Wiener Type Regularity of a Boundary Point for the 3D Lamé System

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Abstract. In this paper, we study the 3D Lamé system and establish its weighted positive definiteness for a certain range of elastic constants. By modifying the general theory developed in [4], we then show, under the assumption of weighted positive definiteness, that the divergence of the classical Wiener integral for a boundary point guarantees the continuity of solutions to the Lamé system at this point.

Keywords: Weighted positivity, 3D Lamé system, Wiener type regularity

1. The Main Results

In our previous work [1], we studied weighted integral inequalities of the type

$$\int_{\Omega} Lu \cdot \Psi u \, dx \geq 0 \tag{1}$$

for general second order elliptic systems L in $\mathbb{R}^n$ ($n \geq 3$). For weights that are smooth and positive homogeneous of order $2 - n$, we have shown that L is positive definite in the sense of (1) only if the weight is the fundamental matrix of L, possibly multiplied by a semi-positive definite constant matrix.

A question that arises naturally is under what conditions are elliptic systems indeed positive definite with such weights. Although it is difficult to answer this question in general, we study, as a special case, the 3D Lamé system

$$Lu = -\Delta u - \alpha \operatorname{grad} \operatorname{div} u, \qquad u = (u_1, u_2, u_3)^T$$

in this paper, deriving sufficient conditions for its weighted positive definiteness and showing that some restrictions on the elastic constants are inevitable. By modifying the general theory developed in [4], we then show that the divergence of the classical Wiener integral for a boundary point guarantees the continuity of solutions to the Lamé system at this point, assuming the weighted positive definiteness.

We first recall the following definition.

DEFINITION 1.1. *Let L be the 3D Lamé system*

$$Lu = -\Delta u - \alpha \operatorname{grad} \operatorname{div} u = -D_{kk}u_i - \alpha D_{ki}u_k \qquad (i = 1, 2, 3),$$

where as usual repeated indices indicate summation. The system L is said to be positive definite with weight $\Psi(x) = (\Psi_{ij}(x))^3_{i,j=1}$ *if*

$$\int_{\mathbb{R}^3} (Lu)^T \Psi u \, dx = -\int_{\mathbb{R}^3} \left[D_{kk}u_i(x) + \alpha D_{ki}u_k(x) \right] u_j(x) \Psi_{ij}(x) \, dx \geq 0 \tag{2}$$

for all real valued, smooth vector functions $u = (u_i)^3_{i=1}$, $u_i \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$. *As usual, D denotes the gradient* $(D_1, D_2, D_3)^T$ *and Du is the Jacobian matrix of u.*

Remark. The 3D Lamé system satisfies the strong elliptic condition if and only if $\alpha > -1$, and we will make this assumption throughout this paper. □

The fundamental matrix of the 3D Lamé system is given by $\Phi = (\Phi_{ij})^3_{i,j=1}$, where

$$\Phi_{ij} = c_\alpha r^{-1} \left(\delta_{ij} + \frac{\alpha}{\alpha+2} \omega_i \omega_j \right) \qquad (i, j = 1, 2, 3), \tag{3}$$

$$c_\alpha = \frac{\alpha+2}{8\pi(\alpha+1)} > 0.$$

As usual, δ_{ij} is the Kronecker delta, $r = |x|$ and $\omega_i = x_i/|x|$.

The first result we shall prove in this paper is the following

THEOREM 1.2. *The 3D Lamé system L is positive definite with weight* Φ *when* $\alpha_- < \alpha < \alpha_+$, *where* $\alpha_- \approx -0.194$ *and* $\alpha_+ \approx 1.524$. *It is not positive definite with weight* Φ *when* $\alpha < \alpha_-^{(c)} \approx -0.902$ *or* $\alpha > \alpha_+^{(c)} \approx 39.450$.

The proof of this theorem is given in Section 2.

Let Ω be an open set in $\mathbb{R}^3$ and consider the Dirichlet problem

$$Lu = f, \qquad f_i \in C_0^\infty(\Omega), \ u_i \in \mathring{H}^1(\Omega). \tag{4}$$

As usual, $\mathring{H}^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the Sobolev norm:

$$\|f\|_{H^1(\Omega)} = \left[\|f\|^2_{L^2(\Omega)} + \|Df\|^2_{L^2(\Omega)} \right]^{1/2}.$$

DEFINITION 1.3. *The point* $P \in \partial\Omega$ *is regular with respect to L if, for any* $f = (f_i)^3_{i=1}$, $f_i \in C_0^\infty(\Omega)$, *the solution of* (4) *satisfies*

$$\lim_{\Omega \ni x \to P} u_i(x) = 0 \qquad (i = 1, 2, 3). \tag{5}$$

DEFINITION 1.4. *The classical harmonic capacity of a compact set K in $\mathbb{R}^3$ is given by:*

$$\mathrm{cap}(K) = \inf\left\{\int_{\mathbb{R}^3} |Df(x)|^2\, dx : f \in \mathcal{A}(K)\right\},$$

where

$$\mathcal{A}(K) = \left\{f \in C_0^\infty(\mathbb{R}^3) : f = 1 \text{ in a neighborhood of } K\right\}.$$

Note that an equivalent definition of $\mathrm{cap}(K)$ can be obtained by replacing $\mathcal{A}(K)$ with $\mathcal{A}_1(K)$ where

$$\mathcal{A}_1(K) = \left\{f \in C_0^\infty(\mathbb{R}^3) : f \geq 1 \text{ on } K\right\}.$$

([3], sec. 2.2.1).

Using Theorem 1.2, we will prove that the divergence of the classical Wiener integral for a boundary point P guarantees its regularity with respect to the Lamé system. To simplify notations we assume, without loss of generality, that $P = O$ is the origin of the space.

THEOREM 1.5. *Suppose the 3D Lamé system L is positive definite with weight Φ. Then $O \in \partial\Omega$ is regular with respect to L if*

$$\int_0^1 \mathrm{cap}(\bar{B}_\rho \setminus \Omega)\rho^{-2}\, d\rho = \infty. \tag{6}$$

As usual, B_ρ is the open ball centered at O with radius ρ.

The proof of this theorem is given in Section 3.

2. Proof of Theorem 1.2

We start the proof of Theorem 1.2 by rewriting the integral

$$\int_{\mathbb{R}^3} (Lu)^T \Phi u\, dx = -\int_{\mathbb{R}^3} \big(D_{kk}u_i + \alpha D_{ki}u_k\big)u_j \Phi_{ij}\, dx$$

into a more revealing form. In the following, we shall write $\int f\, dx$ instead of $\int_{\mathbb{R}^3} f\, dx$, and by u_{ii}^2 we always mean $\sum_{i=1}^3 u_{ii}^2$; to express $\left(\sum_{i=1}^3 u_{ii}\right)^2$ we will write $u_{ii}u_{jj}$ instead. Furthermore, we always assume $u_i \in C_0^\infty(\mathbb{R}^3)$ unless otherwise stated.

LEMMA 2.1.

$$\int (Lu)^T \Phi u \, dx = \frac{1}{2}|u(0)|^2 + \mathscr{B}(u, u) \qquad (7)$$

where

$$\mathscr{B}(u, u) = \frac{\alpha}{2} \int (u_j D_k u_k - u_k D_k u_j) D_i \Phi_{ij} \, dx$$
$$+ \int (D_k u_i D_k u_j + \alpha D_k u_k D_i u_j) \Phi_{ij} \, dx.$$

Proof. By definition,

$$\int (Lu)^T \Phi u \, dx = - \int D_{kk} u_i \cdot u_j \Phi_{ij} \, dx - \alpha \int D_{ki} u_k \cdot u_j \Phi_{ij} \, dx =: I_1 + I_2.$$

Since Φ is symmetric, we have $\Phi_{ij} = \Phi_{ji}$ and

$$I_1 = - \int D_{kk} u_i \cdot u_j \Phi_{ij} \, dx$$
$$= -\frac{1}{2} \int \Big[D_{kk}(u_i u_j) - 2 D_k u_i D_k u_j \Big] \Phi_{ij} \, dx$$
$$= -\frac{1}{2} \int u_i u_j D_{kk} \Phi_{ij} \, dx + \int D_k u_i D_k u_j \cdot \Phi_{ij} \, dx.$$

On the other hand, Φ is the fundamental matrix of L, so we have

$$-D_{kk}\Phi_{ij} - \alpha D_{ki}\Phi_{kj} = \delta_{ij}\delta(x),$$

and

$$-\frac{1}{2} \int u_i u_j D_{kk} \Phi_{ij} \, dx = \frac{1}{2} \int u_i u_j \Big[\delta_{ij}\delta(x) + \alpha D_{ki}\Phi_{kj} \Big] \, dx$$
$$= \frac{1}{2}|u(0)|^2 - \frac{\alpha}{2} \int (D_i u_i \cdot u_j + u_i D_i u_j) D_k \Phi_{kj} \, dx$$
$$= \frac{1}{2}|u(0)|^2 - \frac{\alpha}{2} \int (D_k u_k \cdot u_j + u_k D_k u_j) D_i \Phi_{ij} \, dx.$$

Now I_2 can be written as

$$I_2 = \alpha \int D_k u_k \big(D_i u_j \cdot \Phi_{ij} + u_j D_i \Phi_{ij} \big) \, dx,$$

and the lemma follows by adding up the results.

Remark. With $\Phi(x)$ replaced by $\Phi_y(x) := \Phi(x - y)$, we have

$$\int (Lu)^T \Phi_y u \, dx = \int (Lu_y)^T \Phi u_y \, dx \qquad (u_y(x) = u(x + y))$$
$$= \frac{1}{2}|u_y(0)|^2 + \mathscr{B}(u_y, u_y) =: \frac{1}{2}|u(y)|^2 + \mathscr{B}_y(u, u),$$

where

$$\mathscr{B}_y(u, u) = \frac{\alpha}{2} \int (u_j D_k u_k - u_k D_k u_j) D_i \Phi_{y,ij}\, dx$$
$$+ \int (D_k u_i D_k u_j + \alpha D_k u_k D_i u_j) \Phi_{y,ij}\, dx.$$

$\square$

To proceed, we introduce the following decomposition for $C_0^\infty(I\!R^3)$ functions:

$$f(x) = \bar{f}(r) + g(x), \qquad \bar{f} \in C_0^\infty[0, \infty),\ g \in C_0^\infty(I\!R^3),$$

where

$$\bar{f}(r) = \frac{1}{4\pi} \int_{S^2} f(r\omega)\, d\sigma.$$

Note that

$$\int_{S^2} g(r\omega)\, d\sigma = 0, \qquad \forall r \ge 0,$$

so we may think of $\bar{f}$ as the "0-th order harmonics" of the function f. We shall show below in Lemma 2.2 that all 0-th order harmonics in (7) are canceled out, so it is possible to control u by Du.

LEMMA 2.2. *With the decomposition*

$$u_i(x) = \bar{u}_i(r) + v_i(x) \qquad (i = 1, 2, 3) \tag{8}$$

where

$$\begin{cases} \bar{u}_i(r) = \dfrac{1}{4\pi} \displaystyle\int_{S^2} u_i(r\omega)\, d\sigma \\ \displaystyle\int_{S^2} v_i(r\omega)\, d\sigma = 0 \end{cases} \qquad \forall r \ge 0 \qquad (i = 1, 2, 3),$$

we have

$$\int (Lu)^T \Phi u\, dx = \frac{1}{2}|u(0)|^2 + \mathscr{B}^*(u, u) \tag{9}$$

where

$$\mathscr{B}^*(u, u) = \frac{\alpha}{2} \int (v_j D_k v_k - v_k D_k v_j) D_i \Phi_{ij}\, dx \tag{10}$$
$$+ \int (D_k u_i D_k u_j + \alpha D_k u_k D_i u_j) \Phi_{ij}\, dx.$$

Proof. By Lemma 2.1, it is enough to show

$$\int (u_j D_k u_k - u_k D_k u_j) D_i \Phi_{ij}\, dx = \int (v_j D_k v_k - v_k D_k v_j) D_i \Phi_{ij}\, dx.$$

Since

$$\int (u_j D_k u_k - u_k D_k u_j) D_i \Phi_{ij}\, dx$$

$$= \int (\bar{u}_j D_k \bar{u}_k - \bar{u}_k D_k \bar{u}_j) D_i \Phi_{ij}\, dx + \int (\bar{u}_j D_k v_k - \bar{u}_k D_k v_j) D_i \Phi_{ij}\, dx$$

$$+ \int (v_j D_k \bar{u}_k - v_k D_k \bar{u}_j) D_i \Phi_{ij}\, dx + \int (v_j D_k v_k - v_k D_k v_j) D_i \Phi_{ij}\, dx$$

$$=: I_1 + I_2 + I_3 + I_4,$$

it suffices to show $I_1 = I_2 = I_3 = 0$. Now

$$D_i \Phi_{ij} = D_i \left[c_\alpha r^{-1} \left(\delta_{ij} + \frac{\alpha}{\alpha + 2} \omega_i \omega_j \right) \right]$$

$$= -c_\alpha r^{-2} \omega_i \delta_{ij} + \frac{c_\alpha \alpha}{\alpha + 2} r^{-2} \left[-\omega_i^2 \omega_j + (\delta_{ii} - \omega_i^2)\omega_j + (\delta_{ji} - \omega_j \omega_i)\omega_i \right]$$

$$= -c_\alpha r^{-2} \omega_j + \frac{c_\alpha \alpha}{\alpha + 2} r^{-2} \omega_j =: d_\alpha r^{-2} \omega_j, \qquad (11)$$

where

$$d_\alpha = \frac{-2c_\alpha}{\alpha + 2} = \frac{-1}{4\pi(\alpha + 1)}.$$

We have

$$I_1 = d_\alpha \int r^{-2} \omega_j (\bar{u}_j D_r \bar{u}_k \cdot \omega_k - \bar{u}_k D_r \bar{u}_j \cdot \omega_k)\, dx \qquad (D_r = \partial/\partial r)$$

$$= d_\alpha \int r^{-2} (\bar{u}_j D_r \bar{u}_k \cdot \omega_j \omega_k - \bar{u}_k D_r \bar{u}_j \cdot \omega_k \omega_j)\, dx = 0,$$

$$I_3 = d_\alpha \int r^{-2} (v_j D_r \bar{u}_k \cdot \omega_j \omega_k - v_k D_r \bar{u}_j \cdot \omega_k \omega_j)\, dx = 0.$$

As for I_2, we obtain

$$I_2 = d_\alpha \int r^{-2} (\bar{u}_j D_k v_k \cdot \omega_j - \bar{u}_k D_k v_j \cdot \omega_j)\, dx$$

$$= d_\alpha \int r^{-2} (\bar{u}_j D_k v_k \cdot \omega_j - \bar{u}_j D_j v_k \cdot \omega_k)\, dx$$

$$= - \lim_{\epsilon \to 0^+} d_\alpha \int_{S^2} \left[\bar{u}_j(\epsilon) v_k(\epsilon \omega) \omega_j \omega_k - \bar{u}_j(\epsilon) v_k(\epsilon \omega) \omega_j \omega_k \right] d\sigma$$

$$- \lim_{\epsilon \to 0^+} d_\alpha \int_{\mathbb{R}^3 \setminus B_\epsilon} \left\{ v_k r^{-3} \left[-2\bar{u}_j \omega_j \omega_k + r D_r \bar{u}_j \cdot \omega_j \omega_k + \bar{u}_j (\delta_{jk} - \omega_j \omega_k) \right] \right.$$

$$\left. - v_k r^{-3} \left[-2\bar{u}_j \omega_j \omega_k + r D_r \bar{u}_j \cdot \omega_j \omega_k + \bar{u}_j (\delta_{kj} - \omega_k \omega_j) \right] \right\} dx = 0.$$

The result follows.

Remark. With $\Phi(x)$ replaced by $\Phi_y(x) := \Phi(x - y)$ and (8) replaced by

$$u_i(x) = \bar{u}_i(r_y) + v_i(x) \qquad (i = 1, 2, 3),$$

where $r_y = |x - y|$ and

$$\begin{cases} \bar{u}_i(r_y) = \dfrac{1}{4\pi} \displaystyle\int_{S^2} u_i(y + r_y \omega) \, d\sigma \\ \displaystyle\int_{S^2} v_i(y + r_y \omega) \, d\sigma = 0 \end{cases} \qquad \forall r_y \geq 0 \qquad (i = 1, 2, 3),$$

we have

$$\int (Lu)^T \Phi_y u \, dx = \frac{1}{2} |u(y)|^2 + \mathscr{B}_y^*(u, u)$$

where

$$\mathscr{B}_y^*(u, u) = \frac{\alpha}{2} \int (v_j D_k v_k - v_k D_k v_j) D_i \Phi_{y,ij} \, dx$$

$$+ \int (D_k u_i D_k u_j + \alpha D_k u_k D_i u_j) \Phi_{y,ij} \, dx.$$

$\square$

In the next Lemma, we use the definition of Φ and derive an explicit expression for the bilinear form $\mathscr{B}^*(u, u)$ defined in (10).

LEMMA 2.3.

$$\mathscr{B}^*(u, u) = c_\alpha \int \left\{ \frac{\alpha}{\alpha + 2} r^{-2} \Big[v_k (D_k v) \cdot \omega - (\operatorname{div} v)(v \cdot \omega) \Big] \right. \tag{12}$$

$$+ r^{-1} \Big[|D_r \bar{u}|^2 + \alpha \frac{2\alpha + 3}{\alpha + 2} (D_r \bar{u}_i)^2 \omega_i^2 + |Dv|^2 + \alpha (\operatorname{div} v)^2$$

$$+ \frac{\alpha}{\alpha + 2} |(D_k v) \cdot \omega|^2 + \frac{\alpha^2}{\alpha + 2} (\operatorname{div} v)[\omega_i (D_i v) \cdot \omega]$$

$$\left. + \alpha \frac{3\alpha + 4}{\alpha + 2} (D_r \bar{u} \cdot \omega)(\operatorname{div} v) + \alpha (D_r \bar{u} \cdot \omega)[\omega_i (D_i v) \cdot \omega] \Big] \right\} dx.$$

Before proving this lemma, we need a simple yet important observation that will be useful in the following computation.

LEMMA 2.4. *Let* $g \in C_0^\infty(\mathbb{R}^3)$ *be such that*

$$\int_{S^2} g(r\omega) \, d\sigma = 0, \qquad \forall r \geq 0.$$

Then

$$\begin{cases} \displaystyle\int \int f(r) g(x) \, dx = 0 \\ \displaystyle\int r^{-1} Df(r) \cdot Dg(x) \, dx = 0 \end{cases} \qquad \forall f \in C_0^\infty[0, \infty).$$

Proof. By switching to polar coordinates, we easily see that

$$\int f(r)g(x)\,dx = \int_0^\infty r^2 f(r)\,dr \int_{S^2} g(r\omega)\,d\sigma = 0.$$

On the other hand,

$$\int r^{-1} Df(r) \cdot Dg(x)\,dx = \int r^{-1} D_r f D_i g \cdot \omega_i\,dx$$

$$= -\int g\Big[-r^{-2}(D_r f)\omega_i^2 + r^{-1}(D_{rr} f)\omega_i^2 + r^{-2} D_r f(\delta_{ii} - \omega_i^2)\Big]\,dx$$

$$= -\int g\big(r^{-2} D_r f + r^{-1} D_{rr} f\big)\,dx = 0,$$

where the last equality follows by switching to polar coordinates.

Proof of Lemma 2.3. By definition,

$$\mathscr{B}^*(u,u) = \frac{\alpha}{2} \int (v_j D_k v_k - v_k D_k v_j) D_i \Phi_{ij}\,dx$$

$$+ \int (D_k u_i D_k u_j + \alpha D_k u_k D_i u_j) \Phi_{ij}\,dx =: I_1 + I_2.$$

We have shown in Lemma 2.2 that (see (11))

$$I_1 = 2^{-1}\alpha d_\alpha \int r^{-2}\omega_j(v_j D_k v_k - v_k D_k v_j)\,dx$$

$$= \frac{c_\alpha \alpha}{\alpha+2} \int r^{-2}\Big[v_k(D_k v)\cdot\omega - (\operatorname{div} v)(v\cdot\omega)\Big]\,dx.$$

On the other hand,

$$I_2 = c_\alpha \int r^{-1} D_k u_i D_k u_i\,dx + \frac{c_\alpha \alpha}{\alpha+2} \int r^{-1} D_k u_i D_k u_j \cdot \omega_i \omega_j\,dx$$

$$+ c_\alpha \alpha \int r^{-1} D_k u_k D_i u_i\,dx + \frac{c_\alpha \alpha^2}{\alpha+2} \int r^{-1} D_k u_k D_i u_j \cdot \omega_i \omega_j\,dx$$

$$=: I_3 + I_4 + I_5 + I_6.$$

Substituting $u_i = \bar{u}_i + v_i$ into I_3 and using Lemma 2.4 yields

$$I_3 = c_\alpha \int r^{-1}\big(D_r \bar{u}_i D_r \bar{u}_i \cdot \omega_k^2 + D_k v_i D_k v_i\big)\,dx + 2c_\alpha \int r^{-1} D_k \bar{u}_i D_k v_i\,dx$$

$$= c_\alpha \int r^{-1}\big(|D_r \bar{u}|^2 + |Dv|^2\big)\,dx. \tag{13}$$

Next,

$$I_5 = c_\alpha \alpha \int r^{-1}\big(D_r \bar{u}_k D_r \bar{u}_i \cdot \omega_k \omega_i + 2D_i v_i D_r \bar{u}_k \cdot \omega_k + D_k v_k D_i v_i\big)\,dx.$$

Note that for $k \neq i$,

$$\int r^{-1} D_r \bar{u}_k D_r \bar{u}_i \cdot \omega_k \omega_i \, dx = \int_0^\infty r D_r \bar{u}_k D_r \bar{u}_i \, dr \int_{S^2} \omega_k \omega_i \, d\sigma = 0,$$

and therefore

$$I_5 = c_\alpha \alpha \int r^{-1} \Big[(D_r \bar{u}_i)^2 \omega_i^2 + 2(\operatorname{div} v)(D_r \bar{u} \cdot \omega) + (\operatorname{div} v)^2 \Big] \, dx.$$

As for I_4,

$$
\begin{aligned}
I_4 &= \frac{c_\alpha \alpha}{\alpha + 2} \int r^{-1} D_k(\bar{u}_i + v_i) D_k(\bar{u}_j + v_j) \cdot \omega_i \omega_j \, dx \\
&= \frac{c_\alpha \alpha}{\alpha + 2} \int r^{-1} \big(D_r \bar{u}_i D_r \bar{u}_j \cdot \omega_i \omega_j \omega_k^2 + D_r \bar{u}_i D_k v_j \cdot \omega_i \omega_j \omega_k \\
&\qquad\qquad + D_k v_i D_r \bar{u}_j \cdot \omega_i \omega_j \omega_k + D_k v_i D_k v_j \cdot \omega_i \omega_j \big) \, dx \\
&= \frac{c_\alpha \alpha}{\alpha + 2} \int r^{-1} \Big[(D_r \bar{u}_i)^2 \omega_i^2 + 2(D_r \bar{u} \cdot \omega)[\omega_k(D_k v) \cdot \omega] + |D_k v \cdot \omega|^2 \Big] \, dx.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_6 &= \frac{c_\alpha \alpha^2}{\alpha + 2} \int r^{-1} D_k(\bar{u}_k + v_k) D_i(\bar{u}_j + v_j) \cdot \omega_i \omega_j \, dx \\
&= \frac{c_\alpha \alpha^2}{\alpha + 2} \int r^{-1} \big(D_r \bar{u}_k D_r \bar{u}_j \cdot \omega_i^2 \omega_j \omega_k + D_r \bar{u}_k D_i v_j \cdot \omega_i \omega_j \omega_k \\
&\qquad\qquad + D_r \bar{u}_j D_k v_k \cdot \omega_i^2 \omega_j + D_k v_k D_i v_j \cdot \omega_i \omega_j \big) \, dx \\
&= \frac{c_\alpha \alpha^2}{\alpha + 2} \int r^{-1} \Big[(D_r \bar{u}_j)^2 \omega_j^2 + (D_r \bar{u} \cdot \omega)[\omega_i(D_i v) \cdot \omega] \\
&\qquad\qquad + (D_r \bar{u} \cdot \omega)(\operatorname{div} v) + (\operatorname{div} v)[\omega_i(D_i v) \cdot \omega] \Big] \, dx.
\end{aligned}
$$

The lemma follows by adding up all these integrals. $\qquad\square$

With the help of Lemma 2.3, we now complete the proof of Theorem 1.2.

Proof of Theorem 1.2. By Lemma 2.2 and 2.3,

$$-c_\alpha^{-1} \int (Lu)^T \Phi u \, dx = \frac{1}{2} c_\alpha^{-1} |u(0)|^2 + I_1 + I_2 + I_3,$$

where

$$
\begin{aligned}
I_1 = \int r^{-1} \Big[&|D_r \bar{u}|^2 + \alpha \frac{2\alpha + 3}{\alpha + 2} (D_r \bar{u}_i)^2 \omega_i^2 + |Dv|^2 \\
&+ \alpha(\operatorname{div} v)^2 + \frac{\alpha}{\alpha + 2} |(D_k v) \cdot \omega|^2 \Big] \, dx,
\end{aligned}
$$

$$I_2 = \int r^{-1}\left[\frac{\alpha^2}{\alpha+2}(\operatorname{div} v)[\omega_i(D_i v)\cdot\omega] + \alpha\frac{3\alpha+4}{\alpha+2}(D_r\bar{u}\cdot\omega)(\operatorname{div} v)\right.$$
$$\left. + \alpha(D_r\bar{u}\cdot\omega)[\omega_i(D_i v)\cdot\omega]\right]dx,$$

$$I_3 = \int \frac{\alpha}{\alpha+2}r^{-2}\Big[v_k(D_k v)\cdot\omega - (\operatorname{div} v)(v\cdot\omega)\Big]dx.$$

Consider first the case $\alpha \geq 0$. By switching to polar coordinates, we have

$$I_1 \geq \int r^{-1}\left[|D_r\bar{u}|^2 + \alpha\frac{2\alpha+3}{\alpha+2}(D_r\bar{u}_i)^2\omega_i^2 + |Dv|^2 + \alpha(\operatorname{div} v)^2\right]dx$$
$$= \int_0^\infty r\left[\left(1 + \frac{\alpha}{3}\cdot\frac{2\alpha+3}{\alpha+2}\right)\|D_r\bar{u}\|_\omega^2 + \|Dv\|_\omega^2 + \alpha\|\operatorname{div} v\|_\omega^2\right]dr,$$

where we have written $\|\cdot\|_\omega$ for $\|\cdot\|_{L^2(S^2)}$ and used the fact that

$$\int_{S^2}(D_r\bar{u}_i)^2\omega_i^2\,d\sigma = \frac{4\pi}{3}\sum_{i=1}^3(D_r\bar{u}_i)^2 = \frac{1}{3}\int_{S^2}|D_r\bar{u}|^2\,d\sigma = \frac{1}{3}\|D_r\bar{u}\|_\omega^2.$$

Next,

$$|I_2| \leq \int r^{-1}\left[\frac{\alpha^2}{\alpha+2}|\operatorname{div} v||Dv| + \alpha\frac{3\alpha+4}{\alpha+2}|D_r\bar{u}\cdot\omega||\operatorname{div} v| + \alpha|D_r\bar{u}\cdot\omega||Dv|\right]dx$$
$$\leq \int_0^\infty r\left[\frac{\alpha^2}{\alpha+2}\|\operatorname{div} v\|_\omega\|Dv\|_\omega + \frac{\alpha}{\sqrt{3}}\cdot\frac{3\alpha+4}{\alpha+2}\|D_r\bar{u}\|_\omega\|\operatorname{div} v\|_\omega\right.$$
$$\left. + \frac{\alpha}{\sqrt{3}}\|D_r\bar{u}\|_\omega\|Dv\|_\omega\right]dr,$$

where we have used

$$\|D_r\bar{u}\cdot\omega\|_\omega^2 = \int_{S^2}D_r\bar{u}_i D_r\bar{u}_j\cdot\omega_i\omega_j\,d\sigma$$
$$= D_r\bar{u}_i D_r\bar{u}_j\cdot\frac{4\pi}{3}\delta_{ij} = \frac{4\pi}{3}\sum_{i=1}^3(D_r\bar{u}_i)^2 = \frac{1}{3}\|D_r\bar{u}\|_\omega^2.$$

As for I_3, we note that

$$|I_3| \leq \frac{\alpha}{\alpha+2}\int r^{-2}\big(|v||Dv| + |v||\operatorname{div} v|\big)\,dx$$
$$\leq \frac{\alpha}{\alpha+2}\int_0^\infty \|v\|_\omega\big(\|Dv\|_\omega + \|\operatorname{div} v\|_\omega\big)\,dr.$$

Since 2 is the first non-trivial eigenvalue of the Laplace-Beltrami operator on S^2, we have

$$\|v\|_\omega^2 = \int_{S^2} |v(r\omega)|^2 \, d\sigma \leq \frac{1}{2} \int_{S^2} |D_\omega[v(r\omega)]|^2 \, d\sigma$$
$$= \frac{r^2}{2} \int_{S^2} |(D_\omega v)(r\omega)|^2 \, d\sigma \leq \frac{r^2}{2} \|Dv\|_\omega^2, \tag{14}$$

where D_ω is the gradient operator on S^2. Thus

$$|I_3| \leq \frac{1}{\sqrt{2}} \cdot \frac{\alpha}{\alpha + 2} \int_0^\infty r \Big[\|Dv\|_\omega^2 + \|Dv\|_\omega \| \operatorname{div} v\|_\omega \Big] \, dr,$$

and by putting all pieces together we obtain

$$I_1 + I_2 + I_3 \geq \int_0^\infty r(w^T B_+ w) \, dr, \tag{15}$$

where

$$w = \big(\|D_r \bar{u}\|_\omega, \|Dv\|_\omega, \| \operatorname{div} v\|_\omega \big)^T,$$

$$B_+ = \begin{bmatrix} 1 + \dfrac{\alpha}{3} \cdot \dfrac{2\alpha + 3}{\alpha + 2} & -\dfrac{\alpha}{2\sqrt{3}} & -\dfrac{\alpha}{2\sqrt{3}} \cdot \dfrac{3\alpha + 4}{\alpha + 2} \\ -\dfrac{\alpha}{2\sqrt{3}} & 1 - \dfrac{1}{\sqrt{2}} \cdot \dfrac{\alpha}{\alpha + 2} & -\dfrac{\alpha}{2} \cdot \dfrac{\alpha + 2^{-1/2}}{\alpha + 2} \\ -\dfrac{\alpha}{2\sqrt{3}} \cdot \dfrac{3\alpha + 4}{\alpha + 2} & -\dfrac{\alpha}{2} \cdot \dfrac{\alpha + 2^{-1/2}}{\alpha + 2} & \alpha \end{bmatrix}.$$

Clearly, the weighted positive definiteness of L follows from the positive definiteness of B_+, because the latter implies, for some $c > 0$, that

$$\int_0^\infty r(w^T B_+ w) \, dr \geq c \int_0^\infty r|w|^2 \, dr$$
$$\geq c \int_0^\infty r \big(\|D_r \bar{u}\|_\omega^2 + \|Dv\|_\omega^2 \big) \, dr = c \int r^{-1} |Du|^2 \, dx.$$

The positive definiteness of B_+, on the other hand, is equivalent to the positivity of the determinants of all leading principal minors of B_+:

$$p_{+,1}(\alpha) = \frac{2\alpha^2 + 6\alpha + 6}{3(\alpha + 2)} > 0, \tag{16a}$$

$$p_{+,2}(\alpha) = -\frac{1}{12(\alpha + 2)^2} \Big[\alpha^4 - 4(1 - \sqrt{2})\alpha^3 - 12(3 - \sqrt{2})\alpha^2$$
$$- 12(6 - \sqrt{2})\alpha - 48 \Big] > 0, \tag{16b}$$

$$p_{+,3}(\alpha) = -\frac{\alpha}{12(\alpha+2)^3}\Big[6\alpha^5 + (23+3\sqrt{2})\alpha^4 + (13+19\sqrt{2})\alpha^3$$
$$- (77-38\sqrt{2})\alpha^2 - (157-24\sqrt{2})\alpha - 96\Big] > 0. \tag{16c}$$

With the help of computer algebra packages, we find that (16) holds for $0 \leq \alpha < \alpha_+$, where $\alpha_+ \approx 1.524$ is the largest real root of $p_{+,3}$.

The estimates of I_1, I_2, and I_3 are slightly different when $\alpha < 0$, since now the quadratic term $\alpha\|\operatorname{div}v\|_\omega^2$ in I_1 is negative. This means that it is no longer possible to control the $\|\operatorname{div}v\|_\omega$ terms in I_2, I_3 by $\alpha\|\operatorname{div}v\|_\omega^2$, and in order to obtain positivity we need to bound $\|\operatorname{div}v\|_\omega$ by $\|Dv\|_\omega$ as follows:

$$\|\operatorname{div}v\|_\omega^2 \leq 3\|Dv\|_\omega^2.$$

This leads to the following revised estimates:

$$I_1 \geq \int_0^\infty r\left[\left(1+\frac{\alpha}{3}\cdot\frac{2\alpha+3}{\alpha+2}\right)\|D_r\bar{u}\|_\omega^2 + \|Dv\|_\omega^2 + 3\alpha\|Dv\|_\omega^2 + \frac{\alpha}{\alpha+2}\|Dv\|_\omega^2\right]dr,$$

$$|I_2| \leq \int_0^\infty r\left[\frac{\sqrt{3}\,\alpha^2}{\alpha+2}\|Dv\|_\omega^2 - \alpha\frac{3\alpha+4}{\alpha+2}\|D_r\bar{u}\|_\omega\|Dv\|_\omega - \frac{\alpha}{\sqrt{3}}\|D_r\bar{u}\|_\omega\|Dv\|_\omega\right]dr,$$

$$|I_3| \leq -\frac{1}{\sqrt{2}}\cdot\frac{\alpha}{\alpha+2}\int_0^\infty r\Big[\|Dv\|_\omega^2 + \sqrt{3}\,\|Dv\|_\omega^2\Big]dr.$$

Hence

$$I_1 + I_2 + I_3 \geq \int_0^\infty r\left(w^T B_- w\right)dr, \tag{17}$$

where

$$w = \left(\|D_r\bar{u}\|_\omega, \|Dv\|_\omega\right)^T,$$

$$B_- = \begin{bmatrix} 1+\dfrac{\alpha}{3}\cdot\dfrac{2\alpha+3}{\alpha+2} & \dfrac{\alpha}{2}\cdot\dfrac{3\alpha+4}{\alpha+2}+\dfrac{\alpha}{2\sqrt{3}} \\ \dfrac{\alpha}{2}\cdot\dfrac{3\alpha+4}{\alpha+2}+\dfrac{\alpha}{2\sqrt{3}} & 1+3\alpha+\dfrac{\alpha}{\alpha+2}\Big(1+\dfrac{1+\sqrt{3}}{\sqrt{2}}-\sqrt{3}\,\alpha\Big) \end{bmatrix}.$$

The positive definiteness of B_- is equivalent to:

$$p_{-,1}(\alpha) = \frac{2\alpha^2+6\alpha+6}{3(\alpha+2)} > 0, \tag{18a}$$

$$p_{-,2}(\alpha) = \frac{1}{6(\alpha+2)^2}\Big[-(2+7\sqrt{3})\alpha^4 + 2(15+\sqrt{2}-11\sqrt{3}+\sqrt{6})\alpha^3$$
$$+ 2(57+3\sqrt{2}-10\sqrt{3}+3\sqrt{6})\alpha^2 + 6(20+\sqrt{2}+\sqrt{6})\alpha + 24\Big] > 0, \tag{18b}$$

and (18) holds for $\alpha_- < \alpha < 0$, where $\alpha_- \approx -0.194$ is the smallest real root of $p_{-,2}$.

Now we show that the 3D Lamé system is not positive definite with weight Φ when α is either too close to -1 or too large. By Proposition 3.11 in [1], the 3D Lamé system is positive definite with weight Φ only if

$$\sum_{i,\beta,\gamma} A_{ip}^{\beta\gamma} \xi_\beta \xi_\gamma \Phi_{ip}(\omega) \geq 0, \qquad \forall \xi \in \mathbb{R}^3, \ \forall \omega \in S^2 \qquad (p = 1, 2, 3),$$

where

$$A_{ij}^{\beta\gamma} = \delta_{ij}\delta_{\beta\gamma} + \frac{\alpha}{2}(\delta_{i\beta}\delta_{j\gamma} + \delta_{i\gamma}\delta_{j\beta})$$

and (see equation (3))

$$\Phi_{ij}(\omega) = c_\alpha r^{-1}\Big(\delta_{ij} + \frac{\alpha}{\alpha+2}\omega_i\omega_j\Big) \qquad (i, j = 1, 2, 3).$$

This means, in particular, that the matrix

$$A(\omega; \alpha) := \left(\sum_{i=1}^{3} A_{i1}^{\beta\gamma} \Phi_{i1}(\omega)\right)_{\beta,\gamma=1}^{3}$$

$$= \frac{c_\alpha r^{-1}}{2(\alpha+2)} \begin{bmatrix} 2(\alpha+1)(\alpha+2+\alpha\omega_1^2) & \alpha^2\omega_1\omega_2 & \alpha^2\omega_1\omega_3 \\ \alpha^2\omega_1\omega_2 & 2(\alpha+2+\alpha\omega_1^2) & 0 \\ \alpha^2\omega_1\omega_3 & 0 & 2(\alpha+2+\alpha\omega_1^2) \end{bmatrix}$$

is semi-positive definite for any $\omega \in S^2$ if the 3D Lamé system is positive definite with weight Φ. But $A(\omega; \alpha)$ is semi-positive definite only if the determinant of its leading principal minor

$$d_2(\omega; \alpha) := \det \begin{bmatrix} 2(\alpha+1)(\alpha+2+\alpha\omega_1^2) & \alpha^2\omega_1\omega_2 \\ \alpha^2\omega_1\omega_2 & 2(\alpha+2+\alpha\omega_1^2) \end{bmatrix}$$
$$= 4(\alpha+1)(\alpha+2+\alpha\omega_1^2)^2 - \alpha^4\omega_1^2\omega_2^2$$

is non-negative, and elementary estimate shows that

$$\min_{\omega \in S^2} d_2(\omega; \alpha) \leq d_2\big[(2^{-1/2}, 2^{-1/2}, 0); \alpha\big]$$

$$= (\alpha+1)(3\alpha+4)^2 - \frac{\alpha^4}{4} =: q(\alpha).$$

It follows that the 3D Lamé system is not positive definite with weight Φ when $q(\alpha) < 0$, which holds for $\alpha < \alpha_-^{(c)} \approx -0.902$ or $\alpha > \alpha_+^{(c)} \approx 39.450$. $\qquad\square$

Remark. We have in fact shown that, for $\alpha_- < \alpha < \alpha_+$ and some $c > 0$ depending on α,

$$\int (Lu)^T \Phi u \, dx \geq \frac{1}{2}|u(0)|^2 + c \int |Du(x)|^2 \frac{dx}{|x|}.$$

If we replace $\Phi(x)$ by $\Phi_y(x) := \Phi(x-y)$, then

$$
\begin{aligned}
\int (Lu)^T \Phi_y u \, dx &= \int [Lu(x+y)]^T \Phi u(x+y) \, dx \\
&\geq \frac{1}{2}|u(y)|^2 + c \int |Du(x+y)|^2 \frac{dx}{|x|} \\
&\geq \frac{1}{2}|u(y)|^2 + c \int \frac{|Du(x)|^2}{|x-y|} \, dx. \qquad (19)
\end{aligned}
$$

$\square$

3. Proof of Theorem 1.5

In the next lemma and henceforth, we use the notation

$$
m_\rho(u) = \rho^{-3} \int_{\Omega \cap S_\rho} |u(x)|^2 \, dx, \qquad S_\rho = \{x : \rho < |x| < 2\rho\},
$$

$$
M_\rho(u) = \rho^{-3} \int_{\Omega \cap B_\rho} |u(x)|^2 \, dx.
$$

LEMMA 3.1. *Suppose L is positive definite with weight Φ, and let $u = (u_i)_{i=1}^3$, $u_i \in \overset{\circ}{H}^1(\Omega)$ be a solution of*

$$
Lu = 0 \qquad on \ \Omega \cap B_{2\rho}.
$$

Then

$$
\int_\Omega [L(u\eta_\rho)]^T \Phi_y u\eta_\rho \, dx \leq c m_\rho(u), \qquad \forall y \in B_\rho,
$$

where $\eta_\rho(x) = \eta(x/\rho)$, $\eta \in C_0^\infty(B_{5/3})$, $\eta = 1$ on $B_{4/3}$, and $\Phi_y(x) = \Phi(x-y)$.

   Proof. By definition of u,

$$
\int_\Omega [L(u\eta_\rho)]^T \Phi_y u\eta_\rho \, dx = \int_\Omega [L(u\eta_\rho)]^T \Phi_y u\eta_\rho \, dx - \int_\Omega (Lu)^T \Phi_y u\eta_\rho^2 \, dx,
$$

where the second integral on the right side vanishes and the first one equals

$$
-\int_\Omega \Big[2D_k u_i D_k \eta_\rho + u_i D_{kk}\eta_\rho + \alpha(D_i u_k D_k \eta_\rho + D_k u_k D_i \eta_\rho + u_k D_{ki}\eta_\rho) \Big] u_j \eta_\rho (\Phi_y)_{ij} \, dx.
$$

Note that $D\eta_\rho$, $D^2\eta_\rho$ have compact support in $R := B_{5\rho/3} \setminus B_{4\rho/3}$, $|D^k \eta_\rho| \leq c\rho^{-k}$, and

$$
|\Phi_{y,ij}(x)| \leq \frac{c}{|x-y|} \leq c\rho^{-1}, \qquad \forall x \in R, \ \forall y \in B_\rho.
$$

Thus

$$\int_\Omega [L(u\eta_\rho)]^T \Phi_y u\eta_\rho \, dx \le c \int_{\Omega \cap R} \rho^{-2}|u||Du| \, dx + c \int_{\Omega \cap R} \rho^{-3}|u|^2 \, dx$$

$$\le c\left[\rho^{-3} \int_{\Omega \cap S_\rho} |u|^2 \, dx\right]^{1/2} \left[\rho^{-1} \int_{\Omega \cap R} |Du|^2 \, dx\right]^{1/2} + c\rho^{-3} \int_{\Omega \cap S_\rho} |u|^2 \, dx.$$

The lemma then follows from the well-known local energy estimate [5]

$$\rho^{-1} \int_{\Omega \cap R} |Du|^2 \, dx \le \rho^{-3} \int_{\Omega \cap S_\rho} |u|^2 \, dx.$$

Combining (19) (with u replaced by $u\eta_\rho$) and Lemma 3.1, we arrive at the following local estimate.

COROLLARY 3.2. *Let the conditions of Lemma 3.1 be satisfied. Then*

$$|u(y)|^2 + \int_{\Omega \cap B_\rho} \frac{|Du(x)|^2}{|x-y|} \, dx \le c m_\rho(u), \qquad \forall y \in \Omega \cap B_\rho.$$

To proceed, we need the following Poincaré-type inequality (see [2]).

LEMMA 3.3. *Let $u = (u_i)_{i=1}^3$ be any vector function with $u_i \in \mathring{H}^1(\Omega)$. Then for any $\rho > 0$,*

$$m_\rho(u) \le \frac{c}{\mathrm{cap}(\bar{S}_\rho \setminus \Omega)} \int_{\Omega \cap S_\rho} |Du|^2 \, dx$$

where c is independent of ρ.

The next corollary is a direct consequence of Corollary 3.2 and Lemma 3.3.

COROLLARY 3.4. *Let the conditions of Lemma 3.1 be satisfied. Then*

$$|u(y)|^2 + \int_{\Omega \cap B_\rho} \frac{|Du(x)|^2}{|x-y|} \, dx \le \frac{c}{\mathrm{cap}(\bar{S}_\rho \setminus \Omega)} \int_{\Omega \cap S_\rho} |Du|^2 \, dx, \qquad \forall y \in \Omega \cap B_\rho.$$

We are now in a position to prove the following lemma, which is the key ingredient in the proof of Theorem 1.5.

LEMMA 3.5. *Suppose L is positive definite with weight Φ, and let $u = (u_i)_{i=1}^3$, $u_i \in \mathring{H}^1(\Omega)$ be a solution of $Lu = 0$ on $\Omega \cap B_{2R}$. Then, for all $\rho \in (0, R)$,*

$$\sup_{x \in \Omega \cap B_\rho} |u(x)|^2 + \int_{\Omega \cap B_\rho} |Du(x)|^2 \frac{dx}{|x|} \le c_1 M_{2R}(u) \exp\left[-c_2 \int_\rho^R \mathrm{cap}(\bar{B}_r \setminus \Omega) r^{-2} \, dr\right],$$

$$(20)$$

where c_1, c_2 are independent of ρ.

Proof. Define

$$\gamma(r) := r^{-1}\mathrm{cap}(\bar{S}_r \setminus \Omega).$$

We first claim that $\gamma(r)$ is bounded from above by some absolute constant A. Indeed, The monotonicity of capacity implies that

$$\mathrm{cap}(\bar{S}_r \setminus \Omega) \le \mathrm{cap}(\bar{B}_r).$$

By choosing smooth test functions $\eta_r(x) = \eta(x/r)$ with $\eta \in C_0^\infty(B_2)$ and $\eta = 1$ on $B_{3/2}$, we also have

$$\mathrm{cap}(\bar{B}_r) \le \int_{\mathbb{R}^3} |D\eta_r|^2\, dx \le \sup_{x \in \mathbb{R}^3} |D\eta(x)|^2 \int_{B_{2r}} r^{-2}\, dx$$

$$= \Big[\frac{32}{3}\pi \sup_{x \in \mathbb{R}^3} |D\eta(x)|^2\Big] r.$$

Hence the claim follows.

We next consider the case $\rho \in (0, R/2]$. Denote the first and second terms on the left side of (20) by φ_ρ and ψ_ρ, respectively. From Corollary 3.4, it follows that for $r \le R$,

$$\varphi_r + \psi_r \le \frac{c}{\gamma(r)}(\psi_{2r} - \psi_r) \le \frac{c}{\gamma(r)}(\psi_{2r} - \psi_r + \varphi_{2r} - \varphi_r),$$

which implies that

$$\varphi_r + \psi_r \le \frac{c}{c + \gamma(r)}(\varphi_{2r} + \psi_{2r}) = \frac{ce^{c_0\gamma(r)}}{c + \gamma(r)}\Big[e^{-c_0\gamma(r)}(\varphi_{2r} + \psi_{2r})\Big], \qquad \forall c_0 > 0.$$

Since $\gamma(r) \le A$ and

$$\sup_{s \in [0,A]} \frac{ce^{c_0 s}}{c + s} \le \max\Big\{1, \frac{ce^{c_0 A}}{c + A}, cc_0 e^{1 - cc_0}\Big\},$$

it is possible to choose $c_0 > 0$ sufficiently small so that

$$\sup_{r > 0} \frac{ce^{c_0\gamma(r)}}{c + \gamma(r)} \le 1.$$

It follows, for c_0 chosen this way, that

$$\varphi_r + \psi_r \le e^{-c_0\gamma(r)}(\varphi_{2r} + \psi_{2r}). \tag{21}$$

By setting $r = 2^{-l}R$ ($l \in \mathbb{N}$) and repeatedly applying (21), we obtain

$$\varphi_{2^{-l}R} + \psi_{2^{-l}R} \le \exp\Big[-c_0 \sum_{j=1}^{l} \gamma(2^{-j}R)\Big](\varphi_R + \psi_R).$$

If l is such that $l \leq \log_2(R/\rho) < l+1$, then $\rho \leq 2^{-l}R < 2\rho$ and

$$\varphi_\rho + \psi_\rho \leq \varphi_{2^{-l}R} + \psi_{2^{-l}R} \leq \exp\left[-c_0 \sum_{j=1}^{l} \gamma(2^{-j}R)\right](\varphi_R + \psi_R).$$

Note that by Corollary 3.2,

$$\varphi_R + \psi_R \leq cm_R(u) \leq cM_{2R}(u).$$

In addition, the subadditivity of the harmonic capacity implies that

$$\sum_{j=1}^{l} \gamma(2^{-j}R) \geq \sum_{j=1}^{l} \frac{\mathrm{cap}(\bar{B}_{2^{1-j}R} \setminus \Omega) - \mathrm{cap}(\bar{B}_{2^{-j}R} \setminus \Omega)}{2^{-j}R}$$

$$= \left[\frac{\mathrm{cap}(\bar{B}_R \setminus \Omega)}{2^{-1}R} - \frac{\mathrm{cap}(\bar{B}_{2^{-l}R} \setminus \Omega)}{2^{-l}R}\right] + \sum_{j=1}^{l-1} \frac{\mathrm{cap}(\bar{B}_{2^{-j}R} \setminus \Omega)}{2^{-j}R}$$

$$= \frac{1}{2} \cdot \frac{\mathrm{cap}(\bar{B}_R \setminus \Omega)}{R} - 2\frac{\mathrm{cap}(\bar{B}_{2^{-l}R} \setminus \Omega)}{2^{-l}R} + \sum_{j=1}^{l} \frac{\mathrm{cap}(\bar{B}_{2^{-j}R} \setminus \Omega)}{2^{-j}R}$$

$$\geq -2\frac{\mathrm{cap}(\bar{B}_{2^{-l}R} \setminus \Omega)}{2^{-l}R} + \frac{1}{2}\sum_{j=0}^{l} \frac{\mathrm{cap}(\bar{B}_{2^{-j}R} \setminus \Omega)}{2^{-j}R}.$$

Since

$$\frac{\mathrm{cap}(\bar{B}_{2^{-l}R} \setminus \Omega)}{2^{-l}R} \leq A,$$

$$\sum_{j=0}^{l} \frac{\mathrm{cap}(\bar{B}_{2^{-j}R} \setminus \Omega)}{2^{-j}R} \geq \frac{1}{2}\sum_{j=1}^{l+1} \frac{\mathrm{cap}(\bar{B}_{2^{1-j}R} \setminus \Omega)}{(2^{-j}R)^2} \cdot 2^{-j}R$$

$$\geq \frac{1}{2}\sum_{j=1}^{l+1} \int_{2^{-j}R}^{2^{1-j}R} \mathrm{cap}(\bar{B}_r \setminus \Omega)r^{-2}\,dr$$

$$\geq \frac{1}{2}\int_{\rho}^{R} \mathrm{cap}(\bar{B}_r \setminus \Omega)r^{-2}\,dr,$$

we have

$$\exp\left[-c_0\sum_{j=1}^{l}\gamma(2^{-j}R)\right] \leq \exp\left[-\frac{c_0}{4}\int_{\rho}^{R}\mathrm{cap}(\bar{B}_r \setminus \Omega)r^{-2}\,dr + 2c_0A\right].$$

Hence (20) follows with $c_1 = ce^{2c_0A}$ and $c_2 = c_0/4$.

Finally we consider the case $\rho \in (R/2, R)$. By Corollary 3.2,

$$|u(y)|^2 + \int_{\Omega \cap B_\rho} \frac{|Du(x)|^2}{|x-y|}\,dx \leq cm_\rho(u), \qquad \forall y \in \Omega \cap B_\rho,$$

which implies that

$$\sup_{y\in\Omega\cap B_\rho}|u(y)|^2 + \int_{\Omega\cap B_\rho}|Du(x)|^2\frac{dx}{|x|} \le cM_{2R}(u).$$

In addition,

$$\int_\rho^R \mathrm{cap}(\bar{B}_r\setminus\Omega)r^{-2}\,dr \le A\int_{R/2}^R r^{-1}\,dr = A\log 2,$$

so

$$\left[\sup_{y\in\Omega\cap B_\rho}|u(y)|^2 + \int_{\Omega\cap B_\rho}|Du(x)|^2\frac{dx}{|x|}\right]\exp\left[c_2\int_\rho^R \mathrm{cap}(\bar{B}_r\setminus\Omega)r^{-2}\,dr\right] \le c_1 M_{2R}(u)$$

provided that $c_1 \ge ce^{c_2 A\log 2}$.

Proof of Theorem 1.5. Consider the Dirichlet problem (4)

$$Lu = f, \qquad f_i \in C_0^\infty(\Omega),\ u_i \in \mathring{H}^1(\Omega).$$

Since f vanishes near the boundary, there exists $R > 0$ such that $f = 0$ in $\Omega\cap B_{2R}$. By Lemma 3.5,

$$\sup_{x\in\Omega\cap B_\rho}|u(x)|^2 \le c_1 M_{2R}(u)\exp\left[-c_2\int_\rho^R \mathrm{cap}(\bar{B}_r\setminus\Omega)r^{-2}\,dr\right],$$

and in particular,

$$\limsup_{x\to 0}|u(x)|^2 \le c_1 M_{2R}(u)\exp\left[-c_2\int_0^R \mathrm{cap}(\bar{B}_r\setminus\Omega)r^{-2}\,dr\right] = 0,$$

where the last equation follows from the divergence of the Wiener integral

$$\int_0^1 \mathrm{cap}(\bar{B}_r\setminus\Omega)r^{-2}\,dr = \infty.$$

Hence O is regular with respect to L. $\qquad\square$

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