On the third boundary value problem in domains with cusps

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Abstract

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1 Introduction.

We study the third boundary value problem for the equation

$$\Delta u + \omega u = f \ , \tag{1.1}$$

in a planar domain Ω with an exterior cusp O on $\partial\Omega$. By ω we denote arbitrary real or complex number and f is a given complex valued function. The solutions are subject to the boundary condition,

$$\partial_n u - \rho u = g, \ \text{on} \ \partial\Omega \setminus O, \tag{1.2}$$

where ρ and g are prescribed complex valued functions on $\partial\Omega \setminus O$. Let us describe the domain Ω. We fix a certain Cartesian system $x = (x_1, x_2)$ with the origin O and set $\Omega_\varepsilon := \Omega \cap \{x_1 < \varepsilon\}$, where ε is a small positive number. We assume that Ω_ε coincides with the set

$$\{x : 0 < x_1 < \varepsilon, \ \phi_0(x_1) < x_2 < \phi_1(x_1)\}, \tag{1.3}$$

where ϕ_0 and ϕ_1 are functions from $C^2[0, \varepsilon]$, such that

$$\phi_0(0) = \phi_1(0) = \phi_0'(0) = \phi_1'(0) = 0, \tag{1.4}$$

and

$$\phi_1''(0) > \phi_0''(0) \ . \tag{1.5}$$

Moreover, let ε be so small that $\phi_1 > \phi_0$ on $(0, \varepsilon)$. We assume that $\rho \in C^\infty(\partial\Omega \setminus O)$ and there exist two complex numbers ρ_0 and ρ_1, such that

$$\rho(x) = \rho_0, \ x \in \{x : 0 < x_1 < \varepsilon, \ x_2 = \phi_0(x_1)\}, \tag{1.6}$$

$$\rho(x) = \rho_1, \ x \in \{x : 0 < x_1 < \varepsilon, \ x_2 = \phi_1(x_1)\}, \tag{1.7}$$

Our goal is to describe the asymptotic behavior of solutions to the problem (2.1), (1.2) in the neighborhood of an external cusp O. The solutions we are dealing with belong to a very wide class; to be more precise they may grow as $\exp(cx_1^{-1})$ as $x_1 \to +0$, with a sufficiently small positive constant c.

The problem (1.1), (1.2) is a particular case of an elliptic boundary value problems in cuspidal domains considered in [1], [2], where the Fredholm and other properties of solutions were investigated. The Dirichlet and Neumann problems for the Laplacian and Lamé system were studied from different points of view in [3]-[19] (see also [20], where other references can be found).

It appears that the problem (1.1), (1.2) has special features which make its study more complicated in comparison with Dirichlet and Neumann problems. In fact, the principal term in the asymptotic representation of a solution is determined by the lower order term in the boundary operator. To be more precise, for example, we prove that

$$u(x) \sim c_1 x_1^{-\frac{1}{2}+i\sqrt{\lambda-\frac{1}{4}}} + c_2 x_1^{-\frac{1}{2}-i\sqrt{\lambda-\frac{1}{4}}}, \quad x_1 \to +0, \tag{1.8}$$

where

$$\lambda := 2\frac{\rho_0 + \rho_1}{\phi_1''(0) - \phi_0''(0)}, \tag{1.9}$$

provided $\lambda > \frac{1}{4}$. In the case $\lambda < \frac{1}{4}$ we have

$$u(x) \sim c_1 x_1^{-\frac{1}{2}+\sqrt{\frac{1}{4}-\lambda}} + c_2 x_1^{-\frac{1}{2}-\sqrt{\frac{1}{4}-\lambda}}, \quad x_1 \to +0. \tag{1.10}$$

And, finally, if $\lambda = \frac{1}{4}$,

$$u(x) \sim c_1 x_1^{-\frac{1}{2}} + c_2\, x_1^{-\frac{1}{2}} \ln x_1, \quad x_1 \to +0. \tag{1.11}$$

In the above formulae c_1 and c_2 are some constants.

The asymptotic representations (1.8), (1.10) and (1.11) show, in particular, that for real λ the profile of the solution, in general, depends on the sign of $4\lambda - 1$: the solution exhibits an oscillatory behavior if and only if $4\lambda > 1$.

The paper is organised as follows: Section 2. contains known auxiliary results. In Section 3. we map a small neighborhood of the cusp into a suitable strip and investigate the resulting transformed problem. In the last section we study the asymptotic behavior of solutions near the cusp and their other properties.

2 Formulation of the problem and known results.

Consider the problem:

$$\Delta u + \omega u = f \text{ in } \Omega, \quad \partial_n u - \rho u = g, \text{ on } \partial\Omega \setminus O. \tag{2.1}$$

It is known that the boundary value problem (2.1) is Fredholm in certain weighted spaces, see [2]. Let Υ be a domain and $\kappa > 0$ be fixed. Let β, γ be real and $l = 0, 1, \ldots$. We define weighted Sobolev space $\mathcal{W}^l_{\beta,\gamma}(\Upsilon)$ as the closure of the set $C_0^\infty(\overline{\Upsilon} \setminus O)$ with respect to the norm

$$\|u : \mathcal{W}^l_{\beta,\gamma}(\Upsilon)\|^2 :=$$

$$\sum_{|\delta| \le l} \int_\Upsilon e^{\frac{4\beta}{\kappa x_1}} |x_1|^{4(\gamma - l + |\delta|)} |\partial_x^\delta u|^2 dx, \tag{2.2}$$

where $\delta \in \mathbb{Z}^2_+$ is the usual multi-index. Furthermore, for $l \ge 1$ we define $\mathcal{W}^{l-1/2}_{\beta,\gamma}(\partial\Upsilon)$ as the trace space for $\mathcal{W}^l_{\beta,\gamma}(\Upsilon)$ on the boundary $\partial\Upsilon$. Then one can see that operator A of the boundary value problem (2.1) is continuous from $\mathcal{W}^{l+2}_{\beta,\gamma}(\Omega)$ to $\mathcal{W}^l_{\beta,\gamma}(\Omega) \times \mathcal{W}^{l+1/2}_{\beta,\gamma}(\partial\Omega)$ for any $l = 0, 1, ..$ and any real β and γ.

Theorem 2.1. *Suppose that $\beta\pi^{-1} \notin \mathbb{Z}$. Then the operator A_β of the boundary value problem* (2.1) *is Fredholm from $\mathcal{W}^2_{\beta,\gamma}(\Omega)$ to $\mathcal{W}^0_{\beta,\gamma}(\Omega) \times \mathcal{W}^{1/2}_{\beta,\gamma}(\partial\Omega)$ for any real γ. In particular, for any $\delta > 0$ small enough, every solution of* (2.1) *satisfies the estimate*

$$\|u\|_{\mathcal{W}^2_{\beta,\gamma}(\Omega)} \le c\left(\|f\|_{\mathcal{W}^0_{\beta,\gamma}(\Omega)} + \|g\|_{\mathcal{W}^{1/2}_{\beta,\gamma}(\partial\Omega)} + \|u\|_{\mathcal{W}^0_{\beta,\gamma}(\Omega\setminus B_\delta)}\right). \tag{2.3}$$

Remark 2.1. Let us assume that

$$|\omega| + \|\rho\|_{C^2(\partial\Omega\setminus B_\varepsilon)} + |\rho_0| + |\rho_1| \le K, \tag{2.4}$$

where K is a fixed large positive number. Then, the constant c in (2.3), can be chosen independently of ω and ρ. The condition (2.4) will be assumed throughout the paper.

Theorem 2.2. *Let $-\pi < \beta_1 < 0 < \beta_2 < \pi$, and $u \in \mathcal{W}^2_{\beta_1,\gamma}(\Omega)$ be a solution of the boundary value problem* (2.1) *where $(f, g) \in \mathcal{W}^0_{\beta_2,\gamma}(\Omega) \times \mathcal{W}^{1/2}_{\beta_2,\gamma}(\partial\Omega)$. Then the solution u admits representation*

$$u = c_1 u_1 + c_2 u_2 + \widetilde{u}, \ \text{ in } \Omega_\varepsilon, \tag{2.5}$$

for sufficiently small ε. Here $\widetilde{u} \in \mathcal{W}^2_{\beta_2,\gamma}(\Omega)$, c_j are constants, and $u_j \in \mathcal{W}^2_{\beta_1,\gamma}(\Omega)$, $j = 1, 2$, are linearly independent modulo $\mathcal{W}^2_{\beta_2,\gamma}(\Omega)$ and solve the homogeneous problem (2.1) *in Ω_ε.*

These statements are simple particular cases of Theorem 9.2.1 and Theorem 9.2.2 from [2]. The exact information on the forbidden values of β is due to the known eigenvalues of related operator pencil, which corresponds to the Neumann Laplacian on the interval $[0, 1]$.

The above function spaces are based on *exponential weights* (zero is a forbidden value of β). However they are not sufficient for our purpose to obtain asymptotics of the solutions near the cusp. Below we will construct alternative weighted Sobolev spaces with *power-type weights*, such that the operator will be Fredholm and additionally will have zero index for large range of parameters. On the other hand, we will provide a precise information on u_1 and u_2 appearing in Theorem 2.2, and on their asymptotic behaviour near the singularity point O.

3 Problem in a strip.

3.1 Change of variables and asymptotic properties in the strip

In this section we investigate local properties of the solution of the problem

$$\Delta u + \omega u = f \text{ in } \Omega_\varepsilon; \quad \partial_n u - \rho_0 u = g_0 \text{ on } S_0; \quad \partial_n u - \rho_1 u = g_1 \text{ on } S_1. \tag{3.1}$$

Our approach is based on employing the following transformation:

$$z = \frac{x_2 - \phi_0(x_1)}{\phi(x_1)}, \ t = \frac{2}{\kappa} x_1^{-1}, \tag{3.2}$$

where

$$\phi := \phi_1 - \phi_0, \qquad \kappa := \phi_1''(0) - \phi_0''(0). \tag{3.3}$$

This transformation maps the cusp Ω_ε onto semi-strip $\Pi_T = \{(t, z) | z \in (0, 1), t > T\}$, $T = \frac{2}{\kappa\varepsilon}$.

Then conditions (1.4) on $\phi_j,\ j=1,2$ imply that, for $t\to+\infty$:

$$\phi_j(x_1(t)) - \frac{2\phi_j''(0)}{\kappa^2 t^2} = O(t^{-3}), \tag{3.4}$$

$$\phi_j'(x_1(t)) - \frac{2\phi_j''(0)}{\kappa} t^{-1} = O(t^{-2}), \tag{3.5}$$

$$\phi_j''(x_1(t)) = O(1). \tag{3.6}$$

In order to rewrite (3.1) in the new variables (t,z), we routinely evaluate

$$\partial_{x_1} = -\frac{2}{\kappa x_1^2}\partial_t + \frac{\partial z}{\partial x_1}\partial_z = -\frac{\kappa t^2}{2}\partial_t - \left(\frac{\phi_0'}{\phi} + z\frac{\phi'}{\phi}\right)\partial_z, \tag{3.7}$$

$$\partial_{x_1}^2 = \left(\frac{\partial t}{\partial x_1}\right)^2 \partial_t^2 + \frac{\partial^2 t}{\partial x_1^2}\partial_t + 2\frac{\partial z}{\partial x_1}\frac{\partial t}{\partial x_1}\partial_t\partial_z + \left(\frac{\partial z}{\partial x_1}\right)^2\partial_z^2 + \frac{\partial^2 z}{\partial x_1^2}\partial_z \tag{3.8}$$

$$= \frac{\kappa^2 t^4}{4}\left(\partial_t^2 + 2t^{-1}\partial_t + \frac{4}{\kappa^2 t^4}\left(2\frac{\partial z}{\partial x_1}\frac{\partial t}{\partial x_1}\partial_t\partial_z + \left(\frac{\partial z}{\partial x_1}\right)^2\partial_z^2 + \frac{\partial^2 z}{\partial x_1^2}\partial_z\right)\right),$$

$$\partial_{x_2} = \frac{1}{\phi(x_1)}\partial_z, \quad \partial_{x_2}^2 = \frac{1}{\phi^2(x_1)}\partial_z^2. \tag{3.9}$$

Consequently,

$$\Delta_x + \omega := \partial_{x_1}^2 + \partial_{x_2}^2 + \omega = \frac{\kappa^2 t^4}{4}\left(\partial_t^2 + \partial_z^2 + \mathcal{L}\right), \tag{3.10}$$

where

$$\mathcal{L} = 2t^{-1}\partial_t + \frac{4}{\kappa^2 t^4}\left(2\frac{\partial z}{\partial x_1}\frac{\partial t}{\partial x_1}\partial_t\partial_z + \left(\frac{\partial z}{\partial x_1}\right)^2\partial_z^2 + \frac{\partial^2 z}{\partial x_1^2}\partial_z + \left(\frac{1}{\phi^2(x_1)} - \frac{\kappa^2 t^4}{4}\right)\partial_z^2 + \omega\right). \tag{3.11}$$

In a similar way, using (3.7) and (3.9),we have

$$\partial_{n_x} = (1+(\phi_0')^2)^{-1/2}(\phi_0'\partial_{x_1} - \partial_{x_2}) =$$

$$\left(1+(\phi_0')^2\right)^{-1/2}\left(-\phi_0'\left(\frac{\kappa t^2}{2}\partial_t + \left(\frac{\phi_0'}{\phi} + z\frac{\phi'}{\phi}\right)\partial_z\right) - \phi^{-1}\partial_z\right) =$$

$$= (1+(\phi_0')^2)^{1/2}\phi^{-1}\left(-\partial_z - \frac{\kappa t^2 \phi\phi_0'}{2(1+\phi_0'^2)}\partial_t\right), \quad \text{for } z=0,$$

and

$$\partial_{n_x} = (1+(\phi_1')^2)^{-1/2}(-\phi_1'\partial_{x_1} + \partial_{x_2}) =$$

$$\left(1+(\phi_1')^2\right)^{-1/2}\left(\phi_1'\left(\frac{\kappa t^2}{2}\partial_t + \left(\frac{\phi_0'}{\phi} + z\frac{\phi'}{\phi}\right)\partial_z\right) + \phi^{-1}\partial_z\right) =$$

$$= (1+(\phi_1')^2)^{1/2}\phi^{-1}\left(\partial_z + \frac{\kappa t^2 \phi\phi_1'}{2(1+\phi_1'^2)}\partial_t\right), \quad \text{for } z=1.$$

As a result we have:

$$\left(\partial_t^2+\partial_z^2+\mathcal{L}\right)u=F, \quad \text{in } \Pi_T, \tag{3.12}$$

$$(-\partial_z+\mathcal{N}_0)u=G_0,\ z=0,\ t>T. \tag{3.13}$$

and

$$(\partial_z+\mathcal{N}_1)u=G_1,\ z=1,\ t>T. \tag{3.14}$$

Here

$$\mathcal{N}_0=-\frac{\kappa t^2\phi\phi_0'}{2(1+\phi_0'^2)}\partial_t-\rho_0\phi\left(1+\phi_0'^2\right)^{-1/2}, \tag{3.15}$$

$$\mathcal{N}_1=\frac{\kappa t^2\phi\phi_1'}{2(1+\phi_1'^2)}\partial_t-\rho_1\phi\left(1+\phi_1'^2\right)^{-1/2}, \tag{3.16}$$

and $F=\frac{4}{\kappa^2t^4}f$, $G_0=\phi\left(1+\phi_0'^2\right)^{-1/2}g_0$, $G_1=\phi\left(1+\phi_1'^2\right)^{-1/2}g_1$.

In what follows we explore a more subtle properties of the operators appearing in (3.12)-(3.14), therefore we will need the following representations:

$$\mathcal{N}_0=-2\frac{\phi_0''(0)}{\kappa t}\partial_t-\frac{2\rho_0}{\kappa t^2}+\mathfrak{N}_0,\ \mathfrak{N}_0=2\frac{\phi_0''(0)}{\kappa t}\partial_t+\frac{\kappa t^2\phi\phi_0'}{2(1+\phi_0'^2)}\partial_t+\frac{2\rho_0}{\kappa t^2}-\rho_0\phi\left(1+\phi_1'^2\right)^{-1/2}, \tag{3.17}$$

$$\mathcal{N}_1=2\frac{\phi_1''(0)}{\kappa t}\partial_t-\frac{2\rho_1}{\kappa t^2}+\mathfrak{N}_1,\ \mathfrak{N}_1=-2\frac{\phi_1''(0)}{\kappa t}\partial_t+\frac{\kappa t^2\phi\phi_1'}{2(1+\phi_1'^2)}\partial_t+\frac{2\rho_1}{\kappa t^2}-\rho_1\phi\left(1+\phi_1'^2\right)^{-1/2}. \tag{3.18}$$

Next we are going to employ "method of projections", in a form somewhat similar to [21]. Let us represent the solution to (3.12)-(3.14) in the form of the following decomposition

$$u(t,z)=u_1(t)+u_2(t,z), \tag{3.19}$$

where $u_1(t)=\int_0^1 u(t,z)dz=:P_1u$, $u_2=P_2u:=u-P_1u$. (Hence P_1 and P_2 are appropriate projectors.) Clearly $\int_0^1 u_2(t,z)dz=0$. Substituting (3.19) into (3.12) we get,

$$\partial_t^2u_1+\Delta_{(t,z)}u_2+\mathcal{L}(u_1+u_2)=F, \tag{3.20}$$

where $\Delta_{(t,z)}:=\partial_t^2+\partial_z^2$. Integrating (3.20) with respect to z over $(0,1)$ we obtain

$$\partial_t^2u_1+P_1\Delta u_2+P_1\mathcal{L}(u_1+u_2)=P_1F, \quad \text{in } \Pi_T, \tag{3.21}$$

having henceforth dropped the subscript (t,z) for $\Delta_{(t,z)}$ for ease of notation. Using (3.11) yields

$$\partial_t^2u_1+2t^{-1}\partial_tu_1+\frac{4\omega}{\kappa^2t^4}u_1+P_1\partial_z^2u+P_1\mathcal{L}u_2=P_1F, \quad \text{in } \Pi_T. \tag{3.22}$$

Integrating by parts in the third term in (3.22) and using (3.13)-(3.16) we get

$$\partial_t^2u_1+2t^{-1}\partial_tu_1+\frac{4\omega}{\kappa^2t^4}u_1-\mathcal{N}_1u|_{z=1}-\mathcal{N}_0u|_{z=0}+P_1\mathcal{L}u_2=\mathfrak{F}_1,$$

where $\mathfrak{F}_1 := P_1 F - G_1 - G_0$. Using further (3.17) and (3.18),

$$\partial_t^2 u_1 + 2\frac{\rho_0+\rho_1}{\kappa t^2}u_1 + \frac{4\omega}{\kappa^2 t^4}u_1 - \mathfrak{N}_1 u_1 - \mathfrak{N}_0 u_1 \tag{3.23}$$

$$-\mathcal{N}_1 u_2|_{z=1} - \mathcal{N}_0 u_2|_{z=0} + P_1\mathcal{L}u_2 = \mathfrak{F}_1,\ t > T.$$

On the other hand, subtracting (3.21) from (3.20) and integrating by parts we similarly obtain,

$$\Delta u_2 + (\mathcal{N}_1 + \mathcal{N}_0)u + P_2\mathcal{L}u = P_2 F + G_1 + G_0. \tag{3.24}$$

This equation is supplemented by the boundary conditions, see(3.13) and (3.14) :

$$-\partial_z u_2 + \mathcal{N}_0(u_1+u_2) = G_0,\ z=0 \ \text{ and } \ \partial_z u_2 + \mathcal{N}_1(u_1+u_2) = G_1,\ z=1. \tag{3.25}$$

We then rewrite (3.23) and (3.24), (3.25) as a system of boundary value problems, with anticipated "main order" parts $\mathbf{A}_1$, $\mathbf{A}_2$ and "perturbations" $\mathfrak{B}_{ij}$, $i,j = 1.2$:

$$\mathbf{A}_1 u_1 + \mathfrak{B}_{11}u_1 + \mathfrak{B}_{12}u_2 = \mathfrak{F}_1, \quad t > T, \tag{3.26}$$

$$\mathfrak{B}_{21}u_1 + (\mathbf{A}_2 + \mathfrak{B}_{22})u_2 = \mathfrak{F}_2. \ \ t > T. \tag{3.27}$$

Here

$$\mathbf{A}_1 = \partial_t^2 + \lambda t^{-2}\ ,\ \lambda := \frac{2}{\kappa}(\rho_0+\rho_1),\ \mathfrak{B}_{11} = \frac{4\omega}{\kappa^2 t^4} - \mathfrak{N}_0 - \mathfrak{N}_1,\ \mathfrak{B}_{12} = P_1\mathcal{L} - \mathcal{N}_1 - \mathcal{N}_0, \tag{3.28}$$

and

$$\mathbf{A}_2 u_2 = (\Delta u_2, -\partial_z u_2|_{z=0}, \partial_z u_2|_{z=1}), \tag{3.29}$$

$$\mathfrak{B}_{21} = (\mathcal{N}_0+\mathcal{N}_1, \mathcal{N}_0, \mathcal{N}_1),\ \ \mathfrak{B}_{22}u_2 = \big((P_2\mathcal{L}+\mathcal{N}_0+\mathcal{N}_1)u_2, \mathcal{N}_0 u_2|_{z=0}, \mathcal{N}_1 u_2|_{z=1}\big), \tag{3.30}$$

and $\mathfrak{F}_2 = (P_2 F + G_1 + G_0, G_0, G_1)$. Let us notice that, by our construction, u is a solution to the problem (3.12)-(3.14) if and only if the vector (u_1, u_2) is a solution to (3.26), (3.27).

Let χ be cut-off function such that

$$\chi \in C^\infty(\mathbb{R}),\ \ \chi(t) = 0 \text{ for } t < -1 \text{ and } \chi(t) = 1 \text{ for } t > 1. \tag{3.31}$$

Let $\chi_\varepsilon(t) := \chi(t-T)$ (recall that $T = \frac{2}{\kappa\varepsilon}$). Consider a system

$$\mathbf{A}_1 u_1 + \mathfrak{B}^\varepsilon_{11}u_1 + \mathfrak{B}^\varepsilon_{12}u_2 = \mathfrak{F}^\varepsilon_1, \tag{3.32}$$

$$\mathfrak{B}^\varepsilon_{21}u_1 + (\mathbf{A}_2 + \mathfrak{B}^\varepsilon_{22})u_2 = \mathfrak{F}^\varepsilon_2, \tag{3.33}$$

where $\mathfrak{B}_{ij} = \chi_\varepsilon\mathfrak{B}_{ij}$, $\mathfrak{F}^\varepsilon_i = \chi_\varepsilon\mathfrak{F}^\varepsilon_i$ $i,j = 1,2$. The presence of the cut-off functions allows us to consider a new system (3.32)-(3.33) for $t < T$ as well. On the other hand, any solution of (3.32), (3.33) is a solution of (3.26),(3.27) for $t > T+1$ (since these systems coincide for $t > T+1$).

Let us consider the operator of the main order in (3.32):

$$\mathbf{A}_1 = \partial_t^2 + \lambda t^{-2} \ . \tag{3.34}$$

We are going to consider it as an operator on functions defined on $\mathbb{R}_+$. Let us introduce the functional space $V_\sigma^l(\mathbb{R}_+)$ ($l = 0, 1, ...,$ and $\sigma \in \mathbb{R}$) which we define as the closure of $C_0^\infty(\mathbb{R}_+)$ with respect to the norm

$$\|u : V_\sigma^l(\mathbb{R}_+)\|^2 = \sum_{n=0}^{l} \int_{\mathbb{R}_+} t^{2(\sigma+n-l)} |\partial_t^n u|^2 dt. \tag{3.35}$$

Then, employing e.g. the Mellin's transform, we have

Lemma 3.1. *Let $\sigma \neq 1 \pm Re\big(\frac{1}{4} - \lambda\big)^{1/2}$. Then operator $\boldsymbol{A}_1$ is an isomorphism from $V_\sigma^2(\mathbb{R}_+)$ to $V_\sigma^0(\mathbb{R}_+)$.*

Remark 3.1. Clearly $\mathbf{A}_1$ and $\mathbf{A}_1^{-1}$ depend on parameters λ and σ. In fact we have the following estimates:

$$\|\mathbf{A}_1\| \leq c, \ \ \|\mathbf{A}_1^{-1}\| \leq c \left((\sigma - 1)^2 - \left(\mathrm{Re} \, (1/4 - \lambda)^{1/2} \right)^2 \right)^{-1}, \tag{3.36}$$

where constant c depends only on K (see (2.4)). In particular, norm of $\mathbf{A}_1^{-1}$ remain bounded by a constant dependent only on K, provided

$$\left| \sigma - 1 - \mathrm{Re}\big(\frac{1}{4} - \lambda\big)^{1/2} \right| > K^{-1}, \ \left| \sigma - 1 + \mathrm{Re}\big(\frac{1}{4} - \lambda\big)^{1/2} \right| > K^{-1}. \tag{3.37}$$

The definition of operator of main order in (3.33), namely of Neumann Laplacian, see (3.29), is more subtle due to the presence of the projections P_1 and P_2 in (3.33). We are going to consider this operator as an operator acting on the functions defined on the whole strip $\Pi = \{(t, z) | -\infty < t < +\infty, z \in (0, 1)\}$. To this end we need a Sobolev space with a power-type weight $H_\sigma^l(\Pi)$ ($l = 0, 1, ...$ and $\sigma \in \mathbb{R}$), which we define as the closure of the set $C_0^\infty(\overline{\Pi})$ with respect to the norm

$$\|u : H_\sigma^l(\Pi)\|^2 = \sum_{|\delta| \leq l} \int_{\Pi} (t^2 + 1)^\sigma |\nabla^\delta u|^2 dt dz.$$

In the usual way we define the trace spaces $H_\sigma^{l-1/2}(\partial\Pi)$.

Now we define the domain of the other main order operator $\mathbf{A}_2$ and its range:

$$\mathcal{D}_\sigma^2 = \left\{ u_2 \in H_\sigma^2(\Pi) : \int_0^1 u_2(t, z) dz = 0 \ , \ t \in \mathbb{R} \right\}, \tag{3.38}$$

and

$$\mathcal{R}_\sigma^0 = \left\{ \mathfrak{f} \in L_\sigma^2(\Pi) \times H_\sigma^{1/2}(\mathbb{R}) \times H_\sigma^{1/2}(\mathbb{R}) : \int_0^1 \mathfrak{f}_1(t, z) dz = \mathfrak{f}_2(t) + \mathfrak{f}_3(t) \ , \ t \in \mathbb{R} \right\}. \tag{3.39}$$

Lemma 3.2. *For any $\sigma \in \mathbb{R}$, $\boldsymbol{A}_2$ is an isomorphism from $\mathcal{D}^2_\sigma$ to $\mathcal{R}^0_\sigma$.*

Proof. Obviously $\mathbf{A}_2$ acts continuously for any σ. Let us prove its invertibility. To prove the claim for $\sigma = 0$ we apply Fourier transform $t \to \xi$ and use the explicit Green's function for the resulting operator pencil, see [22] p.27. (The presence of singularity at $\xi = 0$ does not cause problems, due to the orthogonality condition in the definition of the space $\mathcal{R}^0_\sigma$, see (3.39)).

Consider now the problem

$$\mathbf{A}_2\mathfrak{u} = \mathfrak{f}, \tag{3.40}$$

in the space $\mathcal{D}^2_\sigma$ for $\sigma \neq 0$. Multiplying (3.40) by $\langle t\rangle^\sigma := (a + t^2)^{\sigma/2}$, $a > 1$, we get

$$\mathbf{A}_2\langle t\rangle^\sigma\mathfrak{u} + [\langle t\rangle^\sigma, \mathbf{A}_2]\mathfrak{u} = \langle t\rangle^\sigma\mathfrak{f}, \tag{3.41}$$

where $[\cdot,\cdot]$ denotes the commutator. Now $\langle t\rangle^\sigma\mathfrak{f} \in \mathcal{R}^0_0$ and $[\langle t\rangle^\sigma, \mathbf{A}_2]\langle t\rangle^{-\sigma}$ can be directly checked to be small from $\mathcal{D}^2_0$ to $\mathcal{R}^0_0$ for a large enough. Consequently there is a unique solution to (3.41), $\langle t\rangle^\sigma\mathfrak{u} \in \mathcal{D}^2_0$. This is equivalent to $\mathfrak{u} \in \mathcal{D}^2_\sigma$. □

Now we can treat the remaining operators in (3.32) and (3.33) as *perturbations* of $\mathbf{A}_1$ and $\mathbf{A}_2$. Below we use the notation $A \lesssim B$ instead of $A \leq cB$.

Lemma 3.3. *For any $\sigma \in \mathbb{R}$ the following estimates hold:*

$$\|\mathfrak{B}^\varepsilon_{11}\|_{V^2_{\sigma-1}(\mathbb{R}_+)\to V^0_\sigma(\mathbb{R}_+)} \lesssim 1,\ \ \|\mathfrak{B}^\varepsilon_{11}\|_{V^2_\sigma(\mathbb{R}_+)\to V^0_\sigma(\mathbb{R}_+)} \lesssim \varepsilon, \tag{3.42}$$

$$\|\mathfrak{B}^\varepsilon_{12}\|_{\mathcal{D}^2_{\sigma-1}(\Pi)\to V^0_\sigma(\mathbb{R}_+)} \lesssim 1,\ \ \|\mathfrak{B}^\varepsilon_{12}\|_{\mathcal{D}^2_\sigma(\Pi)\to V^0_\sigma(\mathbb{R}_+)} \lesssim \varepsilon, \tag{3.43}$$

$$\|\mathfrak{B}^\varepsilon_{21}\|_{V^2_\sigma(\mathbb{R}_+)\to \mathcal{R}^0_\sigma(\Pi)} \lesssim 1, \tag{3.44}$$

$$\|\mathfrak{B}^\varepsilon_{22}\|_{\mathcal{D}^2_\sigma(\Pi)\to \mathcal{R}^0_{\sigma-1}(\Pi)} \lesssim 1,\ \ \ \|\mathfrak{B}^\varepsilon_{22}\|_{\mathcal{D}^2_\sigma(\Pi)\to \mathcal{R}^0_\sigma(\Pi)} \lesssim \varepsilon. \tag{3.45}$$

Proof. 1. Let us prove first (3.43). We have, see (3.28),

$$\|\mathfrak{B}^\varepsilon_{12}u_2\|^2_{V^0_\sigma(\mathbb{R}_+)} \lesssim \|\chi_\varepsilon P_1\mathcal{L}u_2\|^2_{V^0_\sigma(\mathbb{R}_+)} + \|\chi_\varepsilon(\mathcal{N}_1 + \mathcal{N}_0)u_2\|^2_{V^0_\sigma(\mathbb{R}_+)}. \tag{3.46}$$

Consider the second term on the right hand side of (3.46). Using (3.16), (3.4) and (3.5), we obtain

$$\|\chi_\varepsilon\mathcal{N}_1u_2\|^2_{V^0_\sigma(\mathbb{R}_+)} \lesssim \int_T t^{2\sigma}|\mathcal{N}_1u_2|^2dt \lesssim \int_T t^{2\sigma}|\phi u_2(t,1)|^2 + t^{2\sigma}|\phi_0'\partial_tu_2(t,1)|^2dt \lesssim$$

$$\int_T t^{2\sigma}|t^{-2}u_2(t,1)|^2 + t^{2\sigma}|t^{-1}\partial_tu_2(t,1)|^2dt \lesssim \|u_2\|^2_{H^2_{\sigma-1}(\Pi)},$$

and

$$\|\chi_\varepsilon\mathcal{N}_1u_2\|^2_{V^0_\sigma(\mathbb{R}_+)} \lesssim \varepsilon^2\int_T t^{2\sigma}\big(|t^{-1}u_2(t,1)|^2 + |\partial_tu_2(t,1)|^2\big)dt \lesssim$$

$$\varepsilon^2\int_{-\infty}^{+\infty}(1+t^2)^\sigma\big(|u_2(t,1)|^2 + |\partial_tu_2(t,1)|^2\big)dt \lesssim \varepsilon^2\|u_2\|^2_{H^2_\sigma(\Pi)}.$$

In the same way we obtain

$$\|\chi_\varepsilon \mathcal{N}_0 u_2\|^2_{V^0_\sigma(\mathbb{R}_+)} \lesssim \|u_2\|^2_{H^2_{\sigma-1}(\Pi)}, \quad \|\chi_\varepsilon \mathcal{N}_0 u_2\|^2_{V^0_\sigma(\mathbb{R}_+)} \lesssim \varepsilon^2 \|u_2\|^2_{H^2_\sigma(\Pi)}.$$

The first terms in right hand part of (3.46) can be estimated via (3.11),(3.4)-(3.6) as follows:

$$\|\chi_\varepsilon P_1 \mathcal{L} u_2\|^2_{V^0_\sigma(\mathbb{R}_+)}$$

$$\lesssim \int_T t^{2\sigma} t^{-2} (|\nabla^2 u_2| + |\nabla u_2|)^2 dz dt \lesssim \varepsilon^2 \|u_2\|^2_{H^2_\sigma(\Pi)},$$

and

$$\|\chi_\varepsilon P_1 \mathcal{L} u_2\|^2_{V^0_\sigma(\mathbb{R}_+)} \lesssim \|u_2\|^2_{H^2_{\sigma-1}(\Pi)}.$$

This proves (3.43).

2. Now let us prove the estimate (3.44) for $\mathfrak{B}^\varepsilon_{21}$. We have, via (3.30),

$$\|\mathfrak{B}^\varepsilon_{21} u_1\|^2_{\mathcal{R}^0_\sigma} = \|\chi_\varepsilon (\mathcal{N}_0 + \mathcal{N}_1) u_1\|^2_{L^2_\sigma(\Pi)} + \|\chi_\varepsilon \mathcal{N}_0 u_1\|^2_{H^{1/2}_\sigma(\mathbb{R})} + \|\chi_\varepsilon \mathcal{N}_1 u_1\|^2_{H^{1/2}_\sigma(\mathbb{R})}$$

$$\lesssim \|\chi_\varepsilon \mathcal{N}_0 u_1\|^2_{H^1_\sigma(\Pi)} + \|\chi_\varepsilon \mathcal{N}_1 u_1\|^2_{H^1_\sigma(\Pi)}. \tag{3.47}$$

Consider the first term on the right hand side of (3.47).

$$\|\chi_\varepsilon \mathcal{N}_0 u_1\|^2_{H^1_\sigma(\Pi)} \lesssim \|\chi_\varepsilon \phi (1 + \phi_0'^2)^{-1/2} u_1\|^2_{H^1_\sigma} + \|\chi_\varepsilon t^2 \phi \phi_0' (1 + \phi_0'^2)^{-1} \partial_t u_1\|^2_{H^1_\sigma}. \tag{3.48}$$

Considering the last term in (3.48),

$$\|\chi_\varepsilon t^2 \phi \phi_0' (1 + \phi_0'^2)^{-1} \partial_t u_1\|^2_{H^1_\sigma} \lesssim$$

$$\int_T t^{2\sigma} \big(|\phi_0' \partial_t u_1|^2 + |\phi' \partial_t^2 u_1|^2 + |(\partial_t u_1) \partial_t t^2 \phi \phi_0' (1 + \phi_0'^2)^{-1}|^2 \big) dt,$$

where we have used condition (3.4). Now using (3.5) and (3.6) we get

$$|\phi_0' \partial_t u_1|^2 + |\phi_0' \partial_t^2 u_1|^2 + |(\partial_t u_1) \partial_t t^2 \phi \phi_0' (1 + \phi_0'^2)^{-1}|^2 \lesssim$$
$$t^{-2} |\partial_t u_1|^2 + t^{-2} |\partial_t^2 u_1|^2, \; t > \varepsilon^{-1},$$

and consequently

$$\|\chi_\varepsilon t^2 \phi \phi_0' (1 + \phi_0'^2)^{-1} \partial_t u_1\|^2_{H^1_\sigma} \lesssim \|u_1\|^2_{V^2_\sigma(\mathbb{R}_+)}.$$

Consider now the first term on the right hand side of (3.48). We have

$$\|\chi_\varepsilon \phi (1 + \phi_0'^2)^{-1/2} u_1\|^2_{H^1_\sigma} \lesssim \int_{\Pi_T} t^{2\sigma} \big(|\phi u_1|^2 + |\phi \partial_t u_1|^2 + |u_1 \partial_t \phi (1 + \phi_0'^2)^{-1/2}|^2 \big) dt dz \lesssim$$

$$\int_{\Pi_T} t^{2\sigma} \big(t^{-4} |u_1|^2 + t^{-4} |\partial_t u_1|^2 + t^{-6} |u_1|^2 \big) dt dz \lesssim \|u_1\|^2_{V^2_\sigma(\mathbb{R}_+)}.$$

The second term on the right hand side of (3.47) can be estimated in the same way. Consequently, assembling,

$$\|\mathfrak{B}^\varepsilon_{21} u_1\|^2_{\mathcal{R}^0_\sigma} \lesssim \int_T t^{2\sigma} \big(t^{-4} |u_1|^2 + t^{-2} |\partial_t u_1|^2 + |\partial_t^2 u_1|^2 \big) dt \lesssim \|u_1\|^2_{V^2_\sigma(\mathbb{R}_+)},$$

yielding (3.44).

Remark 3.2. If we separate the main order terms from the operator $\mathfrak{B}^\varepsilon_{21}$ then for the remainder, i.e. operator $\tilde{\mathfrak{B}}^\varepsilon_{21} = \chi_\varepsilon(\mathfrak{N}_0 + \mathfrak{N}_1, \mathfrak{N}_0, \mathfrak{N}_1)$ we will have better estimate, namely

$$\|\tilde{\mathfrak{B}}^\varepsilon_{21}\|_{V^2_{\sigma-1}(\mathbb{R}_+)\to\mathcal{R}^0_\sigma(\Pi)} \lesssim 1, \tag{3.49}$$

which can be proved in the same way.

3. Now let us proof (3.42). We have, via (3.28),

$$\|\mathfrak{B}^\varepsilon_{11}u_1\|^2_{V^0_\sigma(\mathbb{R}_+)} \lesssim \int_T t^{2\sigma}\Big(\left|\frac{4\omega}{\kappa^2 t^4}u_1\right|^2 + |\mathfrak{N}_0 u_1|^2 + |\mathfrak{N}_1 u_1|^2\Big)\,dt. \tag{3.50}$$

Using (3.17), (3.4) and (3.5) we get

$$|\mathfrak{N}_0 u_1| \lesssim t^{-3}|u_1| + t^{-2}|\partial_t u_1|, \tag{3.51}$$

and it follows from (3.18), (3.4) and (3.5) that

$$|\mathfrak{N}_1 u_1| \lesssim t^{-3}|u_1| + t^{-2}|\partial_t u_1|. \tag{3.52}$$

As a result

$$\|\mathfrak{B}^\varepsilon_{11}u_1\|^2_{V^0_\sigma(\mathbb{R}_+)} \lesssim \int_T t^{2\sigma}t^{-2}\big(t^{-4}|u_1|^2 + t^{-2}|\partial_t u_1|^2\big)\,dt \lesssim \|u_1\|^2_{V^2_{\sigma-1}(\mathbb{R}_+)}, \tag{3.53}$$

and

$$\|\mathfrak{B}^\varepsilon_{11}u_1\|^2_{V^0_\sigma(\mathbb{R}_+)} \lesssim \varepsilon^2\|u_1\|^2_{V^2_\sigma(\mathbb{R}_+)}.$$

4. The estimate (3.45) can be obtained in the same way. Indeed

$$\|\mathfrak{B}^\varepsilon_{22}u_2\|_{\mathcal{R}^0_\sigma(\Pi)} \lesssim \|\chi_\varepsilon(P_2\mathcal{L} + \mathcal{N}_0 + \mathcal{N}_1)u_2\|^2_{L^2_\sigma(\Pi)} + \|\chi_\varepsilon\mathcal{N}_0 u_2\||^2_{H^{1/2}_\sigma(\mathbb{R})} + \|\chi_\varepsilon\mathcal{N}_1 u_2\|^2_{H^{1/2}_\sigma(\mathbb{R})},$$

and since $u_2 \in H^2_\sigma(\Pi)$ and all the coefficients are decaying at least as t^{-1}, we easily obtain the desired estimates. □

Remark 3.3. Clearly, the operators $\mathfrak{B}^\varepsilon_{ij}$, $i,j = 1,2$, depend analytically on ω, ρ_0 and ρ_1.

Corollary 3.4. *Operator $\mathbf{A}^\varepsilon_\sigma$, defined by the matrix operator*

$$\mathbf{A}^\varepsilon = \begin{pmatrix} \mathbf{A}_1 + \mathfrak{B}^\varepsilon_{11} & \mathfrak{B}^\varepsilon_{12} \\ \mathfrak{B}^\varepsilon_{21} & \mathbf{A}_2 + \mathfrak{B}^\varepsilon_{22} \end{pmatrix} \tag{3.54}$$

is an isomorphism from $V^2_\sigma(\mathbb{R}_+) \times \mathcal{D}^2_\sigma(\Pi)$ to $V^0_\sigma(\mathbb{R}_+) \times \mathcal{R}_\sigma(\Pi)$ for $\sigma \neq 1 \pm Re(1/4 - \lambda)^{1/2}$, $\lambda = \frac{2}{\kappa}(\rho_0 + \rho_1)$ and ε small enough.

Remark 3.4. Let us clarify the meaning of ε being small enough. In fact ε should satisfy the estimate

$$\varepsilon \le c\left((\sigma-1)^2 - \left(\mathrm{Re}\,(1/4-\lambda)^{1/2}\right)^2\right), \tag{3.55}$$

where c depends only on K. In particular Corollary 3.4 implies that there is a solution u to the problem (3.12)-(3.14) in Π_{T+1} for $T > \frac{1}{\kappa\varepsilon_0}$. This solution can be represented in the form

$$u(t,z) = u_1(t) + u_2(t,z), \quad u_1(t) = \int_0^1 u(t,z)dz,$$

and

$$\|u_1\|_{V^2_\sigma(\mathbb{R}_{T+1})} + \|u_2\|_{H^2_\sigma(\Pi_{T+1})} \le c\left(\|F\|_{L^2_\sigma(\Pi_T)} + \|G\|_{H^{1/2}_\sigma(\mathbb{R}_T)}\right), \tag{3.56}$$

where c does not depend on F and G, and satisfies the estimate

$$c \le c(K,\sigma)\left((\sigma-1)^2 - \left(\mathrm{Re}\,(1/4-\lambda)^{1/2}\right)^2\right)^{-1}. \tag{3.57}$$

In other words if $|\sigma| < K$ and (2.4),(3.37) are satisfied, then c does not depend on σ, ρ_0, ρ_1 either (it depends only on K).

We next describe the asymptotic behavior of the solution of (3.12)-(3.14) with a special right hand side.

Theorem 3.5. *Let $F(t,z) = p(z)t^\alpha \ln^m t$, $G_0(t) = b_0 t^\alpha \ln^m t$, $G_1(t) = b_1 t^\alpha \ln^m t$, where α, b_1 and b_2 are complex-valued constants, $m = 0, 1, ..,$ and $p \in L^2(0,1)$. Then, for sufficiently small ε, there exists a solution of the problem (3.12)-(3.14) u, such that*

$$u(t,z) = u_1(t) + u_2(t,z), \quad \int_0^1 u_2(t,z)dz = 0, \quad t > T,$$
$$u_1(t) = \hat{u}_1(t) + \tilde{u}_1(t), \;\; \tilde{u}_1 \in V^2_\sigma(\mathbb{R}_T), \;\; \forall \sigma < 1/2 - Re\,\alpha, \tag{3.58}$$
$$u_2(t,z) = \hat{u}_2(t,z) + \tilde{u}_2(t,z), \;\; \tilde{u}_2 \in H^2_\sigma(\Pi_T), \;\; \forall \sigma < 1/2 - Re\,\alpha.$$

Here

$$\hat{u}_1(t) = t^{\alpha+2} Q(\ln t),$$
$$\hat{u}_2(t,z) = t^\alpha \ln^m t P(z) + t^\alpha Q(\ln t) P_1(z) + t^\alpha Q'(\ln t) P_2(z),$$

where $P, P_1, P_2 \in H^2(0,1)$, and

1. if $\alpha \ne -\frac{3}{2} \pm \sqrt{\frac{1}{4} - \lambda}$ then

$$Q(\tau) = \sum_{k=0}^{m} \frac{a_k}{k!}\tau^k, \tag{3.59}$$

where a_k are constants;

2. if $\alpha = -\frac{3}{2} \pm \sqrt{\frac{1}{4} - \lambda}$ and $\lambda \ne 1/4$ then

$$Q(\tau) = \sum_{k=1}^{m+1} \frac{a_k}{k!}\tau^k, \tag{3.60}$$

where a_k are constants;

3. if $\alpha = -\frac{3}{2} \pm \sqrt{\frac{1}{4} - \lambda}$ *and* $\lambda = 1/4$ *then*

$$Q(\tau) = \frac{a_{m+2}}{(m+2)!}\tau^{m+2}, \tag{3.61}$$

where a_{m+2} *is a constant.*

Proof. The statement of the theorem is equivalent to the existence of a solution of the equation

$$\mathbf{A}\mathbf{u} = \mathfrak{F}, \; t > T, \tag{3.62}$$

where $\mathbf{A}$ is the matrix block operator appearing in left hand side of (3.26)-(3.27), $\mathbf{u} = (u_1, u_2)$, and

$$\mathfrak{F} = (\mathfrak{F}_1, \mathfrak{F}_2), \; \mathfrak{F}_1 = (p_1 - b_0 - b_1)t^\alpha \ln^m t, \; \mathfrak{F}_2 = ((p_2 + b_0 + b_1), b_0, b_1)\, t^\alpha \ln^m t,$$

$$p_1 = \int_0^1 p(z)dz, \; p_2(z) = p(z) - p_1,$$

$$\mathbf{u} = (\hat{u}_1, \hat{u}_2) + \widetilde{\mathbf{u}}, \quad \widetilde{\mathbf{u}} \in V^2_\sigma(\mathbb{R}_T) \times \mathcal{D}^2_\sigma(\Pi_T), \; \forall \sigma < 1/2 - \operatorname{Re}\alpha. \tag{3.63}$$

Let us notice that

$$(\hat{u}_1, \hat{u}_2) \in V^2_{\sigma-1}(\mathbb{R}_T) \times H^2_{\sigma-1}(\Pi_T), \; \forall \sigma < 1/2 - \operatorname{Re}\alpha, \tag{3.64}$$

and does not belong to $V^2_{-\frac{1}{2}-\operatorname{Re}\alpha}(\mathbb{R}_T) \times H^2_{-\frac{1}{2}-\operatorname{Re}\alpha}(\Pi_T)$, so (3.63) indeed delivers an asymptotics of solution $\mathbf{u}$.

The existence of the above solution follows from the existence of the solution of the following problem,

$$\mathbf{A}^\varepsilon \widetilde{\mathbf{u}} = -\chi_1 \mathbf{A}^\varepsilon \left(\hat{u}_1, \hat{u}_2\right) + \chi_1 \mathfrak{F}, \tag{3.65}$$

in the space $V^2_\sigma(\mathbb{R}_+) \times \mathcal{D}^2_\sigma(\Pi)$ (see Corollary 3.4), since systems (3.62) and (3.65) coincide for $t > T$.

It remains to verify that the right hand side in (3.65) belongs to the space $V^0_\sigma(\mathbb{R}_+) \times \mathcal{R}^0_\sigma(\Pi)$. For the first component of $\chi_1 \mathbf{A}^\varepsilon \left(\hat{u}_1, \hat{u}_2\right) - \chi_1 \mathfrak{F}$ which we denote I_1, we have

$$I_1 = \chi_1 \mathbf{A}_1 t^{\alpha+2} Q(\ln t) + \mathfrak{B}^\varepsilon_{11} t^{\alpha+2} Q(\ln t) + \mathfrak{B}^\varepsilon_{12} \hat{u}_2 - \chi_1 (p_1 - b_0 - b_1) t^\alpha \ln^m t =$$

$$\chi_1 \big((\partial_t^2 + \lambda t^{-2}) t^{\alpha+2} Q(\ln t) - (p_1 - b_0 - b_1) t^\alpha \ln^m t \big) + \mathfrak{B}^\varepsilon_{11} t^{\alpha+2} Q(\ln t) + \mathfrak{B}^\varepsilon_{12} \hat{u}_2. \tag{3.66}$$

The second and third terms in (3.66) are clearly in $V^0_\sigma(\mathbb{R}_+)$, see (3.64) and (3.53), (3.43). As for the first term in (3.66), we choose Q to make it disappear, i.e. Q has to be a solution of the equation,

$$(\partial_t^2 + \lambda t^{-2}) t^{\alpha+2} Q(\ln t) = (p_1 - b_0 - b_1) t^\alpha \ln^m t. \tag{3.67}$$

This equation can be easily solved, and one can directly verify that Q has the form (3.59)-(3.61) (depending on the parameters α and λ).

In particular, if $\alpha \neq -\frac{3}{2} \pm \sqrt{\frac{1}{4}-\lambda}$ we have

$$a_m = m!\,((\alpha+2)(\alpha+1)-\lambda)^{-1}\,(p_1-b_0-b_1),$$

$$a_k = ((\alpha+2)(\alpha+1)-\lambda)^{-1}\,((-2\alpha-3)a_{k+1}-a_{k+2})\,,\ k=m-1,..0.$$

If $\alpha = -\frac{3}{2} \pm \sqrt{\frac{1}{4}-\lambda}$ and $\lambda \neq 1/4$, then

$$a_{m+1} = m!\,(2(\alpha+2)-1)^{-1}\,(p_1-b_0-b_1),$$

$$a_k = -\,(2(\alpha+2)-1)^{-1}\,a_{k+1},\ k=m,m-1,..1.$$

If $\alpha = -\frac{3}{2} \pm \sqrt{\frac{1}{4}-\lambda}$ and $\lambda = 1/4$, then

$$a_{m+2} = m!(p_1-b_0-b_1).$$

As a result we conclude that $I_1 \in V^0_\sigma(\mathbb{R}_+)$.

Now let us estimate the second component of $\chi_1\mathbf{A}^\varepsilon\,(\hat{u}_1,\hat{u}_2)-\mathfrak{F}$ which we denote by I_2. We have

$$I_2 = \chi_1\mathfrak{B}^\varepsilon_{21}\hat{u}_1(t)+\chi_1\mathbf{A}_2\hat{u}_2(t,z)-\chi_1\mathfrak{F}_2+\chi_1\mathfrak{B}^\varepsilon_{22}\hat{u}_2(t,z). \tag{3.68}$$

Clearly the last term in (3.68) belongs to $\mathcal{R}^0_\sigma(\Pi)$, see (3.64) and (3.45). Let us evaluate the remaining terms. We have

$$\mathfrak{B}^\varepsilon_{21} = \chi_\varepsilon(\mathcal{N}_0+\mathcal{N}_1,\mathcal{N}_0,\mathcal{N}_1)$$

$$=\chi_\varepsilon\left(\frac{2}{t}\partial_t-\frac{\lambda}{t^2},-\frac{2\phi_0''(0)}{\kappa t}\partial_t-\frac{2\rho_0}{\kappa t^2},\frac{2\phi_1''(0)}{\kappa t}\partial_t-\frac{2\rho_1}{\kappa t^2}\right)+\chi^\varepsilon(\mathfrak{N}_0+\mathfrak{N}_1,\mathfrak{N}_0,\mathfrak{N}_1),$$

$$\mathbf{A}_2 u = (\partial_t^2 u+\partial_z^2 u,-\partial_z u|_{z=0},\partial_z u|_{z=1}).$$

Therefore,

$$\chi_1\mathfrak{B}^\varepsilon_{21}\hat{u}_1(t) =$$

$$\chi_\varepsilon\left(2(\alpha+2)-\lambda,\frac{2}{\kappa}\left(-\phi_0''(0)(\alpha+2)-\rho_0\right),\frac{2}{\kappa}\left(\phi_1''(0)(\alpha+2)-\rho_1\right)\right)t^\alpha Q(\ln t)$$

$$+\left(2,-2\phi_0''(0)\kappa^{-1},2\phi_1''(0)\kappa^{-1}\right)t^\alpha Q'(\ln t)+\chi^\varepsilon(\mathfrak{N}_0+\mathfrak{N}_1,\mathfrak{N}_0,\mathfrak{N}_1)t^{\alpha+2}Q(\ln t), \tag{3.69}$$

$$\chi_1\mathbf{A}_2\hat{u}_2(t,z) = \chi_1\Big(P_1''(z),-P_1'(0),P_1'(1)\Big)t^\alpha Q(\ln t)$$

$$+\chi_1\Big(P_2''(z),-P_2'(0),P_2'(1)\Big)t^\alpha Q'(\ln t)+\chi_1\Big(P''(z),-P'(0),P'(1)\Big)t^\alpha\ln^m t$$

$$+\chi_1\left(\partial_t^2(t^\alpha\ln^m tP(z)+t^\alpha Q(\ln t)P_1(z)+t^\alpha Q'(\ln t)P_2(z)),0,0\right). \tag{3.70}$$

$$-\chi_1\mathfrak{F}_2 = \chi_1\Big(-p_2-b_0-b_1,-b_0,-b_1\Big)t^\alpha\ln^m t. \tag{3.71}$$

Clearly the last term in (3.70) is in $\mathcal{R}^0_\sigma(\Pi)$, see (3.64). The same is true for the last term in (3.69), see (3.64) and (3.49). We need to pick up P,P_1 and P_2 in such a way that the sum of the remaining terms in (3.69),(3.70) and (3.71) disappears. We achieve this by putting

$$P_2(z) = 2\phi_0''(0)\kappa^{-1}\left(\frac{(z-1)^2}{2} - \frac{1}{6}\right) - 2\phi_1''(0)\kappa^{-1}\left(\frac{z^2}{2} - \frac{1}{6}\right),$$

$$P(z) = b_0\left(\frac{(z-1)^2}{2} - \frac{1}{6}\right) + b_1\left(\frac{z^2}{2} - \frac{1}{6}\right) + \widetilde{P}(z),$$

$$P_1(z) = \frac{2}{\kappa}\left(\phi_0''(0)(\alpha+2) + \rho_0\right)\left(\frac{(z-1)^2}{2} - \frac{1}{6}\right) - \frac{2}{\kappa}\left(\phi_1''(0)(\alpha+2) - \rho_1\right)\left(\frac{z^2}{2} - \frac{1}{6}\right),$$

where $\tilde{P} \in H^2(0,1)$ is a unique solution of the problem

$$\widetilde{P}''(z) = p_2(z),\ z \in (0,1),\ \widetilde{P}(0) = \widetilde{P}(1) = 0,\ \int_0^1 \widetilde{P}(z)dz = 0.$$

As result we conclude that the sum of (3.69)-(3.71) is in $\mathcal{R}^0_\sigma(\Pi)$, and as a result $I_2 \in \mathcal{R}^0_\sigma(\Pi)$. □

As a corollary of the proof of the above theorem we have,

Theorem 3.6. *There exist solutions v^+ and v^- of the homogeneous problem* (3.12)-(3.14) *for small enough ε, such that*

$$v^\pm(t,z) = v_1^\pm(t) + v_2^\pm(t,z),\ \int_0^1 v_2^\pm(t,z)dz = 0,\ t > T,$$

where

$$v_1^\pm(t) = \hat{v}_1^\pm(t) + \tilde{v}_1^\pm(t),\ \tilde{v}_1^\pm \in V^2_{\sigma_\pm}(\mathbb{R}_T),\ \forall \sigma_\pm < 5/2 - Re\,\Lambda^\pm, \tag{3.72}$$

$$v_2^\pm(t,z) = \hat{v}_2^\pm(t,z) + \tilde{v}_2^\pm(t,z),\ \tilde{v}_2^\pm \in H^2_{\sigma_\pm}(\Pi_T),\ \forall \sigma_\pm < 5/2 - Re\,\Lambda^\pm.$$

Here

$$\hat{v}_1^\pm(t) = t^{\Lambda^\pm} Q^\pm(\ln t),$$

$$\hat{v}_2^\pm(t,z) = t^{\Lambda^\pm - 2} Q^\pm(\ln t) P_1^\pm(z) + t^{\Lambda^\pm - 2}(Q^\pm)'(\ln t) P_2(z),$$

where

$$P_2(z) = 2\phi_0''(0)\kappa^{-1}\left(\frac{(z-1)^2}{2} - \frac{1}{6}\right) - 2\phi_1''(0)\kappa^{-1}\left(\frac{z^2}{2} - \frac{1}{6}\right),$$

$$P_1^\pm(z) = \frac{2}{\kappa}\left(\phi_0''(0)\Lambda^\pm + \rho_0\right)\left(\frac{(z-1)^2}{2} - \frac{1}{6}\right) - \frac{2}{\kappa}\left(\phi_1''(0)\Lambda^\pm - \rho_1\right)\left(\frac{z^2}{2} - \frac{1}{6}\right),$$

and

1. If $\lambda \neq 1/4$ then

$$\Lambda^\pm = 1/2 \pm i(\lambda - 1/4)^{1/2},\ Q^\pm(\tau) = 1;$$

2. If $\lambda = 1/4$ then

$$\Lambda^\pm = 1/2,\ Q^+(\tau) = 1,\ Q^-(\tau) = \tau;$$

Proof. We follow the pattern of Theorem 3.5. We define the remainder $(\tilde{v}_1^{\pm}(t), \tilde{v}_1^{\pm}(t))$ as a solution to the the problem

$$\mathbf{A}^{\varepsilon}(\tilde{v}_1^{\pm}(t), \tilde{v}_2^{\pm}(t)) = -\chi_1 \mathbf{A}^{\varepsilon}\left(\hat{v}_1^{\pm}(t), \hat{v}_2^{\pm}(t)\right), \tag{3.73}$$

Notice that $\chi_1 \mathbf{A}^{\varepsilon_0}\left(\hat{v}_1^{\pm}(t), \hat{v}_2^{\pm}(t)\right) \in V_{\sigma}^0(\mathbb{R}_+) \times \mathcal{R}_{\sigma}^0(\Pi),\ \forall \sigma < 5/2 - \operatorname{Re}\Lambda^{\pm}$, since

$$(\partial_t^2 - \lambda t^{-2}) t^{\lambda^{\pm}} Q^{\pm}(\ln t) = 0.$$

Consequently there is a solution $(\tilde{v}_1^{\pm}(t), \tilde{v}_1^{\pm}(t)) \in V_{\sigma_{\pm}}^2(\mathbb{R}_T) \times H_{\sigma_{\pm}}^2(\Pi_T),\ \forall \sigma_{\pm} < 5/2 - \operatorname{Re}\Lambda^{\pm}$.

□

Remark 3.5. There are many other solutions of the homogeneous problem (3.12)-(3.14). Let us demonstrate how we can we fix these solutions. Consider the case λ is real and $\lambda \geq 1/4$. Then $v^{\pm}$ can be chosen in such a way that their norms remain bounded with respect to λ and ω^2. Indeed for main terms $\hat{v}_1^{\pm}(t)$ and $\hat{v}_1^{\pm}(t)$ it follows from the explicit formulae. Let us define

$$(\tilde{v}_1^{\pm}(t), \tilde{v}_1^{\pm}(t)) := -\left(\mathbf{A}^{\varepsilon_0}_{5/2 - \operatorname{Re}\Lambda^{\pm} - \frac{1}{K}}\right)^{-1} \chi_1 \mathbf{A}^{\varepsilon}\left(\hat{v}_1^{\pm}(t), \hat{v}_2^{\pm}(t)\right), \tag{3.74}$$

where ε_0 is chosen to satisfy (3.55). The choice $\sigma_{\pm} = 5/2 - \operatorname{Re}\Lambda^{\pm} - \frac{1}{K}$ ensures that condition (3.37) is satisfied and we can use Remark 3.4 to estimate the remainders $\tilde{v}_1^{\pm}(t)$ and $\tilde{v}_1^{\pm}(t)$. As result our special solutions $v^{\pm} \in V_{p_{\pm}}^2(\mathbb{R}_T) \times H_{p_{\pm}}^2(\Pi_T),\ \forall p_{\pm} < 3/2 - \operatorname{Re}\Lambda^{\pm}$ are determined uniquely by our construction and the remainders are bounded in $V_{\sigma_{\pm}}^2(\mathbb{R}_T) \times H_{\sigma_{\pm}}^2(\Pi_T),\ \forall \sigma_{\pm} < 5/2 - \operatorname{Re}\Lambda^{\pm} - \frac{1}{K}$. The same result remains true if λ has a small imaginary part, say $\lambda \in \{|\mathrm{Im}\lambda| < K^{-1}, \operatorname{Re}\lambda \geq 1/4\}$.

3.2 Problem with additional smoothness of coefficients.

In this subsection we impose additional conditions on the functions $\phi_j,\ j = 1, 2$ describing the cusp. Namely we suppose that for all $N = 0, 1, ...$ the following holds

$$\left|\partial_{x_1}^k\left(\phi_j(x_1) - \sum_{n=2}^{N} b_n^{(j)} x_1^n\right)\right| \leq C_N x_1^{N+1-k},\ k = 0, 1, 2,\ b_2^{(j)} = \phi_j''(0)/2. \tag{3.75}$$

Under the above conditions we have the following refined version of Theorem 3.5:

Theorem 3.7. *Let $F(t, z) = p(z) t^{\alpha} \ln^m t$, $G_0(t) = b_0 t^{\alpha} \ln^m t$, $G_1(t) = b_1 t^{\alpha} \ln^m t$, where α, b_1 and b_2 are complex-valued constants $m = 0, 1, ..$ and $p \in L^2(0, 1)$. Then, for sufficiently small ε and any $M = 0, 1, ...$, there exists a solution of the problem* (3.12)-(3.14) *u, such that*

$$u(t, z) = u_1(t) + u_2(t, z),\ \int_0^1 u_2(t, z) dz = 0, \quad t > T,$$

$$u_1(t) = \hat{u}_1(t) + \tilde{u}_1(t),\ \tilde{u}_1 \in V^2_\sigma(\mathbb{R}_T),\ \forall\sigma < 1/2 - Re\alpha + M, \tag{3.76}$$
$$u_2(t,z) = \hat{u}_2(t,z) + \tilde{u}_2(t,z),\ \tilde{u}_2 \in H^2_\sigma(\Pi_T),\ \forall\sigma < 1/2 - Re\alpha + M.$$

Here

$$\hat{u}_1(t) = \sum_{n=0}^{M} t^{\alpha+2-n} Q_n(\ln t),$$
$$\hat{u}_2(t,z) = \sum_{n=0}^{M} t^{\alpha-n} P_n(z,\ln t),$$

where $P_n(z,\tau)$ *is polynomial in* τ *with coefficients in* $H^2(0,1)$, *and* $Q_n(\tau)$ *is a polynomial.*

Proof. The proof follows immediately, since we under assumptions (3.75) we can iterate the procedure described in Theorem 3.5. □

The following theorem is in turn a refined version of Theorem 3.6. Here we assume that $\lambda \in \mathbb{R}$ in order to formulate more precise results.

Theorem 3.8. *There exist solutions* v^+ *and* v^- *of the homogeneous problem* (3.12)-(3.14) *for small enough* ε, *such that*

$$v^\pm(t,z) = v_1^\pm(t) + v_2^\pm(t,z),\ \int_0^1 v_2^\pm(t,z)dz = 0,\ t > T,$$

and for any $M = 0, 1, ...$

$$v_1^\pm(t) = \hat{v}_1^\pm(t) + \tilde{v}_1^\pm(t),\ \tilde{v}_1^\pm \in V^2_{\sigma_\pm}(\mathbb{R}_T),\ \forall\sigma_\pm < 5/2 - Re\,\Lambda^\pm + M, \tag{3.77}$$
$$v_2^\pm(t,z) = \hat{v}_2^\pm(t,z) + \tilde{v}_2^\pm(t,z),\ \tilde{v}_2^\pm \in H^2_{\sigma_\pm}(\Pi_T),\ \forall\sigma < 5/2 - Re\,\Lambda^\pm + M,$$
$$\Lambda^\pm = 1/2 \pm i(\lambda - 1/4)^{1/2}.$$

Further,
1. If $\lambda > 1/4$ *then*

$$\hat{v}_1^\pm(t) = \sum_{n=0}^{M} t^{\Lambda^\pm - n} q_n,\quad \hat{v}_2^\pm(t,z) = \sum_{n=0}^{M} t^{\Lambda^\pm - n-2} P_n(z),$$

where
$P_n(z)$ *are polynomials and* q_n *are some constants,* q_0 *is arbitrary;*
2. If $\lambda = 1/4$ *then*

$$\hat{v}_1^+(t) = \sum_{n=0}^{M} t^{\frac{1}{2}-n} a_n,\quad \hat{v}_2^+(t,z) = \sum_{n=0}^{M} t^{-\frac{3}{2}-n} P_n(z),$$

$$\hat{v}_1^-(t) = \sum_{n=0}^{M} t^{\frac{1}{2}-n} b_n(\ln t),\quad \hat{v}_2^-(t,z) = \sum_{n=0}^{M} t^{-\frac{3}{2}-n} Q_n(z,\ln t),$$

$P_n(z)$ are polynomials, a_n are some constants, a_0 is arbitrary, $b_n(\tau)$ are linear functions of τ, moreover $b_0(\tau) = const \times \tau$. Finally $Q_n(z,\tau)$ are polynomials in z and linear in τ;
3. If $\lambda < 1/4$ then

$$\hat{v}_1^+(t) = \sum_{n=0}^{M} t^{\frac{1}{2}-\sqrt{\frac{1}{4}-\lambda}-n} a_n, \quad \hat{v}_2^+(t,z) = \sum_{n=0}^{M} t^{-\frac{3}{2}-\sqrt{\frac{1}{4}-\lambda}-n} P_n(z),$$

$$\hat{v}_1^-(t) = \sum_{n=0}^{M} t^{\frac{1}{2}+\sqrt{\frac{1}{4}-\lambda}-n} b_n(\ln t), \quad \hat{v}_2^-(t,z) = \sum_{n=0}^{M} t^{-\frac{3}{2}+\sqrt{\frac{1}{4}-\lambda}-n} Q_n(z,\ln t),$$

$P_n(z)$ are polynomials, a_n are some constants, a_0 is arbitrary, $b_n(\tau)$ are linear functions of τ, moreover $b_0(\tau) = const$. Finally $Q_n(z,\tau)$ are polynomials in z and linear in τ.

Proof. In the Theorem 3.6 we already proved the existence of $v^\pm$ and constructed the main term of asymptotic expansion. Now the existence of lower order terms in the asymptotic expansion of $v^\pm$ follows from Theorem 3.7. □

4 Asymptotics near cuspidal point and the Fredholm property

4.1 Asymptotics near cuspidal point

Returning back to variables (x_1, x_2) we obtain the local solution to the problem (3.1) for ε small enough, which can be represented as

$$u(x_1,x_2) = u_1(x_1) + u_2(x_1,x_2), \quad 0 < x_1 < \varepsilon, \quad 0 < x_2 < \phi(x_1), \tag{4.1}$$

where, see (3.2),

$$u_1(x_1) = \phi(x_1)^{-1} \int_0^{\phi(x_1)} u(x_1,x_2)dx_2, \; u_2 = P_2 u := u - u_1,$$

and from (3.56) we get,

$$\|u_1\|_{V^2_{2-\sigma}(\Omega_{\varepsilon/2})} + \|u_2\|_{\mathcal{W}^2_{1-\sigma/2}(\Omega_{\varepsilon/2})} \lesssim \|f\|_{\mathcal{W}^0_{1-\sigma/2}(\Omega_\varepsilon)} + \|g\|_{\mathcal{W}^{1/2}_{1-\sigma/2}(S_0\cup S_1)} .$$

Here we have used the notation $\mathcal{W}^l_\gamma := \mathcal{W}^l_{0,\gamma}$, see (2.2), and

$$\|u : V^l_\gamma(\Omega)\|^2 = \sum_{|\delta|\le l} \int_\Omega |x_1|^{2(\gamma-l+|\delta|)} |\partial_x^\delta u|^2 dx.$$

Let us consider the space

$$\mathcal{V}^2_\gamma(\Omega) = \{u \in V^2_{2\gamma}(\Omega) : P_2 u \in \mathcal{W}^2_\gamma(\Omega \cap B_\varepsilon)\},$$

with the norm

$$\|u\|_{\mathcal{V}^2_\gamma(\Omega)} = \|u\|_{V^2_{2\gamma}(\Omega)} + \|P_2 u\|_{\mathcal{W}^2_\gamma(\Omega\cap B_\varepsilon)}. \tag{4.2}$$

Obviously the space does not depend on $\varepsilon > 0$ and the norms are equivalent.

As a direct consequence of Corollary 3.4 and Remark 3.4 we have

Theorem 4.1. *Let* $\{f, g\} \in \mathcal{W}^0_\gamma(\Omega) \times \mathcal{W}^{1/2}_\gamma(\partial\Omega)$ *and* $\gamma \neq 1/2 \pm 1/2 Re\sqrt{1/4-\lambda}$. *Then there exists a local solution* $u \in \mathcal{V}^2_\gamma(\Omega_\varepsilon)$ *to the problem* (3.1) *for* ε *small enough, and*

$$\|u\|_{\mathcal{V}^2_\gamma(\Omega_\varepsilon)} \le c\left(\|f\|_{\mathcal{W}^0_\gamma(\Omega_{2\varepsilon})} + \|g\|_{\mathcal{W}^{1/2}_\gamma(S_{2\varepsilon})}\right). \tag{4.3}$$

Moreover, if condition (3.37) *holds, then the constant* c *in* (4.3) *can be chosen independently of* ω^2, q_0 *and* q_1.

The next theorem follows from Theorem 4.1 and Theorem 2.1:

Theorem 4.2. *Let* $\gamma \neq 1/2 \pm 1/2 Re\sqrt{1/4-\lambda}$, *then there exists* $\varepsilon_0 > 0$ *such that, for any* $0 < \varepsilon < \varepsilon_0$, *every solution of* (2.1) *satisfies the estimate*

$$\|u\|_{\mathcal{V}^2_\gamma(\Omega_{\varepsilon/2})} \le c\left(\|f\|_{\mathcal{W}^0_\gamma(\Omega_\varepsilon)} + \|g\|_{\mathcal{W}^{1/2}_\gamma(\partial\Omega\cap B_\varepsilon)} + \|u\|_{L_2(\Omega_\varepsilon\setminus B_{\varepsilon/2})}\right). \tag{4.4}$$

Moreover, if condition (3.37) *holds, then the constant* c *in* (4.3) *can be chosen independently of* ω^2, q_0 *and* q_1.

The next theorem follows from Theorem 3.6 via change of variables (3.2).

Theorem 4.3. *There exist solutions* v^+ *and* v^- *of the homogeneous problem* (3.1) *for small enough* ε, *such that*

$$\boldsymbol{v}^\pm(x) = v_1^\pm(x_1) + v_2^\pm(x), \quad \int_0^{\phi(x_1)} v_2^\pm(x)dx_2 = 0, \;\; x_1 < \varepsilon,$$

where

$$v_1^\pm(x_1) = \hat v_1^\pm(2\kappa^{-1}x_1^{-1}) + \tilde v_1^\pm(x_1), \;\; \tilde v_1^\pm \in V^2_{2\gamma_\pm}(\Omega_\varepsilon), \;\; \forall \gamma_\pm > Re\,\Lambda^\pm/2 - 1/4, \tag{4.5}$$

$$v_2^\pm(x_1,x_2) = \hat v_2^\pm(2\kappa^{-1}x_1^{-1}, z) + \tilde v_2^\pm(x), \;\; \tilde v_2^\pm \in \mathcal{W}^2_{\gamma_\pm}(\Omega_\varepsilon), \;\; \forall \gamma_\pm > Re\,\Lambda^\pm/2 - 1/4.$$

Here

$$z = \frac{x_2 - \phi_0(x_1)}{\phi(x_1)},$$

$$\hat v_1^\pm(t) = t^{\Lambda^\pm} Q^\pm(\ln t),$$

$$\hat v_2^\pm(t,z) = t^{\Lambda^\pm - 2} Q^\pm(\ln t) P_1^\pm(z) + t^{\Lambda^\pm-2}(Q^\pm)'(\ln t) P_2(z),$$

where

$$P_2(z) = 2\phi_0''(0)\kappa^{-1}\left(\frac{(z-1)^2}{2} - \frac{1}{6}\right) - 2\phi_1''(0)\kappa^{-1}\left(\frac{z^2}{2} - \frac{1}{6}\right),$$

$$P_1^\pm(z) = \frac{2}{\kappa}\left(\phi_0''(0)\Lambda^\pm + \rho_0\right)\left(\frac{(z-1)^2}{2} - \frac{1}{6}\right) - \frac{2}{\kappa}\left(\phi_1''(0)\Lambda^\pm - \rho_1\right)\left(\frac{z^2}{2} - \frac{1}{6}\right),$$

and
1. If $\lambda \neq 1/4$ *then*

$$\Lambda^{\pm} = 1/2 \pm i(\lambda - 1/4)^{1/2}, \; Q^{\pm}(\tau) = 1;$$

2. If $\lambda = 1/4$ *then*

$$\Lambda^{\pm} = 1/2, \; Q^{+}(\tau) = 1, \; Q^{-}(\tau) = \tau.$$

Remark 4.1. It will be useful in what follows to use another representation for $\mathbf{v}^{\pm}$ instead of (4.5), namely

$$\boldsymbol{v}^{\pm} = \mathbf{v}_1^{\pm} + \tilde{\mathbf{v}}^{\pm}, \tag{4.6}$$

where

$$\mathbf{v}_1^{\pm} = \hat{v}_1^{\pm}(2\kappa^{-1}x_1^{-1}) + \hat{v}_2^{\pm}(2\kappa^{-1}x_1^{-1}, z), \tag{4.7}$$

and

$$\tilde{\mathbf{v}}^{\pm} \in \mathcal{V}^2_{\gamma_{\pm}}(\Omega_{\varepsilon/2}), \quad \forall \gamma_{\pm} > \mathrm{Re}\, \Lambda^{\pm}/2 - 1/4.$$

Let us mention again that if $\lambda \in \{|\mathrm{Im}\lambda| < K^{-1}, \mathrm{Re}\lambda \geq 1/4\}$, we have uniform boundness of $\mathbf{v}_1^{\pm}$ and $\tilde{\mathbf{v}}^{\pm}$, see Remark 3.5.

The following theorem is a refined version of Theorem 2.2. It follows from Theorems 2.2, 4.1 and 4.3.

Theorem 4.4. *Let* $-\pi < \beta < 0$ *and* $\gamma_1 \neq 1/2 \pm 1/2Re\sqrt{1/4 - \lambda}$, $k = 1, 2$. *Suppose that* $u \in \mathcal{W}^2_{\beta,\gamma}(\Omega)$ *is a solution of the boundary value problem* (2.1), *where* $(f, g) \in \mathcal{W}^0_{\gamma_1}(\Omega) \times \mathcal{W}^{1/2}_{\gamma_1}(\partial\Omega)$. *Then the solution* u *admits representation*

$$u = c^{+}\boldsymbol{v}^{+} + c^{-}\boldsymbol{v}^{-} + \widetilde{u}, \; in \; \Omega_{\varepsilon}, \tag{4.8}$$

for sufficiently small ε. *Here* $\widetilde{u} \in \mathcal{V}^2_{\gamma_1}(\Omega_{\varepsilon})$, $\boldsymbol{v}^{\pm}$ *functions described in Theorem 4.3, and* $c^{\pm}$ *are constants.*

Proof. The proof follows from local solvability given by Theorem 4.1 and application of Theorem 2.2. □

Remark 4.2. We can assume that $c^{\pm}$ are zero if $\boldsymbol{v}^{\pm} \in \mathcal{V}^2_{\gamma_1}(\Omega)$.

4.2 On the indices of the operators

Consider the operator of the boundary value problem (2.1). Obviously it is continuous from $\mathcal{V}^2_{\gamma}(\Omega)$ to $\mathcal{W}^0_{\gamma}(\Omega) \times \mathcal{W}^{1/2}_{\gamma}(\partial\Omega)$. We denote this operator $\mathcal{A}_{\gamma}$.

Now we compare the attributes of the newly introduced operators $\mathcal{A}_{\gamma}$ and of previously studied operators A_{β} (see Theorem 2.1).

Theorem 4.5. *Let* $-\pi < \beta_1 < 0 < \beta_2 < \pi$, *and* $\gamma_1 < 1/2 - 1/2Re\sqrt{1/4 - \lambda}$, $\gamma_2 > 1/2 + 1/2Re\sqrt{1/4 - \lambda}$. *Then*

$$dim\ kerA_{\beta_1} = dim\ ker\mathcal{A}_{\gamma_2}, \quad dim\ cokerA_{\beta_1} = dim\ coker\mathcal{A}_{\gamma_2}, \tag{4.9}$$

and

$$dim\ kerA_{\beta_2} = dim\ ker\mathcal{A}_{\gamma_1}, \quad dim\ cokerA_{\beta_2} = dim\ coker\mathcal{A}_{\gamma_1}. \tag{4.10}$$

Proof. Let us prove (4.12). It follows from Theorem 4.4 that dim kerA_{β_1} = dim ker$\mathcal{A}_{\gamma_2,}$. Let dim coker$A_{\beta_1} = n$. Then there exist n linearly independent functionals $\psi_1, ..., \psi_n \in \big(\mathcal{W}^0_{\beta_1,\gamma}(\Omega) \times \mathcal{W}^{1/2}_{\beta_1,\gamma}(\partial\Omega)\big)^*$, such that, for $\{f, g\} \in \mathcal{W}^0_{\beta_1,\gamma}(\Omega) \times \mathcal{W}^{1/2}_{\beta_1,\gamma}(\partial\Omega)$, conditions

$$\psi_k(\{f, g\}) = 0, \;\; k = 1, ..., n \tag{4.11}$$

are equivalent to existence of a solution of $A_{\beta_1}u = \{f, g\}$. Clearly, conditions (4.11) are necessary for solvability of $\mathcal{A}_{\gamma_2}u = \{f, g\}$ if $\{f, g\} \in \mathcal{W}^0_{\gamma_2}(\Omega) \times \mathcal{W}^{1/2}_{\gamma_2}(\partial\Omega)$. Let us show that these conditions are also sufficient for the solvability of $\mathcal{A}_{\gamma_2}u = \{f, g\}$. Indeed then there exists a solution of $A_{\beta_1}u = \{f, g\}$. Moreover, since $\{f, g\} \in \mathcal{W}^0_{\gamma_2}(\Omega) \times \mathcal{W}^{1/2}_{\gamma_2}(\partial\Omega)$ then, due to Theorem 4.4, $u \in \mathcal{V}^2_{\gamma_2}(\Omega)$ and we get a solution of $\mathcal{A}_{\gamma_2}u = \{f, g\}$. It remains to notice that $\psi_1, ..., \psi_n$ are linearly independent as functionals from $\big(\mathcal{W}^0_{\gamma_2}(\Omega) \times \mathcal{W}^{1/2}_{\gamma_2}(\partial\Omega)\big)^*$ as well, since $\mathcal{W}^0_{\gamma_2}(\Omega) \times \mathcal{W}^{1/2}_{\gamma_2}(\partial\Omega)$ is dense in $\mathcal{W}^0_{\beta_1,\gamma}(\Omega) \times \mathcal{W}^{1/2}_{\beta_1,\gamma}(\partial\Omega)$. Identities (4.10) can be proved in the same way. □

Corollary 4.6. *It follows from Theorems 2.1, 4.2 and 4.5 that for* $\gamma \neq 1/2 \pm 1/2Re\sqrt{1/4 - \lambda}$, *operator of the boundary value problem* (2.1), $\mathcal{A}_\gamma$, *is Fredholm from* $\mathcal{V}^2_\gamma(\Omega)$ *to* $\mathcal{W}^0_\gamma(\Omega) \times \mathcal{W}^{1/2}_\gamma(\partial\Omega)$. *Moreover*

$$ind\, A_{\beta_1} = ind\, \mathcal{A}_{\gamma_2} \tag{4.12}$$

and

$$ind\, A_{\beta_2} = ind\, \mathcal{A}_{\gamma_1}, \tag{4.13}$$

for $-\pi < \beta_1 < 0 < \beta_2 < \pi$, *and* $\gamma_1 < 1/2 - 1/2Re\sqrt{1/4 - \lambda}$, $\gamma_2 > 1/2 + 1/2Re\sqrt{1/4 - \lambda}$.

We can describe the kernel of the adjoint operator $\mathcal{A}^*_\gamma$ in the following way: $\psi \in$ ker$\mathcal{A}^*_\gamma$ iff there exists $u \in$ ker $\mathcal{A}^+_{1-\gamma}$ such that

$$\psi(\{f, g\}) = \int_\Omega uf dx + \int_{\partial\Omega} ugds, \quad \forall\, \{f, g\} \in \mathcal{W}^0_\gamma(\Omega) \times \mathcal{W}^{1/2}_\gamma(\partial\Omega). \tag{4.14}$$

Here $\mathcal{A}^+_{1-\gamma}$ is a formally adjoint operator to $\mathcal{A}_{1-\gamma}$, i.e. operator of the boundary value problem (2.1) with ω and ρ replaced by $\overline{\omega}$ and $\overline{\rho}$, and acting from $\mathcal{V}^2_{1-\gamma}(\Omega)$ to $\mathcal{W}^0_{1-\gamma}(\Omega) \times \mathcal{W}^{1/2}_{1-\gamma}(\partial\Omega)$.

Following [22] (p.148), this representation allows us to evaluate the index of operator $\mathcal{A}_\gamma$, in the case when ρ is real valued function. Indeed since the difference of operators say $\mathcal{A}'_\gamma$ and $\mathcal{A}''_\gamma$ which correspond to different values of ω is a compact operator, it is enough to calculate the index of operator $\mathcal{A}_\gamma$ which corresponds to $\omega \in \mathbb{R}$. In this case $\mathcal{A}^+_{1-\gamma} = \mathcal{A}_{1-\gamma}$ and representation (4.14) implies

$$\text{ind}\, \mathcal{A}_\gamma = -\text{ind}\, \mathcal{A}_{1-\gamma}. \tag{4.15}$$

On the other hand (for definiteness let us consider the case $\lambda \geq 1/4$), we have

$$\text{ind}\, \mathcal{A}_{\gamma_2} = \text{ind}\, \mathcal{A}_{\gamma_1} + 2, \tag{4.16}$$

where $\gamma_1 < 1/2 < \gamma_2$. The analogous identity was proved in [23] (or see monograph [2]) for domains with conical singularities, but actually the proof relies on statements analogous to Theorem 4.4 and representation (4.14). This provides the desired information on index of $\mathcal{A}_\gamma$.

Theorem 4.7. *1. Let $\lambda < 1/4$, then*

$$ind\,\mathcal{A}_\gamma = \begin{cases} -1, & \gamma < 1/2 - 1/2\sqrt{1/4-\lambda} \\ 0, & 1/2 - 1/2\sqrt{1/4-\lambda} < \gamma < 1/2 + 1/2\sqrt{1/4-\lambda} \\ 1, & 1/2 - 1/2\sqrt{1/4+\lambda} < \gamma. \end{cases}$$

2. If $\lambda \geq 1/4$ then

$$ind\,\mathcal{A}_\gamma = \begin{cases} -1, & \gamma < 1/2, \\ 1, & 1/2 < \gamma. \end{cases}$$

Corollary 4.8. *Let $\gamma_1 < 1/2 - 1/2Re\sqrt{1/4-\lambda}$ and $\gamma_2 > 1/2 + 1/2Re\sqrt{1/4-\lambda}$ and ω^2, ρ are real, then dim ker$\mathcal{A}_{\gamma_2}$ − dim ker$\mathcal{A}_{\gamma_1}$ = 1. The corresponding one-dimensional space is described by function η, for which we have the following asymptotic representation*

$$\eta = a^+\boldsymbol{v}^+ + a^-\boldsymbol{v}^- + \tilde{\eta}, \ \text{in } \Omega_\varepsilon, \ \tilde{\eta} \in \mathcal{W}^2_{\beta,\gamma}(\Omega_\varepsilon), \ \beta < \pi.$$

Here functions $\boldsymbol{v}^\pm$ are as described in Theorem 4.3, and $a^\pm$ are constants connected by linear relation, i.e. either $a^- = sa^+$ or $a^+ = sa^-$ with some constant s. Moreover, if $\lambda \leq 1/4$ then $s \in \mathbb{R}$, if $\lambda > 1/4$ then $s \in \mathbb{C}$ and $|s| = 1$ (the last statement follows by simple integration by parts).

Remark 4.3. Theorem 4.7 shows that if $\lambda \geq 1/4$ then the index of operator $\mathcal{A}_\gamma$ is not zero for any admissible value of γ. Bearing this in mind, we can modify our operator introducing a space with *radiation conditions*: let $\lambda \geq 1/4$ and $\gamma < 1/2$, then

$$u \in \mathcal{V}^{2,+}_\gamma(\Omega) \Leftrightarrow u = a\boldsymbol{v}^+ + \tilde{u}, \quad \tilde{u} \in \mathcal{V}^2_\gamma(\Omega), \ a \in \mathbb{C}. \tag{4.17}$$

Then it is clear that the corresponding operator $\mathcal{A}^{rad,+}_\gamma$ maps $\mathcal{V}^{2,+}_\gamma$ into $\mathcal{W}^0_\gamma(\Omega) \times \mathcal{W}^{1/2}_\gamma(\partial\Omega)$ and its index *is* zero. Of course, one can consider different radiation conditions, for example by adding to $\mathcal{V}^2_\gamma(\Omega)$ the one-dimensional subspace generated by $\boldsymbol{v}^-$ (rather than $\boldsymbol{v}^+$) and constructing $\mathcal{A}^{rad,-}_\gamma$ with the same properties. An important feature of these two particular extensions of $\mathcal{A}_\gamma$ is that the dimension of the kernel does not increase, i.e. dim ker$\mathcal{A}_\gamma$ = dim ker$\mathcal{A}^{rad,\pm}_\gamma$ for $\lambda > 1/4$.

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