Estimates for kernels of inverse operators of integral equations of elasticity on surfaces with conic points

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Abstract. Boundary integral equations of linear isotropic elasticity, with the double layer potential generated by the preudo-stress operator, are considered on surfaces with a finite number of conic points. Representations for solutions are obtained in terms of inverse operators of the Dirichlet and Neumann problems in the interior and exterior of the surface. Pointwise estimates for kernels of inverse operators and their derivatives of any order are derived with the help of estimates for fundamental solutions of those boundary value problems. The Laplace operator is contained here as a special case.

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1 Introduction

We consider boundary integral equations of linear isotropic elasticity on surfaces which are smooth everywhere except for a finite number of conic points. Here, similarly to [M1] (see also [M2]) the study of the integral equations is reduced to the study of some auxiliary boundary value problems. Representations for inverse operators of these equations are obtained in terms of inverse operators of the Dirichlet and Neumann problems in the interior and exterior of the surface. Using estimates for fundamental soutions of these boundary value problems, we arrive at estimates for kernels of inverse operators of integral operators in question.

The system of integral equations under consideration contains the boundary integral equations for the Laplace operator as a special case.

We describe the main results. Let G^+ be a simply connected region in $\mathbb{R}^3$ with compact closure. We put $\Gamma = \partial G^+$ and let $0 \in \Gamma$. Further we assume that $\Gamma \backslash \{0\}$ is a smooth surface and that near the origin G^+ coincides with the cone K^+ excising an open set Ω^+ on the unit sphere S^2. Let $G^- = \mathbb{R}^3 \backslash \overline{G^+}$ and $\Omega^- = S^2 \backslash \overline{\Omega^+}$.

In what follows, by $\varkappa$ we denote a real number depending on the shape of the cone K^+ and the boundary conditions. It can be expressed in terms of eihenvalues of some boundary value problems with complex parameter in Ω^+ and Ω^-. In the case of harmonic potentials $\varkappa$ is positive, being equal to the minimum of δ^+ and ν^-, where δ^+ and ν^- are positive numbers such that $\delta^+(\delta^+)$ and $\nu^-(\nu^-+1)$ are the first eigenvalues of the Dirichlet ptoblem in Ω^+ and the Neumann problem in Ω^- for the Beltrami operator. It follows from [KM] that $\varkappa$ is positive for integral equations of elasticity if the cone K^+ can be explicitly described in a Cartesian coordinate system.

Given any $\beta \in \mathbb{R}$, $\alpha \in (0,1)$ and any nonnegative integer l, by $N^{l,\alpha}_\beta(S)$ we denote the space of functions u on S with the norm

$$\sup_{x,y\in S\backslash 0} \frac{|r^\beta(x)\nabla_l u(x) - r^\beta(y)\nabla_l u(y)|}{|x-y|^\alpha} + \sum_{0\le j\le l} r^{\beta-l-\alpha+j}(x)|\nabla_j u(x)|.$$

We assume that the double layer potential $W\varphi$ is generated by the so-called pseudo-stress operator (see [M2]). The case of the double layer potential generated by the stress operator will be considered elsewhere.

Let $W_0\varphi$ stand for the direct value of $W\varphi$ on $S\backslash 0$ and let T be the integral operator defined by the equality

$$T\varphi = 2\, W_0\varphi.$$

It is known that the system of three boundary integral equations of the first boundary value problem for the Lamé equations has the form

$$(1+T)\,\varphi = f. \tag{1.1}$$

Note that in the case of a smooth boundary the kernel of T has a weak singularity.

Let $0 < \beta - \alpha < 1$, $l \ge 1$. Then system (1.1) is uniquely solvable in the space $N^{l,\alpha}_{\beta+l}(S)$ for all $f \in N^{l,\alpha}_{\beta+l}(S)$ and

$$(1+T)^{-1} f = (1+L)\, f. \tag{1.2}$$

Here L is an integral operator with kernel $\mathcal{L}(x,y)$ satisfying the inequalities

$$|\mathcal{L}(x,y)| \le\le \begin{cases} c\,|y|^{-2}(|x|/|y|)^{\varkappa-\varepsilon} + c\,|y|^{\varkappa-1-\varepsilon}, & |x| < |y|/2 \\ c\,|y|^{-1}|x-y|^{-1}, & |y|/2 < |x| < 2|y|, \\ c\,|x|^{-1}|y|^{-1}(|y|/|x|)^{\varkappa-\varepsilon}, & |x| > 2|y|, \end{cases}$$

where ε is a sufficiently small positive number,

$$|\mathcal{L}| = \max_{1\le i\le 3, 1\le j\le 3} |\mathcal{L}_{ij}|,$$

and $\mathcal{L}_{ij}$ are elements of the matrix $\mathcal{L}$.

Let T^* be the operator formally adjoint of T. If $1 < \beta - \alpha < 2$, then the system of equations $(1+T^*)\psi = g$ is uniquely solvable in the space $N^{l,\alpha}_{\beta+l}(S)$ for all $g \in N^{l,\alpha}_{\beta+l}(S)$ and

$$(1+T^*)^{-1} g = (1+M)\, g, \tag{1.3}$$

where the kernel $\mathcal{M}(x,y)$ of M satyisfies the estimates

$$|\mathcal{M}(x,y)| \le \begin{cases} c\,|y|^{-2}(|x|/|y|)^{\varkappa-\varepsilon}, & |x| < |y|/2 \\ c\,|y|^{-1}|x-y|^{-1}, & |y|/2 < |x| < 2|y|, \\ c\,|x|^{-1}|y|^{-1}(|y|/|x|)^{\varkappa-\varepsilon} + c\,|y|^{\varkappa-1-\varepsilon}, & |x| > 2|y|. \end{cases}$$

The paper consists of three sections. In Section 2 we obtain representations for the operators $(1+T)^{-1}$ and $(1+T^*)^{-1}$. Sections 3 and 4 concern pointwise estimates for the kernels of the operators $(1+T)^{-1}$ and $(1+T^*)^{-1}$.

2 Representations for inverse operators of boundary integral equations

2.1 Spaces of functions

We use the notations G^+, G^-, and $N^{l,\alpha}_\beta$ explained in Introduction. It is easily seen that the space $N^{l,\alpha}_\beta(G^+)$ can be supplied with the norm

$$\|u\|_{N^{l,\alpha}_\beta(G^+)} = \sup_{x\in F^+} |x|^\beta [u]^{l+\alpha}_{B(|x|/2,x)\cap G^+} + \sup_{x\in G^+} |x|^{\beta-l-\alpha}|u(x)|, \tag{2.1}$$

where $B(r,x)$ is an open ball of radius r centered at x,

$$[u]^\rho_E = \sup_{x,y\in E} \sum_{|\sigma|=[\rho]} |x-y|^{[\rho]-\rho}|\partial^\sigma_x u(x) - \partial^\sigma_y u(y)|,$$

where E is a subset of $\mathbb{R}^3$, ρ is a positive noninteger, $[\rho]$ is the integer part of ρ, $\sigma = (\sigma_1,\sigma_2,\sigma_3)$ is a multiindex of order $|\sigma| = \sigma_1+\sigma_2+\sigma_3$, and $\partial^\sigma_x = \partial^{|\sigma|}/\partial x_1^{\sigma_1}\partial x_2^{\sigma_2}\partial x_3^{\sigma_3}$.

Besides, one can directly verify (see [MP3]) that the norm (2.2) is equivalent to the norm

$$\sup_{x,y\in G^+} |x-y|^{-\alpha} \sum_{0\le|\sigma|\le l} \big|\, |x|^{\beta-l+|\sigma|}\partial^\sigma_x u(x) - |y|^{\beta-l+|\sigma|}\partial^\sigma_y u(y)\,\big|$$
$$+ \sup_{x\in G^+} \sum_{0\le|\sigma|\le l} |x|^{\beta-l-\alpha+|\sigma|}|\partial^\sigma_x u(x)|.$$

For $0<\beta<l+\alpha$ we define the space $C^{l,\alpha}_\beta(G^+)$ of functions on G^+ with the norm

$$\|u\|_{C^{l,\alpha}_\beta(G^+)} = \sup_{x\in G^+} |x|^\beta [u]^{l+\alpha}_{B(|x|/2,x)\cap G^+} + \sup_{x\in G^+} \sum_{l+\alpha-\beta<|\sigma|\le l} |x|^{\beta-l-\alpha+|\sigma|}|\partial^\sigma_x u(x)|$$
$$+[u]^{l+\alpha-\beta}_{G^+} + \sum_{0\le|\sigma|<l+\alpha-\beta} \sup_{x\in G^+} |\partial^\sigma_x u(x)|. \tag{2.2}$$

Similar spaces $N^{l,\alpha}_\beta(G^-)$ and $C^{l,\alpha}_\beta(G^-)$ are defined for the domain G^-. Suppose that the ball $B(R,0)$ contains $\overline{G^+}$. Let χ denote a function in the space $C^\infty(\mathbb{R}^3)$ equal to one on $B(R,0)$ and to zero on $\mathbb{R}^3\backslash B(R+1,0)$. A function u in G^- belongs to $N^{l,\alpha}_\beta(G^-)$ (resp. $C^{l,\alpha}_\beta(G^-)$) if and only if the norm (2.2) (resp. (2.2)) of the function $u\,\chi$ and the norm

$$\sup_{x\in G^-} |x|^{l+1+\alpha}[v]^{l+\alpha}_{B(|x|/2,x)} + \sup_{x\in G^-} |x|\,|v(x)|$$

of $v=(1-\chi)u$ are finite.

The spaces of traces on Γ of functions from $C^{l,\alpha}_\beta(G^+)$ and $N^{l,\alpha}_\beta(G^+)$ are denoted by $N^{l,\alpha}_\beta(\Gamma)$ and $N^{l,\alpha}_\beta(\Gamma)$.

Finally, let $N^{l,\alpha}_\beta(G^+)$and $C^{l,\alpha}_\beta(G)$ refer to spaces of functions u in $G = G^+\cup G^-$ whose restrictions $u^\pm$ on $G^\pm$ belong to $N^{l,\alpha}_\beta(G^\pm)$ and $C^{l,\alpha}_\beta(G^\pm)$, respectively, and

$$\|u\|_{N^{l,\alpha}_\beta(G)} = \sum_\pm \|u^\pm\|_{N^{l,\alpha}_\beta(G^\pm)}, \qquad \|u\|_{C^{l,\alpha}_\beta(G)} = \sum_\pm \|u^\pm\|_{C^{l,\alpha}_\beta(G^\pm)}.$$

2.2 Boundary value problems

We consider the interior first boundary value problem for the three-dimensional Lamé equation of the linear isotropic elasticity theory

$$\Delta^* u = 0 \quad \text{in } G^+, \qquad u = 0 \quad \text{on } \Gamma. \tag{2.3}$$

Here $u(x) = (u_1(x), u_2(x), u_3(x))^T$ and $f(x) = (f_1(x), f_2(x), f_3(x))^T$ are the displacement vectors in G^+ and on Γ, the notation T stands for the transposition of a matrix,

$$\Delta^* = \mu\Delta + (\lambda + \mu)\nabla \,\mathrm{div}$$

and λ, μ are the Lamé constants.

We introduce the auxiliary problem

$$\Delta^* v = 0 \quad \text{in } G^-, \qquad \mathcal{N} v = g \quad \text{on } \Gamma \backslash 0, \tag{2.4}$$

where v and g are vector-valued functions with three components, $\mathcal{N} = \mathcal{N}(\partial_x, n_x)$ is the matrix differential operator with elements

$$\mathcal{N}(\partial_x, n_x) = \mu\delta_{ij}\partial/\partial n_x + (\lambda + \mu) n_i(x)\partial/\partial x_j$$
$$+\mu(\lambda + \mu)(n_j(x)\partial/\partial x_i - n_i(x)\partial/\partial x_j)/(\lambda + 3\mu),$$

which is called the pseudo-stress operator. Here $n_x = (n_1(x), n_2(x), n_3(x))$ is the normal vector of length 3 to the surface Γ directed outward with respect to G^+, δ_{ij} is the Kroneker delta and $\partial/\partial n_x$ is the normal derivative.

Let Ω^+ and Ω^- be the domains on the unit sphere defined in Introduction. In order to formulate solvability theorems it is necessary to consider certain boundary value problems in Ω^+ and Ω^- depending on a complex parameter γ (see [K], [MP2], [MP5]). These problems can be obtained by substitution of the vector-valued functions

$$u(x) = |x|^\gamma \varphi(x/|x|) \qquad \text{and} \quad v(x) = |x|^\gamma \psi(x/|x|)$$

into equations (2.3) and (2.4) with $f = 0$ and $g = 0$. In what follows these problems will be denoted by $p(\gamma)$ and $q(\gamma)$.

Theorem 1 *(*see [MP4]*) Let δ^+ be the largest number such that the strip $|\Re\gamma + 1/2| < 1/2 + \delta^+$ contains no eigenvalues of problem $p(\gamma)$. If*

$$l \geq 1, \qquad |\beta - \alpha - 1/2| < 1/2 + \delta^+,$$

then there exists a unique solution $u \in N^{l,\alpha}_\beta(G^+)$ of problem (2.3) for all $f \in N^{l,\alpha}_\beta(\Gamma)$. This solution can be written in the form

$$u(x) = \int_\Gamma \mathcal{P}^+(x,\xi)\, f(\xi) ds_\xi. \tag{2.5}$$

The operator

$$P^+ : N^{l,\alpha}_{\beta+l}(\Gamma) \ni f \;\to\; u \in N^{l,\alpha}_{\beta+l}(G^+),$$

defined by (2.5) is bounded. The derivatives of the kernel $\mathcal{P}^+(x,\xi)$ obey the estimates

$$|\partial^\sigma_x \partial^\tau_\xi \mathcal{P}^+(x,\xi)| \leq \begin{cases} c\,|x|^{\delta^+ - |\sigma| - \varepsilon} |\xi|^{-2-\delta^+ - |\tau| + \varepsilon}, & |x| < d\,|\xi|, \\ c\,|x-\xi|^{-|\tau| - |\sigma| - 2}, & d\,|\xi| < |x| < d^{-1}|\xi|, \\ c\,|x|^{-\delta^+ - |\sigma| - 1 - \varepsilon} |\xi|^{\delta^+ - |\tau| - 1 - \varepsilon}, & |x| > d^{-1}|\xi|, \end{cases} \tag{2.6}$$

Here d is a fixed number from $(0,1)$, τ and σ are arbitrary multi-indices, ε is any sufficiently small posiitive number, and $|\mathcal{A}|$ means $\max|\mathcal{A}_{ij}|$, where $\mathcal{A}_{ij}$ are elements of the matrix $\mathcal{A}$.

Remark (see [MP4]) Let $\omega(\omega+1)$ be the first eigenvalue of the Dirichlet problem for the Beltrami operator on Ω^+, $\omega > 0$. Then

$$\delta^+ > \omega(3\mu+\lambda)/(\omega(\lambda+\mu)+2(3\mu+2\lambda)).$$

Lemma 1 *If $0 < \alpha-\beta < 1$, $\alpha-\beta < \delta^+$, $l = 1, 2, \ldots$, then the function u defined by (2.5) is a unique solution of (2.3) in the space $C^{l,\alpha}_{\beta+l}(G^+)$ for all $f \in C^{l,\alpha}_{\beta+l}(\Gamma)$. Moreover,*

$$\|u\|_{C^{l,\alpha}_{\beta+l}(G^+)} \leq C\, \|f\|_{C^{l,\alpha}_{\beta+l}(\Gamma)}.$$

Proof. It easily seen (compare with [MP7]), that the function $u \in C^{l,\alpha}_{\beta+l}(G^+)$, with $0 < \alpha-\beta$, admits the representation

$$u = u(0) + v, \qquad v \in N^{l,\alpha}_{\beta+l}(G^+)$$

and that the norm of u in $N^{l,\alpha}_{\beta+l}(G^+)$ is equivalent to the norm

$$|u(0)| + \|u - u[0]\|_{N^{l,\alpha}_{\beta+l}(G^+)}.$$

A similar representation holds for functions from the space $C^{l,\alpha}_{\beta+l}(G^+)$.

Thus, the existence of the solution $u \in C^{l,\alpha}_{\beta+l}(G^+)$ follows from Theorem 1 and from the fact that $u = c$ is a solution of (2.3) for $f = c$, where c is a constant vector. The uniqueness of the solution $u \in C^{l,\alpha}_{\beta+l}(G^+)$ is the consequence of the inclusion $C^{l,\alpha}_{\beta+l}(G^+) \subset N^{l,\alpha}_{\beta+l}(G^+)$ and of the uniqueness of the solution of (2.3) in the space $N^{l,\alpha}_{\beta+l}(G^+)$ by Theorem 1. The lemma is proved.

The following assertion is essentially contained in [MP6].

Theorem 2 *Let ν^- be the largest number such that the strip $|\Re\gamma + 1/2| < 1/2 + \nu^-$ contains no eigenvalues of problem $q(\gamma)$, except for $\gamma = 0$ and $\gamma = -1$. If*

$$|\beta-\alpha-1/2| < 1/2 + \min\{0, \nu^-\}, \qquad l = 1, 2, \ldots,$$

then there exists a unique solution $v \in N^{l,\alpha}_{\beta+l}(G^-)$ of problem (2.4) for any vector-valued function $g \in N^{l-1,\alpha}_{\beta+l}(\Gamma)$. This solution admits the representation

$$v(x) = \int_\Gamma \mathcal{Q}^-(x,\xi)\, g(\xi) ds(\xi). \tag{2.7}$$

The operator

$$Q^- : N^{l-1,\alpha}_{\beta+l}(\Gamma) \ni g \to v \in N^{l,\alpha}_{\beta+l}(G^-),$$

defined by (2.7) is bounded. The derivatives of the kernel $\mathcal{Q}^-(x,\xi)$ obey the estimates

$$|\partial_x^\sigma \partial_\xi^\tau \mathcal{Q}^-(x,\xi)| \leq \begin{cases} c\,|x|^{-|\sigma|}|\xi|^{-1-|\tau|}(|x|/|\xi|)^{\nu-\varepsilon}, & |x| < d\,|\xi|, \\ c\,|x-\xi|^{-1-|\tau|-|\sigma|}, & d\,|\xi| < |x| < d^{-1}|\xi|, \\ c\,|x|^{-1-|\sigma|}|\xi|^{-|\tau|}(|\xi|/|x|)^{\nu-\varepsilon}, & |x| > d^{-1}|\xi|, \end{cases} \tag{2.8}$$

if the points x and ξ lie in a neighborhood of the vertex 0 and $\nu = \min\{0, \nu^-\}$.

We note that $-1/2 < \nu^- < 1$. The left inequality follows from the solvability of problem (2.4) in the energy space. The right inequality is the consequence of the fact that the rigid displacement vector is the solution of the homogeneous problem (2.4).

Lemma 2 (see [KM]) or Theorem 4.3.1 in [KMR] *Let* Γ *be defined by the equation* $x_3 = h(x_1, x_2)$ *near the vertex of the cone with* h *being a positive homogeneous function of order 1, smooth on* $\mathbb{R}^2\backslash\{0\}$*. Then there are only two eigenvalues* $\gamma_0 = 0$ *and* $\gamma_1 = -1$ *of the problem* $q(\gamma)$ *in the strip* $-1 \le \Re\gamma \le 0$*. These eigenvalues have multiplicity 3 and the Jordan chains for* $q(\gamma)$*, corresponding to them, consist only of eigenfunctions.*

The vector-valued functions $w = const$ *are the only eigenfunctions associated with the eigenvalue* γ_0*.*

In fact, Lemma 2 is proved in [KM] and [KMR] for the Neumann problem generated by the stress operator but the proof holds for the pseudo-stress operator with obvious changes.

Lemma 2 enables one to give a more precise description of behaviour of the kernel $\mathcal{Q}^-(x,\xi)$.

Theorem 3 *Let the surface* Γ *satisfy the assumptions in Lemme 2. Then*

$$\mathcal{Q}^-(x,\xi) = \mathcal{Q}^-(0,\xi) + \mathcal{R}^-(x,\xi), \qquad |x| < d\,|\xi|, \tag{2.9}$$

$$\mathcal{Q}^-(x,\xi) = \mathcal{Q}^-(x,0) + (\mathcal{R}^-(\xi,x))^T, \qquad |x| > d^{-1}|\xi|,$$

$$\mathcal{Q}^-(x,0) = (\mathcal{Q}^-(0,x))^T = |x|^{-1}\mathcal{A}^-(x/|x|)\,\mathcal{B}^- + \mathcal{C}^-(x), \tag{2.10}$$

if the points x *and* y *lie in a neighbourhood of* 0*. Here* $\mathcal{A}^-(x/|x|)$ *is a matrix whose columns are eigenfunctions corresponding to the eigenvalue* $\gamma = -1$*,* $\mathcal{B}^-$ *is a constant matrix and the matrices* $\mathcal{R}^-(x,\xi)$ *and* $\mathcal{C}^-(x)$ *satisfy*

$$|\partial_x^\sigma \partial_\xi^\tau \mathcal{R}^-(x,\xi)| \le c\,|x|^{-|\sigma|}|\xi|^{-1-|\tau|}(|x|/|\xi|)^{\nu^- - \varepsilon}, \tag{2.11}$$

$$|\partial_x^\sigma \mathcal{C}^-(x)| \le c\,|x|^{\nu^- - |\sigma| - \varepsilon}. \tag{2.12}$$

Lemma 3 *The columns of the matrix* $|x|^{-1}\mathcal{A}^-(x/|x|)$ *are linear combinations of columns of the matrix* $\Phi(x,0)$*, where* $\Phi(x,\xi)$ *is the Kelvin-Somigliana matrix with elements*

$$\Phi_{kj}(x,\xi) = \frac{\lambda+3\mu}{2\mu(\lambda+2\mu}\Big(\frac{\delta_{kj}}{|x-\xi|} + \frac{\lambda+\mu}{\lambda+3\mu}\;\frac{(x_k-\xi_k)(x_j-\xi_j)}{|x-\xi|^3}\Big).$$

Proof. Since the columns of the matrix $\Phi(x,0)$ are solutions of the homogeneous problem (2.4) in an infinite cone (see [Ku], [KGBB]), it follows that the colomns of the matrix $|x|\Phi(x/|x|,0)$ are eigenfunctions of of problem $q(\gamma)$ corresponding to the eigenvalue $\gamma = -1$. It remains to notice that the columns of the matrix $|x|\Phi(x,0)$ form a basis in the eigenspace by Lemma 2. The lemma is proved.

Lemma 4 *Let* $0 < \alpha - \beta < \nu^-$ *and let* l *be a positive integer. If* $g \in N^{l-1,\alpha}_{\beta+l}(\Gamma)$*, then the function (2.7) is a unique solution of problem (2.4) in the space* $C^{l,\alpha}_{\beta+l}(G^-)$*. Moreover,*

$$\|v\|_{C^{l,\alpha}_{\beta+l}(G^-)} \le c\,\|f\|_{N^{l-1,\alpha}_{\beta+l}(\Gamma)}.$$

Proof. Since $0 < \beta + 1 - \alpha < 1$, the uniqueness of the solution $v \in C^{l,\alpha}_{\beta+l}(G^-)$ of problem (2.4) follows from the inclusion $C^{l,\alpha}_{\beta+l}(G^-) \subset N^{l,\alpha}_{\beta+l}(G^-)$ and from Theorem 2. Let us prove the existence of the solution $v \in C^{l,\alpha}_{\beta+l}(G^-)$. Suppose that $g \in N^{l-1,\alpha}_{\beta+l}(\Gamma)$. It is clear that $g \in N^{l-1,\alpha}_{\beta+1+l}(\Gamma)$. By Theorem 2 there exists a solution $v \in C^{l,\alpha}_{\beta+1+l}(G^-)$ of (2.4) and

$$\|v\|_{N^{l,\alpha}_{\beta+1+l}(G^-)} \le c\,\|g\|_{N^{l-1,\alpha}_{\beta+1+l}(\Gamma)}.$$

From the asymptotic representation of v near the conic point (see [MP1]) it follows that

$$v = v(0)\eta + w, \qquad w \in N^{l,\alpha}_{\beta+l}(G^-)$$

and

$$|v(0)| + \|w\|_{N^{l,\alpha}_{\beta+l}(G^-)} \le c\,\|g\|_{N^{l-1,\alpha}_{\beta+1+l}(\Gamma)}.$$

Here $\eta \in C_0^\infty(\mathbb{R}^3)$ and $\eta = 1$ in a neighbourhood of the point 0. Thus $v \in C^{l,\alpha}_{\beta+l}(G^-)$. The lemma is proved.

2.3 Representation of the inverse operators of the integral equations

Let $V\varphi$ and $W\psi$ denote the single and double layer potentials with densities φ and ψ defined by

$$\begin{aligned}(V\varphi)(x) &= \frac{1}{4\pi}\int_\Gamma \Psi(x,\xi)\varphi(\xi)ds_\xi, && x \in \mathbb{R}^3,\\ (W\psi)(x) &= -\frac{1}{4\pi}\int_\Gamma (\mathcal{N}(\partial_\xi, n_\xi)\Psi(x,\xi))^T\psi(\xi)ds_\xi, && x \in G.\end{aligned} \tag{2.13}$$

Lemma 5 *Let $0 < \beta - \alpha < 2$ and let l be a positive integer. If $\psi \in N^{l,\alpha}_{\beta+l}(\Gamma)$, then $W\psi \in N^{l,\alpha}_{\beta+l}(G)$ and the equalities*

$$(W\psi)^\pm = W_0\psi \pm \psi/2, \qquad (\mathcal{N}W\psi)^+ = (\mathcal{N}W\psi)^- \tag{2.14}$$

hold on $\Gamma\backslash 0$, where W_0 is the direct value of the double layer potential on Γ and the symbols $\pm$ mean the traces on $\Gamma\backslash 0$ of functions defined on $G^\pm$.

Proof. It can be verified directly that

$$\sup_{x\in B(1,0)} |x|^{\beta-\alpha}|(W\psi)(x)| \le c\sup_{x\in\Gamma}|x|^{\beta-\alpha}|\psi(x)|. \tag{2.15}$$

Consider the problem

$$\Delta^* u = 0 \quad \text{in } G, \qquad u^+ - u^- = \psi \quad \text{on } \Gamma, \tag{2.16}$$

$$(\mathcal{N}u)^+ - (\mathcal{N}u)^- = 0 \qquad \text{on } \Gamma\backslash 0.$$

It follows from properties of the double layer potential on smooth surfaces that the vector-valued function $W\psi$ belongs to the class $C^{l,\alpha}_{\mathrm{loc}}(\overline{G^\pm}\backslash 0)$ and it solves problem (2.16).

We introduce the sets

$$U_k = \{\xi : 1/2 < 2^k\,\|\xi| < 2\}, \qquad V_k = \{\xi : 1/4 < 2^k\,\|\xi| < 4\}$$

for all $k = 1, 2, \ldots$. It is known that solutions of (2.16) obey the estimate

$$2^{-k(l+\alpha)}[u]^{l+\alpha}_{U_k\cap G^\pm} \le c\big(2^{-k(l+\alpha)}[\psi]^{l+\alpha}_{V_k\cap\Gamma} + \sup_{x\in V_k\cap\Gamma}|u(x)|\big).$$

From this and (2.15) we conclude that $W\psi \in N^{l,\alpha}_{\beta+l}(G)$.

The relations (2.14) follow directly from similar relations valid in the case of smooth surfaces (see [Ku], [KGBB]). The lemma is proved.

Lemma 6 *Let $0 < \beta - \alpha < 1$ and let l be a positive integer. If $\varphi \in N^{l-1,\alpha}_{\beta+l}(\Gamma)$, then $V\varphi \in N^{l,\alpha}_{\beta+l}(G)$ and the equalities*

$$(\mathcal{N}V\psi)^{\pm} = -W_0^*\varphi \pm \varphi/2, \qquad (V\varphi)^+ = (V\varphi)^- \tag{2.17}$$

hold on $\Gamma\backslash 0$, where W_0^* is an integral operator on $\Gamma\backslash 0$, formally adjoint of W_0.

Proof. It can be verified directly that

$$\sup_{x\in B(1,0)} |x|^{\beta-\alpha}|(V\varphi)(x)| \le c \sup_{x\in\Gamma} |x|^{\beta-\alpha+1}|\varphi(x)|. \tag{2.18}$$

Consider the problem

$$\Delta^* v = 0 \quad \text{in } G, \qquad v^+ - v^- = 0 \quad \text{on } \Gamma, \tag{2.19}$$
$$(\mathcal{N}v)^+ - (\mathcal{N}v)^- = \varphi \qquad \text{on } \Gamma\backslash 0.$$

It follows from properties of the single layer potential on smooth surfaces that the vector-valued function $V\varphi$ belongs to the class $C^{l,\alpha}_{\text{loc}}(\overline{G^{\pm}}\backslash 0)$ and it solves problem (2.19). Let U_k and V_k be the sets introduced in the proof of Lemma 5. Then the local estimate

$$2^{-k(l+\alpha)}[v]^{l+\alpha}_{U_k\cap G^{\pm}} \le c\big(2^{-k(l+\alpha)}[\varphi]^{l-1+\alpha}_{V_k\cap\Gamma} + \sup_{x\in V_k\cap\Gamma}|v(x)|\big)$$

holds for the solution of (2.19). This and (2.18) imply that $V\varphi \in N^{l,\alpha}_{\beta+l}(G)$.

The relations (2.17) follow directly from similar relations valid in the case of smooth surfaces (see [Ku], [KGBB]). The lemma is proved.

Lemma 7 *Let $0 < \beta - \alpha < 1$. If $u \in N^{1,\alpha}_{\beta+1}(G)$ and $\Delta^* u = 0$ in G, then*

$$u = V((\mathcal{N}u)^+ - (\mathcal{N}u)^-) + W(u^+ - u^-). \tag{2.20}$$

Proof. It is known (cf. [Ku], [KGBB]) that the equalities

$$u(x) = (V(\mathcal{N}u)^+)(x) + (Wu^+)(x), \qquad x \in G^+, \tag{2.21}$$
$$0 = (V(\mathcal{N}u)^+)(x) + (Wu^+)(x), \qquad x \in G^-, \tag{2.22}$$
$$0 = -(V(\mathcal{N}u)^-)(x) - (Wu^-)(x), \qquad x \in G^+, \tag{2.23}$$
$$u(x) = -(V(\mathcal{N}u)^-)(x) - (Wu^-)(x), \qquad x \in G^-, \tag{2.24}$$

hold for all vector-valued functions u in G such that

$$u^{\pm} \in C^{\infty}(\overline{G^{\pm}}), \qquad u^- = O(|x|^{-1}) \quad \text{as } x \to \infty \quad \text{and } \Delta^* u = 0.$$

Here $u^{\pm}$ is the restriction of the function u to $G^{\pm}$. We show that equalities (2.21) - (2.24) can be extended to functions u of the class $N^{1,\alpha}_{\beta+1}(G)$ satisfying the system $\Delta^* u = 0$ in G. Let $\{v_n\}_{n=1}^{\infty}$ stands for a sequence of vector-valued functions in G^+ of the class $C^{\infty}(\overline{G^+})$ which converge to u in the norm of the space $N^{1,\alpha-\varepsilon}_{\beta+1}(G)$, where ε is a small positive number such that $\beta - \alpha + \varepsilon < 1$. As shown in [Ku], [KGBB], we have the relations

$$v_n(x) = -\int_{G^+} \Phi(x,y)\Delta^* v_n(y)\,dy + (Wv_n)(x) + (V\mathcal{N}v_n)(x), \quad x \in G^+$$

for all n. Integrating by parts and passing to the limit as $n \to \infty$, we get (2.21). The equalities (2.22) - (2.24) are proved in a similar manner.

Adding the equalities (2.21), (2.23) and then (2.22), (2.24), we arrive at representation (2.20). The lemma is proved.

Theorem 4 *Let $T = 2W_0$,*

$$|\beta - \alpha - 1/2| < 1/2 + \min\{0, \nu^-\},$$

and let l be a positive integer. If $f \in N^{l,\alpha}_{\beta+l}(\Gamma)$, then the equation

$$(1+T)\varphi = f \tag{2.25}$$

is uniquely solvable in the space $N^{l,\alpha}_{\beta+l}(\Gamma)$ and

$$(1+T)^{-1}f = \frac{1}{2}(1 - Q^-\mathcal{N}P^+)f, \tag{2.26}$$

where P^+ and Q^- are defined by (2.5) and (2.7).

Proof. From Theorems 1 and 2 it follows that the vector-valued function φ defined by

$$\varphi = 1/2(1 - Q^-\mathcal{N}P^+)f$$

belongs to the space $N^{l,\alpha}_{\beta+l}(\Gamma)$. We show that φ is a solution of the equation (2.25).

Consider the vector-valued function $u \in N^{l,\alpha}_{\beta+l}(G)$ which solves the problem

$$\Delta^* u = 0 \quad \text{in } G, \quad u^+ = f \text{ on } \Gamma, \quad (\mathcal{N}u)^+ - (\mathcal{N}u)^- = 0 \text{ on } \Gamma\backslash 0.$$

It is clear that $u^- = Q^-\mathcal{N}P^+ f$. Hence $\varphi = (u^+ - u^-)/2$. By this and Lemmas 3, 5 we arrive at the chain of equalities

$$(1+T)\varphi = 2(W\varphi)^+ = (W(u^+ - u^-))^+ = u^+ = f.$$

It remains to verify the uniqueness of a solution of (2.25). Let $\varphi_0 \in N^{l,\alpha}_{\beta+l}(\Gamma)$ satisfy $(1+T)\varphi_0 = 0$. Consider the vector-valued function $u = W\varphi_0$. Since u is a solution of (2.3) with zero boundary conditions, it follows that $u = 0$ in G^- by the uniqueness property of (2.3). Using

$$(\mathcal{N}W\varphi_0)^+ = (\mathcal{N}W\varphi_0)^- \qquad \text{on} \quad \Gamma\backslash 0$$

and that (2.4) is uniquely solvable, we conclude that $W\varphi_0 = 0$ in G^-. Thus,

$$\varphi_0 = (W\varphi_0)^+ - (W\varphi_0)^- = 0,$$

which completes the proof.

Theorem 5 *Let $T^* = 2W_0^*$,*

$$|\beta - \alpha - 1/2| < 1/2 + \min\{0, \nu^-\},$$

and let l be a positive integer. Then the equation

$$(1+T^*)\psi = g \tag{2.27}$$

is uniquely solvable in the space $N^{l-1,\alpha}_{\beta+l}(\Gamma)$ for all $g \in N^{l-1,\alpha}_{\beta+l}(\Gamma)$ and

$$(1+T^*)^{-1}g = \frac{1}{2}(1 - \mathcal{N}P^+Q^-)\, g. \tag{2.28}$$

Proof. From Theorems 1 and 2 it follows that

$$\psi = 1/2(1 - \mathcal{N}P^+Q^-)\, g \in N^{l-1,\alpha}_{\beta+l}(\Gamma).$$

Let v be a solution of the problem

$$\Delta^* v = 0 \quad \text{in } G, \quad v^+ - v^- = 0 \ \text{ on } \Gamma, \quad (\mathcal{N}v)^- = g \ \text{ on } \Gamma\backslash 0$$

from the class $N^{l,\alpha}_{\beta+l}(G)$. It is clear that

$$(\mathcal{N}v)^+ = \mathcal{N}P^+Q^-\, g \qquad \text{on} \quad \Gamma\backslash 0$$

and

$$\psi = 1/2((\mathcal{N}v)^- - (\mathcal{N}v)^+).$$

Thus, by Lemmas 4 and 5 we get

$$(1 + T^*)\psi = -2(\mathcal{N}V\psi)^- = (\mathcal{N}V((\mathcal{N}v)^+ - (\mathcal{N}v)^-))^- = (\mathcal{N}v)^- = g.$$

It remains to verify the uniqueness of a solution of (2.27). Let $\psi_0 \in N^{l-1,\alpha}_{\beta+l}(\Gamma)$ satisfy the equation $(1 + T^*)\psi_0 = 0$. Since $v = V\psi_0$ solves (2.4) for $g = 0$, it follows that $v = 0$ in G^-. From the equality $(V\psi_0)^* = (V\psi_0)^-$ and from the unique solvability of (2.3) we conclude that $v = 0$ in G^+. Thus,

$$\psi_0 = (\mathcal{N}V\psi_0)^+ - (\mathcal{N}V\psi_0)^- = 0.$$

The proof is complete.

Lemma 8 *Let $|\beta - \alpha - 1/2| < 1/2 + \min\{0, \nu^-\}$ and let l be a positive integer. Then the operators $(1 + T)^{-1}$ and $(1 + T^*)^{-1}$ are continuous in the spaces $N^{l,\alpha}_{\beta+l}(\Gamma)$ and $N^{l-1,\alpha}_{\beta+l}(\Gamma)$, respectively.*

The assertions of this lemma are corollaries of representations (2.26), (2.28) and Theorems 1 and 2.

Lemma 9 *Let the cone K^+ be explicitly described in a Cartesian coordinate system. If $0 < \alpha - \beta < \min\{\delta^+, \nu^-\}$, then the operators $(1 + T)^{-1}$ and $(1 + T^*)^{-1}$ are continuous in the spaces $C^{l,\alpha}_{\beta+l}(\Gamma)$ and $N^{l-1,\alpha}_{\beta+l}(\Gamma)$, respectively.*

The assertion of this lemma is a corollary of representations (2.26), (2.28) and of Lemmas 1 and 4.

3 Estimates for the kernel of the operator $(1 + T)^{-1}$

In what follows we assume that the cone K^+ admits an explicit description in a Cartesian coordinate system. The purpose of this section is to prove the following assertion.

Theorem 6 *Let $0 < \beta - \alpha < 1$ and let l be a positive integer. Then*

$$(1 + T)^{-1} f = (1 + L) f, \qquad f \in N^{l,\alpha}_{\beta+l}(\Gamma). \tag{3.1}$$

Here L is an integral operator on $\Gamma\backslash 0$ with a kernel $\mathcal{L}(x, y)$ satisfying the estimates

$$|\mathcal{L}(x,y)| \le \begin{cases} c\,|y|^{-2}(|x|/|y|)^{\varkappa-\varepsilon} + c\,|y|^{\varkappa-1-\varepsilon}, & |x| < |y|/2, \\ c\,|y|^{-1}|x-y|^{-1}, & |y|/2 < |x| < 2|y|, \\ c\,|x|^{-1}|y|^{-1}(|y|/|x|)^{\varkappa-\varepsilon}, & |x| > 2\,|y|, \end{cases} \tag{3.2}$$

where $\varkappa = \min\{\delta^+, \nu^-\}$ and ε is any sufficiently small positive number.

In what follows by $\{\chi_k\}_{k=1}^3$, η_1, η_2, and $\zeta_\varepsilon^{(j)}$ we mean functions in $C^\infty([0,\infty))$ such that

(1) $$\sum_{1\le k\le 3} \chi_k = 1, \quad \operatorname{supp}\chi_1 \subset [0,5/8), \quad \operatorname{supp}\chi_2 \subset (1/2,2), \quad \chi_3 \subset (8/5,\infty);$$

(2) $\eta_1(t) = 1$ for $t < 1/8$ and $\eta_1(t) = 0$ for $t \ge 1/4$;

(3) $\eta_2(t) = 1$ for $t < 5/6$ and $\eta_2(t) = 0$ for $t \ge 6/7$;

(4) $\zeta_\varepsilon^{(j)}(t) = 1$ for $(2+j\varepsilon)^{-1} < t < 2+j\varepsilon$ and $\zeta_\varepsilon^{(j)}(t) = 0$ outside of the set $(2+(j+1)\varepsilon)^{-1} < t < 2+(j+1)\varepsilon$.

3.1 Estimates for the kernel $\mathcal{L}(x,y)$ with $|y| < 5|x|/8$

Given $x \in \Gamma\backslash 0$, consider the Dirichlet problem

$$\begin{aligned} \Delta_y^* \mathcal{R}^+(y,x) &= 0 \quad \text{in } G^+ \\ \mathcal{R}^+(y,x) &= \eta_2(|y|/|x|)\mathcal{R}^-(y,x) \quad \text{on } \Gamma. \end{aligned} \tag{3.3}$$

Lemma 10 *Let $|\delta-\alpha-1/2| < 1/2+\varkappa$ and let l be a positive integer. Then problem (3.3) is uniquely solvable in the space $N_{\delta+l}^{l,\alpha}(\Gamma)$ for each $x \in \Gamma\backslash 0$ and*

$$|\partial_y^\tau \mathcal{R}^+(y,x)| \le c_\tau\, |x|^{-1}|y|^{-|\tau|}(|y|/|x|)^{\varkappa-\varepsilon}. \tag{3.4}$$

Proof. We set $x = |x|X$, $y = |x|Y$ and let $G_{|x|}$ and $\Gamma_{|x|}$ be the images of the sets G^+ and Γ under the mapping $y \to Y$. The function

$$Y \to \mathcal{R}_{|x|}(Y,X) = |x|\mathcal{R}^+(|x|Y,|x|X)$$

is the solution of the problem

$$\begin{aligned} \Delta_Y^* \mathcal{R}_{|x|}(Y,X) &= 0 \quad \text{in } G_{|x|} \\ \mathcal{R}_{|x|}(Y,X) &= |x|\eta_2(|Y|/|X|)\mathcal{R}^-(|x|Y,|x|X) \quad \text{on } \Gamma_{|x|}. \end{aligned} \tag{3.5}$$

for all X with $|X| = 1$.

By (2.11) we have

$$|\partial_Y^\tau(|x|\eta_2(|Y|/|X|)\mathcal{R}^-(|x|Y,|x|X) \le c\,|Y|^{\varkappa-|\tau|-\varepsilon/2},$$

with a constant c independent of x. Thus

$$\|\,|x|\eta_2\mathcal{R}^-\|_{N^{l,\alpha}_{l+\alpha-\varkappa+\varepsilon}(\Gamma_{|x|})} \le C.$$

Applying Theorem 1 to problem (3.5), we get

$$\|\mathcal{R}_{|x|}(\cdot,X)\|_{N^{l,\alpha}_{l+\alpha-\varkappa+\varepsilon}(G^+_{|x|})} \le C.$$

Hence

$$|\partial_Y^\tau \mathcal{R}_{|x|}(Y,X)| \le c_\tau\,|Y|^{\varkappa-|\tau|-\varepsilon}.$$

Returning back to the variables x and y, we arrive at (3.4). The lemma is proved.

Lemma 11 *Let $|\beta - \alpha - 1/2| < 1/2$ and let l be a positive integer. For any $\varphi \in N^{l,\alpha}_{\beta+l}(\Gamma)$ the representation*

$$\int_\Gamma \mathcal{Q}^-(x,\xi)\big(\mathcal{N}(\partial_\xi, n_\xi)\int_\Gamma \mathcal{P}^+(\xi,y)\chi_1(|y|/|x|)\varphi(y)ds_y\big)ds_\xi$$
$$= \int_\Gamma \mathcal{L}(x,y)\chi_1(|y|/|x|)\varphi(y)ds_y \tag{3.6}$$

is valid, where $x \in \Gamma\backslash 0$ and

$$|\mathcal{L}(x,y)| \le c\,|x|^{-1}|y|^{-1}(|y|/|x|)^{\varkappa-\varepsilon}. \tag{3.7}$$

Proof. Setting

$$v(\xi) = \int_\Gamma \mathcal{P}^+(\xi,y)\chi_1(|y|/|x|)\varphi(y)ds_y$$

and substituting the solution of (3.3) into (2.9), we rewrite the left-hand side of (3.6) in the form

$$\int_\Gamma (\eta_2(|\xi|/|x|)\mathcal{Q}^-(0,x)^T\mathcal{N}v(\xi)ds_\xi + \int_\Gamma (\mathcal{R}^+(\xi,x))^T\mathcal{N}v(\xi)ds_\xi$$
$$+ \int_\Gamma (1-\eta_2(|\xi|/|x|)\mathcal{Q}^-(x,\xi)\mathcal{N}v(\xi)dx_\xi.$$

Applying the Betti formula

$$\int_{G^+}\big((\Delta^* u)^T v - u^T\Delta^* v\big)dx = \int_\Gamma \big((\mathcal{N}u)^T v - u^T\mathcal{N}v\big)ds$$

to the first and second integrals, we arrive at representation (3.6) with

$$\mathcal{L}(x,y) = \sum_{1\le k\le 3}\mathcal{L}_k(x,y), \tag{3.8}$$

where

$$\mathcal{L}_1(x,y) = (\mathcal{N}(\partial_y,n_y)\mathcal{R}^+(y,x))^T,$$
$$\mathcal{L}_2(x,y) = -\int_{G^+}(\Delta^*_\xi\eta_2(|\xi|/|x|)\mathcal{Q}^-(0,x))^T\mathcal{P}^+(\xi,y)d\xi,$$
$$\mathcal{L}_3(x,y) = -\int_\Gamma (1-\eta_2(|\xi|/|x|))\mathcal{Q}^-(x,\xi)\mathcal{N}(\partial_\xi,n_\xi)\mathcal{P}^+(\xi,y)d\xi.$$

Here we used the relations

$$\Delta^* v = 0 \quad \text{in } G^+, \quad v(\xi) = \chi_1(|\xi|/|x|)\,\varphi(\xi) \quad \text{for } \xi\in\Gamma$$

and the fact that $\chi_1(|\xi|/|x|) = 0$ on the support of the function

$$\xi \to \mathcal{N}(\partial_\xi,n_\xi)\eta_2(|\xi|/|x|).$$

Next we estimate each term in (3.8). Inequality (3.7) for $\mathcal{L}_1(x,y)$ follows directly from Lemma 10. Let us estimate $\mathcal{L}_2(x,y)$. It is clear that $|\xi| > 5/6|x|$ on the support of the function $\xi \to \Delta_\xi^* \eta_2(|\xi|/|x|)$. From this and from the inequality $|y| < 5/8|x|$ we conclude that $|\xi| > 4/3|y|$. Hence, by estimates (2.6) and by (2.10) we have

$$|\mathcal{L}_2(x,y)| \le c \int_{\xi \in G^+ : |\xi| > 5/6|x|} |\xi|^{-3}|x|^{-1}|y|^{-1}(|y|/|\xi|)^{\delta^+ - \varepsilon} d\xi$$
$$\le c|x|^{-1}|y|^{-1}(|y|/|x|)^{\delta^+ - \varepsilon}.$$

Finally, in order to obtain the required estimate for $\mathcal{L}_3(x,y)$, we write it as the sum of two integrals over the sets

$$\Gamma_1 = \{\xi \in \Gamma : |\xi| < 2\,|x|\} \qquad \text{and} \qquad \Gamma_2 = \{\xi \in \Gamma : |\xi| > 2\,|x|\}.$$

It remains to use (2.6), (2.8) and representation (2.10) to get:

$$|\mathcal{L}_3(x,y)| \le c \int_{\substack{\xi \in \Gamma : |\xi| > 5/6|x| \\ |x-\xi| < 3|x|}} |x-\xi|^{-1}|\xi|^{-2}|y|^{-1}(|y|/|\xi|)^{\delta^+ - \varepsilon} ds_\xi$$

$$+ c \int_{\xi \in \Gamma : |\xi| > 2|x|} |\xi|^{-3}|y|^{-1}(|y|/|\xi|)^{\delta^+ - \varepsilon} ds_\xi \le c\,|x|^{-1}|y|^{-1}(|y|/|x|)^{\delta^+ - \varepsilon}.$$

The lemma is proved.

3.2 Estimates for the kernel $\mathcal{L}(x,y)$ with $|y| > 8|x|/5$

Consider two Dirichlet problems

$$\Delta_y^* \mathcal{R}^+(y,x) = 0 \quad \text{in } G^+ \qquad \mathcal{R}^+(y,x) = \eta_2(|x|/|y|)(\mathcal{R}^-(y,x))^T \quad \text{on } \Gamma, \tag{3.9}$$

$$\Delta^* \mathcal{C}^+ = 0 \quad \text{in } G^+ \qquad \mathcal{C}^+ = \mathcal{C}^- \quad \text{on } \Gamma, \tag{3.10}$$

where x is a fixed point on Γ.

Lemma 12 *Let $|\delta - \alpha - 1/2| < 1/2 + \varkappa$ and let l be a positive integer. Then problems (3.9) and (3.10) are uniquely solvable in the space $N_{\delta+l}^{l,\alpha}(G^+)$ for each $x \in \Gamma$ and*

$$|\partial_y^\sigma \mathcal{R}^+(y,x)| \le c_\sigma\,|y|^{-1-|\sigma|}(|x|/|y|)^{\varkappa - \varepsilon}, \tag{3.11}$$

$$|\partial_y^\sigma \mathcal{C}^+(y)| \le c_\sigma\,|y|^{\varkappa - |\sigma| - \varepsilon}. \tag{3.12}$$

Proof. Inequality (3.12) is an immediate corollary of Theorem 1 and inequality (3.11) is proved in a similar manner as Lemma 10.

Lemma 13 *Let $0 < \beta - \alpha < 1$ and let l be a positive integer. For any $\varphi \in N_{\beta+l}^{l,\alpha}(\Gamma)$*

$$\begin{aligned} &\int_\Gamma \mathcal{Q}^-(x,\xi)\big(\mathcal{N}(\partial_\xi, n_\xi) \int_\Gamma \mathcal{P}^+(\xi,y)\chi_3(|y|/|x|)\varphi(y) ds_y\big) ds_\xi \\ &= \int_\Gamma \mathcal{L}(x,y)\chi_3(|y|/|x|)\varphi(y) ds_y, \end{aligned} \tag{3.13}$$

where $x \in \Gamma \backslash 0$ and

$$|\mathcal{L}(x,y)| \le c\,(|y|^{\varkappa - 1 - \varepsilon} + |y|^{-2}(|x|/|y|)^{\varkappa - \varepsilon}). \tag{3.14}$$

Proof. Setting

$$v(\xi)=\mathcal{P}^+(\xi,y)\chi_3(|y|/|x|)\varphi(y)ds_y$$

and substituting the solutions of (3.9), (3.10) into (2.9), (2.10), we write the left-hand side of (3.13) in the form

$$\int_\Gamma (\eta_2(|x|/|\xi|)\big(|\xi|^{-1}\mathcal{A}^-(\xi/|\xi|)+\mathcal{B}^-+\mathcal{D}^+(\xi)\big)^T\mathcal{N}v(\xi)ds_\xi$$
$$+\int_\Gamma(\mathcal{R}^+(\xi,x))^T\mathcal{N}v(\xi)ds_\xi+\int_\Gamma(1-\eta_2(|x|/|\xi|)\mathcal{Q}^-(x,\xi)\mathcal{N}v(\xi)dx_\xi.$$

By Lemma 3, the expression $|\xi|^{-1}\mathcal{A}^-(\xi/|\xi|)$ is defined in G^+. Applying Betti's formula, we arrive at representation (3.13) with

$$\mathcal{L}(x,y)=\sum_{1\le k\le 4}\mathcal{L}_k(x,y), \tag{3.15}$$

where

$$\mathcal{L}_1(x,y)=\eta_2\Big(\frac{|x|}{|y|}\Big)\mathcal{N}(\partial_y,n_y)\Big(|y|^{-1}\mathcal{A}^-\Big(\frac{y}{|y|}\Big)+\mathcal{D}^+(y)\Big)^T,$$
$$\mathcal{L}_2(x,y)=-\int_{G^+}(\Delta^*_\xi\eta_2\Big(\frac{|x|}{|\xi|}\Big)\Big(|\xi|^{-1}\mathcal{A}^-\Big(\frac{\xi}{|\xi|}\Big)+\mathcal{B}^-+\mathcal{D}^+(\xi)\Big)^T\mathcal{P}^+(\xi,y)d\xi,$$
$$\mathcal{L}_3(x,y)=(\mathcal{N}(\partial_y,n_y)\mathcal{R}^+(y,x))^T,$$
$$\mathcal{L}_4(x,y)=-\int_\Gamma\Big(1-\eta_2\Big(\frac{|x|}{|\xi|}\Big)\Big)\mathcal{Q}^-(x,\xi)\mathcal{N}(\partial_\xi,n_\xi)\mathcal{P}^+(\xi,y)ds_\xi.$$

In order to prove the assertion of Lemma, it suffices to show that (3.14) holds for each $\mathcal{L}_k(x,y)$. From Lemma 3 it follows that

$$\mathcal{N}(\partial_y,n_y)\big(|y|^{-1}\mathcal{A}^-(y/|y|)=0$$

at the boundary of the cone K^+, $y\neq 0$. This and (3.12) imply (3.14) for $\mathcal{L}_1(x,y)$.

Furthermore, since

$$\Delta^*\big(|\xi|^{-1}\mathcal{A}^-(\xi/|\xi|)+\mathcal{B}^-+\mathcal{D}^+(\xi)\big)=0$$

for $\xi\in G^+$ and since $|y|>4/3$ on the domain of integration of $\mathcal{L}_2(x,y)$, it follows by (2.6) that

$$|\mathcal{L}_3(x,y)|\le c\int_{\xi\in\Gamma:7|x|/6<|\xi|<6|x|/5}|\xi|^{-3}|y|^{-2}(|\xi|/|y|)^{\delta^+-\varepsilon}d\xi$$
$$\le c|y|^{-2}(|x|/|y|)^{\delta^+-\varepsilon}.$$

The required estimate for $\mathcal{L}_3(x,y)$ is a direct corollary of (3.11).

Dividing the domain of integration of $\mathcal{L}_4(x,y)$ onto the sets

$$\Gamma_1=\{\xi\in\Gamma:|x|/2<|\xi|<6|x|/5\}\qquad\text{and}\qquad\Gamma_2=\{\xi\in\Gamma:|\xi<|x|/2\}$$

and taking into account that $|y|>4|\xi|/3$ on $\Gamma_1\cap\Gamma_2$, using (2.6), (2.8) and representations (2.5), (2.10), we arrive at

$$|\mathcal{L}_4(x,y)|\le c\int_{\Gamma_1}|x-\xi|^{-1}|y|^{-2}|\xi|^{-1}\Big(\frac{|\xi|}{|y|}\Big)^{\delta^+-\varepsilon}ds_\xi$$
$$+\int_{\Gamma_2}|\xi|^{-2}|y|^{-2}\Big(\frac{|\xi|}{|y|}\Big)^{\delta^+-\varepsilon}ds_\xi\le c\,|y|^{-2}|\xi|^{-1}\Big(\frac{|x|}{|y|}\Big)^{\delta^+-\varepsilon}.$$

The result follows.

3.3 Estimates for the kernel $\mathcal{L}$ with $|y|/2 < |x| < 2|y|$

The purpose of this subsection is to prove the following assertion.

Lemma 14 *Let $0 < \beta - \alpha < 1$ and let l be a positive integer. For any $\varphi \in N^{l,\alpha}_{\beta+l}(\Gamma)$*

$$\begin{aligned}&\int_\Gamma \mathcal{Q}^-(x,\xi)\big(\mathcal{N}(\partial_\xi, n_\xi)\int_\Gamma \mathcal{P}^+(\xi,y)\chi_2(|y|/|x|)\varphi(y)ds_y\big)ds_\xi\\ &= -\varphi(x) + \int_\Gamma \mathcal{L}(x,y)\zeta_1^{(8)}(|y|/|x|)\varphi(y)ds_y,\end{aligned} \tag{3.16}$$

where $x \in \Gamma\backslash 0$ and

$$|\mathcal{L}(x,y)| \le c\,|y|^{-1}|x-y|^{-1}. \tag{3.17}$$

First we prove several auxiliary assertions. Let D^+ be a bounded open set in $\mathbb{R}^3$ with smooth boundary Γ^+ which coincides with the cone K^+ for $1/18 < |x| < 18$. We assume that $0 \notin \Gamma^+$ and that $D^+ \subset G^+$. Let D^+_ρ denote the set $\{x \in \mathbb{R}^3 : x/\rho \in D^+\}$, $\rho > 0$, and let Γ_ρ be the boundary of D^+_ρ. We introduce the operator T_ρ in the space $C(\Gamma_\rho)$ of continuous functions defined by the equality $T_\rho\varphi = 2W_0\varphi$, where $W_0\varphi$ is the direct value of the double layer potential with density φ on the surface Γ_ρ.

Lemma 15 *Let ρ be a positive number such that the domain G^+ coincides with the cone K^+ for $|x| < 18\rho$. Then*

$$|\partial_x^\sigma\partial_y^\tau(\mathcal{P}^+(x,y) - \mathcal{P}^+_\rho(x,y)| \le c_{\sigma\tau}\rho^{-2-|\tau|-|\sigma|}, \tag{3.18}$$

$$|\partial_x^\sigma\partial_y^\tau(\mathcal{Q}^-(x,y) - \mathcal{Q}^-_\rho(x,y)| \le c_{\sigma\tau}\rho^{-1-|\tau|-|\sigma|}, \tag{3.19}$$

with $\rho/16 < |x| < 16\rho$, $\rho/16 < |y| < 16\rho$, where $\mathcal{P}^+_\rho(x,y)$ and $\mathcal{Q}^-_\rho(x,y)$ are kernels of the integral operators P^+_ρ and Q^-_ρ on Γ_ρ such that the vector-valued function $u = P^+_\rho f$ and $v = Q^-_\rho g$ are solutions of the boundary value problems

$$\Delta^* u = 0 \quad \text{in } D^+_\rho, \qquad u = f \quad \text{on } \Gamma_\rho, \tag{3.20}$$

$$\Delta^* u = 0 \quad \text{in } \mathbb{R}^3\backslash\overline{D^+_\rho}, \qquad \mathcal{N}v = g \quad \text{on } \Gamma_\rho. \tag{3.21}$$

Proof. Since inequalities (3.18) and (3.19) are proved in a similar way, we limit ourselves to (3.18). Let $\mathcal{G}(x,y)$ be Green's kernel of the operator (2.3), i.e. it solves the problem

$$\begin{aligned}&\delta_x^*\mathcal{G}(x,y) = \delta(x-y)\,1, \quad x,y \in G^+,\\ &\mathcal{G}(x,y) = 0, \qquad x \in \Gamma, y \in G^+.\end{aligned} \tag{3.22}$$

Here $\delta(x,y)$ is the Dirac measure concentrated at the point y and 1 is the unit matrix of order 3. Inequalities (3.22) are understood in the sense that for all vector-valued functions $v \in N^{l,\alpha}_{\beta+l}(G^+)$ we have

$$v(y) = \int_{G^+}(\mathcal{G}(x,y))^T\Delta^* v(x)dx + \int_\Gamma \big(\mathcal{N}(\partial_x, n_x)\,\mathcal{G}(x,y)\big)^T v(x)ds_x.$$

We write $\mathcal{G}(x,y)$ in the form

$$\mathcal{G}(x,y) = \zeta_i^{(15)}(|x|/\rho)\mathcal{G}_\rho(x,y) + \tilde{\mathcal{G}}(x,y), \tag{3.23}$$

where $\mathcal{G}_\rho(x, y)$ is Green's kernel of problem (3.20).

We put $x = \rho X$, $y = \rho Y$ and let G_ρ^+ stand for the image of the set G^+ under the mapping $x \to X$. Further, let Γ_ρ be the boundary of G^+. It is clear that $\mathcal{G}_\rho(x, y) = \rho^{-1}\mathcal{G}_1(X, Y)$. It follows from (3.23) that the function w_ρ defined by

$$w_\rho(X, Y) = \partial_Y^\sigma(\rho\,\tilde{\mathcal{G}}(\rho X, \rho Y)),$$

is a solution of the boundary value problem

$$\begin{aligned} &\Delta_x^* w_\rho(X, Y) = \psi(X, Y), \quad X \in G_\rho, \\ &w_\rho(X, Y) = f(X, Y), \qquad X \in \Gamma_\rho \end{aligned} \tag{3.24}$$

for $Y \in G_\rho^+$, $1/16 < |Y| < 16$, where f and ψ are smooth functions with support in $\{x : 1/18 < |x| < 18\}$.

From [ADN] it follows that a solution of (3.24) obeys the estimate

$$|\partial_X^\tau w_\rho(X, Y)| \le c_\tau.$$

It is clear that this inequality is equivalent to

$$|\partial_x^\tau \partial_y^\sigma \tilde{\mathcal{G}}(x, y)| \le c_{\sigma\tau}\,\rho^{-1-|\tau|-|\sigma|}.$$

Hence, using (3.23) and the relations

$$\mathcal{P}^+(x, y) = \big(\mathcal{N}(\partial_y, n_y)\,\mathcal{G}(y, x)\big)^T, \qquad \mathcal{P}_\rho^+(x, y) = \big(\mathcal{N}(\partial_y, n_y)\,\mathcal{G}_\rho(y, x)\big)^T,$$

we arrive at (3.18).

Lemma 16 *Let $\varphi \in C(\Gamma_\rho)$. Then*

$$(1 + T_\rho)^{-1}\varphi = (1 + H_\rho)\varphi, \tag{3.25}$$

where H_ρ is an integral operator on Γ_ρ with the kernel $\mathcal{H}_\rho(x, y)$ satisfying the estimate

$$|\mathcal{H}_\rho(x, y)| \le c\,\rho^{-1}|x - y|^{-1}.$$

Proof. Let $\rho = 1$. It is known (see [Ku], [KGBB]) that the kernel $\mathcal{T}_1(x, y)$ of the operatorT_1 obeys the estimate

$$|\mathcal{T}_1(x, y)| \le c\,|x - y|^{-1}.$$

Hence the kernel of the integral operator T_1^3 is bounded. Noting that the solution u of the equation $(1 + T_1)u = \varphi$ is a solution of the equation $(1 + T_1^3)u = \psi$ with $\psi = (1 - T_1 + T_1^2)\varphi$ and using the fact that $\gamma = -1$ is not in the spectrum of T_1^3, we arrive at the assertion of the present lemma for $\rho = 1$. The estimate (3.25) for an arbitrary $\rho > 0$ follows directly from the estimate for $H_1(x, y)$ and the equality

$$H_\rho(x, y) = \rho^{-2} H_1(x/\rho, y/\rho).$$

Corollary 1 *Let $\varphi \in C^1(\Gamma_\rho)$. Then*

$$Q_\rho^- \mathcal{N}\mathcal{P}_\rho^+\,\varphi = -(1 + 2H_\rho)\,\varphi.$$

Proof. The argument used in the proof of Theorem 4 shows that the equality

$$(1+T_\rho)^{-1}\varphi = 1/2(1-Q_\rho^- \mathcal{N}\mathcal{P}_\rho^+)\,\varphi$$

holds for all $\varphi \in C^1(\Gamma_\rho)$. Here we have used the classic solvability theorems for problems (3.20) and (3.21) (see [Ku], [KGBB]). Comparing the above equality with (3.24), we arrive at the required assertion.

Lemma 17 *Let $x \in \Gamma \cap U_k$, $U_k = \{\xi : 1/2 < 2^k|\xi| < 2\}$. Suppose that the domain G^+ coincides with the cone K^+ for all ξ with $|\xi| < 18\rho$, $\rho = 2^{-k}$. Then the problem*

$$\begin{aligned}
&\Delta_y^* \psi^+(x,y) = 0 \quad \text{in } G^+, \\
&\psi^+(x,y) = \zeta_1^{(1)}(|x|/|y|)\big(\mathcal{Q}^-(x,y) - \mathcal{Q}_\rho^-(x,y)\big)^T \quad \text{on } \Gamma
\end{aligned}$$

is uniquely solvable in $N_{\delta+l}^{l,\alpha}(G^+)$ for

$$|\delta - \alpha - 1/2| < 1/2 + \delta^+, \qquad l = 1, 2, \ldots$$

and the estimate

$$|\partial_y^\sigma \psi^+(x,y)| \le c_\sigma\, \rho^{-1-|\sigma|} \tag{3.26}$$

holds for all y with $d\,|x| < |y| < d^{-1}|x|$, $d \in (0,1)$.

Proof. It is obvious that $\rho/8 < |y| < 8\rho$ on the support of the function $y \to \zeta_1^{(1)}(|x|/|y|)$. The argument used in the proof of Lemma 15 shows that (3.26) follows from (3.19) and from local estimates of solutions to boundary value problems for elliptic differential equations near a smooth part of the boundary.

Proof of Lemma 14. Let $x \in U_k \cap \Gamma$, where $U_k = \{\xi : 1/2 < 2^k|\xi| < 2\}$. We assume that the number k is so large that the domain G^+ coincides with the cone K^+ for all x with $|x| < 18\rho$, $\rho = 2^{-k}$. We write the left-hand side of (3.16) in the form

$$\begin{aligned}
&\int_\Gamma \zeta_1^{(1)}(|x|/|y|)\mathcal{Q}_\rho^-(x,\xi)\mathcal{N}\, v(\xi)\, ds_\xi + \int_\Gamma \big(\psi^+(x,\xi)\big)^T \mathcal{N}\, v(\xi)\, ds_\xi \\
&+ \int_\Gamma (1-\zeta_1^{(1)}(|x|/|\xi|))\mathcal{Q}^-(x,\xi)\mathcal{N}\, v(\xi)\, ds_\xi,
\end{aligned} \tag{3.27}$$

where

$$v(\xi) = \int_\Gamma \mathcal{P}^+(\xi,y)\chi_2(|y|/|x|)\,\varphi(y)\, ds_y.$$

Noting that the equality

$$\eta_1\Big(\frac{|\xi-y|}{|y|}\Big)\zeta_1^{(1)}\Big(\frac{|x|}{|\xi|}\Big) = \eta_1\Big(\frac{|\xi-y|}{|y|}\Big)$$

holds on the support of the function $y \to \chi_2(|y|/|x|)$ and applying Betti's formula to the second integral in (3.27), we express (3.27) in the form

$$
\begin{aligned}
&\int_\Gamma \mathcal{Q}_\rho^-(x,\xi)\mathcal{N}(\partial_\xi, n_\xi) \int_\Gamma \mathcal{P}_\rho^+(\xi,y)\,\eta_1\Big(\frac{|y-\xi|}{|y|}\Big)\chi_2\Big(\frac{|y|}{|x|}\Big)\,\varphi(y)\,ds_y\,ds_\xi \\
&+\int_\Gamma \mathcal{Q}_\rho^-(x,\xi)\mathcal{N}(\partial_\xi, n_\xi) \int_\Gamma (\mathcal{P}_\rho^+(\xi,y)-\mathcal{P}_\rho^+(\xi,y))\,\eta_1\Big(\frac{|y-\xi|}{|y|}\Big)\chi_2\Big(\frac{|y|}{|x|}\Big)\,\varphi(y)\,ds_y\,ds_\xi \\
&+\int_\Gamma \zeta_1^{(1)}\Big(\frac{|x|}{|\xi|}\Big)\;\mathcal{Q}_\rho^-(x,\xi)\mathcal{N}(\partial_\xi, n_\xi)\int_\Gamma \mathcal{P}^+(\xi,y)\Big(1-\eta_1\Big(\frac{|y-\xi|}{|y|}\Big)\Big)\chi_2\Big(\frac{|y|}{|x|}\Big)\,\varphi(y)\,ds_y ds_\xi \\
&+\int_\Gamma \big(\mathcal{N}(\partial_\xi, n_\xi)\psi^+(x,\xi)\big)^T\chi_2\Big(\frac{|\xi|}{|x|}\Big)\,\varphi(\xi)\,ds_\xi \qquad (3.28)\\
&+\int_\Gamma\int_\Gamma \Big(1-\zeta_1^{(1)}\Big(\frac{|x|}{|\xi|}\Big)\Big)\mathcal{Q}^-(x,\xi)\mathcal{N}(\partial_\xi, n_\xi)\mathcal{P}^+(\xi,y)\chi_2\Big(\frac{|y|}{|x|}\Big)\,\varphi(y)\,ds_y\,ds_\xi.
\end{aligned}
$$

Since

$$\chi_2\Big(\frac{|y|}{|x|}\Big)\zeta_1^{(2)}\Big(\frac{|y|}{\rho}\Big) = \chi_2\Big(\frac{|y|}{|x|}\Big),$$

we write the first integral in (3.28) as

$$
\begin{aligned}
&\int_{\Gamma_\rho} \mathcal{Q}_\rho^-(x,\xi)\mathcal{N}(\partial_\xi, n_\xi)\int_{\Gamma_\rho} \mathcal{P}_\rho^+(\xi,y)\,\zeta_1^{(2)}\Big(\frac{|y|}{\rho}\Big)\varphi(y)\,ds_y\,ds_\xi \\
&-\int_{\Gamma_\rho} \mathcal{Q}_\rho^-(x,\xi)\mathcal{N}(\partial_\xi, n_\xi)\int_\Gamma \mathcal{P}_\rho^+(\xi,y)\Big(1-\eta_1\Big(\frac{|y-\xi|}{|y|}\Big)\Big)\zeta_1^{(2)}\Big(\frac{|y|}{\rho}\Big)\,\varphi(y)\,ds_y\,ds_\xi \\
&\qquad (3.29)\\
&-\int_\Gamma \mathcal{Q}_\rho^-(x,\xi)\mathcal{N}(\partial_\xi, n_\xi)\int_\Gamma \mathcal{P}_\rho^+(\xi,y)\,\eta_1\Big(\frac{|y-\xi|}{|y|}\Big)\Big(1-\chi_2\Big(\frac{|y|}{|x|}\Big)\Big)\,\zeta_1^{(2)}\Big(\frac{|y|}{\rho}\Big)\varphi(y)\,ds_y ds_\xi
\end{aligned}
$$

Given any $x \in U_k \cap \Gamma$, let $\mathcal{Q}_\rho^+$ be a solution of the problem

$$\Delta_\xi^* \mathcal{Q}_\rho^+(x,\xi) = 0 \quad \text{in } D_\rho, \qquad \mathcal{Q}_\rho^+(x,\xi) = \Big(1-\eta_1\Big(\frac{|x-\xi|}{|x|}\Big)\Big)\big(\mathcal{Q}_\rho^-(x,\xi)\big)^T \quad \text{on } \Gamma.$$

Clearly,

$$|\partial_\xi^\sigma \mathcal{Q}_\rho^+(x,\xi)| \le c_\sigma \rho^{-1-|\sigma|}. \qquad (3.30)$$

Since $\eta_1(|x-\xi|/|x|) = 0$ on the support of the function

$$\xi \to \eta_1\Big(\frac{|y-\xi|}{|y|}\Big)\Big(1-\chi\Big(\frac{|y|}{|x|}\Big)\Big)\zeta_1^{(2)}\Big(\frac{|y|}{\rho}\Big),$$

we can replace $\mathcal{Q}_\rho^-(x,\xi)$ in the last term of (3.29) by $\big(\mathcal{Q}_\rho^+(x,\xi)\big)^T$. Applying Betti's formula to this term and using Corollary 1, we write (3.29) in the form

$$
\begin{aligned}
&-\varphi(x) - 2\int_\Gamma \mathcal{H}(x,y)\zeta_1^{(1)}\Big(\frac{|y|}{\rho}\Big)\,\varphi(y)\,ds_y \\
&-\int_\Gamma\int_{\Gamma_\rho} \mathcal{Q}_\rho^-(x,\xi)\mathcal{N}(\partial_\xi, n_\xi)\mathcal{P}_\rho^+(\xi,y)\Big(1-\eta_1\Big(\frac{|y-\xi|}{|y|}\Big)\Big)\zeta_1^{(2)}(|y|/\rho)\,\varphi(y)\,ds_y\,ds_\xi \\
&-\int_\Gamma \big(\mathcal{N}(\partial_\xi, n_\xi)\mathcal{Q}^- + \rho(x,\xi)\big)^T\Big(1-\chi_2\Big(\frac{|\xi|}{|x|}\Big)\Big)\zeta_1^{(2)}(|\xi|/\rho)\,\varphi(\xi)\,ds_\xi \\
&-\int_{G^+} \big(\mathcal{Q}_\rho^+(x,\xi)\big)^T\Delta_\xi^*\int_\Gamma \mathcal{P}_\rho^+(\xi,y)\eta_1\Big(\frac{|y-\xi|}{|y|}\Big)\Big(1-\chi_2\Big(\frac{|y|}{|x|}\Big)\Big)\zeta_1^{(2)}\Big(\frac{|y|}{\rho}\Big)\,\varphi(y)ds_y ds_\xi.
\end{aligned}
$$

Combining this with (3.28) and using the equality

$$\zeta_1^{(8)}\Big(\frac{|y|}{|x|}\Big)\,\zeta_1^{(2)}\Big(\frac{|y|}{\rho}\Big) = \zeta_1^{(2)}\Big(\frac{|y|}{\rho}\Big),$$

valid for $x \in U_k \cup \Gamma$, we arrive at (3.15), where

$$\mathcal{L}(x,y) = \sum_{k=1}^{8} \mathcal{L}_k(x,y) \tag{3.31}$$

$$\mathcal{L}_1(x,y) = -2\,\mathcal{H}_\rho(x,y)\zeta_1^{(2)}\Big(\frac{|y|}{\rho}\Big),$$

$$\mathcal{L}_2(x,y) = -\int_{\Gamma_\rho} \mathcal{Q}_\rho^-(x,\xi)\mathcal{N}(\partial_\xi,n_\xi)\mathcal{P}_\rho^+(\xi,y)\Big(1-\eta_1\Big(\frac{|y-\xi|}{|y|}\Big)\Big)\zeta_1^{(2)}\Big(\frac{|y|}{\rho}\Big)ds_\xi$$

$$\mathcal{L}_3(x,y) = -\big(\mathcal{N}(\partial_\xi,n_\xi)\mathcal{Q}^- + \rho(x,\xi)\big)^T\Big(1-\chi_2\Big(\frac{|y|}{|x|}\Big)\Big)\zeta_1^{(2)}\Big(\frac{|y|}{\rho}\Big)$$

$$\mathcal{L}_4(x,y) = -\int_{G^+}\big(\mathcal{Q}_\rho^+(x,\xi)\big)^T\Delta_\xi^*\mathcal{P}_\rho^+(\xi,y)\eta_1\Big(\frac{|y-\xi|}{|y|}\Big)\Big(1-\chi_2\Big(\frac{|y|}{|x|}\Big)\Big)\zeta_1^{(2)}\Big(\frac{|y|}{\rho}\Big)\,ds_\xi$$

$$\mathcal{L}_5(x,y) = -\int_\Gamma \mathcal{Q}_\rho^-(x,\xi)\mathcal{N}(\partial_\xi,n_\xi)\big(\mathcal{P}^+(\xi,y)-\mathcal{P}_\rho^+(\xi,y)\big)\eta_1\Big(\frac{|y-\xi|}{|y|}\Big)\chi_2\Big(\frac{|y|}{|x|}\Big)ds_\xi$$

$$\mathcal{L}_6(x,y) = \big(\mathcal{N}(\partial_\xi,n_\xi)\psi^+(x,y)\big)^T\chi_2\Big(\frac{|y|}{|x|}\Big)$$

$$\mathcal{L}_7(x,y) = \int_\Gamma \zeta_1^{(1)}\Big(\frac{|x|}{|\xi|}\Big)\mathcal{Q}_\rho^-(x,\xi)\mathcal{N}(\partial_\xi,n_\xi)\mathcal{P}^+(\xi,y)\Big(1-\eta_1\Big(\frac{|y-\xi|}{|y|}\Big)\Big)\chi_2\Big(\frac{|y|}{|x|}\Big)\,ds_\xi$$

$$\mathcal{L}_8(x,y) = \int_\Gamma \Big(1-\zeta_1^{(1)}\Big(\frac{|x|}{|\xi|}\Big)\Big)\mathcal{Q}_\rho^-(x,\xi)\mathcal{N}(\partial_\xi,n_\xi)\mathcal{P}^+(\xi,y)\chi_2\Big(\frac{|y|}{|x|}\Big)\,ds_\xi$$

To complete the proof it is sufficient to obtain estimate (3.17) for each term $\mathcal{L}_k(x,y)$. The required estimate for $\mathcal{L}_1(x,y)$ is a direct corollary of (3.25). For $\mathcal{L}_2(x,y)$ we have

$$|\mathcal{L}_2(x,y)| \le c\,\rho^{-2}. \tag{3.32}$$

In fact, we put

$$x = \rho\,X, \quad y = \rho\,Y, \quad \xi = \rho\,\Xi.$$

It is obvious that

$$-\int_{\Gamma_1} \mathcal{Q}_1^-(X,\Xi)\mathcal{N}(\partial_\Xi,n_\Xi)\mathcal{P}_1^+(\Xi,Y)\Big(1-\eta_1\Big(\frac{|Y-\Xi|}{|Y|}\Big)\Big)\zeta_1^{(2)}(|Y|)\,ds_\Xi \le C.$$

Returning to the variables x, y, ξ, we arrive at (3.32).

By (3.30), (3.18), and (3.26), the same holds for $\mathcal{L}_3(x,y)$, $\mathcal{L}_5(x,y)$, and $\mathcal{L}_6(x,y)$.

Noting that $\Delta_\xi^*\mathcal{P}_\rho^+(\xi,y) = 0$ on the set of integration of $\mathcal{L}_4(x,y)$ and that the inequality $|y-\xi| > |y|/8$ is valid on the support of the function $\xi \to \nabla_\xi\eta_1(|y-\xi|/|y|)$, we arrive at the estimate (3.32) for $\mathcal{L}_4(x,y)$.

In order to obtain the estimate for $\mathcal{L}_7(x,y)$, we write it as the sum of two integrals over the sets

$$\Gamma_1 = \{\xi \in \Gamma : |\xi| < |y|\}, \qquad \Gamma_2 = \{\xi \in \Gamma : |\xi| > |y|\}$$

and use estimates (2.6) and (2.8). Then

$$|\mathcal{L}_7(x,y)| \le c\int_{\Gamma_1} |x-\xi|^{-1}|\xi|^{-1}|y|^{-2}(|\xi|/|y|)^{\delta^+-\varepsilon}ds_\xi$$
$$+c\int_{\Gamma_2} |x-\xi|^{-1}|\xi|^{-2}|y|^{-1}(|y|/|\xi|)^{\delta^+-\varepsilon}ds_\xi \le c\,\rho^{-2}.$$

Finally, dividing the domain of integration of $\mathcal{L}_8(x,y)$ into two sets

$$\Gamma_3 = \{\xi \in \Gamma : |\xi| > |x|\}, \qquad \Gamma_4 = \{\xi \in \Gamma : |\xi| < |x|/3\}$$

and noting that $|\xi| > 3/2|y|$ and $|\xi| < 2/3|y|$ on Γ_3 and Γ_4, resepectively, we arrive at estimate (3.32) for the last integral.

The case $x \in \Gamma$, $|x| \ge \varepsilon > 0$ can be considered in a similar manner and is even simpler. The lemma is proved.

Thus, Lemmas 11, 13, and 14 together with Theorem 4 imply Theorem 6.

4 Estimates for the kernel of the operator $(1+T)^{-1}$

The purpose of this section is to obtain the following result.

Theorem 7 *Let $0 < \beta - \alpha < 1$ and let l be a positive integer. For any $\psi \in N^{l-1,\alpha}_{\beta+l}(\Gamma)$*

$$(1+T^*)^{-1}\psi = (1+M)\,\psi, \tag{4.1}$$

where M is an integral operator with the kernel $\mathcal{M}(x,y)$ satisfying the estimates

$$|\mathcal{M}(x,y)| \le \begin{cases} c\,|x|^{-1}\,|y|^{-1}(|x|/|y|)^{\varkappa-\varepsilon}, & |x| < |y|/2, \\ c\,|x|^{-1}|x-y|^{-1}, & |y|/2 < |x| < 2|y|, \\ c\,|x|^{-2}(|y|/|x|)^{\varkappa-\varepsilon} + c\,|x|^{\varkappa-1-\varepsilon}, & |x| > 2\,|y|, \end{cases} \tag{4.2}$$

where $\varkappa = \min\{\delta^+, \nu^-\}$ and ε is any sufficiently small positive number.

4.1 Estimates for the kernel $\mathcal{M}(x,y)$ with $|y| < 5|x|/8$

In this subsection we prove the following assertion with notations for the cut-off functions χ_k, η_j, $\zeta^{(j)}_\varepsilon$ introduced at the beginning of Sect. 2.

Lemma 18 *Let $\psi \in N^{l-1,\alpha}_{\beta+l}(\Gamma)$. Then*

$$\mathcal{N}(\partial_x, n_x)\int_\Gamma \mathcal{P}^+(x,\xi)\int_\Gamma \mathcal{Q}^-(\xi,y)\chi_1(|y|/|x|)\,\psi(y)ds_y\,ds_\xi$$
$$= \int \mathcal{M}(x,y)\chi_1(4|y|/5|x|)\,\psi(y)ds_y, \quad x \in \Gamma\backslash 0, \tag{4.3}$$

where

$$|\mathcal{M}(x,y)| \le c\left(|x|^{\varkappa-1-\varepsilon} + |x|^{-2}(|y|/|x|)^{\varkappa-\varepsilon}\right). \tag{4.4}$$

First we prove some auxiliary assertions.

Lemma 19 *The following relations hold*

$$\lim_{x\to x_0}\int_\Gamma \mathcal{P}^+(x,\xi)\eta_1(|x-\xi|/|x|)\,ds_\xi = 1, \quad x_0\in\Gamma\backslash 0,\ \ x\in G^+, \tag{4.5}$$

$$\Big|\mathcal{N}(\partial_x,n_x)\int_\Gamma \mathcal{P}^+(x,\xi)\,\eta_1(|x-\xi|/|x|)\,ds_\xi\Big| \le c\,|x|^{-1}, \quad x\in\Gamma\backslash 0. \tag{4.6}$$

Proof. The identity (4.5) is a corollary of the equality

$$\mathcal{P}^+(x_0,\xi)\eta_1(|x_0-\xi|/|x_0|) = 0$$

for $x_0,\xi\in\Gamma\backslash 0$ and the identity

$$\lim_{x\to x_0}\int_\Gamma G^+(x,\xi)\,ds_\xi = 1,$$

which holds by the uniqueness of solution of problem (2.3) .

Now we prove (4.6). Since

$$\mathcal{N}(\partial_x,n_x)\int_\Gamma \mathcal{P}^+(x,\xi)ds_\xi = 0,$$

it is sufficient to obtain the inequality

$$\big|\mathcal{N}(\partial_x,n_x)\int_\Gamma \mathcal{P}^+(x,\xi)(1-\eta_1(|x-\xi|/|x|)\,ds_\xi\big| \le c\,|x|^{-1}, \quad x\in\Gamma\backslash 0,$$

which follows from (2.6). The proof is complete.

Lemma 20 *The following relations hold*

$$\Big|\int_\Gamma \mathcal{P}^+(x,\xi)\eta_1(|x-\xi|/|x|)\,(x_j-\xi_j)\,ds_\xi\Big| \le c\,|x|, \quad x\in G^+,\ j=1,2,3, \tag{4.7}$$

$$\Big|\mathcal{N}(\partial_x,n_x)\int_\Gamma \mathcal{P}^+(x,\xi)\,\eta_1(|x-\xi|/|x|)\,(x_j-\xi_j)\,ds_\xi\Big| \le c. \tag{4.8}$$

Proof. It is obvious that (4.7) is a corollary of estimate (2.6).

We prove (4.8). Let $x\in G^+$. By

$$\mathcal{P}^+(x,\xi) = \big(\mathcal{N}(\partial_\xi,n_\xi)\,\mathcal{G}(\xi,x)\big)^T,$$

where $\mathcal{G}(\xi,x) = \big(\mathcal{G}(x,\xi)\big)^T$ is Green's matrix of problem (2.3) (see (3.22)), and using the property $\mathcal{G}(\xi,x)=0$ for $\xi\in\Gamma$ together with Betti's formula, we write the expression on the left-hand side of (4.8) in the form

$$\begin{aligned}&\Big|\mathcal{N}(\partial_x,n_x)\int_{\partial B(\varepsilon,x)}\mathcal{G}(x,\xi)\mathcal{N}(\partial_\xi,n_\xi)\eta_1(|x-\xi|/|x|)\,(x_j-\xi_j)(x_j-\xi_j)\,ds_\xi\\ &-\mathcal{N}(\partial_x,n_x)\int_{\partial B(\varepsilon,x)}\mathcal{P}^+(x,\xi)\eta_1(|x-\xi|/|x|)\,(x_j-\xi_j)(x_j-\xi_j)\,ds_\xi\\ &-\int_{G^+\cap B(\varepsilon,x)}\mathcal{G}(x,\xi)\Delta^*_\xi\big(\eta_1(|x-\xi|/|x|)\big)(x_j-\xi_j)(x_j-\xi_j)\,d\xi\Big|,\end{aligned}$$

where $\partial B(\varepsilon,x)$ is the two-dimensional sphere of radius ε centered at x.

Hence, using the inequality

$$|\partial_x^\sigma \partial_\xi^\tau \mathcal{G}(x,\xi)| \le c_{\sigma\tau} \varepsilon^{-1-|\sigma|-|\tau|}$$

valid on the sphere $\partial B(\varepsilon, x)$ together with the relations $3/4|x| < |\xi| < 5/4|x||$ which hold on the support of the function

$$\xi \to \nabla_\xi \, \eta_1(|x-\xi|/|x|),$$

we arrive at (4.7). The proof is complete.

Lemma 21 *Let $|y| < 5/8|x|$. Then*

$$\begin{aligned} &\Big| \mathcal{N}(\partial_x, n_x) \int_\Gamma \mathcal{P}^+(x,\xi)\big(|\xi|^{-1}\mathcal{A}(\xi/|\xi|) + \mathcal{B}^-\big)\eta_2(|y|/|\xi|) ds_\xi \Big| \\ &\le c\left(1 + |x|^{-2}(|y|/|x|)^{\varkappa-\varepsilon}\right). \end{aligned} \tag{4.9}$$

Proof. Let $y \in \Gamma \backslash 0$. Consider the function which solves the boundary value problem

$$\begin{aligned} &\Delta_x^* u(x,y) = 0 \quad \text{in } G^+ \\ &u(x,y) = \big(|x|^{-1}\mathcal{A}(x/|x|) + \mathcal{B}^-\big)\eta_2(|y|/|x|) \quad \text{for } x \in \Gamma \end{aligned} \tag{4.10}$$

in the space $N^{l,\alpha}_{\delta+l}(G^+)$, $1 < \delta - \alpha < 1 + \delta^+$. By Theorem 1, u admits the representation

$$u(x,y) = \int_\Gamma \mathcal{P}^+(x,\xi)\big(|\xi|^{-1}\mathcal{A}(\xi/|\xi|) + \mathcal{B}^-\big)\eta_2(|y|/|\xi|) ds_\xi. \tag{4.11}$$

We are looking for a solution of problem (4.10) in the form

$$u(x,y) = \big(|x|^{-1}\mathcal{A}(x/|x|) + \mathcal{B}^-\big)\, \eta_2(|y|/|x|) + w(x,y). \tag{4.12}$$

Then, for each $y \in \Gamma \backslash 0$, the function w solves the Dirichlet problem

$$\Delta_x^* w(x,y) = f(x,y) \quad \text{in } G^+, \qquad w(x,y) = 0 \quad \text{on } \Gamma,$$

where the support of f lies in the spherical layer $7/6|y| < |x| < 6/5|y|$ and

$$|\partial_x^\sigma f(x,y)| \le c_\sigma \, |y|^{-1-|\sigma|}.$$

We put $x = |y|X$, $y = |y|Y$ and by $\Gamma_{|y|}$ denote the image of the surface Γ under the mapping $x \to X$. The same argument as in the proof of Lemma 10 shows that

$$\|w_{|y|}(X,Y)\|_{N^{l,\alpha}_{l+\alpha+1+\delta^+-\varepsilon}(\Gamma_{|y|})} \le c$$

with $|Y| = 1$, which implies

$$|\partial_X^\delta w_{|y|}(X,Y)| \le c_\sigma \, |X|^{-1-|\sigma-\delta^+ + \varepsilon}.$$

Returning to the variables x and y, we obtain

$$|\partial_x^\sigma w(x,y)| \le c_\sigma |x|^{-1-|\sigma|}\big(|y|/|x|\big)^{\delta^+-\varepsilon}.$$

Taking into account that

$$\mathcal{N}(\partial_x, n_x)\big(|x|^{-1}\mathcal{A}(x/|x|) + \mathcal{B}^-\big) = 0$$

at the boundary of the cone ($x \neq 0$), from (4.11) and (4.12) we arrive at (4.9).

Proof of Lemma 18. Using (2.5) and (2.10), we write the left-hand side of (4.3) in the form

$$\begin{aligned}
&\mathcal{N}(\partial_x, n_x) \int_\Gamma \mathcal{P}^+(x,\xi) \int_\Gamma \big(\mathcal{R}^-(y,\xi)\big)^T \eta_2\Big(\frac{|y|}{|x|}\Big)\, \psi(y)\, ds_y\, ds_\xi \\
&+\mathcal{N}(\partial_x, n_x) \int_\Gamma \mathcal{P}^+(x,\xi) \int_\Gamma \Big(|\xi|\mathcal{A}^-\Big(\frac{\xi}{|\xi|}\Big) + \mathcal{B}\Big)\, \eta_2\Big(\frac{|y|}{|x|}\Big)\chi_1\Big(\frac{|y|}{|x|}\Big)\psi(y)\, ds_y\, ds_\xi \\
&+\mathcal{N}(\partial_x, n_x) \int_\Gamma \mathcal{P}^+(x,\xi) \int_\Gamma \mathcal{C}^-(\xi)\eta_2\Big(\frac{|y|}{|x|}\Big)\chi_1\Big(\frac{|y|}{|x|}\Big)\psi(y)\, ds_y\, ds_\xi \\
&+\mathcal{N}(\partial_x, n_x) \int_\Gamma \mathcal{P}^+(x,\xi) \int_\Gamma \mathcal{Q}^-(\xi,y)\Big(1-\eta_2\Big(\frac{|y|}{|x|}\Big)\Big)\chi_1\Big(\frac{|y|}{|x|}\Big)\psi(y)\, ds_y\, ds_\xi
\end{aligned} \tag{4.13}$$

Let ∂K^+ stand for the boundary of the cone K^+. By N we denote a number such that for all $x \in \partial K^+$ with $|x| = 1$ there exists a diffeomorphism Z_x which maps $K^+ \cap B((2N)^{-1}, x)$ onto a subset F_x of the half space $\mathbb{R}^3_+$, while the surface $\partial K + \cap B((2N)^{-1}, x)$ is mapped onto a sunset of a plane, and the derivatives of order $l \geq 1$ of the matrices $(Z_x)'$ and $(Z_x^{-1})'$ are bounded on $\overline{K^+} \cap B((2N)^{-1}, x)$ and on $\overline{F_x}$ by some constant c_l independent of x.

For other points $x \in \partial K^+$ with $|x| = \rho > 0$ the mapping $Z_x : K^+ \cap B((2N)^{-1}, x) \to F_x$ is defined by th equality

$$Z_x\xi = \rho\, Z_{x/\rho}(\xi/\rho).$$

Obviously, for $b = Z_x\xi$,

$$|\partial_\xi^\sigma b| \leq c_\sigma\, |x|^{1-|\sigma|}, \qquad |\partial_b^\tau \xi| \leq c_\tau\, |x|^{1-|\tau|}. \tag{4.14}$$

Let $x \in \Gamma \cap B(|x_0|/4N, x_0)$, where x_0 is a point on $\Gamma\backslash 0$. We assume that x_0 is plkaced so close to the origin that the surface Γ coincides with ∂K^+ in the ball $B(|x_0|/2N, x_0)$. For each point $y \in \Gamma$ we define the function $b \to \mathcal{R}(b, y)$, $b \in F_{x_0}$ by

$$\mathcal{R}(b, y) = \mathcal{R}^+(Z_{x_0}^{-1}(b), y),$$

where $\mathcal{R}^+(\xi, y)$ is a solution of problem (3.9). By Taylor's formula we have

$$\mathcal{R}(b, y) = \mathcal{R}(a, y) + \sum_{|\sigma|=1} (b-a)^\sigma \partial_b^\sigma \mathcal{R}(a, y) + \frac{1}{2} \sum_{|\sigma|=2} (b-a)^\sigma \partial_b^\sigma \mathcal{R}(c, y),$$

where $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ are images of the points

$$x \in \overline{G^+} \cap B(|x_0|/4N, x_0), \qquad \xi \in \Gamma \cap B(|x_0|/4N, x_0)$$

and c is a point lying on the segment with end points a and b,

$$(b-a)^\sigma = \prod_{1\leq k\leq 3} (b_k - a_k)^{\sigma_k}.$$

Thus, we can represent the first term of (4.13) as

$$\frac{1}{2}\mathcal{N}(\partial_x, n_x)\int_\Gamma \mathcal{P}^+(x,\xi)\eta_1\Big(\frac{N|x-\xi|}{|x|}\Big)\int_\Gamma \sum_{|\sigma|=2}(b-a)^\sigma \partial_b^\sigma \mathcal{R}(c,y)\chi_1\Big(\frac{|y|}{|x|}\Big)\psi(y)dx_y ds_\xi$$

$$+\mathcal{N}(\partial_x, n_x)\int_\Gamma \mathcal{P}^+(x,\xi)\eta_1\Big(\frac{N|x-\xi|}{|x|}\Big)\int_\Gamma \mathcal{R}^+(x,y)\chi_1\Big(\frac{|y|}{|x|}\Big)\psi(y)dx_y ds_\xi$$

$$+\mathcal{N}(\partial_x, n_x)\int_\Gamma \mathcal{P}^+(x,\xi)\eta_1\Big(\frac{N|x-\xi|}{|x|}\Big)\int_\Gamma \sum_{|\sigma|=1}(b-a)^\sigma \partial_b^\sigma \mathcal{R}(a,y)\chi_1\Big(\frac{|y|}{|x|}\Big)\psi(y)dx_y ds_\xi$$

$$+\mathcal{N}(\partial_x, n_x)\int_\Gamma \mathcal{P}^+(x,\xi)\Big(1-\eta_1\Big(\frac{N|x-\xi|}{|x|}\Big)\Big)\int_\Gamma \mathcal{R}^+(\xi,y)\chi_1\Big(\frac{|y|}{|x|}\Big)\psi(y)dx_y ds_\xi.$$

Combining this with (4.13) and using

$$\chi_1(t)\,\chi(4t/5) = \chi_1(t),$$

we arrive at (4.3), where

$$\mathcal{M}(x,y) = \sum_{1\le k\le 7}\mathcal{M}_k(x,y),$$

and

$$\mathcal{M}_1 = \frac{1}{2}\mathcal{N}(\partial_x, n_x)\int_\Gamma \mathcal{P}^+(x,\xi)\eta_1\Big(\frac{N|x-\xi|}{|x|}\Big)\sum_{|\sigma|=2}(b-a)^\sigma \partial_b^\sigma \mathcal{R}(c,y)\chi_1\Big(\frac{|y|}{|x|}\Big)\,ds_\xi$$

$$\mathcal{M}_2 = \mathcal{N}(\partial_x, n_x)\int_\Gamma \mathcal{P}^+(x,\xi)\eta_1\Big(\frac{N|x-\xi|}{|x|}\Big)\mathcal{R}^+(x,y)\chi_1\Big(\frac{|y|}{|x|}\Big)\,ds_\xi$$

$$\mathcal{M}_3 = \mathcal{N}(\partial_x, n_x)\int_\Gamma \mathcal{P}^+(x,\xi)\eta_1\Big(\frac{N|x-\xi|}{|x|}\Big)\sum_{|\sigma|=1}(b-a)^\sigma \partial_b^\sigma \mathcal{R}(a,y)\chi_1\Big(\frac{|y|}{|x|}\Big)ds_\xi$$

$$\mathcal{M}_4 = \mathcal{N}(\partial_x, n_x)\int_\Gamma \mathcal{P}^+(x,\xi)\Big(1-\eta_1\Big(\frac{N|x-\xi|}{|x|}\Big)\Big)\mathcal{R}^+(\xi,y)\chi_1\Big(\frac{|y|}{|x|}\Big)ds_\xi$$

$$\mathcal{M}_5 = \mathcal{N}(\partial_x, n_x)\int_\Gamma \mathcal{P}^+(x,\xi)\big(|\xi|\mathcal{A}^-(\xi/|\xi|)+\mathcal{B}\big)\,\eta_2\Big(\frac{|y|}{|\xi|}\Big)\chi_1\Big(\frac{|y|}{|x|}\Big)ds_\xi$$

$$\mathcal{M}_6 = \mathcal{N}(\partial_x, n_x)\int_\Gamma \mathcal{P}^+(x,\xi)\mathcal{C}^-(\xi)\chi_1\Big(\frac{|y|}{|x|}\Big)ds_\xi$$

$$\mathcal{M}_7 = \mathcal{N}(\partial_x, n_x)\int_\Gamma \mathcal{P}^+(x,\xi)\big(\mathcal{Q}^-(\xi,y)-\mathcal{C}^-(\xi)\big)\Big(1-\eta_2\Big(\frac{|y|}{|\xi|}\Big)\Big)\chi_1\Big(\frac{|y|}{|x|}\Big)ds_\xi,$$

Now we obtain estimates for each $\mathcal{M}_k$. Let $z = Z_x^{-1}c$. Since $|y| < 5|x|/8$ and $|z-x| < |x|/4$, we have $|z| > 3|x|/4$ and $|z| > 6|y|/5$. Hence, using estimates (4.14), (3.11), and (2.6), we conclude that

$$|\mathcal{M}_1(x,y)| \le c\,|x|^{-2}(|y|/|x|)^{\varkappa-\varepsilon}. \tag{4.15}$$

Noting that

$$b_k - a_k = (\xi_j - x_j)\frac{\partial (Z_{x_0})_k}{\partial \xi_j}(x) + O(|\xi - x|^2),$$

by Lemmas 19, 20 and estimates (4.14), (3.11), (2.6) we arrive at (4.15) for $\mathcal{M}_2(x,y)$ and $\mathcal{M}_3(x,y)$.

The same estimate for $\mathcal{M}_4(x,y)$ and $\mathcal{M}_7(x,y)$ is an immediate corollary of Theorems 1 and 3.

By Lemma 21 estimate (4.8) holds for $\mathcal{M}_5(x,y)$.

Finally, the inequality

$$|\mathcal{M}_6(x,y)| \le c\,|y|^{\varkappa-1-\varepsilon}$$

follows directly from (3.2) since the function

$$u(x) = \int_\Gamma \mathcal{P}^+(x,\xi)\mathcal{C}^-(\xi)\,ds_\xi$$

is a solution of problem (3.10).

The case $x_0 \in \Gamma$, $|x_0| \ge \varepsilon$, where ε is a fixed positive number, may be considered in a similar manner and even simpler. The proof is complete.

4.2 Estimates for the kernel $\mathcal{M}(x,y)$ with $|y| > 8|x|/5$

Lemma 22 *Let $\psi \in N^{l-1,\alpha}_{\beta+l}(\Gamma)$. Then*

$$\begin{aligned}&\mathcal{N}(\partial_x,n_x)\int_\Gamma \mathcal{P}^+(x,\xi)\mathcal{Q}^-(\xi,y)\chi_3(|y|/|x|)\,\psi(y)\,ds_y\,ds_\xi\\ &= \int_\Gamma \mathcal{M}(x,y)\,\chi_3(5|y|/4|x|)\psi(y)\,ds_y, \quad x\in\Gamma\backslash 0,\end{aligned} \tag{4.16}$$

where

$$|\mathcal{M}(x,y)| \le c\,|x|^{-1}|y|^{-1}(|x|/|y|)^{\varkappa-\varepsilon}.$$

Proof. By (2.9) we can write the left-hand side of (4.13) as

$$\begin{aligned}&\mathcal{N}(\partial_x,n_x)\int_\Gamma \mathcal{P}^+(x,\xi)\int_\Gamma \mathcal{Q}^-(0,y)\eta\Big(\frac{|\xi|}{|y|}\Big)\chi_3\Big(\frac{|y|}{|x|}\Big)\,ds_y\,ds_\xi\\ &+\mathcal{N}(\partial_x,n_x)\int_\Gamma \mathcal{P}^+(x,\xi)\int_\Gamma \mathcal{R}^-(\xi,y)\eta\Big(\frac{|\xi|}{|y|}\Big)\chi_3\Big(\frac{|y|}{|x|}\Big)\,ds_y\,ds_\xi\\ &+\mathcal{N}(\partial_x,n_x)\int_\Gamma \mathcal{P}^+(x,\xi)\int_\Gamma \mathcal{Q}^-(\xi,y)\Big(1-\eta\Big(\frac{|\xi|}{|y|}\Big)\Big)\chi_3\Big(\frac{|y|}{|x|}\Big)\,ds_y\,ds_\xi.\end{aligned}$$

Furthermore, keeping the notation introduced in the proof of Lemma 18 (with the only difference that here $\mathcal{R}^+(\xi,y)$ is a solution of problem (3.3) for a fixed $y\in\Gamma\backslash 0$), in a similar manner we arrive at (4.16) in a neighbourhood of $x_0\in\Gamma\backslash 0$, where

$$\begin{aligned}\mathcal{M}(x,y) &= \mathcal{N}(\partial_x,n_x)\int_\Gamma \mathcal{P}^+(x,\xi)\mathcal{Q}^-(0,y)\chi_3\Big(\frac{|y|}{|x|}\Big)\,ds_\xi\\ &+\mathcal{N}(\partial_x,n_x)\int_\Gamma \mathcal{P}^+(x,\xi)\mathcal{Q}^-(0,y)\Big(1-\eta_2\Big(\frac{|\xi|}{|y|}\Big)\Big)\chi_3\Big(\frac{|y|}{|x|}\Big)\,ds_\xi\\ &+\mathcal{N}(\partial_x,n_x)\int_\Gamma \mathcal{P}^+(x,\xi)\mathcal{R}^+(x,y)\eta_1\Big(\frac{N|x-\xi|}{|x|}\Big)\chi_3\Big(\frac{|y|}{|x|}\Big)\,ds_\xi\\ &+\mathcal{N}(\partial_x,n_x)\int_\Gamma \sum_{|\sigma|=1}\mathcal{P}^+(x,\xi)\,\eta_1\Big(\frac{N|x-\xi|}{|x|}\Big)(b-a)^\sigma\partial_b^\sigma\mathcal{R}(a,y)\chi_3\Big(\frac{|y|}{|x|}\Big)ds_\xi\\ &+\int_\Gamma \mathcal{N}(\partial_x,n_x)\sum_{|\sigma|=2}\mathcal{P}^+(x,\xi)\eta_1\Big(\frac{N|x-\xi|}{|x|}\Big)(b-a)^\sigma\partial_b^\sigma\mathcal{R}(c,y)\chi_3\Big(\frac{|y|}{|x|}\Big)ds_\xi\\ &+\int_\Gamma \mathcal{N}(\partial_x,n_x)\mathcal{P}^+(x,\xi)\Big(1-\eta_1\Big(\frac{N|x-\xi|}{|x|}\Big)\Big)\mathcal{R}^+(\xi,y)\chi_3\Big(\frac{|y|}{|x|}\Big)\,ds_\xi\\ &+\int_\Gamma \mathcal{N}(\partial_x,n_x)\mathcal{P}^+(x,\xi)\mathcal{Q}^+(\xi,y)\Big(1-\eta_2\Big(\frac{|\xi|}{|y|}\Big)\Big)\chi_3\Big(\frac{|y|}{|x|}\Big)\,ds_\xi.\end{aligned}$$

Estimating each term by Lemmas 19, 20, 21, 10 and Theorems 1, 3, we complete the proof of the lemma.

4.3 Estimates for the kernel $\mathcal{M}(x,y)$ with $|x|/2 < |y| < 2|x|$

We use the notation introduced in Subsection 2.3. Let T_ρ^* be the operator in the space $C(\Gamma_\rho)$ of vector-valued functions ψ defined by

$$T_\rho^* \psi = 2\, W_0 \psi,$$

where W_0^* stands for the integral operator on Γ_ρ with the kernel

$$-\mathcal{N}(\partial_x, n_x)\, \Phi(x,\xi)/4\pi.$$

Lemma 23 *Let $\psi \in C(\Gamma_\rho)$, $\rho > 0$. Then*

$$\mathcal{N} P_\rho^+ \, Q_\rho^- \psi = -(1 + 2H_\rho^*)\psi,$$

where H_ρ^* is the integral operator with kernel $\mathcal{H}_\rho^*(x,y)$ satisfying

$$|\mathcal{H}_\rho^*(x,y)| \le c\, \rho^{-1} |x-y|^{-1}. \tag{4.17}$$

Proof. Using the relation

$$(1 + T_\rho^*)^{-1} \psi = \frac{1}{2}(1 - \mathcal{N} P_\rho^+ \, Q_\rho^-)\psi$$

and duplicating the argument in the proof of (3.25), we arrive at the assertion of the lemma.

Lemma 24 *Let $\psi \in N_{\beta+l}^{l-1,\alpha}(\Gamma)$. Then*

$$\begin{aligned} &\mathcal{N}(\partial_x, n_x) \int_\Gamma \mathcal{P}^+(x,\xi) \int_\Gamma \mathcal{Q}^-(\xi,y) \chi_3\Big(\frac{|y|}{|x|}\Big) \psi(y)\, ds_y\, ds_\xi \\ &= -\psi(x) + \int_\Gamma \mathcal{M}(x,y)\, \zeta_1^{(8)}\Big(\frac{|y|}{|x|}\Big)\, \psi(y) ds_y, \quad x \in \Gamma\backslash 0, \end{aligned} \tag{4.18}$$

where

$$|\mathcal{M}(x,y)| \le c\, |y|^{-1} |x-y|^{-1}. \tag{4.19}$$

Proof. Let $x \in \Gamma \cap U_k$, $\rho = 2^{-k}$. We assume that k is a sufficiently large integer. Since $\rho/4 < |y| < 4\rho$ and $\rho/16 < |\xi| < 16\rho$ on the support of the function

$$(\xi, y) \to \zeta_1^{(1)}\Big(\frac{|\xi|}{|y|}\Big)\chi\Big(\frac{|y|}{|x|}\Big),$$

(4.13) can be written in the form

$$\begin{aligned} &\mathcal{N}(\partial_x, n_x) \int_\Gamma \mathcal{P}^+(x,\xi) \int_\Gamma \mathcal{Q}_\rho^-(\xi,x)\, \zeta_1^{(1)}\Big(\frac{|\xi|}{|y|}\Big) \chi_2\Big(\frac{|y|}{|x|}\Big) \psi(y)\, ds_y\, ds_\xi \\ &+ \mathcal{N}(\partial_x, n_x) \int_\Gamma \mathcal{P}^+(x,\xi) \int_\Gamma (\mathcal{Q}^-(\xi,y) - \mathcal{Q}_\rho^-(\xi,y))\, \zeta_1^{(1)}\Big(\frac{|\xi|}{|y|}\Big) \chi_2\Big(\frac{|y|}{|x|}\Big) \psi(y)\, ds_y\, ds_\xi \\ &+ \mathcal{N}(\partial_x, n_x) \int_\Gamma \mathcal{P}^+(x,\xi) \int_\Gamma \mathcal{Q}^-(\xi,y)(1 - \zeta_1^{(1)}\Big(\frac{|\xi|}{|y|}\Big)) \chi_2\Big(\frac{|y|}{|x|}\Big) \psi(y)\, ds_y\, ds_\xi. \end{aligned} \tag{4.20}$$

Using

$$\zeta_1^{(1)}\Big(\frac{|\xi|}{|y|}\Big)\,\eta_1\Big(\frac{|\xi-x|}{|x|}\Big)=\eta_1\Big(\frac{|\xi-x|}{|x|}\Big)$$

on the support of the function $y\to\chi_2(|y|/|x|)$, we write the first term in (4.20) as the sum

$$\begin{aligned}&\mathcal{N}(\partial_x,n_x)\int_\Gamma \mathcal{P}^+(x,\xi)\,\eta_1\Big(\frac{|\xi-x|}{|x|}\Big)\int_\Gamma \mathcal{Q}^-_\rho(\xi,y)\,\chi_2\Big(\frac{|y|}{|x|}\Big)\psi(y)\,ds_y\,ds_\xi\\&+\mathcal{N}(\partial_x,n_x)\int_\Gamma(\mathcal{P}^+(x,\xi)-\mathcal{P}^+_\rho(x,\xi))\eta_1\Big(\frac{|\xi-x|}{|x|}\Big)\int_\Gamma \mathcal{Q}^-_\rho(\xi,x)\,\chi_2\Big(\frac{|y|}{|x|}\Big)\psi(y)ds_yds_\xi\\&+\mathcal{N}(\partial_x,n_x)\int_\Gamma \mathcal{P}^+(x,\xi)\Big(1-\eta_1\Big(\frac{|\xi-x|}{|x|}\Big)\Big)\int_\Gamma \mathcal{Q}^-_\rho(\xi,y)\zeta_1^{(1)}\Big(\frac{|\xi|}{|y|}\Big)\chi_2\Big(\frac{|y|}{|x|}\Big)\psi(y)ds_yds_\xi\end{aligned}\tag{4.21}$$

Since the identity

$$\chi_2\Big(\frac{|y|}{|x|}\Big)\,\zeta_1^{(2)}\Big(\frac{|y|}{\rho}\Big)=\chi_2\Big(\frac{|y|}{|x|}\Big)$$

holds for $x\in U_k$, the first integral in (4.21) can be represented in the form

$$\begin{aligned}&\mathcal{N}(\partial_x,n_x)\int_{\Gamma_\rho}\mathcal{P}^+_\rho(x,\xi)\int_\Gamma \mathcal{Q}^-_\rho(\xi,y)\,\zeta_1^{(2)}\Big(\frac{|y|}{\rho}\Big)\psi(y)\,ds_y\,ds_\xi\\&-\mathcal{N}(\partial_x,n_x)\int_{\Gamma_\rho}\mathcal{P}^+_\rho(x,\xi)\Big(1-\eta_1\Big(\frac{|x-\xi|}{|x|}\Big)\Big)\int_\Gamma \mathcal{Q}^-_\rho(\xi,y)\,\zeta_1^{(2)}\Big(\frac{|y|}{\rho}\Big)\psi(y)\,ds_y\,ds_\xi\\&-\mathcal{N}(\partial_x,n_x)\int_{\Gamma_\rho}\mathcal{P}^+_\rho(x,\xi)\eta_1\Big(\frac{|x-\xi|}{|x|}\Big)\int_\Gamma \mathcal{Q}^-_\rho(\xi,y)\Big(1-\chi_2\Big(\frac{|y|}{|x|}\Big)\Big)\zeta_1^{(2)}\Big(\frac{|y|}{\rho}\Big)\psi(y)\,ds_y\,ds_\xi\end{aligned}$$

Hence, by (4.20), (4.21) and Lemma 23 we arrive at (4.13), where

$$\mathcal{M}(x,y)=\sum_{1\le k\le 7}\mathcal{M}_k(x,y)$$

and

$$\begin{aligned}\mathcal{M}_1(x,y)&=-2\mathcal{H}^*_\rho(x,y)\,\zeta_1^{(2)}\Big(\frac{|y|}{\rho}\Big)\\ \mathcal{M}_2(x,y)&=-\mathcal{N}(\partial_x,n_x)\int_{\Gamma_\rho}\mathcal{P}^+_\rho(x,\xi)\Big(1-\eta_1\Big(\frac{|x-\xi|}{|x|}\Big)\Big)\mathcal{Q}^-_\rho(\xi,y)\,\zeta_1^{(2)}\Big(\frac{|y|}{\rho}\Big)\psi(y)ds_\xi\\ \mathcal{M}_3(x,y)&=-\mathcal{N}(\partial_x,n_x)\int_{\Gamma_\rho}\mathcal{P}^+_\rho(x,\xi)\eta_1\Big(\frac{|x-\xi|}{|x|}\Big)\mathcal{Q}^-_\rho(\xi,y)\Big(1-\chi_2\Big(\frac{|y|}{|x|}\Big)\Big)\zeta_1^{(2)}\Big(\frac{|y|}{\rho}\Big)ds_\xi\\ \mathcal{M}_4(x,y)&=\mathcal{N}(\partial_x,n_x)\int_\Gamma\mathcal{P}^+_\rho(x,\xi)\eta_1\Big(\frac{|x-\xi|}{|x|}\Big)\mathcal{Q}^-_\rho(\xi,y)\,\chi_2\Big(\frac{|y|}{|x|}\Big)\,ds_\xi\\ \mathcal{M}_5(x,y)&=\mathcal{N}(\partial_x,n_x)\int_\Gamma\mathcal{P}^+_\rho(x,\xi)\Big(1-\eta_1\Big(\frac{|x-\xi|}{|x|}\Big)\Big)\mathcal{Q}^-_\rho(\xi,y)\,\zeta_1^{(1)}\Big(\frac{|\xi}{|y|}\Big)\chi_2\Big(\frac{|y|}{|x|}\Big)ds_\xi\\ \mathcal{M}_6(x,y)&=-\mathcal{N}(\partial_x,n_x)\int_\Gamma\mathcal{P}^+_\rho(x,\xi)(\mathcal{Q}^-(\xi,y)-\mathcal{Q}^-_\rho(\xi,y))\,\zeta_1^{(1)}\Big(\frac{|\xi|}{|y|}\Big)\chi_2\Big(\frac{|y|}{|x|}\Big)\,ds_\xi\\ \mathcal{M}_7(x,y)&=-\mathcal{N}(\partial_x,n_x)\int_\Gamma\mathcal{P}^+_\rho(x,\xi)(\mathcal{Q}^-(\xi,y)\Big(1-\zeta_1^{(1)}\Big(\frac{|\xi|}{|y|}\Big)\Big)\chi_2\Big(\frac{|y|}{|x|}\Big)\,ds_\xi.\end{aligned}$$

Now we evaluate each term $\mathcal{M}_k(x,y)$. The estimate (4.19) for $\mathcal{M}_1(x,y)$ is an immediate corollary of (4.17). It is clear that $\mathcal{M}_2(x,y)$ satisfies the stronger estimate

$$|\mathcal{M}_2(x,y)|\le c\,\rho^{-2}.\tag{4.22}$$

Since the function $\xi \to 1 - \eta_1(|y-\xi|/|y|)$ is identically equal to 1 on the support of the function

$$\xi \to \eta_1(|x-\xi|/|x|)(1-\chi_2(|y|/|x|)),$$

we write $\mathcal{M}_3(x,y)$ in the form

$$\begin{aligned}\mathcal{M}_3(x,y) &= -\mathcal{N}(\partial_x, n_x)\int_{\Gamma_\rho} \mathcal{P}^+_\rho(x,\xi)\mathcal{Q}^-_\rho(\xi,y)\Big(1-\eta_1\Big(\frac{|y-\xi|}{|y|}\Big)\Big) \\ &\times\Big(1-\chi_2\Big(\frac{|y|}{|x|}\Big)\Big)\zeta_1^{(2)}\Big(\frac{|y|}{\rho}\Big)\, ds_\xi \\ &+\mathcal{N}(\partial_x, n_x)\int_{\Gamma_\rho} \mathcal{P}^+_\rho(x,\xi)\Big(1-\eta_1\Big(\frac{|x-\xi|}{|x|}\Big)\Big)\mathcal{Q}^-_\rho(\xi,y)\Big(1-\eta_1\Big(\frac{|y-\xi|}{|y|}\Big)\Big) \\ &\times\Big(1-\chi_2\Big(\frac{|y|}{|x|}\Big)\Big)\zeta_1^{(2)}\Big(\frac{|y|}{\rho}\Big)\, ds_\xi.\end{aligned}$$

The validity of (4.22) for the second term is obvious. To obtain the estimate (4.22) for the first term, it is sufficient to note that the function u defined by

$$u(x,y) = \int_{\Gamma_\rho} \mathcal{P}^+_\rho(x,\xi)\mathcal{Q}^-_\rho(\xi,y)(1-\eta_1(|y-\xi|/|y|))ds_\xi$$

is a solution of the boundary value problem

$$\Delta^*_x u = 0 \quad \text{in } D_\rho, \qquad u(x,y) = \mathcal{Q}^-_\rho(x,y)(1-\eta_1(|y-x|/|y|)) \quad \text{on } \Gamma_\rho$$

for $y \in \Gamma$, $\rho/5 < |y| < 5\rho$.

The estimate (4.22) for $\mathcal{M}_6(x,y)$ follows from the fact that the function

$$v(x,y) = \int_{\Gamma} \mathcal{P}^+_\rho(x,\xi)(\mathcal{Q}^-(\xi,y) - \mathcal{Q}^-_\rho(\xi,y))\,\zeta_1^{(1)}(|\xi|/|y|)\, ds_\xi$$

is a solution of the boundary value problem

$$\Delta^*_x v = 0 \quad \text{in } G^+, \qquad v(x,y) = (\mathcal{Q}^-(x,y) - \mathcal{Q}^-_\rho(x,y))\zeta_1^{(1)}(|x|/|y|)$$

for $y \in \Gamma$, $\rho/4 < |y| < 4\rho$ and therefore this solution satisfies

$$|\partial^\sigma_x v(x,y)| \le c_\sigma \rho^{-1-|\sigma|}$$

(compare with Lemma 17).

The remaining terms can be estimated in a similar manner as in Lemma 3.5. The proof is complete.

Thus, Lemmas 18, 22 and 24 together with Theorem 5 lead to Theorem 7.

Remark If we do not require the cone K^+ to be described in Cartesian coordinates, Theorems 6 and 7 are valid provided $\varkappa$ is replaced by $\min\{0,\varkappa\}$ and the role of the inequality $0 < \beta - \alpha < 1$ is played by

$$|\beta - \alpha - 1/2| < 1/2 + \min\{0, \varkappa\}.$$

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