= 2^{-1} \int_{B_R\backslash\bar{\omega}} e_{st}(P^{(j)}(\boldsymbol{\xi}))\sigma_{st}(P^{(k)}(\boldsymbol{\xi}))\, dS_{\boldsymbol{\xi}} \\ &= 2^{-1} \int_{\partial(B_R\backslash\bar{\omega})} P^{(j)}(\boldsymbol{\xi}) \cdot T_n(P^{(k)}(\boldsymbol{\xi}))\, dS_{\boldsymbol{\xi}} \, , \end{aligned} \tag{72}$$

where we have used Betti's formula and the fact that the columns of P satisfy the homogeneous Lamé equation. Noting the boundary condition (58), the preceding equation may be written as

$$\begin{aligned} E_{jk}(P(\boldsymbol{\xi}), B_R\backslash\bar{\omega}) &= 2^{-1} \Big\{ \int_{\partial B_R} P^{(j)}(\boldsymbol{\xi}) \cdot T_n(P^{(k)}(\boldsymbol{\xi}))\, dS_{\boldsymbol{\xi}} \\ &\quad + \int_{\partial\omega} \mathbf{e}^{(j)} \cdot T_n(P^{(k)}(\boldsymbol{\xi}))\, dS_{\boldsymbol{\xi}} \Big\} \, . \end{aligned} \tag{73}$$

Applying Betti's formula once more to the vectors $\mathbf{e}^{(j)}$ and $P^{(k)}(\boldsymbol{\xi})$, we have

$$\begin{aligned} E_{jk}(P(\boldsymbol{\xi}), B_R\backslash\bar{\omega}) &= 2^{-1} \Big\{ \int_{\partial B_R} P^{(j)}(\boldsymbol{\xi}) \cdot T_n(P^{(k)}(\boldsymbol{\xi}))\, dS_{\boldsymbol{\xi}} \\ &\quad - \int_{\partial B_R} \mathbf{e}^{(j)} \cdot T_n(P^{(k)}(\boldsymbol{\xi}))\, dS_{\boldsymbol{\xi}} \Big\} \, , \end{aligned} \tag{74}$$

which holds for all R. Using the asymptotic representation for P given in Lemma 6, we pass to the limit as $R \to \infty$ yielding

$$\begin{aligned} E_{jk}(P(\boldsymbol{\xi}), C\bar{\omega}) &= -2^{-1} \lim_{R\to\infty} \int_{\partial B_R} B_{rk}\sigma_{jp}(\Gamma^{(r)}(\boldsymbol{\xi}, \mathbf{O}))n_p\, dS_{\boldsymbol{\xi}} \\ &= 2^{-1}B_{jk} \, , \end{aligned} \tag{75}$$

where (75) has been obtained via Betti's formula applied to the vectors $\mathbf{e}^{(j)}$ and $\Gamma^{(r)}(\boldsymbol{\xi}, \mathbf{O})$ in B_R. Thus we have proved relation (71).

ii) Now we prove the symmetry of the matrix B. Again using Lemma 6, we take the limit in (73) as $R \to \infty$, then comparing to (75), we have

$$\int_{\partial\omega} \mathbf{e}^{(j)} \cdot T_n(P^{(k)}(\boldsymbol{\xi}))\, dS_{\boldsymbol{\xi}} = B_{jk} \, . \tag{76}$$

Then, interchanging the indices k and j, and subtracting the result from (76) gives

$$B_{jk} - B_{kj} = \int_{\partial\omega} \{\mathbf{e}^{(j)} \cdot T_n(P^{(k)}(\boldsymbol{\xi})) - \mathbf{e}^{(k)} \cdot T_n(P^{(j)}(\boldsymbol{\xi}))\}\, dS_{\boldsymbol{\xi}} \, . \tag{77}$$

Recalling that on $\partial\omega$ we have $P^{(j)}(\boldsymbol{\xi}) = \mathbf{e}^{(j)}$, for $j = 1, 2, 3$, we see that the right-hand side is the result of application of the Betti formula to vectors $P^{(j)}(\boldsymbol{\xi})$ and $P^{(k)}(\boldsymbol{\xi})$ in $C\bar{\omega}$. Namely in (77) we have

$$B_{jk} - B_{kj} = \int_{C\bar{\omega}} \{P^{(j)}(\boldsymbol{\xi}) \cdot L(P^{(k)}(\boldsymbol{\xi})) - P^{(k)}(\boldsymbol{\xi}) \cdot L(P^{(j)}(\boldsymbol{\xi}))\}\, dS_{\boldsymbol{\xi}} \, . \tag{78}$$

Since the columns of P are solutions to the homogeneous Lamé equation the right-hand side in (78) is zero and

$$B_{jk} = B_{kj} \, ,$$

i.e. the capacity matrix B is symmetric.

□

Next we prove that the elastic capacity matrix B represents a tensor.

Lemma 8 *The elastic capacity matrix is a Cartesian tensor of rank* 2.

Proof. Let $l = [l_{mk}]_{m,k=1}^3$ be a arbitrary matrix of rotation and consider the matrix $\mathfrak{P}$ with columns $\mathfrak{P}^{(m)} = l_{mk} P^{(k)}$, where $P^{(k)}$, $k = 1, 2, 3$, are columns of the elastic capacitary potential. By definition of the vectors $P^{(k)}$, the vector functions $\mathfrak{P}^{(m)}$ solve the problem

$$\mu \Delta_{\boldsymbol{\xi}} \mathfrak{P}^{(m)}(\boldsymbol{\xi}) + (\lambda + \mu) \nabla_{\boldsymbol{\xi}} (\nabla_{\boldsymbol{\xi}} \cdot \mathfrak{P}^{(m)}(\boldsymbol{\xi})) = \mathbf{O} \quad \text{in } C\bar{\omega} \, , \tag{79}$$

$$\mathfrak{P}^{(m)}(\boldsymbol{\xi}) = (l^T)^{(m)} \quad \text{on } \partial C\bar{\omega} \, , \tag{80}$$

$$\mathfrak{P}^{(m)}(\boldsymbol{\xi}) \to \mathbf{O} \quad \text{as} \quad |\boldsymbol{\xi}| \to \infty \, . \tag{81}$$

In a similar way to the proof of Lemma 6, the asymptotic representation for $\mathfrak{P}^{(m)}$ is given as

$$\mathfrak{P}^{(m)}(\boldsymbol{\xi}) = \Gamma(\boldsymbol{\xi}, \mathbf{O}) \mathfrak{B}^{(m)} + O(|\boldsymbol{\xi}|^{-2}) \, , \tag{82}$$

where $\mathfrak{B}^{(m)}$, $m = 1, 2, 3$ are the columns of the elastic capacity matrix of the set ω in the rotated system, and for this we have

$$E_{mn}(\mathfrak{P}(\boldsymbol{\xi}), C\bar{\omega}) = 2^{-1}\mathfrak{B}_{mn} \ , \tag{83}$$

as in Lemma 7.

Also, by definition of $\mathfrak{P}^{(m)}$, the following representation holds

$$\mathfrak{P}^{(m)}(\boldsymbol{\xi}) = l_{mk}\Gamma(\boldsymbol{\xi}, \mathbf{O})B^{(k)} + O(|\boldsymbol{\xi}|^{-2}) \ , \tag{84}$$

obtained by using Lemma 6 for the columns of P.

Considering the entry E_{mn} of the elastic energy matrix in the domain $B_R\backslash\bar{\omega}$ and using the representation (84) and the same procedure as used in the proof of (71), we obtain that

$$E_{mn}(\mathfrak{P}(\boldsymbol{\xi}), C\bar{\omega}) = 2^{-1}l_{mq}l_{nk}B_{qk} \ . \tag{85}$$

Comparing (83), (85) we deduce that the elastic capacity matrix is a Cartesian tensor of rank 2. □

4.2.2 Upper and lower elastic capacity versus electrostatic capacity

Let S denote set of vector functions $\mathbf{u}$, such that

$$\mu\Delta_{\boldsymbol{\xi}}\mathbf{u}(\boldsymbol{\xi}) + (\lambda + \mu)\nabla_{\boldsymbol{\xi}}(\nabla_{\boldsymbol{\xi}} \cdot \mathbf{u}(\boldsymbol{\xi})) = \mathbf{O} \quad \text{in } C\bar{\omega} \ , \tag{86}$$

$$\mathbf{u}(\boldsymbol{\xi}) = \mathbf{c} \quad \text{on } \partial C\bar{\omega} \ , \tag{87}$$

$$\mathbf{u}(\boldsymbol{\xi}) \to \mathbf{O} \quad \text{as} \quad |\boldsymbol{\xi}| \to \infty \ , \tag{88}$$

and for $|\boldsymbol{\xi}| > 2$ has the asymptotic representation

$$\mathbf{u}(\boldsymbol{\xi}) = \Gamma(\boldsymbol{\xi}, \mathbf{O})B\mathbf{c} + O(|\boldsymbol{\xi}|^{-2}) \ , \tag{89}$$

where $\mathbf{c} = \{c_j\}_{j=1}^3$ is a constant vector with $|\mathbf{c}| = 1$.

We define the lower elastic capacity, of the set $C\bar{\omega}$, to be

$$\underline{\text{cap}}_{elast}\omega = \inf_{\substack{\mathbf{u}\in S \\ \mathbf{c},|\mathbf{c}|=1}} \mathscr{E}(\mathbf{u}, C\bar{\omega}) \ , \tag{90}$$

and upper elastic capacity as

$$\overline{\text{cap}}_{elast}\omega = \sup_{\mathbf{c},|\mathbf{c}|=1} \inf_{\mathbf{u},\in S} \mathscr{E}(\mathbf{u}, C\bar{\omega}) \ . \tag{91}$$

The following Lemma shows that upper and lower elastic capacity are equivalent to electrostatic capacity.

Lemma 9 *For the upper and lower capacities the following inequalities hold*

$$\overline{\text{cap}}_{elast}\omega \quad \le \quad k_2 \,\text{cap}\;\omega \;, \tag{92}$$

$$k_1 \,\text{cap}\;\omega \quad \le \quad \underline{\text{cap}}_{elast}\omega \;, \tag{93}$$

where $k_1 = \min\{\mu, \lambda + 2\mu\}$ *and* $k_2 = \mu + |\lambda + \mu|$*. (From which it follows*

$$\overline{\text{cap}}_{elast}\omega \le k_3 \,\underline{\text{cap}}_{elast}\omega \;, \tag{94}$$

where $k_3 = k_2/k_1$*.)*

In order that we prove the preceding Lemma, we shall need the following auxiliary inequality

Lemma 10 *For any vector function* $\mathbf{v}$ *in* $C\bar{\omega}$*, constant on* $\partial\omega$*, the elastic energy functional* $\mathscr{E}$ *satisfies the inequality*

$$k_1 \int_{C\bar{\omega}} \|\nabla \mathbf{v}\|^2 \, d\boldsymbol{\xi} \le \mathscr{E}(\mathbf{v}, C\bar{\omega}) \le k_2 \int_{C\bar{\omega}} \|\nabla \mathbf{v}\|^2 \, d\boldsymbol{\xi} \;, \tag{95}$$

where the constants k_1*,* k_2 *are the best possible.*

Proof. We take an arbitrary vector function $\mathbf{v}|_{\partial\omega} = \mathbf{b}$, where $\mathbf{b}$ is a constant vector, and consider the elastic energy for this in the domain $C\bar{\omega}$

$$\mathscr{E}(\mathbf{v}, C\bar{\omega}) = 2^{-1} \int_{C\bar{\omega}} e_{ij}(\mathbf{v})\sigma_{ij}(\mathbf{v}) \, d\boldsymbol{\xi} \;. \tag{96}$$

We may rewrite this in the following way

$$\mathscr{E}(\mathbf{v}, C\bar{\omega}) = \mu \int_{C\bar{\omega}} \|\nabla \mathbf{v}\|^2 \, d\boldsymbol{\xi} + (\lambda + \mu) \int_{C\bar{\omega}} (\nabla \cdot \mathbf{v})^2 \, d\boldsymbol{\xi} \;. \tag{97}$$

Extending $\mathbf{v}$ by $\mathbf{b}$ over the domain ω, we have using Parseval's identity and the Schwarz inequality,

$$\int_{C\bar{\omega}} (\nabla \cdot \mathbf{v})^2 \, d\boldsymbol{\xi} = \int_{\mathbb{R}^3} |\mathcal{F}(\nabla \cdot \mathbf{v})|^2 \, d\boldsymbol{\nu} \le \int_{\mathbb{R}^3} |\boldsymbol{\nu}|^2 |\mathcal{F}(\mathbf{v})|^2 \, d\boldsymbol{\nu} = \int_{C\bar{\omega}} \|\nabla \mathbf{v}\|^2 \, d\boldsymbol{\xi} \;, \tag{98}$$

where $\mathcal{F}$ is the Fourier transform and $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$ is the Fourier transform variable.

Thus using (98) in (97) we deduce that

$$\mathscr{E}(\mathbf{v}, C\bar{\omega}) \le (\mu + |\lambda + \mu|) \int_{C\bar{\omega}} \|\nabla \mathbf{v}\|^2 \, d\boldsymbol{\xi} \;. \tag{99}$$

We now consider two cases. When $\lambda + \mu > 0$, then it is clear from (97) that

$$\mathscr{E}(\mathbf{v}, C\bar{\omega}) \geq \mu \int_{C\bar{\omega}} \|\nabla \mathbf{v}\|^2 \, d\boldsymbol{\xi} \ . \tag{100}$$

Hence from (99) and (100) we have

$$\mu \int_{C\bar{\omega}} \|\nabla \mathbf{v}\|^2 \, d\boldsymbol{\xi} \leq \mathscr{E}(\mathbf{v}, C\bar{\omega}) \leq (\mu + |\lambda + \mu|) \int_{C\bar{\omega}} \|\nabla \mathbf{v}\|^2 \, d\boldsymbol{\xi} \ . \tag{101}$$

When $\lambda + \mu < 0$, we obtain from (98)

$$(\lambda + \mu) \int_{C\bar{\omega}} (\nabla \cdot \mathbf{v})^2 \, d\boldsymbol{\xi} \geq (\lambda + \mu) \int_{C\bar{\omega}} \|\nabla \mathbf{v}\|^2 \, d\boldsymbol{\xi} \ , \tag{102}$$

and so from (97) we obtain

$$(\lambda + 2\mu) \int_{C\bar{\omega}} \|\nabla \mathbf{v}\|^2 \, d\boldsymbol{\xi} \leq \mathscr{E}(\mathbf{v}, C\bar{\omega}) \ . \tag{103}$$

Therefore, from (99) and (103) we have

$$(\lambda + 2\mu) \int_{C\bar{\omega}} \|\nabla \mathbf{v}\|^2 \, d\boldsymbol{\xi} \leq \mathscr{E}(\mathbf{v}, C\bar{\omega}) \leq (\mu + |\lambda + \mu|) \int_{C\bar{\omega}} \|\nabla \mathbf{v}\|^2 \, d\boldsymbol{\xi} \ . \tag{104}$$

Combining (101) and (104) to cover both cases we obtain (95). □

Now we are in a position to prove Lemma 9.

Proof of Lemma 9. We first take $\mathbf{u} \in S$, and consider the elastic energy for this vector function in the domain $B_R \backslash \bar{\omega}$. Repeating the same procedure as in the proof (71) we obtain for the vector $\mathbf{u}$, that

$$\mathscr{E}(\mathbf{u}, C\bar{\omega}) = 2^{-1}(\mathbf{c}, B\mathbf{c}) \ . \tag{105}$$

Let α be an eigenvalue of the matrix B and $\mathbf{c}$ the corresponding eigenvector, i.e.

$$B\mathbf{c} = \alpha \mathbf{c} \ , \quad \text{where } |\mathbf{c}| = 1 \ . \tag{106}$$

From (106), we obtain that $\alpha = (\mathbf{c}, B\mathbf{c})$, this means that for (105), we have

$$\mathscr{E}(\mathbf{u}, C\bar{\omega}) = 2^{-1}\alpha \ . \tag{107}$$

Moreover, by the definition of upper and lower elastic capacity (90) we have that upper and lower elastic capacity are the maximum, minimum eigenvalues, respectively, of the elastic capacity matrix B.

We shall obtain the inequality (92) first. Let the vector $\mathbf{u}^{(1)}$ be sought in the form $\mathbf{u}^{(1)} = \mathcal{P}(\boldsymbol{\xi})\mathbf{c}$ where $\mathcal{P}$ is the electrostatic potential. Considering the Dirichlet integral for $\mathbf{u}^{(1)}$ in $C\bar{\omega}$, we obtain

$$\int_{C\bar{\omega}} \|\nabla \mathbf{u}^{(1)}\|^2 \, d\boldsymbol{\xi} = \sum_{j=1}^{3} \int_{C\bar{\omega}} c_j^2 |\nabla \mathcal{P}|^2 \, d\boldsymbol{\xi} = \text{cap } \omega \ , \tag{108}$$

since the function $\mathcal{P}$ minimises the electrostatic energy functional and $|\mathbf{c}| = 1$. Applying now the upper inequality of (95) of Lemma 10 to the vector function $\mathbf{u}^{(1)}$ we have

$$\inf_{\mathbf{u} \in S} \mathscr{E}(\mathbf{u}, C\bar{\omega}) \leq \mathscr{E}(\mathbf{u}^{(1)}, C\bar{\omega}) \leq k_2 \, \text{cap } \omega \ . \tag{109}$$

Then taking the supremum on the left hand side with respect to $\mathbf{c}$, with $|\mathbf{c}| = 1$ to arrive at

$$\overline{\text{cap}}_{elast} \omega \leq k_2 \, \text{cap } \omega \ , \tag{110}$$

which is (92) proved.

Next, we take a vector function $\mathbf{u}^{(2)} \in S$, with boundary condition $\mathbf{u}^{(2)} = \mathbf{c}^{(2)}$ on $C\bar{\omega}$ that minimises the elastic energy in $\mathbf{u}$ and $\mathbf{c}$. Applying the lower inequality of (95) to $\mathbf{u}^{(2)}$, we have

$$k_1 \int_{C\bar{\omega}} \|\nabla \mathbf{u}^{(2)}\|^2 \, d\boldsymbol{\xi} \leq \underline{\text{cap}}_{elast} \omega \ . \tag{111}$$

However the vector $\mathbf{u}^{(2)}$ is not a minimizer of the Dirichlet integral (we have seen that $\mathbf{u}^{(1)}$ is such a vector). Thus

$$k_1 \, \text{cap } \omega = k_1 \int_{C\bar{\omega}} \|\nabla \mathbf{u}^{(1)}\|^2 \, d\boldsymbol{\xi} \leq k_1 \int_{C\bar{\omega}} \|\nabla \mathbf{u}^{(2)}\|^2 \, d\boldsymbol{\xi} \leq \underline{\text{cap}}_{elast} \omega \ , \tag{112}$$

completing the proof of (93).

Combining inequalities (92) and (93), we arrive at the proof of (94). □

Hence from Lemma 9 we have the elastic capacity and the electrostatic capacity are equivalent.

4.3 Asymptotic estimates for the regular part h of Green's tensor in an unbounded domain

We now give an auxiliary result concerning an asymptotic estimate for the tensor h, which we shall make use of in the algorithm.

Lemma 11 *For all $\boldsymbol{\eta} \in C\bar{\omega}$ and $\boldsymbol{\xi}$ with $|\boldsymbol{\xi}| > 2$ the estimate holds*

$$|h^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta}) - \Gamma(\boldsymbol{\xi}, \mathbf{O}) P^{T(j)}(\boldsymbol{\eta})| \leq \text{const}\, |\boldsymbol{\xi}|^{-2} |\boldsymbol{\eta}|^{-1} , \tag{113}$$

where $j = 1, 2, 3$.

Proof. From the definition of $h(\boldsymbol{\xi}, \boldsymbol{\eta})$ in (55), the columns of $h(\boldsymbol{\xi}, \boldsymbol{\eta})$ satisfy

$$\mu \Delta_{\boldsymbol{\xi}} h^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta}) + (\lambda + \mu) \nabla_{\boldsymbol{\xi}} (\nabla_{\boldsymbol{\xi}} \cdot h^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta})) = \mathbf{O} \quad \boldsymbol{\xi}, \boldsymbol{\eta} \in C\bar{\omega} , \tag{114}$$

$$h^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \Gamma^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta}) , \quad \boldsymbol{\xi} \in \partial C\bar{\omega} \text{ and } \boldsymbol{\eta} \in C\bar{\omega} , \tag{115}$$

$$h^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta}) \to \mathbf{O} \quad \text{ as } |\boldsymbol{\xi}| \to \infty \text{ and } \boldsymbol{\eta} \in C\bar{\omega} , \tag{116}$$

for $j = 1, 2, 3$.

From Lemma 5, we see that $g^{(i)}(\boldsymbol{\xi}, \boldsymbol{\eta})$, $i = 1, 2, 3$ for sufficiently large $|\boldsymbol{\xi}|$ can be approximated by a linear combination of columns of the fundamental solution as follows

$$|\boldsymbol{\xi}|(g^{(i)}(\boldsymbol{\xi}, \boldsymbol{\eta}) - C_{ji}(\boldsymbol{\eta}) \Gamma^{(j)}(\boldsymbol{\xi}, \mathbf{O})) \xrightarrow{|\boldsymbol{\xi}| \to \infty} \mathbf{O} . \tag{117}$$

We now apply Betti's formula to tensors $g^{(k)}(\boldsymbol{\xi}, \boldsymbol{\eta})$ and $\mathbf{e}^{(l)} - P^{(l)}(\boldsymbol{\xi})$, $k, l = 1, 2, 3$, in the domain $B_R \backslash \bar{\omega}$ where $B_R = \{\boldsymbol{\xi} : |\boldsymbol{\xi}| < R\}$ is a ball with sufficiently large radius R. Recalling $P^{(j)}(\boldsymbol{\xi}) = \mathbf{e}^{(j)}$ and $g^{(k)}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \mathbf{O}$ when $\boldsymbol{\xi} \in \partial C\bar{\omega}$, we have

$$\int_{B_R \backslash \bar{\omega}} e_{ij}(g^{(k)}(\boldsymbol{\xi}, \boldsymbol{\eta})) \sigma_{ij}(P^{(l)}(\boldsymbol{\xi}))\, d\boldsymbol{\xi}$$

$$= P_{kl}(\boldsymbol{\eta}) - \delta_{kl} - \int_{\partial B_R} (\delta_{il} - P_{il}(\boldsymbol{\xi})) \sigma_{ij}(g^{(k)}(\boldsymbol{\xi}, \boldsymbol{\eta})) n_j \, dS_{\boldsymbol{\xi}} , \tag{118}$$

and

$$\int_{B_R \backslash \bar{\omega}} e_{ij}(g^{(k)}(\boldsymbol{\xi}, \boldsymbol{\eta})) \sigma_{ij}(P^{(l)}(\boldsymbol{\xi}))\, d\boldsymbol{\xi} = \int_{\partial B_R} g_{ik}(\boldsymbol{\xi}, \boldsymbol{\eta}) \sigma_{ij}(P^{(l)}(\boldsymbol{\xi})) n_j \, dS_{\boldsymbol{\xi}} , \tag{119}$$

for $k, l = 1, 2, 3$.

Then from (118), (119) we have

$$\begin{aligned} \delta_{kl} - P_{kl}(\boldsymbol{\eta}) \;\; = \;\; & - \int_{\partial B_R} \big\{ (\delta_{il} - P_{il}(\boldsymbol{\xi})) \sigma_{ij}(g^{(k)}(\boldsymbol{\xi}, \boldsymbol{\eta})) n_j \\ & + g_{ik}(\boldsymbol{\xi}, \boldsymbol{\eta}) \sigma_{ij}(P^{(l)}(\boldsymbol{\xi})) n_j \big\} \, dS_{\boldsymbol{\xi}} . \end{aligned} \tag{120}$$

Using the asymptotic representation for g given in (117) and that for P given in Lemma 6, we take the limit in (120) as $R \to \infty$ and obtain

$$\delta_{kl} - P_{kl}(\boldsymbol{\eta}) = -\lim_{R\to\infty} \int_{\partial B_R} C_{rk}(\boldsymbol{\eta})\sigma_{lj}(\Gamma^{(r)}(\boldsymbol{\xi}, \mathbf{O}))n_j \, dS_{\boldsymbol{\xi}} \, . \tag{121}$$

Computing the above integral, by applying integration by parts to $\mathbf{e}^{(l)}$ and $\Gamma^{(r)}(\boldsymbol{\xi}, \mathbf{O})$ in B_R, yields

$$\delta_{kl} - P_{kl}(\boldsymbol{\eta}) = C_{lk}(\boldsymbol{\eta}) \, , \tag{122}$$

or equivalently in the form of matrices

$$I_3 - P^T(\boldsymbol{\eta}) = C(\boldsymbol{\eta}) \, . \tag{123}$$

Let $|\boldsymbol{\xi}| > 2$. Then for $\boldsymbol{\eta} \in \partial C\bar{\omega}$

$$|h^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta}) - \Gamma(\boldsymbol{\xi}, \mathbf{O})P^{T(j)}(\boldsymbol{\eta})| = |h^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta}) - \Gamma^{(j)}(\boldsymbol{\xi}, \mathbf{O})|$$

$$= |\Gamma^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta}) - \Gamma^{(j)}(\boldsymbol{\xi}, \mathbf{O})| \leq \text{const } |\boldsymbol{\eta}||\boldsymbol{\xi}|^{-2} \leq \text{const } |\boldsymbol{\xi}|^{-2} \, , \tag{124}$$

here we have used that for $\boldsymbol{\eta} \in \partial C\bar{\omega}$, $|\boldsymbol{\eta}| \leq 1$. By Lemma 1 for functions satisfying the Lamé equation in $\boldsymbol{\eta}$, we have from (124) that

$$|h^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta}) - \Gamma(\boldsymbol{\xi}, \mathbf{O})P^{T(j)}(\boldsymbol{\eta})| \leq \text{const } |\boldsymbol{\xi}|^{-2}|\boldsymbol{\eta}|^{-1} \, , \tag{125}$$

for $\boldsymbol{\eta} \in C\bar{\omega}$ and $|\boldsymbol{\xi}| > 2$. □

4.4 A uniform asymptotic formula for Green's function G_ε in three dimensions

Now we present the main result concerning the uniform approximation of Green's tensor G_ε in the case of 3-dimensions.

Theorem 1 *Green's tensor $G_\varepsilon(\mathbf{x}, \mathbf{y})$ for the Lamé operator in $\Omega_\varepsilon \subset \mathbb{R}^3$ admits the representation*

$$\begin{aligned} G_\varepsilon(\mathbf{x}, \mathbf{y}) \; = \; & G(\mathbf{x}, \mathbf{y}) + \varepsilon^{-1}g(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) - \Gamma(\mathbf{x}, \mathbf{y}) + P(\varepsilon^{-1}\mathbf{x})H(\mathbf{O}, \mathbf{y}) \\ & + H(\mathbf{x}, \mathbf{O})P^T(\varepsilon^{-1}\mathbf{y}) - P(\varepsilon^{-1}\mathbf{x})H(\mathbf{O}, \mathbf{O})P^T(\varepsilon^{-1}\mathbf{y}) \\ & - \varepsilon H(\mathbf{x}, \mathbf{O})BH(\mathbf{O}, \mathbf{y}) + O(\varepsilon^2(\min\{|\mathbf{x}|, |\mathbf{y}|\})^{-1}) \, , \end{aligned} \tag{126}$$

uniformly with respect to $\mathbf{x}$, $\mathbf{y} \in \Omega_\varepsilon$.

As in [6], we present a formal argument concerning the structure of $G_\varepsilon(\mathbf{x},\mathbf{y})$.

Let G_ε be represented in the form

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = \Gamma(\mathbf{x},\mathbf{y}) - H_\varepsilon(\mathbf{x},\mathbf{y}) - h_\varepsilon(\mathbf{x},\mathbf{y}) \; , \tag{127}$$

where the columns of $H_\varepsilon(\mathbf{x},\mathbf{y}) = [H_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})]$, $h_\varepsilon(\mathbf{x},\mathbf{y}) = [h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})]$, $j = 1,2,3$, satisfy the Dirichlet problems

$$\begin{gathered}\mu\Delta_{\mathbf{x}} H_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) + (\lambda+\mu)\nabla_{\mathbf{x}}(\nabla_{\mathbf{x}}\cdot H_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})) = \mathbf{O} \; , \quad \mathbf{x},\mathbf{y}\in\Omega_\varepsilon \; , \\ H_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = \Gamma^{(j)}(\mathbf{x},\mathbf{y}) \; , \quad \mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon \; , \\ H_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = \mathbf{O} \; , \quad \mathbf{x}\in\partial C\bar{\omega}_\varepsilon, \mathbf{y}\in\Omega_\varepsilon \; ,\end{gathered}$$

and

$$\begin{gathered}\mu\Delta_{\mathbf{x}} h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) + (\lambda+\mu)\nabla_{\mathbf{x}}(\nabla_{\mathbf{x}}\cdot h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})) = \mathbf{O} \; , \quad \mathbf{x},\mathbf{y}\in\Omega_\varepsilon \; , \\ h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = \Gamma^{(j)}(\mathbf{x},\mathbf{y}) \; , \quad \mathbf{x}\in\partial C\bar{\omega}_\varepsilon, \mathbf{y}\in\Omega_\varepsilon \\ h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = \mathbf{O} \; , \quad \mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon \; .\end{gathered} \tag{128}$$

From (127), it is enough to approximate the columns of H_ε and h_ε, to obtain the asymptotic formula for G_ε.

Approximation of $H_\varepsilon(\mathbf{x},\mathbf{y})$. Consider $H_\varepsilon(\mathbf{x},\mathbf{y}) - H(\mathbf{x},\mathbf{y})$, which satisfies the homogeneous Lamé equation and has zero boundary value when $\mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon$. When $\mathbf{x}\in\partial C\bar{\omega}_\varepsilon$, the leading part of $H_\varepsilon(\mathbf{x},\mathbf{y}) - H(\mathbf{x},\mathbf{y})$ is given by $-H(\mathbf{O},\mathbf{y})$. We extend $-H(\mathbf{O},\mathbf{y})$ onto $C\bar{\omega}_\varepsilon$ to a tensor that satisfies the homogeneous Lamé equation in variable $\mathbf{x}$, in the form $-P(\varepsilon^{-1}\mathbf{x})H(\mathbf{O},\mathbf{y})$, whose leading order part is $-\varepsilon\Gamma(\mathbf{x},\mathbf{O})BH(\mathbf{O},\mathbf{y})$ for $\mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon$. Thus

$$\begin{aligned}H_\varepsilon(\mathbf{x},\mathbf{y}) - H(\mathbf{x},\mathbf{y}) \;&=\; -P(\varepsilon^{-1}\mathbf{x})H(\mathbf{O},\mathbf{y}) + \varepsilon H(\mathbf{x},\mathbf{O})BH(\mathbf{O},\mathbf{y}) \\ &\quad + \mathfrak{H}_\varepsilon(\mathbf{x},\mathbf{y}) \; , \quad \mathbf{x},\mathbf{y}\in\Omega_\varepsilon \; ,\end{aligned} \tag{129}$$

where $\mathfrak{H}_\varepsilon(\mathbf{x},\mathbf{y})$ is the remainder term produced by this approximation.

Approximation of $h_\varepsilon(\mathbf{x},\mathbf{y})$. Using the definition of h and (128) of h_ε, we have

$$h_\varepsilon(\mathbf{x},\mathbf{y}) - \varepsilon^{-1}h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) = \mathbf{O} \quad \text{for } \mathbf{x}\in\partial C\bar{\omega}_\varepsilon \; . \tag{130}$$

Then from Lemma 11, we have

$$h_\varepsilon(\mathbf{x},\mathbf{y}) - \varepsilon^{-1}h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) = -\Gamma(\mathbf{x},\mathbf{O})P^T(\boldsymbol{\eta}) + O(\varepsilon^2(|\mathbf{x}|^2|\mathbf{y}|)^{-1}) \; ,$$

for $\mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon$. The tensor that satisfies the homogeneous Lamé equation in $\mathbf{x}$ and has boundary data $\Gamma(\mathbf{x},\mathbf{O})P^T(\boldsymbol{\eta})$ when $\mathbf{x}\in\partial\Omega$ is

$$H(\mathbf{x},\mathbf{O})P^T(\boldsymbol{\eta}) \; .$$

Thus, we have

$$h_\varepsilon(\mathbf{x},\mathbf{y})-\varepsilon^{-1}h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) = -H(\mathbf{x},\mathbf{O})P^T(\boldsymbol{\eta})+\chi_\varepsilon(\mathbf{x},\mathbf{y}) \text{ for } \mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon \ ,$$

where $\chi_\varepsilon(\mathbf{x},\mathbf{y})$ is the remainder. For $\mathbf{x}\in\partial C\bar{\omega}_\varepsilon$, $\chi_\varepsilon(\mathbf{x},\mathbf{y}) = H(\mathbf{x},\mathbf{O})P^T(\boldsymbol{\eta})$. Since the components of $H(\mathbf{x},\mathbf{O})$ are smooth for $\mathbf{x}$, $\mathbf{y}\in\Omega$, we may approximate the latter by $H(\mathbf{O},\mathbf{O})P^T(\boldsymbol{\eta})$. However this tensor is not necessarily small. Making an extension of $H(\mathbf{O},\mathbf{O})P^T(\boldsymbol{\eta})$ to a tensor which satisfies the homogeneous Lamé equation for $\mathbf{x}\in C\bar{\omega}_\varepsilon$, and is small for $\mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon$, we have

$$\chi_\varepsilon(\mathbf{x},\mathbf{y}) = P(\varepsilon^{-1}\mathbf{x})H(\mathbf{O},\mathbf{O})P^T(\varepsilon^{-1}\mathbf{y}) + \mathfrak{h}_\varepsilon(\mathbf{x},\mathbf{y}) \ ,$$

where $\mathfrak{h}_\varepsilon(\mathbf{x},\mathbf{y})$ is the new remainder. Hence we may now assume the asymptotic representation

$$\begin{aligned} h_\varepsilon(\mathbf{x},\mathbf{y}) - \varepsilon^{-1}h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) \ &= \ -H(\mathbf{x},\mathbf{O})P^T(\varepsilon^{-1}\mathbf{y}) \\ &\quad +P(\varepsilon^{-1}\mathbf{x})H(\mathbf{O},\mathbf{O})P^T(\varepsilon^{-1}\mathbf{y}) \\ &\quad +\mathfrak{h}_\varepsilon(\mathbf{x},\mathbf{y}) \ , \end{aligned} \tag{131}$$

for $\mathbf{x},\mathbf{y}\in\Omega_\varepsilon$.

Combined formula. Combining (131) and (129) in (127), yields

$$\begin{aligned} G_\varepsilon(\mathbf{x},\mathbf{y}) \ &= \ \Gamma(\mathbf{x},\mathbf{y}) - H(\mathbf{x},\mathbf{y}) + P(\varepsilon^{-1}\mathbf{x})H(\mathbf{O},\mathbf{y}) \\ &\quad -\varepsilon H(\mathbf{x},\mathbf{O})BH(\mathbf{O},\mathbf{y}) - \varepsilon^{-1}h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) \\ &\quad +H(\mathbf{x},\mathbf{O})P^T(\varepsilon^{-1}\mathbf{y}) - P(\varepsilon^{-1}\mathbf{x})H(\mathbf{O},\mathbf{O})P^T(\varepsilon^{-1}\mathbf{y}) \\ &\quad +R_\varepsilon(\mathbf{x},\mathbf{y}) \ , \end{aligned} \tag{132}$$

where $R_\varepsilon(\mathbf{x},\mathbf{y})$ is the sum of the remainders $\mathfrak{H}_\varepsilon(\mathbf{x},\mathbf{y})$ and $\mathfrak{h}_\varepsilon(\mathbf{x},\mathbf{y})$, which we shall estimate. Recalling the definition of G and g from (54) and (55), the preceding expression is equivalent to

$$\begin{aligned} G_\varepsilon(\mathbf{x},\mathbf{y}) \ &= \ G(\mathbf{x},\mathbf{y}) + \varepsilon^{-1}g(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) - \Gamma(\mathbf{x},\mathbf{y}) \\ &\quad +P(\varepsilon^{-1}\mathbf{x})H(\mathbf{O},\mathbf{y}) + H(\mathbf{x},\mathbf{O})P^T(\varepsilon^{-1}\mathbf{y}) \\ &\quad -P(\varepsilon^{-1}\mathbf{x})H(\mathbf{O},\mathbf{O})P^T(\varepsilon^{-1}\mathbf{y}) - \varepsilon H(\mathbf{x},\mathbf{O})BH(\mathbf{O},\mathbf{y}) \\ &\quad +R_\varepsilon(\mathbf{x},\mathbf{y}) \ . \end{aligned} \tag{133}$$

Next we give a rigorous proof of (126).

4.4.1 Proof of Theorem 1

The columns of $R_\varepsilon(\mathbf{x},\mathbf{y})$ solve the problem

$$\mu\Delta_{\mathbf{x}} R_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) + (\lambda+\mu)\nabla_{\mathbf{x}}(\nabla_{\mathbf{x}}\cdot R_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})) = \mathbf{O} \quad \mathbf{x},\mathbf{y}\in\Omega_\varepsilon\ , \tag{134}$$

$$\begin{aligned} R_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) &= \varepsilon^{-1}h^{(j)}(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) - H(\mathbf{x},\mathbf{O})P^{T(j)}(\varepsilon^{-1}\mathbf{y}) \\ &\quad - P(\varepsilon^{-1}\mathbf{x})H^{(j)}(\mathbf{O},\mathbf{y}) + P(\varepsilon^{-1}\mathbf{x})H(\mathbf{O},\mathbf{O})P^{T(j)}(\varepsilon^{-1}\mathbf{y}) \\ &\quad + \varepsilon H(\mathbf{x},\mathbf{O})BH^{(j)}(\mathbf{O},\mathbf{y})\ , \quad \mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon\ , \end{aligned} \tag{135}$$

$$\begin{aligned} R_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) &= H^{(j)}(\mathbf{x},\mathbf{y}) - H^{(j)}(\mathbf{O},\mathbf{y}) - H(\mathbf{x},\mathbf{O})P^{T(j)}(\varepsilon^{-1}\mathbf{y}) \\ &\quad + H(\mathbf{O},\mathbf{O})P^{T(j)}(\varepsilon^{-1}\mathbf{y}) + \varepsilon H(\mathbf{x},\mathbf{O})BH^{(j)}(\mathbf{O},\mathbf{y})\ , \\ &\quad \mathbf{x}\in\partial\omega_\varepsilon, \mathbf{y}\in\Omega_\varepsilon\ . \end{aligned} \tag{136}$$

Both $H^{(j)}(\mathbf{x},\mathbf{O})$ and $H^{(j)}(\mathbf{O},\mathbf{y})$ are columns of H (see (54)), and $H^{(j)}(\mathbf{x},\mathbf{O})$ is bounded on $\partial\Omega$. They are also bounded for $\mathbf{x}\in\partial\omega_\varepsilon$, $\mathbf{y}\in\Omega_\varepsilon$. The term $\varepsilon H(\mathbf{x},\mathbf{O})BH^{(j)}(\mathbf{O},\mathbf{y})$ is bounded by const ε in (135) and (136). Since the components of $H(\mathbf{x},\mathbf{y})$ are smooth for $\mathbf{x},\mathbf{y}\in\Omega$ and by Lemma 4 the entries of the tensor $P(\boldsymbol{\xi})$ are bounded, from (136) we have

$$|H^{(j)}(\mathbf{x},\mathbf{y}) - H^{(j)}(\mathbf{O},\mathbf{y}) - (H(\mathbf{x},\mathbf{O}) - H(\mathbf{O},\mathbf{O}))P^{T(j)}(\boldsymbol{\eta})| \le \text{const}\ \varepsilon\ , \tag{137}$$

for $\mathbf{x}\in\partial\omega_\varepsilon, \mathbf{y}\in\Omega_\varepsilon$. Thus when $\mathbf{x}\in\partial\omega_\varepsilon$ and $\mathbf{y}\in\Omega_\varepsilon$

$$|R_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})| \le \text{const}\ \varepsilon\ ,$$

for $j=1,2,3$.

Next we estimate $|R_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})|$ when $\mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon$. By Lemma 4, the columns of capacitary potential satisfy the following inequality

$$|P^{(j)}(\varepsilon^{-1}\mathbf{x})| \le \text{const}\ \varepsilon|\mathbf{x}|^{-1}\ , \quad j=1,2,3, \quad \text{for } \mathbf{x}\in\Omega_\varepsilon\ . \tag{138}$$

Now, (62) of Lemma 6 and the definition of $H(\mathbf{x},\mathbf{y})$ imply

$$\begin{aligned} & |\varepsilon H(\mathbf{x},\mathbf{O})BH^{(j)}(\mathbf{O},\mathbf{y}) - P(\varepsilon^{-1}\mathbf{x})H^{(j)}(\mathbf{O},\mathbf{y})| \\ =\ & |(\Gamma(\varepsilon^{-1}\mathbf{x},\mathbf{O})B - P(\varepsilon^{-1}\mathbf{x}))H^{(j)}(\mathbf{O},\mathbf{y})| \le \text{const}\ \varepsilon^2\ , \end{aligned} \tag{139}$$

for $\mathbf{x}\in\partial\Omega$, $\mathbf{y}\in\Omega_\varepsilon$. We also have, using Lemma 11 and (138), the following estimate

$$\begin{aligned} & |\varepsilon^{-1}h^{(j)}(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) - H(\mathbf{x},\mathbf{O})P^{T(j)}(\varepsilon^{-1}\mathbf{y})| \\ =\ & \varepsilon^{-1}|h^{(j)}(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) - \Gamma(\boldsymbol{\xi},\mathbf{O})P^{T(j)}(\varepsilon^{-1}\mathbf{y})| \\ \le\ & \text{const}\ \varepsilon^2|\mathbf{x}|^{-2}|\mathbf{y}|^{-1} \le \text{const}\ \varepsilon^2|\mathbf{y}|^{-1}\ , \quad \mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon\ , \end{aligned} \tag{140}$$

where we have used the estimate (113) and for $\mathbf{x} \in \partial\Omega$, $|\mathbf{x}| \geq 1$. Combining (138), (139) and (140) in (135) we obtain

$$|R_\varepsilon^{(j)}(\mathbf{x}, \mathbf{y})| \leq \text{const } \varepsilon^2 |\mathbf{y}|^{-1} \quad \text{for } \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega_\varepsilon , \tag{141}$$

for $j = 1, 2, 3$.

Therefore, by Lemma 1, we have

$$|R_\varepsilon^{(j)}(\mathbf{x}, \mathbf{y})| \leq \text{const } \max\left\{\varepsilon^2|\mathbf{x}|^{-1} , \varepsilon^2|\mathbf{y}|^{-1}\right\} , \tag{142}$$

for $j = 1, 2, 3$, and $\mathbf{x}, \mathbf{y} \in \Omega_\varepsilon$. Thus,

$$|R_\varepsilon^{(j)}(\mathbf{x}, \mathbf{y})| \leq \text{const } \varepsilon^2 (\min\{|\mathbf{x}|, |\mathbf{y}|\})^{-1} . \tag{143}$$

The proof is complete. □

5 Green's tensor for a planar domain with a small hole

Now we present the uniform approximation of the tensor $G_\varepsilon(\mathbf{x}, \mathbf{y})$ for the case of a planar domain with a small hole, formulated in Section 2. We once again introduce model domains and governing equations needed for the study related to this case.

5.1 Green's kernels for model domains in two dimensions

Let $G(\mathbf{x}, \mathbf{y}) = [G^{(1)}(\mathbf{x}, \mathbf{y}), G^{(2)}(\mathbf{x}, \mathbf{y})]$ and $g(\boldsymbol{\xi}, \boldsymbol{\eta}) = [g^{(1)}(\boldsymbol{\xi}, \boldsymbol{\eta}), g^{(2)}(\boldsymbol{\xi}, \boldsymbol{\eta})]$ denote Green's tensor for the Lamé operator in the bounded domain Ω and $C\bar{\omega} = \mathbb{R}^2 \backslash \bar{\omega}$ respectively. The tensor G is a solution the following problem

$$\mu\Delta_{\mathbf{x}} G(\mathbf{x}, \mathbf{y}) + (\lambda+\mu)\nabla_{\mathbf{x}}(\nabla_{\mathbf{x}} \cdot G(\mathbf{x}, \mathbf{y})) + \delta(\mathbf{x}-\mathbf{y}) I_2 = 0 I_2 , \quad \mathbf{x}, \mathbf{y} \in \Omega , \tag{144}$$

$$G(\mathbf{x}, \mathbf{y}) = 0 I_2 , \quad \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega , \tag{145}$$

and the tensor g solves

$$\mu\Delta_{\boldsymbol{\xi}} g(\boldsymbol{\xi}, \boldsymbol{\eta}) + (\lambda+\mu)\nabla_{\boldsymbol{\xi}}(\nabla_{\boldsymbol{\xi}} \cdot g(\boldsymbol{\xi}, \boldsymbol{\eta})) + \delta(\boldsymbol{\xi}-\boldsymbol{\eta}) I_2 = 0 I_2 , \quad \boldsymbol{\xi}, \boldsymbol{\eta} \in C\bar{\omega} , \tag{146}$$

$$g(\boldsymbol{\xi}, \boldsymbol{\eta}) = 0 I_2 , \quad \boldsymbol{\xi} \in \partial C\bar{\omega}, \boldsymbol{\eta} \in C\bar{\omega} , \tag{147}$$

$$|g^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta})| \text{ is bounded as } |\boldsymbol{\xi}| \to \infty , \boldsymbol{\eta} \in C\bar{\omega} \text{ for } j = 1, 2 . \tag{148}$$

We represent $G(\mathbf{x},\mathbf{y})$ as

$$G(\mathbf{x},\mathbf{y}) = \gamma(\mathbf{x},\mathbf{y}) - H(\mathbf{x},\mathbf{y}) \ , \qquad (149)$$

and $g(\boldsymbol{\xi},\boldsymbol{\eta})$ as

$$g(\boldsymbol{\xi},\boldsymbol{\eta}) = \gamma(\boldsymbol{\xi},\boldsymbol{\eta}) - h(\boldsymbol{\xi},\boldsymbol{\eta}) \ , \qquad (150)$$

where H and h are the regular parts of G and g respectively, and $\gamma(\mathbf{x},\mathbf{y}) = [\gamma_{ij}(\mathbf{x},\mathbf{y})]_{i,j=1}^2$, is the fundamental solution of the Lamé operator in two dimensions with components

$$\begin{aligned}\gamma_{ij}(\mathbf{x},\mathbf{y}) &= (\lambda+3\mu)(4\pi\mu(\lambda+2\mu))^{-1}(-\log|\mathbf{x}-\mathbf{y}|\delta_{ij} \\ &\quad +(\lambda+\mu)(\lambda+3\mu)^{-1}(x_i-y_i)(x_j-y_j)|\mathbf{x}-\mathbf{y}|^{-2}) \ , \end{aligned} \qquad (151)$$

for $i,j=1,2$. We introduce the tensor ζ as

$$\zeta(\boldsymbol{\eta}) = \lim_{|\boldsymbol{\xi}|\to\infty} g(\boldsymbol{\xi},\boldsymbol{\eta}) \ , \qquad (152)$$

and the constant matrix

$$\zeta^\infty = \lim_{|\boldsymbol{\eta}|\to\infty} \{\zeta(\boldsymbol{\eta}) + \gamma(\boldsymbol{\eta},\mathbf{O})\} \ , \qquad (153)$$

where it will be shown that ζ^∞ is a symmetric matrix.

5.2 Auxiliary properties of the regular part h of Green's tensor for an unbounded planar domain and the tensor ζ

In the present subsection, we shall formulate and prove an asymptotic representation for the regular part h of Green's tensor g, in the unbounded domain. For this we shall need the following Lemma which is the two dimensional analog of Lemma 5.

Lemma 12 *Suppose the columns $u^{(j)}(\boldsymbol{\xi})$ of the matrix $u(\boldsymbol{\xi})$ are solutions of*

$$\mu\Delta u^{(j)}(\boldsymbol{\xi}) + (\lambda+\mu)\nabla(\nabla\cdot u^{(j)}(\boldsymbol{\xi})) = \mathbf{O} \ , \qquad \textit{in } C\bar{\omega} \ ,$$

and that $|u^{(j)}(\boldsymbol{\xi})| \leq \text{const}\ (1+|\boldsymbol{\xi}|)^k$, $k \geq 0$, *for* $j=1,2$.

Then for $|\boldsymbol{\xi}| > 2$

$$u^{(j)}(\boldsymbol{\xi}) = \mathscr{P}_k^{(j)}(\boldsymbol{\xi}) + \gamma(\boldsymbol{\xi},\mathbf{O})C^{(j)} + O(|\boldsymbol{\xi}|^{-1}) \ , \qquad (154)$$

where $\mathscr{P}_k^{(j)}(\boldsymbol{\xi}) = \{\mathscr{P}_i^{(j,k)}(\boldsymbol{\xi})\}_{i=1}^2$, $\mathscr{P}_i^{(j,k)}(\boldsymbol{\xi})$ *are polynomials of order not greater than* k, $C^{(j)} = \{C_i^{(j)}\}_{i=1}^2$, *where* $C_i^{(j)}$ *are constants.*

We now formulate a result related to the approximation of the regular part of Green's tensor g needed for our algorithm.

Lemma 13 *Let $|\boldsymbol{\xi}| > 2$, $\boldsymbol{\eta} \in C\bar{\omega}$. Then the columns of the regular part $h^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta})$ of Green's tensor in $C\bar{\omega}$ admit the asymptotic representation*

$$h^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \gamma^{(j)}(\boldsymbol{\xi}, \mathbf{O}) - \zeta^{(j)}(\boldsymbol{\eta}) + O(|\boldsymbol{\xi}|^{-1}) \ . \tag{155}$$

Proof. By definition of g (cf. (146)–(148)), the columns $h^{(j)}$ of its regular part satisfies

$$\mu \Delta_{\boldsymbol{\xi}} h^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta}) + (\lambda + \mu) \nabla_{\boldsymbol{\xi}} (\nabla_{\boldsymbol{\xi}} \cdot h^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta})) = \mathbf{O} \ , \quad \boldsymbol{\xi}, \boldsymbol{\eta} \in C\bar{\omega} \ , \tag{156}$$

$$h^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \gamma^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta}) \ , \quad \boldsymbol{\xi} \in \partial C\bar{\omega}, \boldsymbol{\eta} \in C\bar{\omega} \ , \tag{157}$$

and by (152)

$$h^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta}) \sim \gamma^{(j)}(\boldsymbol{\xi}, \mathbf{O}) - \zeta^{(j)}(\boldsymbol{\eta}) \ , \qquad \text{as } |\boldsymbol{\xi}| \to \infty, \boldsymbol{\eta} \in C\bar{\omega} \ , \tag{158}$$

for $j = 1, 2$.

Setting $U^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta}) = h^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta}) - \gamma^{(j)}(\boldsymbol{\xi}, \mathbf{O})$, we have that $U^{(j)}$ solves

$$\mu \Delta_{\boldsymbol{\xi}} U^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta}) + (\lambda + \mu) \nabla_{\boldsymbol{\xi}} (\nabla_{\boldsymbol{\xi}} \cdot U^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta})) = \mathbf{O} \ , \quad \boldsymbol{\xi}, \boldsymbol{\eta} \in C\bar{\omega} \ , \tag{159}$$

$$U^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \gamma^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta}) - \gamma^{(j)}(\boldsymbol{\xi}, \mathbf{O}) \ , \quad \boldsymbol{\xi} \in \partial C\bar{\omega}, \boldsymbol{\eta} \in C\bar{\omega} \ , \tag{160}$$

and by (152)

$$U^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta}) \sim -\zeta^{(j)}(\boldsymbol{\eta}) \ , \qquad \text{as } |\boldsymbol{\xi}| \to \infty, \boldsymbol{\eta} \in C\bar{\omega} \ . \tag{161}$$

Consulting Lemma 12, we see that for $|\boldsymbol{\xi}| > 2$ the following representation for $U^{(j)}$ holds

$$U^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta}) = K^{(j)} + \gamma(\boldsymbol{\xi}, \mathbf{O}) C^{(j)} + O(|\boldsymbol{\xi}|^{-1}) \ . \tag{162}$$

where $K^{(j)}$ and $C^{(j)}$ are vector functions of $\boldsymbol{\eta}$ only.

Then in order that condition (161) be satisfied we must take $K^{(j)} = -\zeta^{(j)}(\boldsymbol{\eta})$ and $C^{(j)} = \mathbf{O}$. Thus, recalling the definition of $U^{(j)}$, we obtain (155). □

We also have the following asymptotic representation of the tensor ζ.

Lemma 14 *For $|\boldsymbol{\xi}| > 2$, the following representation for $\zeta^{(j)}$, $j = 1, 2$, holds*

$$\zeta^{(j)}(\boldsymbol{\xi}) = -\gamma^{(j)}(\boldsymbol{\xi}, \mathbf{O}) + \zeta^{(\infty, j)} + O(|\boldsymbol{\xi}|^{-1}) \ . \tag{163}$$

Proof. The columns $\zeta^{(j)}(\boldsymbol{\xi})$ are solutions of

$$\mu\Delta\zeta^{(j)}(\boldsymbol{\xi}) + (\lambda+\mu)\nabla(\nabla\cdot\zeta^{(j)}(\boldsymbol{\xi})) = \mathbf{O}\ , \quad \boldsymbol{\xi}\in C\bar{\omega}\ , \tag{164}$$

$$\zeta^{(j)}(\boldsymbol{\xi}) = \mathbf{O}\ , \quad \boldsymbol{\xi}\in\partial C\bar{\omega}\ , \tag{165}$$

$$\zeta^{(j)}(\boldsymbol{\xi}) \sim -\gamma^{(j)}(\boldsymbol{\xi},\mathbf{O}) + \zeta^{(\infty,j)} \quad \text{as} \quad |\boldsymbol{\xi}|\to\infty\ , \tag{166}$$

for $j=1,2$, where $\zeta^{(\infty,j)}$ are the columns of ζ^{∞} and the preceding boundary value problem is consistent with (152), (153).

Setting $U^{(j)} = \zeta^{(j)}(\boldsymbol{\xi}) + \gamma^{(j)}(\boldsymbol{\xi},\mathbf{O})$, and in the same way as in the proof of the previous lemma, we deduce (163). □

We also have the following property of the matrix function ζ.

Lemma 15 *The tensor $\zeta(\boldsymbol{\eta})$ is symmetric.*

Proof. We begin by applying the Betti formula to the vectors $-\zeta^{(k)}(\boldsymbol{\xi})$ and $g^{(l)}(\boldsymbol{\xi},\boldsymbol{\eta})$ (noting that $\zeta^{(k)}(\boldsymbol{\xi})$ is a solution of the homogeneous Lamé equation), in the domain $B_R(\mathbf{O})\backslash\bar{\omega}$ for sufficiently large R, so that we obtain

$$\begin{aligned}-\int_{B_R\backslash\bar{\omega}}\zeta^{(k)}(\boldsymbol{\xi})\cdot Lg^{(l)}(\boldsymbol{\xi},\boldsymbol{\eta})\,d\boldsymbol{\xi} \ &= \int_{\partial(B_R\backslash\bar{\omega})}\{-\zeta^{(k)}(\boldsymbol{\xi})\cdot T_n(g^{(l)}(\boldsymbol{\xi},\boldsymbol{\eta}))\\ &\quad + g^{(l)}(\boldsymbol{\xi},\boldsymbol{\eta})\cdot T_n(\zeta^{(k)}(\boldsymbol{\xi}))\}\,dS_{\boldsymbol{\xi}}\ .\end{aligned} \tag{167}$$

Now using the definition of g and the fact that $\zeta^{(k)}(\boldsymbol{\xi})=\mathbf{O}$ and $g^{(l)}(\boldsymbol{\xi},\boldsymbol{\eta})=\mathbf{O}$ on $\partial C\bar{\omega}$, we have from the preceding equation

$$\zeta_{lk}(\boldsymbol{\eta}) = \int_{\partial B_R}\{-\zeta^{(k)}(\boldsymbol{\xi})\cdot T_n(g^{(l)}(\boldsymbol{\xi},\boldsymbol{\eta})) + g^{(l)}(\boldsymbol{\xi},\boldsymbol{\eta})\cdot T_n(\zeta^{(k)}(\boldsymbol{\xi}))\}\,dS_{\boldsymbol{\xi}}\ , \tag{168}$$

which holds for all R. Using the asymptotic representation for $\zeta^{(j)}$ and that for $h^{(j)}$ given in Lemmas 13 and 14 respectively, $j=1,2$, we take the limit in (168) as R tends to infinity and obtain

$$\zeta_{lk}(\boldsymbol{\eta}) = -\lim_{R\to\infty}\int_{\partial B_R}\zeta^{(l)}(\boldsymbol{\eta})\cdot T_n(\gamma^{(k)}(\boldsymbol{\xi},\mathbf{O}))\,dS_{\boldsymbol{\xi}}\ . \tag{169}$$

Computing the above integral, by applying Betti's formula to the vectors $\zeta^{(l)}(\boldsymbol{\eta})$ and $\gamma^{(k)}(\boldsymbol{\xi},\mathbf{O})$ in B_R, gives

$$\zeta_{lk}(\boldsymbol{\eta}) = \zeta_{kl}(\boldsymbol{\eta})\ . \tag{170}$$

Hence from (170) we have the tensor $\zeta(\boldsymbol{\eta})$ is symmetric.

□

It also follows from this Lemma and the definition of the constant matrix ζ^{∞}, (cf. (153)), that this matrix is also symmetric.

5.3 A uniform asymptotic approximation of an elastic capacitary potential matrix

Let $P_\varepsilon(\mathbf{x}) = [P_\varepsilon^{(1)}(\mathbf{x}), P_\varepsilon^{(2)}(\mathbf{x})]$ denote the elastic capacitary potential of the set ω_ε, whose columns are a solution of the following problem

$$\mu\Delta P_\varepsilon^{(j)}(\mathbf{x}) + (\lambda+\mu)\nabla(\nabla\cdot P_\varepsilon^{(j)}(\mathbf{x})) = \mathbf{O}, \quad \mathbf{x}\in\Omega_\varepsilon\,, \tag{171}$$

$$P_\varepsilon^{(j)}(\mathbf{x}) = \mathbf{O}\,, \quad \mathbf{x}\in\partial\Omega\,, \tag{172}$$

$$P_\varepsilon^{(j)}(\mathbf{x}) = \mathbf{e}^{(j)}\,, \quad \mathbf{x}\in\partial C\bar{\omega}_\varepsilon\,, \tag{173}$$

for $j = 1, 2$.

Lemma 16 *The asymptotic approximation of $P_\varepsilon(\mathbf{x})$ is given by the formula*

$$P_\varepsilon(\mathbf{x}) = (G(\mathbf{x},\mathbf{O}) - \zeta(\boldsymbol{\xi}) - \gamma(\boldsymbol{\xi},\mathbf{O}) + \zeta^\infty)D + p(\mathbf{x})\,, \tag{174}$$

where D is the matrix given by (23) − (25) *and $p(\mathbf{x}) = [p^{(1)}(\mathbf{x}), p^{(2)}(\mathbf{x})]$ is such that*

$$|p^{(j)}(\mathbf{x})| \le \mathrm{const}\,\varepsilon(\log\varepsilon)^{-1}, \quad j = 1, 2\,, \tag{175}$$

uniformly with respect to $\mathbf{x}\in\Omega_\varepsilon$.

Proof. Let $\varepsilon\to 0$, then $\Omega_\varepsilon\to\Omega\backslash\{\mathbf{O}\}$. In this limit domain, it is suitable to approximate the columns $P_\varepsilon^{(j)}(\mathbf{x})$ of the elastic capacitary potential, by $V^{(j)}(\mathbf{x})$, which solves the boundary value problem

$$\mu\Delta V^{(j)}(\mathbf{x}) + (\lambda+\mu)\nabla(\nabla\cdot V^{(j)}(\mathbf{x})) + \delta(\mathbf{x})\mathbf{e}^{(j)} = \mathbf{O}, \quad \mathbf{x}\in\Omega\backslash\{\mathbf{O}\}\,, \tag{176}$$

$$V^{(j)}(\mathbf{x}) = \mathbf{O}\,, \quad \mathbf{x}\in\partial\Omega\,, \tag{177}$$

for $j = 1, 2$. Let $V^{(j)}(\mathbf{x})$ be sought in the form

$$V^{(j)}(\mathbf{x}) = D_{1j}G^{(1)}(\mathbf{x},\mathbf{O}) + D_{2j}G^{(2)}(\mathbf{x},\mathbf{O})\;, j = 1, 2\,. \tag{178}$$

The representation of $V^{(j)}(\mathbf{x})$ by (178) does not satisfy the boundary conditions on $\partial C\bar{\omega}_\varepsilon$. Therefore, we construct a boundary layer $M^{(j)}(\boldsymbol{\xi})$, which is a solution of

$$\mu\Delta M^{(j)}(\boldsymbol{\xi}) + (\lambda+\mu)\nabla(\nabla\cdot M^{(j)}(\boldsymbol{\xi})) = \mathbf{O}, \quad \boldsymbol{\xi}\in C\bar{\omega}\,, \tag{179}$$

$$M^{(j)}(\boldsymbol{\xi}) = \mathbf{e}^{(j)} - D_{1j}G^{(1)}(\mathbf{x},\mathbf{O}) - D_{2j}G^{(2)}(\mathbf{x},\mathbf{O})\,, \quad \boldsymbol{\xi}\in\partial\omega\,, \tag{180}$$

$$M^{(j)}(\boldsymbol{\xi}) \to \mathbf{O} \text{ as } |\boldsymbol{\xi}|\to\infty\,, \tag{181}$$

for $j = 1, 2$.

Since ω_ε is a small void, we may rewrite the boundary condition (180) for $M^{(j)}(\boldsymbol{\xi})$ by considering $G^{(j)}(\mathbf{x},\mathbf{O})$, $j=1,2$ as follows. Using

$$G^{(j)}(\mathbf{x},\mathbf{O}) = \gamma^{(j)}(\mathbf{x},\mathbf{O}) - H^{(j)}(\mathbf{x},\mathbf{O}) \ , \quad j=1,2 \ , \tag{182}$$

where $\gamma^{(j)}$ is the j^{th} column of $\{\gamma_{ij}\}_{i=1}^2$ and the fact the components of $H^{(j)}(\mathbf{x},\mathbf{O})$ are smooth functions for $\mathbf{x}$, $\mathbf{y} \in \Omega$, on $\partial C\bar{\omega}_\varepsilon$ we may expand these about $\mathbf{O}$, to give

$$\begin{aligned} G^{(j)}(\mathbf{x},\mathbf{O}) &= -K_2 \log\varepsilon\, \mathbf{e}^{(j)} + \gamma^{(j)}(\boldsymbol{\xi},\mathbf{O}) - H^{(j)}(\mathbf{O},\mathbf{O}) \\ &\quad +O(\varepsilon) \ , \quad j=1,2 \ . \end{aligned} \tag{183}$$

Then using (183) we have from (180)

$$\begin{aligned} M^{(j)}(\boldsymbol{\xi}) &= \mathbf{e}^{(j)} + D_{1j}\left(K_2\log\varepsilon\,\mathbf{e}^{(1)} - \gamma^{(1)}(\boldsymbol{\xi},\mathbf{O}) + H^{(1)}(\mathbf{O},\mathbf{O})\right) \\ &\quad + D_{2j}\left(K_2\log\varepsilon\,\mathbf{e}^{(2)} - \gamma^{(2)}(\boldsymbol{\xi},\mathbf{O}) + H^{(2)}(\mathbf{O},\mathbf{O})\right) \\ &\quad +O(\varepsilon) \ , \end{aligned} \tag{184}$$

for $\boldsymbol{\xi} \in \partial\omega$, where K_2 is the constant given in (25).

The tensors $\zeta^{(j)}(\boldsymbol{\xi})$ satisfy (164)–(166). Setting

$$\mathring{\zeta}^{(j)}(\boldsymbol{\xi}) = \zeta^{(j)}(\boldsymbol{\xi}) + \gamma^{(j)}(\boldsymbol{\xi},\mathbf{O}) - \zeta^{(\infty,j)} \ , j=1,2 \ , \tag{185}$$

we have that $\mathring{\zeta}^{(j)}(\boldsymbol{\xi})$ satisfies

$$\mu\Delta\mathring{\zeta}^{(j)}(\boldsymbol{\xi}) + (\lambda+\mu)\nabla(\nabla\cdot\mathring{\zeta}^{(j)}(\boldsymbol{\xi})) = \mathbf{O}, \quad \boldsymbol{\xi} \in C\bar{\omega} \ , \tag{186}$$

$$\mathring{\zeta}^{(j)}(\boldsymbol{\xi}) = \gamma^{(j)}(\boldsymbol{\xi},\mathbf{O}) - \zeta^{(\infty,j)} \ , \quad \boldsymbol{\xi} \in \partial C\bar{\omega} \ , \tag{187}$$

$$\mathring{\zeta}^{(j)}(\boldsymbol{\xi}) \to \mathbf{O} \quad \text{as} \quad |\boldsymbol{\xi}| \to \infty \ , \tag{188}$$

for $j=1,2$.

Substituting the boundary condition (187), for $\mathring{\zeta}^{(j)}(\boldsymbol{\xi})$ on $\partial C\bar{\omega}$, into (184) we have

$$\begin{aligned} M^{(j)}(\boldsymbol{\xi}) &= \mathbf{e}^{(j)} + D_{1j}\Big(K_2\log\varepsilon\,\mathbf{e}^{(1)} - (\mathring{\zeta}^{(1)}(\boldsymbol{\xi}) + \zeta^{(\infty,1)}) + H^{(1)}(\mathbf{O},\mathbf{O})\Big) \\ &\quad + D_{2j}\Big(K_2\log\varepsilon\,\mathbf{e}^{(2)} - (\mathring{\zeta}^{(2)}(\boldsymbol{\xi}) + \zeta^{(\infty,2)}) + H^{(2)}(\mathbf{O},\mathbf{O})\Big) \\ &\quad +O(\varepsilon) \ , \end{aligned} \tag{189}$$

for $\boldsymbol{\xi} \in \partial C\bar{\omega}$. The boundary layer $M^{(j)}(\boldsymbol{\xi})$ is sought in the form

$$M^{(j)}(\boldsymbol{\xi}) = -D_{1j}\mathring{\zeta}^{(1)}(\boldsymbol{\xi}) - D_{2j}\mathring{\zeta}^{(2)}(\boldsymbol{\xi}) + W^{(j)}(\boldsymbol{\xi}) \ , \quad j=1,2 \ , \tag{190}$$

where $W^{(j)}(\boldsymbol{\xi})$ is a solution of

$$\mu\Delta W^{(j)}(\boldsymbol{\xi}) + (\lambda+\mu)\nabla(\nabla\cdot W^{(j)}(\boldsymbol{\xi})) = \mathbf{O}, \quad \boldsymbol{\xi} \in C\bar{\omega} \ , \tag{191}$$

$$\begin{aligned} W^{(j)}(\boldsymbol{\xi}) &= \mathbf{e}^{(j)} + D_{1j}\left(K_2 \log\varepsilon\, \mathbf{e}^{(1)} - \zeta^{(\infty,1)} + H^{(1)}(\mathbf{O},\mathbf{O})\right) \\ &\quad + D_{2j}\left(K_2 \log\varepsilon\, \mathbf{e}^{(2)} - \zeta^{(\infty,2)} + H^{(2)}(\mathbf{O},\mathbf{O})\right) \ , \end{aligned} \tag{192}$$

for $\boldsymbol{\xi} \in \partial C\bar{\omega}$, and

$$W^{(j)}(\boldsymbol{\xi}) \to \mathbf{O} \quad \text{as} \quad |\boldsymbol{\xi}| \to \infty \ . \tag{193}$$

In order that we satisfy the condition (193) we must choose D_{ij}, $i,j = 1,2$ as follows,

$$D = [D^{(1)}, D^{(2)}] = -A^{-1} \ , \tag{194}$$

where $A = [A_{ij}]$, whose entries are given by

$$A_{ij} = K_2 \log\varepsilon\, \delta_{ij} - \zeta^{\infty}_{ij} + H_{ij}(\mathbf{O},\mathbf{O}) \ , \quad i,j = 1,2 \ . \tag{195}$$

Choosing D as in (194) we have from (191)–(193), $W^{(j)}(\boldsymbol{\xi}) \equiv \mathbf{O}$, $j = 1,2$, and the form of the constant matrix D (given by (23)–(25)) has been proved.

Combining (178) and (190) in

$$P^{(j)}_{\varepsilon}(\mathbf{x}) = V^{(j)}(\mathbf{x}) + M^{(j)}(\boldsymbol{\xi}) + p^{(j)}(\mathbf{x}) \ ,$$

where $p^{(j)}(\mathbf{x})$ is the remainder term, we have (174).

5.3.1 Estimating the remainder term

The remainder $p(\mathbf{x}) = [p^{(1)}(\mathbf{x}), p^{(2)}(\mathbf{x})]$ satisfies

$$\mu\Delta p(\mathbf{x}) + (\lambda+\mu)\nabla(\nabla\cdot p(\mathbf{x})) = 0I_2 \ , \quad \mathbf{x} \in \Omega_\varepsilon \ , \tag{196}$$

$$p(\mathbf{x}) = (\zeta(\boldsymbol{\xi}) + \gamma(\boldsymbol{\xi},\mathbf{O}) - \zeta^{\infty})D \ , \quad \mathbf{x} \in \partial\Omega \ , \tag{197}$$

$$p(\mathbf{x}) = I_2 - \left(-K_2 \log\varepsilon I_2 + \zeta^{\infty} - H(\mathbf{x},\mathbf{O})\right) D \ , \quad \mathbf{x} \in \partial C\bar{\omega}_\varepsilon \ . \tag{198}$$

For the boundary condition on $\partial C\bar{\omega}_\varepsilon$, using (194) and (195)

$$p(\mathbf{x}) = (H(\mathbf{x},\mathbf{O}) - H(\mathbf{O},\mathbf{O}))D \ , \quad \mathbf{x} \in \partial C\bar{\omega}_\varepsilon \ . \tag{199}$$

Since the components of $H(\mathbf{x},\mathbf{O})$ are smooth for $\mathbf{x}$, $\mathbf{y} \in \Omega$

$$H(\mathbf{x},\mathbf{O}) - H(\mathbf{O},\mathbf{O}) = O(\varepsilon) \ , \ \text{as } \mathbf{x} \in \partial C\bar{\omega}_\varepsilon \ .$$

Next we consider the matrix D. Comparing to (24) we have $K_1^{-1} = (\det A)^{-1}$, is of $O(\log^{-2}\varepsilon)$, from which we see $D = O((\log\varepsilon)^{-1})$. Thus we have the right-hand side of (198) is $O(\varepsilon(\log\varepsilon)^{-1})$.

Using Lemma 14, we have

$$\zeta(\boldsymbol{\xi}) + \gamma(\boldsymbol{\xi}, \mathbf{O}) - \zeta^{\infty} = O(\varepsilon) , \quad \text{for } \mathbf{x} \in \partial\Omega , \tag{200}$$

and therefore again we have the right-hand side of (197) is $O(\varepsilon(\log\varepsilon)^{-1})$.

Thus by the Lemma 1 we have

$$p(\mathbf{x}) = O(\varepsilon(\log\varepsilon)^{-1}) \quad \text{for} \quad \mathbf{x} \in \Omega_{\varepsilon} .$$

□

5.4 A uniform asymptotic formula for Green's function G_{ε} in two dimensions

We are now in a position to formulate and prove our result concerning the uniform approximation of the tensor G_{ε} for the case of two dimensions.

Theorem 2 *Green's tensor G_{ε} for the Lamé operator in $\Omega_{\varepsilon} \subset \mathbb{R}^2$ admits the representation*

$$\begin{aligned} G_{\varepsilon}(\mathbf{x}, \mathbf{y}) &= G(\mathbf{x}, \mathbf{y}) + g(\boldsymbol{\xi}, \boldsymbol{\eta}) - \gamma(\boldsymbol{\xi}, \boldsymbol{\eta}) \\ &\quad + P_{\varepsilon}(\mathbf{x}) A P_{\varepsilon}^{T}(\mathbf{y}) - \zeta(\boldsymbol{\eta}) - \zeta(\boldsymbol{\xi}) + \zeta^{\infty} + O(\varepsilon) , \end{aligned} \tag{201}$$

which is uniform with respect to $(\mathbf{x}, \mathbf{y}) \in \Omega_{\varepsilon} \times \Omega_{\varepsilon}$.

Proof. Let G_{ε} be given by

$$G_{\varepsilon}(\mathbf{x}, \mathbf{y}) = \gamma(\mathbf{x}, \mathbf{y}) - H_{\varepsilon}(\mathbf{x}, \mathbf{y}) - h_{\varepsilon}(\mathbf{x}, \mathbf{y}) , \tag{202}$$

where the columns of $H_{\varepsilon}(\mathbf{x}, \mathbf{y})$ and $h_{\varepsilon}(\mathbf{x}, \mathbf{y})$ are solutions of the boundary value problems

$$\mu \Delta_{\mathbf{x}} H_{\varepsilon}^{(j)}(\mathbf{x}, \mathbf{y}) + (\lambda + \mu) \nabla_{\mathbf{x}} (\nabla_{\mathbf{x}} \cdot H_{\varepsilon}^{(j)}(\mathbf{x}, \mathbf{y})) = \mathbf{O} , \quad \mathbf{x}, \mathbf{y} \in \Omega_{\varepsilon} , \tag{203}$$

$$H_{\varepsilon}^{(j)}(\mathbf{x}, \mathbf{y}) = \gamma^{(j)}(\mathbf{x}, \mathbf{y}) , \quad \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega_{\varepsilon} , \tag{204}$$

$$H_{\varepsilon}^{(j)}(\mathbf{x}, \mathbf{y}) = \mathbf{O} , \quad \mathbf{x} \in \partial\omega_{\varepsilon}, \mathbf{y} \in \Omega_{\varepsilon} , \tag{205}$$

and

$$\mu \Delta_{\mathbf{x}} h_{\varepsilon}^{(j)}(\mathbf{x}, \mathbf{y}) + (\lambda + \mu) \nabla_{\mathbf{x}} (\nabla_{\mathbf{x}} \cdot h_{\varepsilon}^{(j)}(\mathbf{x}, \mathbf{y})) = \mathbf{O} , \quad \mathbf{x}, \mathbf{y} \in \Omega_{\varepsilon} , \tag{206}$$

$$h_{\varepsilon}^{(j)}(\mathbf{x}, \mathbf{y}) = \mathbf{O} , \quad \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega_{\varepsilon} , \tag{207}$$

$$h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = \gamma^{(j)}(\mathbf{x},\mathbf{y}) \ , \quad \mathbf{x} \in \partial\omega_\varepsilon, \mathbf{y} \in \Omega_\varepsilon \ , \tag{208}$$

for $j = 1, 2$.

The approximation of $H_\varepsilon(\mathbf{x},\mathbf{y})$. Let $H_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})$ be represented in the form

$$\begin{aligned} H_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) &= S_{1j}(\mathbf{y},\log\varepsilon)G^{(1)}(\mathbf{x},\mathbf{O}) + S_{2j}(\mathbf{y},\log\varepsilon)G^{(2)}(\mathbf{x},\mathbf{O}) \\ &\quad + H^{(j)}(\mathbf{x},\mathbf{y}) + R_\varepsilon^{(j)}(\mathbf{x},\mathbf{y},\log\varepsilon) \ , \end{aligned} \tag{209}$$

where $S_{ij}(\mathbf{y},\log\varepsilon)$, $i,j = 1,2$ are to be determined. In (209), the term $R_\varepsilon^{(j)}(\mathbf{x},\mathbf{y},\log\varepsilon)$ satisfies the boundary value problem

$$\mu\Delta_{\mathbf{x}}R_\varepsilon^{(j)}(\mathbf{x},\mathbf{y},\log\varepsilon) + (\lambda+\mu)\nabla_{\mathbf{x}}(\nabla_{\mathbf{x}}\cdot R_\varepsilon^{(j)}(\mathbf{x},\mathbf{y},\log\varepsilon)) = \mathbf{O} \ , \mathbf{x},\mathbf{y} \in \Omega_\varepsilon \ , \tag{210}$$

$$R_\varepsilon^{(j)}(\mathbf{x},\mathbf{y},\log\varepsilon) = \mathbf{O} \ , \quad \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega_\varepsilon \ , \tag{211}$$

$$\begin{aligned} R_\varepsilon^{(j)}(\mathbf{x},\mathbf{y},\log\varepsilon) &= -S_{1j}G^{(1)}(\mathbf{x},\mathbf{O}) - S_{2j}G^{(2)}(\mathbf{x},\mathbf{O}) - H^{(j)}(\mathbf{x},\mathbf{y}) \ , \\ &\quad \mathbf{x} \in \partial C\bar{\omega}_\varepsilon, \mathbf{y} \in \Omega_\varepsilon \ , \end{aligned} \tag{212}$$

and is approximated by $R^{(j)}(\boldsymbol{\xi},\mathbf{y},\log\varepsilon)$, which is a solution of

$$\mu\Delta_{\boldsymbol{\xi}}R^{(j)}(\boldsymbol{\xi},\mathbf{y},\log\varepsilon) + (\lambda+\mu)\nabla_{\boldsymbol{\xi}}(\nabla_{\boldsymbol{\xi}}\cdot R^{(j)}(\boldsymbol{\xi},\mathbf{y},\log\varepsilon)) = \mathbf{O} \ , \boldsymbol{\xi} \in C\bar{\omega} \ , \tag{213}$$

$$\begin{aligned} R^{(j)}(\boldsymbol{\xi},\mathbf{y},\log\varepsilon) &= S_{1j}\left(K_2\log\varepsilon\,\mathbf{e}^{(1)} - \gamma^{(1)}(\boldsymbol{\xi},\mathbf{O}) + H^{(1)}(\mathbf{O},\mathbf{O})\right) \\ &\quad + S_{2j}\left(K_2\log\varepsilon\,\mathbf{e}^{(2)} - \gamma^{(2)}(\boldsymbol{\xi},\mathbf{O}) + H^{(2)}(\mathbf{O},\mathbf{O})\right) \\ &\quad - H^{(j)}(\mathbf{O},\mathbf{y}) \ , \qquad \boldsymbol{\xi} \in \partial C\bar{\omega} \ , \end{aligned} \tag{214}$$

$$R^{(j)}(\boldsymbol{\xi},\mathbf{y},\log\varepsilon) \to \mathbf{O} \text{ as } |\boldsymbol{\xi}| \to \infty \ , \tag{215}$$

where $\mathbf{y} \in \Omega_\varepsilon$. We represent the solution of (213), (214) and (215) as

$$\begin{aligned} R^{(j)}(\boldsymbol{\xi},\mathbf{y},\log\varepsilon) &= S_{1j}\left(K_2\log\varepsilon\,\mathbf{e}^{(1)} - \gamma^{(1)}(\boldsymbol{\xi},\mathbf{O}) + H^{(1)}(\mathbf{O},\mathbf{O}) - \zeta^{(1)}(\boldsymbol{\xi})\right) \\ &\quad + S_{2j}\left(K_2\log\varepsilon\,\mathbf{e}^{(2)} - \gamma^{(2)}(\boldsymbol{\xi},\mathbf{O}) + H^{(2)}(\mathbf{O},\mathbf{O}) - \zeta^{(2)}(\boldsymbol{\xi})\right) \\ &\quad - H^{(j)}(\mathbf{O},\mathbf{y}) \ . \end{aligned} \tag{216}$$

Now, using the boundary condition (166) of $\zeta(\boldsymbol{\xi})$, in (216), we deduce that in order that (215) be satisfied we must choose the columns of S as follows

$$S(\mathbf{y},\log\varepsilon) = [S^{(1)}(\mathbf{y},\log\varepsilon), S^{(2)}(\mathbf{y},\log\varepsilon)] = -DH(\mathbf{O},\mathbf{y}) \ , \tag{217}$$

where the entries of D are given by (23)–(25).

Combining (217), (214) and (209), we have

$$\begin{aligned} H_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) &= S_{1j}G^{(1)}(\mathbf{x},\mathbf{O}) + S_{2j}G^{(2)}(\mathbf{x},\mathbf{O}) \\ &\quad + S_{1j}\left(K_2\log\varepsilon\,\mathbf{e}^{(1)} - \gamma^{(1)}(\boldsymbol{\xi},\mathbf{O}) + H^{(1)}(\mathbf{O},\mathbf{O}) - \zeta^{(1)}(\boldsymbol{\xi})\right) \\ &\quad + S_{2j}\left(K_2\log\varepsilon\,\mathbf{e}^{(2)} - \gamma^{(2)}(\boldsymbol{\xi},\mathbf{O}) + H^{(2)}(\mathbf{O},\mathbf{O}) - \zeta^{(2)}(\boldsymbol{\xi})\right) \\ &\quad - H^{(j)}(\mathbf{O},\mathbf{y}) + H^{(j)}(\mathbf{x},\mathbf{y}) + \mathfrak{H}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) \\ &= -P_\varepsilon(\mathbf{x})H^{(j)}(\mathbf{O},\mathbf{y}) + H^{(j)}(\mathbf{x},\mathbf{y}) + \mathfrak{H}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})\;. \end{aligned} \tag{218}$$

Here $\mathfrak{H}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})$ satisfies

$$\mu\Delta_{\mathbf{x}}\mathfrak{H}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) + (\lambda+\mu)\nabla_{\mathbf{x}}(\nabla_{\mathbf{x}}\cdot\mathfrak{H}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})) = \mathbf{O}\;, \quad \mathbf{x},\mathbf{y}\in\Omega_\varepsilon\;, \tag{219}$$

$$\mathfrak{H}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = H^{(j)}(\mathbf{O},\mathbf{y}) - H^{(j)}(\mathbf{x},\mathbf{y})\;, \quad \mathbf{x}\in\partial C\bar{\omega}_\varepsilon\;, \mathbf{y}\in\Omega_\varepsilon\;, \tag{220}$$

$$\mathfrak{H}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = \mathbf{O}\;, \quad \mathbf{x}\in\partial\Omega\;, \mathbf{y}\in\Omega_\varepsilon\;, \tag{221}$$

where the right-hand side of the boundary condition (220) is $O(\varepsilon)$, uniformly with respect to $\mathbf{x}\in\partial C\bar{\omega}_\varepsilon$ and $\mathbf{y}\in\Omega_\varepsilon$.

Using Lemma 1 we obtain $\mathfrak{H}_\varepsilon(\mathbf{x},\mathbf{y}) = O(\varepsilon)$ for $\mathbf{x},\mathbf{y}\in\Omega_\varepsilon$.

The approximation of $h_\varepsilon(\mathbf{x},\mathbf{y})$. Now we shall proceed to approximate h_ε. The columns of $h_\varepsilon(\mathbf{x},\mathbf{y})$ satisfy the homogeneous Dirichlet condition on $\partial\Omega$ and for $\mathbf{x}\in\partial C\bar{\omega}_\varepsilon$ we rewrite the boundary condition (208) as

$$h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = -K_2\log\varepsilon\,\mathbf{e}^{(j)} + \gamma^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta})\;, \quad \mathbf{x}\in\partial C\bar{\omega}_\varepsilon\;, \mathbf{y}\in\Omega_\varepsilon\;.$$

Let $h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})$ be sought in the form

$$h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = -K_2\log\varepsilon\,\mathbf{e}^{(j)} + h^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta}) + \chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})\;, \tag{222}$$

where the vector field $\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})$ satisfies

$$\mu\Delta\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) + (\lambda+\mu)\nabla(\nabla\cdot\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})) = \mathbf{O}\;, \quad \mathbf{x},\mathbf{y}\in\Omega_\varepsilon\;, \tag{223}$$

$$\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = \mathbf{O}\;, \quad \mathbf{x}\in\partial C\bar{\omega}_\varepsilon, \mathbf{y}\in\Omega_\varepsilon\;, \tag{224}$$

$$\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = K_2\log\varepsilon\,\mathbf{e}^{(j)} - h^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta})\;, \quad \mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon\;. \tag{225}$$

Using Lemma 13, we rewrite (225) as

$$\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = -\gamma^{(j)}(\mathbf{x},\mathbf{O}) + \zeta^{(j)}(\boldsymbol{\eta}) + O(\varepsilon)\;, \quad \mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon\;. \tag{226}$$

From the definition of $H(\mathbf{x},\mathbf{y})$ and the elastic capacitary potential we write $\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})$ as

$$\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = -H^{(j)}(\mathbf{x},\mathbf{O}) + (I_2 - P_\varepsilon(\mathbf{x}))\zeta^{(j)}(\boldsymbol{\eta}) + \mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})\;, \mathbf{x},\mathbf{y}\in\Omega_\varepsilon\;, \tag{227}$$

where $\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})$ satisfies the homogeneous Lamé equation; by Lemma 13 is $O(\varepsilon)$ for $\mathbf{x}\in\partial\Omega$, $\mathbf{y}\in\Omega_\varepsilon$ and

$$\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = H^{(j)}(\mathbf{x},\mathbf{O}) = H^{(j)}(\mathbf{O},\mathbf{O}) + O(\varepsilon)\;, \tag{228}$$

for $\mathbf{x}\in\partial C\bar{\omega}_\varepsilon$, $\mathbf{y}\in\Omega_\varepsilon$. Therefore, using the elastic capacitary potential, $P_\varepsilon(\mathbf{x})$, we write

$$\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = P_\varepsilon(\mathbf{x})H^{(j)}(\mathbf{O},\mathbf{O}) + O(\varepsilon)\;, \tag{229}$$

which is uniform with respect to $\mathbf{x},\mathbf{y}\in\Omega_\varepsilon$, by Lemma 1.

Collecting now (227), (229) in (222) we have

$$\begin{aligned} h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) &= h^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta}) - K_2\log\varepsilon\,\mathbf{e}^{(j)} \\ &\quad -H^{(j)}(\mathbf{x},\mathbf{O}) + (I_2 - P_\varepsilon(\mathbf{x}))\zeta^{(j)}(\boldsymbol{\eta}) \\ &\quad +P_\varepsilon(\mathbf{x})H^{(j)}(\mathbf{O},\mathbf{O}) + O(\varepsilon)\;. \end{aligned} \tag{230}$$

Combined formula. Substituting (218), (230) in (202) we have the columns of Green's tensor for the domain Ω_ε

$$\begin{aligned} G_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) &= \gamma^{(j)}(\mathbf{x},\mathbf{y}) - H^{(j)}(\mathbf{x},\mathbf{y}) - h^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta}) \\ &\quad +K_2\log\varepsilon\,\mathbf{e}^{(j)} + H^{(j)}(\mathbf{x},\mathbf{O}) - \zeta^{(j)}(\boldsymbol{\eta}) \\ &\quad -P_\varepsilon(\mathbf{x})(H^{(j)}(\mathbf{O},\mathbf{O}) - \zeta^{(j)}(\boldsymbol{\eta}) - H^{(j)}(\mathbf{O},\mathbf{y})) + O(\varepsilon) \\ &= \gamma^{(j)}(\mathbf{x},\mathbf{y}) - H^{(j)}(\mathbf{x},\mathbf{y}) - h^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta}) + K_2\log\varepsilon\,\mathbf{e}^{(j)} \\ &\quad +(I_2 - P_\varepsilon(\mathbf{x}))(H^{(j)}(\mathbf{O},\mathbf{O}) - \zeta^{(j)}(\boldsymbol{\eta}) - H^{(j)}(\mathbf{O},\mathbf{y})) \\ &\quad +H^{(j)}(\mathbf{x},\mathbf{O}) + H^{(j)}(\mathbf{O},\mathbf{y}) - H^{(j)}(\mathbf{O},\mathbf{O}) + O(\varepsilon)\;. \end{aligned} \tag{231}$$

Using the relation

$$H(\mathbf{O},\mathbf{O}) - \zeta(\boldsymbol{\eta}) - H(\mathbf{O},\mathbf{y}) = A(I_2 - P_\varepsilon^T(\mathbf{y}))\;, \tag{232}$$

obtained from the leading part of P_ε, we have

$$\begin{aligned} G_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) &= \gamma^{(j)}(\mathbf{x},\mathbf{y}) - H^{(j)}(\mathbf{x},\mathbf{y}) - h^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta}) \\ &\quad +K_2\log\varepsilon\,\mathbf{e}^{(j)} + (I_2 - P_\varepsilon(\mathbf{x}))A(\mathbf{e}^{(j)} - P_\varepsilon^{T(j)}(\mathbf{y})) \\ &\quad +H^{(j)}(\mathbf{x},\mathbf{O}) + H^{(j)}(\mathbf{O},\mathbf{y}) - H^{(j)}(\mathbf{O},\mathbf{O}) + O(\varepsilon) \\ &= \gamma^{(j)}(\mathbf{x},\mathbf{y}) - H^{(j)}(\mathbf{x},\mathbf{y}) - h^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta}) \\ &\quad +P_\varepsilon(\mathbf{x})AP_\varepsilon^{T(j)}(\mathbf{y}) - \zeta^{(j)}(\boldsymbol{\eta}) - \zeta^{(j)}(\boldsymbol{\xi}) \\ &\quad +\zeta^{(\infty,j)} + O(\varepsilon)\;, \end{aligned} \tag{233}$$

which is (201). The proof is complete.

□

6 Simplified asymptotic formulae subject to constraints on independent variables

It is now of interest to see how the asymptotic formulae obtained in Theorems 1 and 2, simplify under constraints on the points $\mathbf{x}$, $\mathbf{y} \in \Omega_\varepsilon$, where $\Omega_\varepsilon \subset \mathbb{R}^n$, $n = 2, 3$. We consider two situations, the first is when these points are outside a small neighborhood of the hole, the second is when the points are in the vicinity of the hole.

We now consider the case of three dimensions.

Corollary 1 *a) Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be points of* $\Omega_\varepsilon \subset \mathbb{R}^3$*, such that*

$$\min\{|\mathbf{x}|, |\mathbf{y}|\} > 2\varepsilon \ . \tag{234}$$

Then $G_\varepsilon(\mathbf{x}, \mathbf{y})$ *admits the representation*

$$G_\varepsilon(\mathbf{x}, \mathbf{y}) = G(\mathbf{x}, \mathbf{y}) - \varepsilon G(\mathbf{x}, \mathbf{O}) B G(\mathbf{O}, \mathbf{y}) + O(\varepsilon^2(|\mathbf{x}||\mathbf{y}| \min\{|\mathbf{x}|, |\mathbf{y}|\})^{-1}) \ . \tag{235}$$

b) If $\max\{|\mathbf{x}|, |\mathbf{y}|\} < 1/2$*, then*

$$\begin{aligned} G_\varepsilon(\mathbf{x}, \mathbf{y}) &= \varepsilon^{-1} g(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) - (I_3 - P(\varepsilon^{-1}\mathbf{x})) H(\mathbf{O}, \mathbf{O})(I_3 - P^T(\varepsilon^{-1}\mathbf{y})) \\ &\quad + O(\max\{|\mathbf{x}|, |\mathbf{y}|\}) \ . \end{aligned} \tag{236}$$

Both (235) *and* (236) *are uniform with respect to* $\mathbf{x}, \mathbf{y} \in \Omega_\varepsilon$.

Proof. *a)* We may rewrite (126) as follows

$$\begin{aligned} G_\varepsilon(\mathbf{x}, \mathbf{y}) &= G(\mathbf{x}, \mathbf{y}) - \varepsilon^{-1} h(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) \\ &\quad + P(\varepsilon^{-1}\mathbf{x}) H(\mathbf{O}, \mathbf{y}) + H(\mathbf{x}, \mathbf{O}) P^T(\varepsilon^{-1}\mathbf{y}) \\ &\quad - P(\varepsilon^{-1}\mathbf{x}) H(\mathbf{O}, \mathbf{O}) P^T(\varepsilon^{-1}\mathbf{y}) - \varepsilon H(\mathbf{x}, \mathbf{O}) B H(\mathbf{O}, \mathbf{y}) \\ &\quad + O\left(\varepsilon^2 (\min\{|\mathbf{x}|, |\mathbf{y}|\})^{-1}\right) \ . \end{aligned} \tag{237}$$

From Lemma 4, we have for $|\mathbf{x}| > 2\,\varepsilon$

$$P(\varepsilon^{-1}\mathbf{x}) = \varepsilon \Gamma(\mathbf{x}, \mathbf{O}) B + O\left(\varepsilon^2 |\mathbf{x}|^{-2}\right) \ . \tag{238}$$

Also, by Lemma 11 we have

$$\begin{aligned} \varepsilon^{-1} h(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) &= \varepsilon^{-1} \Gamma(\varepsilon^{-1}\mathbf{x}, \mathbf{O}) P^T(\varepsilon^{-1}\mathbf{y}) + O\left(\varepsilon^2 (|\mathbf{x}|^2 |\mathbf{y}|)^{-1}\right) \\ &= \varepsilon^{-1} \Gamma(\varepsilon^{-1}\mathbf{x}, \mathbf{O}) B \Gamma(\varepsilon^{-1}\mathbf{y}, \mathbf{O}) \\ &\quad + O\left(\varepsilon^2 (|\mathbf{x}||\mathbf{y}| \min\{|\mathbf{x}|, |\mathbf{y}|\})^{-1}\right) \end{aligned} \tag{239}$$

By substitution of (238) and (239) into (237) we have

$$\begin{aligned} G_\varepsilon(\mathbf{x},\mathbf{y}) &= G(\mathbf{x},\mathbf{y}) - \varepsilon^{-1}\Gamma(\varepsilon^{-1}\mathbf{x},\mathbf{O})B\Gamma(\varepsilon^{-1}\mathbf{y},\mathbf{O}) \\ &\quad +\varepsilon\Gamma(\mathbf{x},\mathbf{O})BH(\mathbf{O},\mathbf{y}) + \varepsilon H(\mathbf{x},\mathbf{O})B\Gamma(\mathbf{y},\mathbf{O}) \\ &\quad -\varepsilon H(\mathbf{x},\mathbf{O})BH(\mathbf{O},\mathbf{y}) \\ &\quad +O(\varepsilon^2(|\mathbf{x}||\mathbf{y}|\min\{|\mathbf{x}|,|\mathbf{y}|\})^{-1}) \,, \end{aligned} \tag{240}$$

which is equivalent to

$$\begin{aligned} G_\varepsilon(\mathbf{x},\mathbf{y}) &= G(\mathbf{x},\mathbf{y}) - \Gamma(\varepsilon^{-1}\mathbf{x},\mathbf{O})BG(\mathbf{O},\mathbf{y}) \\ &\quad +\varepsilon H(\mathbf{x},\mathbf{O})BG(\mathbf{O},\mathbf{y}) \\ &\quad +O\left(\varepsilon^2(|\mathbf{x}||\mathbf{y}|\min\{|\mathbf{x}|,|\mathbf{y}|\})^{-1}\right) \,, \end{aligned} \tag{241}$$

which is equivalent to (235).

b) Since the components of $H(\mathbf{x},\mathbf{y})$ are smooth for $\mathbf{x}$, $\mathbf{y} \in \Omega$, in the vicinity of $(\mathbf{O},\mathbf{O})$ in $\Omega_\varepsilon \times \Omega_\varepsilon$ we may rewrite (126) as

$$\begin{aligned} G_\varepsilon(\mathbf{x},\mathbf{y}) &= \varepsilon^{-1}g(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) - H(\mathbf{O},\mathbf{O}) \\ &\quad +(H(\mathbf{O},\mathbf{O})+O(|\mathbf{x}|))P^T(\varepsilon^{-1}\mathbf{y}) + P(\varepsilon^{-1}\mathbf{x})(H(\mathbf{O},\mathbf{O})+O(|\mathbf{y}|)) \\ &\quad -P(\varepsilon^{-1}\mathbf{x})H(\mathbf{O},\mathbf{O})P^T(\varepsilon^{-1}\mathbf{y}) + O(\max\{|\mathbf{x}|,|\mathbf{y}|\}) \,, \end{aligned} \tag{242}$$

from which (236) follows.

□

Next we shall simplify the asymptotic formula given in (201) for the case of two dimensions under the same conditions on the points $\mathbf{x}$ and $\mathbf{y}$.

Corollary 2 *a) Let* $\mathbf{x},\mathbf{y} \in \Omega_\varepsilon \subset \mathbb{R}^2$ *such that*

$$\min\{|\mathbf{x}|,|\mathbf{y}|\} > 2\varepsilon \,. \tag{243}$$

Then

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}) - G(\mathbf{x},\mathbf{O})DG(\mathbf{O},\mathbf{y}) + O(\varepsilon(\min\{|\mathbf{x}|,|\mathbf{y}|\})^{-1}) \,. \tag{244}$$

b) If $\max\{|\mathbf{x}|,|\mathbf{y}|\} < 1/2$, *then*

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = g(\boldsymbol{\xi},\boldsymbol{\eta}) - \zeta(\boldsymbol{\xi})D\zeta(\boldsymbol{\eta}) + O(\max\{|\mathbf{x}|,|\mathbf{y}|\}) \,. \tag{245}$$

Both (244) *and* (245) *are uniform with respect to* ε *and* $(\mathbf{x},\mathbf{y}) \in \Omega_\varepsilon \times \Omega_\varepsilon$.

Proof. *a*) By Lemma 13,

$$h(\boldsymbol{\xi},\boldsymbol{\eta}) = \gamma(\boldsymbol{\xi},\mathbf{O}) - \zeta(\boldsymbol{\eta}) + O(|\boldsymbol{\xi}|^{-1}) \; . \tag{246}$$

Also from (166),

$$\zeta(\boldsymbol{\xi}) = -\gamma(\boldsymbol{\xi},\mathbf{O}) + \zeta^{\infty} + O(|\boldsymbol{\xi}|^{-1}) \text{ as } |\boldsymbol{\xi}| \to \infty \; . \tag{247}$$

Substituting (247) into (174) we obtain

$$P_{\varepsilon}(\mathbf{x}) = \left(G(\mathbf{x},\mathbf{O}) + O\left(\varepsilon|\mathbf{x}|^{-1}\right)\right) D \; . \tag{248}$$

Combining (246), (247) and (248) in (201), we have

$$G_{\varepsilon}(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}) - (G(\mathbf{x},\mathbf{O}) + O(\varepsilon^{-1}|\mathbf{x}|))D(G(\mathbf{O},\mathbf{y}) + O(\varepsilon^{-1}|\mathbf{y}|)) + O(\varepsilon) \; , \tag{249}$$

from which we obtain (244).

b) Rewriting formula (232) in the form

$$P_{\varepsilon}(\mathbf{x}) = I_2 - (H(\mathbf{O},\mathbf{O}) - \zeta(\boldsymbol{\xi}) - H(\mathbf{x},\mathbf{O}))A^{-1} \; , \tag{250}$$

and substituting this into (201) for G_{ε}, we have

$$\begin{aligned} G_{\varepsilon}(\mathbf{x},\mathbf{y}) \;\; = \;\; & g(\boldsymbol{\xi},\boldsymbol{\eta}) - H(\mathbf{x},\mathbf{y}) \\ & -(H(\mathbf{O},\mathbf{O}) - \zeta(\boldsymbol{\xi}) - H(\mathbf{x},\mathbf{O}))D(H(\mathbf{O},\mathbf{O}) - \zeta(\boldsymbol{\eta}) - H(\mathbf{O},\mathbf{y})) \\ & +H(\mathbf{x},\mathbf{O}) + H(\mathbf{O},\mathbf{y}) - H(\mathbf{O},\mathbf{O}) + O(\varepsilon) \; . \end{aligned} \tag{251}$$

Using the fact that the components of $H(\mathbf{x},\mathbf{y})$ are smooth for $\mathbf{x}$, $\mathbf{y} \in \Omega$, in the vicinity of the origin we have from (251)

$$G(\mathbf{x},\mathbf{y}) = g(\boldsymbol{\xi},\boldsymbol{\eta}) - (O(|\mathbf{x}|) - \zeta(\boldsymbol{\xi}))D(O(|\mathbf{y}|) - \zeta(\boldsymbol{\eta})) + O(\max\{|\mathbf{x}|,|\mathbf{y}|\}) \; . \tag{252}$$

Since from (247), $\zeta(\boldsymbol{\xi}) = O\left(\log(\varepsilon^{-1}|\mathbf{x}|)\right)$ we have

$$G(\mathbf{x},\mathbf{y}) = g(\boldsymbol{\xi},\boldsymbol{\eta}) - \zeta(\boldsymbol{\xi})D\zeta(\boldsymbol{\eta}) + O(\max\{|\mathbf{x}|,|\mathbf{y}|\}) \; . \tag{253}$$

□

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