Asymptotics of a singular solution to the Dirichlet problem for an elliptic equation with discontinuous coefficients near the boundary

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Dedicated to Prof. Hans Triebel on the occasion of his 65th birthday

Abstract

We consider the Dirichlet problem for elliptic equations of arbitrary order and prove an asymptotic formula for a singular solution near a boundary point. The only a priori assumption on the coefficients of the principal part of the equation is the smallness of the local oscillation near the point.

1 Introduction

In this article, we are interested in the behavior of solutions to the Dirichlet problem for arbitrary even order $2m$ strongly elliptic equations in divergence form near a point $\mathcal{O}$ at the smooth boundary. We require only that the coefficients of the principal part of the operator have small oscillation near this point and the coefficients in lower order terms are allowed to have singularities at the boundary. In our recent paper [4], we derived an explicit asymptotic formula near $\mathcal{O}$ for solutions with finite energy integral. Here, our objective is to obtain an analogous asymptotic representation for solutions with infinite energy integral which have the least possible singularity. Since this representation is new even in the case of the second order equations, we start with describing it for this particular case.

Let us consider the uniformly elliptic equation

$$-\mathrm{div}\,(A(x)\,\mathrm{grad}\,u(x)) = f(x) \quad \text{in } G \tag{1}$$

complemented by the Dirichlet condition

$$u = 0 \quad \text{on } \partial G \setminus \{\mathcal{O}\}, \tag{2}$$

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where G is a domain in $\mathbb{R}^n$ with smooth boundary. We assume that the elements of the $n \times n$-matrix $A(x)$ are measurable and bounded complex-valued functions. We deal with a solution u having a finite Dirichlet integral outside any neighborhood of $\mathcal{O}$ and require, for simplicity, that $f = 0$ in a certain δ-neighborhood $G_\delta = \{x \in G : |x| < \delta\}$ of the origin. We suppose that there exists a constant symmetric matrix A with positive definite real part such that the function

$$\varkappa(r) := \sup_{G_r} ||A(x) - A||$$

is sufficiently small for $r < \delta$. We introduce the function

$$\mathcal{R}(x) = \frac{\langle (A(x) - A)\nu, \nu\rangle - n\langle \nu, (A(x) - A)A^{-1}x\rangle\langle \nu, x\rangle\langle A^{-1}x, x\rangle^{-1}}{|S^{n-1}|(\det A)^{1/2}\langle A^{-1}x, x\rangle^{n/2}}, \qquad (3)$$

where $\langle z, \zeta\rangle = z_1\zeta_1 + \ldots + z_n\zeta_n$ and ν is the interior unit normal at $\mathcal{O}$. (For the notation $(\det A)^{1/2}$ and $\langle A^{-1}x, x\rangle^{n/2}$ see [2], Sect.6.2.)

The following asymptotic formula is a corollary of our main Theorem 1

$$u(x) = \exp\Big(\int_{G_\delta \setminus G_{|x|}} \mathcal{R}(y)dy + O\Big(\int_{|x|}^{\delta} \varkappa(\rho)^2 \frac{d\rho}{\rho}\Big)\Big)$$
$$\times\Big(C\Big(\frac{\mathrm{dist}(x, \partial G)}{\langle A^{-1}x, x\rangle^{n/2}} + O\Big(|x|^{2-n-\varepsilon}\int_{|x|}^{\delta} \varkappa(\rho)\frac{d\rho}{\rho^{2-\varepsilon}}\Big)\Big) + O\Big(|x|^{1-\varepsilon}\Big)\Big), \quad (4)$$

where $C = \mathrm{const}$ and ε is a small positive number.

In Theorem 1 we obtain a general asymptotic formula similar to (4) for solutions of the Dirichlet problem for the uniformly strongly elliptic equation with complex-valued measurable coefficients

$$\sum_{0 \le |\alpha|, |\beta| \le m} (-\partial_x)^\alpha(\mathcal{L}_{\alpha\beta}(x)\partial_x^\beta u(x)) = f(x) \quad \text{on } B_\delta^+, \qquad (5)$$

where $B_\delta^+ = \mathbb{R}_+^n \cap B_\delta$, $\mathbb{R}_+^n = \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$ and $B_\delta = \{x \in \mathbb{R}^n : |x| < \delta\}$. Here and elsewhere by ∂_x we mean the vector of partial derivatives $(\partial_{x_1}, \ldots, \partial_{x_n})$. The only a priori assumption on the coefficients $\mathcal{L}_{\alpha\beta}$ is smallness of the function

$$\sum_{|\alpha|=|\beta|=m} |\mathcal{L}_{\alpha\beta}(x) - L_{\alpha\beta}| + \sum_{|\alpha+\beta|<2m} x_n^{2m-|\alpha+\beta|}|\mathcal{L}_{\alpha\beta}(x)|,$$

where $x \in B_\delta^+$ and $L_{\alpha\beta}$ are constants.

The proof of Theorem 1 follows the same lines as that of the main result in [4]. In order to make our exposition self-contained, we give complete formulations of intermediate results and their detailed proofs instead of referring repeatedly to analogous statements in [4].

2 Function spaces

Let $1 < p < \infty$ and let $W^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ denote the space of functions u defined on $\mathbb{R}^n_+$ and such that $\eta u \in W^{m,p}(\mathbb{R}^n_+)$ for all smooth η with compact support in $\overline{\mathbb{R}^n} \setminus \mathcal{O}$. Also let $\mathring{W}^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ be the subspace of $W^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$, which contains functions subject to

$$\partial^k_{x_n} u = 0 \quad \text{on } \partial\mathbb{R}^n_+ \setminus \mathcal{O} \text{ for } k = 0, \dots, m-1. \tag{6}$$

We introduce a family of seminorms in $\mathring{W}^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ by

$$\mathfrak{M}^m_p(u; K_{ar,br}) = \Big(\sum_{k=0}^{m} \int_{K_{ar,br}} |\nabla_k u(x)|^p |x|^{pk-n} dx \Big)^{1/p}, \quad r > 0, \tag{7}$$

where $K_{\rho,r} = \{x \in \mathbb{R}^n_+ \, : \, \rho < |x| < r\}$, a and b are positive constants, $a < b$ and $\nabla_k u$ is the vector $\{\partial^\alpha_x u\}_{|\alpha|=k}$. One can easily see that (6) implies the equivalence of $\mathfrak{M}^m_p(u; K_{ar,br})$ and the seminorm

$$\Big(\int_{K_{ar,br}} |\nabla_m u(x)|^p |x|^{pm-n} dx \Big)^{1/p} .$$

With another choice of a and b we arrive at an equivalent family of seminorms. Clearly,

$$\mathfrak{M}^m_p(u; K_{a'r,b'r}) \le c_1(a, b, a', b') \int_{a'r/b}^{b'r/a} \mathfrak{M}^m_p(u; K_{a\rho,b\rho}) \frac{d\rho}{\rho} , \tag{8}$$

where c_1 is a continuous function of its arguments.

We say that a function v belongs to the space $\mathring{W}^{m,q}_{\rm comp}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$, $pq = p + q$, if $v \in \mathring{W}^{m,q}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ and v has a compact support in $\overline{\mathbb{R}^n_+} \setminus \mathcal{O}$. By $W^{-m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ we denote the dual of $\mathring{W}^{m,q}_{\rm comp}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ with respect to the inner product in $L^2(\mathbb{R}^n_+)$. We supply $W^{-m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ with the seminorms

$$\mathfrak{M}^{-m}_p(f; K_{ar,br}) = \sup \Big| \int_{\mathbb{R}^n_+} f \, \overline{v} \, |x|^{-n} dx \Big| , \tag{9}$$

where the supremum is taken over all functions $v \in \mathring{W}^{m,q}_{\rm comp}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ supported by $ar \le |x| \le br$ and such that $\mathfrak{M}^m_p(v; K_{ar,br}) \le 1$. By a standard argument it follows from (8) that

$$\mathfrak{M}^{-m}_p(f; K_{a'r,b'r}) \le c_2(a, b, a'b') \int_{a'r/b}^{b'r/a} \mathfrak{M}^{-m}_p(f; K_{a\rho,b\rho}) \frac{d\rho}{\rho} , \tag{10}$$

where c_2 depends continuously on its arguments.

3 Statement of the Dirichlet problem in $\mathbb{R}^n_+$

We consider the Dirichlet problem

$$\mathcal{L}(x, \partial_x)u = f(x) \quad \text{in } \mathbb{R}^n_+, \tag{11}$$

$$\partial^k_{x_n} u\big|_{x_n=0} = 0 \quad \text{for } k = 0, 1, \dots, m-1 \quad \text{on } \mathbb{R}^{n-1} \setminus \mathcal{O} \tag{12}$$

for the differential operator

$$\mathcal{L}(x, \partial_x)u = \sum_{|\alpha|,|\beta| \le m} (-\partial_x)^\alpha \big(\mathcal{L}_{\alpha\beta}(x)\partial^\beta_x u\big) \tag{13}$$

with measurable complex valued coefficients $\mathcal{L}_{\alpha\beta}$ in $\mathbb{R}^n_+$.

We also need a differential operator with constant coefficients

$$L(\partial_x) = (-1)^m \sum_{|\alpha|=|\beta|=m} L_{\alpha\beta}\partial^{\alpha+\beta}_x\,, \tag{14}$$

where $\Re L(\xi) > 0$ for $\xi \in \mathbb{R}^n \setminus \mathcal{O}$. It will be convenient to require that the coefficient of $L(\partial_x)$ in $\partial^{2m}_{x_n}$ is equal to $(-1)^m$.

We treat $\mathcal{L}(x, \partial_x)$ as a perturbation of $L(\partial_x)$ and characterize this perturbation by the function

$$\Omega(r) = \sup_{x \in K_{r/e,r}} \Big(\sum_{|\alpha|=|\beta|=m} |\mathcal{L}_{\alpha\beta}(x) - L_{\alpha\beta}| + \sum_{|\alpha+\beta|<2m} x_n^{2m-|\alpha+\beta|}|\mathcal{L}_{\alpha\beta}(x)|\Big)\,, \tag{15}$$

which is assumed to be smaller than a certain positive constant depending on n, m, p and the coefficients $L_{\alpha\beta}$. It is straightforward that

$$\Omega(r) \le \int_{r/e}^{re} \Omega(t)\frac{dt}{t}\,. \tag{16}$$

By the classical Hardy inequality

$$\mathfrak{M}^{-m}_p((\mathcal{L} - L)(u); K_{r/e,r}) \le c\,\Omega(r)\mathfrak{M}^m_p(u; K_{r/e,r})\,, \tag{17}$$

where c depends only on n, m and p. Therefore, the boundedness of $\Omega(r)$ implies that the operator $\mathcal{L}(x, \partial_x)$ maps $\mathring{W}^{m,p}_{\text{loc}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ into $W^{-m,p}_{\text{loc}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$.

In what follows we always require that the right-hand side f in (11) belongs to $W^{-m,p}_{\text{loc}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ and consider a solution u of (11) in the space $\mathring{W}^{m,p}_{\text{loc}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$. This solution satisfies

$$\int_{\mathbb{R}^n_+} \sum_{|\alpha|,|\beta| \le m} \mathcal{L}_{\alpha\beta}(x)\partial^\beta_x u(x)\partial^\alpha_x \overline{v}(x)dx = \int_{\mathbb{R}^n_+} f\overline{v}(x)dx \tag{18}$$

for all $v \in \mathring{W}^{m,q}_{\text{comp}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$, $pq = p + q$. The integral on the right is understood in the distribution sense.

4 Formulation of the main result

In the next statement we make use of the notation introduced in Sect. 3. We also need the Poisson kernel E of the equation

$$\sum_{|\alpha|=|\beta|=m} L_{\alpha\beta}\partial_x^{\alpha+\beta}E(x)=0 \quad \text{in } \mathbb{R}^n_+, \tag{19}$$

which is positive homogeneous of degree $m-n$ and subject to the Dirichlet conditions on the hyperplane $x_n=0$:

$$\partial^j_{x_n}E=0 \quad \text{for} \quad 0\le j\le m-2\;, \quad \text{and} \quad \partial^{m-1}_{x_n}E=\delta(x')\;, \tag{20}$$

where δ is the Dirac function. In principle, the function E can be calculated using the Fourier transform in x' (see [1], Ch.1, Sect.2).

In the case of the polyharmonic operator $(-\Delta)^m$ one verifies directly that $E(x)=\text{const}\,x_n^m|x|^{-n}$. The constant factor can be evaluated by the identity

$$(\partial^m_{x_n}E)(x',0)=m\Gamma(n/2)\pi^{-n/2}|x'|^{-n}$$

which is found in Example 11.6.4[5]. Eventually,

$$E(x)=\frac{\Gamma(n/2)}{(m-1)!\pi^{n/2}}\,\frac{x_n^m}{|x|^n}\,. \tag{21}$$

In what follows, by c and $\mathcal{C}$ (sometimes enumerated) we denote different positive constants which depend only on m, n, p and the coefficients $L_{\alpha,\beta}$.

Theorem 1 *Assume that $\Omega(r)$ does not exceed a sufficiently small positive constant depending on m, n, p and $L_{\alpha\beta}$. There exist positive constants $\mathcal{C}$ and c depending on the same parameters such that the following assertions are valid.*

(i) *There exists $\mathcal{Z}\in \mathring{W}^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+}\setminus\mathcal{O})$ subject to $\mathcal{L}(x,\partial_x)\mathcal{Z}=0$ on B^+_e and satisfying*

$$(r\partial_r)^k\mathcal{Z}(x)=\exp\Big(\int_r^1\big(\mathcal{T}(\rho)+\Upsilon(\rho)\big)\frac{d\rho}{\rho}\Big)\big((m-n)^kE(x)+r^{m-n}v_k(x)\big), \tag{22}$$

where $k=0,1,\dots,m$, $r=|x|<1$ and Υ is a measurable function on $(0,1)$ satisfying

$$|\Upsilon(r)| \le \mathcal{C}\,\Omega(r)\Big(r^{-n}\int_0^r e^{\mathcal{C}\int_\rho^r\Omega(s)\frac{ds}{s}}\Omega(\rho)\rho^{n-1}d\rho+r\int_r^e e^{\mathcal{C}\int_r^\rho\Omega(s)\frac{ds}{s}}\Omega(\rho)\rho^{-2}d\rho\Big), \tag{23}$$

and

$$\begin{aligned}\mathcal{T}(\rho) &= \rho^n\int_{S^{n-1}_+}\sum_{|\beta|=m}(\mathcal{L}_{(0',m),\beta}(\xi)-L_{(0',m),\beta})E^{(\beta)}(\xi)d\theta_\xi\\ &+ \rho^n\int_{S^{n-1}_+}\sum_{|\beta|+k<2m}\mathcal{L}_{(0',k),\beta}(\xi)\frac{\xi_n^{m-k}}{(m-k)!}E^{(\beta)}(\xi)d\theta_\xi\end{aligned}$$

with $\rho = |\xi|$, $\theta = \xi/|\xi|$. *The functions* v_k *belong to* $L^p_{\rm loc}((0,\infty); \mathring{W}^{m-k,p}(S^{n-1}_+))$ *and satisfy*

$$\Big(\int_{r/e}^r \big(||v_k(\rho,\cdot)||^p_{W^{m-k,p}(S^{n-1}_+)} + ||\rho \partial_\rho v_k(\rho,\cdot)||^p_{W^{m-k-1,p}(S^{n-1}_+)} \big) \frac{d\rho}{\rho} \Big)^{1/p}$$

$$\leq c \Big(r^{-1} \int_0^r e^{\mathcal{C} \int_\rho^r \Omega(s) \frac{ds}{s}} \Omega(\rho) d\rho + r^n \int_r^e e^{\mathcal{C} \int_r^\rho \Omega(s) \frac{ds}{s}} \Omega(\rho) \rho^{-n-1} d\rho \Big), \quad (24)$$

where $k = 0, \dots, m-1$, S^{n-1}_+ *is the upper hemisphere and* $\mathring{W}^{m-k,p}(S^{n-1}_+)$ *is the completion of* $C_0^\infty(S^{n-1}_+)$ *in the norm of the Sobolev space* $W^{m,p}(S^{n-1}_+)$. *In the case* $k = m$ *estimate* (24) *holds without the second norm in the left-hand side.*

(ii) *Let*

$$I_f := \int_0^e \rho^{m+n} \exp \Big(\mathcal{C} \int_\rho^1 \Omega(s) \frac{ds}{s} \Big) \mathfrak{M}_p^{-m}(f; K_{\rho/e,\rho}) \frac{d\rho}{\rho} < \infty \quad (25)$$

and let $u \in \mathring{W}^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ *be a solution of* $\mathcal{L}(x,\partial_x)u = f$ *on* B^+_e *subject to*

$$\Big(\int_{K_{r/e,r}} |u(x)|^p |x|^{-n} dx \Big)^{1/p} = o \Big(r^{m-n-1} \exp \Big(-\mathcal{C} \int_r^1 \Omega(\rho) \frac{d\rho}{\rho} \Big) \Big) \quad (26)$$

as $r \to 0$. *Then for* $x \in B^+_1$

$$u(x) = C \mathcal{Z}(x) + w(x)\,, \quad (27)$$

where the constant C *satisfies*

$$|C| \leq c(I_f + ||u||_{L^p(K_{1,e})}) \quad (28)$$

and the function $w \in \mathring{W}^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ *is subject to*

$$\mathfrak{M}_p^m(w; K_{r/e,r}) \leq c\, r^m \Big(\int_0^r r^{-n} \rho^{m+n} e^{\mathcal{C} \int_\rho^r \Omega(s) \frac{ds}{s}} \mathfrak{M}_p^{-m}(f; K_{\rho/e,\rho}) \frac{d\rho}{\rho}$$

$$+ \int_r^e \rho^m e^{\mathcal{C} \int_r^\rho \Omega(s) \frac{ds}{s}} \mathfrak{M}_p^{-m}(f; K_{\rho/e,\rho}) \frac{d\rho}{\rho} + e^{\mathcal{C} \int_r^1 \Omega(s) \frac{ds}{s}} ||u||_{L^p(K_{1,e})} \Big) \quad (29)$$

for $r < 1$.

The proof of this theorem will be given in Sect. 5–18.

5 Reduction of problem (11), (12) to the Dirichlet problem in a cylinder

We write problem (11), (12) in the variables

$$t = -\log|x| \quad \text{and} \quad \theta = x/|x|. \quad (30)$$

The mapping $x \to (t, \theta)$ transforms $\mathbb{R}^n_+$ onto the cylinder $\Pi = S^{n-1}_+ \times \mathbb{R}$.

We shall need the spaces $\mathring{W}^{m,p}_{\mathrm{loc}}(\Pi)$ and $W^{-m,p}_{\mathrm{loc}}(\Pi)$ which are the images of $\mathring{W}^{m,p}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ and $W^{-m,p}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ under mapping (30). They can be defined independently as follows.

The space $\mathring{W}^{m,p}_{\mathrm{loc}}(\Pi)$ consists of functions whose derivatives up to order m belong to $L^p(D)$ for every compact subset D of $\overline{\Pi}$ and whose derivatives up to order $m-1$ vanish on $\partial\Pi$. The seminorm $\mathfrak{M}^m_p(u; K_{e^{-1-t}, e^{-t}})$ in $\mathring{W}^{m,p}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ is equivalent to the seminorm $||u||_{W^{m,p}(\Pi_t)}$, $t \in \mathbb{R}$, where $\Pi_t = \{(\theta, \tau) \in \Pi : \tau \in (t, t+1)\}$. The space $W^{-m,p}_{\mathrm{loc}}(\Pi)$ consists of the distributions f on Π such that the seminorm

$$||f||_{W^{-m,p}(\Pi_t)} = \sup \Big| \int_{\Pi_t} f\, \overline{v} d\tau d\theta \Big| \tag{31}$$

is finite for every $t \in \mathbb{R}$. The supremum in (31) is taken over all $v \in \mathring{W}^{m,q}_{\mathrm{loc}}(\Pi)$, $pq = p + q$, supported by $\overline{\Pi_t}$ and subject to $||v||_{W^{m,q}(\Pi_t)} \le 1$. The seminorm (31) is equivalent to $\mathfrak{M}^{-m}_p(f; ; K_{e^{-1-t}, e^{-t}})$.

In the variables (t, θ) the operator L takes the form

$$L(\partial_x) = e^{2mt} \mathbf{A}(\theta, \partial_\theta, -\partial_t) \ , \tag{32}$$

where $\mathbf{A}$ is an elliptic partial differential operator of order $2m$ on Π with smooth coefficients. We introduce the operator $\mathbb{N}$ by

$$L(\partial_x) - \mathcal{L}(x, \partial_x) = e^{2mt} \mathbb{N}(\theta, t, \partial_\theta, -\partial_t). \tag{33}$$

Now problem (11), (12) can be written as

$$\begin{cases} \mathbf{A}(\theta, \partial_\theta, -\partial_t) u = \mathbb{N}(\theta, t, \partial_\theta, -\partial_t) u + e^{-2mt} f \quad \text{on } \Pi \\ u \in \mathring{W}^{p,m}_{\mathrm{loc}}(\Pi) \,, \end{cases} \tag{34}$$

where $f \in W^{-m,p}_{\mathrm{loc}}(\Pi)$. We do not mark the dependence on the new variables t, θ in u and f.

Let $W^{-m,p}(S^{n-1}_+)$ denote the dual of $\mathring{W}^{m,q}(S^{n-1}_+)$ with respect to the inner product in $L^2(S^{n-1}_+)$. We introduce the operator pencil

$$\mathcal{A}(\lambda) : \mathring{W}^{m,p}(S^{n-1}_+) \to W^{-m,p}(S^{n-1}_+) \tag{35}$$

by

$$\mathcal{A}(\lambda) U(\theta) = r^{-\lambda + 2m} L(\partial_x) r^\lambda U(\theta) = \mathbf{A}(\theta, \partial_\theta, \lambda) U(\theta). \tag{36}$$

The following properties of $\mathcal{A}$ and its adjoint are standard and their proofs can be found, for example in [5], Sect. 10.3. The operator (35) is Fredholm for all $\lambda \in \mathbb{C}$ and its spectrum consists of eigenvalues with finite geometric multiplicities. These eigenvalues are

$$m, m+1, m+2, \dots \quad \text{and} \quad m-n, m-n-1, m-n-2, \dots \ , \tag{37}$$

and there are no generalized eigenvectors. The only eigenvector (up to a constant factor) corresponding to the eigenvalue $m - n$ is $|x|^{n-m}E(x) = E(\theta)$, where E is the Poisson kernel defined in Sect. 4.

We introduce the operator pencil $\overline{\mathcal{A}}(\lambda)$ defined on $\mathring{W}^{m,p}(S_+^{n-1})$ by the formula $\overline{\mathcal{A}}(\lambda)U(\theta) = r^{-\lambda+2m}\overline{L}(\partial_x)r^\lambda U(\theta)$. This pencil has the same eigenvalues as the pencil $\mathcal{A}(\lambda)$. The only eigenvector (up to a constant factor) of $\overline{\mathcal{A}}$ corresponding to the eigenvalue m is $|x|^{-m}x_n^m = \theta_n^m$.

Using the definitions of the above pencils and Green's formula for L and $\overline{L}$ one can show that $(\mathcal{A}(\lambda))^* = \overline{\mathcal{A}}(2m - n - \overline{\lambda})$, where $*$ denotes passage to the adjoint operator in $L^2(S_+^{n-1})$.

6 Properties of the unperturbed Dirichlet problems in $\mathbb{R}_+^n$ and Π

Let us consider the Dirichlet problem

$$\left\{ \begin{array}{ll} L(\partial_x)u = f & \text{in } \mathbb{R}_+^n, \\ u \in \mathring{W}^{p,m}_{\text{loc}}(\overline{\mathbb{R}_+^n} \setminus \mathcal{O})\,. & \end{array} \right. \tag{38}$$

Proposition 1 (i) *Let $f \in W^{-m,p}_{\text{loc}}(\overline{\mathbb{R}_+^n} \setminus \mathcal{O})$ be subject to*

$$\int_0^1 \rho^{m+n}\mathfrak{M}_p^{-m}(f; K_{\rho/e,\rho})\frac{d\rho}{\rho} + \int_1^\infty \rho^m \mathfrak{M}_p^{-m}(f : K_{\rho/e,\rho})\frac{d\rho}{\rho} < \infty\,. \tag{39}$$

Then problem (38) *has a solution $u \in \mathring{W}^{m,p}_{\text{loc}}(\overline{\mathbb{R}_+^n} \setminus \mathcal{O})$ satisfying*

$$\begin{aligned} &\mathfrak{M}_p^m(u; K_{r/e,r}) \\ &\leq c\Big(\int_0^r r^{m-n}\rho^{m+n}\mathfrak{M}_p^{-m}(f; K_{\rho/e,\rho})\frac{d\rho}{\rho} + \int_r^\infty r^m\rho^m\mathfrak{M}_p^{-m}(f; K_{\rho/e,\rho})\frac{d\rho}{\rho}\Big). \end{aligned} \tag{40}$$

Estimate (40) *implies*

$$\mathfrak{M}_p^m(u; K_{r/e,r}) = \left\{ \begin{array}{ll} o(r^{m-n}) & \textit{if } r \to 0 \\ o(r^m) & \textit{if } r \to \infty. \end{array} \right. \tag{41}$$

Solution $u \in \mathring{W}^{m,p}_{\text{loc}}(\overline{\mathbb{R}_+^n} \setminus \mathcal{O})$ of problem (38) *subject to* (41) *is unique.*

(ii) *Let $f \in W^{-m,p}_{\text{loc}}(\overline{\mathbb{R}_+^n} \setminus \mathcal{O})$ be subject to*

$$\int_0^1 \rho^{m+n+1}\mathfrak{M}_p^{-m}(f; K_{\rho/e,\rho})\frac{d\rho}{\rho} + \int_1^\infty \rho^{m+n}\mathfrak{M}_p^{-m}(f; K_{\rho/e,\rho})\frac{d\rho}{\rho} < \infty\,. \tag{42}$$

Then problem (38) *has a solution $u \in \mathring{W}^{m,p}_{\text{loc}}(\overline{\mathbb{R}_+^n} \setminus \mathcal{O})$ satisfying*

$$\begin{aligned} &\mathfrak{M}_p^m(u; K_{r/e,r}) \\ &\leq c\Big(\int_0^r r^{m-n-1}\rho^{m+n+1}\mathfrak{M}_p^{-m}(f; K_{\rho/e,\rho})\frac{d\rho}{\rho} + \int_r^\infty r^{m-n}\rho^{m+n}\mathfrak{M}_p^{-m}(f; K_{\rho/e,\rho})\frac{d\rho}{\rho}\Big). \end{aligned} \tag{43}$$

Estimate (43) *implies*

$$\mathfrak{M}_p^m(u; K_{r/e,r}) = \begin{cases} o(r^{m-n-1}) & \textit{if } r \to 0 \\ o(r^{m-n}) & \textit{if } r \to \infty. \end{cases} \tag{44}$$

Solution $u \in \mathring{W}^{m,p}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ *of problem* (38) *subject to* (44) *is unique.*

Proof. (i) Let us assume that f is supported by $\{x \in \overline{\mathbb{R}^n_+} : 1/2 \le |x| \le 4\}$. We set

$$u(x) = \int_{\mathbb{R}^n_+} \mathcal{G}(x,y) f(y) dy\,, \tag{45}$$

where $\mathcal{G}$ is Green's function of problem (38). Using standard estimates of $\mathcal{G}$ and its derivatives, one arrives at

$$\mathfrak{M}_p^m(u; K_{r/e,r}) \le c(r^m + r^{m-n})\mathfrak{M}_p^{-m}(f; K_{1/4,8})\,.$$

By (10) this inequality can be written in the form (40). We check by dilation that the same holds for f supported by $\rho/2 \le |x| \le 4\rho$ where ρ is an arbitrary positive number.

Now, we remove the restriction on the support of f. By a partition of unity we represent f as the series $f = \sum f_k$, where $f_k \in W^{-m,p}(\mathbb{R}^n_+)$ is supported by $2^{k-1} \le |x| \le 2^{k+2}$, $k = 0, \pm 1, \dots$, and

$$\sum_{k=-\infty}^{\infty} \mathfrak{M}_p^{-m}(f_k; K_{\rho/e,\rho}) \le c\mathfrak{M}_p^{-m}(f; K_{\rho/e,\rho})\,. \tag{46}$$

Denote by u_k the solution of problem (38) given by (45) with f replaced by f_k. It follows from (46) that the series $u = \sum u_k$ satisfies (40). Hence, u is a required solution.

The uniqueness of u follows from Theorem 3.9.1 in [3], where $k_+ = m$ and $k_- = m - n$.

(ii) The proof of existence is the same as in (i) with the only difference that representation (45) is replaced by

$$u(x) = \int_{\mathbb{R}^n_+} \mathcal{G}(x,y) f(y) dy + \frac{E(x)}{m!} \int_{\mathbb{R}^n_+} y_n^m f(y) dy\,.$$

Uniqueness is a consequence of Theorem 3.9.1 in [3] where $k_+ = m - n$ and $k_- = m - n - 1$. The proof is complete.

Let us turn to the Dirichlet problem

$$\begin{cases} \mathbf{A}(\theta, \partial_\theta, -\partial_t) u = e^{-2mt} f & \text{on } \Pi \\ u \in \mathring{W}^{p,m}_{\mathrm{loc}}(\Pi)\,. \end{cases} \tag{47}$$

The next statement follows directly from Proposition 1 by the change of variables (30).

Proposition 2 (i) *Let* $f \in W^{-m,p}_{\rm loc}(\Pi)$ *be subject to*

$$\int_0^\infty e^{-(m+n)\tau}||f||_{W^{-m,p}(\Pi_\tau)}d\tau + \int_{-\infty}^0 e^{-m\tau}||f||_{W^{-m,p}(\Pi_\tau)}d\tau < \infty \ . \tag{48}$$

Then problem (47) *has a solution* $u \in \mathring{W}^{m,p}_{\rm loc}(\Pi)$ *satisfying the estimate*

$$\begin{aligned} ||u||_{W^{m,p}(\Pi_t)} &\leq c\Big(\int_t^\infty e^{(n-m)t-(m+n)\tau}||f||_{W^{-m,p}(\Pi_\tau)}d\tau \\ &+ \int_{-\infty}^t e^{-m(t+\tau)}||f||_{W^{-m,p}(\Pi_\tau)}d\tau\Big). \end{aligned} \tag{49}$$

Estimate (49) *implies*

$$||u||_{W^{m,p}(\Pi_t)} = \begin{cases} o(e^{(n-m)t}) & \textit{if } t \to +\infty \\ o(e^{-mt}) & \textit{if } t \to -\infty. \end{cases} \tag{50}$$

The solution $u \in \mathring{W}^{m,p}_{\rm loc}(\Pi)$ *of problem* (47) *subject to* (50) *is unique.*

(ii) *Let* $f \in W^{-m,p}_{\rm loc}(\Pi)$ *be subject to*

$$\int_0^\infty e^{-(m+n+1)\tau}||f||_{W^{-m,p}(\Pi_\tau)}d\tau + \int_{-\infty}^0 e^{-(m+n)\tau}||f||_{W^{-m,p}(\Pi_\tau)}d\tau < \infty \ . \tag{51}$$

Then problem (47) *has a solution* $u \in \mathring{W}^{m,p}_{\rm loc}(\Pi)$ *satisfying the estimate*

$$\begin{aligned} ||u||_{W^{m,p}(\Pi_t)} &\leq c\Big(\int_t^\infty e^{(n+1-m)t-(m+n+1)\tau}||f||_{W^{-m,p}(\Pi_\tau)}d\tau \\ &+ \int_{-\infty}^t e^{(n-m)t-(m+n)\tau}||f||_{W^{-m,p}(\Pi_\tau)}d\tau\Big). \end{aligned} \tag{52}$$

Estimate (52) *implies*

$$||u||_{W^{m,p}(\Pi_t)} = \begin{cases} o(e^{(n-m+1)t}) & \textit{if } t \to +\infty \\ o(e^{(n-m)t}) & \textit{if } t \to -\infty. \end{cases} \tag{53}$$

The solution $u \in \mathring{W}^{m,p}_{\rm loc}(\Pi)$ *of problem* (47) *subject to* (53) *is unique.*

The following assertion can be interpreted as a description of the asymptotic behavior of solutions to problem (47) at $\pm\infty$.

Proposition 3 *Let* $f \in W^{-m,p}_{\rm loc}(\Pi)$ *be subject to*

$$\int_{\mathbb{R}} e^{-(m+n)\tau}||f||_{W^{-m,p}(\Pi_\tau)}d\tau < \infty \ . \tag{54}$$

Also let u_1 *and* u_2 *be solutions from* Proposition 2 (i) *and* (ii) *respectively. Then*

$$u_2 - u_1 = Ce^{(n-m)t}E(\theta) \ , \tag{55}$$

where C *is a constant.*

Proof. By Proposition 2 (i) and (ii)

$$||u_2 - u_1||_{W^{m,p}(\Pi_t)} = o(e^{(n-m+1)t}) \quad \text{as } t \to +\infty$$

and

$$||u_2 - u_1||_{W^{m,p}(\Pi_t)} = o(e^{-mt}) \quad \text{as } t \to -\infty.$$

By the local regularity result (see [1], Sect.15) the same relations remain valid for $||u_2 - u_1||_{W^{2m,2}(\Pi_t)}$. Now (55) follows from Proposition 3.8.1 in [KM].

Returning to the variables x we derive from Proposition 3 the following description of the asymptotic behavior of solutions to problem (38) both at infinity and near the origin.

Proposition 4 *Let $f \in W^{-m,p}_{\text{loc}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ be subject to*

$$\int_0^\infty \rho^{m+n} \mathfrak{M}_p^{-m}(f; K_{\rho/e,\rho}) \frac{d\rho}{\rho} < \infty \ . \tag{56}$$

Also let u_1 and u_2 be solutions from (i) *and* (ii) *in Proposition* 1 *respectively. Then*

$$u_2(x) - u_1(x) = C\, E(x) \ , \tag{57}$$

where C is a constant.

We show that the constant C in (57) can be found explicitly.

Proposition 5 *The constant C in* (57) *is given by*

$$C = \frac{-1}{m!} \int_{\mathbb{R}^n_+} f(x) x_n^m dx \ . \tag{58}$$

Proof. Integrating by parts we check the identity

$$\int_{\mathbb{R}^n_+} L\big(\partial_x\big)(\zeta E(x))\, x_n^m dx = -m! \ , \tag{59}$$

where ζ is a smooth function equal to 1 in a neighborhood of the origin and zero for large $|x|$.

By (57)

$$\int_{\mathbb{R}^n_+} L(\zeta(u_2 - u_1)) x_n^m dx = C \int_{\mathbb{R}^n_+} L(\zeta E(x)) x_n^m dx. \tag{60}$$

It follows from (59) that the right-hand side is equal to $-C\, m!$. Using (44), we see that

$$\int_{\mathbb{R}^n_+} L(\zeta u_1) x_n^m dx = 0.$$

Similarly, by (41)

$$\int_{\mathbb{R}^n_+} L((\zeta - 1) u_2) x_n^m dx = 0,$$

which together with (60) lead to (58).

Proposition 6 *The constant C in* (55) *is given by*

$$C = \frac{-1}{m!} \int_\Pi e^{-(m+n)t} f(t,\theta)\, \theta_n^m \, dtd\theta \; .$$

Proof results from Proposition 5.

The following uniqueness result is a consequence of Proposition 3.

Corollary 1 *Let $u \in \overset{\circ}{W}{}^{m,p}_{\rm loc}(\Pi)$ be a solution of* (47) *with $f = 0$. Suppose that u is subject to*

$$||u||_{W^{m,p}(\Pi_t)} = \begin{cases} o(e^{(n-m+1)t}) & \textit{if } t \to +\infty \\ o(e^{-mt}) & \textit{if } t \to -\infty. \end{cases} \tag{61}$$

Then $u = {\rm const}\, e^{(n-m)t} E(\theta)$.

Proof. Let $\zeta = \zeta(t)$ be a smooth function on $\mathbb{R}$ equal to 1 for $t > 1$ and 0 for $t < 0$. Then $u = u_2 - u_1$, where $u_2 = \zeta u$ and $u_1 = (\zeta - 1)u$. The functions u_1 and u_2 satisfy (47) with $f = \mathbf{A}(\zeta u) - \zeta \mathbf{A} u$. Now the result follows from Proposition 3.

7 Properties of the perturbed Dirichlet problems in Π and $\mathbb{R}^n_+$

Now the turn to the Dirichlet problem (34). By (17), the perturbation $\mathbb{N}$ of the operator $\mathbf{A}$ satisfies

$$||\mathbb{N}||_{\overset{\circ}{W}{}^{m,p}(\Pi_t) \to W^{-m,p}(\Pi_t)} \le c\,\omega(t)\,, \tag{62}$$

where we use the notation $\omega(t) = \Omega(e^{-t})$. As before, we assume that Ω does not exceed a sufficiently small constant depending on n, m, p and $L_{\alpha\beta}$.

The next statement generalizes Proposition 2.

Proposition 7 *There exist positive constants c and $\mathcal{C}$ such that the following two assertions hold:*

(i) *Let $f \in W^{-m,p}_{\rm loc}(\Pi)$ be subject to*

$$\int_0^\infty e^{-(m+n)\tau + \mathcal{C}\int_0^\tau \omega(s)ds} ||f||_{W^{-m,p}(\Pi_\tau)} d\tau + \int_{-\infty}^0 e^{-m\tau + \mathcal{C}\int_\tau^0 \omega(s)ds} ||f||_{W^{-m,p}(\Pi_\tau)} d\tau < \infty\,. \tag{63}$$

Then problem (34) *has a solution $u \in \overset{\circ}{W}{}^{m,p}_{\rm loc}(\Pi)$ satisfying the estimate*

$$||u||_{W^{m,p}(\Pi_t)} \le c\Big(\int_t^\infty e^{(n-m)t-(m+n)\tau + \mathcal{C}\int_t^\tau \omega(s)ds} ||f||_{W^{-m,p}(\Pi_\tau)} d\tau + \int_{-\infty}^t e^{-m(t+\tau)+\mathcal{C}\int_\tau^t \omega(s)ds} ||f||_{W^{-m,p}(\Pi_\tau)} d\tau \Big). \tag{64}$$

Estimate (64) *implies*

$$||u||_{W^{m,p}(\Pi_t)} = \begin{cases} o\big(e^{(n-m)t-\mathcal{C}\int_0^t \omega(s)ds}\big) & \textit{if } t \to +\infty \\ o\big(e^{-mt-\mathcal{C}\int_t^0 \omega(s)ds}\big) & \textit{if } t \to -\infty. \end{cases} \tag{65}$$

Solution $u \in \mathring{W}^{m,p}_{\mathrm{loc}}(\Pi)$ *of problem* (34) *subject to* (65) *is unique.*

(ii) *Let* $f \in W^{-m,p}_{\mathrm{loc}}(\Pi)$ *be subject to*

$$\int_0^\infty e^{-(m+n+1)\tau+\mathcal{C}\int_0^\tau \omega(s)ds}||f||_{W^{-m,p}(\Pi_\tau)}d\tau$$
$$+\int_{-\infty}^0 e^{-(m+n)\tau+\mathcal{C}\int_\tau^0 \omega(s)ds}||f||_{W^{-m,p}(\Pi_\tau)}d\tau < \infty\,. \tag{66}$$

Then problem (34) *has a solution* $u \in \mathring{W}^{m,p}_{\mathrm{loc}}(\Pi)$ *satisfying the estimate*

$$\begin{aligned} ||u||_{W^{m,p}(\Pi_t)} \quad &\leq \quad c\Big(\int_t^\infty e^{(n+1-m)t-(m+n+1)\tau+\mathcal{C}\int_t^\tau \omega(s)ds}||f||_{W^{-m,p}(\Pi_\tau)}d\tau \\ &+ \quad \int_{-\infty}^t e^{(n-m)t-(m+n)\tau+\mathcal{C}\int_\tau^t \omega(s)ds}||f||_{W^{-m,p}(\Pi_\tau)}d\tau\Big). \end{aligned} \tag{67}$$

Estimate (67) *implies*

$$||u||_{W^{m,p}(\Pi_t)} = \begin{cases} o\big(e^{(n-m+1)t-\mathcal{C}\int_0^t \omega(s)ds}\big) & \textit{if } t \to +\infty \\ o\big(e^{(n-m)t-\mathcal{C}\int_t^0 \omega(s)ds}\big) & \textit{if } t \to -\infty. \end{cases} \tag{68}$$

Solution $u \in \mathring{W}^{m,p}_{\mathrm{loc}}(\Pi)$ *of problem* (34) *subject to* (68) *is unique.*

Proof. Let $k_+ = m$, $k_- = m-n$ in the case (i) and $k_+ = m-n$, $k_- = m-n-1$ in the case (ii). Repeating the proof of Theorem 5.3.2 in [3] with Proposition 2 playing the role of Theorem 3.5.5 in [KM1], we construct a solution u satisfying

$$||u||_{W^{m,p}(\Pi_t)} \leq c\int_{\mathbb{R}} g_\omega(t,\tau)||f||_{W^{-m,p}(\Pi_\tau)}d\tau\,,$$

where g_ω is a certain positive Green's function of the ordinary differential operator

$$-(\partial_t + k_+)(\partial_t + k_-) - c\omega(t)\,.$$

According to Proposition 6.3.1 in [3] this Green's function satisfies

$$g_\omega(t,\tau) \leq c\, e^{k_\pm(\tau-t)\pm\mathcal{C}\int_\tau^t \omega(s)ds} \quad \text{for } t \gtrless \tau,$$

which completes the proof of existence.

We turn to the proof of uniqueness. Let $u \in \mathring{W}^{m,p}_{\mathrm{loc}}(\Pi)$ be a solution of problem (34) with $f=0$ subject either to estimate (65) or (68). Clearly, these estimates are valid for $p=2$. The result follows from Theorem 10.8.13 in [3], where $\ell = 2m$ and $q = m$.

By the change of variables (30) on can formulate Proposition 7 as follows

Proposition 8 *There exists a positive constant $\mathcal{C}$ such that the following two assertions hold:*

(i) *Let $f \in W^{-m,p}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ be subject to*

$$\int_0^1 \rho^{m+n} e^{\mathcal{C}\int_\rho^1 \Omega(s)\frac{ds}{s}} \mathfrak{M}^{-m}_p(f; K_{\rho/e,\rho}) \frac{d\rho}{\rho} + \int_1^\infty \rho^m e^{\mathcal{C}\int_1^\rho \Omega(s)\frac{ds}{s}} \mathfrak{M}^{-m}_p(f : K_{\rho/e,\rho}) \frac{d\rho}{\rho} < \infty\,. \tag{69}$$

Then problem (11), (12) *has a solution $u \in \mathring{W}^{m,p}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ satisfying*

$$\mathfrak{M}^m_p(u; K_{r/e,r}) \le c\Big(\int_0^r r^{m-n} \rho^{m+n} e^{\mathcal{C}\int_\rho^r \Omega(s)\frac{ds}{s}} \mathfrak{M}^{-m}_p(f; K_{\rho/e,\rho}) \frac{d\rho}{\rho} + \int_r^\infty r^m \rho^m e^{\mathcal{C}\int_r^\rho \Omega(s)\frac{ds}{s}} \mathfrak{M}^{-m}_p(f; K_{\rho/e,\rho}) \frac{d\rho}{\rho}\Big). \tag{70}$$

Estimate (70) *implies*

$$\mathfrak{M}^m_p(u; K_{r/e,r}) = \begin{cases} o\big(r^{m-n} e^{-\mathcal{C}\int_r^1 \Omega(s)\frac{ds}{s}}\big) & \textit{if } r \to 0 \\ o\big(r^m e^{-\mathcal{C}\int_1^r \Omega(s)\frac{ds}{s}}\big) & \textit{if } r \to \infty. \end{cases} \tag{71}$$

Solution $u \in \mathring{W}^{m,p}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ of problem (11), (12) *subject to* (71) *is unique.*

(ii) *Let $f \in W^{-m,p}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ be subject to*

$$\int_0^1 \rho^{m+n+1} e^{\mathcal{C}\int_\rho^1 \Omega(s)\frac{ds}{s}} \mathfrak{M}^{-m}_p(f; K_{\rho/e,\rho}) \frac{d\rho}{\rho} + \int_1^\infty \rho^{m+n} e^{\mathcal{C}\int_1^\rho \Omega(s)\frac{ds}{s}} \mathfrak{M}^{-m}_p(f; K_{\rho/e,\rho}) \frac{d\rho}{\rho} < \infty\,. \tag{72}$$

Then problem (11), (12) *has a solution $u \in \mathring{W}^{m,p}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ satisfying*

$$\mathfrak{M}^m_p(u; K_{r/e,r}) \le c\Big(\int_0^r r^{m-n-1} \rho^{m+n+1} e^{\mathcal{C}\int_\rho^r \Omega(s)\frac{ds}{s}} \mathfrak{M}^{-m}_p(f; K_{\rho/e,\rho}) \frac{d\rho}{\rho} + \int_r^\infty r^{m-n} \rho^{m+n} e^{\mathcal{C}\int_r^\rho \Omega(s)\frac{ds}{s}} \mathfrak{M}^{-m}_p(f; K_{\rho/e,\rho}) \frac{d\rho}{\rho}\Big). \tag{73}$$

Estimate (73) *implies*

$$\mathfrak{M}^m_p(u; K_{r/e,r}) = \begin{cases} o\big(r^{m-n-1} e^{-\mathcal{C}\int_r^1 \Omega(s)\frac{ds}{s}}\big) & \textit{if } r \to 0 \\ o\big(r^{m-n} e^{-\mathcal{C}\int_1^r \Omega(s)\frac{ds}{s}}\big) & \textit{if } r \to \infty. \end{cases} \tag{74}$$

Solution $u \in \mathring{W}^{m,p}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ of problem (11), (12) *subject to* (74) *is unique.*

8 Reduction of problem (34) to a first order system in t

Let

$$\int_{\mathbb{R}} e^{-(m+n)\tau+\mathcal{C}|\int_0^\tau \omega(s)ds|}||f||_{W^{-m,p}(\Pi_\tau)}d\tau < \infty\,. \tag{75}$$

This condition implies both (66) and (63) hence there exist the solutions u_1 and u_2 from Proposition 7 (i) and (ii) respectively. Clearly, the difference $u_1 - u_2$ satisfies the homogeneous problem (34) and the relation

$$||u_1 - u_2||_{W^{m,p}(\Pi_t)} = \begin{cases} o(e^{(n-m+1)t-\mathcal{C}\int_0^t \omega(s)ds}) & \text{if } t \to +\infty \\ o(e^{-mt-\mathcal{C}\int_t^0 \omega(s)ds}) & \text{if } t \to -\infty. \end{cases} \tag{76}$$

Here and in sections 9-18 we show that there exists a solution $\mathcal{Z}$ of the homogeneous problem (34), unique up to a constant factor, such that $u_1 - u_2 = C_f\mathcal{Z}$, where C_f is a constant depending on f. We also give an asymptotic representation of $\mathcal{Z}$ at infinity. We start with reducing problem (34) to a first order system in t. To this end we write (34) in a slightly different form. First we obtain a representation of the right-hand side f by using the following standard assertion

Lemma 1 *One can represent $f \in W^{-m,p}_{\text{loc}}(\Pi)$ as*

$$f = e^{2mt}\sum_{j=0}^{m}(-\partial_t)^{m-j}f_j\ , \tag{77}$$

where $f_j \in L^p_{\text{loc}}(\mathbb{R}; W^{-j,p}(S^{n-1}_+))$. This representation can be chosen to satisfy

$$c_1\mathfrak{M}^{-m}_p(f; K_{e^{-1-t},e^{-t}}) \le e^{2mt}\sum_{j=0}^{m}||f_j||_{W^{-j,p}(\Pi_t)} \le c_2\mathfrak{M}^{-m}_p(f; K_{e^{-2-t},e^{1-t}})\,,$$

where c_1 and c_2 are constants depending only on n, m and p.

One verifies directly that

$$r^{|\alpha|}\partial^\alpha_x u = \sum_{l=0}^{|\alpha|} Q_{\alpha l}(\theta, \partial_\theta)(r\partial_r)^l u$$

and

$$r^{2m}\partial^\alpha_x(r^{-2m+|\alpha|}u) = \sum_{l=0}^{|\alpha|} P_{\alpha l}(\theta, \partial_\theta)(r\partial_r)^l u$$

where $Q_{\alpha l}(\theta, \partial_\theta)$ and $P_{\alpha l}(\theta, \partial_\theta)$ are differential operators of order $|\alpha| - l$ with smooth coefficients. Furthermore, integrating by parts in

$$\int_{\mathbb{R}^n_+} \partial^\alpha_x(r^{-2m+|\alpha|}u)r^{2m-n}\overline{v}dx$$

we obtain

$$(-1)^{|\alpha|}\sum_{l=0}^{|\alpha|}Q_{\alpha l}(r\partial_r+2m-n)^l=\sum_{l=0}^{|\alpha|}P^*_{\alpha l}(-r\partial_r)^l\,. \tag{78}$$

Now we write $\mathbf{A}$ in the form

$$\mathbf{A}(\theta,\partial_\theta,-\partial_t)=\sum_{j=0}^{m}(-\partial_t)^{m-j}\mathcal{A}_j(-\partial_t)\,,$$

where

$$\mathcal{A}_j(-\partial_t)=\sum_{k=0}^{m}A_{jk}(-\partial_t)^{m-k}$$

with

$$A_{jk}=(-1)^m\sum_{|\alpha|=|\beta|=m}P_{\alpha,m-j}(\theta,\partial_\theta)L_{\alpha\beta}Q_{\beta,m-k}(\theta,\partial_\theta)\,.$$

It is clear that

$$A_{jk}:\mathring{W}^{k,p}(S^{n-1}_+)\to W^{-j,p}(S^{n-1}_+) \tag{79}$$

are differential operators of order $\le j+k$ on S^{n-1}_+ with smooth coefficients. Since $Q_{\alpha,|\alpha|}=P_{\alpha,|\alpha|}=\theta^\alpha$, we have

$$A_{00}=L(\theta). \tag{80}$$

We also write

$$\mathbb{N}(\theta,t,\partial_\theta,-\partial_t)u=\sum_{j=0}^{m}(-\partial_t)^{m-j}\big(\mathcal{N}_j(t,-\partial_t)u\big)\,, \tag{81}$$

where

$$\mathcal{N}_j(t,-\partial_t)=\sum_{k=0}^{m}\mathcal{N}_{jk}(t)(-\partial_t)^{m-k} \tag{82}$$

with

$$\mathcal{N}_{jk}=\sum_{m-j\le|\alpha|\le m}\;\sum_{m-k\le|\beta|\le m}(-1)^{|\alpha|}P_{\alpha,m-j}N_{\alpha\beta}Q_{\beta,m-k}\,. \tag{83}$$

We use the notation $N_{\alpha\beta}(e^{-t}\theta)=L_{\alpha\beta}-\mathcal{L}_{\alpha\beta}(e^{-t}\theta)$ if $|\alpha|=|\beta|=m$ and $N_{\alpha\beta}(e^{-t}\theta)=-e^{(|\alpha+\beta|-2m)t}\mathcal{L}_{\alpha\beta}(e^{-t}\theta)$ if $|\alpha+\beta|<2m$. By (83) the operators

$$\mathcal{N}_{jk}(t)\,:\,\mathring{W}^{k,p}(S^{n-1}_+)\to W^{-j,p}(S^{n-1}_+)$$

are continuous. By (83) and (78), for almost all $r > 0$

$$\int_{S_+^{n-1}} \sum_{j,k\le m} \mathcal{N}_{jk}(-\partial_t)^{m-k} u \partial_t^{m-j}(e^{(2m-n)t}\overline{v}) d\theta$$
$$= \int_{S_+^{n-1}} \sum_{j,k\le m} \sum_{m-j\le|\alpha|\le m} \sum_{m-k\le|\beta|\le m} (-1)^{|\alpha|} N_{\alpha\beta} Q_{\beta,m-k}(-\partial_t)^{m-k}u$$
$$\times P^*_{\alpha,m-j}\partial_t^{m-j}(e^{(2m-n)t}\overline{v})d\theta$$
$$= r^n \int_{S_+^{n-1}} \sum_{|\alpha|,|\beta|\le m} (L_{\alpha\beta} - \mathcal{L}_{\alpha\beta}(x))\partial_x^\beta u \partial_x^\alpha \overline{v} d\theta$$
$$-r^n \int_{S_+^{n-1}} \sum_{|\alpha+\beta|<2m} \mathcal{L}_{\alpha\beta}(x)\partial_x^\beta u \partial_x^\alpha \overline{v} d\theta\,, \tag{84}$$

where u and v are in $\mathring{W}^{m,p}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+})$.

Using the operators $\mathcal{A}_j(-\partial_t)$ and $\mathcal{N}_j(t,-\partial_t)$, and (77) we write problem (34) in the form

$$\sum_{j=0}^m (-\partial_t)^{m-j}\mathcal{A}_j(-\partial_t)u(t) = \sum_{j=0}^m (-\partial_t)^{m-j}\big(\mathcal{N}_j(t,-\partial_t)u + f_j(t)\big) \quad \text{on } \mathbb{R}, \tag{85}$$

where we consider u and f_j as functions on $\mathbb{R}$ taking values in function spaces on S^{n-1}_+. By (15) and (78)

$$||\mathcal{N}_{jk}(t)||_{\mathring{W}^{k,p}(S_+^{n-1})\to W^{-j,p}(S_+^{n-1})} \le\; c\,\omega(t)\;. \tag{86}$$

Clearly, $\mathcal{N}_j$ acts from $W^{m,p}_{\mathrm{loc}}(\Pi)$ to $L^p_{\mathrm{loc}}(\mathbb{R}; W^{-j,p}(S^{n-1}_+))$.

Let $\mathcal{U} = \mathrm{col}(\mathcal{U}_1, \dots, \mathcal{U}_{2m})$, where

$$\mathcal{U}_k = (-\partial_t)^{k-1}u, \quad k = 1, \dots, m, \tag{87}$$

$$\mathcal{U}_{m+1} = \mathcal{A}_0(-\partial_t)u - \mathcal{N}_0(t,-\partial_t)u - f_0 \tag{88}$$

and

$$\mathcal{U}_{m+j} = -\partial_t\mathcal{U}_{m+j-1} + \mathcal{A}_{j-1}(-\partial_t)u - \mathcal{N}_{j-1}(t,-\partial_t)u - f_{j-1} \tag{89}$$

for $j = 2, \dots, m$. With this notation (85) takes the form

$$-\partial_t\mathcal{U}_{2m} + \mathcal{A}_m(-\partial_t)u - \mathcal{N}_m(t,-\partial_t)u - f_m = 0. \tag{90}$$

Using (87) we write (88) as

$$(A_{00} - \mathcal{N}_{00}(t))(-\partial_t)^m u = \mathcal{U}_{m+1} - \sum_{k=0}^{m-1}(A_{0,m-k} - \mathcal{N}_{0,m-k}(t))\mathcal{U}_{k+1} + f_0\;. \tag{91}$$

Since the function

$$\mathcal{N}_{00}(t) = \sum_{|\alpha|=|\beta|=m} N_{\alpha\beta}(e^{-t}\theta)\theta^{\alpha+\beta}$$

is bounded by $c\,\omega(t)$, equation (91) is uniquely solvable with respect to $(-\partial_t)^m u$ and

$$(-\partial_t)^m u = \mathcal{S}(t)\mathcal{U} + (A_{00} - \mathcal{N}_{00}(t))^{-1} f_0 \ , \tag{92}$$

where

$$\mathcal{S}(t)\mathcal{U} = (A_{00} - \mathcal{N}_{00}(t))^{-1}\Big(\mathcal{U}_{m+1} - \sum_{k=0}^{m-1}(A_{0,m-k} - \mathcal{N}_{0,m-k}(t))\mathcal{U}_{k+1}\Big). \tag{93}$$

From (87) it follows that

$$-\partial_t \mathcal{U}_k = \mathcal{U}_{k+1} \quad \text{for } k = 1, \dots, m-1. \tag{94}$$

By (92) we have

$$-\partial_t \mathcal{U}_m = \mathcal{S}(t)\mathcal{U} + (A_{00} - \mathcal{N}_{00}(t))^{-1} f_0. \tag{95}$$

Using (92), we write (89) as

$$\begin{aligned} -\partial_t \mathcal{U}_{m+j} &= \mathcal{U}_{m+j+1} - \sum_{k=0}^{m-1}(A_{j,m-k} - \mathcal{N}_{j,m-k}(t))\mathcal{U}_{k+1} \\ &-(A_{j0} - \mathcal{N}_{j0}(t))(\mathcal{S}(t)\mathcal{U} + (A_{00} - \mathcal{N}_{00}(t))^{-1} f_0) + f_j \end{aligned} \tag{96}$$

for $j = 1, \dots, m-1$ and (90) takes the form

$$\begin{aligned} &-\partial_t \mathcal{U}_{2m} + \sum_{k=0}^{m-1}(A_{m,m-k} - \mathcal{N}_{m,m-k}(t))\mathcal{U}_{k+1} \\ &+(A_{m0} - \mathcal{N}_{m0}(t))(\mathcal{S}(t)\mathcal{U} + (A_{00} - \mathcal{N}_{00}(t))^{-1} f_0) - f_m = 0. \end{aligned} \tag{97}$$

The relations (94), (95)-(97) can be written as the first order evolution system

$$(-\mathcal{I}\partial_t - \mathfrak{A})\mathcal{U}(t) - \mathfrak{N}(t)\mathcal{U}(t) = \mathcal{F}(t) \quad \text{on } \mathbb{R}, \tag{98}$$

where

$$\mathcal{F}(t) = \mathrm{col}(0, \dots, 0, \mathcal{F}_m(t), \mathcal{F}_{m+1}(t), \dots, \mathcal{F}_{2m}(t)) \tag{99}$$

with

$$\mathcal{F}_m(t) = (A_{00} - \mathcal{N}_{00}(t))^{-1} f_0(t), \tag{100}$$

$$\mathcal{F}_{m+j}(t) = f_j(t) - (A_{j0} - \mathcal{N}_{j0}(t))(A_{00} - \mathcal{N}_{00}(t))^{-1} f_0(t), \quad j = 1, \dots, m\,. \tag{101}$$

The operator $\mathfrak{N}$ is given by

$$\mathfrak{N}(t)\mathcal{U} = \mathrm{col}(0, \dots, 0, \mathfrak{N}_m(t)\mathcal{U}, \mathfrak{N}_{m+1}(t)\mathcal{U}, \dots, \mathfrak{N}_{2m}(t)\mathcal{U}), \tag{102}$$

where

$$\mathfrak{N}_m(t)\mathcal{U} = (A_{00} - \mathcal{N}_{00}(t))^{-1} \Big(\sum_{k=0}^{m-1} \mathcal{N}_{0,m-k}(t))\mathcal{U}_{k+1} + \mathcal{N}_{00}(t)\mathcal{S}(t)\mathcal{U} \Big) \tag{103}$$

and

$$\begin{aligned} \mathfrak{N}_{m+j}(t)\mathcal{U} = \sum_{k=0}^{m-1} \mathcal{N}_{j,m-k}(t)\mathcal{U}_{k+1} + \mathcal{N}_{j0}(t)\mathcal{S}(t)\mathcal{U} \\ -A_{j0}(A_{00} - \mathcal{N}_{00}(t))^{-1} \Big(\sum_{k=0}^{m-1} \mathcal{N}_{0,m-k}(t)\mathcal{U}_{k+1} + \mathcal{N}_{00}(t)\mathcal{S}(t)\mathcal{U} \Big) \end{aligned} \tag{104}$$

for $j = 1, \dots, m$.

In (98), by $\mathcal{I}$, we denote the identity operator. We also use the operator matrix

$$\mathfrak{A} = \mathcal{J} - \mathfrak{L} \tag{105}$$

with $\mathcal{J} = \{(\mathcal{J})_{jk}\}_{j,k=1}^{2m}$ given by

$$\begin{array}{cc} & \begin{array}{ccccccc} & & & & (m+1) & & \end{array} \\ \begin{array}{c} \\ \\ \\ (m) \\ \\ \\ \\ \end{array} & \begin{pmatrix} 0 & I & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & I & \cdots & \vdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \vdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & A_{00}^{-1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & I \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \end{array}$$

and with $\mathfrak{L} = \{\mathfrak{L}_{jk}\}_{j,k=1}^{2m}$ equal to

$$\begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ A_{00}^{-1}A_{0,m} & \cdots & A_{00}^{-1}A_{0,1} & 0 & \cdots & 0 \\ A_{1,m} - A_{1,0}A_{00}^{-1}A_{0,m} & \cdots & A_{1,1} - A_{1,0}A_{00}^{-1}A_{0,1} & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \ddots & \vdots \\ A_{m,m} - A_{m,0}A_{00}^{-1}A_{0,m} & \cdots & A_{m,1} - A_{m,0}A_{00}^{-1}A_{0,1} & 0 & \cdots & 0 \end{pmatrix}$$

We put

$$\mathcal{D} = \mathring{W}^{m,p}(S_+^{n-1}) \times \cdots \times \mathring{W}^{1,p}(S_+^{n-1}) \times L^p(S_+^{n-1}) \times (W^{-m,p}(S_+^{n-1}))^{m-1}$$

and

$$\mathcal{R} = \mathring{W}^{m-1,p}(S_+^{n-1}) \times \cdots \times \mathring{W}^{1,p}(S_+^{n-1}) \times L^p(S_+^{n-1}) \times (W^{-m,p}(S_+^{n-1}))^{m} \,.$$

By (79) the operator $\mathfrak{A} : \mathcal{D} \to \mathcal{R}$ is continuous.

9 Linearization of the pencil $\mathcal{A}(\lambda)$

Here we find a correspondence between $\mathcal{A}(\lambda)$ and the linear pencil $\lambda\mathcal{I} - \mathfrak{A}$.

Lemma 2 *Let the row vector*

$$e(\lambda) = (e_1(\lambda), \dots , e_{2m}(\lambda))$$

be given by

$$e_{2m-j}(\lambda) = \lambda^j, \quad j = 0, \dots , m-1, \tag{106}$$

$$e_m(\lambda) = \sum_{j=0}^{m} \lambda^{m-j} A_{j0} \,, \tag{107}$$

$$e_{m-k}(\lambda) = \sum_{s=0}^{k} \sum_{j=0}^{m} \lambda^{k+m-s-j} A_{js}, \quad k = 1, \dots , m-1. \tag{108}$$

Then for all $\lambda \in \mathbb{C}$ the equality

$$e(\lambda)(\lambda\mathcal{I} - \mathfrak{A}) = (\mathcal{A}(\lambda), 0, \dots , 0) \tag{109}$$

is valid.

Proof follows by direct substitution of (106)–(108) in (109).

We introduce the operator matrix $\mathcal{E}(\lambda) = \{\mathcal{E}_{pq}(\lambda)\}_{p,q=1}^{2m}$ as

$$\begin{array}{cc}
 & \begin{array}{ccccccc} & & & (m) & & \phantom{e_{2m-1}(\lambda)} & \phantom{e_{2m}(\lambda)} \end{array} \\
\begin{array}{c} \\ \\ \\ \\ (m+1) \\ \\ \\ \end{array} &
\begin{pmatrix}
e_1(\lambda) & e_2(\lambda) & \cdots & e_m(\lambda) & \cdots & e_{2m-1}(\lambda) & e_{2m}(\lambda) \\
-I & 0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & -I & \cdots & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & -A_{00} & \cdots & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & -I & 0
\end{pmatrix}
\end{array} \tag{110}$$

One can check directly that $\mathcal{E}^{-1}(\lambda)$ is given by

$$\begin{array}{cc} & \begin{array}{ccccccc} & & & (m+1) & & & \end{array} \\ (m) & \begin{pmatrix} 0 & -I & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -A_{00}^{-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & -I \\ I & e_1(\lambda) & \cdots & e_m(\lambda) & \cdots & e_{2m-2}(\lambda) & e_{2m-1}(\lambda) \end{pmatrix} \end{array}$$

Lemma 3 *For all $\lambda \in \mathbb{C}$*

$$\begin{aligned} &\mathcal{E}(\lambda)(\lambda\mathcal{I} - \mathfrak{A}) \\ &= \mathrm{diag}(\mathcal{A}(\lambda), I, \ldots, I) \begin{pmatrix} J(\lambda) & 0 \\ -\mathcal{B}(\lambda) & J(\lambda) - \mathcal{M} \end{pmatrix} \end{aligned} \tag{111}$$

where the $m \times m$ matrices $J(\lambda)$, $\mathcal{M}$ and $\mathcal{B}(\lambda)$ are defined by

$$J(\lambda) = \begin{pmatrix} I & 0 & \ldots & 0 & 0 \\ -\lambda & I & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & I & 0 \\ 0 & 0 & \ldots & -\lambda & I \end{pmatrix}, \ \mathcal{M} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ A_{10}A_{00}^{-1} & 0 & \ldots & 0 & 0 \\ A_{20}A_{00}^{-1} & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ A_{m-1,0}A_{00}^{-1} & 0 & \ldots & 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned} \mathcal{B}(\lambda) = &\begin{pmatrix} A_{0m} & A_{0,m-1} & \ldots & A_{02} & A_{01} \\ A_{1,m} & A_{1,m-1} & \ldots & A_{12} & A_{11} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ A_{m-1,m} & A_{m-1,m-1} & \ldots & A_{m-1,2} & A_{m-1,1} \end{pmatrix} \\ - &\begin{pmatrix} 0 & 0 & \ldots & 0 & -\lambda A_{00} \\ A_{10}A_{00}^{-1}A_{0,m} & A_{10}A_{00}^{-1}A_{0,m-1} & \ldots & A_{10}A_{00}^{-1}A_{02} & A_{10}A_{00}^{-1}A_{01} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ A_{m-1,0}A_{00}^{-1}A_{0,m} & A_{m-1,0}A_{00}^{-1}A_{0,m-1} & \ldots & A_{m-1,0}A_{00}^{-1}A_{02} & A_{m-1,0}A_{00}^{-1}A_{01} \end{pmatrix} \end{aligned}$$

Proof. By Lemma 2 the left-hand side of (111) is a triangular matrix with the diagonal $\mathcal{A}(\lambda), I, \ldots, I$. One can directly verify that it is equal to the right-hand side in (111).

Clearly, the matrix $J(\lambda)$ has the inverse

$$J(\lambda)^{-1} = \begin{pmatrix} I & 0 & 0 & \dots & 0 & 0 \\ \lambda & I & 0 & \dots & \vdots & \vdots \\ \lambda^2 & \lambda & I & \dots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \dots & I & 0 \\ \lambda^{m-1} & \lambda^{m-2} & \lambda^{m-3} & \dots & \lambda & I \end{pmatrix}. \tag{112}$$

In the next lemma we evaluate the inverse of the last matrix in (111). We show, in particular, that this inverse is a polynomial operator matrix.

Lemma 4 *The folowing formula is valid:*

$$\begin{pmatrix} J(\lambda) & 0 \\ -\mathcal{B}(\lambda) & J(\lambda) - \mathcal{M} \end{pmatrix}^{-1} = \begin{pmatrix} J^{-1}(\lambda) & 0 \\ Q(\lambda) & J^{-1}(\lambda)(I + \mathcal{M}) \end{pmatrix}, \tag{113}$$

where the elements of the matrix $Q(\lambda) = \{Q_{jk}(\lambda)\}_{j,k=1}^m$ *are given by*

$$Q_{jk}(\lambda) = \sum_{l=0}^{j-1} \sum_{q=k-1}^{m} \lambda^{q+l+1-k} A_{j-l-1,m-q}. \tag{114}$$

Proof. Let us look for $(J(\lambda) - \mathcal{M})^{-1}$ in the form $J^{-1}(\lambda) + S(\lambda)$, where $S(\lambda)$ has non-zero elements only in the first column and $S_{11}(\lambda) = 0$. We have

$$(J(\lambda) - \mathcal{M})(J^{-1}(\lambda) + S(\lambda)) = I + J(\lambda)S(\lambda) - \mathcal{M}J^{-1}(\lambda).$$

Hence

$$S(\lambda) = J^{-1}(\lambda)\mathcal{M}.$$

Therefore we arrive at (113) with

$$Q(\lambda) = (J^{-1}(\lambda) + S(\lambda))\mathcal{B}(\lambda)J^{-1}(\lambda).$$

One can check that the last equality gives (114).

Lemma 5 (i) *The operator*

$$\lambda\mathcal{I} - \mathfrak{A} : \mathcal{D} \to \mathcal{R} \tag{115}$$

is Fredholm for all $\lambda \in \mathbb{C}$.

(ii) *The spectra of the operator* $\mathfrak{A}$ *and the pencil* $\mathcal{A}(\lambda)$ *coincide and consist of eigenvalues of the same multiplicity.*

Proof. Let

$$\mathfrak{B} = W^{-m,p}(S_+^{n-1}) \times \mathring{W}^{m-1,p}(S_+^{n-1}) \times \dots \times \mathring{W}^{1,p}(S_+^{n-1}) \times L^p(S_+^{n-1}) \times (W^{-m,p}(S_+^{n-1}))^{m-1}.$$

The operator

$$\mathcal{E}(\lambda) : \mathcal{R} \to \mathfrak{B}$$

is an isomorphism for all $\lambda \in \mathbb{C}$. Analogously, one verifies that the operator

$$\left\{ \begin{array}{cc} J(\lambda) & 0 \\ -\mathcal{B}(\lambda) & J(\lambda) - \mathcal{M} \end{array} \right\} : \mathcal{D} \to \mathcal{D}$$

is isomorphic for all $\lambda \in \mathbb{C}$. Hence and by (111) the polynomial operator functions

$$\lambda\mathcal{I} - \mathfrak{A} : \mathcal{D} \to \mathcal{R}$$

and

$$\mathrm{diag}(\mathcal{A}(\lambda), I, \dots, I) : \mathcal{D} \to \mathfrak{B}$$

are equivalent and therefore these functions have the same spectrum, and the geometric, partial and algebraic multiplicities of their eigenvalues coincide (see, for example, [KM], Appendix).

10 Spectral properties of $\mathfrak{A}$

We put

$$\phi(\theta) = E(\theta) \quad \text{and} \quad \psi(\theta) = -(m!)^{-1}\theta_n^m .$$

By (59) and (36)

$$\int_\Pi \mathcal{A}(-\partial_t)(\eta(t)e^{(n-m)t}\phi(\theta))\, e^{(m-n)t}\psi(\theta)d\theta dt = 1 \, , \tag{116}$$

where η is a smooth function equal to 1 for large positive t and 0 for large negative t. The equality (116) can be written as

$$\int_{S_+^{n-1}} \mathcal{A}'(m-n)\phi(\theta)\, \psi(\theta)d\theta = -1 \, . \tag{117}$$

We introduce the vector

$$\Phi = \mathrm{col}(\Phi_k)_{k=1}^{2m} = \begin{pmatrix} J^{-1}(m-n) & 0 \\ Q(m-n) & J^{-1}(m-n)(I+\mathcal{M}) \end{pmatrix} \mathrm{col}(\phi, 0, \dots, 0). \tag{118}$$

Owing to (111) and (113) we obtain

$$((m-n)\mathcal{I} - \mathfrak{A})\Phi = 0 \, . \tag{119}$$

Using (112) and the definitions of the matrices $\mathcal{M}$ and $\mathcal{B}$ we get

$$\Phi_k = (m-n)^{k-1}\phi, \quad k = 1, \dots, m, \tag{120}$$

$$\Phi_{m+k} = \sum_{p=0}^{k-1} \sum_{q=0}^{m} A_{k-p-1,m-q}(m-n)^{p+q}\phi \tag{121}$$

for $k = 1, \dots, m$.

We introduce the vector $\Psi = \mathrm{col}(\Psi_k)_{k=1}^{2m}$, by

$$\Psi = \mathcal{E}^*(m-n)\mathrm{col}(\psi, 0, \dots, 0) \tag{122}$$

where $\mathcal{E}^*(\lambda)$ is the adjoint of $\mathcal{E}(\overline{\lambda})$. Since ψ is the eigenfunction of the pencil $(\mathcal{A}(\lambda))^*$ corresponding to the eigenvalue $\lambda = m$, it follows from (111) that

$$((m-n)\mathcal{I} - \mathfrak{A}^*)\Psi = 0. \tag{123}$$

By (110)

$$\Psi_k = \sum_{p=0}^{m-k} \sum_{q=0}^{m} A_{qp}^*(m-n)^{2m-k-q-p}\psi$$

for $k = 1, \dots, m-1$,

$$\Psi_m = \sum_{q=0}^{m} A_{q0}^*(m-n)^{m-q}\psi$$

and $\Psi_{m+k} = (m-n)^{m-k}\psi$ for $k = 1, \dots, m$.

Clearly, $\Phi \in \mathcal{D}$, $\quad \Psi \in \mathcal{R}^*$, where

$$\mathcal{R}^* = W^{1-m,q}(S_+^{n-1}) \times \cdots \times W^{-1,q}(S_+^{n-1}) \times L^q(S_+^{n-1}) \times (\mathring{W}^{m,q}(S_+^{n-1}))^m .$$

Proposition 9 *The biorthogonality condition*

$$(\Phi, \Psi)_{L^2(S_+^{n-1})} = -1 \tag{124}$$

is valid.

Proof. By (111) and (113)

$$\big((\lambda\mathcal{I} - \mathfrak{A})\Phi_\lambda, \Psi_\lambda\big) = (\mathcal{A}(\lambda)\phi, \psi)_{(L^2(S_+^{n-1}))}\,, \tag{125}$$

where

$$\Phi_\lambda = \begin{pmatrix} J^{-1}(\lambda) & 0 \\ Q(\lambda) & J^{-1}(\lambda)(I + \mathcal{M}) \end{pmatrix} \mathrm{col}(\phi, 0, \dots, 0)$$

and $\Psi_\lambda = \mathcal{E}^*(\lambda)\mathrm{col}(\psi_k, 0, \dots, 0)$. Taking the first derivative of (125) with respect to λ, setting $\lambda = m - n$ and using (119) and (123) together with (117) we arrive at (124).

We introduce the spectral projector $\mathcal{P}$ corresponding to the eigenvalue $\lambda = m - n$:

$$\mathcal{P}\mathcal{F} = -(\mathcal{F}, \Psi)_{L^2(S_+^{n-1})}\Phi\,. \tag{126}$$

This operator maps $\mathcal{R}$ into $\mathcal{D}$.

11 Equivalence of equation (85) and system (98)

We introduce some vector function spaces to be used in the subsequent study of system (98).

Let $\mathbb{S}(a,b)$ be the space of vector functions $\mathcal{U}$ on the interval (a,b) with values in $\mathcal{D}$ such that

$$||\mathcal{U}||_{\mathbb{S}(a,b)} = \Big(\int_a^b (||\mathcal{U}(\tau)||^p_{\mathcal{D}} + ||\partial_\tau \mathcal{U}(\tau)||^p_{\mathcal{R}}) d\tau\Big)^{1/p} < \infty \,.$$

More explicitly:

$$||\mathcal{U}||_{\mathbb{S}(a,b)} = \Big(\int_a^b \Big(\sum_{j=1}^{m+1} ||\mathcal{U}_j(\tau)||^p_{\mathring{W}^{m+1-j,p}(S_+^{n-1})} + \sum_{j=m+2}^{2m} ||\mathcal{U}_j(\tau)||^p_{W^{-m,p}(S_+^{n-1})}$$

$$+ \sum_{j=1}^{m} ||\partial_\tau \mathcal{U}_j(\tau)||^p_{\mathring{W}^{m-j,p}(S_+^{n-1})} + \sum_{j=m+1}^{2m} ||\partial_\tau \mathcal{U}_j(\tau)||^p_{W^{-m,p}(S_+^{n-1})}\Big) d\tau\Big)^{1/p} .$$

By $\mathbb{S}_{\rm loc}(\mathbb{R})$ we denote the space of functions defined on $\mathbb{R}$ with finite seminorms $||\mathcal{U}||_{\mathbb{S}(t,t+1)}$, $t \in \mathbb{R}$. Let $\mathcal{P}$ be the projector given by (126). Clearly,

$$||\mathcal{P}\mathcal{U}||_{\mathbb{S}(a,b)} \le c||\mathcal{U}||_{\mathbb{S}(a,b)} \,. \tag{127}$$

By $L^p(a,b;B)$ and $L^p_{\rm loc}(\mathbb{R};B)$ we denote the L^p and $L^p_{\rm loc}$ spaces of vector functions on (a,b) and $\mathbb{R}$ which take values in a Banach space B.

Let $W_0^{m,p}((a,b)\times S_+^{n-1})$ be the subspace of the Sobolev space $W^{m,p}((a,b)\times S_+^{n-1})$ containing functions vanishing on $(a,b)\times \partial S_+^{n-1}$ together with their derivatives up to order $m-1$. The space of vector functions

$$\mathcal{U}(t) = \mathrm{col}(u(t), \dots, \partial_t^{m-1}u(t), u_{m+1}, \dots, u_{2m}(t)) \tag{128}$$

with $u \in W_0^{m,p}((a,b)\times S_+^{n-1})$,

$$u_{m+1} \in L^p(a,b; L^p(S_+^{n-1})), \quad \partial_t u_{m+1} \in L^p(a,b; W^{-m,p}(S_+^{n-1}))$$

and

$$u_{m+j},\ \partial_t u_{m+j} \in L^p(a,b; W^{-m,p}(S_+^{n-1})), \quad j = 2, \dots, m,$$

will be denoted by $\mathbf{S}(a,b)$. The norm in $\mathbf{S}(a,b)$ is defined by

$$||\mathcal{U}||_{\mathbf{S}(a,b)} = ||u||_{W^{m,p}((a,b)\times S_+^{n-1})} + ||u_{m+1}||_{L^p(a,b;L^p(S_+^{n-1}))}$$

$$+ \sum_{j=2}^{m} ||u_{m+j}||_{L^p(a,b;W^{-m,p}(S_+^{n-1}))} + \sum_{j=1}^{m} ||\partial_t u_{m+j}||_{L^p(a,b;W^{-m,p}(S_+^{n-1}))}.$$

The space $\mathbf{S}(a,b)$ is embedded into $\mathbb{S}(a,b)$ and for $\mathcal{U} \in \mathbf{S}$

$$c_1||\mathcal{U}||_{\mathbf{S}(a,b)} \le ||\mathcal{U}||_{\mathbb{S}(a,b)} \le c_2||\mathcal{U}||_{\mathbf{S}(a,b)}.$$

The space $\mathbf{S}_{\rm loc}(\mathbb{R})$ is defined as the set of vector functions $\mathcal{U}$ such that their restrictions to every finite interval (a,b) belong to $\mathbf{S}(a,b)$. The seminorms in this space are $||\mathcal{U}||_{\mathbf{S}(t,t+1)}$, $t\in\mathbb{R}$.

By $\mathbb{X}(a,b)$ we denote the space

$$\mathbb{X}(a,b)=\{\mathcal{V}\,:\,\mathcal{V}=(\mathcal{I}-\mathcal{P})\mathcal{U},\ \mathcal{U}\in\mathbf{S}(a,b)\} \tag{129}$$

endowed with the norm

$$||\mathcal{V}||_{\mathbb{X}(a,b)}=\inf||\mathcal{U}||_{\mathbf{S}(a,b)}\,,$$

where the infinum is taken over all $\mathcal{U}$ in (129).

We use the space $\mathbb{X}_{\rm loc}(\mathbb{R})=\{\mathcal{V}\ :\ \mathcal{V}=(\mathcal{I}-\mathcal{P})\mathcal{U},\ \ \mathcal{U}\in\mathbf{S}_{\rm loc}(\mathbb{R}))\}$ and finally, we introduce the space $\mathbb{Y}_{\rm loc}(\mathbb{R})$ which consists of the vector functions $\mathcal{F}={\rm col}(0,\dots,0,\mathcal{F}_m,\mathcal{F}_{m+1},\dots,\mathcal{F}_{2m})$ with finite seminorms

$$||\mathcal{F}||_{\mathbb{Y}(t,t+1)}=\Big(\sum_{j=0}^m\int_t^{t+1}||\mathcal{F}_{m+j}(\tau)||^p_{W^{-j,p}(S^{n-1}_+)}d\tau\Big)^{1/p},\quad t\in\mathbb{R}\,.$$

We return to system (98). By (86) the operator $\mathfrak{N}(t)\,:\,\mathbb{S}_{\rm loc}(\mathbb{R})\to\mathbb{Y}_{\rm loc}(\mathbb{R})$ is continuous and

$$||\mathfrak{N}||_{\mathbb{S}(t,t+1)\to\mathbb{Y}(t,t+1)}\le c\,\omega(t)\,. \tag{130}$$

Furthermore,

$$c_1||\mathcal{F}||_{\mathbb{Y}(t,t+1)}\le\sum_{j=0}^m||f_j||_{W^{-j,p}(\Pi_t)}\le c_2||\mathcal{F}||_{\mathbb{Y}(t,t+1)}\,, \tag{131}$$

where c_1 and c_2 are positive constant.

We prove that equation (85) and system (98) is equivalent in a certain sense.

Lemma 6 *Let the functions $f_j\in W^{-j,p}_{\rm loc}(\Pi)$ and the vector function $\mathcal{F}\in\mathbb{Y}_{\rm loc}(\mathbb{R})$ be connected by* (99)–(101).

(i) *If $u\in\mathring{W}^{m,p}_{\rm loc}(\Pi)$ is a solution of* (85) *then the vector function $\mathcal{U}\in\mathbb{S}_{\rm loc}(\mathbb{R})$ given by* (87)–(89) *solves* (98).

(ii) *If $\mathcal{U}\in\mathbb{S}_{\rm loc}(\mathbb{R})$ is a solution of* (98) *then $\mathcal{U}\in\mathbf{S}_{\rm loc}(\mathbb{R})$ and the function $u=\mathcal{U}_1$ solves* (85).

Proof. (i) This assertion follows directly from the above reduction of (85) to the first order system (98).

(ii) By (94) and (95) we obtain $\mathcal{U}_k=(-\partial_t)^{k-1}\mathcal{U}_1$ for $k=1,\dots,m$ and $\mathcal{S}(t)\mathcal{U}=(-\partial_t)^m\mathcal{U}_1$. Now (96) takes the form

$$-\partial_t\mathcal{U}_{m+j}=\mathcal{U}_{m+j+1}-\mathcal{A}_j(-\partial_t)\mathcal{U}_1+\mathcal{N}_j(t,-\partial_t)\mathcal{U}_1+\mathcal{F}_j$$

and (97) can be written as

$$-\partial_t\mathcal{U}_{2m}+\mathcal{A}_m(-\partial_t)\mathcal{U}_1-\mathcal{N}_m(t,-\partial_t)\mathcal{U}_1-\mathcal{F}_m=0.$$

The last two equations imply (85) for $u=\mathcal{U}_1$.

12 Spectral splitting of the first order system (98)

Let $\mathcal{P}$ be the spectral projector (126). Applying $\mathcal{P}$ and $\mathcal{I}-\mathcal{P}$ to system (98) we arrive at

$$(\mathcal{I}\partial_t + \mathfrak{A})\mathbf{u} + \mathcal{P}\mathfrak{N}(t)(\mathbf{u}+\mathbf{v}) = -\mathcal{P}\mathcal{F} \quad \text{on } \mathbb{R} \tag{132}$$

and

$$(\mathcal{I}\partial_t + \mathfrak{A})\mathbf{v} + (\mathcal{I}-\mathcal{P})\mathfrak{N}(t)\mathbf{v} = (\mathcal{P}-\mathcal{I})(\mathcal{F}+\mathfrak{N}(t)\mathbf{u}) \quad \text{on } \mathbb{R}, \tag{133}$$

where

$$\mathbf{u}(t) = \mathcal{P}\mathcal{U}(t), \quad \mathbf{v}(t) = (\mathcal{I}-\mathcal{P})\mathcal{U}(t). \tag{134}$$

Clearly, $\mathbf{u}$ can be represented as $\mathbf{u}(t) = \kappa(t)\Phi$, where Φ is given by (118). Furthermore, $\mathbf{u} \in \mathbb{S}_{\rm loc}(\mathbb{R})$ if and only if $\kappa \in W^{1,p}_{\rm loc}(\mathbb{R})$. Thus we have split system (98) into the scalar equation (132) and the infinite-dimensional system (133). Equation (132) can be written as

$$\frac{d\kappa}{dt}(t) + (m-n)\kappa(t) - (\mathfrak{N}(t)(\mathbf{u}+\mathbf{v})(t), \Psi) = (\mathcal{F}(t), \Psi)$$

where Ψ is defined in Sect. 10.

In the next lemma we establish the equivalence of equation (85) and the split system (132), (133).

Lemma 7 (i) *Let $f_j \in L^p_{\rm loc}(\mathbb{R}; W^{-j,p}(S^{n-1}_+))$, $j = 0,\dots,m$, and let $u \in \mathring{W}^{m,p}_{\rm loc}(\Pi)$ be a solution of* (85). *Then the vector function $\mathcal{U}$ given by* (87)-(89) *belongs to $\mathbf{S}_{\rm loc}(\mathbb{R})$ and the vector functions* (134) *satisfy* (132) *and* (133) *with $\mathcal{F}$ given by* (99)–(101).

(ii) *Let $\mathcal{F} \in \mathbb{Y}_{\rm loc}(\mathbb{R})$. Assume that*

$$\mathbf{u}(t) = (\mathbf{u}_1(t), \dots, \mathbf{u}_{2m}(t)) = \kappa(t)\Phi\,,$$

$\kappa \in W^{1,2}_{\rm loc}(\mathbb{R})$, and $\mathbf{v} = (\mathbf{v}_1,\dots,\mathbf{v}_{2m}) \in \mathbb{S}_{\rm loc}(\mathbb{R})$, such that $\mathcal{P}\mathbf{v}(t) = 0$ for all $t \in \mathbb{R}$, satisfy (132) *and* (133). *Then $\mathbf{u}+\mathbf{v} \in \mathbf{S}_{\rm loc}(\mathbb{R})$ and*

$$u = \mathbf{u}_1 + \mathbf{v}_1 \in \mathring{W}^{m,p}_{\rm loc}(\Pi)$$

solves (85) *with $f_0 = (A_{00} - \mathcal{N}_{00})\mathcal{F}_m$ and*

$$f_j = \mathcal{F}_{m+j} + (A_{j0} - \mathcal{N}_{j0})\mathcal{F}_m, \quad j = 1,\dots,m\,.$$

Moreover, $(-\partial_t)^j u = \mathbf{u}_{j+1} + \mathbf{v}_{j+1}$ for $j = 1,\dots,m-1$.

Proof. (i) It suffices to use Lemma 6(i) and to apply the projectors $\mathcal{P}$ and $\mathcal{I}-\mathcal{P}$ to system (98).

(ii) We put $\mathcal{U} = \mathbf{u}+\mathbf{v}$. Clearly, $\mathcal{U} \in \mathbb{S}_{\rm loc}(\mathbb{R})$ and equalities (98) and (134) hold. Now the result follows from Lemma 6(ii).

13 Solvability of the unperturbed infinite-dimensional part of the split system

We consider the case $\mathfrak{N} = 0$. In other words, we deal with the system

$$(-\mathcal{I}\partial_t - \mathfrak{A})\mathbf{v} = (\mathcal{I} - \mathcal{P})\mathcal{F} \quad \text{on } \mathbb{R}. \tag{135}$$

Lemma 8 (i) (Existence) *Let $\mathcal{F} \in \mathbb{Y}_{\rm loc}(\mathbb{R})$. Suppose that*

$$\int_0^\infty e^{(m-n-1)\tau}||\mathcal{F}||_{\mathbb{Y}(\tau,\tau+1)}d\tau + \int_{-\infty}^0 e^{m\tau}||\mathcal{F}||_{\mathbb{Y}(\tau,\tau+1)}d\tau < \infty\,. \tag{136}$$

Then equation (135) *has a solution $\mathbf{v} \in \mathbb{X}_{\rm loc}(\mathbb{R})$ satisfying*

$$\begin{aligned} ||\mathbf{v}||_{\mathbb{X}(t,t+1)} \quad &\le \quad c\Big(\int_t^\infty e^{(n-m+1)(t-\tau)}||\mathcal{F}||_{\mathbb{Y}(\tau,\tau+1)}d\tau \\ &\qquad + \int_{-\infty}^t e^{-m(t-\tau)}||\mathcal{F}||_{\mathbb{Y}(\tau,\tau+1)}d\tau\Big), \end{aligned} \tag{137}$$

where c is a constant independent of $\mathcal{F}$.

(ii) (Uniqueness) *Let $\mathbf{v} \in \mathbb{X}_{\rm loc}(\mathbb{R})$ satisfy* (135) *with $\mathcal{F} = 0$. Also let*

$$||\mathbf{v}||_{\mathbb{S}(t,t+1)} = \begin{cases} o(e^{(n-m+1)t}) & \text{if } t \to +\infty \\ o(e^{-mt}) & \text{if } t \to -\infty. \end{cases} \tag{138}$$

be valid. Then $\mathbf{v} = 0$.

Proof. (i) Let $f_0 = A_{00}\mathcal{F}_m$ and

$$f_j = \mathcal{F}_{m+j} + A_{j0}\mathcal{F}_m\,, \quad j = 1, \dots, m.$$

Clearly, $f_j \in L^p_{\rm loc}(\mathbb{R}; W^{-j,p}(S^{n-1}_+))$ and

$$\sum_{j=0}^m ||f_j||_{L^p_{\rm loc}(t,t+1;W^{-j,p}(S^{n-1}_+))} \le c||\mathcal{F}||_{L^p(t,t+1;\mathbb{Y})}\,.$$

Let ζ be a smooth function on $\mathbb{R}$, equal to 1 for $t > 1$ and 0 for $t < 0$. For a fixed $a \in \mathbb{R}$ we represent f_j as $f^{(-)}_{ja} + f^{(+)}_{ja}$, where

$$f^{(-)}_{ja}(t) = \zeta(t-a)f_j(t), \quad f^{(+)}_{ja}(t) = (1 - \zeta(t-a))f_j(t)\,.$$

Then the functions

$$f^{(\pm)}_a(t) = e^{2mt}\sum_{j=0}^m (-\partial_t)^{m-j} f^{(\pm)}_{ja}(t)$$

satisfy (51) and (48) respectively because of (136). By Proposition 2 there exist solutions $u_a^{(\pm)} \in W^{m,p}_{\rm loc}(\mathbb{R})$ subject to (52) and (49) with f replaced by $f_a^{(\pm)}$. We put $u_a = u_a^{(-)} + u_a^{(+)}$. Then u_a satisfies (34) and

$$||u_a||_{W^{m,p}(a,a+1)} \leq c\Big(\int_a^\infty e^{(m-n-1)(a-\tau)}||\mathcal{F}||_{\mathbb{Y}(\tau,\tau+1)}d\tau + \int_{-\infty}^a e^{-m(a-\tau)}||\mathcal{F}||_{\mathbb{Y}(\tau,\tau+1)}d\tau\Big), \tag{139}$$

We introduce the vector function $\mathcal{U}_a = \mathrm{col}(\mathcal{U}_1, \dots, \mathcal{U}_{2m})$ by (87)–(89), where $\mathcal{N} = 0$ and u is replaced by u_a, and put $\mathbf{v}_a = (\mathcal{I} - \mathcal{P})\mathcal{U}_a$. Clearly, $\mathbf{v}_a$ belongs to $\mathbb{X}_{\rm loc}(\mathbb{R})$ and satisfies (138). Let us show that $\mathbf{v}_a$ does not depend on a. In fact, let a and b be different real numbers. Then the function $u_a - u_b$ satisfies the homogeneous problem (34) and relations (61). Hence and by Corollary 1 we have $u_a - u_b = Ce^{(n-m)t}E(+theta)$. The last equality implies $\mathbf{v}_a = \mathbf{v}_b$ because of the definition of $\mathcal{P}$. Thus, we can use the notation $\mathbf{v}$ for the vector function $\mathbf{v}_a$. Since $||\mathbf{v}||_{\mathbb{S}(a,a+1)} \leq c||u_a||_{W^{m,p}(a,a+1)}$, estimate (137) follows from (139).

(ii) We put $u = \mathbf{v}_1$. Since $(\mathcal{I}\partial_t + \mathfrak{A})\mathbf{v} = 0$ it follows by (94) and (95) with $\mathcal{N} = 0$ that $\mathbf{v}_k = (-\partial_t)^{k-1}u$ for $k = 1, \dots, m$ and

$$(-\partial_t)^m u = A_{00}^{-1}\big(\mathbf{v}_{m+1} - \sum_{k=0}^{m-1} A_{0,m-k}(-\partial_t)^k u\big)\ .$$

Hence $\mathbf{v}_{m+1} = \mathcal{A}_0(-\partial_t)u$. (Note that $u \in W^{m,p}_{\rm loc}(\Pi)$ because $\mathbf{v} \in \mathbb{S}_{\rm loc}(\mathbb{R})$) Now relation (96) with $\mathcal{N} = 0$ takes the form

$$-\partial_t \mathbf{v}_{m+j} = \mathbf{v}_{m+j+1} - \mathcal{A}_j(-\partial_t)u\ , \tag{140}$$

where $j = 1, \dots, m-1$, and (97) becomes

$$-\partial_t \mathbf{v}_{2m} + \mathcal{A}_m(-\partial_t)u = 0. \tag{141}$$

Using (140) and (141) we obtain $\mathcal{A}(-\partial_t)u = 0$. Furthermore, by (138) the function u satisfies (61). By Corollary 1 we arrive at $u(t) = Ce^{(n-)mt}E(\theta)$ and using the definition (126) of $\mathcal{P}$ we get $\mathbf{v} = 0$. The proof is complete.

14 Solvability of the infinite-dimensional part of the perturbed split system

Here we study the system

$$(\mathcal{I}\partial_t + \mathfrak{A})\mathbf{v} + (\mathcal{I} - \mathcal{P})\mathfrak{N}(t)\mathbf{v} = (\mathcal{I} - \mathcal{P})F \quad \text{on } \mathbb{R}. \tag{142}$$

We introduce the operator $\mathfrak{L}$ which assigns the solution $\mathbf{v} \in \mathbb{S}_{\rm loc}(\mathbb{R})$ subject to (138) to the right-hand side in (135) satisfying the conditions of Lemma 8.

Estimate (137) can be written as

$$||\mathfrak{L}(\mathcal{I}-\mathcal{P})\mathcal{F}||_{\mathbb{X}(t,t+1)} \le c\int_{-\infty}^{\infty} g(t-\tau)||\mathcal{F}||_{\mathbb{Y}(\tau,\tau+1)}d\tau, \tag{143}$$

where

$$g(t) = \begin{cases} e^{-mt} & \text{for } t \ge 0 \\ e^{-(m-n-1)t} & \text{for } t < 0. \end{cases} \tag{144}$$

Lemma 9 *Let c, $c > 0$, and*

$$\delta := \sup_{\tau\in\mathbb{R}} \omega(\tau) \tag{145}$$

satisfy the inequality $(1+c)\delta \le (n+1)/8$. Then the series

$$g_\omega(t,\tau) = g(t-\tau) + \sum_{k=1}^{\infty} c^k \int_{\mathbb{R}^k} g(t-\tau_1)\omega(\tau_1)g(\tau_1-\tau_2)\omega(\tau_2)$$
$$\ldots\omega(\tau_k)g(\tau_k-\tau)d\tau_1\tau_2\ldots\tau_k \tag{146}$$

is convergent and admits the estimate

$$g_\omega(t,\tau) \le \begin{cases} c_1 e^{-m(t-\tau)+c_1\int_\tau^t \omega(s)ds} & \mathit{for}\ t \ge \tau \\ c_1 e^{(n-m+1)(t-\tau)+c_1\int_t^\tau \omega(s)ds} & \mathit{for}\ t < \tau, \end{cases} \tag{147}$$

where $c_1 = 2(1+c)$.

Proof. We denote the right-hand side in (147) by $g_*(t,\tau)$ and justify the inequality

$$g_*(t,\tau) \ge g(t-\tau) + c\int_{\mathbb{R}} g(t-s)\omega(s)g_*(s,\tau)ds\ . \tag{148}$$

Consider the case $t \ge \tau$. We have

$$\int_\tau^t g(t-s)\omega(s)g_*(s,\tau)ds = e^{-m(t-\tau)}\Big(e^{c_1\int_\tau^t \omega(s)ds} - 1\Big).$$

Furthermore,

$$\int_t^\infty g(t-s)\omega(s)g_*(s,\tau)ds \le \frac{c_1\delta}{n+1-c_1\delta}e^{-m(t-\tau)+c_1\int_\tau^t \omega(s)ds}$$

and

$$\int_{-\infty}^{\tau} g(t-s)\omega(s)g_*(s,\tau)ds \le \frac{c_1\delta}{n+1-c_1\delta}e^{-m(t-\tau)}\ .$$

From the last three relations we derive (148), taking into account that

$$c_1\delta < n+1 \quad \text{and} \quad c_1 \ge 1 + c + 2\frac{c_1 c\delta}{n+1-c_1\delta} \tag{149}$$

by the assumptions of Lemma. The case $\tau > t$ is considered analogously.

Now, iterating (148) we arrive at (147).

The following assertion which concerns the variable coefficient case is similar to Lemma 8.

Lemma 10 *There exist positive constants δ_0 and c_0 depending only on n, m, p and L such that for all $\delta \leq \delta_0$, where δ is given by* (145)*, the following assertions hold:*

(i) *Let F belong to $\mathbb{Y}_{\text{loc}}(\mathbb{R})$ and be subject to*

$$\int_0^\infty e^{(m-n-1)\tau + c_0 \int_0^\tau \omega(s)ds} ||F||_{\mathbb{Y}(\tau,\tau+1)} d\tau + \int_{-\infty}^0 e^{m\tau + c_0 \int_\tau^0 \omega(s)ds} ||F||_{\mathbb{Y}(\tau,\tau+1)} d\tau < \infty . \tag{150}$$

Then system (142) *has a solution $\mathbf{v} \in \mathbb{X}_{\text{loc}}(\mathbb{R})$ satisfying*

$$\begin{aligned} ||\mathbf{v}||_{\mathbb{S}(t,t+1)} \quad \leq \quad & c \int_t^\infty e^{(n-m+1)(t-\tau) + c_0 \int_t^\tau \omega(s)ds} ||F||_{\mathbb{Y}(\tau,\tau+1)} d\tau \\ & \int_{-\infty}^t e^{-m(t-\tau) + c_0 \int_\tau^t \omega(s)ds} ||F||_{\mathbb{Y}(\tau,\tau+1)} d\tau \; . \end{aligned} \tag{151}$$

(ii) *The solution $\mathbf{v} \in \mathbb{X}_{\text{loc}}(\mathbb{R})$ to* (142) *subject to*

$$||\mathbf{v}||_{\mathbb{S}(t,t+1)} = \begin{cases} o\Big(e^{(n-m+1)t - c_0 \int_0^t \omega(\tau)d\tau}\Big) & \text{as } t \to +\infty \\ o\Big(e^{-mt - c_0 \int_t^0 \omega(\tau)d\tau}\Big) & \text{as } t \to -\infty \end{cases} \tag{152}$$

is unique. (*We note that* (150) *together with* (151) *imply* (152).)

Proof. Let c be the constant in (143). Then one can take

$$\delta_0 = \frac{n+1}{8(1+c)} \quad \text{and} \quad c_0 = 4(1+c).$$

(i) Formally, the solution $\mathcal{U}$ of (142) can be written as the series

$$\sum_{k=0}^\infty (\mathfrak{L}(\mathcal{I} - \mathcal{P})\mathfrak{N})^k \mathfrak{L}(\mathcal{I} - \mathcal{P})F, \tag{153}$$

where $\mathfrak{L}$ is the operator defined at the end of Sect.13. We introduce the sequence

$$F^{(k)} = \mathfrak{N}\, \mathfrak{L}\big((\mathcal{I} - \mathcal{P})\mathfrak{N}\, \mathfrak{L}\big)^k (\mathcal{I} - \mathcal{P})F\,, \quad k = 0, 1, \ldots \tag{154}$$

Clearly, $F^{(k)} = \text{col}(0, \ldots, 0, F_m^{(k)}, \ldots, F_{2m}^{(k)})$ and by (143) and (130) $F^{(k)} \in \mathbb{Y}_{\text{loc}}(\mathbb{R})$. Now, (153) can be written as

$$\mathfrak{L}(\mathcal{I} - \mathcal{P})F + \mathfrak{L}(\mathcal{I} - \mathcal{P}) \sum_{k=0}^\infty F^{(k)}. \tag{155}$$

We show that the series

$$\sum_{k=0}^{\infty} F^{(k)} \tag{156}$$

converges in $\mathbb{Y}_{\rm loc}(\mathbb{R})$. We have $F^{(0)} = \mathfrak{N}\,\mathfrak{L}\,(\mathcal{I}-\mathcal{P})F$ and

$$F^{(k)} = \mathfrak{N}\ \mathfrak{L}(\mathcal{I}-\mathcal{P})F^{(k-1)},\ \ k=1,\dots$$

By (143) and (130)

$$\| \ F^{(k)} \ \|_{\mathbb{Y}(t,t+1)} \leq c\omega(t) \int_{\mathbb{R}} g(t-\tau) \ \| \ F^{(k-1)} \ \|_{\mathbb{Y}(\tau,\tau+1)} \ d\tau.$$

Therefore,

$$\begin{aligned}
&\| \ F^{(k)} \ \|_{\mathbb{Y}(t,t+1)} \\
&\leq c^{k+1}\omega(t) \int_{\mathbb{R}^{k+1}} g(t-\tau_1)\omega(\tau_1)g(\tau_1-\tau_2)\omega(\tau_2)\dots\omega(\tau_k)g(\tau_k-\tau) \\
&\times \| \ F \ \|_{\mathbb{Y}(\tau,\tau+1)} \ d\tau_1\dots d\tau_k d\tau,\ \ k=0,1,\dots
\end{aligned} \tag{157}$$

This implies

$$\sum_{k=0}^{\infty} ||F^{(k)}||_{\mathbb{Y}(t,t+1)} \leq c\,\omega(t) \int_{\mathbb{R}} g_\omega(t,\tau) \ \| \ F \ \|_{\mathbb{Y}(\tau,\tau+1)} \ d\tau \ , \tag{158}$$

where g_ω is given by (146). Hence, series (156) converges in $\mathbb{Y}_{\rm loc}(\mathbb{R})$ to a function F_*. Since

$$g_\omega(t,\tau) = g(t-\tau) + c\int_{\mathbb{R}} g(t-s)\omega(s)g_\omega(s,\tau)ds \ , \tag{159}$$

it follows from (158), (147) and (150) that

$$\int_{\mathbb{R}} g(-\tau) \ \| \ F_* \ \|_{\mathbb{Y}(\tau,\tau+1)} \ d\tau < \infty.$$

Therefore, $(\mathcal{I}-\mathcal{P})F_*$ belongs to the domain of $\mathfrak{L}$. Thus, series (155) is well defined, and we denote it by $\mathbf{v}$. Estimates (143), (158) together with (159) imply

$$\| \ \mathbf{v} \ \|_{\mathbb{X}(t,t+1)} \leq c \ \int_{\mathbb{R}} g_\omega(t,\tau) \ \| \ F \ \|_{\mathbb{Y}(\tau,\tau+1)} \ d\tau \ .$$

Owing to (147) we arrive at (151). Clearly, $\mathbf{v}$ is a solution of (142).

(ii) Let $\mathbf{v} \in \mathbb{X}_{\rm loc}(\mathbb{R})$ solve the equation

$$(\mathcal{I}\partial_t + \mathfrak{A})\mathbf{v} = (\mathcal{P}-\mathcal{I})\mathfrak{N}(t)\mathbf{v} \quad \text{on } \mathbb{R}. \tag{160}$$

Using (152) one checks directly that the right-hand side in (160) satisfies the conditions of Lemma 8(i). Therefore, by the same lemma and by (130) we arrive at

$$||\mathbf{v}||_{\mathbb{S}(t,t+1)} \le c \int_{\mathbb{R}} g(t-\tau)\omega(\tau)||\mathbf{v}||_{\mathbb{S}(\tau,\tau+1)} d\tau \ , \tag{161}$$

where g is given by (144).

By (152) there exists the least constants A_+ and A_- in

$$||\mathbf{v}||_{\mathbb{S}(t,t+1)} \le \begin{cases} A_+ e^{(n-m+1)t - c_0 \int_0^t \omega(\tau)d\tau} & \text{as } t \ge 0 \\ A_- e^{-mt - c_0 \int_t^0 \omega(\tau)d\tau} & \text{as } t < 0 \end{cases} \tag{162}$$

Without loss of generality we assume that $A_+ \le A_-$. Suppose that $A_+ > 0$ and let $t \ge 0$. Using (162) we estimate the right-hand side in (161) by

$$\begin{aligned} & c\, A_+ \Big(e^{(n-m+1)t} \int_t^\infty \omega(\tau) e^{-c_0 \int_0^\tau \omega(s)ds} d\tau + e^{-mt} \int_{-\infty}^0 \omega(\tau) e^{-c_0 \int_\tau^0 \omega(s)ds} d\tau \\ & + e^{-mt} \int_0^t \omega(\tau) e^{(n+1)\tau - c_0 \int_0^\tau \omega(s)ds} d\tau \Big) \\ & \le c\, A_+ e^{(n-m+1)t - c_0 \int_0^t \omega(s)ds} \Big(\frac{1}{c_0} + \frac{1}{c_0} e^{-(n+1)t + c_0 \int_0^t \omega(s)ds} + \frac{\delta}{n+1-c_0\delta} \Big) \end{aligned}$$

provided $c_0\delta < n+1$. By the above assumptions

$$c(2/c_0 + \delta/(n+1-c_0\delta)) < 1 \ .$$

Therefore the constant A_+ in (162) can be diminished. Thus, $A_+ = 0$ and therefore, $\mathbf{v} = 0$.

15 Scalar integro–differential equation

Lemma 10 enables one to introduce the operator $\mathfrak{M}$ whose domain consists of the vector functions $(\mathcal{I} - \mathcal{P})F$ with $F \in \mathbb{Y}_{\rm loc}(\mathbb{R})$ subject to (150). The vector function $\mathfrak{M}(\mathcal{I} - \mathcal{P})F$ is equal to the solution $\mathbf{v}$ from the same lemma. Using this operator one can write (132) as

$$\begin{aligned} & (\mathcal{I}\partial_t + \mathfrak{A})\mathbf{u} + \mathcal{P}\mathfrak{N}(t)\mathbf{u} + \mathcal{P}\mathfrak{N}(t)\mathfrak{M}\mathfrak{N}(t)\mathbf{u} \\ & = -\mathcal{P}\big(\mathcal{F} + \mathfrak{N}(t)\mathfrak{M}(\mathcal{P} - \mathcal{I})(\mathcal{F}\big) \quad \text{on } \mathbb{R}. \end{aligned} \tag{163}$$

Representing $\mathbf{u}$ as

$$\mathbf{u}(t) = \exp\Big((n-m)t + \int_0^t \lambda(\tau)d\tau\Big)\ h(t)\Phi \ ,$$

where

$$\lambda(t) = (\mathfrak{N}(t)\Phi, \Psi) \ ,$$

we derive from (163) the following integro-differential equation for h:

$$\dot{h}(t) - \mathcal{K}(h)(t) = \mathfrak{f}(t) \,, \tag{164}$$

where

$$\mathcal{K}(h)(t) = (\mathfrak{N}(t)\mathfrak{M}_{\tau\to t}(e^{(m-n)(t-\tau)-\int_\tau^t \lambda(s)ds}\mathfrak{N}(\tau)h(\tau)\Phi)(t), \Psi)$$

and

$$\mathfrak{f}(t) = e^{(m-n)t-\int_0^t \lambda(\tau)d\tau}(\mathcal{F}(t) + \mathfrak{N}(t)\mathfrak{M}(\mathcal{P}-\mathcal{I})\mathcal{F}(t), \Psi)\,. \tag{165}$$

Using (151) together with (130) we obtain the estimates

$$||\mathcal{K}(h)||_{W^{1,p}(t,t+1)} \le c\,\omega(t)\int_{\mathbb{R}} \sigma(t,\tau)\omega(\tau)||h||_{L^\infty(\tau,\tau+1)}d\tau \tag{166}$$

and

$$||\mathfrak{f}||_{L^p(t,t+1)} \le c\left(||\mathcal{F}||_{\mathbb{Y}(t,t+1)} + \omega(t)\int_{\mathbb{R}} \sigma(t,\tau)\omega(\tau)||\mathcal{F}||_{\mathbb{Y}(\tau,\tau+1)}d\tau\right)\,, \tag{167}$$

where

$$\sigma(t,\tau) = \left\{ \begin{array}{ll} e^{-n(t-\tau)+c_2\int_\tau^t \omega(s)ds} & \text{for } t \ge \tau \\ e^{t-\tau+c_2\int_t^\tau \omega(s)ds} & \text{for } t < \tau \end{array} \right. \tag{168}$$

Here c_2 is a positive constant, which depends on n, m, p and the coefficients of the operator L.

Lemma 11 *The function $\lambda(t) = (\mathfrak{N}(t)\Phi, \Psi)$ admits the representation*

$$\lambda(t) = \sum_{j=0}^m \sum_{k=0}^m \left(\mathcal{N}_{m-j,m-k}(t)(m-n)^k\phi, (m-n)^j\psi\right) + O(\omega(t)^2), \tag{169}$$

where ϕ and ψ are the same functions as in Sect. 10*.*

Proof. By (93) and by (120), (121)

$$\begin{aligned} \mathcal{S}(t)\Phi &= (A_{00} - \mathcal{N}_{00}(t))^{-1}\Big(A_{00}(m-n)^m\phi + \sum_{k=0}^{m-1}\mathcal{N}_{0,m-k}(t)(m-n)^k\phi\Big) \\ &= (m-n)^m\phi + (A_{00} - \mathcal{N}_{00}(t))^{-1}\sum_{k=0}^{m}\mathcal{N}_{0,m-k}(t)(m-n)^k\phi \,. \end{aligned}$$

Now, using (102)-(104) we obtain

$$\mathfrak{N}_m(t)\Phi = A_{00}^{-1}\sum_{k=0}^m \mathcal{N}_{0,m-k}(t)(m-n)^k\phi + O(\omega^2(t))$$

and

$$
\begin{aligned}
\mathfrak{N}_{m+j}(t)\Phi &= \sum_{k=0}^{m} \mathcal{N}_{j,m-k}(t)(m-n)^k \phi \\
&- A_{j0}A_{00}^{-1} \sum_{k=0}^{m} \mathcal{N}_{0,m-k}(t)(m-n)^k \phi + O(\omega^2(t)) \, .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\big(\mathfrak{N}(t)\Phi, \Psi\big) = \sum_{j=0}^{m}\sum_{k=0}^{m} \big(A_{00}^{-1}\mathcal{N}_{0,m-k}(t)(m-n)^k\phi, A_{j0}^{*}(m-n)^{m-j}\psi\big) \\
&+ \sum_{j=1}^{m}\sum_{k=0}^{m} \big((\mathcal{N}_{j,m-k}(t) - A_{j0}A_{00}^{-1}\mathcal{N}_{0,m-k})(m-n)^k\phi, (m-n)^{m-j}\psi\big) + O(\omega^2(t)) \\
&= \sum_{j=0}^{m}\sum_{k=0}^{m} \big(\mathcal{N}_{j,m-k}(t)(m-n)^k\phi, (m-n)^{m-j}\psi\big) + O(\omega^2(t)) \, .
\end{aligned}
$$

Clearly, the right-hand sides in the last equality and (169) coincide.

16 Homogeneous equation (164)

We start with a uniqueness result for the equation

$$
\dot z(t) + (\mathcal{K}z)(t) = 0 \quad t \in \mathbb{R}. \tag{170}
$$

Lemma 12 *There exist positive constants δ_0 and c_3 depending only on n, m, p and L such that: if $\delta \le \delta_0$ and $z \in W^{1,p}_{\mathrm{loc}}(\mathbb{R})$ is a solution of* (170) *subject to*

$$
z(t) = \left\{ \begin{array}{ll} o\Big(e^{t - c_3 \int_0^t \omega(s)ds}\Big) & \text{as } t \to +\infty \\ o\Big(e^{-nt - c_3 \int_t^0 \omega(s)ds}\Big) & \text{as } t \to -\infty \end{array} \right. \tag{171}
$$

and $z(t_0) = 0$ for some t_0 then $z(t) = 0$ for all $t \in \mathbb{R}$.

Proof. Without loss of generality we set $t_0 = 0$. Integrating (170) and using (166) we obtain

$$
\nu(t) \le c \Big| \int_0^t \omega(\tau) \int_{\mathbb{R}} \sigma(\tau, s)\omega(s)\nu(s)dsd\tau \Big| \, , \tag{172}
$$

where $\nu(t) = ||z||_{L^\infty(t,t+1)}$. We set

$$
A = \sup_{t\ge 0} e^{-t + c_3 \int_0^t \omega(s)ds} \nu(t) + \sup_{t<0} e^{nt + c_3 \int_t^0 \omega(s)ds} \nu(t) \, .
$$

Let c_2 be the same constant as in (168). We may suppose that $c_3 > c_2$.

For $t \geq 0$ we estimate the right-hand side of (172) by

$$cA \int_0^t \omega(\tau) \Big\{ \int_\tau^\infty \omega(s) e^{\tau + c_2 \int_\tau^s \omega(x)dx - c_3 \int_0^s \omega(x)dx} ds$$
$$+ \int_0^\tau \omega(s) e^{n(s-\tau)+s+c_2 \int_s^\tau \omega(x)dx - c_3 \int_0^s \omega(x)dx} ds$$
$$+ \int_{-\infty}^0 \omega(s) e^{-n\tau + c_2 \int_s^\tau \omega(x)dx - c_3 \int_s^0 \omega(x)dx} ds \Big\} d\tau$$

Direct calculations give that the right hand-side is majorized by

$$cA \int_0^t \omega(\tau) \Big\{ \frac{1}{c_3 - c_2} e^{\tau - c_3 \int_0^\tau \omega(x)dx}$$
$$+ \frac{1}{c_3 + c_2} e^{-n\tau + c_2 \int_0^\tau \omega(x)dx} + \frac{\delta}{n + 1 - (c_2 + c_3)\delta} e^{\tau - c_3 \int_0^\tau \omega(x)dx} \Big\} d\tau\ .$$

Supposing that $(c_2 + c_3)\delta < n + 1$ we conclude that the right-hand side is less than

$$cA \Big\{ \frac{1}{c_3 - c_2} + \frac{1}{c_2 + c_3} + \frac{\delta}{n + 1 - (c_2 + c_3)\delta} \Big\} \int_0^t \omega(\tau) e^{\tau - c_3 \int_0^\tau \omega(x)dx} d\tau$$
$$\leq cA \Big\{ \frac{1}{c_3 - c_2} + \frac{1}{c_2 + c_3} + \frac{\delta}{n + 1 - (c_2 + c_3)\delta} \Big\} \frac{\delta}{n - c_3\delta} e^{t - c_3 \int_0^t \omega(x)dx}\ .$$

Therefore, assuming that δ is sufficiently small, one can choose $c_2 \geq 4(1 + c)$ and c_3 satisfying the above restrictions and such that

$$\sigma_+ = c \Big\{ \frac{1}{c_3 - c_2} + \frac{1}{c_2 + c_3} + \frac{\delta}{n + 1 - (c_2 + c_3)\delta} \Big\} \frac{\delta}{n - c_3\delta} < 1.$$

This implies

$$\sup_{t \geq 0} e^{-t + c_3 \int_0^t \omega(s)ds} \nu(t) \leq \sigma_+ A\ .$$

Analogously, one verifies that

$$\sup_{t < 0} e^{nt + c_3 \int_t^0 \omega(s)ds} \nu(t) \leq \sigma_- A$$

with some $\sigma_- < 1$. Therefore, $A = 0$.

Lemma 13 *Equation* (170) *has a solution* $z \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ *given by*

$$z(t) = \exp\Big(\int_{t_0}^t \Lambda(\tau) d\tau \Big)\ , \tag{173}$$

where Λ *is a locally summable function satisfying*

$$|\Lambda(t)| \leq c\, \chi(t)\,, \tag{174}$$

where

$$\chi(\tau) = \omega(t)\Big(\int_{-\infty}^{t} e^{n(\tau-t)+\mathcal{C}\int_\tau^t \omega(s)ds}\omega(\tau)d\tau + \int_t^{\infty} e^{t-\tau+\mathcal{C}\int_t^\tau \omega(s)ds}\omega(\tau)d\tau\Big).$$

Proof. Let ε be a sufficiently small number depending on n, m and L and let $\mathcal{B}_\varepsilon = \{\Lambda \in L^\infty(\mathbb{R}) : |\Lambda(t)| \le \varepsilon\omega(t)\}$. Inserting (173) into (170) we arrive at the equation for Λ:

$$\Lambda(t) + G(\Lambda)(t) = 0\ , \quad t \in \mathbb{R}\ , \tag{175}$$

where

$$G(\Lambda)(t) = \mathcal{K}_{\tau\to t}\Big(\exp\Big(\int_t^\tau \Lambda(s)ds\Big)\Big)\ .$$

Using (166) with $p = 2$ and assuming that δ is sufficiently small we obtain for $\Lambda \in \mathcal{B}_\varepsilon$:

$$|G(\Lambda)(t)| \le c\omega(t)\int_{\mathbb{R}} \sigma(t,\tau)\omega(\tau)e^{\varepsilon|\int_t^\tau \omega(s)ds|}d\tau \le c_1\delta\omega(t), \tag{176}$$

where c_1 is a constant depending only on n, m and L. We suppose that $c_1\delta \le \varepsilon$. This guarantees, in particular, that G maps $\mathcal{B}_\varepsilon$ into itself.

Now let Λ_1 and Λ_2 be functions from $\mathcal{B}_\varepsilon$. By (166) we have

$$\begin{aligned}&|G(\Lambda_2)(t) - G(\Lambda_1)(t)|\\ &\le c\omega(t)\int_{\mathbb{R}} \sigma(t,\tau)\omega(\tau)\sup_{\tau\in(t,t+1)}\Big|\exp\Big(\int_t^\tau \Lambda_2(s)ds\Big) - \exp\Big(\int_t^\tau \Lambda_1(s)ds\Big)\Big|d\tau\ .\end{aligned}$$

Since

$$\begin{aligned}&\Big|\exp\Big(\int_t^\tau \Lambda_2(s)ds\Big) - \exp\Big(\int_t^\tau \Lambda_1(s)ds\Big)\Big|\\ &\le e^{\varepsilon\delta|t-\tau|}\Big|\int_t^\tau \omega(s)ds\Big|\sup_{s\in\mathbb{R}}\frac{|\Lambda_2(s) - \Lambda_1(s)|}{\omega(s)}\ ,\end{aligned}$$

we obtain

$$|G(\Lambda_2)(t) - G(\Lambda_1)(t)| \le c_2\delta\omega(t)\sup_{s\in\mathbb{R}}\frac{|\Lambda_2(s) - \Lambda_1(s)|}{\omega(s)}$$

with some constant c_2 depending on n, m and L. Assuming that $c_2\delta < 1$ we get the existence of $\Lambda \in \mathcal{B}_\varepsilon$ satisfying (175) by the Banach fixed point theorem.

Estimate (174) results from (176) and (175).

The next statement directly follows from Lemma 13.

Corollary 2 *Suppose that*

$$\Big|\int_{\mathbb{R}} \chi(\tau)d\tau\Big| < \infty\ .$$

Then the solution z *from* Lemma 13 *admits the asymptotic representation*

$$z(t) = 1 + O\Big(\int_t^\infty \chi(\tau) d\tau \Big) \quad as\ t \to +\infty\ .$$

We denote by $z(t,\tau)$ the solution of (170) subject to (171) and such that $z(\tau,\tau) = 1$. By Lemma 12 this solution is unique and by Lemma 13 such a solution exists and satisfies

$$e^{-c|\int_\tau^t \chi(s)ds|} \le |z(t,\tau)| \le e^{c|\int_\tau^t \chi(s)ds|} \tag{177}$$

with c depending only on n, m and L.

17 Representation of solutions of the homogeneous problem (34)

Lemma 14 *There exists a nontrivial solution* $\mathfrak{z} \in \mathring{W}^{m,p}_{\text{loc}}(\Pi)$ *to the homogeneous problem* (34) *subject to*

$$||\mathfrak{z}||_{W^{m,p}(\Pi_t)} = \begin{cases} o(e^{(n-m+1)t - \mathcal{C}\int_0^t \omega(s)ds}) & \text{if } t \to +\infty \\ o(e^{-mt - \mathcal{C}\int_t^0 \omega(s)ds}) & \text{if } t \to -\infty. \end{cases} \tag{178}$$

This solution is unique up to a constant factor and

$$(-\partial_t)^k \mathfrak{z}(t,\theta) = C(m-n)^k \exp\Big((n-m)t + \int_0^t \lambda(\tau)d\tau\Big) z(t)(\phi(\theta) + v_k(t,\theta))\,, \tag{179}$$

where $C = \text{const}$, $k = 0, \dots, m$, *and* z *is the function from* Lemma 13. *For* $k < m$ *the remainder* v_k *satisfies*

$$\begin{aligned} &||v_k||_{W^{m-k,p}(\Pi_t)} + ||\partial_t v_k||_{W^{m-k-1,p}(\Pi_t)} \\ &\le c\Big(\int_{-\infty}^t e^{n(\tau - t + \mathcal{C}\int_\tau^t \omega(s)ds}\omega(\tau)d\tau + \int_t^\infty e^{t-\tau+\mathcal{C}\int_t^\tau \omega(s)ds}\omega(\tau)d\tau \Big). \end{aligned} \tag{180}$$

If $k = m$, *then the second term on the right in the last inequality should be omitted.*

Proof. We introduce the vector function $\mathcal{U} = (\mathcal{U}_1, \dots, \mathcal{U}_{2m})$ with $\mathcal{U}_k$ given by (87)–(89) where $f_0 = f_1 = \ldots = f_{m-1} = 0$. By Lemma 6 the function $u \in \mathring{W}^{m,p}_{\text{loc}}(\Pi)$ solves the homogeneous equation (85) (or equivalently (34)) if and only if $\mathcal{U} \in \mathbb{S}_{\text{loc}}(\mathbb{R})$ is a solution of (98) with $\mathcal{F} = 0$.

(i) *Existence.* Let

$$\mathbf{u}(t) = \exp\big((n-m)t + \int_0^t \lambda(\tau)d\tau\big) z(t)\Phi\,, \tag{181}$$

where $z(t)$ is the solution of (170) from Lemma 13. We are looking for a solution $\mathcal{U}$ of the homogeneous system (98) in the form $\mathcal{U}(t) = \mathbf{u}(t)+\mathbf{v}(t)$, where $\mathcal{P}\mathbf{v}(t) = 0$. Then $\mathbf{v}$ satisfies (133) with $\mathcal{F} = 0$. By (130) and Lemma 13 the $\mathbb{Y}(t, t+1)$-seminorm of $\mathcal{N}\mathbf{u}$ is majorized by

$$c\omega(t)\exp\big((n-m)t + \Re\int_o^t \lambda(\tau)d\tau + c\int_0^t \chi(\tau)d\tau\big)\,.$$

By Lemma 10 system (133) has a solution $\mathbf{v}$ satisfying

$$||\mathbf{v}||_{\mathbb{S}(t,t+1)} \le ce^{(n-m)t+\Re\int_0^t \lambda(\tau)d\tau} \times$$
$$\Big(\int_{-\infty}^t e^{n(\tau-t)+\mathcal{C}\int_\tau^t \omega(s)ds}\omega(\tau)d\tau + \int_t^\infty e^{t-\tau+\mathcal{C}\int_t^\tau \omega(s)ds}\omega(\tau)d\tau\Big).$$

Hence, $\mathfrak{Z} = \mathcal{U}_1$ is the required solution of equation (34) The solution of the homogeneous system (98) constructed above will be denoted by $\mathcal{U}_* = \mathbf{u}_* + \mathbf{v}_*$ where $\mathcal{P}\mathbf{v}_* = 0$.

(ii) *Uniqueness.* Suppose that the $W^{m,p}(\Pi_t)$-seminorm of a solution $u = \mathfrak{Z}$ of the homogeneous equation (34) is subject to (178). Consider the vector function $\mathcal{U} - c\mathcal{U}_*$, where c is a arbitrary constant. We represent $\mathbf{u}$ in the form (181) with a certain z. Similarly, let $\mathbf{u}_*$ be given by (181) with z_* instead of z. Clearly, $z - cz_*$ satisfies (170) and (171). Choosing c to satisfy $z(0) - cz_*(0) = 0$ and using Lemma 12 one obtains $z(t) - cz_*(t) = 0$ for all t. Now, applying Lemma 10(ii) to the vector function $\mathbf{v} - c\mathbf{v}_*$, which solves the homogeneous system (142), we conclude that $\mathbf{v} - c\mathbf{v}_* = 0$. The proof is complete.

Corollary 3 *Let $f \in W^{-m,p}_{\text{loc}}(\Pi)$ be subject to*

$$\mathcal{J}_f := \int_{\mathbb{R}} e^{-(n+m)\tau + c|\int_0^\tau \omega(s)ds|}||f||_{W^{-m,p}(\Pi_\tau)}d\tau < \infty\,, \tag{182}$$

Also let u_1 and u_2 be solutions of problem (34) *from* Proposition 7 (i) *and* (ii) *respectively. Then*

$$u_2 - u_1 = CZ(t), \tag{183}$$

where C is a constant satisfying

$$|C| \le c\mathcal{J}_f\,. \tag{184}$$

Proof. It follows from Proposition 7 that $u_2 - u_1$ is the solution of the homogeneous equation (34) satisfying (178). Now, (183) holds by Lemma 14. In order to prove (184) we write

$$|C|\,||\mathfrak{Z}||_{L^p(\Pi_0)} \le ||u_2||_{L^p(\Pi_0)} + ||u_1||_{L^p(\Pi_0)}\,.$$

Using estimates (67) and (64) we see that the right-hand side is majorized by $c\mathcal{J}_f$. By (179) and (177)

$$||\mathfrak{Z}||_{L^p(\Pi_0)} \ge c||z||_{L^p(0,1)} \ge c_1.$$

The proof is complete.

Lemma 15 *There exists a nontrivial solution $\mathcal{Z} \in \mathring{W}^{m,p}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ to the homogeneous problem* (11), (12) *subject to*

$$\mathfrak{M}_p^m(\mathcal{Z}; K_{r/e,r}) = \begin{cases} o(r^{m-n-1} e^{-\mathcal{C}\int_r^1 \Omega(\rho)\frac{d\rho}{\rho}}) & \text{if } r \to 0 \\ o(r^m e^{-\mathcal{C}\int_1^r \Omega(\rho)\frac{d\rho}{\rho}}) & \text{if } r \to \infty. \end{cases} \tag{185}$$

This solution is unique up to a constant factor and admits the representation

$$\begin{aligned} &(r\partial_r)^k \mathcal{Z}(x) \\ &= C \, \exp\Big(\int_r^1 \big(\mathcal{T}(\rho) + \Upsilon(\rho)\big)\frac{d\rho}{\rho}\Big)\big((m-n)^k E(x) + r^(m-n) v_k(x)\big) \end{aligned} \tag{186}$$

with the same notation as in the statement of Theorem 1 *and with v_k subject to* (24).

Proof. By (169)

$$\lambda(t) = \sum_{j=0}^m \sum_{k=0}^m \big(\mathcal{N}_{j,k}(t)(-\partial_t)^{m-k})(e^{(n-m)t}\phi), \partial_t^{m-j}(e^{(m-n)t}\psi)\big) + O(\omega(t)^2), \tag{187}$$

where ϕ and ψ are the same functions as in Sect. 10. Setting $u = e^{(n-m)t}\phi$ and $v = e^{-mt}\psi$ in (84) we arrive at

$$\begin{aligned} \lambda(-\log r) = \frac{r^n}{m!} \int_{S_+^{n-1}} \sum_{|\alpha|,|\beta| \le m} (\mathcal{L}_{\alpha\beta} - L_{\alpha\beta}(x)) \partial_x^\beta E(x) \partial_x^\alpha x_n^m d\theta \\ + \frac{r^n}{m!} \int_{S_+^{n-1}} \sum_{|\alpha+\beta|<2m} \mathcal{L}_{\alpha\beta}(x) \partial_x^\beta E(x) \partial_x^\alpha x_n^m d\theta + O(\Omega(r)^2) \,. \end{aligned}$$

This can be written as $\lambda(-\log r) = \mathcal{T}(r) + O(\Omega(r)^2)$. The result follows from Lemma 14 by the change of variables $(t, \theta) \to x$.

Corollary 4 *Let $f \in W_{\mathrm{loc}}^{-m,p}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ be subject to*

$$\mathcal{I}_f := \int_0^\infty \rho^{m+n} e^{-\mathcal{C}|\int_\rho^1 \Omega(s)\frac{ds}{s}|} \mathfrak{M}_p^{-m}(f; K_{\rho/e,\rho}) \frac{d\rho}{\rho} < \infty \,. \tag{188}$$

Also let u_1 and u_2 be solutions of problem (11), (12) *from* (i) *and* (ii) *in Proposition* 8 *respectively. Then*

$$u_2(x) - u_1(x) = C\mathcal{Z}(x) \,, \tag{189}$$

where $\mathcal{Z}$ is defined in Lemma 15 *and C is a constant subject to*

$$|C| \le c\mathcal{I}_f \,.$$

Proof follows directly from Corollary 3.

18 End of proof of Theorem 1

Assertion (i) follows from Lemma 15. In order to obtain (ii) we introduce the cut-off function $\eta \in C_0^\infty(B_2)$, $\eta(x) = 1$ for $|x| \le 3/2$. The function $\eta\, u$ satisfies the zero Dirichlet conditions on $\mathbb{R}^{n-1} \setminus \mathcal{O}$ and the equation $\mathcal{L}(x, \partial_x)(\eta\, u) = f_1$ on $\mathbb{R}^n_+$ with $f_1 = \eta\, f + [\mathcal{L}, \eta] u$. Clearly,

$$\mathfrak{M}_p^{-m}(f_1; K_{r/e,r}) = \mathfrak{M}_p^{-m}(f; K_{r/e,r}) \tag{190}$$

if $r < 3/2$ and $r > 2e$. By the standard local estimate for solutions of the Dirichlet problem

$$\mathfrak{M}_p^{m}(u; K_{3r/2,2r}) \le c\,(r^{2m}\mathfrak{M}_p^{-m}(f; K_{r,er}) + r^{-n/p}||u||_{L^p(K_{r,er})}) \tag{191}$$

we have

$$\mathfrak{M}_p^{-m}([\mathcal{L}, \eta]u; K_{3/2,2}) \le c\,(\mathfrak{M}_p^{-m}(f; K_{1,e}) + ||u||_{L^p(K_{1,e})}).$$

Hence, for $r \in (3/2, 2e)$

$$\mathfrak{M}_p^{-m}(f_1; K_{r/e,r}) \le c(\mathfrak{M}_p^{-m}(f; K_{1/2,e}) + ||u||_{L^p(K_{1,e})}). \tag{192}$$

Therefore,

$$\mathcal{I}_{f_1} \le c(I_f + ||u||_{L^p(K_{1,e})}).$$

By (10) and finiteness of I_f,

$$\begin{aligned}\mathfrak{M}_p^{-m}(f; K_{\rho/e^2, e\rho}) &\le c \int_{r/e^2}^{e^2 r} \mathfrak{M}_p^{-m}(f; K_{\rho/e,\rho}) \frac{d\rho}{\rho} \\ &= o\Big(r^{-m-n} \exp\Big(-\mathcal{C}\int_r^1 \Omega(\rho)\frac{d\rho}{\rho}\Big)\Big) \quad \text{as } r \to 0.\end{aligned}$$

This along with (191) and (26) implies (71) with u replaced by $\eta\, u$. Therefore $\eta\, u$ is the solution of problem (11), (12) (with f_1 instead of f) from Proposition 8(ii). The result follows from Corollary 4.

19 Corollaries of the main result

Clearly, Theorem 1 remains valid if Ω is replaced by its nondecreasing majorant

$$\Omega^\diamond(r) = \sup_{x \in B_r^+} \Big(\sum_{|\alpha|=|\beta|=m} |\mathcal{L}_{\alpha\beta}(x) - L_{\alpha\beta}| + \sum_{|\alpha+\beta|<2m} x_n^{2m-|\alpha+\beta|} |\mathcal{L}_{\alpha\beta}(x)|\Big). \tag{193}$$

Corollary 5 *Let $\mathcal{Z}$ be the same solution as in* Theorem 1. *Then*

$$\begin{aligned}\partial_x^\alpha Z(x) &= \exp\Big(\int_r^1 \mathcal{T}(\rho)\frac{d\rho}{\rho} + \Psi^\diamond(r)\Big) \\ &\times \big(\partial_x^\alpha E(x) + r^{m-n-|\alpha|} v_\alpha(x)\big),\end{aligned} \tag{194}$$

where $|x|<1$ *and the function* $\Psi^\diamond$ *satisfies*

$$\Psi^\diamond(r) \le \mathcal{C}\int_r^e \Omega^\diamond(\rho)^2 \frac{d\rho}{\rho}$$

and

$$|\partial_r \Psi^\diamond(r)| \le \mathcal{C}\Omega^\diamond(r)\int_r^e e^{\mathcal{C}\int_r^\rho \Omega^\diamond(s)\frac{ds}{s}}\Omega^\diamond(\rho)\frac{d\rho}{\rho^2}\,.$$

For $|\alpha| \le m-1$ *the function* v_α *belongs to* $\mathring{W}^{1,p}_{\rm loc}(\overline{\mathbb{R}^n_+}\setminus \mathcal{O})$ *and satisfies*

$$\Big(r^{-n}\int_{K_{r/e,r}} (r|\nabla v_\alpha(x)| + |v_\alpha(x)|)^p dx\Big)^{1/p} \le cr^{1-\varepsilon}\int_r^e \Omega^\diamond(\rho)\frac{d\rho}{\rho^{2-\varepsilon}} \tag{195}$$

for $r<1$. *If* $|\alpha|=m$, *the term* $r|\nabla v_\alpha(x)|$ *should be removed. By* ε, *we denote a sufficiently small number depending on* n, m, p *and* $L_{\alpha\beta}$.

Proof is the same as that of Corollary 5 in [4].

Corollary 6 *If* $\Omega(r)\to 0$ *as* $r\to 0$ *then the right-hand side in* (24) *tends to* 0 *as* $r\to 0$ *and*

$$\mathfrak{M}^m_p(w;K_{r/e,r}) = o(r^{m-n}e^{-\mathcal{C}\int_r^1 \Omega(s)\frac{ds}{s}})\,.$$

In the case $p>n$ *the solution* u *in* Theorem 1 (ii) *satisfies*

$$\begin{aligned}\partial_x^\alpha u(x) &= \exp\Big(\int_{|x|}^1 (\mathcal{T}(\rho)+\Upsilon(\rho))\frac{d\rho}{\rho}\Big)\\ &\times \Big(C\partial_x^\alpha E(x) + o(|x|^{m-n-|\alpha|})\Big) \quad \text{as } |x|\to 0\end{aligned} \tag{196}$$

uniformly with respect to $x/|x|$. *Here* α *is an arbitrary multi-index of order* $\le m-1$. *The function* $\Psi^\diamond(r)$ *is the same as in Corollary* 5. *Moreover,* (196) *remains valid also for* $|\alpha|=m$ *but then* $\Phi=o(1)$ *should be understood as*

$$r^{-n/p}||\Phi||_{L^p(K_{r/e,r})}\to 0 \quad \text{as } r\to 0.$$

Proof is the same as that of Corollary 7 in [4].

Corollary 7 *Let* $p>n$ *and*

$$\int_0^1 \Omega(\rho)^2\frac{d\rho}{\rho} < \infty\,. \tag{197}$$

Then the solution u *from* Theorem 1 (ii) *satisfies*

$$\begin{aligned}&\partial_x^\alpha u(x)\\ &= \exp\Big(\int_{|x|}^1 \mathcal{T}(\rho)\frac{d\rho}{\rho}\Big)\Big(C\partial_x^\alpha E(x) + o(|x|^{m-n-|\alpha|})\Big)\end{aligned} \tag{198}$$

for $|\alpha|\le m-1$ *uniformly with respect to* $x/|x|$. *The same is true for* $|\alpha|=m$ *if the symbol* $o(1)$ *is understood as in* Theorem 1 (ii).

Proof is the same as that of Corollary 8 in [4].

20 Second order elliptic equations

Example 1. Consider the equation with complex-valued measurable coefficients

$$-\sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(x)\partial_{x_j}u) = 0 \quad \text{in } B_3^+$$

complemented by the boundary condition

$$u(x',0) = 0 \quad \text{for } |x'| < 3. \tag{199}$$

We assume that there exists a constant symmetric matrix $\{a_{ij}\}_{i,j=1}^n$ with positive definite real part such that the function

$$\Omega^\diamond(r) = \sup_{B_r^+} \sum_{i,j=1}^{n} |a_{ij}(x) - a_{ij}|$$

is sufficiently small in B_3^+. In view of Theorem 1(ii) and Corollary 5

$$u(x) = \exp\Big\{ \int_{r<|y|<1} \sum_{i=1}^{n}(a_{ni}(y) - a_{ni})\partial_{y_i}E(y)dy + O\Big(\int_{|x|}^{1} \Omega^\diamond(\rho)^2 \frac{d\rho}{\rho}\Big)\Big\}$$
$$\times\Big(C\Big(E(x) + O\big(|x|^{2-n-\varepsilon}\int_{|x|}^{3} \Omega^\diamond(\rho)\rho^{\varepsilon-2}d\rho\big)\Big) + O\big(|x|^{1-\varepsilon}\big)\Big), \tag{200}$$

where ε is a small positive number depending on n and the coefficients a_{ij}. Here $E(x)$ stands for the Poisson kernel of the equation

$$\sum_{i,j=1}^{n} a_{ij}\partial_{x_i}\partial_{x_j}v = 0 \quad \text{in } \mathbb{R}_+^n,$$

i.e.

$$E(x) = (\det\{a_{ij}\})^{-1/2}|S^{n-1}|^{-1}x_n\Big(\sum_{k,l=1}^{n} b_{kl}x_kx_l\Big)^{-n/2}$$

where $\{b_{lj}\}$ is the inverse of $\{a_{ij}\}$ (see [2], Sect.6.2). Setting this expression of $E(x)$ into (200), we arrive at (4) where $\delta = 1$, $G = \mathbb{R}_+^n$ and $\mathcal{Q}$ is given by (3). The case of a domain with smooth boundary mentioned in the introduction can be easily reduced to the present one by changing variables.

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