Sharp pointwise estimates for directional derivatives of harmonic functions in a multidimensional ball

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In memory of Israel Gohberg

Abstract. A representation of the sharp constant in a pointwise estimate for the absolute value of directional derivative of a harmonic function in a multidimensional ball is obtained under the assumption that the function's boundary values belong to L^p. This representation is concretized for the cases of radial and tangential derivatives. It is proved for $p = 1$ and $p = 2$ that the maximum of the absolute value of the directional derivative of a harmonic function with a fixed L^p-norm of its boundary values is attained at the radial direction. This proves D. Khavinson's hypothesis for $p = 1$ and $p = 2$.

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1 Introduction

Sharp estimates for harmonic functions are important in problems relating electrostatics as well as hydrodynamics of ideal fluid (see Protter and Weinberger [9]), elasticity and hydrodynamics of the viscous incompressible fluid.

A number of estimates for derivatives of harmonic functions are known, to mention only a few. The following estimate for the absolute value of the gradient of a harmonic function is borrowed from the book by Protter and Weinberger [9]

$$|\nabla u(x)| \leq \frac{n\omega_{n-1}}{(n-1)\omega_n d}\ \mathrm{osc}_{\mathcal{D}}(u)\ . \tag{1.1}$$

Here u is harmonic function in $\mathcal{D} \subset \mathbb{R}^n,\ n \geq 2$, ω_n is the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, $\mathrm{osc}_{\mathcal{D}}(u)$ is the oscillation of u on $\mathcal{D}$, and d is the distance from $x \in \mathcal{D}$ to $\partial\mathcal{D}$.

Inequality (1.1) is a corollary of the estimate for the modulus of the gradient of a harmonic function at the center of the ball $\mathbb{B}_R = \{x \in \mathbb{R}^n : |x| < R\}$:

$$|\nabla u(0)| \leq \frac{n\omega_{n-1}}{(n-1)\omega_n R}\ \mathrm{osc}_{\mathbb{B}_R}(u)\ .$$

Note that the last estimate is equivalent to the inequality

$$|\nabla u(0)| \leq \frac{2n\omega_{n-1}}{(n-1)\omega_n R} \sup_{|x|<R} |u(x)| \tag{1.2}$$

with the best possible constant (see Khavinson [5], Burgeth [1]).

In the analytic functions theory, there exist sharp estimates for derivatives of an analytic function f prescribed on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : r = |z| < 1\}$ with different characteristics of the real part of f in the majorant part. We mention, in particular, two equivalent inequalities for the first derivative (see Khavinson [5]; another proof can be found in the authors' book [6])

$$|f'(z)| \leq \frac{4}{\pi(1-r^2)} \sup_{|\zeta|<1} |\Re f(\zeta)| \tag{1.3}$$

and

$$|f'(z)| \leq \frac{2}{\pi(1-r^2)}\ \mathrm{osc}_{\mathbb{D}}(\Re f), \tag{1.4}$$

where z is an arbitrary point in $\mathbb{D}$.

Inequalities for analytic functions with certain characteristics of its real part as majorants, are called *real-part theorems* in reference to the first assertion of such a kind, the celebrated Hadamard real-part theorem [3]. The collection of real-part theorems and related assertions is rather broad. It involves assertions of various form (see, e.g., Kresin and Maz'ya [6] and the bibliography collected there).

Obviously, the inequalities for the first derivative of an analytic function can be restated as estimates for the gradient of a harmonic function. For example, inequality (1.3) can be written in the form

$$|\nabla u(z)| \leq \frac{4}{\pi(1-r^2)} \sup_{|\zeta|<1} |u(\zeta)|, \tag{1.5}$$

where u is a harmonic function in the disk $\mathbb{D} = \{z = (x, y) \in \mathbb{R}^2 : r = |z| < 1\}$.

At the moment, there are no analogues of the sharp inequality (1.5) for harmonic functions in the unit n-dimensional ball. The sharp constant in the inequality

$$\left|\frac{\partial u(x)}{\partial |x|}\right| \leq K(x) \sup_{|y|<1} |u(y)|,$$

where u is a harmonic function in the three-dimensional unit ball B, $x \in B$, $r = |x|$, was found by Khavinson [5], who suggested, in a private conversation, that the same constant should appear in the stronger inequality

$$|\nabla u(x)| \leq K(x) \sup_{|y|<1} |u(y)|.$$

In the present work we find a representation for the sharp coefficient $\mathcal{K}_p(x; \boldsymbol{\ell})$ in the inequality

$$\big|\big(\nabla u(x),\ \boldsymbol{\ell}\big)\big| \leq \mathcal{K}_p(x; \boldsymbol{\ell}) ||u||_p \ , \tag{1.6}$$

where x is an arbitrary point in the multidimensional unit ball $\mathbb{B} = \{x \in \mathbb{R}^n : r = |x| < 1\}$, $\boldsymbol{\ell}$ is an arbitrary unit vector in $\mathbb{R}^n$, u is a harmonic function in $\mathbb{B}$, represented by the Poisson integral with boundary values in $L^p(\partial\mathbb{B})$, $|| \cdot ||_p$ is the norm in $L^p(\partial\mathbb{B})$, $1 \leq p \leq \infty$. The representation in question is specified for the normal and tangent directions with respect to the sphere $|x| = r < 1$ at an arbitrary point. As a consequence, we obtain explicit formulas for the sharp coefficient $\mathcal{K}_p(x)$ in the inequality

$$|\nabla u(x)| \leq \mathcal{K}_p(x) ||u||_p \tag{1.7}$$

for the cases $p = 1$ and $p = 2$. We show that for $p = 1, 2$ and for any n the maximum value of $\mathcal{K}_p(x; \boldsymbol{\ell})$ with respect to $\boldsymbol{\ell}$ is attained at the radial direction. In other words, we show that the sharp constant $\mathcal{K}_p(x)$ in (1.7) coincides with the sharp constant $\mathcal{K}_p(x; \boldsymbol{\ell})$ in (1.6) for $p = 1, 2$, where $\boldsymbol{\ell}$ is the radial direction. This fact proves the generalized Khavinson's hypothesis concerning the role of the radial direction in (1.6) for $p = 1$ and $p = 2$.

Note that all estimates we obtain can be written in terms of integral mean boundary values of a harmonic function on the sphere $\mathbb{S}^{n-1}$. In a similar way, the norm $||u||_p$ in (1.6) and (1.7) can be replaced by the best approximation of the boundary value by a constant in the norm of $L^p(\mathbb{S}^{n-1})$:

$$\Lambda_p(u) = \min_{c \in \mathbb{R}} ||u - c||_p \ .$$

In particular, the norms $||u||_2$ and $||u||_\infty$ can be replaced by

$$\Lambda_2(u) = ||u - u(0)||_2 \quad \text{and} \quad \Lambda_\infty(u) = \frac{1}{2} \operatorname{osc}_{\partial\mathbb{B}}(u),$$

respectively.

In Section 2 we find a representation for the best constant in (1.6) as an integral over the sphere $\mathbb{S}^{n-1}$. As a corollary, we characterize the sharp constant

$\mathcal{K}_p(x)$ in inequality (1.7) in terms of an extremal problem on the unit sphere in $\mathbb{R}^n$.

In Section 3 we obtain the following generalization of (1.2)

$$|\nabla u(0)| \leq \mathcal{K}_p(0)||u||_p, \tag{1.8}$$

where $\mathcal{K}_1(0) = n/\omega_n$, and

$$\mathcal{K}_p(0) = \frac{n}{\omega_n}\left\{\frac{2\pi^{(n-1)/2}\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{n+q}{2}\right)}\right\}^{1/q}, \tag{1.9}$$

where $1 < p \leq \infty$, $1/q + 1/p = 1$. We also consider the case $p = 1$ and show that

$$\mathcal{K}_1(x) = \mathcal{K}_1(x;\ \boldsymbol{\nu}_x) = \frac{n + (n-2)r}{\omega_n(1-r)^n}\ ,$$

where $\boldsymbol{\nu}_x$ is the normal to the sphere $|x| = r < 1$ at a point x.

Section 4 is devoted to a simplification of the representation of $\mathcal{K}_p(x;\boldsymbol{\ell})$, reducing the integral over $\mathbb{S}^{n-1}$ to a double integral depending on the tangential and normal projections of a unit vector $\boldsymbol{\ell}$. As a corollary, we reduce the extremal problem for $\mathcal{K}_p(x)$ on the unit sphere in $\mathbb{R}^n$ to that of finding of the supremum of a certain double integral, depending on a scalar parameter. We also obtain a representation of $\mathcal{K}_p(x;\boldsymbol{\nu}_x)$ as a definite integral as well as a formula for the best constant in the estimate of the modulus of derivative in a tangent direction $\boldsymbol{\tau}_x$:

$$\mathcal{K}_p(x;\boldsymbol{\tau}_x)=\mathcal{K}_p(0)(1-r^2)\left\{F\left(\frac{n(q-1)+q+2}{2},\ \frac{(n+2)q}{2};\ \frac{n+q}{2};\ r^2\right)\right\}^{1/q},$$

where $F(a,b;c;x)$ is the hypergeometric Gauss function, and $\mathcal{K}_p(0)$ is defined by formula (1.9).

In Section 5 we show that

$$\mathcal{K}_2(x) = \mathcal{K}_2(x;\ \boldsymbol{\nu}_x) = \left(\frac{n + (n^2+n-4)r^2 + (n-2)^2r^4}{\omega_n(1-r^2)^{n+1}}\right)^{1/2},$$

and

$$\mathcal{K}_2(x;\boldsymbol{\tau}_x) = \left\{\frac{n(1+r^2)}{\omega_n(1-r^2)^{n+1}}\right\}^{1/2}.$$

Moreover, $\mathcal{K}_2(x;\boldsymbol{\tau}_x) \leq \mathcal{K}_2(x;\ \boldsymbol{\ell}) \leq \mathcal{K}_2(x;\ \boldsymbol{\nu}_x)$.

We conclude this paper by giving explicit formulas for $\mathcal{K}_\infty(x;\ \boldsymbol{\nu}_x)$ and $\mathcal{K}_\infty(x;\boldsymbol{\tau}_x)$. In particular, we find

$$\mathcal{K}_\infty(x;\boldsymbol{\nu}_x) = \frac{1}{r^2}\left(\frac{\left(1+r^2/3\right)^{3/2}}{1-r^2} - 1\right), \tag{1.10}$$

$$\mathcal{K}_\infty(x;\boldsymbol{\tau}_x) = \frac{2}{\pi(1-r^2)}\left[2\mathrm{K}(r) - (1+r^2)\mathrm{D}(r)\right]$$

for $n=3$, and

$$\mathcal{K}_\infty(x;\boldsymbol{\nu}_x)=\frac{1}{\pi r^3}\left(\frac{r(2+r^2)(4-r^2)^{1/2}}{(1-r^2)}-2\arccos\frac{(2-r^2)^2-2}{2}\right),$$

$$\mathcal{K}_\infty(x;\boldsymbol{\tau}_x)=\frac{8(1-r^2)}{\pi r^2}\left[\frac{1+r^2}{(1-r^2)^2}-\frac{1}{2r}\log\frac{1+r}{1-r}\right]$$

for $n=4$. Here $\mathrm{K}(r)$ is the complete elliptic integral of the first kind, and $\mathrm{D}(r)$ is the complete elliptic integral.

Note that the constant (1.10) was found by Khavinson [5]. In the paper by Kresin and Maz'ya [8] an analogue of Kavinson's extremal problem was solved for harmonic functions in the multidimensional half-space for the cases $p=1,2$, and ∞. It is shown by Kresin [7] that the Khavinson's hypothesis on the role of the radial direction holds true in estimates for the gradient of bounded or semibounded harmonic functions.

In connection with the Khavinson problem, we mention another type of sharp estimates (see Kresin and Maz'ya [6]) for functions which are harmonic in the disk $\mathbb{D}$:

$$\left|\frac{\partial u(z)}{\partial l_{\alpha+\beta}}\right|\le\mathcal{C}_p(r,\alpha)\left\|\frac{\partial u}{\partial l_\beta}\right\|_p, \tag{1.11}$$

where $1\le p\le\infty$, l_β is a unit vector such that the angle β between l_β and the radial direction is constant. In particular,

$$\mathcal{C}_1(r,\alpha)=\frac{1}{\pi(1-r^2)}\,,\qquad \mathcal{C}_2(r,\alpha)=\frac{1}{\sqrt{\pi(1-r^2)}}\,.$$

Unlike the cases $p=1$ and $p=2$, the coefficient $\mathcal{C}_\infty(r,\alpha)$ depends on α and

$$\max_\alpha\mathcal{C}_\infty(r,\alpha)=\mathcal{C}_\infty(r,\pi/2)=\frac{2}{\pi r}\log\left(\frac{1+r}{1-r}\right)\,, \tag{1.12}$$

$$\min_\alpha\mathcal{C}_\infty(r,\alpha)=\mathcal{C}_\infty(r,0)=\frac{4}{\pi r}\arctan r\,.$$

Suppose that $|\partial u(\zeta)/\partial l_\beta|\le 1$ for $|\zeta|=1$. If, in view of (1.12), we put $\beta=0,p=\infty$ in (1.11), we conclude that the maximum modulus of the directional derivative is attained at the direction which is tangent to the circumference $|z|=r$.

Putting $\beta=\pi/2,p=\infty$ in (1.11), we find by (1.12) that the maximum modulus of the directional derivative is attained at the direction which is normal to the circumference $|z|=r$. In connection with this we mention the paper by Hile and Stanoyevitch [4] which proves that $|\nabla u(z)|$ has logarithmic growth as z approaches the smooth boundary ∂G of a bounded domain $G\in\mathbb{R}^n$ under the assumption that the boundary values of a harmonic function are Lipschitz.

Summarizing our comments, we describe the generalization of Khavinson's problem as follows. Let $\mathcal{H}$ be a class of harmonic functions in the ball $\mathbb{B}$, $x\in\mathbb{B}$, and let $\Phi(u)$ be a positively homogeneous functional defined on boundary values of functions in $\mathcal{H}$ and such that

$$\left|\left(\nabla u(x),\ \boldsymbol{\ell}\right)\right| \leq C(x)\Phi(u)$$

for all $|\boldsymbol{\ell}|=1$ and $u \in \mathcal{H}$. One is looking for directions $\boldsymbol{\ell}$ for which the sharp constant in

$$\left|\left(\nabla u(x),\ \boldsymbol{\ell}\right)\right| \leq \mathcal{K}_\Phi(x;\boldsymbol{\ell})\Phi(u), \qquad u \in \mathcal{H},$$

attains its minimal and maximal values.

A similar problem can be considered in other domains (for example, in the half-space).

2 Representation for $\mathcal{K}_p(x;\boldsymbol{\ell})$ as an integral over $\mathbb{S}^{n-1}$

We introduce some notation used henceforth. Let $|\cdot|$ be the length of a vector in the n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathbb{B} = \left\{x \in \mathbb{R}^n : |x| < 1\right\}$, and $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

By $||\cdot||_p$ we denote the norm in the space $L^p(\mathbb{S}^{n-1})$, that is

$$||f||_p = \left\{\int_{\mathbb{S}^{n-1}} |f(x)|^p d\sigma_x\right\}^{1/p},$$

if $1 \leq p < \infty$, and $||f||_\infty = \text{ess sup}\{|f(x)| : x \in \mathbb{S}^{n-1}\}$.

Next, by $h^p(\mathbb{B})$ we denote the Hardy space of harmonic functions on $\mathbb{B}$, which can be represented as the Poisson integral

$$u(x) = \frac{1}{\omega_n}\int_{\mathbb{S}^{n-1}} \frac{1-r^2}{|y-x|^n} u(y) d\sigma_y \tag{2.1}$$

with boundary values in $L^p(\mathbb{S}^{n-1})$, $1 \leq p \leq \infty$, where $r = |x| < 1$.

In what follows, we assume that the Cartesian coordinates with origin $\mathcal{O}$ at the center of the ball are chosen in such a way that $x = r\boldsymbol{e}_n$. By $\boldsymbol{\ell}$ we denote an arbitrary unit vector in $\mathbb{R}^n$ and by $\boldsymbol{\nu}_x$ we mean the unit vector of exterior normal to the sphere $|x| = r$ at a point x. Let $\boldsymbol{\ell}_\tau$ be the orthogonal projection of $\boldsymbol{\ell}$ on the tangent hyperplane to the sphere $|x| = r$ at x. If $\boldsymbol{\ell}_\tau \neq \boldsymbol{0}$, we set $\boldsymbol{\tau}_x = \boldsymbol{\ell}_\tau/|\boldsymbol{\ell}_\tau|$, otherwise $\boldsymbol{\tau}_x$ is an arbitrary unit vector tangent to the sphere $|x| = r$ at x. Hence $\boldsymbol{\ell} = \ell_\tau \boldsymbol{\tau}_x + \ell_\nu \boldsymbol{\nu}_x$, where $\ell_\tau = |\boldsymbol{\ell}_\tau|$ and $\ell_\nu = (\boldsymbol{\ell}, \boldsymbol{\nu}_x)$.

Now, we find a representation for the best coefficient $\mathcal{K}_p(x;\boldsymbol{\ell})$ in the inequality for the absolute value of derivative of $u \in h^p(\mathbb{B})$ in an arbitrary direction $\boldsymbol{\ell}$ at a point $x \in \mathbb{B}$. In particular, we obtain a formula for the constant in a similar inequality for the modulus of the gradient.

Theorem 1. *Let $u \in h^p(\mathbb{B})$, and let x be an arbitrary point in $\mathbb{B}$. The sharp coefficient $\mathcal{K}_p(x;\boldsymbol{\ell})$ in the inequality*

$$|(\nabla u(x), \boldsymbol{\ell})| \leq \mathcal{K}_p(x;\boldsymbol{\ell})||u||_p \tag{2.2}$$

is given by

$$\mathcal{K}_p(x;\boldsymbol{\ell}) = \frac{N(r)}{\omega_n}\left\{\int_{\mathbb{S}^{n-1}} \frac{\big|\big(\mathcal{A}(r)\big(y-\lambda(r)x\big),\ \boldsymbol{\ell}\big)\big|^q}{|y-x|^{(n+2)q}}\, d\sigma_y\right\}^{1/q}. \tag{2.3}$$

Here $1/p+1/q=1$,

$$N(r) = n\left(1-r^2\right)+4r^2, \qquad \lambda(r) = 1+2d(r), \tag{2.4}$$

and $\mathcal{A}(r)$ *is the* $(n\times n)$*-matrix of the form*

$$\mathcal{A}(r) = \mathrm{diag}\{nd(r),\ \dots, nd(r),\ 1\}\ , \tag{2.5}$$

where

$$d(r) = \frac{1-r^2}{n\,(1-r^2)+4r^2}\ . \tag{2.6}$$

In particular, the sharp coefficient $\mathcal{K}_p(x)$ *in*

$$|\nabla u(x)| \le \mathcal{K}_p(x)\big\|u\big\|_p \tag{2.7}$$

is given by

$$\mathcal{K}_p(x) = \max_{|\boldsymbol{\ell}|=1}\mathcal{K}_p(x;\boldsymbol{\ell})\ . \tag{2.8}$$

Proof. Fix a point $x\in\mathbb{B}$. By (2.1) we have

$$\frac{\partial u}{\partial x_i} = \frac{1}{\omega_n}\int_{\mathbb{S}^{n-1}}\left[\frac{-2x_i}{|y-x|^n}+\frac{n\left(1-r^2\right)(y_i-x_i)}{|y-x|^{n+2}}\right]u(y)d\sigma_y,$$

that is

$$\nabla u(x) = \frac{1}{\omega_n}\int_{\mathbb{S}^{n-1}}\frac{n\left(1-r^2\right)(y-x)-2|y-x|^2x}{|y-x|^{n+2}}\,u(y)d\sigma_y. \tag{2.9}$$

Since $y-x=y-r\boldsymbol{e}_n=\big(y-y_n\boldsymbol{e}_n\big)+\big(y_n-r\big)\boldsymbol{e}_n$, we can express the numerator of the integrand in (2.9) as the sum

$$\begin{aligned} n\left(1-r^2\right)(y-x)-2|y-x|^2x &= n\left(1-r^2\right)\left(y-y_n\boldsymbol{e}_n\right) \\ &+ \left(n\left(1-r^2\right)\left(y_n-r\right)-2r|y-x|^2\right)\boldsymbol{e}_n\ . \end{aligned} \tag{2.10}$$

Introducing the notation

$$T(r) = n\left(1-r^2\right), \qquad N(r) = n\left(1-r^2\right)+4r^2\ , \tag{2.11}$$

and

$$\lambda(r) = \frac{n\left(1-r^2\right)+2(1+r^2)}{n\,(1-r^2)+4r^2}\ , \tag{2.12}$$

using (2.10) and the equality

$$n\left(1-r^2\right)\left(y_n-r\right)-2r|y-x|^2=N(r)\big(y_n-\lambda(r)r\big)\ ,$$

we write (2.9) as

$$\nabla u(x)=\frac{1}{\omega_n}\int_{\mathbb{S}^{n-1}}\frac{T(r)\big(y-y_n\boldsymbol{e}_n\big)+N(r)\big(y_n-\lambda(r)r\big)\boldsymbol{e}_n}{|y-x|^{n+2}}\,u(y)d\sigma_y. \tag{2.13}$$

Since the vectors $y-y_n\boldsymbol{e}_n$ and $\boldsymbol{e}_n$ are orthogonal and $x=r\boldsymbol{e}_n$, it follows that

$$y-y_n\boldsymbol{e}_n=\sum_{i=1}^{n-1}\big(y-y_n\boldsymbol{e}_n,\ \boldsymbol{e}_i\big)\boldsymbol{e}_i=\sum_{i=1}^{n-1}\big(y-\lambda(r)x,\ \boldsymbol{e}_i\big)\boldsymbol{e}_i\ ,$$

which together with

$$\big(y_n-\lambda(r)r\big)\boldsymbol{e}_n=\big(y-\lambda(r)x,\ \boldsymbol{e}_n\big)\boldsymbol{e}_n$$

implies (2.13) written in the form

$$\nabla u(x)=\frac{1}{\omega_n}\int_{\mathbb{S}^{n-1}}\frac{\Lambda(r)\big(y-\lambda(r)x\big)}{|y-x|^{n+2}}\,u(y)d\sigma_y\ , \tag{2.14}$$

where the $(n\times n)$-matrix $\Lambda(r)$ is defined by

$$\Lambda(r)=\mathrm{diag}\{T(r),\ \ldots,\ T(r),\ N(r)\}\ . \tag{2.15}$$

Using (2.11), (2.15) as well as notations introduced in Lemma, we can write (2.14) as

$$\nabla u(x)=\frac{N(r)}{\omega_n}\int_{\mathbb{S}^{n-1}}\frac{\mathcal{A}(r)\big(y-\lambda(r)x\big)}{|y-x|^{n+2}}\,u(y)d\sigma_y\ . \tag{2.16}$$

Thus,

$$(\nabla u(x),\boldsymbol{\ell})=\frac{N(r)}{\omega_n}\int_{\mathbb{S}^{n-1}}\frac{\big(\mathcal{A}(r)\big(y-\lambda(r)x\big),\ \boldsymbol{\ell}\big)}{|y-x|^{n+2}}\,u(y)d\sigma_y\ , \tag{2.17}$$

and (2.3) follows.

By (2.17) we have

$$\big|\nabla u(x)\big|=\frac{N(r)}{\omega_n}\max_{|\boldsymbol{\ell}|=1}\int_{\mathbb{S}^{n-1}}\frac{\big(\mathcal{A}(r)\big(y-\lambda(r)x\big),\ \boldsymbol{\ell}\big)}{|y-x|^{n+2}}\,u(y)d\sigma_y\ .$$

Since the suprema commute, this implies that the sharp constant in (2.7) has the form

$$\mathcal{K}_p(x)=\frac{N(r)}{\omega_n}\max_{|\boldsymbol{\ell}|=1}\left\{\int_{\mathbb{S}^{n-1}}\frac{\big|\big(\mathcal{A}(r)\big(y-\lambda(r)x\big),\ \boldsymbol{\ell}\big)\big|^q}{|y-x|^{(n+2)q}}\,d\sigma_y\right\}^{1/q}\ . \tag{2.18}$$

Taking into account (2.3) and (2.18), we arrive at (2.8). □

3 The case $p = 1$ and the formula for $\mathcal{K}_p(0)$

The next assertion concerns sharp coefficient in inequality (2.2) with $p = 1$ for the cases $\boldsymbol{\ell} = \boldsymbol{\nu}_x$ and $\boldsymbol{\ell} = \boldsymbol{\tau}_x$. It is shown that $\mathcal{K}_1(x) = \mathcal{K}_1(x; \boldsymbol{\nu}_x)$, that is the radial direction is extremal for $p = 1$.

Corollary 1. *Let $u \in h^1(\mathbb{B})$, and x be an arbitrary point in $\mathbb{B}$. The sharp coefficient $\mathcal{K}_1(x; \boldsymbol{\ell})$ in the inequality*

$$|(\nabla u(x), \boldsymbol{\ell})| \le \mathcal{K}_1(x; \boldsymbol{\ell}) ||u||_1 \tag{3.1}$$

is given by

$$\mathcal{K}_1(x; \boldsymbol{\ell}) = \frac{N(r)}{\omega_n} \max_{\alpha} \frac{\big| nd(r)\ell_\tau \sin\alpha + \big(\cos\alpha - \lambda(r)r\big)\ell_\nu \big|}{\big(1 - 2r\cos\alpha + r^2\big)^{(n+2)/2}}\ , \tag{3.2}$$

where $\boldsymbol{\ell} = \ell_\tau \boldsymbol{\tau}_x + \ell_\nu \boldsymbol{\nu}_x$.

In particular,

$$\mathcal{K}_1(x) = \mathcal{K}_1(x; \boldsymbol{\nu}_x) = \frac{n + (n-2)r}{\omega_n (1-r)^n}\ , \tag{3.3}$$

and

$$\mathcal{K}_1(x; \boldsymbol{\tau}_x) = \frac{n(1-r^2)(1-\rho^2(r))^{1/2}}{\omega_n (1 - 2r\rho(r) + r^2)^{(n+2)/2}}\ , \tag{3.4}$$

where

$$\rho(r) = \frac{2(n+2)r}{1 + r^2 + \big((1+r^2)^2 + 4n(n+2)r^2\big)^{1/2}}\ .$$

Proof. Passing to the limit as $q \to \infty$ in the right-hand side of (2.18), and using the diagonality of the matrix $\mathcal{A}$, we find

$$\begin{aligned} \mathcal{K}_1(x,\ \boldsymbol{\ell}) &= \frac{N(r)}{\omega_n} \max_{|y|=1} \frac{\big|\mathcal{A}(r)\big(y - \lambda(r)x\big),\ \boldsymbol{\ell})\big|}{|y-x|^{n+2}} \\ &= \frac{N(r)}{\omega_n} \max_{|y|=1} \frac{\big|\big(y - \lambda(r)x\big),\ \mathcal{A}(r)\boldsymbol{\ell})\big|}{|y-x|^{n+2}}\ . \end{aligned} \tag{3.5}$$

Taking into account (2.5) and the choice of the coordinate system, we obtain

$$\mathcal{A}(r)\boldsymbol{\ell} = \mathcal{A}(r)\big(\ell_\tau \boldsymbol{\tau}_x + \ell_\nu \boldsymbol{\nu}_x\big) = nd(r)\ell_\tau \boldsymbol{\tau}_x + \ell_\nu \boldsymbol{\nu}_x\ , \tag{3.6}$$

which together with (3.5) and the orthogonality of $x = r\boldsymbol{e}_n$ and $\boldsymbol{\tau}_x$ implies

$$\mathcal{K}_1(x; \boldsymbol{\ell}) = \frac{N(r)}{\omega_n} \max_{|y|=1} \frac{\big| nd(r)\big(y,\ \boldsymbol{\tau}_x\big)\ell_\tau + \big((y,\ \boldsymbol{\nu}_x) - \lambda(r)r\big)\ell_\nu \big|}{|y-x|^{n+2}}\ . \tag{3.7}$$

Let α stand for the angle between y and $\boldsymbol{\nu}_x$, and let $y_\tau = y - y_n \boldsymbol{e}_n$. By β we denote the angle between y_τ and $\boldsymbol{\tau}_x$. We have

$$(y, \boldsymbol{\tau}_x) = (y_\tau, \boldsymbol{\tau}_x) = |y_\tau|\cos\beta = \sin\alpha\cos\beta,$$

which allows to write (3.7) as

$$\mathcal{K}_1(x;\boldsymbol{\ell})=\frac{N(r)}{\omega_n}\max_{\alpha\in[0,\pi]}\max_{\beta\in[0,2\pi]}\frac{\left|nd(r)\ell_\tau\sin\alpha\cos\beta+\left(\cos\alpha-\lambda(r)r\right)\ell_\nu\right|}{\left(1-2r\cos\alpha+r^2\right)^{(n+2)/2}}\ .$$

Since the maximum in β is attained either at $\beta=0$ or at $\beta=\pi$, we arrive at

$$\mathcal{K}_1(x;\boldsymbol{\ell})=\frac{N(r)}{\omega_n}\max_{\alpha\in[0,2\pi]}\frac{\left|nd(r)\ell_\tau\sin\alpha+\left(\cos\alpha-\lambda(r)r\right)\ell_\nu\right|}{\left(1-2r\cos\alpha+r^2\right)^{(n+2)/2}}\ ,$$

which proves (3.2).

Next we prove (3.3). Using (3.2) together with (2.4) and (2.6), we find

$$\begin{aligned}\mathcal{K}_1(x)\quad&\geq\quad\mathcal{K}_1(x;\ \boldsymbol{\nu}_x)=\frac{N(r)}{\omega_n}\max_{\alpha\in[0,2\pi]}\frac{\left|\cos\alpha-\lambda(r)r\right|}{\left(1-2r\cos\alpha+r^2\right)^{(n+2)/2}}\\&\geq\quad\frac{N(r)}{\omega_n}\frac{\left|1-\lambda(r)r\right|}{\left(1-r\right)^{n+2}}=\frac{n+(n-2)r}{\omega_n(1-r)^n}\ .\end{aligned}\tag{3.8}$$

Now we obtain an upper estimate for $\mathcal{K}_1(x)$. Taking into account (3.2) and the equality $\ell_\tau^2+\ell_\nu^2=1$, we obtain the estimate

$$\mathcal{K}_1(x;\boldsymbol{\ell})\leq\frac{N(r)}{\omega_n}\max_{\alpha\in[0,2\pi]}\left\{\frac{n^2d^2(r)\sin^2\alpha+(\cos\alpha-\lambda(r)r)^2}{\left(1-2r\cos\alpha+r^2\right)^{n+2}}\right\}^{1/2}\ ,$$

which is the same as

$$\mathcal{K}_1(x;\boldsymbol{\ell})\leq\frac{N(r)}{\omega_n}\max_{|t|\leq1}\left\{\frac{n^2d^2(r)(1-t^2)+(t-\lambda(r)r)^2}{\left(1-2rt+r^2\right)^{n+2}}\right\}^{1/2}$$

as $t=\cos\alpha$. Thus,

$$\mathcal{K}_1(x;\boldsymbol{\ell})\leq\frac{N(r)}{\omega_n(1-r)^n}\max_{|t|\leq1}\left\{\frac{n^2d^2(r)(1-t^2)+(t-\lambda(r)r)^2}{\left(1-2rt+r^2\right)^2}\right\}^{1/2}\ .\tag{3.9}$$

Let

$$f(t)=\frac{n^2d^2(r)\left(1-t^2\right)+\left(t-\lambda(r)r\right)^2}{\left(1-2rt+r^2\right)^2}\ .$$

Since

$$f'(t)=\frac{2n(n-2)r(1-r^2)^2}{\left(n\left(1-r^2\right)+4r^2\right)^2\left(1-2rt+r^2\right)^2}\geq0\ ,$$

we have

$$\max_{|t|\leq1}f(t)=f(1)=\frac{\left(n+(n-2)r\right)^2}{\left(n\left(1-r^2\right)+4r^2\right)^2}=\frac{\left(n+(n-2)r\right)^2}{N^2(r)}.$$

This and (3.9) imply

$$\mathcal{K}_1(x) \le \frac{n+(n-2)r}{\omega_n (1-r)^n} \ ,$$

which together with (3.8) proves (3.3).

Now, by (2.4), (2.6) and (3.2) we obtain

$$\mathcal{K}_1(x;\boldsymbol{\tau}_x) = \frac{n(1-r^2)}{\omega_n} \max_{|t|\le 1} \frac{\sqrt{1-t^2}}{(1-2rt+r^2)^{(n+2)/2}} \ .$$

Since the maximum in t is attained at

$$t = \frac{2(n+2)r}{1+r^2+\big((1+r^2)^2+4n(n+2)r^2\big)^{1/2}} \ ,$$

we arrive at (3.4). □

Corollary 2. *Let $u \in h^p(\mathbb{B})$. The sharp coefficient $\mathcal{K}_p(0)$ in the inequality*

$$|\nabla u(0)| \le \mathcal{K}_p(0)||u||_p$$

is given by

$$\mathcal{K}_1(0) = \frac{n}{\omega_n} \ , \tag{3.10}$$

and

$$\mathcal{K}_p(0) = \frac{n}{\omega_n} \left\{ \frac{2\pi^{(n-1)/2}\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{n+q}{2}\right)} \right\}^{1/q} \tag{3.11}$$

for $1 < p \le \infty$.

In particular,

$$\mathcal{K}_2(0) = \left(\frac{n}{\omega_n}\right)^{1/2}, \qquad \mathcal{K}_\infty(0) = \frac{2n\omega_{n-1}}{(n-1)\omega_n} \ .$$

Proof. Equality (3.10) follows from (3.3) with $r=0$. By (2.3) and (2.8),

$$\mathcal{K}_p(0) = \frac{n}{\omega_n} \max_{|\boldsymbol{z}|=1} \left\{ \int_{\mathbb{S}^{n-1}} |(y,\boldsymbol{\ell})|^q \, d\sigma_y \right\}^{1/q}$$

with $1 < p \le \infty$. Since the integral does not depend on $\boldsymbol{\ell}$, we find

$$\mathcal{K}_p(0) = \frac{n}{\omega_n} \left\{ \int_{\mathbb{S}^{n-1}} |(y,\boldsymbol{e}_n)|^q \, d\sigma_y \right\}^{1/q} . \tag{3.12}$$

Using spherical coordinates in (3.12), we obtain

$$\begin{aligned}
\mathcal{K}_p(0) &= \frac{n}{\omega_n} \left\{ \int_0^\pi \dots \int_0^\pi \int_0^{2\pi} |\cos\vartheta_1|^q \sin^{n-2}\vartheta_1 \dots \sin\vartheta_{n-2} d\vartheta_1 \dots d\vartheta_{n-2} d\vartheta_{n-1} \right\}^{1/q} \\
&= \frac{n(2\omega_{n-1})^{1/q}}{\omega_n} \left\{ \int_0^{\pi/2} \cos^q\vartheta_1 \sin^{n-2}\vartheta_1 \, d\vartheta \right\}^{1/q} \\
&= \frac{n(\omega_{n-1})^{1/q}}{\omega_n} B^{1/q}\left(\frac{q+1}{2}, \frac{n-1}{2}\right) \ ,
\end{aligned}$$

which implies (3.11). □

Note that the constant $\mathcal{K}_\infty(0)$ in (1.2) is well known (cf. Protter and Weinberger [9], Khavinson [5], Burgeth [1]).

4 Representations for $\mathcal{K}_p(x;\boldsymbol{\ell})$ as double integrals

In the next assertion we reduce the evaluation of $\mathcal{K}_p(x;\boldsymbol{\ell})$, found as an integral over $\mathbb{S}^{n-1}$, to a double integral.

Proposition 1. *Let $u \in h^p(\mathbb{B})$, $1 < p \le \infty$, and x be an arbitrary point in $\mathbb{B}$. The sharp coefficient $\mathcal{K}_p(x;\boldsymbol{\ell})$ in the inequality*

$$|(\nabla u(x), \boldsymbol{\ell})| \le \mathcal{K}_p(x;\boldsymbol{\ell})||u||_p \tag{4.1}$$

is given by

$$\mathcal{K}_p(x;\boldsymbol{\ell}) = \frac{N(r)(\omega_{n-2})^{1/q}}{\omega_n} \left\{ \int_0^\pi \sin^{n-2}\vartheta d\vartheta \int_0^\pi \mathcal{F}_{n,q}(\vartheta,\varphi;\boldsymbol{\ell},r) \sin^{n-3}\varphi d\varphi \right\}^{1/q},$$

where $1/p + 1/q = 1$, $\boldsymbol{\ell} = \ell_\tau \boldsymbol{\tau}_x + \ell_\nu \boldsymbol{\nu}_x$,

$$\mathcal{F}_{n,q}(\vartheta,\varphi;r,\boldsymbol{\ell}) = \frac{\big|nd(r)\ell_\tau \sin\vartheta\cos\varphi + (\cos\vartheta - \lambda(r)r\big)\ell_\nu\big|^q}{\big(1-2r\cos\vartheta + r^2\big)^{(n+2)q/2}}, \tag{4.2}$$

with $N(r), \lambda(r) = 1 + 2d(r)$ and $d(r)$ defined by (2.4) *and* (2.6).

In particular,

$$\mathcal{K}_p(x;\boldsymbol{\nu}_x) = \frac{N(r)(\omega_{n-1})^{1/q}}{\omega_n} \left\{ \int_0^\pi \frac{|\cos\vartheta - \lambda(r)r|^q \sin^{n-2}\vartheta}{\big(1-2r\cos\vartheta + r^2\big)^{(n+2)q/2}} d\vartheta \right\}^{1/q}, \tag{4.3}$$

and

$$\mathcal{K}_p(x;\boldsymbol{\tau}_x) = \mathcal{K}_p(0)(1-r^2)\left\{ F\left(\frac{n(q-1)+q+2}{2}, \frac{(n+2)q}{2}; \frac{n+q}{2}; r^2 \right) \right\}^{1/q}. \tag{4.4}$$

Here $F(a,b;c;x)$ is the hypergeometric Gauss function, and $\mathcal{K}_p(0)$ is defined by formula (3.11).

Proof. Using (3.6) and diagonality of the matrix $\mathcal{A}$, we write (2.3) as

$$\begin{aligned}\mathcal{K}_p(x;\boldsymbol{\ell}) &= \frac{N(r)}{\omega_n} \left\{ \int_{\mathbb{S}^{n-1}} \frac{\big|nd(r)\big(y,\ \boldsymbol{\tau}_x\big)\ell_\tau + \big((y,\ \boldsymbol{\nu}_x) - \lambda(r)r\big)\ell_\nu\big|^q}{\big|y-x\big|^{(n+2)q/2}} d\sigma_y \right\}^{1/q} \\ &= \frac{N(r)}{\omega_n} \left\{ \int_{\mathbb{S}^{n-1}} \frac{\big|nd(r)\big(y_\tau,\ \boldsymbol{\tau}_x\big)\ell_\tau + \big(y_n - \lambda(r)r\big)\ell_\nu\big|^q}{\big(1-2ry_n + r^2\big)^{(n+2)q/2}} d\sigma_y \right\}^{1/q},\end{aligned} \tag{4.5}$$

where $y_\tau = (y_1, \dots, y_{n-1}, 0)$.

Introducing the function

$$\mathcal{H}_{n,q}(s,t;r,\boldsymbol{\ell}) = \frac{\big|nd(r)s\ell_\tau + \big(t-\lambda(r)r\big)\ell_\nu\big|^q}{\big(1-2rt+r^2\big)^{(n+2)q/2}}\ , \tag{4.6}$$

we write the integral in (4.5) as the sum

$$\int_{\mathbb{S}^{n-1}_+} \mathcal{H}_{n,q}((y_\tau,\boldsymbol{\tau}_x),y_n;r,\boldsymbol{\ell})\ d\sigma_y + \int_{\mathbb{S}^{n-1}_-} \mathcal{H}_{n,q}((y_\tau,\boldsymbol{\tau}_x),y_n;r,\boldsymbol{\ell})\ d\sigma_y, \tag{4.7}$$

where $\mathbb{S}^{n-1}_+ = \{y \in \mathbb{S}^{n-1} : (y,\boldsymbol{e}_n) > 0\}$, $\mathbb{S}^{n-1}_- = \{y \in \mathbb{S}^{n-1} : (y,\boldsymbol{e}_n) < 0\}$.

Let $y' = (y_1, \dots, y_{n-1}) \in \mathbb{B}^{n-1} = \{y' \in \mathbb{R}^{n-1} : |y'| < 1\}$. We put

$$\boldsymbol{\tau}'_x = \sum_{i=1}^{n-1} (\boldsymbol{\tau}_x, \boldsymbol{e}_i)\boldsymbol{e}_i.$$

Since $y_n = \sqrt{1-|y'|^2}$ for $y \in \mathbb{S}^{n-1}_+$ and $y_n = -\sqrt{1-|y'|^2}$ for $y \in \mathbb{S}^{n-1}_-$ and since $d\sigma_y = dy'/\sqrt{1-|y'|^2}$, each of integrals in (4.7) can be written as

$$\int_{\mathbb{S}^{n-1}_+} \mathcal{H}_{n,q}((y_\tau,\boldsymbol{\tau}_x),y_n;r,\boldsymbol{\ell})d\sigma_y = \int_{\mathbb{B}^{n-1}} \frac{\mathcal{H}_{n,q}\Big((y',\boldsymbol{\tau}'_x),\sqrt{1-|y'|^2};r,\boldsymbol{\ell}\Big)}{\sqrt{1-|y'|^2}} dy', \tag{4.8}$$

$$\int_{\mathbb{S}^{n-1}_-} \mathcal{H}_{n,q}((y_\tau,\boldsymbol{\tau}_x),y_n;r,\boldsymbol{\ell})d\sigma_y = \int_{\mathbb{B}^{n-1}} \frac{\mathcal{H}_{n,q}\Big((y',\boldsymbol{\tau}'_x),-\sqrt{1-|y'|^2};r,\boldsymbol{\ell}\Big)}{\sqrt{1-|y'|^2}} dy'. \tag{4.9}$$

Putting

$$\mathcal{M}_{n,q}(s,t;r,\boldsymbol{\ell}) = \mathcal{H}_{n,q}(s,t;r,\boldsymbol{\ell}) + \mathcal{H}_{n,q}(s,-t;r,\boldsymbol{\ell}), \tag{4.10}$$

and using (4.6)-(4.9), we rewrite (4.5) as

$$\mathcal{K}_p(x;\boldsymbol{\ell}) = \frac{N(r)}{\omega_n} \left\{ \int_{\mathbb{B}^{n-1}} \frac{\mathcal{M}_{n,q}\Big((y',\boldsymbol{\tau}'_x),\sqrt{1-|y'|^2};r,\boldsymbol{\ell}\Big)}{\sqrt{1-|y'|^2}}\ dy' \right\}^{1/q}. \tag{4.11}$$

Taking into account the equality

$$\int_{\mathbb{B}^n} g\big((\boldsymbol{y},\boldsymbol{\xi}),\ |\boldsymbol{y}|\big)dy = \omega_{n-1} \int_0^1 \rho^{n-1} d\rho \int_0^\pi g\big(|\boldsymbol{\xi}|\rho\cos\varphi,\ \rho\big)\sin^{n-2}\varphi\ d\varphi$$

(see, e.g., Prudnikov, Brychkov and Marichev [10], **3.3.2(3)**), we express the integral in (4.11) as

$$\begin{aligned}
&\int_{\mathbb{B}^{n-1}} \frac{\mathcal{M}_{n,q}\Big((y',\boldsymbol{\tau}'_x),\sqrt{1-|y'|^2};r,\boldsymbol{\ell}\Big)}{\sqrt{1-|y'|^2}}\ dy' \\
&\quad = \omega_{n-2} \int_0^1 \frac{\rho^{n-2}}{\sqrt{1-\rho^2}}\ d\rho \int_0^\pi \mathcal{M}_{n,q}\Big(\rho\cos\varphi,\ \sqrt{1-\rho^2};r,\boldsymbol{\ell}\Big)\sin^{n-3}\varphi d\varphi.
\end{aligned} \tag{4.12}$$

The change of variable $\rho = \sin\vartheta$ in (4.12) gives

$$\int_{\mathbb{B}^{n-1}} \frac{\mathcal{M}_{n,q}\Big((y', \boldsymbol{\tau}'_x), \sqrt{1-|y'|^2}; r, \boldsymbol{\ell}\Big)}{\sqrt{1-|y'|^2}}\, dy' \tag{4.13}$$

$$= \omega_{n-2}\int_0^{\pi/2} \sin^{n-2}\vartheta d\vartheta \int_0^{\pi} \mathcal{M}_{n,q}\Big(\sin\vartheta\cos\varphi,\ \cos\vartheta; r, \boldsymbol{\ell}\Big)\sin^{n-3}\varphi\, d\varphi\, .$$

Taking into account (4.6), (4.10) and introducing the notation

$$\begin{aligned}\mathcal{F}_{n,q}(\vartheta, \varphi; r, \boldsymbol{\ell}) &= \mathcal{H}_{n,q}\Big(\sin\vartheta\cos\varphi,\ \cos\vartheta; r, \boldsymbol{\ell}\Big)\\ &= \frac{\big|nd(r)\ell_\tau \sin\vartheta\cos\varphi + \big(\cos\vartheta - \lambda(r)r\big)\ell_\nu\big|^q}{\big(1-2r\cos\vartheta + r^2\big)^{(n+2)q/2}}\,,\end{aligned}$$

we write (4.13) as

$$\int_{\mathbb{B}^{n-1}} \frac{\mathcal{M}_{n,q}\Big((y', \boldsymbol{\tau}'_x), \sqrt{1-|y'|^2}; r, \boldsymbol{\ell}\Big)}{\sqrt{1-|y'|^2}}\, dy' \tag{4.14}$$

$$= \omega_{n-2}\int_0^{\pi/2} \sin^{n-2}\vartheta d\vartheta \int_0^{\pi}\Big(\mathcal{F}_{n,q}(\vartheta, \varphi; r, \boldsymbol{\ell}) + \mathcal{F}_{n,p}(\pi-\vartheta, \varphi; r, \boldsymbol{\ell})\Big)\sin^{n-3}\varphi\, d\varphi\, .$$

Changing the variable $\psi = \pi - \vartheta$, we obtain

$$\int_0^{\pi/2} \sin^{n-2}\vartheta\ d\vartheta \int_0^{\pi} \mathcal{F}_{n,q}\big(\pi-\vartheta, \varphi; r, \boldsymbol{\ell}\big)\sin^{n-3}\varphi\, d\varphi$$

$$= \int_{\pi/2}^{\pi} \sin^{n-2}\psi\ d\psi \int_0^{\pi} \mathcal{F}_{n,q}\big(\psi, \varphi; r, \boldsymbol{\ell}\big)\sin^{n-3}\varphi\, d\varphi,$$

which together with (4.14) leads to the representation of (4.11) as

$$\mathcal{K}_p(x; \boldsymbol{\ell}) = S_{n,q}(r)\left\{\int_0^{\pi} \sin^{n-2}\vartheta\ d\vartheta \int_0^{\pi} \mathcal{F}_{n,q}(\vartheta, \varphi; r, \boldsymbol{\ell})\sin^{n-3}\varphi\, d\varphi\right\}^{1/q}, \tag{4.15}$$

where

$$S_{n,q}(r) = \frac{N(r)(\omega_{n-2})^{1/q}}{\omega_n}\,. \tag{4.16}$$

Equality (4.15) proves the representation for $\mathcal{K}_p(x; \boldsymbol{\ell})$, given in the present Proposition.

Formula (4.3) follows directly from (4.2) and (4.15) for $\boldsymbol{\ell} = \boldsymbol{\nu}_x$. Similarly, (4.4) results from (4.2), (4.15) and the equality

$$\int_0^{\pi} \frac{\sin^{\mu-1}x}{(1-2r\cos x + r^2)^{\nu}}\, dx = B\left(\frac{\mu}{2},\ \frac{1}{2}\right) F\left(\nu,\ \nu + \frac{1-\mu}{2};\ \frac{1+\mu}{2};\ r^2\right)$$

(see, e.g., Prudnikov, Brychkov and Marichev [10], **2.5.16(43)**), where $B(u, v)$ is the Beta-function, and $F(a, b; c; x)$ is the hypergeometric Gauss function. □

Remark 1. Since the function (4.2) is even and 2π-periodic in φ, (4.15) can be written as

$$\mathcal{K}_p(x;\boldsymbol{\ell})=\frac{S_{n,q}(r)}{2^{1/q}}\left\{\int_0^\pi \sin^{n-2}\vartheta d\vartheta \int_0^{2\pi}\mathcal{F}_{n,q}(\vartheta,\varphi;r,\boldsymbol{\ell})|\sin^{n-3}\varphi|d\varphi\right\}^{1/q}. \quad (4.17)$$

We introduce a Cartesian coordinate system with origin $\mathcal{O}$ of the sphere $\mathbb{S}^2$ with orthonormal vectors $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$ of the axes $\mathcal{O}x_1, \mathcal{O}x_2, \mathcal{O}x_3$, respectively. We also use spherical coordinates ϑ and φ on $\mathbb{S}^2$, measuring the angle ϑ from the axis $\mathcal{O}x_3$ and the angle φ from the plane $x_1\mathcal{O}x_3$.

The double integral in (4.17) can be now written as the integral over $\mathbb{S}^2$:

$$\mathcal{K}_p(x;\boldsymbol{\ell})=\frac{S_{n,q}(r)}{2^{1/q}}\left\{\int_{\mathbb{S}^2}\frac{\left|\big(\boldsymbol{\eta}-\lambda(r)r\boldsymbol{k},\ nd(r)\ell_\tau\boldsymbol{i}+\ell_\nu\boldsymbol{k}\ \big)\right|^q}{|\boldsymbol{\eta}-r\boldsymbol{k}|^{(n+2)q}}\ |(\boldsymbol{\eta},\boldsymbol{j})|^{n-3}d\sigma_\eta\right\}^{1/q}.$$

This corresponds to the two-dimensional sphere in (2.3).

In the next assertion, the integral in ϑ in (4.17) is reduced to an integral with algebraic integrand. The expression for $\mathcal{K}_p(x;\boldsymbol{\ell})$, given below, contains two factors one of which is an explicitely given function increasing as $r\to 1$ to infinity and the second factor (the double integral) is a bounded function for $r\in[0,\ 1]$.

Lemma 1. *Let $u\in h^p(\mathbb{B})$, $1<p\le\infty$, and x be an arbitrary point in $\mathbb{B}$. The sharp coefficient $\mathcal{K}_p(x;\boldsymbol{\ell})$ in the inequality*

$$|\big(\nabla u(x),\boldsymbol{\ell}\big)|\le \mathcal{K}_p(x;\boldsymbol{\ell})||u||_p \quad (4.18)$$

is given by

$$\mathcal{K}_p(x;\boldsymbol{\ell})=\mathcal{B}_{n,q}(r)\left\{\int_0^\pi \sin^{n-3}\varphi\, d\varphi\int_0^\infty \mathcal{T}_{n,q}(t,\varphi;r,\boldsymbol{\ell})dt\right\}^{1/q}, \quad (4.19)$$

where $1/p+1/q=1$,

$$\mathcal{B}_{n,q}(r)=\frac{n+(n-2)r}{\omega_n}\left\{\frac{2^{n-1}\omega_{n-2}\big(a(r)\big)^{n-1}}{(1+r)^{n-1}(1-r)^{n(q-1)+1}}\right\}^{1/q}, \quad (4.20)$$

and

$$\mathcal{T}_{n,q}(t,\varphi;r,\boldsymbol{\ell})=\frac{\big|2c(r)\ell_\tau t\cos\varphi+(1-t^2)\ell_\nu\big|^q t^{n-2}}{[1+a^2(r)t^2]^{(n+2)q/2}[1+b^2(r)t^2]^{n-1-(nq)/2}}. \quad (4.21)$$

Here $\boldsymbol{\ell}=\ell_\tau\boldsymbol{\tau}_x+\ell_\nu\boldsymbol{\nu}_x$,

$$a(r)=\left(\frac{n+(n-2)r}{n-(n-2)r}\right)^{1/2},\qquad b(r)=\frac{1-r}{1+r}\left(\frac{n+(n-2)r}{n-(n-2)r}\right)^{1/2}, \quad (4.22)$$

and

$$c(r) = \frac{n}{\left(n^2 - (n-2)^2 r^2\right)^{1/2}} \ . \tag{4.23}$$

In particular,

$$\mathcal{K}_p(x) = \sup_{\gamma \geq 0} \frac{\mathcal{C}_{n,q}(r)}{\sqrt{c^2(r) + \gamma^2}} \left\{ \int_0^\pi \sin^{n-3}\varphi \, d\varphi \int_0^\infty \mathcal{P}_{n,q}(t, \varphi; r, \gamma)\, dt \right\}^{1/q}, \tag{4.24}$$

where

$$\mathcal{C}_{n,q}(r) = \frac{n}{\omega_n} \left\{ \frac{2^{n-1}\omega_{n-2}\big(a(r)\big)^{n+q-1}}{(1+r)^{n-1}(1-r)^{n(q-1)+1}} \right\}^{1/q}, \tag{4.25}$$

and

$$\mathcal{P}_{n,q}(t, \varphi; r, \gamma) = \frac{\left|1 + 2\gamma t \cos\varphi - t^2\right|^q t^{n-2}}{[1 + a^2(r)t^2]^{(n+2)q/2}[1 + b^2(r)t^2]^{n-1-(nq)/2}} \ . \tag{4.26}$$

Proof. Changing the order of integration in (4.15), we obtain

$$\mathcal{K}_p(x; \boldsymbol{\ell}) = S_{n,q}(r) \left\{ \int_0^\pi \sin^{n-3}\varphi \, d\varphi \int_0^\pi \mathcal{F}_{n,q}(\vartheta, \varphi; r, \boldsymbol{\ell}) \sin^{n-2}\vartheta \, d\vartheta \right\}^{1/q} \tag{4.27}$$

with the constant $S_{n,q}(r)$ defined by (4.16) and the function $\mathcal{F}_{n,q}(\vartheta, \varphi; r, \boldsymbol{\ell})$ given by (4.2). Making the change of variable

$$\vartheta = 2 \arctan\big(b(r)t\big)$$

in (4.27), we arrive at (4.19).

Since the integrand in (2.3) does not change when the unit vector $\boldsymbol{\ell}$ is replaced by $-\boldsymbol{\ell}$, we may assume that $\ell_\nu = (\boldsymbol{\ell}, \boldsymbol{\nu}_x) > 0$ in (2.8). Introducing the parameter $\gamma = c(r)\ell_\tau/\ell_\nu$ in (4.19) and using the equality $\ell_\tau^2 + \ell_\nu^2 = 1$ together with (2.8), we arrive at (4.24). □

5 The cases $p = 2$ and $p = \infty$

The next assertion gives expressions for $\mathcal{K}_2(x; \boldsymbol{\ell})$ for any direction $\boldsymbol{\ell}$, in particular, the sharp values for $\mathcal{K}_2(x; \boldsymbol{\nu}_x)$ and $\mathcal{K}_2(x; \boldsymbol{\tau}_x)$. It is also shown that the value of $\mathcal{K}_2(x; \boldsymbol{\ell})$ is squeezed between the minimal value $\mathcal{K}_2(x; \boldsymbol{\tau}_x)$ and the maximal value of $\mathcal{K}_2(x) = \mathcal{K}_2(x; \boldsymbol{\nu}_x)$.

Corollary 3. *Let $u \in h^2(\mathbb{B})$, and x be an arbitrary point in $\mathbb{B}$. The sharp coefficient $\mathcal{K}_2(x; \boldsymbol{\ell})$ in the inequality*

$$|\big(\nabla u(x), \boldsymbol{\ell}\big)| \leq \mathcal{K}_2(x; \boldsymbol{\ell}) ||u||_2 \tag{5.1}$$

is given by

$$\mathcal{K}_2(x; \boldsymbol{\ell}) = \left\{ \frac{n(1+r^2)\ell_\tau^2 + \left[n + (n^2+n-4)r^2 + (n-2)^2 r^4\right]\ell_\nu^2}{\omega_n (1-r^2)^{n+1}} \right\}^{1/2} . \tag{5.2}$$

In particular,

$$\mathcal{K}_2(x;\boldsymbol{\tau}_x) \le \mathcal{K}_2(x;\boldsymbol{\ell}) \le \mathcal{K}_2(x;\boldsymbol{\nu}_x), \tag{5.3}$$

where

$$\mathcal{K}_2(x;\boldsymbol{\nu}_x) = \left\{ \frac{n+(n^2+n-4)r^2+(n-2)^2r^4}{\omega_n(1-r^2)^{n+1}} \right\}^{1/2}, \tag{5.4}$$

and

$$\mathcal{K}_2(x;\boldsymbol{\tau}_x) = \left\{ \frac{n(1+r^2)}{\omega_n(1-r^2)^{n+1}} \right\}^{1/2}. \tag{5.5}$$

The equality sign on the left-hand side of (5.3) *is attained for* $\boldsymbol{\ell} = \boldsymbol{\tau}_x$ *and that on the right-hand side for* $\boldsymbol{\ell} = \boldsymbol{\nu}_x$.

Proof. By (4.19)-(4.21),

$$\mathcal{K}_2(x;\boldsymbol{\ell}) = \mathcal{B}_{n,2}(r) \left\{ \int_0^{\pi} \sin^{n-3}\varphi \, d\varphi \int_0^{\infty} \mathcal{T}_{n,2}(t,\varphi;r,\boldsymbol{\ell})dt \right\}^{1/2}, \tag{5.6}$$

where

$$\mathcal{B}_{n,2}(r) = \frac{n+(n-2)r}{\omega_n} \left\{ \frac{2^{n-1}\omega_{n-2}\big(a(r)\big)^{n-1}}{(1+r)^{n-1}(1-r)^{n+1}} \right\}^{1/2}, \tag{5.7}$$

and

$$\mathcal{T}_{n,2}(t,\varphi;r,\boldsymbol{\ell}) = \frac{\big[2c(r)\ell_\tau t\cos\varphi + (1-t^2)\ell_\nu\big]^2[1+b^2(r)t^2]t^{n-2}}{[1+a^2(r)t^2]^{n+2}[1+b^2(r)t^2]^{-1}}. \tag{5.8}$$

Using the equalities

$$\int_0^{\pi} \sin^{n-3}\varphi\cos\varphi \, d\varphi = 0,$$

$$\int_0^{\pi} \sin^{n-3}\varphi \, d\varphi = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}, \qquad \int_0^{\pi} \sin^{n-3}\varphi\cos^2\varphi \, d\varphi = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)}{(n-1)\Gamma\left(\frac{n-1}{2}\right)},$$

we write (5.6) as

$$\mathcal{K}_2(x;\boldsymbol{\ell}) = \mathcal{D}_n(r) \left\{ \int_0^{\infty} \mathcal{G}_n(t;r,\boldsymbol{\ell}) \, dt \right\}^{1/2}, \tag{5.9}$$

where

$$\mathcal{D}_n(r) = \mathcal{B}_{n,2}(r) \left(\frac{\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)}{(n-1)\Gamma\left(\frac{n-1}{2}\right)} \right)^{1/2}, \tag{5.10}$$

and

$$\mathcal{G}_n(t;r,\boldsymbol{\ell}) = \frac{\big[4c^2(r)t^2\ell_\tau^2 + (n-1)(1-t^2)^2\ell_\nu^2\big]\big[1+b^2(r)t^2\big]t^{n-2}}{\big[1+a^2(r)t^2\big]^{n+2}}. \tag{5.11}$$

Introducing the notations

$$\mathcal{J}_1(r) = 4c^2(r)\int_0^\infty \frac{\left[1+b^2(r)t^2\right]t^n}{\left[1+a^2(r)t^2\right]^{n+2}}\,dt\ , \tag{5.12}$$

$$\mathcal{J}_2(r) = (n-1)\int_0^\infty \frac{(1-t^2)^2\left[1+b^2(r)t^2\right]t^{n-2}}{\left[1+a^2(r)t^2\right]^{n+2}}\,dt\ , \tag{5.13}$$

by (5.9) and (5.11) we have

$$\mathcal{K}_2(x;\boldsymbol{\ell}) = \mathcal{D}_n(r)\left\{\mathcal{J}_1(r)\ell_\tau^2 + \mathcal{J}_2(r)\ell_\nu^2\right\}^{1/2}\ . \tag{5.14}$$

In view of the formula

$$\int_0^\infty \frac{t^{k-1}}{(1+p^2t^2)^m}\,dt = \frac{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{2m-k}{2}\right)}{2p^k(m-1)!}$$

(see e.g., Gradshtein and Ryzhik [2], **3.241(4)**), we find that (5.12), (5.13) become

$$\mathcal{J}_1(r) = \frac{2n^2\Gamma^2\left(\frac{n+1}{2}\right)(1+r^2)}{n!(1+r)^2\big(a(r)\big)^{n+1}\left[n^2-(n-2)^2r^2\right]}\ , \tag{5.15}$$

$$\mathcal{J}_2(r) = \frac{2n\Gamma^2\left(\frac{n+1}{2}\right)\left[n+(n^2+n-4)r^2+(n-2)r^4\right]}{n!(1+r)^2\big(a(r)\big)^{n+1}\left[n^2-(n-2)^2r^2\right]}\ . \tag{5.16}$$

Using (5.15), (5.16) in (5.14), taking into account (5.7) and (5.10), as well as the identity

$$2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2z)$$

with $z=(n-1)/2$, we arrive at (5.2).

Formulas (5.4) and (5.5) follow directly from (5.2) for $\ell_\nu = 1, \ell_\tau = 0$ and $\ell_\tau = 1, \ell_\nu = 0$, respectively.

Further, since $\ell_\tau^2 + \ell_\nu^2 = 1$ and

$$n(1+r^2) \le n + (n^2+n-4)r^2 + (n-2)^2r^4$$

for $n \ge 2$, it follows from (5.2) that

$$\left\{\frac{n(1+r^2)}{\omega_n(1-r^2)^{n+1}}\right\}^{1/2} \le \mathcal{K}_2(x;\boldsymbol{\ell}) \le \left\{\frac{n+(n^2+n-4)r^2+(n-2)^2r^4}{\omega_n(1-r^2)^{n+1}}\right\}^{1/2},$$

i.e. $\mathcal{K}_2(x;\boldsymbol{\tau}_x) \le \mathcal{K}_2(x;\boldsymbol{\ell}) \le \mathcal{K}_2(x;\boldsymbol{\nu}_x)$, which proves (5.3).

The monotonicity of $\mathcal{K}_2(x;\boldsymbol{\ell})$ as a function of ℓ_ν follows from (5.2) and the equality $\ell_\tau^2 + \ell_\nu^2 = 1$. □

The next assertion concerns particular cases of representations for $\mathcal{K}_p(x;\boldsymbol{\nu}_x)$ and $\mathcal{K}_p(x;\boldsymbol{\tau}_x)$ for $p=\infty$.

Corollary 4. *Let $u \in h^\infty(\mathbb{B})$, and let x be an arbitrary point in $\mathbb{B}$. The sharp coefficient $\mathcal{K}_\infty(x;\boldsymbol{\nu}_x)$ in the inequality*

$$|(\nabla u(x),\ \boldsymbol{\nu}_x)| \le \mathcal{K}_\infty(x;\boldsymbol{\nu}_x)\big\|u\big\|_\infty \tag{5.17}$$

is given by

$$\mathcal{K}_\infty(x;\boldsymbol{\nu}_x) = \mathcal{D}_n(r)\int_0^1 \frac{(1-t^2)t^{n-2}}{\big[1+a^2(r)t^2\big]^{(n+2)/2}\big[1+b^2(r)t^2\big]^{(n-2)/2}}\,dt\ , \tag{5.18}$$

where

$$\mathcal{D}_n(r) = \frac{2^n\omega_{n-1}\big[n+(n-2)r\big]\big(a(r)\big)^{n-1}}{\omega_n(1+r)^{n-1}(1-r)}\ ,$$

and $a(r)$, $b(r)$ are defined by (4.22).

In particular,

$$\mathcal{K}_\infty(x;\boldsymbol{\nu}_x) = \frac{1}{r^2}\left(\frac{\big(1+r^2/3\big)^{3/2}}{1-r^2} - 1\right) \tag{5.19}$$

for $n=3$, and

$$\mathcal{K}_\infty(x;\boldsymbol{\nu}_x) = \frac{1}{\pi r^3}\left(\frac{r(2+r^2)(4-r^2)^{1/2}}{(1-r^2)} - 2\arccos\frac{(2-r^2)^2-2}{2}\right) \tag{5.20}$$

for $n=4$.

The sharp coefficient $\mathcal{K}_\infty(x;\boldsymbol{\tau}_x)$ in the inequality

$$|(\nabla u(x),\ \boldsymbol{\tau}_x)| \le \mathcal{K}_\infty(x;\boldsymbol{\tau}_x)\big\|u\big\|_\infty \tag{5.21}$$

is given by

$$\mathcal{K}_\infty(x;\boldsymbol{\tau}_x) = \frac{2n\omega_{n-1}(1-r^2)}{(n-1)\omega_n}\ F\left(\frac{3}{2},\ \frac{n+2}{2};\ \frac{n+1}{2};\ r^2\right), \tag{5.22}$$

where $F(a,b;c;x)$ is the hypergeometric Gauss function.

In particular,

$$\mathcal{K}_\infty(x;\boldsymbol{\tau}_x) = \frac{2}{\pi(1-r^2)}\left[2\mathrm{K}(r) - (1+r^2)\mathrm{D}(r)\right] \tag{5.23}$$

for $n=3$, and

$$\mathcal{K}_\infty(x;\boldsymbol{\tau}_x) = \frac{8(1-r^2)}{\pi r^2}\left[\frac{1+r^2}{(1-r^2)^2} - \frac{1}{2r}\log\frac{1+r}{1-r}\right] \tag{5.24}$$

for $n=4$. Here $\mathrm{K}(r)$ is the complete elliptic integral of the first kind, and $\mathrm{D}(r)$ is the complete elliptic integral.

Proof. We put $\ell_\tau = 0, \ell_\nu = 1$ in (4.21) and we pass to the limit as $p \to \infty$ $(q \to 1)$ in (4.19). This results in

$$\mathcal{K}_\infty(x; \boldsymbol{\nu}_x) = \mathcal{B}_{n,1}(r) \int_0^\pi \sin^{n-3}\varphi \, d\varphi \int_0^\infty \mathcal{T}_{n,1}(t, \varphi; r, \boldsymbol{\nu}_x) dt \,, \tag{5.25}$$

where

$$\mathcal{B}_{n,1}(r) = \frac{2^{n-1}\omega_{n-2}\big(n + (n-2)r\big)\big(a(r)\big)^{n-1}}{\omega_n (1+r)^{n-1}(1-r)} \,, \tag{5.26}$$

and

$$\mathcal{T}_{n,1}(t, \varphi; r, \boldsymbol{\nu}_x) = \frac{\big|1 - t^2\big| t^{n-2}}{[1 + a^2(r)t^2]^{(n+2)/2}[1 + b^2(r)t^2]^{(n-2)/2}} \,. \tag{5.27}$$

Note that

$$\int_0^\infty \frac{\big(1 - t^2\big) t^{n-2}}{(1 + a^2(r)t^2)^{(n+2)/2}(1 + b^2(r)t^2)^{(n-2)/2}} \, dt = 0 \,. \tag{5.28}$$

In fact, setting $u \equiv 1$, $\boldsymbol{\ell} = \boldsymbol{\nu}_x$ in (2.17), we have

$$\int_{\mathbb{S}^{n-1}} \frac{\big(y, \ \boldsymbol{\nu}_x\big) - \lambda(r) r}{|y - x|^{n+2}} \, d\sigma_y = 0.$$

Writing the last integral in the same manner as we did in the proof of Proposition 1, we find

$$\int_0^\pi \frac{\cos\vartheta - \lambda(r) r}{\big(1 - 2r\cos\vartheta + r^2\big)^{n+2}} \sin^{n-2} d\vartheta = 0. \tag{5.29}$$

Making the change of variable $\vartheta = 2\arctan\big(b(r)t\big)$ in (5.29) as in Lemma 1, we arrive at (5.28).

By (5.28)

$$\int_0^1 \frac{\big(1 - t^2\big) t^{n-2}}{(1 + a^2(r)t^2)^{(n+2)/2}(1 + b^2(r)t^2)^{(n-2)/2}} \, dt$$

$$= -\int_1^\infty \frac{\big(1 - t^2\big) t^{n-2}}{(1 + a^2(r)t^2)^{(n+2)/2}(1 + b^2(r)t^2)^{(n-2)/2}} \, dt \,.$$

Therefore, (5.25) implies

$$\mathcal{K}_\infty(x; \boldsymbol{\nu}_x) = 2\mathcal{B}_{n,1}(r) \int_0^\pi \sin^{n-3}\varphi \, d\varphi \int_0^1 \mathcal{T}_{n,1}(t, \varphi; r, \boldsymbol{\nu}_x) dt.$$

Evaluating the integral in φ, we obtain (5.18).

Integrating in (5.18), we arrive at (5.19) and (5.20) for $n = 3$ and $n = 4$, respectively.

Equality (5.22) results from (4.4) with $q = 1$. Formulas (5.23) and (5.24) follow from the table of values of the hypergeometric Gauss function (Prudnikov, Brychkov and Marichev [11]). □

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