Global Lipschitz regularity for a class of quasilinear elliptic equations

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Abstract

The Lipschitz continuity of solutions to Dirichlet and Neumann problems for nonlinear elliptic equations, including the p-Laplace equation, is established under minimal integrability assumptions on the data and on the curvature of the boundary of the domain. The case of arbitrary bounded convex domains is also included. The results have new consequences even for the Laplacian.

1 Introduction and main results

We deal with boundary value problems for a class of quasilinear elliptic equations in an open bounded subset Ω of $\mathbb{R}^n$, $n \geq 3$. Neumann problems of the form

$$\begin{cases} -\mathrm{div}(a(|\nabla u|)\nabla u) = f(x) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

are the main objective of the present paper, although our results for the Dirichlet problem

$$\begin{cases} -\mathrm{div}(a(|\nabla u|)\nabla u) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

will also be new. Here, ∇u denotes the gradient of u, and ν is the outward unit normal to $\partial\Omega$. The function $a : (0,\infty) \to (0,\infty)$ is assumed to be of class $C^1(0,\infty)$, and to fulfill

$$-1 < i_a \leq s_a < \infty, \tag{1.3}$$

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where

$$i_a = \inf_{t>0} \frac{ta'(t)}{a(t)} \qquad \text{and} \qquad s_a = \sup_{t>0} \frac{ta'(t)}{a(t)}. \tag{1.4}$$

In particular, the standard p-Laplace operator, corresponding to the choice $a(t) = t^{p-2}$, with $p > 1$, is included in this framework, since $i_a = s_a = p - 2$ in this case.
Let us point out that the approach of this paper is applicable to equations with a more general structure, including lower-order terms. We limit ourselves to the model situations (1.1) and (1.2) to avoid additional technicalities.

Note that problems (1.1) and (1.2) are the Euler equations of the strictly convex functional

$$J(u) = \int_\Omega B(|\nabla u|) - fu\, dx\,, \tag{1.5}$$

in suitable Orlicz-Sobolev spaces, where the function $B : [0, \infty) \to [0, \infty)$ is given by

$$B(t) = \int_0^t b(\tau)\, d\tau \qquad \text{for } t \geq 0, \tag{1.6}$$

and $b : [0, \infty) \to [0, \infty)$ is defined as

$$b(t) = a(t)t \qquad \text{if } t > 0, \tag{1.7}$$

and $b(0) = 0$. By the first inequality in (1.3), the function b is strictly increasing, and hence the function B is strictly convex (see Section 2 for details).

We are concerned with the boundedness of the gradient of solutions to (1.1) and (1.2), and hence with the Lipschitz continuity of solutions. This classical problem has been extensively investigated especially in the case of polynomial type nonlinearities; contributions to this topic include [Be, LU1, LU2, Ur, Ul, Di, Ev, Le, To, Li1, Li2, Li4, Li5]. Equations with non-necessarily power type growth have also been considered – see e.g. [Si, Mar, Ko, Li3]. Needless to say, these references do not exhaust the rich literature on the subject.

The Lipschitz continuity of solutions is well known to depend both on the integrability of f and on the regularity of Ω. Minimal integrability assumptions on f and on the curvature of $\partial\Omega$ will be exhibited under which the solutions to problems (1.1) and (1.2) are Lipschitz continuous. Let us emphasize that the results to be presented sharpen those available in the literature even for the Lapalce operator.

Bounded domains Ω whose boundary $\partial\Omega \in W^2L^{n-1,1}$ will be allowed. This means that Ω is locally the subgraph of a function of $n-1$ variables whose second-order weak derivatives belong to the Lorentz space $L^{n-1,1}$. This is the weakest possible integrability assumption on second-order derivatives for the first-order derivatives to be continuous, and hence for $\partial\Omega \in C^{1,0}$ [CP]. Note that, by contrast, classical results on global boundedeness of the gradient of solutions require $\partial\Omega \in C^1_\omega$ for some continuity modulus ω fulfilling a Dini condition – see [Li1, Section 3] and [Li2, Theorem 5.1] for a proof, and also [An, Remarks on Lemma A3.1].
The case of arbitrary convex domains will also be included in our discussion. This case is of special interest for the Neumann problem (1.1), no general result seems to be available in this case for nonlinear equations even for smooth right-hand sides f. Partial contributions in this connection can be found in [Li4, Example, page 58, and Remark, page 62]. Observe that, for Dirichlet problems, the Lipschitz continuity of solutions in convex domains and for bounded f follows via standard barrier arguments.

As far as the integrability of f is concerned, [Di, Theorem 1 and subsequent Remark] tell us that if $p > 1$ and $f \in L^q(\Omega)$ for some $q > \frac{pn}{p-1}$, then the gradient of weak solutions to p-Laplacian type equations is (locally) bounded. An improvement of this result can be found in [Li3], where the boundedness of the gradient is derived under the weaker assumption that $f \in L^q(\Omega)$ for some $q > n$. An assumption of a slightly different nature on f is considered in [Li5, Section 5], where the boundedness of the gradient of solutions is shown to follow from the membership of f in any Morrey space $L_{1,s}(\Omega)$ with $s > n$.
Very recent contributions of [Mi1] and [DM1, DM2] provide precise pointwise bounds for the gradient of weak solutions to p-Laplacian type equations, with $p \geq 2$, in terms of the nonlinear potentials of f introduced in [HM]. Combined with rearrangement estimates for the relevant nonlinear potentials, these bounds lead to gradient bounds in classes of rearrangement invariant spaces [Ci6]. In particular, this approach tells us that the gradient is locally bounded provided that f belongs to the Lorentz space $L^{n,\frac{1}{p-1}}(\Omega)$. Since $L^q(\Omega) \subsetneqq L^{n,\frac{1}{p-1}}(\Omega)(\subsetneqq L^n(\Omega))$ for every $q > n$ and $p \geq 2$, the result of [Ci6] improves the earlier results mentioned above. Incidentally, note that the assumption $f \in L^{n,\frac{1}{p-1}}(\Omega)$ has also been shown to ensure the continuity of the gradient of solutions to p-Laplacian equations [DM3].

It was however noted in [Mi2] that the condition $f \in L^{n,\frac{1}{p-1}}(\Omega)$ could probably be further weakened. This guess was motivated by the observation that, owing to the inclusion relations between Lorentz spaces, increasing p causes the assumption $f \in L^{n,\frac{1}{p-1}}(\Omega)$ to be more stringent, whereas the solutions to the p-Laplacian equation tend to be more regular as p increases. In the linear case corresponding to $a \equiv 1$, the differential operator is the Laplacian, and, since $p = 2$, we have that $L^{n,\frac{1}{p-1}}(\Omega) = L^{n,1}(\Omega)$. In this case, the assumption $f \in L^{n,1}(\Omega)$ is known to be optimal, in the class of all rearrangement invariant spaces, for the boundedness of the gradient of the solution to the Dirichlet problem with homogeneous boundary condition when Ω is a ball [Ci2]. It was conjectured in [Mi2] that the same result should hold also in the nonlinear case, and hence that the membership of f in $L^{n,1}(\Omega)$ should be a sharp assumption for the boundedness of the gradient of solutions to the p-Laplace equation for any $p \in (1, \infty)$.

In this paper we show that the gradient of global solutions to equations of the more general form appearing in (1.1) and (1.2) is actually bounded if $f \in L^{n,1}(\Omega)$. Our result for weak solutions to the Neumann problem (1.1) in domains with $\partial\Omega \in W^2L^{n-1,1}$ is contained in the following theorem.

Theorem 1.1 *Let Ω be a bounded subset of $\mathbb{R}^n$, $n \geq 3$, such that $\partial\Omega \in W^2L^{n-1,1}$. Assume that $f \in L^{n,1}(\Omega)$. Let u be a weak solution to problem* (1.1). *Then there exists a constant $C = C(i_a, s_a, \Omega)$ such that*

$$\|\nabla u\|_{L^\infty(\Omega)} \leq Cb^{-1}\big(\|f\|_{L^{n,1}(\Omega)}\big). \tag{1.8}$$

In particular, u is Lipschitz continuous on $\overline{\Omega}$.

The next result tells us that the regularity assumption $\partial\Omega \in W^2L^{n-1,1}$ can be replaced by the convexity of Ω. Such a result is new even for smooth f. Only the case of the Laplace equation, with $f \in L^q$ for some $q > n$, is known, having recently been established in [Ma5, Ma6].

Theorem 1.2 *Replace the assumption $\partial\Omega \in W^2L^{n-1,1}$ by the assumption that Ω is convex in the statement of Theorem 1.1. Then the same conclusions hold.*

The counterparts of Theorems 1.1 and 1.2 for weak solutions to the Dirichlet problem (1.2) read as follows.

Theorem 1.3 *Let Ω be a bounded subset of $\mathbb{R}^n$, $n \geq 3$, such that $\partial\Omega \in W^2L^{n-1,1}$. Assume that $f \in L^{n,1}(\Omega)$. Let u be a weak solution to problem (1.2). Then there exists a constant $C = C(i_a, s_a, \Omega)$ such that*

$$\|\nabla u\|_{L^\infty(\Omega)} \leq Cb^{-1}\big(\|f\|_{L^{n,1}(\Omega)}\big). \tag{1.9}$$

In particular, u is Lipschitz continuous on $\overline{\Omega}$.

Theorem 1.4 *Replace the assumption $\partial\Omega \in W^2L^{n-1,1}$ by the assumption that Ω is convex in the statement of Theorem 1.3. Then the same conclusions hold.*

The outline of our approach is the following. We start by integrating the differential equation in (1.1) or (1.2), multiplied by the Laplacian Δu of u, over the level sets of $|\nabla u|$. This step is related to a method exploited in [Ma2, Ma6] for linear equations. Integration over the level sets of the function u is a quite classical and effective technique in the study of integrability properties of u. On the other hand, information on $|\nabla u|$ from integration on its level sets is harder to derive, especially in the nonlinear case. One reason is related to the presence of boundary terms. In this connection, a observation in our proof is that the nonlinear expression $\Delta u \,\mathrm{div}(a(|\nabla u|)\nabla u)$ can be pointwise estimated by the sum of terms in divergence form plus a signed term. This enables us to call the boundary condition into play. Eventually, we derive a differential inequality for the distribution function of $|\nabla u|$, which can be handled to obtain the desired gradient bound. The derivation of this differential inequality requires various ingredients, such as the coarea formula, the relative isoperimetric inequality in Ω, and certain rearrangement inequalities, and is reminiscent of techniques introduced for estimates of u in [Ma1, Ma3, Ta1, Ta3].

A detailed proof of Theorem 1.1, dealing with the Neumann problem, is given in Section 4, where the necessary changes needed to treat the same problem in arbitrary convex domains (Theorem 1.2), and the Dirichlet problem (Theorems 1.3 and 1.4) are also described. Some technical results to be used in the proofs are collected in Section 3. Section 2 contains the background on function spaces and on weak solutions to (1.1) and (1.2) which are non-standard in the generality of this paper. This part can be essentially skipped by the reader who is just interested in the customary case when the differential operator in (1.1) and (1.2) is the plain p-Laplace operator, i.e. $a(t) = t^{p-2}$ with $p \in (1, \infty)$, and who is familiar with classical properties of weak solutions to p-Laplacian type equations.

2 Function spaces and basic properties of solutions

An appropriate functional framework for the solutions to problems (1.1) and (1.2) is provided by the Orlicz-Sobolev spaces, which extend the classical Sobolev spaces. The results form the theory of Orlicz-Sobolev spaces to be used in our proofs are collected in Subsection 2.2. Subsection 2.3 deals with precise definitions of weak solutions to (1.1) and (1.2), with their existence and uniqueness properties and with a priori bounds. We begin with definitions and basic properties of Lorentz and Lorentz-Sobolev spaces, which come into play in our description of the regularity of the datum f and of the domain Ω.

2.1 Lorentz and Lorentz-Sobolev spaces

Let $(\mathcal{R}, m)$ be a positive, finite, non-atomic measure space. Let u be a measurable function on $\mathcal{R}$. The distribution function $\mu_u : [0, \infty) \to [0, m(\mathcal{R})]$ is given by

$$\mu_u(t) = m(\{x \in \mathcal{R} : |u(x)| > t\}) \qquad \text{for } t \geq 0. \tag{2.1}$$

The decreasing rearrangement $u^*:[0,\infty)\to[0,\infty]$ of u is defined as

$$u^*(s)=\sup\{t\geq 0:\mu_u(t)>s\}\qquad \text{for } s\in[0,\infty),$$

and is the unique right-continuous non-increasing function in $[0,\infty)$ equidistributed with u. Note that $u^*(s)=0$ if $s\geq m(\mathcal{R})$.
The function $u^{**}:(0,\infty)\to[0,\infty)$, defined by

$$u^{**}(s)=\frac{1}{s}\int_0^s u^*(r)\,dr\qquad \text{for } s>0,$$

is nondecreasing, and fulfills $u^*(s)\leq u^{**}(s)$ for $s>0$.
The Hardy-Littlewood inequality states that

$$\int_{\mathcal{R}}|u(x)v(x)|dm(x)\leq\int_0^\infty u^*(s)v^*(s)ds \tag{2.2}$$

for all measurable functions u and v in $\mathcal{R}$.
Given $q\in(1,\infty)$ and $\sigma\in(0,\infty]$, the Lorentz space $L^{q,\sigma}(\mathcal{R})$ consists of all measurable functions $u:\mathcal{R}\to\mathbb{R}$ for which the quantity

$$\|u\|_{L^{q,\sigma}(\mathcal{R})}=\|s^{\frac{1}{q}-\frac{1}{\sigma}}u^*(s)\|_{L^\sigma(0,m(\mathcal{R}))} \tag{2.3}$$

is finite. If $\sigma\in[1,\infty]$, then $L^{q,\sigma}(\mathcal{R})$ is a Banach space, equipped with the norm, equivalent to $\|\cdot\|_{L^{q,\sigma}(\mathcal{R})}$, obtained on replacing u^* by u^{**} on the right-hand side of (2.3).
One has that

$$L^{q,q}(\mathcal{R})=L^q(\mathcal{R})\qquad \text{for } q\in(1,\infty),$$
$$L^{q,\sigma_1}(\mathcal{R})\subsetneqq L^{q,\sigma_2}(\mathcal{R})\qquad \text{if } \sigma_1<\sigma_2,$$

and

$$L^{q_1,\sigma_1}(\mathcal{R})\subsetneqq L^{q_2,\sigma_2}(\mathcal{R})\qquad \text{if } q_1>q_2 \text{ and } \sigma_1,\sigma_2\in(0,\infty].$$

Let Ω be an open bounded subset of $\mathbb{R}^n$. Let $m\in\mathbb{N}$, and let $q\in(1,\infty)$ and $\sigma\in[1,\infty]$. The Lorentz-Sobolev space $W^mL^{q,\sigma}(\Omega)$ is the Banach space defined as

$$W^mL^{q,\sigma}(\Omega)=\{u\in L^{q,\sigma}(\Omega): \text{is } m\text{-times weakly differentiable in } \Omega$$
$$\text{and } |\nabla^k u|\in L^{q,\sigma}(\Omega) \text{ for } 1\leq k\leq m\},$$

and is equipped with the norm

$$\|u\|_{W^mL^{q,\sigma}(\Omega)}=\|u\|_{L^{q,\sigma}(\Omega)}+\sum_{k=1}^m\|\nabla^k u\|_{L^{q,\sigma}(\Omega)}.$$

Here, $\nabla^k u$ denotes the vector of all weak derivatives of u of order k. When $k=1$ we simply write ∇u instead of $\nabla^1 u$.
If $\sigma<\infty$, the space $C_0^\infty(\Omega)$ is dense in $L^{q,\sigma}(\Omega)$, as is shown by an adaptation of a classical result for Lebesgue spaces. As a consequence, by a standard convolution argument relying upon a version of Young convolution inequality in Lorentz spaces by O'Neil (see e.g. [Zi, Theorem 2.10.1]), $C^\infty(\Omega)$ is dense in $W^mL^{q,\sigma}(\Omega)$.
A limiting case of the Sobolev embedding theorem asserts that if Ω has a Lipschitz boundary, then $W^1L^{n,1}(\Omega)\to C^0(\Omega)$. Here, the arrow " $\to$ " stands for continuous embedding. Moreover,

$L^{n,1}(\Omega)$ is the smallest space having this property among all rearrangement invariant spaces, namely, roughly speaking, all Banach function spaces whose norm depends only on the decreasing rearrangement of functions [CP]. Hence, in particular,

$$W^2L^{n,1}(\Omega) \to C^{1,0}(\Omega), \tag{2.4}$$

and $L^{n,1}(\Omega)$ is optimal in the same sense as above.

2.2 Orlicz and Orlicz-Sobolev spaces

A function $B : [0,\infty) \to [0,\infty]$ is called a Young function if it is convex and $B(0) = 0$. If, addition, $0 < B(t) < \infty$ for $t > 0$ and

$$\lim_{t\to 0} \frac{B(t)}{t} = 0 \qquad \text{and} \qquad \lim_{t\to\infty} \frac{B(t)}{t} = \infty,$$

then B is called an N-function. The Young conjugate of a Young function B is the Young function $\widetilde{B}$ defined as

$$\widetilde{B}(t) = \sup\{st - B(s) : s \geq 0\} \qquad \text{for } t \geq 0.$$

In particular, if B is an N-function, then $\widetilde{B}$ is an N-function as well.
A Young function (and, more generally, an increasing function) B is said to belong to the class Δ_2 if there exists a constant $C > 1$ such that

$$B(2t) \leq CB(t) \qquad \text{for } t > 0.$$

Let $(\mathcal{R}, m)$ be a positive, finite, non-atomic measure space. The Orlicz space $L^B(\mathcal{R})$ is the Banach function space of those measurable functions $u : \mathcal{R} \to \mathbb{R}$ whose Luxemburg norm

$$\|u\|_{L^B(\mathcal{R})} = \inf\left\{\lambda > 0 : \int_{\mathcal{R}} B\left(\frac{|u(x)|}{\lambda}\right) dm(x) \leq 1\right\}$$

is finite. The inequalities

$$\|u\|_{L^{\widetilde{B}}(\mathcal{R})} \leq \sup_{v\in L^B(\mathcal{R})} \frac{\int_{\mathcal{R}} |u(x)v(x)|\, dx}{\|v\|_{L^B(\mathcal{R})}} \leq 2\|u\|_{L^{\widetilde{B}}(\mathcal{R})} \tag{2.5}$$

hold for every $u \in L^{\widetilde{B}}(\mathcal{R})$. One has that
(2.6)

$$L^{B_1}(\mathcal{R}) \to L^{B_2}(\mathcal{R}) \text{ if and only if there exist } c, t_0 > 0 \text{ such that } B_2(t) \leq B_1(ct) \text{ for } t > t_0.$$

Let Ω be an open bounded set in $\mathbb{R}^n$. The Orlicz-Sobolev space $W^{1,B}(\Omega)$ is the Banach space defined as

$$W^{1,B}(\Omega) = \{u \in L^B(\Omega) : \text{is weakly differentiable in } \Omega \text{ and } |\nabla u| \in L^B(\Omega)\},$$

and is equipped with the norm

$$\|u\|_{W^{1,B}(\Omega)} = \|u\|_{L^B(\Omega)} + \|\nabla u\|_{L^B(\Omega)}.$$

Higher-order Orlicz-Sobolev spaces are defined accordingly. The space $W^{1,B}_0(\Omega)$ is the Banach subspace of $W^{1,B}(\Omega)$ given by

$$W^{1,B}_0(\Omega) = \{u \in W^{1,B}(\Omega) : \text{the continuation of } u \text{ by } 0 \text{ outside } \Omega \text{ is weakly differentiable in } \mathbb{R}^n\}.$$

The Banach subspace $W^{1,B}_{\perp}(\Omega)$ of $W^{1,B}(\Omega)$ is defined as

$$W^{1,B}_{\perp}(\Omega) = \{u \in W^{1,B}(\Omega) : u_\Omega = 0\},$$

where

$$u_\Omega = \frac{1}{|\Omega|} \int_\Omega u(x)\, dx\,,$$

the mean value of u over Ω. Here, $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^n$.

Theorem 2.1 [DT] *Let Ω be an open bounded set in $\mathbb{R}^n$, $n \geq 2$. Assume that B is a Young function such that $B \in \Delta_2$. Then the space the space $C^\infty_0(\Omega)$ is dense in $W^{1,B}_0(\Omega)$.*
If, in addition, Ω has a Lipschitz boundary, then $C^\infty(\overline{\Omega})$ is dense in $W^{1,B}(\Omega)$.

Theorem 2.2 [DT] *Let Ω be an open bounded set in $\mathbb{R}^n$, $n \geq 2$. Let B be a Young function such that B and $\widetilde{B} \in \Delta_2$. Then the spaces $W^{1,B}(\Omega)$, $W^{1,B}_0(\Omega)$ and $W^{1,B}_{\perp}(\Omega)$ are reflexive.*

Let B be a Young function such that

$$\int_0 \left(\frac{t}{B(t)}\right)^{\frac{1}{n-1}} dt < \infty. \tag{2.7}$$

The Sobolev conjugate of B, introduced in [Ci4](and, in an equivalent form, in [Ci3]), is the Young function B_n defined as

$$B_n(t) = B\big({H_n}^{-1}(t)\big) \qquad \text{for } t \geq 0, \tag{2.8}$$

where

$$H_n(s) = \left(\int_0^s \left(\frac{t}{B(t)}\right)^{\frac{1}{n-1}} dt\right)^{1/n'} \qquad \text{for } s \geq 0, \tag{2.9}$$

and ${H_n}^{-1}$ denotes the (generalized) left-continuous inverse of H_n.

Theorem 2.3 [Ci3, Ci4] *Let Ω be an open bounded set in $\mathbb{R}^n$, $n \geq 2$. Let B be a Young function fulfilling* (2.7)*. Then there exists a constant $C = C(n, |\Omega|)$ such that*

$$\|u\|_{L^{B_n}(\Omega)} \leq C \|\nabla u\|_{L^B(\Omega)} \tag{2.10}$$

for every $u \in W^{1,B}_0(\Omega)$. If, in addition, Ω has a Lipschitz boundary, then there exists a constant $C = C(\Omega)$ such that (2.10) *holds for every $u \in W^{1,B}_{\perp}(\Omega)$. The space $L^{B_n}(\Omega)$ is optimal in* (2.10) *among all Orlicz spaces.*

Remark 2.4 Assumption (2.7) is immaterial in Theorem 2.3. In fact, owing to (2.6), the Young function B can be replaced, if necessary, by another Young function fulfilling (2.7) in such a way that $W^{1,B}(\Omega)$ remains unchanged (up to equivalent norms).

Remark 2.5 If

$$\int^\infty \left(\frac{t}{B(t)}\right)^{\frac{1}{n-1}} dt < \infty, \tag{2.11}$$

then ${H_n}^{-1}(t) = \infty$ for large t, and hence $B_n(t) = \infty$ for large t as well. Thus, by (2.6), $\|u\|_{L^{B_n}(\Omega)} = L^\infty(\Omega)$, up to equivalent norms.

Remark 2.6 It is easily verified that there exist constants $c, t_0 > 0$ such that $B(t) \le B_n(ct)$ for $t > t_0$. Thus, by Theorem 2.3 and (2.6), there exists a constant $C = C(n, |\Omega|, B)$ such that

$$\|u\|_{L^B(\Omega)} \le C\|\nabla u\|_{L^B(\Omega)} \tag{2.12}$$

for every $u \in W_0^{1,B}(\Omega)$. If Ω has a Lipschitz boundary, then (2.12) holds with $C = C(\Omega, B)$ for every $u \in W_\perp^{1,B}(\Omega)$.

Remark 2.7 Given any Young function B, there exist constants $c, t_0 > 0$ such that $t \le B(ct)$ for $t > t_0$. As a consequence, one can show that there exist constants k and t_1 such that $t^{n'} \le B_n(kt)$ for some $t > t_1$. Hence, by (2.6),

$$L^{B_n}(\Omega) \to L^{n'}(\Omega). \tag{2.13}$$

It is easily verified that $\lim_{s\to\infty} \frac{B_n^{-1}(s)}{B^{-1}(s)} = 0$. Hence, by [Ci3, Theorem3], the following compact embedding holds.

Theorem 2.8 *Let Ω be an open bounded subset of $\mathbb{R}^n$, $n \ge 2$. Let B be a Young function fulfilling* (2.7)*. Then the embedding*

$$W_0^{1,B}(\Omega) \to L^B(\Omega) \tag{2.14}$$

is compact. If, in addition, Ω has a Lipschitz boundary, then also the embedding

$$W_\perp^{1,B}(\Omega) \to L^B(\Omega) \tag{2.15}$$

is compact.

2.3 Weak solutions

In this section, and in the remaining part of the paper, B will denote the Young function defined by (1.6). The following basic properties of the function B are relevant in connection with the definitions of weak solutions to the boundary value problems (1.1) and (1.2) given below.

Proposition 2.9 *Assume that $a \in C^1(0, \infty)$ and* (1.3) *holds. Let B be the function defined by* (1.6)*. Then B is a strictly convex N-function, and*

$$B \in \Delta_2 \qquad \text{and} \qquad \widetilde{B} \in \Delta_2. \tag{2.16}$$

Moreover, there exists a constant $C = C(i_a, s_a)$ such that

$$\widetilde{B}(b(t)) \le CB(t) \qquad \text{for } t \ge 0. \tag{2.17}$$

Proof. By the second inequality in (1.3) we have that

$$\sup_{t>0} \frac{tb'(t)}{b(t)} \le s_a + 1\,. \tag{2.18}$$

Inequality (2.18) implies that there exists a constant $C = C(s_a)$ such that

$$\sup_{t>0} \frac{tb(t)}{B(t)} \le C, \tag{2.19}$$

and (2.19) in turn implies that $B \in \Delta_2$ – see e.g. [RR, Chapter 2, Theorem 3]. Integration by parts yields

$$\begin{aligned} B(t) &= \int_0^t sa(s)\,ds = \frac{t^2 a(t)}{2} - \int_0^t \frac{s^2 a'(s)}{2}\,ds \\ &\leq \frac{t^2 a(t)}{2} - i_a \int_0^t \frac{sa(s)}{2}\,ds = \frac{tb(t)}{2} - i_a \frac{B(t)}{2} \quad \text{for } t \geq 0. \end{aligned}$$

Thus, $B(t)(2+i_a) \leq tb(t)$ for $t \geq 0$, and hence, owing to (1.3), $\inf_{t>0} \frac{tb(t)}{B(t)} > 1$. By [RR, Chapter 2, Theorem 3], the last inequality ensures that $\widetilde{B} \in \Delta_2$. Equation (2.16) is thus established.
By [MSZ, Proposition 2.6], equation (2.16) implies that B is an N-function. The first inequality in (1.3) entails that B is strictly convex. Indeeed, if B were a linear function in some interval, then $\frac{ta'(t)}{a(t)} = -1$ for every t in such interval.
As for (2.17), since $\widetilde{B}' = b^{-1}$, an increasing function, we have that

$$\widetilde{B}(b(t)) \leq \widetilde{B}'(b(t))b(t) = b^{-1}(b(t))b(t) = tb(t) \leq CB(t) \quad \text{for } t \geq 0,$$

where the last inequality holds by (2.19). □

Let Ω be an open bounded set in $\mathbb{R}^n$. Assume that $f \in L^{\widetilde{B_n}}(\Omega)$. A weak solution to problem (1.2) is a function $u \in W^{1,B}_0(\Omega)$ such that

$$\int_\Omega a(|\nabla u|)\nabla u \cdot \nabla\phi\,dx = \int_\Omega f\phi dx \tag{2.20}$$

for every $\phi \in W^{1,B}_0(\Omega)$. Here, the dot " $\cdot$ " stands for scalar product.
Assume in addition that Ω has a Lipschitz boundary. A weak solution to problem (1.1) is a function $u \in W^{1,B}(\Omega)$ such that

$$\int_\Omega a(|\nabla u|)\nabla u \cdot \nabla\phi\,dx = \int_\Omega f\phi dx \tag{2.21}$$

for every $\phi \in W^{1,B}(\Omega)$.

Remark 2.10 The left-hand sides of (2.20) and (2.21) are well defined owing to (2.17) and to the second inequality in (2.5). The right-hand sides are also well defined, owing to Theorem 2.3 and to the second inequality in (2.5) with B replaced by B_n.

Remark 2.11 By (2.16) and Theorem 2.1, taking test functions $\phi \in C^\infty_0(\Omega)$ and $\phi \in C^\infty(\overline{\Omega})$, instead of $W^{1,B}_0(\Omega)$ and $W^{1,B}(\Omega)$, in (2.20) and (2.21), respectively, results in equivalent definitions.

Remark 2.12 Weak solutions to (1.1) and (1.2) are well defined, in particular, when $f \in L^{n,1}(\Omega)$. Indeed,

$$L^{n,1}(\Omega) \to L^{\widetilde{B_n}}(\Omega).$$

This is a consequence of the chain

$$\begin{aligned} \|u\|_{L^{\widetilde{B_n}}(\Omega)} &\leq \sup_{v\in L^{B_n}(\Omega)} \frac{\int_\Omega |u(x)v(x)|\,dx}{\|v\|_{L^{B_n}(\Omega)}} \\ &\leq C \sup_{v\in L^{n'}(\Omega)} \frac{\int_\Omega |u(x)v(x)|\,dx}{\|v\|_{L^{n'}(\Omega)}} = C\|u\|_{L^n(\Omega)} \leq C'\|u\|_{L^{n,1}(\Omega)}, \end{aligned}$$

which holds for suitable constants C and C' and for every $u \in L^{n,1}(\Omega)$. Note that the first inequality is a consequence of the first inequality in (2.5) with B replaced by B_n, the second one of (2.13), and the last one of the embedding $L^{n,1}(\Omega) \to L^n(\Omega)$.

An existence and uniqueness result for solutions to problems (1.1) and (1.2) is provided by the following theorem.

Theorem 2.13 *Let Ω be an open bounded set in $\mathbb{R}^n$, $n \geq 2$. Assume that $a \in C^1(0,\infty)$ and that (1.3) is in force. Let $f \in L^{\widetilde{B_n}}(\Omega)$.*
(i) There exist a unique solution $u \in W_0^{1,B}(\Omega)$ to problem (1.2).
(ii) Assume, in addition, that Ω has a Lipschitz boundary, and that $\int_\Omega f(x)dx = 0$. Then there exists a unique solution $u \in W_\perp^{1,B}(\Omega)$ to problem (1.1).

Proof. (i) The functional J given by (1.5) is finite-valued on $W_0^{1,B}(\Omega)$ since $B \in \Delta_2$. Inasmuch as B is strictly convex, J is strictly convex and lower semicontinuous with respect to the weak convergence in $W_0^{1,B}(\Omega)$. By Theorem 2.2, the space $W_0^{1,B}(\Omega)$ is reflexive. Thus, by a standard result in the calculus of variations, we can conclude that the functional J has a unique minimizer if we show that

$$\lim_{\|u\|_{W_0^{1,B}(\Omega)}\to\infty} J(u) = \infty. \tag{2.22}$$

To verify (2.22), note that, since $\widetilde{B} \in \Delta_2$, there exist constants $\delta > 0$ and $C > 0$ such that

$$B(t) \geq Cs^{1+\delta}B(t/s) \qquad \text{if } t \geq 0 \text{ and } s \geq 1, \tag{2.23}$$

[MSZ, Equation 2.5]. On making use of the second inequality in (2.5), Theorem 2.3 and Remark 2.4, and inequalities (2.10), (2.12) and (2.23), one has that if $\|\nabla u\|_{L^B(\Omega)} \geq 1$, then

(2.24)
$$\begin{aligned} J(u) &\geq \int_\Omega B(|\nabla u|)dx - 2\|u\|_{L^{B_n}(\Omega)}\|f\|_{L^{\widetilde{B_n}}(\Omega)} \geq \int_\Omega B(|\nabla u|)dx - C\|\nabla u\|_{L^B(\Omega)}\|f\|_{L^{\widetilde{B_n}}(\Omega)} \\ &= \|\nabla u\|_{L^B(\Omega)}\bigg(\frac{1}{\|\nabla u\|_{L^B(\Omega)}}\int_\Omega B(|\nabla u|)dx - C\|f\|_{L^{\widetilde{B_n}}(\Omega)}\bigg) \\ &\geq \|\nabla u\|_{L^B(\Omega)}\bigg(\|\nabla u\|^\delta_{L^B(\Omega)}\int_\Omega B\bigg(\frac{|\nabla u|}{\|\nabla u\|_{L^B(\Omega)}}\bigg)dx - C\|f\|_{L^{\widetilde{B_n}}(\Omega)}\bigg) \end{aligned}$$

for some positive constant $C = C(n,|\Omega|)$. Since $B \in \Delta_2$,

$$\int_\Omega B\bigg(\frac{|\nabla u|}{\|\nabla u\|_{L^B(\Omega)}}\bigg)dx = 1,$$

see e.g. [RR, Chapter 3, Proposition 6]. Hence, (2.22) follows.
A standard argument, exploiting the fact that $B \in \Delta_2$, tells us that the functional J is differentiable. Furthermore, its differential vanishes at the minimizer of J, and hence such a minimizer is a weak solution to problem (1.2), the Euler equation of J. On the other hand, by the convexity of J, any weak solution to (1.2) is a minimizer of J. Thus, the solution to (1.2) is unique.
(ii) The same argument applies, since embedding (2.10) and inequality (2.12) hold for every $u \in W_\perp^{1,B}(\Omega)$ under the present assumptions on Ω. □

The energy estimate for solutions to (1.1) and (1.2) contained in the next proposition relies upon results from [Ta2, Ci1].

Proposition 2.14 *Let Ω be an open bounded subset of $\mathbb{R}^n$, $n \geq 2$. Assume that $a \in C^1(0,\infty)$ and that* (1.3) *is in force. Let $f \in L^{\widetilde{B_n}}(\Omega)$.*
(i) Let $u \in W_0^{1,B}(\Omega)$ be the weak solution to problem (1.2). *Then there exists a constant $C = C(n, i_a, s_a)$ such that*

$$\int_\Omega B(|\nabla u|)dx \leq C|\Omega| \|f\|_{L^{n,1}(\Omega)} b^{-1}\big(\|f\|_{L^{n,1}(\Omega)}\big). \tag{2.25}$$

(ii) Assume, in addition, that Ω has a Lipschitz boundary, and that $\int_\Omega f(x)dx = 0$. Let $u \in W^{1,B}(\Omega)$ be a weak solution to problem (1.1). *Then there exists a constant $C = C(\Omega, i_a, s_a)$ such that*

$$\int_\Omega B(|\nabla u|)dx \leq C(\Omega) \|f\|_{L^{n,1}(\Omega)} b^{-1}\big(\|f\|_{L^{n,1}(\Omega)}\big). \tag{2.26}$$

In the proof of Theorem 2.14, and of our main results, we shall need additional properties of the function B given by (1.6), and of the functions $\widehat{B}(t) : [0,\infty) \to [0,\infty)$, defined as

$$\widehat{B}(t) = \frac{B(t)}{t} \qquad \text{for } t > 0, \tag{2.27}$$

$\widehat{B}(0) = 0$, and $F : [0,\infty) \to [0,\infty)$, given by

$$F(t) = \int_0^t b(\tau)^2 \, d\tau \qquad \text{for } t \geq 0. \tag{2.28}$$

They are the content of the following proposition.

Proposition 2.15 *Assume that $a \in C^1(0,\infty)$ and* (1.3) *holds Let B, $\widehat{B}$ and F be the functions defined by* (1.6), (2.27), *and* (2.28), *respectively. Then:*
(i)

$$a(1)t^{i_a} \leq a(t) \leq a(1)t^{s_a} \quad \textit{for } t > 0; \tag{2.29}$$

(ii)

$$\lim_{t\to 0} b(t) = 0; \tag{2.30}$$

(iii) For every $C > 0$, there exists a positive constant $C' = C'(s_a, C) > 0$ such that

$$Cb^{-1}(s) \leq b^{-1}(C's) \qquad \textit{for } s > 0, \tag{2.31}$$

and a positive constant $C'' = C''(i_a, C) > 0$ such that

$$b^{-1}(Cs) \leq C''b^{-1}(s) \qquad \textit{for } s > 0. \tag{2.32}$$

(iv) There exists a positive constant $C = C(s_a)$ such that

$$B(t) \leq tb(t) \leq CB(t) \qquad \textit{for } t \geq 0; \tag{2.33}$$

(v) There exists a positive constant $C = C(i_a, s_a)$ such that

$$\widehat{B}^{-1}(s) \leq Cb^{-1}(s) \qquad \textit{for } s > 0; \tag{2.34}$$

(vi) There exists a positive constant $C = C(i_a, s_a)$ such that

$$F(t) \leq tb(t)^2 \leq CF(t) \qquad \textit{for } t \geq 0. \tag{2.35}$$

Proof. (i) Integrating the inequality

$$\frac{i_a}{t} \leq \frac{a'(t)}{a(t)} \leq \frac{s_a}{t},$$

which holds for $t > 0$, yields (2.29).
(ii) Since $s_a > -1$, equation (2.30) holds by the second inequality in (2.29).
(iii) Inequality (2.31) is equivalent to $b(tC) \leq C'b(t)$ for $t \geq 0$, and the latter holds as a consequence of the fact that $B \in \Delta_2$ ([MSZ, Proposition 2.6]). Similarly, (2.32) is equivalent to $C(t) \leq b(C''t)$ for $t \geq 0$, which holds since $\widetilde{B} \in \Delta_2$ ([MSZ, Proposition 2.6]).
(iv) The first inequality in (2.33) holds since b is increasing. The second one is a consequence of (2.19).
(v) Inequality (2.34) is equivalent to $B(t) \geq tb(t/C)$ for $t \geq 0$. Thus, by the second inequality in (2.33), inequality (2.34) will follow if we show that, for every $C \geq 1$ there exists $C' \geq 1$ such that $Cb(t) \leq b(C't)$ for $t \geq 0$. This inequality is a consequence of the fact that $\widetilde{B} \in \Delta_2$ ([MSZ, Proposition 2.6]).
(vi) The first inequality in (2.35) holds since b^2 is an increasing function. On the other hand, integration by parts and (2.18) yield

$$F(t) = \int_0^t b(s)^2\, ds = tb(t)^2 - \int_0^t 2sb(s)b'(s)\, ds \geq tb(t)^2 - 2(1+s_a)\int_0^t b(s)^2\, ds \quad \text{for } t \geq 0.$$

Hence, the second inequality in (2.35) follows with $C = 3 + 2s_a$. □

Proof of Theorem 2.14. (i) By [Ta2, Equations (2.32)–(2.36)], there exists a constant $C = C(n)$ such that

$$\int_\Omega B(|\nabla u|)dx \leq C \int_0^{|\Omega|} \widehat{B}^{-1}\Big(s^{-1/n'}\int_0^s f^*(r)dr\Big)s^{-1/n'}\int_0^s f^*(r)drds. \tag{2.36}$$

Hence, owing to (2.34), there exists a constant $C = C(i_a, s_a, n)$ such that

$$\int_\Omega B(|\nabla u|)dx \leq C \int_0^{|\Omega|} \Big(s^{-1/n'}\int_0^s f^*(r)dr\Big)b^{-1}\Big(s^{-1/n'}\int_0^s f^*(r)dr\Big)ds. \tag{2.37}$$

Since

$$s^{-1/n'}\int_0^s f^*(r)dr \leq \int_0^s f^*(r)r^{-1/n'}dr \leq \|f\|_{L^{n,1}(\Omega)} \quad \text{for } s \in (0, |\Omega|),$$

inequality (2.25) follows.
(ii) The proof in analogous, save that inequality (2.36) holds with $C = C(\Omega)$ – see [Ci1, Equations (3.12)–(3.14)]. □

3 Preliminary results

Here, we enucleate some miscellaneous technical facts needed in the proofs of our main results.

We begin with an isoperimetric inequality, which tells us that if Ω is an open subset of $\mathbb{R}^n$, $n \geq 2$, with a Lipschitz boundary, then there exists a constant C such that

$$|E|^{1/n'} \leq C\mathcal{H}^{n-1}(\Omega \cap \partial E) \tag{3.1}$$

for every open set $E \subset \Omega$ with $\partial E \cap \Omega \in C^1$ such that $|E| \leq |\Omega|/2$. Here, $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. Inequality (3.1) can be deduced from the classical isoperimetric inequality in $\mathbb{R}^n$ and from the inequality

$$\mathcal{H}^{n-1}(\partial\Omega \cap \overline{E}) \leq C\mathcal{H}^{n-1}(\Omega \cap \partial E) \tag{3.2}$$

which holds for some constant C and for every open set $E \subset \Omega$ with $\partial E \cap \Omega \in C^1$ such that $|E| \leq |\Omega|/2$. Inequality (3.2) in turn follows via a trace inequality for functions of bounded variation (see e.g. [Ma4, Chapter 6]).

The following differential inequality involving level sets of Sobolev function relies upon the coarea formula and the relative isoperimetric inequality (3.1), and can be found in [Ma3].

Lemma 3.1 *Let Ω be an open bounded set in $\mathbb{R}^n$, $n \geq 2$, with a Lipschitz boundary. Let v be a nonnegative function from $W^{1,2}(\Omega)$. Then there exists a constant $C = C(\Omega)$ such that*

$$1 \leq C(-\mu_v'(t))^{1/2}\mu_v(t)^{-1/n'}\Big(-\frac{d}{dt}\int_{\{v>t\}}|\nabla v|^2dx\Big)^{1/2} \text{for a.e. } t \geq v^*(|\Omega|/2). \tag{3.3}$$

The next lemma provides us with a pointwise bound for the product between the left-hand side of the equation in (1.1) and (1.2) and Δu. The main point of this bound is that it only involves terms in divergence form and a signed term.

Lemma 3.2 *Assume that $a \in C^1(0,\infty)$ and that the first inequality in (1.3) holds. Let Ω be an open set in $\mathbb{R}^n$, $n \geq 2$, and let $u \in C^3(\Omega)$. Then*

$$\begin{aligned}\Delta u \operatorname{div}(a(|\nabla u|)\nabla u) \geq{}& \operatorname{div}(\Delta u\, a(|\nabla u|)\nabla u) \\ &- \sum_{i,j}^{n}\big(u_{x_ix_j}a(|\nabla u|)u_{x_i}\big)_{x_j} + (1+\min\{i_a,0\})a(|\nabla u|)|\nabla^2 u|^2\end{aligned} \tag{3.4}$$

in $\{\nabla u \neq 0\}$.

Proof. In $\{\nabla u \neq 0\}$, we have that

$$\begin{aligned}\Delta u \operatorname{div}(a(|\nabla u|)\nabla u) &= \operatorname{div}(\Delta u\, a(|\nabla u|)\nabla u) - \sum_{i,j} u_{x_ix_jx_j}a(|\nabla u|)u_{x_i} \\ &= \operatorname{div}(\Delta u\, a(|\nabla u|)\nabla u) - \sum_{i,j}\big(u_{x_ix_j}a(|\nabla u|)u_{x_i}\big)_{x_j} \\ &\quad + \sum_{i,j}\big(u_{x_ix_j}\big)^2 a(|\nabla u|) + \sum_{i,j} u_{x_ix_j}a(|\nabla u|)_{x_j}u_{x_i}\,.\end{aligned} \tag{3.5}$$

Now,

$$\begin{aligned}&\sum_{i,j}\big(u_{x_ix_j}\big)^2 a(|\nabla u|) + \sum_{i,j} u_{x_ix_j}a(|\nabla u|)_{x_j}u_{x_i} \\ &\quad= \sum_{i,j}\big(u_{x_ix_j}\big)^2 a(|\nabla u|) + \sum_{i,j,k} u_{x_ix_j}a'(|\nabla u|)\frac{u_{x_k}}{|\nabla u|}u_{x_kx_j}u_{x_i} \\ &\quad= a(|\nabla u|)\Big(\sum_{i,j}(u_{x_ix_j})^2 + \sum_{i,j,k}\frac{a'(|\nabla u|)|\nabla u|}{a(|\nabla u|)}\frac{u_{x_k}}{|\nabla u|}\frac{u_{x_i}}{|\nabla u|}u_{x_ix_j}u_{x_kx_j}\Big).\end{aligned} \tag{3.6}$$

On setting

$$U^j = (u_{x_1x_j}, \cdots, u_{x_nx_j})$$

and

$$\omega = \frac{\nabla u}{|\nabla u|},$$

and making use of the first inequality in (1.3), one obtains that

$$\begin{aligned}
(3.7)\qquad & a(|\nabla u|)\Big(\sum_{i,j}(u_{x_ix_j})^2 + \sum_{i,j,k}\frac{a'(|\nabla u|)|\nabla u|}{a(|\nabla u|)}\frac{u_{x_k}}{|\nabla u|}\frac{u_{x_i}}{|\nabla u|}u_{x_ix_j}u_{x_kx_j}\Big)\\
&= a(|\nabla u|)\sum_j\left(|U^j|^2 + \frac{a'(|\nabla u|)|\nabla u|}{a(|\nabla u|)}(U^j\cdot\omega)^2\right)\\
&\geq a(|\nabla u|)\sum_j\left(|U^j|^2 + i_a(U^j\cdot\omega)^2\right)\\
&\geq a(|\nabla u|)(1+\min\{i_a,0\})\sum_j|U^j|^2.
\end{aligned}$$

Inequality (3.4) follows from (3.5)-(3.7). □

In the following lemma, any function a as in the statements of Theorems 1.1–1.4 is approximated by a family $\{a_\varepsilon\}$ of smooth, strictly positive functions in $[0,\infty)$ whose indices i_{a_ε} and s_{a_ε} are estimated in terms of i_a and s_a, respectively.

Lemma 3.3 *Assume that the function $a : (0,\infty) \to (0,\infty)$ belongs to $C^1(0,\infty)$ and fulfills (1.3). Then there exists a family of functions $\{a_\varepsilon\}_{\varepsilon\in(0,1)}$ such that:*

$$(3.8)\qquad a_\varepsilon : [0,\infty) \to (0,\infty);$$

$$(3.9)\qquad a_\varepsilon \in C^\infty([0,\infty));$$

$$(3.10)\qquad \min\{i_a,0\} \leq i_{a_\varepsilon} \leq s_{a_\varepsilon} \leq \max\{s_a,0\};$$

$$(3.11)\qquad \lim_{\varepsilon\to 0} b_\varepsilon = b \qquad \textit{uniformly in } [0,M] \textit{ for every } M>0,$$

and hence

$$(3.12)\qquad \lim_{\varepsilon\to 0} B_\varepsilon = B \qquad \textit{uniformly in } [0,M] \textit{ for every } M>0.$$

Here, b_ε and B_ε are defined as in (1.7) and (1.6), respectively, with a replaced by a_ε. Moreover,

$$(3.13)\qquad \lim_{\varepsilon\to 0} a_\varepsilon(|\xi|)\xi = a(|\xi|)\xi \qquad \textit{uniformly in } \{\xi\in\mathbb{R}^n : |\xi|\leq M\} \textit{ for every } M>0.$$

Proof. Let $A : \mathbb{R} \to [0,\infty)$ be the function given by

$$A(s) = a(e^s) \qquad \text{for } s\in\mathbb{R}.$$

Note that

$$(3.14)\qquad i_aA(s) \leq A'(s) \leq s_aA(s) \qquad \text{for } s\in\mathbb{R}.$$

For $\varepsilon \in (0,1)$, let $A_\varepsilon : \mathbb{R} \to [0,\infty)$ denote the convolution of A with a nonnegative smooth kernel ρ_ε such that $\int_{\mathbb{R}} \rho_\varepsilon(t)dt = 1$ and $\operatorname{supp} k_\varepsilon \subset (-\varepsilon, \varepsilon)$. From (3.14) one easily deduces that

$$i_a A_\varepsilon(s) \le A'_\varepsilon(s) \le s_a A_\varepsilon(s) \qquad \text{for } s \in \mathbb{R}. \tag{3.15}$$

Next, define $\widehat{a}_\varepsilon : (0,\infty) \to [0,\infty)$ as

$$\widehat{a}_\varepsilon(t) = A_\varepsilon(\log t) \qquad \text{for } t > 0.$$

By (3.15),

$$i_a \le \inf_{t>0} \frac{t\,\widehat{a}'_\varepsilon(t)}{\widehat{a}_\varepsilon(t)} \le \sup_{t>0} \frac{t\,\widehat{a}'_\varepsilon(t)}{\widehat{a}_\varepsilon(t)} \le s_a\,. \tag{3.16}$$

Moreover, since $\lim_{\varepsilon\to 0} A_\varepsilon = A$ uniformly on compact subsets of $\mathbb{R}$, we have that

$$\lim_{\varepsilon\to 0} \widehat{a}_\varepsilon = \widehat{a} \quad \text{uniformly in } [L,M] \text{ for every } M > L > 0. \tag{3.17}$$

Now, define $a_\varepsilon : [0,\infty) \to (0,\infty)$ as

$$a_\varepsilon(t) = \widehat{a}_\varepsilon\big(\sqrt{\varepsilon + t^2}\big) \qquad \text{for } t \ge 0.$$

Equation (3.8) holds since $a_\varepsilon(t) > 0$ for every $t \ge 0$: indeed, $a(t) > 0$ for $t > 0$, and hence $\widehat{a}_\varepsilon(t) > 0$ for $t > 0$ as well.
Equation (3.9) is fulfilled by a standard property of convolutions.
By (3.16), we have that

$$\frac{t\,a'_\varepsilon(t)}{a_\varepsilon(t)} = \frac{\sqrt{\varepsilon+t^2}\,\widehat{a}'_\varepsilon(\sqrt{\varepsilon+t^2})}{\widehat{a}_\varepsilon(\sqrt{\varepsilon+t^2})}\,\frac{t^2}{\varepsilon+t^2} \ge i_a \frac{t^2}{\varepsilon+t^2} \ge \min\{i_a, 0\} \quad \text{for } t \ge 0. \tag{3.18}$$

Similarly,

$$\frac{t\,a'_\varepsilon(t)}{a_\varepsilon(t)} \le \max\{s_a, 0\} \quad \text{for } t \ge 0. \tag{3.19}$$

Equation (3.10) follows from (3.18) and (3.19).
As far as (3.11) is concerned, from (3.17) we deduce that

$$\lim_{\varepsilon\to 0} b_\varepsilon = b \quad \text{uniformly in } [L,M] \text{ for every } M > L > 0. \tag{3.20}$$

On the other hand, by (2.29) with a replaced with a_ε and by (3.19),

$$0 \le b_\varepsilon(t) = t a_\varepsilon(t) \le a_\varepsilon(1) t^{1+\max\{s_a,0\}} \qquad \text{for } t \ge 0. \tag{3.21}$$

Hence,

$$\lim_{t\to 0} b_\varepsilon(t) = 0 \qquad \text{uniformly for } \varepsilon \in (0,1). \tag{3.22}$$

Combining (3.20), (3.22) and (2.30) yields (3.11).
The proof of (3.13) is analogous. □

In the next proposition we are concerned with a property of the so-called pseudo-rearrangement of a function with respect to another. Variants of this property are known in the literature [AT, FV]. We present a short proof for completeness.

Proposition 3.4 *Let Ω be a measurable set in $\mathbb{R}^n$, $n \geq 1$. Let $w : \Omega \to [0, \infty)$ be a measurable function and let $g \in L^1(\Omega)$. Let $\varphi : (0, |\Omega|) \to [0, \infty)$ be the function defined by*

$$\varphi(s) = \frac{d}{ds} \int_{\{w > w^*(s)\}} |g(x)| dx \qquad \text{for a.e. } s \in (0, |\Omega|). \tag{3.23}$$

Then

$$\int_0^s \varphi^*(r) dr \leq \int_0^s g^*(r) dr \qquad \text{for } s \in (0, |\Omega|). \tag{3.24}$$

Proof. Owing to (2.2),

$$\int_0^s \varphi^*(r) dr = \sup_{|E|=s} \int_E \varphi(r) dr. \tag{3.25}$$

Thus, it suffices to show that

$$\int_E \varphi(r) dr \leq \int_0^{|E|} g^*(r) dr \qquad \text{for every measurable set } E \subset (0, |\Omega|). \tag{3.26}$$

In turn, it is easily verified that we may limit ourselves to proving (3.26) in the case when E is an open set. Thus, we may assume that $E = \cup_{k \in \mathcal{K}} (r_k, s_k)$, where $\mathcal{K} \subset \mathbb{N}$ and the intervals (r_k, s_k) are pairwise disjoint. One has that

$$\begin{aligned} \int_E \varphi(r) dr &= \sum_{k \in \mathcal{K}} \int_{r_k}^{s_k} \varphi(r) dr = \sum_{k \in \mathcal{K}} \int_{r_k}^{s_k} \left(\frac{d}{dr} \int_{\{w > w^*(r)\}} |g(x)| dx \right) dr \\ &\leq \sum_{k \in \mathcal{K}} \int_{\{w^*(s_k) < w < w^*(r_k)\}} |g(x)| dx = \int_{\cup_{k \in \mathcal{K}} \{w^*(s_k) < w < w^*(r_k)\}} |g(x)| dx. \end{aligned} \tag{3.27}$$

Note that integration in the last two integrals is extended just over $\{w^*(s_k) < w < w^*(r_k)\}$, instead of $\{w^*(s_k) < w \leq w^*(r_k)\}$, since $\frac{d}{dr} \int_{\{w > w^*(r)\}} |g(x)| dx = 0$ in any interval where w^* is constant. Since $|\{w^*(s_k) < w < w^*(r_k)\}| \leq (s_k - r_k)$, one has that

$$| \cup_{k \in \mathcal{K}} \{w^*(s_k) < w < w^*(r_k)\}| \leq \sum_k (s_k - r_k) = |E|.$$

Thus, by the Hardy-Littlewood inequality (2.2),

$$\int_{\cup_{k \in \mathcal{K}} \{w^*(s_k) < w < w^*(r_k)\}} |g(x)| dx \leq \int_0^{|E|} g^*(r) dr. \tag{3.28}$$

Inequality (3.26) is a consequence of (3.27) and (3.28). □

The next result amounts to a weighted integral inequality between one-dimensional functions via an integral estimate between the squares of their rearrangements.

Lemma 3.5 *Let $L \in (0,\infty]$ and let $\varphi, \psi : (0,L) \to [0,\infty)$ be measurable functions such that*

$$\int_0^s \varphi^*(r)^2 dr \le \int_0^s \psi^*(r)^2 dr \qquad \text{for } s \in (0,L). \tag{3.29}$$

Then, for every $\gamma > \frac{1}{2}$, there exists a constant $C = C(\gamma)$ such that

$$\int_0^L \varphi(s) s^{-\gamma} ds \le C(\gamma) \int_0^L \psi^*(s) s^{-\gamma} ds. \tag{3.30}$$

Proof. We have that

$$\begin{aligned} \int_0^L \varphi(s) s^{-\gamma} ds &\le \int_0^L \varphi^*(s) s^{-\gamma} ds = \int_0^L \big(\varphi^*(s)^2\big)^{1/2} s^{-\gamma} ds \\ &\le \int_0^L \left(\frac{1}{s}\int_0^s \varphi^*(r)^2 dr\right)^{1/2} s^{-\gamma} ds \le \int_0^L \left(\frac{1}{s}\int_0^s \psi^*(r)^2 dr\right)^{1/2} s^{-\gamma} ds, \end{aligned} \tag{3.31}$$

where the first inequality is a consequence of the Hardy-Littlewood inequality (2.2), the second inequality holds since φ^* is non increasing, and the last inequality relies upon (3.29). A Hardy-type inequality for non-increasing functions (see e.g. [CPSS, Theorem 4.1]) tells us that the rightmost side of (3.31) does not exceed the right-hand side of (3.30) for a suitable constant $C = C(\gamma)$. Inequality (3.30) follows. $\square$

We conclude this section with a special case of a Hardy type inequality for monotone functions.

Lemma 3.6 *Let $L \in (0,\infty]$ and let $\gamma \in (\frac{1}{2}, 1)$. Then there exists a constant $C = C(\gamma)$ such that*

$$\left(\int_0^L s^{-2\gamma} \int_0^s \varphi(r)^2 dr ds\right)^{1/2} \le C(\gamma) \int_0^L s^{-\gamma} \varphi(s) ds \tag{3.32}$$

for every non-increasing function $\varphi : (0,L) \to [0,\infty)$.

Proof. By Fubini's theorem,

$$\int_0^L s^{-2\gamma} \int_0^s \varphi(r)^2 dr ds = \int_0^L \varphi(r)^2 \int_r^L s^{-2\gamma} ds dr \le \frac{1}{2\gamma - 1} \int_0^L \varphi(r)^2 r^{1-2\gamma} dr. \tag{3.33}$$

By a Hölder type inequality for non-increasing functions (see e.g. [CPSS, Theorem 3.1]), the square root of the last integral in (3.33) does not exceed a constant depending on γ, times the integral on the right-hand side of (3.32). Hence (3.32) follows. $\square$

4 Proof of the main results

We are now in a position to prove Theorems 1.1–1.4.

Proof of Theorem 1.1. We split the proof in steps. The core of the argument is contained in Step 1, where the statement is proved under additional regularity assumptions on Ω, a and f. These assumptions are removed via approximation in the remaining three steps.

Step 1. Here we prove the statement under the additional assumptions that:

$$\partial\Omega \in C^\infty, \tag{4.1}$$

$$a \in C^\infty([0,\infty)), \tag{4.2}$$

$$a(0) > 0, \tag{4.3}$$

and

$$f \in C_0^\infty(\Omega). \tag{4.4}$$

Note that, owing to (2.29) and (4.3), for every $M > 0$ there exist positive constants $C_1 = C_1(a, M)$ and $C_2 = C_2(a, M)$ such that

$$C_1 \leq a(t) \leq C_2 \qquad \text{for } t \in [0, M]. \tag{4.5}$$

By [Li3, Theorem 1.7 and subsequent remarks], there exist constants $\alpha = \alpha(i_a, s_a, n, \Omega) \in (0,1]$ and $C = C(i_a, s_a, n, \Omega, \|f\|_{L^\infty(\Omega)}) > 0$ such that $u \in C^{1,\alpha}(\overline{\Omega})$ and

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leq C. \tag{4.6}$$

We claim that, in fact,

$$u \in C^3(\overline{\Omega}). \tag{4.7}$$

To verify this assertion, define the function $V : \mathbb{R}^n \to \mathbb{R}$ and, for $i = 1, \dots, n$, the functions $A^i : \mathbb{R}^n \to \mathbb{R}$ as

$$V(\xi) = \sqrt{1 + |\xi|^2} \qquad \text{for } \xi \in \mathbb{R}^n,$$

and

$$A^i(\xi) = a(|\xi|)\xi_i \qquad \text{for } \xi \in \mathbb{R}^n.$$

Observe that, owing to (1.3) and (4.5), for every $M > 0$ there exist constants C and λ such that

$$\sum_i |A^i(\xi)| + V(\xi) \sum_{i,j} |A^i_{\xi_j}(\xi)| \leq C V(\xi) \qquad \text{if } |\xi| \leq M, \tag{4.8}$$

$$\sum_i A^i(\xi)\xi_i \geq \lambda V(\xi)^2 - C \qquad \text{if } |\xi| \leq M, \tag{4.9}$$

and

$$\sum_{i,j} A^i_{\xi_j}(\xi)\eta_i\eta_j \geq \lambda|\eta|^2 \qquad \text{if } |\xi| \leq M \text{ and } \eta \in \mathbb{R}^n. \tag{4.10}$$

Classical regularity results ensure that, if (4.8)–(4.10) are fulfilled for every $\xi, \eta \in \mathbb{R}^n$, weak solutions to the equation in (1.1) belong to $W^{2,2}(\Omega)$ see e.g. [Gi, Section 8.2] or [LU2, Chapter 4, Section 5]. An inspection of the proof of these results reveals that (4.8)–(4.10) are in fact applied with $\xi = \nabla u$. Thus, by (4.6), the same proof tells us that $u \in W^{2,2}(\Omega)$ under the assumption that (4.8)–(4.10) just hold for $|\xi| \leq M$ and $\eta \in \mathbb{R}^n$.
Since, by (4.2), $A^i \in C^3(\mathbb{R}^n)$ and we already know that $u \in C^{1,\alpha}(\overline{\Omega}) \cap W^{2,2}(\Omega)$ for some $\alpha \in (0,1]$, a standard iteration argument leads to (4.7) (see e.g. [LU2, Chapter 4, Section 6]).

The level set $\{|\nabla u| > t\}$ is open for $t > 0$. Moreover, for a.e. $t > 0$, $\partial\{|\nabla u| > t\}$ is an $(n-1)$-dimensional manifold of class C^1 outside a set of $\mathcal{H}^{n-1}$ measure zero, and

$$\partial\{|\nabla u| > t\} = \{|\nabla u| = t\} \cup \big(\partial\Omega \cap \overline{\{|\nabla u| > t\}}\big).$$

By inequality (3.4) and the divergence theorem we have that

$$
\begin{aligned}
&\int_{\{|\nabla u|>t\}} \Delta u f dx \\
&= \int_{\{|\nabla u|>t\}} \Delta u \operatorname{div}(a(|\nabla u|)\nabla u) dx \geq \int_{\{|\nabla u|>t\}} \operatorname{div}(\Delta u\, a(|\nabla u|)\nabla u) dx \\
&- \int_{\{|\nabla u|>t\}} \sum_j \big(\sum_i u_{x_i x_j} a(|\nabla u|) u_{x_i}\big)_{x_j} dx + (1+\min\{i_a,0\}) \int_{\{|\nabla u|>t\}} a(|\nabla u|)|\nabla^2 u|^2 dx \\
&= \int_{\partial\{|\nabla u|>t\}} \Delta u\, a(|\nabla u|) \frac{\partial u}{\partial \nu} d\mathcal{H}^{n-1}(x) - \int_{\partial\{|\nabla u|>t\}} \sum_{i,j} u_{x_i x_j} a(|\nabla u|) u_{x_i} \nu_j d\mathcal{H}^{n-1}(x) \\
&+ (1+\min\{i_a,0\}) \int_{\{|\nabla u|>t\}} a(|\nabla u|)|\nabla^2 u|^2 dx \qquad \text{for a.e. } t>0.
\end{aligned} \tag{4.11}
$$

Here, ν_j denotes the $j-th$ component of the normal vector ν to $\partial\{|\nabla u|>t\}$. Now, observe that $\nu = -\frac{\nabla|\nabla u|}{|\nabla|\nabla u||}$ on $\Omega \cap \{|\nabla u| = t\}$ for a.e. $t>0$. Moreover,

$$
\sum_i u_{x_i x_j} u_{x_i} = |\nabla u|_{x_j} |\nabla u|.
$$

Thus,

$$
\begin{aligned}
&\int_{\partial\{|\nabla u|>t\}} \Delta u\, a(|\nabla u|) \frac{\partial u}{\partial \nu} d\mathcal{H}^{n-1}(x) - \int_{\partial\{|\nabla u|>t\}} \sum_{ij} u_{x_i x_j} a(|\nabla u|) u_{x_i} \nu_j d\mathcal{H}^{n-1}(x) \\
&\qquad + (1+\min\{i_a,0\}) \int_{\{|\nabla u|>t\}} a(|\nabla u|)|\nabla^2 u|^2 dx \\
&= a(t) \int_{\{|\nabla u|=t\}} \Delta u \frac{\partial u}{\partial \nu} d\mathcal{H}^{n-1} + a(t) t \int_{\{|\nabla u|=t\}} |\nabla|\nabla u||\, d\mathcal{H}^{n-1}(x) \\
&+ \int_{\partial\Omega \cap \overline{\{|\nabla u|>t\}}} a(|\nabla u|) \Big(\Delta u \frac{\partial u}{\partial \nu} - \sum_{ij} u_{x_i x_j} u_{x_i} \nu_j\Big) d\mathcal{H}^{n-1}(x) \\
&+ (1+\min\{i_a,0\}) \int_{\{|\nabla u|>t\}} a(|\nabla u|)|\nabla^2 u|^2 dx \qquad \text{for a.e. } t>0.
\end{aligned} \tag{4.12}
$$

Let us focus on the integrals on the right-hand side of (4.12). Since

$$
f = \operatorname{div}(a(|\nabla u|)\nabla u) = a(|\nabla u|)\Delta u + a'(|\nabla u|) \nabla u \cdot \nabla |\nabla u|\,, \tag{4.13}
$$

one has that

(4.14)

$$
\begin{aligned}
a(t) \int_{\{|\nabla u|=t\}} \Delta u \frac{\partial u}{\partial \nu} d\mathcal{H}^{n-1} &= \int_{\{|\nabla u|=t\}} f \frac{\partial u}{\partial \nu} d\mathcal{H}^{n-1}(x) - a'(t) \int_{\{|\nabla u|=t\}} \nabla u \cdot \nabla|\nabla u| \frac{\partial u}{\partial \nu} d\mathcal{H}^{n-1}(x) \\
&= \int_{\{|\nabla u|=t\}} f \frac{\partial u}{\partial \nu} d\mathcal{H}^{n-1}(x) + a'(t) \int_{\{|\nabla u|=t\}} \frac{(\nabla u \cdot \nabla|\nabla u|)^2}{|\nabla|\nabla u||} d\mathcal{H}^{n-1}(x) \\
&= \int_{\{|\nabla u|=t\}} f \frac{\partial u}{\partial \nu} d\mathcal{H}^{n-1}(x) + a'(t) \int_{\{|\nabla u|=t\}} |\nabla|\nabla u|| \Big(\frac{\partial u}{\partial \nu}\Big)^2 d\mathcal{H}^{n-1}(x) \qquad \text{for a.e. } t>0.
\end{aligned}
$$

Next recall that

(4.15)

$$\Delta u \frac{\partial u}{\partial \nu} - \sum_{ij} u_{x_i x_j} u_{x_i} \nu_j$$
$$= \mathrm{div}_T\Big(\frac{\partial u}{\partial \nu}\nabla_T u\Big) - (\mathrm{tr}\mathcal{B})\Big(\frac{\partial u}{\partial \nu}\Big)^2 - \mathcal{B}(\nabla_T u, \nabla_T u) - 2\nabla_T u \cdot \nabla_T \frac{\partial u}{\partial \nu} \qquad \text{on } \partial\Omega \ ,$$

where $\mathcal{B}$ denotes the second fundamental form on $\partial\Omega$, $\mathrm{tr}\mathcal{B}$ is its trace, and div_T and ∇_T denote the divergence operator and the gradient operator on $\partial\Omega$, respectively [Gr, Equation (3,1,1,2)]. Hence, owing to the boundary condition in (1.1),

$$(4.16) \qquad \Delta u \frac{\partial u}{\partial \nu} - \sum_{ij} u_{x_i x_j} u_{x_i} \nu_j = -\mathcal{B}(\nabla_T u, \nabla_T u) \qquad \text{on } \partial\Omega \, .$$

Since $\partial\Omega \in W^2 L^{n-1,1}$, there exists a nonnegative function $k \in L^{n-1,1}(\partial\Omega)$, which is pointwise estimated, up to a multiplicative constant depending on $\partial\Omega$, by the second-order derivatives of the $(n-1)$-dimensional functions which locally represent $\partial\Omega$, such that

$$(4.17) \qquad \mathcal{B}(\nabla_T u, \nabla_T u) \leq k(x)|\nabla_T u|^2 \qquad \text{on } \partial\Omega \, .$$

Thus,

$$(4.18) \quad \int_{\partial\Omega\cap\overline{\{|\nabla u|>t\}}} a(|\nabla u|)\Big(\Delta u \frac{\partial u}{\partial \nu} - \sum_{i,j} u_{x_i x_j} u_{x_i} \nu_j\Big) d\mathcal{H}^{n-1}(x)$$
$$\geq -\int_{\partial\Omega\cap\overline{\{|\nabla u|>t\}}} a(|\nabla u|)|\nabla u|^2 k(x)\, d\mathcal{H}^{n-1}(x) \qquad \text{for a.e. } t > 0.$$

Furthermore, for every $\varepsilon > 0$,

(4.19)

$$\int_{\{|\nabla u|>t\}} f \Delta u dx \leq \varepsilon \int_{\{|\nabla u|>t\}} a(|\nabla u|)(\Delta u)^2 dx + \frac{1}{\varepsilon}\int_{\{|\nabla u|>t\}} f^2 \frac{1}{a(|\nabla u|)} dx$$
$$\leq \varepsilon \int_{\{|\nabla u|>t\}} a(|\nabla u|)|\nabla^2 u|^2 dx + \frac{1}{\varepsilon}\frac{\|\nabla u\|_{L^\infty(\Omega)}}{b(t)}\int_{\{|\nabla u|>t\}} f^2 dx \qquad \text{for a.e. } t > 0.$$

Note that we have exploited the fact that the function b is increasing.
Combining (4.11), (4.12), (4.14), (4.18) and (4.19) yields

(4.20)

$$\int_{\{|\nabla u|=t\}} |\nabla|\nabla u||\bigg(a(t)t + a'(t)\Big(\frac{\partial u}{\partial \nu}\Big)^2\bigg) d\mathcal{H}^{n-1}(x) + (1+\min\{i_a,0\}-\varepsilon)\int_{\{|\nabla u|>t\}} a(|\nabla u|)|\nabla^2 u|^2 dx$$
$$\leq t\int_{\{|\nabla u|=t\}} |f| d\mathcal{H}^{n-1}(x) + \frac{1}{\varepsilon}\frac{\|\nabla u\|_{L^\infty(\Omega)}}{b(t)}\int_{\{|\nabla u|>t\}} f^2 dx + \int_{\partial\Omega\cap\overline{\{|\nabla u|>t\}}} a(|\nabla u|)|\nabla u|^2 k(x)\, d\mathcal{H}^{n-1}(x)$$

for a.e. $t > 0$. Set $a'_-(t) = \min\{0, a'(t)\}$, and note that

$$(4.21) \qquad a(t)t + a'(t)\Big(\frac{\partial u}{\partial \nu}\Big)^2 \geq a(t)t + a'_-(t)t^2 \geq (1+\min\{i_a,0\})b(t) \qquad \text{on } \{|\nabla u| = t\}.$$

Choose

$$\varepsilon = \frac{1+\min\{i_a, 0\}}{2}.$$

From (4.20) and (4.21) we deduce that

$$
\begin{aligned}
(4.22)\qquad 2\varepsilon b(t)\int_{\{|\nabla u|=t\}} |\nabla|\nabla u||\, d\mathcal{H}^{n-1}(x) + \varepsilon \int_{\{|\nabla u|>t\}} a(|\nabla u|)|\nabla^2 u|^2 dx& \\
\leq t\int_{\{|\nabla u|=t\}} |f| d\mathcal{H}^{n-1}(x) + \frac{1}{\varepsilon}\frac{\|\nabla u\|_{L^\infty(\Omega)}}{b(t)}\int_{\{|\nabla u|>t\}} f^2 dx& \\
+\int_{\partial\Omega\cap\{|\nabla u|>t\}} a(|\nabla u|)|\nabla u|^2 k(x)\, d\mathcal{H}^{n-1}(x) \qquad \text{for a.e. } t>0.&
\end{aligned}
$$

Now, owing to the Hardy-Littlewood inequality (2.2),

$$
\begin{aligned}
(4.23)\quad \int_{\partial\Omega\cap\overline{\{|\nabla u|>t\}}} & a(|\nabla u|)|\nabla u|^2 k(x)\, d\mathcal{H}^{n-1}(x) \\
&\leq a\big(\|\nabla u\|_{L^\infty(\Omega)}\big)\|\nabla u\|^2_{L^\infty(\Omega)}\int_{\partial\Omega\cap\overline{\{|\nabla u|>t\}}} k(x)\, d\mathcal{H}^{n-1}(x) \\
&\leq a\big(\|\nabla u\|_{L^\infty(\Omega)}\big)\|\nabla u\|^2_{L^\infty(\Omega)}\int_0^{\mathcal{H}^{n-1}(\partial\Omega\cap\overline{\{|\nabla u|>t\}})} k^*(r)dr \qquad \text{for a.e. } t>0.
\end{aligned}
$$

By (3.2), there exists a constant $C = C(\Omega)$ such that

$$(4.24)\qquad \mathcal{H}^{n-1}(\partial\Omega\cap\overline{\{|\nabla u|>t\}}) \leq C\mathcal{H}^{n-1}(\{|\nabla u|=t\}) \qquad \text{for a.e. } t\geq |\nabla u|^*(|\Omega|/2).$$

Denote simply by $\mu : [0,\infty) \to [0, |\Omega|]$ the distribution function $\mu_{|\nabla u|}$ of $|\nabla u|$ defined as in (2.1). By (3.1), there exists a constant $C = C(\Omega)$ such that

$$(4.25)\qquad \mu(t)^{1/n'} \leq C\mathcal{H}^{n-1}(\{|\nabla u|=t\}) \qquad \text{for a.e. } t\geq |\nabla u|^*(|\Omega|/2).$$

From (4.24) and (4.25), we obtain that

$$
\begin{aligned}
(4.26)\qquad \int_0^{\mathcal{H}^{n-1}(\partial\Omega\cap\overline{\{|\nabla u|>t\}})} k^*(r)dr &\leq \int_0^{C\mathcal{H}^{n-1}(\{|\nabla u|=t\})} k^*(r)dr \\
&= C\mathcal{H}^{n-1}(\{|\nabla u|=t\})k^{**}(C\mathcal{H}^{n-1}(\{|\nabla u|=t\})) \\
&\leq C\mathcal{H}^{n-1}(\{|\nabla u|=t\})k^{**}\big(C'\mu(t)^{1/n'}\big) \\
&\qquad \text{for a.e. } t\geq |\nabla u|^*(|\Omega|/2),
\end{aligned}
$$

for some constants $C = C(\Omega)$ and $C' = C'(\Omega)$. Combining (4.22), (4.23) and (4.26) yields

(4.27)

$$
\begin{aligned}
2\varepsilon b(t)\int_{\{|\nabla u|=t\}} |\nabla|\nabla u||\, d\mathcal{H}^{n-1}(x) &\leq t\int_{\{|\nabla u|=t\}} |f| d\mathcal{H}^{n-1}(x) + \frac{1}{\varepsilon}\frac{\|\nabla u\|_{L^\infty(\Omega)}}{b(t)}\int_{\{|\nabla u|>t\}} f^2 dx \\
&+ Ca\big(\|\nabla u\|_{L^\infty(\Omega)}\big)\|\nabla u\|^2_{L^\infty(\Omega)}\mathcal{H}^{n-1}(\{|\nabla u|=t\})k^{**}\big(C'\mu(t)^{1/n'}\big) \\
&\qquad \text{for a.e. } t\geq |\nabla u|^*(|\Omega|/2).
\end{aligned}
$$

By the coarea formula, for every Borel function $g : \Omega \to [0,\infty)$,

$$(4.28)\qquad \int_{\{|\nabla u|=t\}} g d\mathcal{H}^{n-1}(x) = -\frac{d}{dt}\int_{\{|\nabla u|>t\}} |\nabla|\nabla u|| g dx \qquad \text{for a.e. } t>0.$$

Thus,

$$
\begin{aligned}
&\int_{\{|\nabla u|=t\}} |f| d\mathcal{H}^{n-1}(x) \leq \Big(\int_{\{|\nabla u|=t\}} \frac{|f|}{|\nabla|\nabla u||} d\mathcal{H}^{n-1}(x)\Big)^{1/2} \Big(\int_{\{|\nabla u|=t\}} |\nabla|\nabla u|| d\mathcal{H}^{n-1}(x)\Big)^{1/2} \\
&\qquad = \Big(-\frac{d}{dt}\int_{\{|\nabla u|>t\}} f^2 dx\Big)^{1/2} \Big(-\frac{d}{dt}\int_{\{|\nabla u|>t\}} |\nabla|\nabla u||^2 dx\Big)^{1/2} \qquad \text{for a.e. } t>0.
\end{aligned}
\tag{4.29}
$$

Similarly,

$$
\mathcal{H}^{n-1}(\{|\nabla u|=t\}) \leq (-\mu'(t))^{1/2}\Big(-\frac{d}{dt}\int_{\{|\nabla u|>t\}} |\nabla|\nabla u||^2 dx\Big)^{1/2} \qquad \text{for a.e. } t>0.
\tag{4.30}
$$

By the Hardy-Littlewwod inequality (2.2),

$$
\int_{\{|\nabla u|>t\}} f^2 dx \leq \int_0^{\mu(t)} f^*(r)^2 dr \qquad \text{for } t>0.
\tag{4.31}
$$

Inequalities (4.27), (4.29), (4.30), (4.31), and inequality (3.3) applied with $v = |\nabla u|$ entail that

$$
\begin{aligned}
&2\varepsilon b(t)\Big(-\frac{d}{dt}\int_{\{|\nabla u|>t\}} |\nabla|\nabla u||^2 dx\Big) \\
&\quad \leq t\Big(-\frac{d}{dt}\int_{\{|\nabla u|>t\}} f^2 dx\Big)^{1/2}\Big(-\frac{d}{dt}\int_{\{|\nabla u|>t\}} |\nabla|\nabla u||^2 dx\Big)^{1/2} \\
&\qquad + \frac{C}{\varepsilon}\frac{\|\nabla u\|_{L^\infty(\Omega)}}{b(t)}(-\mu'(t))^{1/2}\mu(t)^{-1/n'}\int_0^{\mu(t)} f^*(r)^2 dr \Big(-\frac{d}{dt}\int_{\{|\nabla u|>t\}} |\nabla|\nabla u||^2 dx\Big)^{1/2} \\
&\qquad + Ca\big(\|\nabla u\|_{L^\infty(\Omega)}\big)\|\nabla u\|^2_{L^\infty(\Omega)}(-\mu'(t))^{1/2}k^{**}\big(C'\mu(t)^{1/n'}\big)\Big(-\frac{d}{dt}\int_{\{|\nabla u|>t\}} |\nabla|\nabla u||^2 dx\Big)^{1/2}
\end{aligned}
\tag{4.32}
$$

for a.e. $t \geq |\nabla u|^*(|\Omega|/2)$, and for some positive constants $C = C(\Omega)$ and $C' = C'(\Omega)$. Dividing through in (4.32) by $-\frac{d}{dt}\int_{\{|\nabla u|>t\}} |\nabla|\nabla u||^2 dx$, and exploiting again (3.3) with $v = |\nabla u|$ yield

$$
\begin{aligned}
2\varepsilon b(t) &\leq Ct(-\mu'(t))^{1/2}\mu(t)^{-1/n'}\Big(-\frac{d}{dt}\int_{\{|\nabla u|>t\}} f^2 dx\Big)^{1/2} \\
&\quad + \frac{C}{\varepsilon}\frac{\|\nabla u\|_{L^\infty(\Omega)}}{b(t)}(-\mu'(t))\mu(t)^{-2/n'}\int_0^{\mu(t)} f^*(r)^2 dr \\
&\quad + Ca\big(\|\nabla u\|_{L^\infty(\Omega)}\big)\|\nabla u\|^2_{L^\infty(\Omega)}(-\mu'(t))\mu(t)^{-1/n'}k^{**}\big(C'\mu(t)^{1/n'}\big) \\
&\qquad\qquad \text{for a.e. } t \in [|\nabla u|^*(|\Omega|/2), \|\nabla u\|_{L^\infty(\Omega)}],
\end{aligned}
\tag{4.33}
$$

for some positive constants $C = C(\Omega)$ and $C' = C'(\Omega)$. Now, define the function $\phi : (0,|\Omega|) \to [0,\infty)$ as

$$
\phi(s) = \Big(\frac{d}{ds}\int_{\{|\nabla u|>|\nabla u|^*(s)\}} f^2 dx\Big)^{1/2} \qquad \text{for a.e. } s \in (0,|\Omega|).
\tag{4.34}
$$

Then

$$(4.35) \qquad \Big(-\frac{d}{dt}\int_{\{|\nabla u|>t\}} f^2 dx\Big)^{1/2} = (-\mu'(t))^{1/2}\phi(\mu(t)) \qquad \text{for a.e. } t>0,$$

and, by Lemma 3.4,

$$(4.36) \qquad \int_0^s \phi^*(r)^2 dr \le \int_0^s f^*(r)^2 dr \qquad \text{for } s\in(0,|\Omega|).$$

Since b is an increasing function, we deduce from inequality (4.33) that

$$\begin{aligned}(4.37)\quad 2\varepsilon b(t)^2 \le{}& Cb\big(\|\nabla u\|_{L^\infty(\Omega)}\big)\|\nabla u\|_{L^\infty(\Omega)}(-\mu'(t))\mu(t)^{-1/n'}\phi(\mu(t))\\ &+\frac{C}{\varepsilon}\|\nabla u\|_{L^\infty(\Omega)}(-\mu'(t))\mu(t)^{-2/n'}\int_0^{\mu(t)} f^*(r)^2dr\\ &+Cb\big(\|\nabla u\|_{L^\infty(\Omega)}\big)^2\|\nabla u\|_{L^\infty(\Omega)}(-\mu'(t))\mu(t)^{-1/n'}k^{**}\big(C'\mu(t)^{1/n'}\big)\\ &\qquad\qquad\qquad \text{for a.e. } t\in[|\nabla u|^*(|\Omega|/2),\|\nabla u\|_{L^\infty(\Omega)}].\end{aligned}$$

Since $|\nabla u|$ is a Sobolev function, the function $|\nabla u|^*$ is (locally absolutely) continuous [CEG, Lemma 6.6], and $|\nabla u|^*(\mu(t))=t$ for $t>0$. Given $t_0\in[|\nabla u|^*(|\Omega|/2),\|\nabla u\|_{L^\infty(\Omega)}]$, integration in (4.37) thus yields

$$\begin{aligned}(4.38)\quad 2\varepsilon F(|\nabla u|^*(s)) \le{}& 2\varepsilon F(t_0)+Cb\big(\|\nabla u\|_{L^\infty(\Omega)}\big)\|\nabla u\|_{L^\infty(\Omega)}\int_s^{\mu(t_0)} r^{-1/n'}\phi(r)dr\\ &+\frac{C}{\varepsilon}\|\nabla u\|_{L^\infty(\Omega)}\int_s^{\mu(t_0)} r^{-2/n'}\int_0^r f^*(\rho)^2d\rho\,dr\\ &+Cb\big(\|\nabla u\|_{L^\infty(\Omega)}\big)^2\|\nabla u\|_{L^\infty(\Omega)}\int_s^{\mu(t_0)} k^{**}\big(C'r^{1/n'}\big)r^{-1/n'}dr\\ \le{}& 2\varepsilon F(t_0)+Cb(\|\nabla u\|_{L^\infty(\Omega)})\|\nabla u\|_{L^\infty(\Omega)}\int_0^{\mu(t_0)} r^{-1/n'}\phi(r)dr\\ &+\frac{C}{\varepsilon}\|\nabla u\|_{L^\infty(\Omega)}\int_0^{\mu(t_0)} r^{-2/n'}\int_0^r f^*(\rho)^2d\rho\,dr\\ &+C''F\big(\|\nabla u\|_{L^\infty(\Omega)}\big)\int_0^{\mu(t_0)^{1/n'}} k^{**}(C'r)r^{\frac{1}{n-1}}\frac{dr}{r} \quad \text{for } s\in[0,\mu(t_0)),\end{aligned}$$

for some constants $C=C(\Omega)$, $C'=C'(\Omega)$ and $C''=C''(\Omega,i_a)$. Note that in the second inequality we have made use of (2.35).
Now, define the function $G:[0,\infty)\to[0,\infty)$ as

$$G(s)=C''\int_0^{s^{1/n'}} k^{**}(C'r)r^{\frac{1}{n-1}}\frac{dr}{r} \qquad \text{for } s\ge 0,$$

and note that G is strictly increasing. Set $s_0=\min\{\frac{|\Omega|}{2},G^{-1}(\varepsilon)\}$, and choose

$$t_0=|\nabla u|^*(s_0).$$

Since $\mu(t_0)\le G^{-1}(\varepsilon)$, we have that

$$C''\int_0^{\mu(t_0)^{1/n'}} k^{**}(C'r)r^{\frac{1}{n-1}}\frac{dr}{r}\le\varepsilon.$$

From (4.38) with $s = 0$ we thus infer that

$$F\big(\|\nabla u\|_{L^\infty(\Omega)}\big) \le CF(t_0) + C\|\nabla u\|_{L^\infty(\Omega)} b(\|\nabla u\|_{L^\infty(\Omega)}) \int_0^{\mu(t_0)} r^{-1/n'}\phi(r)dr \\ + C\|\nabla u\|_{L^\infty(\Omega)} \int_0^{\mu(t_0)} r^{-2/n'} \int_0^r f^*(\rho)^2 d\rho\, dr \tag{4.39}$$

for some constant $C = C(\Omega, i_a)$. By (4.36) and Lemma 3.5 with $\gamma = 1/n'$, there exists a constant $C = C(n)$ such that

$$\int_0^{|\Omega|} r^{-1/n'}\phi(r)dr \le C\|f\|_{L^{n,1}(\Omega)}. \tag{4.40}$$

By Lemma 3.6 with $\gamma = 1/n'$, there exists a constant $C = C(n)$ such that

$$\int_0^{|\Omega|} r^{-2/n'} \int_0^r f^*(\rho)^2 d\rho\, dr \le C\|f\|^2_{L^{n,1}(\Omega)}\,. \tag{4.41}$$

Owing (4.40), (4.41) and (2.35), we obtain from (4.39) that there exists a constant $C = C(\Omega, i_a)$ such that

$$b(\|\nabla u\|_{L^\infty(\Omega)})^2 \le Cb(t_0)^2 + Cb(\|\nabla u\|_{L^\infty(\Omega)})\|f\|_{L^{n,1}(\Omega)} + C\|f\|^2_{L^{n,1}(\Omega)}. \tag{4.42}$$

Hence,

$$b(\|\nabla u\|_{L^\infty(\Omega)}) \le Cb(t_0) + C\|f\|_{L^{n,1}(\Omega)} \tag{4.43}$$

for some constant $C = C(\Omega, i_a)$. Next, let $\beta, \psi : [0, \infty) \to [0, \infty)$ be the functions defined by $\beta(t) = b(t)t$ for $t \ge 0$ and $\psi(s) = sb^{-1}(s)$ for $s \ge 0$. Proposition 2.14 and inequality (2.33) ensure that

$$C\psi\big(\|f\|_{L^{n,1}(\Omega)}\big) \ge \int_\Omega \beta(|\nabla u|)dx \ge \int_{\{|\nabla u| \ge t_0\}} \beta(|\nabla u|)dx \ge \beta(t_0) \lim_{t\to t_0^-} \mu(t) \ge \beta(t_0)s_0, \tag{4.44}$$

for some constant $C = C(\Omega, i_a, s_a)$, whence, by (2.31),

$$\beta(t_0) \le \psi\big(C\|f\|_{L^{n,1}(\Omega)}\big), \tag{4.45}$$

for some constant $C = C(\Omega, i_a, s_a)$. Since $b(\beta^{-1}(\psi(s))) = s$ for $s \ge 0$, inequality (4.45) implies that

$$b(t_0) \le C\|f\|_{L^{n,1}(\Omega)}. \tag{4.46}$$

Inequality (1.8) follows from (4.43), (4.46) and (2.32).
Step 2 Here we remove the additional assumption (4.1), but keep (4.2), (4.3) and (4.4) still in force.
Since smooth functions are dense in $W^2L^{n-1,1}$, there exists a sequence of domains $\Omega_m \supset \Omega$ such that $\partial\Omega_m \in C^\infty$, $|\Omega_m \setminus \Omega| \to 0$ and $\|k_m\|_{L^{n-1,1}} \le C\|k\|_{L^{n-1,1}}$ for some constant C independent of m, where k_m fulfills (4.17) with Ω replaced by Ω_m. The sequence $\{\Omega_m\}_{m\in\mathbb{N}}$ can be chosen in such a way that the constants appearing in (3.1), (3.2) and (3.3) with Ω replaced by Ω_m are estimated, up to a multiplicative constant, by the same constants corresponding to Ω. Let f be continued by 0 in $\Omega_m \setminus \Omega$. Let u_m be the solution to (1.1) with Ω replaced by Ω_m. Owing to the

$C^{1,\alpha}$ estimates for u_m, which by our choice of Ω_m are uniform in m, we have that there exists a function $u \in C^{1,\alpha}(\overline{\Omega})$ such that $u_m \to u$ and $\nabla u_m \to \nabla u$ uniformly in $\overline{\Omega}$. Passing to the limit in the equation

$$\int_{\Omega_m} a(|\nabla u_m|)\nabla u_m \cdot \nabla \phi \, dx = \int_{\Omega_m} f_m \phi dx \tag{4.47}$$

for every $\phi \in W^{1,\infty}(\Omega_m)$ leads to (2.21) for every $\phi \in W^{1,\infty}(\Omega)$, since any function from $W^{1,\infty}(\Omega)$ can be continued to a function in $W^{1,\infty}(\mathbb{R}^n)$. Hence, by Theorem 2.1, equation (2.21) holds for every $\phi \in W^{1,B}(\Omega)$. Hence, (1.8) follows by Step 1 applied to u_m.

Step 3 Here we remove assumptions (4.2) and (4.3), but keep (4.4) in force.

Let $\{a_\varepsilon\}_{\varepsilon \in (0,1)}$ be the family of functions approximating the function a given by Lemma 3.3, and let b_ε and B_ε be as in its statement. By Proposition 2.13, there exists a unique solution $u_\varepsilon \in W^{1,B_\varepsilon}_\perp(\Omega)$ to the problem

$$\begin{cases} -\mathrm{div}(a_\varepsilon(|\nabla u_\varepsilon|)\nabla u_\varepsilon) = f(x) & \text{in } \Omega \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{4.48}$$

By the result of [Li3] and (3.10), for every open set $\Omega' \subset\subset \Omega$, there exist $\alpha \in (0,1]$ and a constant C, independent of ε, such that

$$\|u_\varepsilon\|_{C^{1,\alpha}(\Omega')} \le C \tag{4.49}$$

for $\varepsilon \in (0,1)$. Hence, there exists a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ and a function $u \in C^1(\Omega)$ such that

$$\lim_{k\to\infty} u_{\varepsilon_k} = u \tag{4.50}$$

$$\lim_{k\to\infty} \nabla u_{\varepsilon_k} = \nabla u \tag{4.51}$$

uniformly in every compact subset Ω. Moreover, by the previous steps, there exists a constant C such that

$$|\nabla u_{\varepsilon_k}| \le C \qquad \text{a.e. in } \Omega. \tag{4.52}$$

By (4.50)–(4.52) and Fatou's Lemma, one easily infers that $u \in W^{1,B}_\perp(\Omega)$. Furthermore, since

$$\int_\Omega a_{\varepsilon_k}(|\nabla u_{\varepsilon_k}|)\nabla u_{\varepsilon_k} \cdot \nabla \phi \, dx = \int_\Omega f \phi dx \tag{4.53}$$

for every $\phi \in W^{1,\infty}(\Omega)$, equations (4.51) and (4.52) ensure that u fulfils (2.21) for every $\phi \in W^{1,\infty}(\Omega)$, and hence for every $\phi \in W^{1,B}(\Omega)$, owing to Theorem 2.1. Thus, u is the solution to (1.1). By the preceding steps applied to u_ε and by (4.51) and (3.11), inequality (1.8) follows.

Step 4 We conclude by removing the additional assumption (4.4).

Since the set $C^\infty_0(\Omega)$ is dense in $L^{n,1}(\Omega)$, there exists a sequence $\{f_k\}_{k\in\mathbb{N}} \subset C^\infty_0(\Omega)$ such that

$$f_k \to f \qquad \text{in } L^{n,1}(\Omega).$$

Let u_k be the solution in $W^{1,B}_\perp(\Omega)$ to problem

$$\begin{cases} -\mathrm{div}(a(|\nabla u_k|)\nabla u_k) = f_k(x) & \text{in } \Omega \\ \dfrac{\partial u_k}{\partial \nu} = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{4.54}$$

By Lemma 2.14, there exists a constant C independent of k such that

$$\int_\Omega B(|\nabla u_k|)\,dx \le C. \tag{4.55}$$

Owing to inequalities (4.55) and (2.12), the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $W^{1,B}(\Omega)$. Since the space $W^{1,B}(\Omega)$ is reflexive (Theorem 2.2), and the embedding $W^{1,B}(\Omega)\to L^B(\Omega)$ is compact (Theorem 2.8), there exist a function $u\in W^{1,B}_\perp(\Omega)$ and subsequence of u_k, still denoted by u_k, such that

$$u_k \to u \qquad \text{in } L^B(\Omega), \tag{4.56}$$

and

$$u_k \rightharpoonup u \qquad \text{weakly in } W^{1,B}(\Omega). \tag{4.57}$$

We claim that, up to subsequences,

$$\nabla u_k \to \nabla u \qquad \text{a.e. in } \Omega. \tag{4.58}$$

To verify (4.58), we exploit an argument from [BBGGPV]. Let $\varepsilon>0$. Given $t,\tau>0$, we have that

$$\begin{aligned}|\{|\nabla u_k-\nabla u_m|>t\}| & \\ \le |\{|\nabla u_k|>\tau\}| &+|\{|\nabla u_m|>\tau\}|+|\{|\nabla u_k-\nabla u_m|>t, |\nabla u_k|\le\tau, |\nabla u_m|\le\tau\}|\end{aligned} \tag{4.59}$$

for $k,m\in\mathbb{N}$. Estimate (4.55) entails that, if τ is sufficiently large, then

$$|\{|\nabla u_k|>\tau\}|<\varepsilon, \qquad \text{for } k\in\mathbb{N}. \tag{4.60}$$

Next, define

$$\vartheta=\inf\{[a(\xi)\xi-a(\eta)\eta]\cdot(\xi-\eta): |\xi-\eta|>t, |\xi|\le\tau, |\eta|\le\tau\},$$

and observe that $\vartheta>0$. Making use of u_k-u_m as test function in the weak formulation of (4.54) and of its analogue with k replaced by m and subtracting the resulting equations yield

$$\begin{aligned}\vartheta|\{|\nabla u_k-\nabla u_m|>t, &|\nabla u_k|\le\tau, |\nabla u_m|\le\tau\}| \\ &\le \int_\Omega [a(\nabla u_k)\nabla u_k-a(\nabla u_m)\nabla u_m]\cdot(\nabla u_k-\nabla u_m)\,dx \\ &\le \int_\Omega (f_k-f_m)(u_k-u_m)dx \le \|u_k-u_m\|_{L^{n'}(\Omega)}\|f_k-f_m\|_{L^n(\Omega)} \\ &\le C\|\nabla u_k-\nabla u_m\|_{L^1(\Omega)}\|f_k-f_m\|_{L^{n,1}(\Omega)} \\ &\le C'\|\nabla u_k-\nabla u_m\|_{L^B(\Omega)}\|f_k-f_m\|_{L^{n,1}(\Omega)} \\ &\le C'\big(\|\nabla u_k\|_{L^B(\Omega)}+\|\nabla u_m\|_{L^B(\Omega)}\big)\|f_k-f_m\|_{L^{n,1}(\Omega)}\end{aligned} \tag{4.61}$$

for some constants $C=C(\Omega)$ and $C'=C'(\Omega,B)$. Thus,

$$|\{|\nabla u_k-\nabla u_m|>t, |\nabla u_k|\le\tau, |\nabla u_m|\le\tau\}|<\varepsilon, \tag{4.62}$$

provided that k and m are sufficiently large. Combining (4.59), (4.60) for k and m, and (4.62) tells us that ∇u_k is a Cauchy sequence in measure. Hence, (4.58) follows.
Owing to (4.56), (4.57) and (4.58) one can pass to the limit as $k\to\infty$ in the weak formulation of problem (4.54), and deduce that u is the solution to (1.1). Finally, by Step 3 estimate (1.8) holds with f and u replaced by f_k and u_k. Passing to the limit in this estimate as $k\to\infty$ and making use of (4.58) leads to (1.8). □

Proof of Theorem 1.2. **Step 1**. Assume that Ω, a and f are as in Step 1 of the proof of Theorem 1.1 and, in addition, that Ω is convex. One can begin as in the proof of Theorem 1.1, and exploit the fact that, owing to the convexity of Ω,

$$-\mathcal{B}(\nabla_T u, \nabla_T u) \geq 0 \qquad \text{on } \partial\Omega \ .$$

Thus, inequality (4.20) can be replaced by a stronger inequality, where the term

$$\int_{\partial\Omega\cap\overline{\{|\nabla u|>t\}}} a(|\nabla u|)|\nabla u|^2 k(x)\, d\mathcal{H}^{n-1}(x)$$

is missing. Starting from this inequality, the same argument leads to an inequality analogous to (4.37), where the last summand on the right-hand side is missing. Estimate (1.8) follows analogously.
Step 2. The proof is analogous to that of Theorem 1.1, save that the approximating domains Ω_m have to be chosen convex.
Steps 3 and 4. The proofs are the same as in Theorem 1.1. □

Proof of Theorem 1.3 The proof proceeds along the same lines as that of Theorem 1.1. One has just to note that equation (4.15) now yields

$$\Delta u \frac{\partial u}{\partial \nu} - \sum_{ij} u_{x_i x_j} u_{x_i} \nu_j = -(\mathrm{tr}\mathcal{B}) \left(\frac{\partial u}{\partial \nu}\right)^2 \qquad \text{on } \partial\Omega$$

instead of (4.16), and to make use of the fact that

$$\mathrm{tr}\mathcal{B} \left(\frac{\partial u}{\partial \nu}\right)^2 \leq k(x)|\nabla u|^2 \qquad \text{on } \partial\Omega \tag{4.63}$$

for some nonnegative function $k \in L^{n-1,1}$, instead of (4.17). Also, Parts (i) of Theorem 2.13 and Proposition 2.14 have to be employed in the place of Parts (ii). □

Proof of Theorem 1.4 The proof is a simplification of that of Theorem 1.3 in same spirit as the proof of Theorem 1.2 is a simplification of that of Theorem 1.1. In particular, in Step 1, the convexity of Ω ensures that

$$\mathrm{tr}\mathcal{B} \leq 0 \qquad \text{on } \partial\Omega$$

and hence that the right-hand side of (4.63) can be replaced by 0. □

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Note. When this paper was completed, we were informed by G.Mingione that he and F.Duzaar were about to submit a new paper [DM4] dealing with local estimates for the gradient of local solutions to nonlinear elliptic equations and systems (from a more general class than that considered here). In particular, the result of [DM4] yields the local Lipschitz continuity of solutions when the datum is locally in $L^{n,1}$. The results of [DM4] differ from those of the present paper in that they have a local nature, whereas ours are global and hold for solutions to boundary value problems in possibly non-smooth domains. Also, the approach of [DM4] is different and relies upon nonlinear potential techniques.

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