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Asymptotic formula for solutions to elliptic equations near the Lipschitz boundary

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1. Introduction

During the last thirty years, in the works by R. Hunt, R. Wheeden, A. Calderón, E. Fabes, B. Dahlberg, D. Jerison, C. Kenig, J. Pipher, G. Verchota et al., considerable progress has been made in the study of elliptic boundary-value problems on Lipschitz and more general non-tangentially accessible domains (see [Ken] for a comprehensive survey of this development). In particular, classes of solvability and estimates of solutions received considerable attention in this area.

The new issue we address in the present paper is an explicit description of the asymptotic behavior of solutions near a point $\mathcal{O}$ of the Lipschitz boundary. As corollaries of this description new results on the boundary behavior of solutions to linear and non-linear elliptic equations in convex domains are obtained.

We consider the Lipschitz graph domain

$$G = \left\{x = (x', x_n) \in \mathbb{R}^n : x_n > \varphi(x')\right\},$$

where $\varphi(0) = 0$. The sole a priori assumption on the function φ is the smallness of its Lipschitz constant. We study solutions of an arbitrary strongly elliptic equation of order $2m$ with constant complex-valued coefficients

$$L(\partial_x)u(x) = f(x) \quad \text{on } B_3 \cap G \tag{1}$$

complemented by zero Dirichlet data on $(B_3 \cap \partial G) \setminus \mathcal{O}$. Here and elsewhere $B_\rho = \{x : |x| < \rho\}$ and by ∂_x we mean the vector of partial derivatives $(\partial_{x_1}, \dots, \partial_{x_n})$. We suppose that the operator L has no lower-order terms and the coefficient in $\partial_{x_n}^{2m}$ is equal to $(-1)^m$.

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In Theorem 1 we prove the existence of a solution $\mathfrak{z}$ of the homogeneous equation (1) which admits the asymptotic representation

$$\mathfrak{z}(x) = \exp\left\{ -\int\limits_{|x|<|y'|<1} \varphi(y')\partial_{y_n}^m E(y',0)dy' + O\Big(\int\limits_{|x|}^{1} \varkappa^2(\rho)\frac{d\rho}{\rho} \Big) \right\} \times \left\{ (x_n - \varphi(x'))^m + O\left(|x|^{m+1-\varepsilon} \Big(\int\limits_{|x|}^{1} \varkappa(\rho) \frac{d\rho}{\rho^{2-\varepsilon}} + 1 \Big) \right) \right\}. \tag{2}$$

Here ε is a positive constant,

$$\varkappa(\rho) = \sup_{|y'|<\rho} |\nabla\varphi(y')| , \tag{3}$$

and E is the Poisson solution of the equation $L(\partial_x)E(x) = 0$ in the upper half-space $\mathbb{R}^n_+$, which is positive homogeneous of degree $m-n$ and subject to the Dirichlet conditions on the hyperplane $x_n = 0$:

$$\partial_{x_n}^j E = 0 \quad \text{for} \quad 0 \le j \le m-2 , \quad \text{and} \quad \partial_{x_n}^{m-1} E = \delta(x') , \tag{4}$$

where δ is the Dirac function.

In Theorem 2 we claim that a multiple of $\mathfrak{z}$ is the main term in the asymptotic representation of an arbitrary solution u if both u and f are subject to mild growth conditions near $\mathcal{O}$. The class of solutions dealt with in Theorem 2 includes those having a finite Dirichlet integral.

Further, solutions with a singularity at $\mathcal{O}$ are studied. In Theorem 3 we present a solution $\mathfrak{Z}$ of the homogeneous equation (1) which is subject to the asymptotic formula

$$\mathfrak{Z}(x) = \exp\left\{ \int\limits_{|x|<|y'|<1} \varphi(y')\partial_{y_n}^m E(y',0)dy' + O\Big(\int\limits_{|x|}^{1} \varkappa^2(\rho)\frac{d\rho}{\rho} \Big) \right\} \times \left\{ E(x', x_n - \varphi(x')) + O\left(|x|^{m-n+1-\varepsilon} \Big(\int\limits_{|x|}^{1} \varkappa(\rho) \frac{d\rho}{\rho^{2-\varepsilon}} + 1 \Big) \right) \right\}. \tag{5}$$

Theorem 4, similar in spirit to Theorem 2, contains conditions on f and u ensuring the asymptotic relation $u \sim C\mathfrak{Z}$, where C is a constant factor.

Clearly, the asymptotic formulae (2) and (5) can be simplified under additional conditions on $\varkappa(\rho)$. Let, in particular,

$$\int_0^1 \varkappa^2(\rho)\frac{d\rho}{\rho} < \infty .$$

Then, in the special case of the polyharmonic equation $(-\Delta)^m u = 0$ on $B_3 \cap G$, any solution u satisfying $|u(x)| = O(|x|^{m-n-1+\varepsilon})$ is subject to the following alternatives:

either

$$u(x) \sim C\frac{(x_n-\varphi(x'))^m}{|x|^n}\exp\left\{m\frac{\Gamma(n/2)}{\pi^{n/2}}\int_{|x|<|y'|<1}\varphi(y')\frac{dy'}{|y'|^n}\right\} \tag{6}$$

or

$$u(x) \sim C(x_n-\varphi(x'))^m\exp\left\{-m\frac{\Gamma(n/2)}{\pi^{n/2}}\int_{|x|<|y'|<1}\varphi(y')\frac{dy'}{|y'|^n}\right\} \tag{7}$$

(see Example 2 in Sect. 5.3).

Proofs of the results mentioned rely upon our recent paper [KM3] on the asymptotic behavior near the origin of solutions to the Dirichlet problem for elliptic equations with variable coefficients in $\mathbb{R}^n_+$.

In Sect. 6 we consider the Dirichlet problem for elliptic equations in plane convex domains. The results obtained here were formulated in [KM2].

Boundedness of the first derivatives of solutions to the Dirichlet problem for the Poisson equation in any n-dimensional convex domain is a classical fact. Also, it is well known that the convexity of the domain implies the square summability of the second derivatives of solutions to the same problem ([L], [Kad]). Both properties fail in the presence of re-entrant corners. Recently, considerable progress was made in the study of other differentiability properties of solutions to the Poisson equation in arbitrary convex domains [A], [AJ], [F], [FJ]. But to our knowledge, no results of the same nature have been obtained for higher-order elliptic equations so far.

We study solutions to the Dirichlet problem for elliptic equations of order $2m$ with constant coefficients in an arbitrary bounded plane convex domain Ω. We prove that the m-th order derivatives of these solutions are bounded if the coefficients of the equation are real. In the case of strongly elliptic operators with complex coefficients we obtain the same result under the additional (and also necessary in general) assumption that the angles on $\partial\Omega$ are sufficiently close to π. As a corollary we establish the boundedness of the gradient of the velocity vector satisfying the Navier–Stokes system, as well as the boundedness of the second derivatives of solutions to the Dirichlet problem for the system of von Kármán equations in Ω.

2. Preliminaries

2.1. Function spaces

Let $1 < p < \infty$ and let $W^{m,p}_{\mathrm{loc}}(\overline{G}\setminus\mathcal{O})$ denote the space of functions u defined on G and such that $\eta u \in W^{m,p}(G)$ for all smooth η with compact support in $\overline{G}\setminus\mathcal{O}$. Also let $\mathring{W}^{m,p}_{\mathrm{loc}}(\overline{G}\setminus\mathcal{O})$ be the subspace of $W^{m,p}_{\mathrm{loc}}(\overline{G}\setminus\mathcal{O})$, which contains functions subject to

$$\nabla_k u = 0 \quad \text{on } \partial G\setminus\mathcal{O} \text{ for } k = 0,\dots,m-1, \tag{8}$$

where $\nabla_k u$ is the vector $\{\partial_x^\alpha u\}_{|\alpha|=k}$. We introduce a family of seminorms in $\overset{\circ}{W}{}^{m,p}_{\rm loc}(\overline{G}\setminus\mathcal{O})$ by

$$\mathfrak{M}_p^m(u;G_{ar,br}) = \left(\sum_{k=0}^m \int_{G_{ar,br}} |\nabla_k u(x)|^p |x|^{pk-n} dx\right)^{1/p}, \quad r>0, \tag{9}$$

where $G_{\rho,r} = \{x\in G \,:\, \rho<|x|<r\}$, a and b are positive constants, $a<b$. One can easily see that (8) implies the equivalence of $\mathfrak{M}_p^m(u;G_{ar,br})$ and the seminorm

$$\left(\int_{G_{ar,br}} |\nabla_m u(x)|^p |x|^{pm-n} dx\right)^{1/p}.$$

With another choice of a and b we arrive at an equivalent family of seminorms. Clearly,

$$\mathfrak{M}_p^m(u;G_{a'r,b'r}) \le c_1(a,b,a',b') \int_{a'r/b}^{b'r/a} \mathfrak{M}_p^m(u;G_{a\rho,b\rho}) \frac{d\rho}{\rho}, \tag{10}$$

where c_1 is a continuous function of its arguments.

We say that a function v belongs to the space $\overset{\circ}{W}{}^{m,q}_{\rm comp}(\overline{G}\setminus\mathcal{O})$, $pq=p+q$, if $v\in\overset{\circ}{W}{}^{m,q}_{\rm loc}(\overline{G}\setminus\mathcal{O})$ and v has a compact support in $\overline{G}\setminus\mathcal{O}$. By $W^{-m,p}_{\rm loc}(\overline{G}\setminus\mathcal{O})$ we denote the dual of $\overset{\circ}{W}{}^{m,q}_{\rm comp}(\overline{G}\setminus\mathcal{O})$ with respect to the inner product in $L^2(G)$. We supply $W^{-m,p}_{\rm loc}(\overline{G}\setminus\mathcal{O})$ with the seminorms

$$\mathfrak{M}_p^{-m}(f;G_{ar,br}) = \sup\left|\int_G f\,\overline{v}\,|x|^{-n}dx\right|, \tag{11}$$

where the supremum is taken over all functions $v\in\overset{\circ}{W}{}^{m,q}_{\rm comp}(\overline{G}\setminus\mathcal{O})$ supported by $ar\le|x|\le br$ and such that $\mathfrak{M}_p^m(v;G_{ar,br})\le 1$. By a standard argument it follows from (10) that

$$\mathfrak{M}_p^{-m}(f;G_{a'r,b'r}) \le c_2(a,b,a'b') \int_{a'r/b}^{b'r/a} \mathfrak{M}_p^{-m}(f;G_{a\rho,b\rho}) \frac{d\rho}{\rho}, \tag{12}$$

where c_2 depends continuously on its arguments.

In the case $G=\mathbb{R}^n_+$ we shall denote $G_{r,\rho}$ by $K_{r,\rho}$, i.e. $K_{r,\rho}=\{x\in\mathbb{R}^n_+ : \rho<|x|<r\}$.

2.2. *A mapping of $\mathbb{R}^n_+$ to G and its properties*

Let φ be the Lipschitz function used in the definition of the domain G. We need the extension of φ to $\mathbb{R}^n_+$ given by

$$\Phi(\xi) = \int_{\mathbb{R}^{n-1}} \nu(\tau)\varphi(\tau\xi_n+\xi')d\tau, \tag{13}$$

where $\nu(\tau) = c\exp((|\tau|^2 - 1)^{-1})$ for $|\tau| \le 1$ and $\nu(\tau) = 0$ for $|\tau| > 1$. The constant c is chosen to satisfy

$$\int_{\mathbb{R}^{n-1}} \nu(\tau)d\tau = 1. \tag{14}$$

Clearly, the function Φ is continuous in $\overline{\mathbb{R}^n_+}$ and belongs to $C^\infty(\mathbb{R}^n_+)$, and also

$$\partial_{x_j} = \partial_{\xi_j} - \frac{\partial_{\xi_j}\Phi}{1 + \partial_{\xi_n}\Phi}\,\partial_{\xi_n}. \tag{15}$$

We introduce the mapping $\mathbb{R}^n_+ \ni \xi \to x = x(\xi) \in G$ by

$$x' = \xi'\,, \quad x_n = \xi_n + \Phi(\xi). \tag{16}$$

Lemma 1. *Let the function* $\varkappa$ *be defined by* (3). *The mapping* (16) *has the following properties:*

(i) $\Phi(\xi', 0) = \varphi(\xi')$ *and* $|\Phi(\xi) - \varphi(\xi')| \le c\varkappa(\sqrt{2}|\xi|)\xi_n$.
(ii) *For every* $\xi \in \mathbb{R}^n_+$,

$$|\Phi(\xi)| \le c\varkappa(\sqrt{2}|\xi|)|\xi|, \tag{17}$$

with c *independent of* ξ.
(iii) *For every* $\xi \in \mathbb{R}^n_+$ *and multi-indices* α, $|\alpha| \ge 1$,

$$\left|\partial^\alpha_\xi \Phi(\xi)\right| \le c_\alpha \varkappa(\sqrt{2}|\xi|)\xi_n^{1-|\alpha|}, \tag{18}$$

where the constants c_α *do not depend on* ξ.
(iv) *Let* $\sup\varkappa < 1$. *Then transformation* (16) *is a bi-Lipschitz isomorphism of* $\mathbb{R}^n_+$ *onto* G.
(v) *Let* $u \in \mathring{W}^{l,p}_{\mathrm{loc}}(\overline{G} \setminus \mathcal{O})$, *where* $1 < p < \infty$ *and* $l = 0, 1, \dots$ *Suppose that* $\sup\varkappa$ *does not exceed a sufficiently small absolute constant depending on* n. *Then the function* $v(\xi) = u(x(\xi))$ *belongs to* $\mathring{W}^{l,p}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ *and, for* $r > 0$,

$$\|v\|_{W^{l,p}(K_{e^{-1}r,r})} \le c\|u\|_{W^{l,p}(G_{e^{-2}r,er})}, \tag{19}$$

where c *is a constant depending on* n, l *and* p.
(vi) *Let* $f \in W^{-l,p}_{\mathrm{loc}}(\overline{G} \setminus \mathcal{O})$, *where* $1 < p < \infty$ *and* $l = 0, 1, \dots$ *Suppose that* $\sup\varkappa$ *does not exceed a sufficiently small constant depending on* n. *Then the function* $g(\xi) = u(\xi', \xi_n + \Phi(\xi))$ *belongs to* $W^{-l,p}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ *and, for* $r > 0$,

$$\|g\|_{W^{-l,p}(K_{e^{-1}r,r})} \le c\|f\|_{W^{-l,p}(G_{e^{-2}r,er})}, \tag{20}$$

where c *is a constant depending on* n, l *and* p.

Proof. The assertion (i) is a direct consequence of (14) and the continuity of Φ. Since

$$\sup_{|\tau|\le 1} |\tau\xi_n + \xi'|^2 \le 2|\xi|^2 ,$$

(ii) follows from (13). We turn to (iii). Using

$$\partial_{\xi_j}\varphi(\tau\xi_n+\xi') = \xi_n^{-1}\partial_{\tau_j}\varphi(\tau\xi_n+\xi')$$

and

$$\partial_{\xi_n}\varphi(\tau\xi_n+\xi') = \xi_n^{-1}\sum_{j=1}^{n-1}\tau_j\partial_{\tau_j}\varphi(\tau\xi_n+\xi'),$$

we obtain

$$\partial_\xi^\alpha \Phi(\xi) = \xi_n^{1-|\alpha|}\sum_{j=1}^{n-1}\int_{\mathbb{R}^{n-1}}\nu_{\alpha j}(\tau)\varphi^{(j)}(\tau\xi_n+\xi')d\tau, \tag{21}$$

where $\nu_{\alpha j}$ are smooth functions with supports in $\{\xi' : |\xi'| \le 1\}$ and $\varphi^{(j)}(\xi') = \partial_{\xi_j}\varphi(\xi')$. By

$$\sup_{|\tau|\le 1}|\varphi^{(j)}(\tau\xi_n+\xi')| \le \rho(\sqrt{2}|\xi|)$$

and by (21) we arrive at (18).

(iv) By (i) the restriction of (16) to $\xi_n = 0$: $x' = \xi'$, $x_n = \varphi(\xi')$ maps $\partial\mathbb{R}^n_+$ into ∂G. If $\xi_n > 0$ then

$$\partial_{\xi_n} x_n(\xi) = 1 + \int_{\mathbb{R}^{n-1}}\nu(\tau)\sum_{j=1}^{n-1}\tau_j\varphi^{(j)}(\tau\xi_n+\xi')d\tau.$$

Therefore, $|\partial_{\xi_n}x_n(\xi) - 1| \le \sup\varkappa$. Hence, $\partial_{\xi_n}x_n(\xi) \ge 1 - \sup\varkappa$. Thus the function $x_n(\xi)$ increases with respect to ξ_n. This proves that (16) is a one-to-one mapping from $\overline{\mathbb{R}^n_+}$ to $\overline{G}$. Since $|\partial_{\xi_n}\Phi| \le \sup\varkappa$, the Jacobi matrix of transformation (16) is invertible. This implies that the inverse mapping to (16) is Lipschitz.

(v) Let x and ξ be related by (16). Then $|\xi|/e \le |x| \le e|\xi|$. Let $x \to \xi = \xi(x)$ be the inverse of (16). Clearly, $\xi(K_{e^{-1},r}) \subset G_{e^{-2}r,er}$ and by (iv) we have

$$\|v\|_{L^p(K_{e^{-1}r,r})} \le c\|u\|_{L^p(G_{e^{-2}r,er})}. \tag{22}$$

Using (18) we obtain, for $l \ge |\alpha| \ge 1$,

$$\left|\partial_\xi^\alpha v(\xi)\right| \le c\sum_{\beta\le\alpha}\xi_n^{-|\alpha-\beta|}|u^{(\beta)}(x(\xi))|, \tag{23}$$

where $u^{(\beta)}(x) = \partial_x^\beta u(x)$. Taking into account that $u \in \overset{\circ}{W}{}^{l,p}_{\mathrm{loc}}(\overline{G}\setminus\mathcal{O})$ we can use Hardy's inequality

$$\left(\int_{K_{e^{-1}r,r}}\left(\xi_n^{|\beta|-l}|u^{(\beta)}(x(\xi))|\right)^p d\xi\right)^{1/p} \le c\|u\|_{W^{l,p}(G_{e^{-2}r,er})},$$

which, being combined with (22) and (23), leads to (19).

(vi) Using (15) and (18) we check the estimate

$$\left|\partial_x^\alpha u(x)\right| \le c \sum_{\beta\le\alpha} (x_n - \varphi(x'))^{-|\alpha-\beta|} |v^{(\beta)}(\xi(x))| ,$$

where $v^{(\beta)}(\xi) = \partial_\xi^\beta v(\xi)$. Now, reasoning in the same way as in (v), one obtains

$$\|u\|_{W^{l,p}(G_{e^{-1}r,r})} \le c\|v\|_{W^{l,p}(K_{e^{-2}r,er})} ,$$

which implies (20) by duality. The lemma is proved. □

2.3. Transformation of the Dirichlet problem

Although we are interested in the local behavior of solution to equation (1) near the origin, we assume (to simplify the notation) that the solutions are extended to G as functions from $\mathring{W}^{m,p}_{\mathrm{loc}}(\overline{G} \setminus \mathcal{O})$. This implies that f is also extended to G as an element of $W^{-m,p}_{\mathrm{loc}}(\overline{G} \setminus \mathcal{O})$.

In order to give a weak formulation of the boundary-value problem we represent the operator $L(\partial_x)$ as

$$L(\partial_x) = (-1)^m \sum_{|\alpha|,|\beta|=m} L_{\alpha\beta}\partial_x^{\alpha+\beta} .$$

We suppose that $f \in W^{-m,p}_{\mathrm{loc}}(\overline{G} \setminus \mathcal{O})$, $p \in (1,\infty)$, and consider a weak solution u of (1) in the space $\mathring{W}^{m,p}_{\mathrm{loc}}(\overline{G} \setminus \mathcal{O})$. This means that

$$\int_G \sum_{|\alpha|,|\beta|=m} L_{\alpha\beta}\partial_x^\beta u(x)\partial_x^\alpha \overline{v}(x)dx = \int_G f\overline{v}(x)dx, \tag{24}$$

for all $v \in \mathring{W}^{m,q}_{\mathrm{comp}}(\overline{G} \setminus \mathcal{O})$, $pq = p + q$. We characterize u and f by the functions $\mathfrak{M}^m_p(u; G_{ar,br})$ and $\mathfrak{M}^{-m}_p(f; G_{ar,br})$.

Since the functional determinant of mapping (16) is equal to $1 + \partial_{\xi_n}\Phi$, the variational equation (24) written in the new variables takes the form

$$\begin{aligned}&\int_{\mathbb{R}^n_+} \sum_{|\alpha|,|\beta|=m} L_{\alpha\beta}\partial_x^\alpha U(\xi)\partial_x^\beta \overline{V}(\xi)(1 + \partial_{\xi_n}\Phi(\xi))d\xi \\ &= \int_{\mathbb{R}^n_+} f(x(\xi))\overline{V}(\xi)(1 + \partial_{\xi_n}\Phi(\xi))d\xi,\end{aligned} \tag{25}$$

with ∂_x acting by (15), for all functions $V \in \mathring{W}^{m,q}_{\mathrm{comp}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$. Clearly, the solution u of problem (24) is connected with the solution U of (25) by $U(\xi) = u(x(\xi))$. Using (15) we define the sesquilinear form

$$\begin{aligned}&\int_{\mathbb{R}^n_+} \sum_{|\alpha|,|\beta|\le m} \mathcal{L}_{\alpha\beta}(\xi)\partial_\xi^\alpha U(\xi)\partial_\xi^\beta \overline{V}(\xi)d\xi \\ &:= \int_{\mathbb{R}^n_+} \sum_{|\alpha|,|\beta|=m} L_{\alpha\beta}\partial_x^\alpha U(\xi)\partial_x^\beta \overline{V}(\xi)(1 + \partial_{\xi_n}\Phi(\xi))d\xi .\end{aligned} \tag{26}$$

Thus, the Dirichlet problem (1) and (8) has been reduced to

$$\mathcal{L}(\xi, \partial_\xi)U(\xi) = F(\xi) \quad \text{in } \mathbb{R}^n_+ \tag{27}$$

$$\partial^k_{\xi_n} u = 0 \quad \text{on } \partial\mathbb{R}^n_+ \setminus \mathcal{O} \text{ for } k = 0, \dots m-1, \tag{28}$$

where $F(\xi) = f(x(\xi))(1 + \partial_{\xi_n}\Phi(\xi)$ and

$$\mathcal{L}(\xi, \partial_\xi)U(\xi) = \sum_{|\alpha|,|\beta|\le m} (-\partial_\xi)^\beta \big(\mathcal{L}_{\alpha\beta}(\xi)\partial^\alpha_\xi U(\xi)\big)\,.$$

3. A particular solution to the homogeneous equation (1)

3.1. Formulation of the result

Theorem 1. *Assume that the function $\varkappa$ defined by* (3) *does not exceed a sufficiently small constant depending on m, n, p and the coefficients $L_{\alpha\beta}$. There exist positive constants c and $\mathfrak{C}$ depending on the same parameters such that the following assertion holds.*

There exists $\mathfrak{z} \in \mathring{W}^{m,p}_{\mathrm{loc}}(\overline{G} \setminus \mathcal{O})$ subject to $L(\partial_x)\mathfrak{z} = 0$ on $G \cap B^n_3$ and satisfying

$$\partial^\alpha_x \mathfrak{z}(x) = \exp\Big(-\int_{|x|}^1 \varXi(\rho)\frac{d\rho}{\rho} + \wp(x)\Big)$$
$$\times\Big(\delta^0_{\alpha'}\frac{m!}{(m-|\alpha|)!}(x_n - \varphi(x'))^{m-|\alpha|} + |x|^{m-|\alpha|}\mathcal{V}_\alpha(x)\Big), \tag{29}$$

where $\alpha = (\alpha', \alpha_n)$, $|\alpha| \le m$, $|x| < 1$. The function $\varXi$ is defined by

$$\varXi(\rho) = \rho \int_{|y'|=\rho} \varphi(y')\partial^m_{y_n} E(y', 0)ds_{y'}\,, \tag{30}$$

and the function $\wp$ is subject to the inequalities

$$|\wp(x)| \le c\int_{|x|}^3 \varkappa(\rho)^2\frac{d\rho}{\rho}\,. \tag{31}$$

and

$$|\nabla\wp(x)| \le c\varkappa(2|x|)\int_{|x|}^3 e^{\mathfrak{C}\int_{|x|}^\rho \varkappa(s)\frac{ds}{s}}\varkappa(\rho)\frac{d\rho}{\rho^2}\,. \tag{32}$$

The functions $\mathcal{V}_\alpha$ belong to $\mathring{W}^{1,p}_{\mathrm{loc}}(\overline{G} \setminus \mathcal{O})$ and satisfy

$$\Big(r^{-n}\int_{G_{r/e,r}} \big((1-\delta^m_{|\alpha|})|x||\nabla\mathcal{V}_\alpha(x)| + |\mathcal{V}_\alpha(x)|\big)^p dx\Big)^{1/p} \tag{33}$$
$$\le cr\Big\{\int_r^3 \exp\Big(\mathfrak{C}\int_r^\rho \varkappa(s)\frac{ds}{s}\Big)\frac{\varkappa(\rho)}{\rho^2}d\rho + \exp\Big(\mathfrak{C}\int_r^3 \varkappa(\rho)\frac{d\rho}{\rho}\Big)\Big\},$$

where $r < 1$.

The remaining part of this section deals with the proof of Theorem 1.

3.2. Preliminary version of the asymptotic formula (29)

Since $\mathfrak{z} \in \overset{\circ}{W}{}^{m,p}_{\rm loc}(\overline{G} \setminus \mathcal{O})$, it suffices to obtain formula (29) for x in a small neighborhood of the origin.

Owing to Lemma 1 (i) and (16) we note that

$$x_n - \varphi(x') = \xi_n(1 + O(\varkappa(\sqrt{2}|\xi|))) \tag{34}$$

and

$$|x| = |\xi|(1 + O(\varkappa(\sqrt{2}|\xi|)))\,. \tag{35}$$

The smallness assumption on $\sup \varkappa$, along with (18), implies

$$\begin{aligned} \Omega^\diamond(r) := \sup_{|\xi|<r} \Bigg(\sum_{|\alpha|=|\beta|=m} |\mathcal{L}_{\alpha\beta}(\xi) - L_{\alpha\beta}| + \sum_{|\alpha+\beta|<2m} \xi_n^{2m-|\alpha+\beta|} |\mathcal{L}_{\alpha\beta}(\xi)| \Bigg) \\ \le c\varkappa(\sqrt{2}r)\,. \end{aligned} \tag{36}$$

Hence, Ω does not exceed a sufficiently small constant depending on m, n, p and $L_{\alpha\beta}$, which is one of the conditions of Theorem 1 in [KM3]. We put $\mathfrak{z}(x) = Z(\xi(x))$ where Z is the solution of the equation $\mathcal{L}(\xi, \partial_\xi)Z(\xi) = 0$ in $\mathbb{R}^n_+ \cap B^n_e$ from Theorem 1(i) in [KM3]. By Corollary 5 in [KM3], for $|\xi| < 1$ there holds

$$\begin{aligned} \partial^\alpha_\xi Z(\xi) = c \exp\Bigg(- \int_{|\xi|}^1 \Theta(\rho) \frac{d\rho}{\rho} + \Psi^\diamond(|\xi|) \Bigg) \\ \times \Bigg(\delta^0_{\alpha'} \frac{m!}{(m-|\alpha|)!} \xi_n^{m-|\alpha|} + |\xi|^{m-|\alpha|} v_\alpha(\xi) \Bigg)\,. \end{aligned} \tag{37}$$

Here the function Θ is given by

$$\begin{aligned} \Theta(\rho) = \rho^n \int_{S^{n-1}_+} \sum_{|\alpha|=|\beta|=m} (\mathcal{L}_{\alpha,(0',m)}(\xi) - L_{\alpha,(0',m)}) E^{(\alpha)}(\xi) d\theta_\xi \\ + \rho^n \int_{S^{n-1}_+} \sum_{|\alpha|+k<2m} \mathcal{L}_{\alpha,(0',k)}(\xi) \frac{\xi_n^{m-k}}{(m-k)!} E^{(\alpha)}(\xi) d\theta_\xi\,, \end{aligned} \tag{38}$$

with $\rho = |\xi|$, $\theta = \xi/|\xi|$. The function $\Psi^\diamond$ satisfies the inequalities

$$|\Psi^\diamond(r)| \le c \int_r^2 \varkappa(\rho)^2 \frac{d\rho}{\rho}\,, \tag{39}$$

and

$$|\partial_r \Psi^\diamond(r)| \le c\varkappa(\sqrt{2}r) \int_r^2 e^{\mathfrak{C} \int_r^\rho \varkappa(s) \frac{ds}{s}} \varkappa(\rho) \frac{d\rho}{\rho^2}\,. \tag{40}$$

The function v_α is subject to

$$\left(r^{-n} \int_{K_{r/e,r}} (r|\nabla v_\alpha(x)| + |v_\alpha(x)|)^p dx \right)^{1/p} \le c r^{1-\varepsilon} \int_r^{2e} \varkappa(\rho) \frac{d\rho}{\rho^{2-\varepsilon}}\,, \tag{41}$$

for $r < 1$ and $|\alpha| < m$. The term $r|\nabla v_\alpha(x)|$ should be removed if $|\alpha| = m$.

3.3. Transformation of the exponential in (37)

By (26) we can rewrite (38) as

$$m!\rho^{-n}\Theta(\rho) = \int_{S_+^{n-1}} \sum_{|\alpha|=|\beta|=m} L_{\alpha\beta}\partial_x^\beta \xi_n^m \ \partial_x^\alpha E(\xi)(1+\partial_{\xi_n}\Phi(\xi))d\theta_\xi$$
$$-\int_{S_+^{n-1}} \sum_{|\alpha|=|\beta|=m} L_{\alpha\beta}\partial_\xi^\beta \xi_n^m \ \partial_\xi^\alpha E(\xi)d\theta_\xi \, . \tag{42}$$

We simplify the expression on the right-hand side.

Lemma 2. (i) *There holds:*

$$m!\rho^{-n}\Theta(\rho) = \sum_{|\alpha|=|\beta|=m} \int_{S_+^{n-1}} L_{\alpha\beta}\partial_\xi^\beta\left(\xi_n^m\right)\sum_{j=1}^n \partial_{\xi_j} \sum_{|\gamma|+|\sigma|=m} C_{\gamma\sigma}^{(j\alpha)}\partial_\xi^\gamma\Phi\partial_\xi^\sigma E(\xi)d\theta_\xi$$
$$-\, m\sum_{|\alpha|=|\beta|=m}\int_{S_+^{n-1}} L_{\alpha\beta}\partial_\xi^\beta\left(\Phi(\xi)\xi_n^{m-1}\right)\partial_\xi^\alpha E(\xi)d\theta_\xi$$
$$+\,\rho^{1-n}O\left(\varkappa^2(\sqrt{2}\rho)\right) . \tag{43}$$

Moreover, the coefficients $C_{\gamma\sigma}^{(n\alpha)}$ are equal to zero for $\sigma = (0, \ldots , 0, m)$.
(ii) *The function Θ satisfies*

$$\int_r^1 \Theta(\rho)\frac{d\rho}{\rho} = \int_{\mathfrak{C}_r} \varphi(\xi')\partial_{\xi_n}^m E(\xi)|_{\xi_n=0}d\xi'$$
$$+C + \int_r^1 O\left(\varkappa^2(\sqrt{2}\rho)\right)\frac{d\rho}{\rho} + \mu(r)\, , \tag{44}$$

where $\mathfrak{C}_r = \{\xi' \ : \ r < |\xi'| < 1\}$ and

$$|\mu(r)| + r|\partial_r\mu(r)| \le c\varkappa(\sqrt{2}r)\, . \tag{45}$$

Proof. (i) First, we check by induction that

$$\partial_x^\beta \xi_n^{|\beta|} = \partial_\xi^\beta\left(\xi_n^{|\beta|} - |\beta|\Phi(\xi)\xi_n^{|\beta|-1}\right) + O(\varkappa^2(\sqrt{2}\,|\xi|)). \tag{46}$$

If $|\beta| = 1$ and $\partial_x^\beta = \partial_{x_j}$ then, using (15) we obtain

$$\partial_{x_j}\xi_n = \left(\partial_{\xi_j} - \frac{\partial_{\xi_j}\Phi}{1+\partial_{\xi_n}\Phi}\partial_{\xi_n}\right)\xi_n = \partial_{\xi_j}(\xi_n - \Phi) + \frac{\partial_{\xi_j}\Phi\partial_{\xi_n}\Phi}{1+\partial_{\xi_n}\Phi},$$

which implies (46) for $|\beta| = 1$. Let (46) be proved for a certain multi-index β. Applying ∂_{x_j} to both sides of (46) and using (15) we arrive at

$$\partial_{x_j}\partial_x^\beta\xi_n^{|\beta|+1} = (|\beta|+1)\partial_x^\beta\left(\delta_j^n\xi_n^{|\beta|} - \partial_{\xi_j}\Phi\,\xi_n^{|\beta|} + \frac{\partial_{\xi_j}\Phi\partial_{\xi_n}\Phi}{1+\partial_{\xi_n}\Phi}\xi_n^{|\beta|}\right)$$
$$= (|\beta|+1)\partial_\xi^\beta\left(\delta_j^n\left(\xi_n^{|\beta|} - |\beta|\Phi(\xi)\xi_n^{|\beta|-1}\right) - \xi_n^{|\beta|}\partial_{\xi_j}\Phi\right) + O(\varkappa^2(\sqrt{2}\,|\xi|))$$
$$= \partial_{\xi_j}\partial_\xi^\beta\left(\xi_n^{|\beta|+1} - (|\beta|+1)\Phi(\xi)\xi_n^{|\beta|}\right) + O(\varkappa^2(\sqrt{2}\,|\xi|)).$$

Thus, (46) is proved.

Next, we show that

$$\partial_x^\alpha E(\xi) = \partial_\xi^\alpha E(\xi) - \partial_{\xi_n}\Phi(\xi)\partial_\xi^\alpha E(\xi) + \sum_{j=1}^{n} \partial_{\xi_j}\Big(\sum_{|\gamma|+|\sigma|=m} C_{\gamma\sigma}^{(j\alpha)} \partial_\xi^\gamma \Phi(\xi)\partial_\xi^\sigma E(\xi)\Big) + V_\alpha(\xi), \tag{47}$$

where $C_{\gamma\sigma}^{(n\alpha)} = 0$ for $\sigma = (0, \dots , 0, m)$ and $V_\alpha(\xi)$ satisfies

$$\partial_\xi^\beta V_\alpha(\xi) = O\Big(|\xi|^{m-n-|\beta|-|\alpha|}\varkappa^2(\sqrt{2}|\xi|)\Big). \tag{48}$$

Let $|\alpha| = 1$, i.e. $\partial_x^\alpha = \partial_{x_j}$ for some index j. Then

$$\partial_{x_j} E(\xi) = \left(\partial_{\xi_j} - \frac{\partial_{\xi_j}\Phi}{1+\partial_{\xi_n}\Phi}\partial_{\xi_n}\right)E(\xi) = \partial_{\xi_j}E(\xi) - \partial_{\xi_n}\Phi(\xi)\partial_{\xi_j}E(\xi) + \partial_{\xi_n}(\Phi\partial_{\xi_j}E) - \partial_{\xi_j}(\Phi\partial_{\xi_n}E) + \frac{\partial_{\xi_j}\Phi\partial_{\xi_n}\Phi}{1+\partial_{\xi_n}\Phi}\partial_{\xi_n}E. \tag{49}$$

If we denote the last term on the right-hand side by V_α, then the above expression implies (47), and estimate (48) follows from (18).

Suppose that (47) is proved for a certain α. In order to obtain an analogous representation for $\partial_{x_j}\partial_x^\alpha E$ it suffices to differentiate (47) with respect to x_j and use (49) with E replaced by ∂_ξ^α.

Now, using (46) and (47) one arrives at (43).

(ii) Let

$$M(\xi) = \sum_{|\alpha|=|\beta|=m} L_{\alpha\beta}\partial_\xi^\beta\big(\xi_n^m\big)\sum_{j=1}^{n}\partial_{\xi_j}\sum_{|\gamma|+|\sigma|=m} C_{\gamma\sigma}^{(j\alpha)}\partial_\xi^\gamma\Phi\partial_\xi^\sigma E(\xi)$$

and

$$N(\xi) = \sum_{|\alpha|=|\beta|=m} L_{\alpha\beta}\partial_\xi^\beta\big(\Phi(\xi)\xi_n^{m-1}\big)\partial_\xi^\alpha E(\xi)\ .$$

Then

$$m!\int_r^1 \Theta(\rho)\frac{d\rho}{\rho} = \int_{K_{r,1}}(M(\xi) - mN(\xi))d\xi + \int_r^1 O\big(\varkappa^2(\sqrt{2}\rho)\big)\frac{d\rho}{\rho}\ .$$

Integrating by parts we obtain that the integral $\int_{K_{r,1}} M(\xi)d\xi$ is a linear combination of the terms

$$\int_{\partial K_{r,1}} \nu_j\partial_\xi^\gamma\Phi\partial_\xi^\sigma E(\xi)d\xi, \tag{50}$$

with $|\gamma|+|\sigma| = m$. Here $\nu = (\nu_1, \dots , \nu_n)$ is the outward normal to ∂B_r. Moreover, there are no terms in the linear combination with $\sigma = (0, \dots , 0, m)$. A part of the boundary $\partial K_{r,1}$ lies in the plane $\xi_n = 0$. Since the normal to this part is equal to $(0, \dots , 0, 1)$ and since $\partial_{\xi_n}^q E(\xi) = 0$ for $\xi_n = 0$ and for $q = 0, \dots , m-1$, it follows that the integral in (50) over the plane part of $\partial K_{r,1}$ is zero. Therefore, integral (50) is equal to

$$\int_{\mathbb{R}^n_+\cap\partial B^n_1} \nu_j \partial^\gamma_\xi \Phi \partial^\sigma_\xi E(\xi) d\xi + \int_{\mathbb{R}^n_+\cap\partial B^n_r} \nu_j \partial^\gamma_\xi \Phi \partial^\sigma_\xi E(\xi) d\xi.$$

Let us denote the integral over $\mathbb{R}^n_+ \cap \partial B^n_1$ by C. Taking into account (18) we arrive at

$$m! \int_r^1 \Theta(\rho)\frac{d\rho}{\rho} = -m \int_{K_{r,1}} N(\xi)d\xi + C + \int_r^1 O\big(\varkappa^2(\sqrt{2}\rho))\big)\frac{d\rho}{\rho} + \mu_1(r),$$

where $\mu_1(r)$ is subject to (45).

Repeatedly integrating by parts in the integral $\int_{K_{r,1}} N(\xi)d\xi$, we represent this integral as the sum $I_1 + I_2 + I_3$ of integrals extended over $\mathbb{R}^n_+ \cap \partial B^n_1$, $\mathbb{R}^n_+ \cap \partial B^n_r$ and $\mathfrak{C}_r$. Now, $I_1 = \text{const}$ and by (17) and (18) $|I_2| + r|I_2'(r)| \le c\varkappa(\sqrt{2}r)$. Furthermore, by (18) all terms in I_3, containing derivatives of $\Phi(\xi)$, vanish, which implies

$$I_3 = -(m-1)! \int_{\mathfrak{C}_r} \varphi(\xi')\partial^m_{\xi_n} E(\xi)|_{\xi_n=0} d\xi' .$$

Thus, we arrive at (44). The proof is complete. □

3.4. Modification of the asymptotic formula (37)

Using (44) we write (37) in the form

$$\partial^\alpha_\xi Z(\xi) = c \exp\Big(- \int_{|\xi|}^1 \Xi(\rho)\frac{d\rho}{\rho} + \Psi_1(|\xi|)\Big) \\ \times \Big(\delta^0_{\alpha'} \frac{m!}{(m-|\alpha|)!} \xi_n^{m-|\alpha|} + |\xi|^{m-|\alpha|} v^{(1)}_\alpha(\xi)\Big) . \tag{51}$$

Here, the function Ξ is defined by (30), the function Ψ_1 admits estimates (39) and (40). Finally, $v^{(1)}_\alpha$ is subject to (41). Note that $v^{(1)}_\alpha$ depends both on v_α and the last term μ in (44).

3.5. End of the proof of Theorem 1

By (15) and (18)

$$\partial^\alpha_x = \partial^\alpha_\xi + \sum_{0<\beta\le\alpha} \pi_\beta(\xi)\partial^\beta_\xi , \tag{52}$$

where

$$\xi_n|\nabla\pi_\beta(\xi)| + |\pi_\beta(\xi)| \le c\varkappa(\sqrt{2}|\xi|)\xi_n^{|\beta|-|\alpha|} . \tag{53}$$

Hence, and by (51),

$$\partial_x^\alpha \mathfrak{z}(x) = c \exp\Big(- \int_{|\xi|}^1 \Xi(\rho)\frac{d\rho}{\rho} + \Psi_1(|\xi|)\Big) \times \Big(\delta^0_{\alpha'} \frac{m!}{(m-|\alpha|)!}\xi_n^{m-|\alpha|} + |\xi|^{m-|\alpha|} v^{(2)}_\alpha(\xi)\Big), \tag{54}$$

where

$$v^{(2)}_\alpha(\xi) = v^{(1)}_\alpha(\xi) + \sum_{0<\beta\le\alpha} \pi_\beta(\xi)\Big(\delta^0_{\alpha'} \frac{m!}{(m-|\alpha|)!}\xi_n^{m-|\beta|}|\xi|^{|\alpha|-m} + |\xi|^{|\alpha-\beta|} v^{(1)}_\beta(\xi)\Big).$$

Therefore, and by (53), the function $v^{(2)}_\alpha$ satisfies the same estimate (41) as $v^{(1)}_\alpha$.

We set

$$\wp(x) := \int_{|x|}^{|\xi|} \Xi(\rho)\frac{d\rho}{\rho} + \Psi_1(|\xi|),$$

and note that $\wp$ satisfies (31) and (32) owing to estimates (39) and (40) for Ψ_1 combined with (35) and the definition of Ξ. We have arrived at formula (29) with

$$\mathcal{V}_\alpha(x) = |x|^{|\alpha|-m}\Big(\delta^0_{\alpha'} \frac{m!}{(m-|\alpha|)!}\big(\xi_n^{m-|\alpha|} - (x_n - \varphi(x'))^{m-|\alpha|}\big) + |\xi|^{m-|\alpha|} v^{(2)}_\alpha(\xi)\Big).$$

Using (41) for $v^{(2)}_\alpha$, together with (34), (35) and other properties of the mapping $x \to \xi$ obtained in Lemma 1, we check that $\mathcal{V}_\alpha$ satisfies (33). The proof of Theorem 1 is complete.

4. Asymptotic properties of solutions to the nonhomogeneous equation (1)

4.1. Asymptotic representation of solutions

In the next theorem we deal with a solution of (1) with zero Dirichlet data on ∂G. We assume that this solution has a somewhat weaker singularity than r^{m-n} near $\mathcal{O}$ and claim that it behaves asymptotically like a multiple of the solution $\mathfrak{z}$ constructed in Theorem 1.

Since $\mathfrak{z}(x) = Z(\xi(x))$, where Z is the same as in the proof of Theorem 1, the following assertion directly results from Theorem 1(ii) in [KM3] by the mapping $x \to \xi$.

Theorem 2. *Assume that $\varkappa(r)$, defined by* (3)*, does not exceed a sufficiently small constant depending on m, n, p and the coefficients $L_{\alpha\beta}$. There exist positive constants c and $\mathfrak{C}$ depending on the same parameters such that the following assertions hold.*

Let $f \in W^{-m,p}(\overline{G} \setminus \mathcal{O})$ such that

$$J_f := \int_0^9 \rho^m \exp\Big(\mathfrak{C}\int_\rho^1 \varkappa(s)\frac{ds}{s}\Big)\mathfrak{M}_p^{-m}(f; G_{\rho/e,\rho})\frac{d\rho}{\rho} < \infty\,. \tag{55}$$

Further, let u be a solution of (1), (8) *such that*

$$\left(\int_{G_{r/e,r}} |u(x)|^p|x|^{-n}dx\right)^{1/p} = o\left(r^{m-n}\exp\left(-\mathfrak{C}\int_r^1 \varkappa(\rho)\frac{d\rho}{\rho}\right)\right) \quad \textit{as } r\to 0. \tag{56}$$

Then for $x\in G\cap B_1^n$

$$u(x)=C\mathfrak{z}(x)+w(x)\,, \tag{57}$$

where $\mathfrak{z}$ is the solution from Theorem 1. *The constant C in* (57) *is subject to*

$$|C|\le c\,(J_f+\|u\|_{L^p(G_{1/3,3})})\,. \tag{58}$$

The function w belongs to $\mathring{W}^{m,p}_{\mathrm{loc}}(\overline{G}\setminus\mathcal{O})$ and satisfies

$$\begin{aligned}\mathfrak{M}_p^m(w;G_{r/e,r})\le cr^m\Big\{ & r\exp\left(\mathfrak{C}\int_r^1 \varkappa(\rho)\frac{d\rho}{\rho}\right)\|u\|_{L^p(G_{1,e})}\\ &+r\int_r^{2e}\rho^{m-2}\exp\left(\mathfrak{C}\int_r^\rho \varkappa(\rho)\frac{d\rho}{\rho}\right)\mathfrak{M}_p^{-m}(f;G_{\rho/e,\rho})d\rho\\ &+\int_0^r\rho^{m-1}\exp\left(\mathfrak{C}\int_\rho^r\varkappa(s)\frac{ds}{s}\right)\mathfrak{M}_p^{-m}(f;G_{\rho/e,\rho})d\rho\Big\},\end{aligned} \tag{59}$$

for all $r<1$.

Remark. Needless to say, one can replace the constant $\mathfrak{C}$ by an arbitrary larger constant without violating the above theorem. An elementary argument shows that (59) implies

$$\mathfrak{M}_p^m(w;G_{r/e,r})=o\left(r^m\exp\left(-\mathfrak{C}\int_r^1\varkappa(\rho)\frac{d\rho}{\rho}\right)\right), \tag{60}$$

as $r\to 0$. Comparing this estimate with (29) we see that w plays the role of the remainder term provided $\mathfrak{C}$ is taken sufficiently large.

4.2. Corollaries of Theorems 1 *and* 2

Corollary 1. *Let u be a solution from Theorem* 2. *Then, for all $r<1$, the estimate holds:*

$$\begin{aligned}&\mathfrak{M}_p^m(u;G_{r/e,r})\\ &\le cr^m\exp\left(-\int_r^1\Re\,\Xi(\rho)\frac{d\rho}{\rho}+\mathfrak{C}\int_r^1\varkappa^2(\rho)\frac{d\rho}{\rho}\right)(J_f+\|u\|_{L^p(G_{1/3,3})})\,,\end{aligned} \tag{61}$$

where J_f is defined by (55). *The constants c and $\mathfrak{C}$ depend on n, m, p and the coefficients $L_{\alpha\beta}$.*

Proof. Since the right-hand side of (33) is bounded, it follows from (29) that

$$\mathfrak{M}_p^m(\mathfrak{z}; G_{r/e,r}) \le c r^m \exp\Big(- \int_r^1 \Re\, \Xi(\rho) \frac{d\rho}{\rho} + \mathfrak{C} \int_r^1 \varkappa^2(\rho) \frac{d\rho}{\rho} \Big) . \tag{62}$$

By (59), we obtain

$$\mathfrak{M}_p^m(w; G_{r/e,r}) \le c r^m \exp\Big(- \mathfrak{C} \int_r^1 \varkappa(\rho) \frac{d\rho}{\rho} \Big) (J_f + \|u\|_{L^p(G_{1/3,3})}) . \tag{63}$$

The result follows from (57) combined with (58). □

Corollary 2. *Let $p > n$ and*

$$\int_0^1 \varkappa(\rho)^2 \frac{d\rho}{\rho} < \infty . \tag{64}$$

Then the solution u from Theorem 2 satisfies

$$\begin{aligned} &\partial_x^\alpha u(x) \\ &= \exp\Big(- \int_{|x|}^1 \Xi(\rho) \frac{d\rho}{\rho} \Big) \Big(\frac{C \delta_{\alpha'}^0 m!}{(m - \alpha_n)!} (x_n - \varphi(x'))^{m-\alpha_n} + o(|x|^{m-|\alpha|}) \Big) , \end{aligned}$$

for $|\alpha| \le m - 1$ uniformly with respect to $x/|x|$. The same is true for $|\alpha| = m$ if the notation $g(x) = o(1)$ is understood in the sense $r^{-n/p} \|g\|_{L^p(G_{r/e,r})} = o(1)$. The symbol $o(1)$ is understood as in Theorem 2.

Proof. By (64) and (33) combined with the Sobolev embedding theorem, one has $\mathcal{V}_\alpha(x) = o(|x|^{m-|\alpha|})$. Hence, and by Theorem 1, we arrive at (65) for $\partial^\alpha \mathfrak{z}$. The result follows from (57), (60) together with the Sobolev embedding theorem. □

5. Asymptotics of singular solutions

5.1. Main results

In this section we formulate analogs of Theorems 1 and 2 which concern solutions with an infinite Dirichlet integral in a neighborhood of the origin. Their proofs repeat those of Theorems 1 and 2. The only difference is that the references to Theorem 1 and Corollary 5 in [KM3] should be replaced by those to Theorem 1 and Corollary 5 in [KM4].

Theorem 3. *Assume that the function $\varkappa(r)$ defined by (3) does not exceed a sufficiently small constant depending on m, n, p and the coefficients $L_{\alpha\beta}$. There exist positive constants c and $\mathfrak{C}$ depending on the same parameters such that the following assertion holds.*

There exists $\mathfrak{Z} \in \overset{\circ}{W}{}^{m,p}_{\mathrm{loc}}(\overline{G} \setminus \mathcal{O})$ *subject to* $L(\partial_x)\mathfrak{Z} = 0$ *on* $G \cap B_3^n$ *and satisfying*

$$\partial_x^\alpha \mathfrak{Z}(x) = \exp\left(\int_{|x|}^1 \Xi(\rho) \frac{d\rho}{\rho} + \wp(x) \right) \times \Big((\partial^\alpha E)(x', x_n - \varphi(x')) + |x|^{m-n-|\alpha|} \mathcal{V}_\alpha(x) \Big), \tag{65}$$

where $\alpha = (\alpha', \alpha_n)$, $|\alpha| \le m$, $|x| < 1$, Ξ *is defined by* (30), *and the function* $\wp$ *and* $\mathcal{V}_\alpha$ *satisfy the same conditions as in Theorem* 3.

Theorem 4. *Assume that* $\varkappa(r)$ *defined by* (3) *does not exceed a sufficiently small constant depending on* m, n, p *and the coefficients* $L_{\alpha\beta}$. *There exist positive constants* c *and* $\mathfrak{C}$ *depending on the same parameters such that the following assertions hold.*

Let $f \in W^{-m,p}(\overline{G} \setminus \mathcal{O})$ *such that*

$$\mathcal{J}_f := \int_0^9 \rho^{m+n} \exp\left(\mathfrak{C} \int_\rho^1 \varkappa(s) \frac{ds}{s} \right) \mathfrak{M}_p^{-m}(f; G_{\rho/e,\rho}) \frac{d\rho}{\rho} < \infty \,. \tag{66}$$

Further, let u *be a solution of* (1), (8) *such that*

$$\left(\int_{G_{r/e,r}} |u(x)|^p |x|^{-n} dx \right)^{1/p} = o\left(r^{m-n-1} \exp\left(-\mathfrak{C} \int_r^1 \varkappa(\rho) \frac{d\rho}{\rho} \right) \right) \quad \text{as } r \to 0. \tag{67}$$

Then, for $x \in G \cap B_1^n$,

$$u(x) = C\mathfrak{Z}(x) + w(x)\,, \tag{68}$$

where $\mathfrak{Z}$ *is the solution from Theorem* 3. *The constant* C *in* (57) *is subject to*

$$|C| \le c\,(\mathcal{J}_f + \|u\|_{L^p(G_{1/3,3})})\,. \tag{69}$$

The function w *belongs to* $\overset{\circ}{W}{}^{m,p}_{\mathrm{loc}}(\overline{G} \setminus \mathcal{O})$ *and satisfies*

$$\begin{aligned} \mathfrak{M}_p^m(w; G_{r/e,r}) \le c r^m \Bigg\{ & \exp\left(\mathfrak{C} \int_r^1 \varkappa(\rho) \frac{d\rho}{\rho} \right) \|u\|_{L^p(G_{1,e})} \\ & + \int_r^{2e} \rho^{m-1} \exp\left(\mathfrak{C} \int_r^\rho \varkappa(\rho) \frac{d\rho}{\rho} \right) \mathfrak{M}_p^{-m}(f; G_{\rho/e,\rho}) d\rho \\ & + \int_0^r r^{-n} \rho^{m+n-1} \exp\left(\mathfrak{C} \int_\rho^r \varkappa(s) \frac{ds}{s} \right) \mathfrak{M}_p^{-m}(f; G_{\rho/e,\rho}) d\rho \Bigg\}, \end{aligned} \tag{70}$$

for all $r < 1$.

5.2. *Corollaries of Theorems* 3 *and* 4

The following two corollaries of Theorems 3 and 4 are similar to Corollaries 1 and 2, and are proved verbatim.

Corollary 3. *Let u be a solution from Theorem* 4*. Then, for all $r < 1$, the estimate holds:*

$$\begin{aligned}&\mathfrak{M}_p^m(u; G_{r/e,r}) \\ &\le c r^{m-n} \exp\left(\int_r^1 \Re\,\Xi(\rho)\frac{d\rho}{\rho} + \mathfrak{C}\int_r^1 \varkappa^2(\rho)\frac{d\rho}{\rho}\right)(\mathcal{J}_f + \|u\|_{L^p(G_{1/3,3})}),\end{aligned} \tag{71}$$

where $\mathcal{J}_f$ is defined by (66)*. The constants c and $\mathfrak{C}$ depend on n, m, p and the coefficients $L_{\alpha\beta}$.*

Corollary 4. *Let $p > n$ and*

$$\int_0^1 \varkappa(\rho)^2\frac{d\rho}{\rho} < \infty\,. \tag{72}$$

Then the solution u from Theorem 2 *satisfies*

$$\begin{aligned}&\partial_x^\alpha u(x) \\ &= \exp\left(\int_{|x|}^1 \Xi(\rho)\frac{d\rho}{\rho}\right)\Big(C(\partial^\alpha E)(x', x_n - \varphi(x')) + o(|x|^{m-n-|\alpha|})\Big),\end{aligned}$$

for $|\alpha| \le m-1$ uniformly with respect to $x/|x|$. The same is true for $|\alpha| = m$ if the notation $g(x) = o(1)$ is understood as $r^{-n/p}\|g\|_{L^p(G_{r/e,r})} = o(1)$. The symbol $o(1)$ is understood as in Theorem 2*.*

The last assertion of this section shows that the asymptotic formula (68) can be improved if the right-hand side f satisfies (55) instead of (66).

Corollary 5. *Let $f \in W^{-m,p}(\overline{G}\setminus\mathcal{O})$ such that $J_f < \infty$, where J_f is defined by* (55)*. Further, let u be a solution of* (1), (8) *such that*

$$\left(\int_{G_{r/e,r}} |u(x)|^p|x|^{-n}dx\right)^{1/p} = o\left(r^{m-n-1}\exp\left(-\mathfrak{C}\int_r^1 \varkappa(\rho)\frac{d\rho}{\rho}\right)\right), \tag{73}$$

as $r \to 0$. Then for $x \in G \cap B_1^n$

$$u(x) = C_1\,\mathfrak{Z}(x) + C_2\,\mathfrak{z}(x) + w(x)\,, \tag{74}$$

where $\mathfrak{Z}$ and $\mathfrak{z}$ are the solutions from Theorem 1 *and Theorem* 3*, respectively. The constants C_1 and C_2 are subject to*

$$|C_1| + |C_2| \le c\,(J_f + \|u\|_{L^p(G_{1/3,3})})\,. \tag{75}$$

The function w belongs to $\mathring{W}^{m,p}_{\mathrm{loc}}(\overline{G}\setminus\mathcal{O})$ and satisfies (59)*.*

Proof. Follows directly from Theorems 2 and 4. $\square$

5.3. Examples

Example 1. The function Ξ in the asymptotic formulae (29) and (65), defined by (30), takes a particularly simple form if $n = 2$ and $\mathcal{L}(x, \partial_x) = L(\partial_x)$. Let us write $L(\xi)$ as $L(\xi_1, \xi_2)$ and by $\zeta_1^+, \dots, \zeta_m^+$ and $\zeta_1^-, \dots, \zeta_m^-$ we denote the roots of the polynomial $L(1, \zeta) = 0$ with positive and negative imaginary parts, respectively. By formula (11.6.22) in [KMR2] for $x_1 \neq 0$

$$\partial_{x_2}^m E(x_1, 0) = \frac{-i}{2\pi x_1^2} \sum_{k=1}^m \left(\zeta_k^+ - \zeta_k^-\right). \tag{76}$$

Hence

$$\Xi(\rho) = -i \sum_{k=1}^m \left(\zeta_k^+ - \zeta_k^-\right) \frac{\varphi(\rho) + \varphi(-\rho)}{2\pi\rho}. \tag{77}$$

If, for example, $L(\partial_x) = a\partial_{x_1}^2 + 2b\partial_{x_1}\partial_{x_2} + c\partial_{x_1}^2$ with real a, b and c one has

$$\Xi(\rho) = \sqrt{ac - b^2}\, \frac{\varphi(\rho) + \varphi(-\rho)}{\pi\rho}. \tag{78}$$

Example 2. In the case $L(\partial_x) = (-\Delta)^m$ we have

$$\partial_{\xi_n}^m E(\xi', \xi_n)\Big|_{\xi_n = 0} = m\Gamma(n/2)\pi^{-n/2}|\xi'|^{-n},$$

by Example 1, Sect. 11.6.4 in [KMR2]. Therefore,

$$\Xi(\rho) = \frac{m\Gamma(n/2)}{\pi^{n/2}\rho} \int_{S^{n-2}} \varphi(\rho\theta')d\theta'.$$

This, being combined with (29), (65) and Corollary 5, leads to the alternatives (6) and (7) mentioned in the introduction.

Remark. S. Warschawski [W] obtained an asymptotic formula for conformal mappings of curvilinear strips under rather weak restrictions to their boundaries. Here we state a corollary of Warschawski's result.

Let G be the domain $\{z = x + iy : y > \varphi(x)\}$ in the complex plane, where φ is Lipschitz and $\varphi(0) = 0$. By w we denote a conformal mapping of G onto the half-plane $\{w = u + iv : v > 0\}$, $w(0) = 0$. Suppose that $\varphi'(x) \to 0$ as $x \to 0$ and that

$$\int_0^1 (|\varphi'(\rho)|^2 + |\varphi'(-\rho)|^2)\frac{d\rho}{\rho} < \infty.$$

Then the imaginary part of ζ admits the asymptotic representation as $z \to 0$:

$$u(z) \sim c\,(y - \varphi(x)) \exp\left\{ -\frac{1}{\pi} \int_{|z|}^1 (\varphi(\rho) + \varphi(-\rho))\frac{d\rho}{\rho^2} \right\}, \tag{79}$$

where c is a constant. This relation can be interpreted as an asymptotic representation of a solution to the Laplace equation in a neighbourhood of the origin with zero Dirichlet data on the curve $\{z : y = \varphi(x)\}$. Warschawski's proof of (79) is based on methods of geometric function theory and cannot be extended even to harmonic functions of n variables, $n > 2$. Note that Corollary 2 and (78) imply (79).

6. Boundedness of the m-th order derivatives of solutions to the Dirichlet problem for elliptic equations of order $2m$ in plane convex domains

6.1. Local estimates

The case of a small local Lipschitz constant of the boundary. Let Ω be a bounded plane domain in $\mathbb{R}^2$. We introduce a strongly elliptic operator with constant coefficients

$$L(\partial_x) = \sum_{0\le k\le 2m} a_k\, \partial^k_{x_1} \partial^{2m-k}_{x_2}\,,$$

and by w we denote a weak solution to the Dirichlet problem

$$L(\partial_x)w = f\,, \qquad w \in \overset{\circ}{W}{}^{m,2}(\Omega)\,. \tag{80}$$

If $f \in W^{-m,2}(\Omega)$ this problem is uniquely solvable.

In the following we make use of the following asymptotic estimate for the solution w of problem (80) in a neighborhood $B_{\delta_0} = \{x \in \mathbb{R}^2 : |x| < \delta_0\}$ of the point $\mathcal{O} \in \partial\Omega$, which is contained in Corollary 1 and Example 1. We assume that $\Omega \cap B_{2\delta_0}$ is described by the inequalities $x_2 > \varphi(x_1)$, $|x| < 2\delta_0$, where φ is a Lipschitz function on $[-2\delta_0, 2\delta_0]$ and $\varphi(0) = 0$. Note that here we do not require the convexity of φ.

Lemma 3. *Suppose that the Lipschitz norm of φ on $[-2\delta_0, 2\delta_0]$ does not exceed a certain constant depending only on the coefficients of L. Let f be equal to zero in $\Omega \cap B_{2\delta}$. Then for all $\delta \in (0, \delta_0)$, $x \in \Omega \cap B_\delta$ and $k = 1, \dots, m-1$,*

$$\begin{aligned} &|\nabla_k w(x)| \\ &\le c\, A(2\delta)|x|^{m-k} \exp\left(-a \int_{|x|}^{\delta} \frac{\varphi(\rho)+\varphi(-\rho)}{\rho^2} d\rho + b \int_{|x|}^{\delta} \max_{|t|<\rho} |\varphi'(t)|^2 \frac{d\rho}{\rho} \right). \end{aligned} \tag{81}$$

Here ∇_k is the vector of all partial derivatives of order k. We use the notation

$$a = \frac{1}{2\pi} \Im \sum_{1\le k\le m} \left(\zeta_k^+ - \zeta_k^-\right),$$

where $\zeta_1^+, \dots, \zeta_m^+$ and $\zeta_1^-, \dots, \zeta_m^-$ are roots of the polynomial $L(1,\zeta)$ with positive and negative imaginary parts, respectively. This value of a is best possible. By b and c we denote a positive constant depending only on m and the coefficients of L, and we put

$$A(\delta) = \delta^{-1-m} \|w\|_{L^2(\Omega\cap B_\delta)}\,. \tag{82}$$

Note that for the operator Δ^m we have $\zeta_k^\pm = \pm i$, which implies $a = m/\pi$.

The next assertion is a co-sequence of Lemma 3 when the function φ is convex.

Lemma 4. *Suppose that the function φ describing the domain Ω near the point $\mathcal{O}$ is non-negative and convex, and $|\varphi'(\pm 2\delta_0)|$ does not exceed a sufficiently small constant ℓ_0 depending on m and the coefficients of $L(\partial_x)$. Furthermore, let f be zero in $\Omega \cap B_{2\delta}$ and let w be a solution of* (80)*, which is extended by zero outside Ω. Then*

$$\|\nabla_m w\|_{L^\infty(B_\delta)} \le c\, A(8\delta)\,, \tag{83}$$

where $\delta < \delta_0/8$ and $A(\delta)$ is given by (82)*.*

Proof. Let x be a point of $\Omega \cap B_\delta$ situated at the distance r from $\partial\Omega$. We shall use the local estimate

$$\|\nabla_m w\|_{L^\infty(B_{r/4}(x))} \le c\, r^{-m} \|w\|_{L^\infty(B_{r/2}(x))} \tag{84}$$

(see Sect. 15 in [ADN]). By (84) it is sufficient to prove the estimate

$$\sup_{x \in \Omega \cap B_{2\delta}} \frac{|w(x)|}{\operatorname{dist}(x, \partial\Omega)^m} \le c\, A(8\delta)\,. \tag{85}$$

Denote by z the point in $\partial\Omega \cap B_\delta$ such that $|x - z| = r$. We are going to apply Lemma 3, where the role of the origin is played by z. For simplicity, we preserve the notation φ for the function describing the domain Ω in the Cartesian system with the new origin z. Clearly, we may assume that φ is non-negative and convex.

By the convexity of φ we have, for $\tau \in (0, \delta)$,

$$\max_{|t| \le \tau} \varphi'(t)^2 = \max(\varphi'(\tau)^2, \varphi'(-\tau)^2)\,.$$

Taking into account the inequality

$$\int_r^\delta \frac{\varphi'(\pm\rho)^2}{\rho} d\rho \le \ell_0 \Big(\varphi(\pm\delta) + \int_r^\delta \frac{\varphi(\pm\rho)}{\rho^2} d\rho \Big)$$

and using Lemma 3, we obtain

$$r^{-m} |w(x)| \le c\, \delta^{-1-m} \|w\|_{L^2(\Omega \cap B_{2\delta}(z))} \le cA(8\delta),$$

which completes the proof of (85). The lemma is proved. □

Perturbation of an angle. We introduce some notation. Let an angle K be given in the polar coordinates (r, ψ) by

$$K = \{(r, \psi) : 0 < r < \infty,\ 0 < \psi < \alpha\}\,.$$

Also let $K_\rho = \{x \in K\ :\ \rho/e < |x| < \rho\}$.

We shall prove a pointwise estimate for solutions assuming that the domain is a small perturbation of an angle. Here, no convexity assumption is required.

Lemma 5. *Let the coefficients of $L(\partial_x)$ be real. We introduce the domain $\mathcal{G}$ described in the polar coordinates r, θ by the inequalities*

$$\theta_0(r) < \theta < \alpha + \theta_1(r), \qquad 0 < r < 8\tau,$$

where $\alpha \in (0, \pi)$, the functions θ_0 and θ_1 are non-negative and Lipschitz on $(0, 8\tau]$, $\theta_i(+0) = 0$, for $i = 0, 1$. Let a positive number τ satisfy

$$|\theta_i(r)| + r\left|\theta_i'(r)\right| \le \ell_1 \quad \textit{for } r \le 8\tau,$$

where ℓ_1 is a sufficiently small constant depending on α, m, q and the coefficients of $L(\partial_x)$. Furthermore, let $f \in W^{1-m,q}(\mathcal{G})$ with some $q > 2$ and let u be a solution of $L(\partial_x)u = f$ from $W^{m,2}(\mathcal{G})$ subject to zero Dirichlet conditions on the arcs $\theta = \theta_0(r)$ and $\theta = \alpha + \theta_1(r)$. Then

$$|u(x)| \le C\,|x|^{m+\beta}\big(\tau^{-m-1-\beta}\|u\|_{L^2(\mathcal{G}\cap B_{4\tau})} + \tau^{-\beta-q/2}\|f\|_{W^{1-m,q}(\mathcal{G}\cap B_{4\tau})}\big), \tag{86}$$

for all $x \in \mathcal{G} \cap B_\tau$. Here β and C are positive constants, depending on α, m, q and the coefficients of $L(\partial_x)$.

Proof. It suffices to prove (86) for $\tau = 1$. We extend the functions θ_i by the values $\theta_i(8)$ onto $(8, \infty)$ and define the domain $\mathcal{G}_\infty = \{x : \theta_0(r) < \theta < \alpha + \theta_1(r), 0 < r < \infty\}$. We verify that there exists a function $\Theta(r, \psi)$ given on the angle K such that $\Theta(r, 0) = \theta_0(r)$, $\Theta(r, \alpha) = \theta_1(r)$, and subject to the estimate

$$\sum_{0\le k+\ell\le m} (\alpha\psi - \psi^2)^\ell \left|(r\partial_r)^k \partial_\psi^\ell \Theta(r, \psi)\right| \le c(m)\delta_0, \tag{87}$$

where $c(m)$ is a constant depending only on m. In order to construct Θ we use the following procedure. By the change of variables $t = \log r^{-1}$, $\psi = \psi$ we map the angle K onto the strip $\{(t, \psi) : t \in \mathbb{R}, 0 < \psi < \alpha\}$. Next, we extend the functions $\theta_0(e^{-t})$ and $\theta_1(e^{-t})$ onto the half-planes $\psi > 0$ and $\psi < \alpha$, respectively, by using a standard extension operator (see, for instance, [MSh], Sect. 5.1.1). We call these extensions by $\zeta_0(t, \psi)$ and $\zeta_1(t, \psi)$, and note that they are subject to

$$\sum_{0\le k+\ell\le m} (\alpha\psi - \psi^2)^\ell \left|\partial_t^k \partial_\psi^\ell \zeta_i(t, \psi)\right| \le c(m)\delta_0.$$

We introduce a smooth function η on $[0, \alpha]$ equal to one near zero and to zero near $\psi = \alpha$, and we set $\zeta = \eta\zeta_0 + (1 - \eta)\zeta_1$. Now, the function Θ given by $\Theta(r, \psi) = \zeta(\log r^{-1}, \psi)$ satisfies the conditions required.

By the one-to-one mapping $\kappa : (r, \varphi) \to (r, \psi)$ defined by $\varphi = \psi + \Theta(r, \psi)$ we transform $\mathcal{G}$ onto K. Let $y = (y_1, y_2)$ be the point with the polar coordinates (r, ψ) and let $\mathcal{L}(y, \partial_y)$ be the operator L written in the coordinates y and interpreted in the sense of the corresponding sesquilinear form. By (87) and Hardy's inequality we conclude that, for all $v \in \mathring{W}^{m,2}_{\text{loc}}(\overline{K} \setminus \mathcal{O})$,

$$\|(\mathcal{L} - L)v\|_{W^{-m,2}(K_\rho)} \le c\ell_1 \|v\|_{W^{m,2}(K_\rho)}\,.$$

By $\mathcal{A}(\lambda) : \overset{\circ}{W}{}^{m,2}(0,\alpha) \to W^{-m,2}(0,\alpha)$ we denote the pencil of the Dirichlet problem for $L(\partial_y)$ in the angle K defined by $\mathcal{A}(\lambda)u(\psi) = r^{2m-\lambda}L(\partial_y)(r^\lambda u(\psi))$. We need one more cut-off function $\Gamma \in C_0^\infty(B_3)$, $\Gamma = 1$ on B_2. We put $U = (\Gamma u)\circ\kappa^{-1}$ and $F = (L(\partial_x)(\Gamma u))\circ\kappa^{-1}$.

From Theorem 1 in [Koz] (see also [KMR2], Sect. 8.4.2) it follows that the strip $m-2-\sigma \le \Re\lambda \le m+\sigma$, where $\sigma > 0$ is a certain positive number, contains no eigenvalues of $\mathcal{A}(\lambda)$. Since

$$\mathfrak{M}_2^{-m}(F; K_\rho) \le c\,\mathfrak{M}_q^{1-m}(F; K_\rho) \le c\rho^{-m+1-2/q}\|F\|_{W^{1-m,q}(K)}\,, \tag{88}$$

the function F satisfies

$$\int_0^\infty \rho^{m+1+\sigma}\mathfrak{M}_2^{-m}(F; K_\rho)d\rho < \infty\,,$$

and we are in a position to apply Theorem 10.8.16 [KM1] with $\omega(t) = c\ell_1$ and g_ω being Green's function of the differential operator

$$(\partial_t + m + \sigma)(-\partial_t + 2 - m - \sigma) - c\ell_1 \quad \text{on } \mathbb{R},$$

subject to

$$g_\omega(t,s) \le c\,e^{(m+\sigma-c_0\ell_1)(s-t)} \quad \text{if } t > s \tag{89}$$

and

$$g_\omega(t,s) \le c\,e^{(m-2-\sigma+c_0\ell_1)(s-t)} \quad \text{if } s > t. \tag{90}$$

The theorem just mentioned along with (89), (90) and (88) guarantees the estimates

$$\begin{aligned}\mathfrak{M}_2^m(U; K_r) &\le c\Big(\int_0^r \Big(\frac{r}{\rho}\Big)^{m-2-\sigma+c_0\ell_1}\rho^{2m}\mathfrak{M}_2^{-m}(F; K_\rho)\frac{d\rho}{\rho}\\ &\qquad + \int_r^\infty \Big(\frac{r}{\rho}\Big)^{m+\sigma-c_0\ell_1}\rho^{2m}\mathfrak{M}_2^{-m}(F; K_\rho)\frac{d\rho}{\rho}\Big)\\ &\le C(r^{m+1-2/q} + r^{m-c_0\ell_1+\sigma/2})\|F\|_{W^{1-m,q}(K)},\end{aligned} \tag{91}$$

for $r < 3$. Setting $\beta = \min\{1-2/q, -c_0\ell_1 + \sigma/2\}$ and using the standard local estimate (see [ADN], Ch. 5, Sect. 15), we obtain

$$\mathfrak{M}_q^m(U; K_r) \le Cr^{m+\beta}\|F\|_{W^{1-m,q}(K)}.$$

Hence and by the Sobolev embedding theorem, $|U(r,\psi)|$ is majorized by the right-hand side of the last inequality. Coming back to the coordinates x we obtain

$$\begin{aligned}|u(x)| &\le cr^{m+\beta}\|(L(\partial_x)\Gamma u)\|_{W^{1-m,q}(\mathcal{G}_\infty)}\\ &\le Cr^{m+\beta}(\|\Gamma f\|_{W^{1-m,q}(\mathcal{G}_\infty)} + \|u\|_{W^{m,q}(\mathcal{G}\cap(B_3\setminus\overline{B}_2))})\,.\end{aligned}$$

It remains to note that the second term on the right does not exceed

$$cr^{m+\beta}(\|\Gamma f\|_{W^{-m,q}(\mathcal{G}\cap(B_4\setminus B_1))} + \|u\|_{L^2(\mathcal{G}\cap(B_4\setminus B_1))})$$

(see Proposition 7.5.2/2 [MSh]). The result follows. □

The assertion of Lemma 5 can be improved if, additionally, the domain is convex.

Lemma 6. *Let $\mathcal{G}$, u and f be the same as in Lemma 5 and let $\mathcal{G}$ be convex near the origin. We assume that $f(x) = 0$ for $|x| < 8\tau$, and τ is so small that, for all $r \le 8\tau$,*

$$|\theta_i(r)| + r|\theta_i'| \le c_* \min(\ell_0, \ell_1),$$

where c_* is a sufficiently small constant depending on α, m and the coefficients of $L(\partial_x)$. Then the estimate

$$|\nabla_m u(x)| \le C|x|^\beta \, \tau^{-m-1-\beta} \|u\|_{L^2(\mathcal{G}\cap B_{4\tau})} \tag{92}$$

holds for $x \in \mathcal{G} \cap B_\tau$. Here β and C are positive constants, depending on α, m and the coefficients of $L(\partial_x)$.

Proof. It is sufficient to obtain (92) with $\tau = 1$. By local estimate (84) we only need to prove the inequality

$$|u(x)| \le C|x|^\beta \mathrm{dist}(x, \partial G)^m \, \|u\|_{L^2(\mathcal{G}\cap B_4)}, \tag{93}$$

for all $x \in \mathcal{G} \cap B_1$. Let $c(\alpha)$ be a sufficiently small constant depending only on α. If $\mathrm{dist}(x, \partial\mathcal{G}) \ge c(\alpha)|x|$, estimate (93) follows from Lemma 5. Suppose that x satisfies the opposite inequality $\mathrm{dist}(x, \partial\mathcal{G}) < c(\alpha)|x|$. We denote by z the only point at $\partial\mathcal{G}$ nearest to x. By Lemma 4

$$|x - z|^{-m} |u(x)| \le C\,|x|^{-1-m} \|u\|_{L^2(\mathcal{G}\cap B_{|x|/2}(z))}\,. \tag{94}$$

We shall not violate this inequality replacing $B_{|x|/2}(z)$ by the disk $B_{2|x|}$ centered at $\mathcal{O}$. By Lemma 5

$$\|u\|_{L^2(\mathcal{G}\cap B_{4|x|})} \le C\,|x|^{m+\beta+1} \|u\|_{L^2(\mathcal{G}\cap B_4)}\,.$$

The proof is complete. □

6.2. *Estimates for Green's function*

In this section and elsewhere, we denote by $G(x, y)$ Green's function of problem (80).

Theorem 5. *Let Ω be an arbitrary bounded convex domain and let the coefficients of L be real. Then for, all x, y in Ω,*

$$\sum_{|\alpha|=|\beta|=m} \left|\partial_x^\alpha \partial_y^\beta G_L(x, y)\right| \le C\,|x - y|^{-2}, \tag{95}$$

where C is a positive constant depending on Ω.

Proof. Let F be a fundamental solution of the operator L:

$$F(\xi) = P_{2m-2}(\xi)\log|\xi| + Q_{2m-2}\,,$$

where P_{2m-2} is a homogeneous polynomial of degree $2m-2$ and Q_{2m-2} is a positive homogeneous function which is smooth on $\mathbb{R}^2 \setminus \mathcal{O}$ (see [J]). Clearly, for any multi-index α of order $m-1$,

$$\partial_\xi^\alpha F(\xi) = P_{m-1}^{(\alpha)}(\xi)\log|\xi| + Q_{m-1}^{(\alpha)}(\xi)\,.$$

Here, $P_{m-1}^{(\alpha)}$ is a homogeneous polynomial of degree $m-1$ and $Q_{m-1}^{(\alpha)}$ is a positive homogeneous function of order $m-1$.

We set

$$\begin{aligned} &R_\alpha(x,y) \qquad (96)\\ &= \partial_y^\alpha G(x,y) - \eta\Big(\frac{x-y}{d}\Big)\Big(P_{m-1}^{(\alpha)}(x-y)\log\frac{|x-y|}{d} + Q_{m-1}^{(\alpha)}(x-y)\Big), \end{aligned}$$

where $d = \operatorname{dist}(y, \partial\Omega)$, $\eta \in C_0^\infty(B_{1/2})$ and $\eta = 1$ on $B_{1/4}$.

The function $x \to R_\alpha(x,y)$ solves problem (80) with

$$f(x) = -\Big[L(\partial_x), \eta\Big(\frac{x-y}{d}\Big)\Big]\Big(P_{m-1}^{(\alpha)}(x-y)\log\frac{|x-y|}{d} + Q_{m-1}^{(\alpha)}(x-y)\Big),$$

where the square brackets denote the commutator. Obviously, $f \in C_0^\infty(B_{d/2}(y) \setminus B_{d/4}(y))$ and

$$|f(x)| \le Cd^{-1-m},$$

with C independent of x and y. We have

$$\|\nabla_m R_\alpha\|_{L^2(\Omega)} \le C\|f\|_{W^{-m,2}(\Omega)}\,.$$

Let z_0 be a point on $\partial\Omega$ such that $d = |y - z_0|$. By a standard induction argument one shows that the Hardy-type inequality

$$\int_\Omega \frac{|v(z)|^2}{|z-z_0|^{2m}}dz \le \int_\Omega |\nabla_m v(z)|^2 dz, \qquad (97)$$

holds for all $v \in \mathring{W}^{m,2}(\Omega)$. Hence,

$$\begin{aligned} \|f\|_{W^{-m,2}(\Omega)} &= \sup\Big\{\Big|\int_\Omega f(z)\overline{v}(z)dz\Big| \,:\, \|v\|_{\mathring{W}^{m,2}(\Omega)} = 1\Big\}\\ &\le c\,d^m\|f\|_{L^2(B_{d/2}(y))}\sup\Big\{\Big(\int_{B_d(z_0)}\frac{|v(z)|^2}{|z-z_0|^{2m}}dz\Big)^{1/2} : \|v\|_{\mathring{W}^{m,2}(\Omega)} = 1\Big\}\\ &\le c_1\,d^m\|f\|_{L^2(B_{d/2}(y))} \le c_2. \end{aligned}$$

Therefore we have

$$\|\nabla_m R_\alpha(\cdot,y)\|_{L^2(\Omega)} \le c. \qquad (98)$$

Let $r = |x - y|$ and let $\mathrm{dist}(x, \partial\Omega) < r/N$, where N is a sufficiently large positive fixed number. By Lemmata 4 and 6

$$\left|\partial_x^\beta \partial_y^\alpha G(x, y)\right| \le C r^{-1-m} \left\|\partial_y^\alpha G(\cdot, y)\right\|_{L^2(\Omega \cap B_{r/8}(x))},$$

for all multi-indices $|\beta|$ of order m. Hence, and by Hardy's inequality (97),

$$\left|\partial_x^\beta \partial_y^\alpha G(x, y)\right| \le C\, r^{-1} \left\|\nabla_m \partial_y^\alpha G(\cdot, y)\right\|_{L^2(\Omega \cap B_{r/8}(x))}.$$

In the case $\mathrm{dist}(x, \partial\Omega) \ge r/N$ we use the classical interior local estimate (see [ADN], Ch. 5, Sect. 15) in order to obtain

$$\left|\partial_x^\beta \partial_y^\alpha G(x, y)\right| \le C\, r^{-1} \left\|\nabla_m \partial_y^\alpha G(\cdot, y)\right\|_{L^2(\Omega \cap B_{r/8}(x))}.$$

Combining the two last inequalities with (96) and (98) we obtain

$$\left|\partial_x^\beta \partial_y^\alpha G(x, y)\right| \le c|x - y|^{-1}, \tag{99}$$

where $|\alpha| = m - 1$ and $|\beta| = m$.

Now consider the function

$$y \to G_\beta(y) = \partial_x^\beta G(x, y),$$

with $|\beta| = m$. It satisfies the Dirichlet problem in Ω for the equation

$$L(\partial_y) G_\beta(y) = \partial_x^\beta \delta(x - y).$$

Using Lemmata 4 and 6, and the classical interior local estimate, once more we arrive at the inequality

$$\left|\partial_y^\alpha G_\beta(y)\right| \le C r^{-2} \|\nabla_{m-1} G_\beta\|_{L^2(B_{r/2}(y))},$$

where $|\alpha| = |\beta| = m$. The result follows from (99). □

Theorem 6. *Let L be an arbitrary strongly elliptic operator with complex coefficients. Suppose that Ω is a bounded convex domain such that the jumps of all angles between the exterior normal vector to $\partial\Omega$ and the x-axis be smaller than a constant depending on m and the coefficients of $L(\partial_x)$. Then for all x, y in Ω estimate* (95) *holds.*

Proof. This is the same as that of Theorem 5. The only difference is that Lemma 6 is not used in the present case. □

Corollary 6. *Let Ω be an arbitrary bounded convex domain in $\mathbb{R}^2$ and let the coefficients of $L(\partial_x)$ be real. Then the solution w of problem* (80) *with $f \in W^{1-m,q}(\Omega)$, $q > 2$, satisfies*

$$\|w\|_{C^{m-1,1}(\Omega)} \le C\, \|f\|_{W^{1-m,q}(\Omega)}, \tag{100}$$

where the constant C depends on Ω, m, q and the coefficients of $L(\partial_x)$.

Proof. We represent the right-hand side in (80) in the form

$$f = \sum_{|\gamma|=m-1} \partial^\gamma f_\gamma ,$$

where $f_\gamma \in L^q(\Omega)$. Clearly, for almost all $x \in \Omega$,

$$\partial_x^\beta w(x) = (-1)^{m-1} \sum_{|\gamma|=m-1} \int_\Omega \partial_x^\beta \partial_y^\gamma G(x, y) f_\gamma(y) dy \,, \tag{101}$$

where β is any multi-index of order m and G is Green's function of problem (80). By Theorem 5

$$\left|\partial_x^\beta \partial_y^\gamma G(x, y)\right| \le c\, |x - y|^{-1} .$$

Hence, and by (101), the result follows. □

Generally, this assertion does not hold for differential operators with complex coefficients. More precisely, if there exists an angle vertex on the boundary of a convex domain Ω, one can construct a second-order strongly elliptic operator $L(\partial_x)$ with complex coefficients such that the Dirichlet problem (80) with $f \in C^\infty(\overline{\Omega})$ has a solution with unbounded gradient ([Koz], Sect. 9, see also [KMR2], Sect. 8.4.3). Our next result shows, in particular, that for L with complex coefficients the statement of Theorem 7 holds if the jumps of the normal vector are absent or small.

Corollary 7. *Let Ω be a bounded convex domain in $\mathbb{R}^2$ such that all jumps of the angle between the exterior normal vector to $\partial\Omega$ and the x-axis do not exceed a sufficiently small constant depending on m and the coefficients of $L(\partial_x)$. Then the conclusion of Corollary* 6 *holds.*

Proof. This is the same as that of Corollary 6, with Theorem 6 in the role of Theorem 5. □

6.3. The Dirichlet problem for the equations of hydrodynamics and elasticity in plane convex domains

We deduce some corollaries concerning the differentiability of solutions to classical problems of mathematical physics in a plane convex domain.

Stokes system. Let us start with the Dirichlet problem for the system:

$$\begin{cases} -\nu\Delta \mathbf{v} + \nabla p = \mathbf{f} & \text{in } \Omega \\ \nabla \cdot \mathbf{v} = 0 & \text{in } \Omega \\ \mathbf{v} = 0 & \text{on } \partial\Omega, \end{cases} \tag{102}$$

where $\mathbf{f} \in \left(W^{-1,2}(\Omega)\right)^2$ and $(\mathbf{v}, p) \in \left(\overset{\circ}{W}{}^{1,2}(\Omega)\right)^2 \times L^2(\Omega)$.

Proposition 1. *Let Ω be a bounded convex two-dimensional domain and let* $\mathbf{f} \in (L^q(\Omega))^2$*, for some* $q > 2$*. Then* $\mathbf{v} \in (C^{0,1}(\overline{\Omega}))^2$ *and*

$$\|\mathbf{v}\|_{C^{0,1}(\overline{\Omega})} \leq C\, \|\mathbf{f}\|_{(L^q(\Omega))^2}\,,$$

where C depends only on Ω.

Proof. Let $\mathbf{v} = (v_1, v_2)$. Introducing a stream function Φ by $v_1 = \partial_{x_2}\Phi$ and $v_2 = -\partial_{x_1}\Phi$, we arrive at the Dirichlet problem

$$\begin{cases} -\nu\Delta^2\Phi = \partial_{x_2} f_1 - \partial_{x_1} f_2 & \text{in } \Omega \\ \Phi \in \mathring{W}^{2,2}(\Omega). \end{cases}$$

By Corollary 6, $\Phi \in C^{1,1}(\overline{\Omega})$ and

$$\|\Phi\|_{C^{1,1}(\overline{\Omega})} \leq C\|\mathbf{f}\|_{(L^q(\Omega))^2},$$

which completes the proof. □

The Navier–Stokes system. Let $(\mathbf{v}, p) \in (\mathring{W}^{1,2}(\Omega))^2 \times L^2(\Omega)$ solve the Dirichlet problem:

$$\begin{cases} -\nu\Delta\mathbf{v} + \nabla p + \sum_{k=1}^2 v_k\partial_{x_k}\mathbf{v} = \mathbf{f} & \text{in } \Omega \\ \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega \\ \mathbf{v} = 0 & \text{on } \partial\Omega, \end{cases} \tag{103}$$

where $\mathbf{f} \in (W^{-1,2}(\Omega))^2$.

Proposition 2. *Let Ω be a bounded convex two-dimensional domain and let* $\mathbf{f} \in (L^q(\Omega))^2$*, for some* $q > 2$*. Then* $\mathbf{v} \in (C^{0,1}(\overline{\Omega}))^2$.

Proof. By Sobolev's embedding theorem the non-linear term in the Navier–Stokes system belongs to $(L^s(\Omega))^2$, for all $s < 2$. We note that the operator $\mathbf{f} \to \mathbf{v}$ corresponding to problem (102) continuously maps $(L^t(\Omega))^2$ into $(\mathring{W}^{1,2}(\Omega))^2$ and $(L^q(\Omega))^2$ into $(W^{1,r}(\Omega))^2$ with arbitrary $t > 1$, $q > 2$ and $r < \infty$. (Continuity of the former mapping is a consequence of the embedding $L^t(\Omega) \subset W^{-1,2}(\Omega)$, the continuity of the latter follows from Proposition 1 even with $r = \infty$.) Interpolating, one concludes that the operator $\mathbf{f} \to \mathbf{v}$ continuously maps $(L^s(\Omega)^2 \to (\mathring{W}^{1,\sigma}(\Omega))^2$ with an arbitrary $\sigma < 2s/(2-s)$. This implies $\mathbf{v} \in (\mathring{W}^{1,\sigma}(\Omega))^2$, with any $\sigma < \infty$. Hence and by $\mathbf{f} \in (L^q(\Omega))^2$, we conclude the proof by referring to Proposition 1. □

Von Kármán equations. Now, we deal with the Dirichlet problem for the system describing the non-linear bending of a thin plate whose boundary is clamped in the transversal direction and free in the horizontal direction [Ciarlet]

$$\begin{cases} \Delta^2 u_1 = [u_1, u_2] + f_1 & \text{in } \Omega \\ \Delta^2 u_2 = [u_1, u_1] + f_2 & \text{in } \Omega \\ \mathbf{u} := (u_1, u_2) \in \left(\mathring{W}^{2,2}(\Omega)\right)^2, \quad \mathbf{f} := (f_1, f_2) \in \left(W^{-2.2}(\Omega)\right)^2, \end{cases} \tag{104}$$

where

$$[u, v] = \partial_{x_1}^2 u\; \partial_{x_2}^2 v + \partial_{x_2}^2 u\; \partial_{x_1}^2 v - 2\partial_{x_1}\partial_{x_2} u\; \partial_{x_1}\partial_{x_2} v\,.$$

Proposition 3. *Let Ω be a bounded convex two-dimensional domain and let* $\mathbf{f} \in (W^{-1,q}(\Omega))^2$, *for some* $q > 2$. *Then* $\mathbf{u} \in (C^{1,1}(\overline{\Omega}))^2$.

Proof. Since

$$[u, v] = \partial_{x_1}\left(\partial_{x_1}u\ \partial^2_{x_2}v - \partial_{x_2}u\ \partial_{x_1}\partial_{x_2}v\right) + \partial_{x_2}\left(\partial_{x_2}u\ \partial^2_{x_1}v - \partial_{x_1}u\ \partial_{x_1}\partial_{x_2}v\right),$$

the non-linear term in (104) belongs to $(W^{-1,s}(\Omega))^2$, for all $s < 2$. It follows from the embedding $W^{-1,t}(\Omega) \subset W^{-2,2}(\Omega)$, for all $t > 1$, that the inverse operator of the Dirichlet problem for Δ^2 maps $W^{-1,t}(\Omega)$ into $\overset{\circ}{W}{}^{2,2}(\Omega)$. By Corollary 6 the same inverse maps $W^{-1,q}(\Omega)$ into $\overset{\circ}{W}{}^{2,r}(\Omega)$, for all $q > 2$ and $r < \infty$. Interpolating we obtain that this inverse continuously maps $(W^{-1,s}(\Omega))^2$ into $(\overset{\circ}{W}{}^{2,\sigma}(\Omega))^2$ with an arbitrary $\sigma < 2s/(2-s)$. Hence $\mathbf{u} \in (\overset{\circ}{W}{}^{2,\sigma}(\Omega))^2$ with any $\sigma < \infty$. This, in combination with $\mathbf{f} \in (W^{-1,q}(\Omega))^2$ and Corollary 6, completes the proof. □

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