Invertibility of boundary integral operators of elasticity on surfaces with conic points

Nikolai Grachev

Department of Mathematics
Technical University
180680 Pskov, Russia

Vladimir Maz'ya

Department of Mathematical Sciences
University of Liverpool, M&O Building
Liverpool, L69 3BX, UK

Department of Mathematics
Linköping University
SE-58183 Linköping, Sweden

Abstract. The system of boundary integral equations of linear isotropic elasticity, with the double layer potential generated by the preudo-stress operator, is considered on surfaces with a finite number of conic points. The solvability of the system is established in various function spaces. Representation for the inverse operator of the system in question is obtained in terms of inverse operators of some boundary value problems. Pointwise estimates for the kernel of the inverse operator of the system and their derivatives of any order are derived together with 'quasilocal" estimates for solutions of the integral equations. The Laplace operator is contained here as a special case.

2000 MSC. Primary: 35G15, 45p05, 47b38, 47G10

Keywords: Potential theory, boundary integral equations of elasticity, conic points, Dirichlet and Neumann problems

1 Introduction

This paper is a developement of our article [MG]. Here we study boundary integral equations corresponding to the system of linear isotropic elasticity

$$\mu\,\Delta u + (\lambda+\mu)\,\nabla \mathrm{div}\, u = 0$$

in the case when the double layer potential is generated by the pseudo-stress operator. The integral equations are considered on a closed bounded two-dimensional surface which is smooth outside a finite number of conic vertices.

In Section 2 the so called quasilocal estimates for solutions of boundary integral equations are obtained. Using these estimates and pointwise estimates for kernels of inverse operators obtained in [MG], we prove theorems on invertibility for the above mentioned integral equations in various function spaces and obtain pointwise estimates for derivatives of kernels of corresponding inverse operators. Some of the results relating the harmonic, elastic, and hydrodynamic potentials were stated in [M1] and [M2], Sect. 1.6. Solvability of boundary integral equations of elasticity theory on Lipschitz surfaces in L_p, $1 < p \leq \infty$, was independently established in [DKV] by a different method.

We describe main results of the article. Let Γ be the boundary of a closed simply connected region G^+ in $\mathbb{R}^3$ which coincides with an open cone K^+ near the origin. We assume $\Gamma\backslash 0$ to be a smooth (of the class C^∞) surface. Let $K^- = \mathbb{R}^3\backslash\overline{K^+}$ and $\Omega^\pm = \{x \in K^\pm : |x| = 1\}$. The notations used in [MG] have the same meaning in the present paper.

We consider the system of integral equations associated with the first boundary value problem of linear isotropic elasticity in G^+

$$(1 + T)\,\varphi = f, \tag{1.1}$$

where 1 is the identity matrix and $T = 2\,W_0$ with $W_0\sigma$ being the direct value of the double layer potential on Γ generated by the pseudo-stress operator (see [KM]).

First we formulate our results on the solvability of system (1.1) in spaces $V^l_{p,\beta}(\Gamma)$ and $N^{l,\alpha}_\beta(\Gamma)$, where $1 < p < \infty$, $\alpha \in (0, 1)$, $\beta \in \mathbb{R}^1$, and $l = 0, 1, \ldots$. The spaces are defined as follows.

Given a function u supported by an arbitrary coordinate neighbourhood on $\Gamma\backslash 0$, the norm of u in $V^l_{p,\beta}(\Gamma)$ is defined by

$$\sum_{0\leq j\leq l} \|r^{\beta+j-l}\nabla_j u\|_{L_p(\Gamma)},$$

where $r(x) = |x|$ and ∇_j is the vector of all derivatives of order j. Similarly, the norm in $N^{l,\alpha}_\beta(\Gamma)$ is introduced as

$$\sup_{x\in\Gamma\backslash 0} r^{\beta-l-\alpha}(x)\,|u(x)| + \sup_{x,y\in\Gamma\backslash 0} \frac{|r^\beta(x)\nabla_l u(x) - r^\beta(y)\nabla_l u(y)|}{|x - y|^\alpha}.$$

The operator $1 + T$ *in (1.1) is an isomorphism of the space* $V^l_{p,l+t}(\Gamma)$ *onto itself for all* p, t, *and* l *such that*

$$1 < p < \infty, \qquad 0 < t + 2/p < 1 + \varkappa, \qquad l = 0, 1, \ldots \tag{1.2}$$

Here $\varkappa$ is a real number depending on the shape of the cone K^+. In the case of harmonic potentials $\varkappa$ is positive, being equal to the minimum of δ^+ and ν^-, where δ^+ and ν^- are positive numbers such that $\delta^+(\delta^+ + 1)$ and $\nu^-(\nu^- + 1)$ are the first eigenvalues of the Dirichlet ptoblem in Ω^+ and the Neumann problem in Ω^- for the Beltrami operator. It follows from [KM] that $\varkappa$ is positive for integral equations of elasticity if the cone K^+ can be explicitly described in a Cartesian coordinate system.

A similar assertion is valid for the space $N^{l,\alpha}_{\delta+l}(\Gamma)$ *with (1.2) replaced by*

$$0 \leq \delta - \alpha < 1 + \varkappa, \qquad \alpha \in (0, 1), \qquad l = 0, 1, \ldots$$

We show also that the operators $1 + T$ *and* $(1 + T)^{-1}$ *are continuous in the spaces* $C(\Gamma)$, $C^{0,\alpha}(\Gamma)$ *with* $0 < \alpha < \varkappa$, $L_{1,t}(\Gamma)$ *for* $0 < t + 2 < 1 + \varkappa$ *and* $L_{\infty,t}(\Gamma)$ *for* $0 < t < 1 + \varkappa$.

Here C and $C^{0,\alpha}$ are the spaces of continuous and Hölder continuous functions and the norm in $L_{p,t}(\Gamma)$ is introduced by

$$\|u\|_{L_{p,t}(\Gamma)} = \|r^t\, u\|_{L_p(\Gamma)}.$$

Along with (1.1) we consider the formally adjoint system

$$(1+T^*)\psi = g \tag{1.3}$$

and show that *the operator* $1+T^*$ *is an isomorphism of the space* $V^l_{p,t+l}(\Gamma)$ *onto itself if*

$$1 < p < \infty, \qquad 1-\varkappa < t+2/p < 2, \qquad l = 0, 1, \dots$$

Similarly, *the operator* $1+T^*$ *is an isomorphism of the space* $N^{l,\alpha}_{\delta+l}(\Gamma)$ *onto itself if*

$$\alpha \in (0,1), \qquad 1-\varkappa < \delta < 2, \qquad l = 0, 1, \dots$$

Moreover, *the operators* $1+T^*$ *and* $(1+T^*)^{-1}$ *are continuos in the space* $L_{1,t}(\Gamma)$ *for* $1-\varkappa < t+2/p < 2$ *and in* $L_{\infty,t}(\Gamma)$ for $1-\varkappa < t < 2$.

We shall prove that in the case of all above mentioned spaces, *the inverse operators of systems (1.1) and (1.3) can be written as*

$$(1+T)^{-1} = 1+L, \qquad (1+T^*)^{-1} = 1+L^*. \tag{1.4}$$

Here L *is an integral operator on* Γ *with the kernel* $\mathcal{L}(x,y)$ *satisfying the estimates*

$$|\mathcal{L}(x,y)| \le \begin{cases} c\,|x|^{-|\sigma|}|y|^{-2-|\tau|}(|x|/|y|)^{\varkappa-\varepsilon} + c\,|y|^{\varkappa-1-\varepsilon}\delta^0_{|\sigma|}, & 2|x| < |y|, \\ c\,|y|^{-1}|x-y|^{-1-|\sigma|-|\tau|}, & |y| < 2|x| < 4|y|, \\ c\,|x|^{-1-|\sigma|}|y|^{-1-|\tau|}(|y|/|x|)^{\varkappa-\varepsilon}, & |x| > 2|y|, \end{cases}$$

where δ *and* τ *are multiindeces of orders* $|\sigma|$ *and* $|\tau|$, δ^j_i *is the Kronecker index, and* ε *ia a sufficiently small positive number.*

2 Quasilocal estimates for solutions of integral equations and theorems on isomorphisms

Let K^+ be an open cone in $\mathbb{R}^3$ with vertex at the origin, bounded by the surface ∂K^+. We assume that K^+ can be explicitly described in a Cartesian coordinate system. We also suppose that the subset $\Omega^+ = \{x \in K^+ : |x| = 1\}$ of the unit sphere has smooth boundary.

In what follows, by $\{U_j\}_{1\le j\le N}$ we denote a finite covering of $\partial K^+\backslash 0$ by open sets $U_j \subset \partial K^+\backslash 0$ such that

1. for each U_j there exists a homeomorphism γ_j onto a plane angle V_j and

$$\gamma_j(tx) = t\,\gamma_j(x) \quad \text{for all } x \in U_j,\ t \in \mathbb{R}^+,$$

2. if $U_i \cap U_j = \emptyset$, then the mapping $\gamma_j \circ \gamma_i^{-1}$:

$$\gamma_i(U_i \cap U_j) \to \gamma_j(U_i \cap U_j) \quad \text{is infinitely differentiable.}$$

Moreover, we assume that $|x| = |\gamma_j x|$ for all $x \in U_j$, $j = 1, \dots, N$, where $|\cdot|$ on the left-hand side means the norm in $\mathbb{R}^3$ and the same symbol on the right-hand side means the norm in $\mathbb{R}^2$. Let $\{\xi_j\}_{1\le j\le N}$ be a partition of unity on $\partial K^+\backslash 0$ subordinate to the covering $\{U_j\}_{1\le j\le N}$. Suppose that the functions ξ_j are smooth and positive homogeneous of order 0. We let $V^l_{p,\beta}(\partial K^+\backslash 0)$, with $1 < p < \infty$, $\beta \in \mathbb{R}$, and $l = 0, 1, \dots$, stand for the space of functions with the norm

$$\sum_{1\le j\le N} \Big(\sum_{0\le|\sigma|\le l} \int_{\mathbb{R}^2} |x|^{p(\beta-l+|\sigma|)} |\partial^\sigma_x u_j(x)| p dx \Big)^{1/p},$$

where

$$u_j = \xi_j u \circ \gamma_j^{-1} \qquad \text{on} \;\; V_j$$

and $u_j = 0$ in the exterior of V_j. Using an equivalent atlas and another partition of unity, we arrive at an equivalent norm.

Further, let $N_\delta^{l,\alpha}(\partial K^+)$, with $\alpha \in (0,1)$, $\delta \in \mathbb{R}$, $l = 0, 1, \ldots$, be the space of functions with the finite norm

$$\sum_{1 \le j \le N} \Big(\sup_{x \in \mathbb{R}^2} |x|^\delta [u_j]^{l+\alpha}_{B(|x|/2,x)} + \sup_{x \in \mathbb{R}^2} |x|^{\sigma - l - \alpha} |u_j(x)| \Big),$$

where $B(r,x)$ is an open ball of radius r centered at x,

$$[u]^\rho_\Omega = \sup_{x,y \in \Omega} \sum_{|\sigma| = [\rho]} |x-y|^{[\rho]-\rho} |\partial_x^\sigma u(x) - \partial_y^\sigma u(y)|,$$

$[\rho]$ is the integer part of ρ, $\sigma = (\sigma_1, \sigma_2)$ is a multiindex of order $|\sigma| = \sigma_1 + \sigma_2$, and $\partial_x^\sigma = \partial^{|\sigma|}/\partial x_1^{\sigma_1} \partial x_2^{\sigma_2}$.

Let Γ be a simply connected domain in $\mathbb{R}^3$ with compact closure. We assume that $0 \in \Gamma$ and that $\Gamma \backslash 0$ is a smooth surface. Moreover, let Γ coincide with ∂K^+ in the ball B_ε of radius ε centered at 0.

We note that the spaces $V_{p,\beta}^l(\Gamma)$ and $N_\delta^{l,\alpha}(\Gamma)$, defined in Introduction, admit equivalent with norms

$$\|u\|_{V_{p,\beta}^l(\Gamma)} = \|\eta u\|_{V_{p,\beta}^l(\partial K^+)} + \|(1-\eta)u\|_{W_p^l(\Gamma \backslash B_{\varepsilon/2})},$$

$$\|u\|_{N_\delta^{l,\alpha}(\Gamma)} = \|\eta u\|_{N_\delta^{l,\alpha}(\partial K^+)} + \|(1-\eta)u\|_{C^{l,\alpha}(\Gamma \backslash B_{\varepsilon/2})},$$

where η is a function in the class $C^\infty(\mathbb{R}^3)$ such that $\eta = 1$ in $B_{\varepsilon/2}$ and $\eta = 0$ outside B_ε.

2.1 Quasilocal estimates

Let $\frac{1}{2}T$ be the operator of the direct value of the elastic double layer potential on Γ (see [KM]), defined for almost all $x \in \Gamma$ by

$$(T\psi)(x) = \int_\Gamma \mathcal{T}(x,\xi)\, \psi(\xi) ds_\xi.$$

Here $\psi \in L_1(\Gamma)$ and $\mathcal{T}(x,\xi)$ is the matrix with elements

$$(\mathcal{T}(x,\xi))_{i,j} = \frac{-1}{2\pi(\lambda + 3\mu)} \Big(2\mu \delta_i^j + 3(\lambda+\mu) \frac{\partial |x-\xi|}{\partial \xi_i} \, \frac{\partial |x-\xi|}{\partial \xi_j} \Big) \, \frac{\partial}{\partial n_\xi} \frac{1}{|x-\xi|},$$

where λ and μ are the Lamé constants, $\partial/\partial n_\xi$ is the normal derivative with respect to the outward normal to $\Gamma \backslash 0$ at ξ.

It is easily verified that for $\psi \in C(\Gamma)$ the last integral is a continuous function on $\Gamma \backslash 0$. Moreover, for $\psi \in C(\Gamma)$ there exists the limit

$$\begin{aligned}
&\lim_{\Gamma \backslash 0 \ni x \to 0} \int_\Gamma \mathcal{T}(x,\xi)\, \psi(\xi) ds_\xi \\
&= \int_\Gamma \mathcal{T}(0,\xi)\, (\psi(\xi - \psi(0)) ds_\xi + \lim_{\Gamma \backslash 0 \ni x \to 0} \int_\Gamma \mathcal{T}(x,\xi)\, ds_\xi \, \psi(0) \\
&= \Big(1 - \int_\Gamma \mathcal{T}(0,\xi)\, ds_\xi \Big) \psi(0) + \int_\Gamma \mathcal{T}(0,\xi)\, \psi(\xi) ds_\xi,
\end{aligned}$$

where 1 is the identity matrix.

So, in the case $\psi \in C(\Gamma)$ we shall use the following definition of $(T\psi)(x)$ for all $x \in \Gamma$:

$$(T\psi)(x) = \begin{cases} \displaystyle\int_\Gamma \mathcal{T}(x,\xi)\,\psi(\xi) ds_\xi, & x \in \Gamma\backslash 0, \\ \displaystyle\Big(1 - \int_\Gamma \mathcal{T}(0,\xi)\,ds_\xi\Big)\psi(0) + \int_\Gamma \mathcal{T}(0,\xi)\,\psi(\xi) ds_\xi, & x = 0. \end{cases}$$

Defined in this way, the operator T maps the space $C(\Gamma)$ continuously into itself.

Let ζ and χ stand for nonnegative functions in the class $C^\infty(\mathbb{R}^3)$ which are equal to 1 in the ring $\{x : \rho < 2|x| < 4\rho\}$ for some $\rho > 0$ and vanishing outside the ring $\{x : \rho < 4|x| < 16\rho\}$. Besides, we assume the following two properties to hold:

(*i*) $|\partial_x^\sigma \zeta(x)| \le C_\sigma\, \rho^{-|\sigma|}, \qquad |\partial_x^\sigma \chi(x)| \le C_\sigma\, \rho^{-|\sigma|}$

(*ii*) one of the inequalities $|x| < d\,|\xi|$ or $|\xi| < d\,|x|$ with some $d \in (0,1)$ are valid on the support of the function

$$(x,\xi) \to \zeta(x)\big(1 - \chi(\xi)\big).$$

Lemma 1 *Let $1 < p < \infty$. If $u \in W_p^l(\Gamma\backslash 0, \mathrm{loc}) \cap L_1(\Gamma)$ is a solution of the equation $(1+T)u = \varphi$, then*

$$\|\zeta\, u\|_{V^l_{p,l}(\Gamma)} \le c\Big(\|\chi\varphi\|_{V^l_{p,l}(\Gamma)} + \Big(\int_\Gamma\Big(\int_\Gamma \frac{|\chi(x)||u(\xi)|}{(|x|+|\xi|)^2} ds_\xi\Big)^p ds_x\Big)^{1/p}\Big).$$

Proof. Let ρ be so small that Γ coincides with ∂K^+ on the ring $\{x : \rho < 4|x| < 16\rho\}$. For any $j = 0, 1, \dots, l$, let $\zeta_{\rho,j}$ stand for functions in the class $C^\infty(\mathbb{R}^3)$ such that for each pair $\zeta_{\rho,j}$, $\zeta_{\rho,j+1}$ the properties (*i*) and (*ii*) are valid. Multiplying the equality $(1+T)u = \varphi$ by $\zeta_{\rho,0}$, we obtain

$$\zeta_{\rho,0} u + \zeta_{\rho,0} T \zeta_{\rho,1} u = \varphi_\rho, \tag{2.1}$$

where

$$\varphi_\rho = \zeta_{\rho,0}\,\varphi - \zeta_{\rho,0}\,T\,(1 - \zeta_{\rho,1})\,u.$$

Putting

$$x = \rho X, \qquad v(X) = u(\rho X), \qquad \psi(x) = \varphi_\rho(\rho X)$$

and using (2.1), we conclude that the vector-valued function $h(X) = v(X)\,\zeta_{1,0}(X)$ satisfies the equation

$$h + T_1 h = \psi + \big(T_1\,\zeta_{1,0} - \zeta_{1,0}\,T_1\big)\zeta_{1,1} v \tag{2.2}$$

on Γ_1. Here Γ_1 is a smooth surface without boundary, coinciding with ∂K^+ in the spherical layer $\{x : 1 < 4|x| < 16\}$, and $\frac{1}{2}T_1$ is the operator of the direct value of the double layer potential on Γ_1.

From the explicit formula for the kernel $\mathcal{T}_1(x,y)$ of T_1 one obtains the estimate:

$$|\partial_{x'}^\sigma\,\partial_{y'}^\tau \mathcal{T}_1(x,y)| \le c\,|x-y|^{-2-|\sigma|-|\tau|}. \tag{2.3}$$

Hence

$$|\partial_{x'}^\sigma\,\partial_{y'}^\tau \mathcal{K}_1(x,y)| \le c\,|x-y|^{-1-|\sigma|-|\tau|},$$

where $\mathcal{K}_1(x,y)$ is the kernel of

$$[T_1, \zeta_{1,0}] = T_1\,\zeta_{1,0} - \zeta_{1,0}\,T_1.$$

Here and in what follows, the symbol $\partial_{x'}^{\sigma}$ means the derivative of order $|\sigma|$ in a local coordinate system (x' is the coordinate of the point x).

Using the Calderon-Zygmund theorem on the continuity of a singular operator in the space L_p, we conclude that the mapping

$$W_p^k(\Gamma_1) \ni \varphi \to [T_1, \zeta_{1,0}]\varphi \in W_p^{k+1}(\Gamma_1) \tag{2.4}$$

is continuous for all $k = 0, 1, \ldots$. Here W_p^k is the Sobolev space of functions whose derivatives up to order k are in L_p.

This and the fact that $(1+T_1)^{-1}$ continuously maps $W_p^k(\Gamma_1)$ onto itself imply the estimate

$$\|h\|_{W_p^l(\Gamma_1)} \le C\Big(\|\psi\|_{W_p^l(\Gamma_1)} + \|\zeta_{1,1}v\|_{W_p^{l-1}(\Gamma_1)}\Big), \qquad l = 1, 2, \ldots$$

for solutions of (2.2). Returning back to the variable $x = \rho\, X$, we arrive at the inequality

$$\|\zeta_{\rho,0}u\|_{V^l_{p,l}(\Gamma)} \le C\Big(\|\varphi_\rho\|_{V^l_{p,l}(\Gamma)} + \|\zeta_{\rho,1}u\|_{V^{l-1}_{p,l-1}(\Gamma)}\Big).$$

If $l \ge 1$, we subsequently multiply $(1+T)u = \varphi$ by the functions $\zeta_{\rho,j}$, $j = 1, 2, \ldots, l-1$, and use similar arguments. Then, after $l-1$ steps, we obtain

$$\begin{aligned} &\|\zeta_{\rho,0}u\|_{V^l_{p,l}(\Gamma)} \\ &\le C\Big(\|\zeta_{p,l}\varphi\|_{V^l_{p,l}(\Gamma)} + \|\zeta_{p,l}u\|_{L_p(\Gamma)} + \sum_{0\le j\le l-1} \|\zeta_{\rho,0}\, T(1-\zeta_{\rho,j+1})u\|_{V^{l-j}_{p,l-j}(\Gamma)}\Big) \end{aligned} \tag{2.5}$$

If $|x| < d\,|\xi|$, $d \in (0,1)$, then $|x-\xi| \ge |\xi| - |x| > (1-d)\,|\xi|$.

Thus, the estimate

$$|x-\xi| > (1-d)(|x| + |\xi|)/2$$

holds on the set

$$\{(x,\xi) : |x| < d\,|\xi|\} \cup \{(x,\xi) : |\xi| < d\,|x|\}.$$

Estimating the last sum on the right-hand side of (2.5) by (2.3), we complete the proof of lemma.

Lemma 2 *If $u \in C^{l,\alpha}(\Gamma\backslash 0, \mathrm{loc})$ is a solution of the equation $(1+T)u = \varphi$, then*

$$\|\zeta\, u\|_{N^{l,\alpha}_{l+\alpha}(\Gamma)} \le c\Big(\|\chi\varphi\|_{N^{l,\alpha}_{l+\alpha}(\Gamma)} + \int_\Gamma (1 + \rho\,(|\xi| + \rho)^{-3})\,|u(\xi)|\,ds_\xi\Big).$$

Proof. It is similar to that of Lemma 1. The only difference is that one should use the continuity of the mappings

$$C^{k,\alpha}(\Gamma_1) \ni \varphi \;\to\; [T_1, \zeta_{1,0}]\,\varphi \in C^{k+1,\alpha}(\Gamma_1),$$

$$C(\Gamma_1) \ni \varphi \;\to\; [T_1, \zeta_{1,0}]\,\varphi \in C^{0,\alpha}(\Gamma_1)$$

instead of (2.4). The inequality

$$\|\zeta_{\rho,j} T(1-\zeta_{\rho,j+1})u\|_{N^{k,\alpha}_{k+\alpha}(\Gamma)} \le c\int_\Gamma \big(1+\rho\,(|\xi|+\rho)^{-3}\big)\,|u(\xi)|\,ds_\xi$$

follows from (2.3) and the estimate

$$|\mathcal{T}(x,\xi)| \le C\big(1+|x|\,|\xi|^{-3}\big) \quad \text{on} \quad \{(x,\xi)\in\Gamma : 2|x|<|\xi|\}. \tag{2.6}$$

The last estimate results from

$$|\cos(n_y, x-y)| \le |\sin(y,x-y)| = |\sin(x,y)|\,|x|\,|x-y|^{-1} \le |x|\,|x-y|^{-1},$$

where $x\in\partial K^+$, $y\in\partial K^+$, and n_y is the normal vector to ∂K^+ at y. The lemma is proved.

In what follows, $*$ denotes the passage to the formally adjoint operator.

Lemma 3 *Let $1<p<\infty$. If $u\in W^l_p(\Gamma\backslash 0,\mathrm{loc})$ is a solution of the equation*

$$(1+T^*)v=\psi,$$

then

$$\|\zeta\, v\|_{V^l_{p,l}(\Gamma)} \le C\Big(\|\chi\psi\|_{V^l_{p,l}(\Gamma)} + \Big(\int_\Gamma\Big(\int_\Gamma |\chi(x)|(|x|+|\xi|)^{-2}|v(\xi|\,ds_\xi\Big)^p ds_x\Big)^{1/p}\Big).$$

This assertion can be proved similarly to the proof of Lemma 1 with T replaced by T^*.

Lemma 4 *If $u\in C^{l,\alpha}(\Gamma\backslash 0,\mathrm{loc})$ is a solution of the equation $(1+T^*)v=\psi$, then*

$$\|\zeta\, v\|_{N^{l,\alpha}_{l+\alpha}(\Gamma)} \le C\Big(\|\chi\psi\|_{N^{l,\alpha}_{l+\alpha}(\Gamma)} + \int_\Gamma (|\xi|+\rho)^{-2}|v(\xi)|\,ds_\xi\Big).$$

The proof of this lemma is similar to that of Lemma 3.

2.2 Invertibility theorems for $1+T$ and $1+T^*$

Let $L_{p,t}(\Gamma)$ be the space defined in Introduction.

Lemma 5 *The operators T and T^* are continuous in the space $L_{p,t}(\Gamma)$ for all p and t such that*

$$1<p\le\infty, \qquad 0<t+2/p<2.$$

Moreover, the operator T is continuous in $L_\infty(\Gamma)$.

Proof. We write the operator T defined by

$$(T\varphi)(x) = \int_\Gamma \mathcal{T}(x,\xi)\,\varphi(\xi)\,ds_\xi, \tag{2.7}$$

as the sum of three integrals T_k, $k=1,2,3$, over the sets Γ_k:

$\Gamma_1=\{\xi\in\Gamma : 2\,|\xi|<|x|\}$, $\Gamma_2=\{\xi\in\Gamma : |x|<2\,|\xi|<4|x|\}$, $\Gamma_3=\{\xi\in\Gamma : |\xi|>2\,|x|\}$.

By the definition of $\mathcal{T}(x,\xi)$ we have

$$|\mathcal{T}(x,\xi)| \le \begin{cases} c\,|\xi|^{-2}, & 2\,|x| < |\xi|, \\ c\,|x|^{-1}|x-\xi|^{-1}, & |\xi| < 2\,|x| < 4\,|\xi|, \\ c\,|x|^{-2}, & |x| > 2\,|\xi|, \end{cases} \tag{2.8}$$

We prove the continuity of each T_k. Clearly, it suffices to show that

$$\|T_k\varphi\|_{L_{p,t}(\Gamma_0)} \le C\,\|\varphi\|_{L_{p,t}(\Gamma_0)}, \qquad k=1,2,3, \tag{2.9}$$

for φ supported by $\Gamma_0 = \Gamma \cap \partial K^+$.

Let

$$F(r) = \int_0^r f(t)\,dt \qquad \text{for } \alpha < -1/p,$$

and

$$F(r) = \int_r^\infty f(t)\,dt \qquad \text{for } \alpha > -1/p.$$

Then Hardy's inequality

$$\|F\|_{L_{p,\alpha}(\mathbb{R}^1_+)} \le C\,\|f\|_{L_{p,\alpha+1}(\mathbb{R}^1_+)} \tag{2.10}$$

is valid. Let $\gamma = \partial K^+ \cap S^2$, where S^2 is the unit sphere in $\mathbb{R}^3$ centered at the origin. We introduce the function $\overline{\varphi}$ on $\mathbb{R}^1_+$ by

$$\overline{\varphi} = \int_\gamma \varphi(ry)\,dl_y.$$

Setting

$$F(r) = \int_0^r \tau\,|\overline{\varphi}(\tau)|\,d\tau,$$

by (2.8) we have

$$\|T_1\varphi\|_{L_{p,t}(\Gamma_0)} \le C\,\|F\|_{L_{p,t-2+1/p}(\mathbb{R}^1_+)}.$$

Together with (2.10) this implies (2.9) for $k=1$, $t < 2-2/p$.

In a similar way, setting

$$F(r) = \int_r^\infty \tau^{-1}|\overline{\varphi}(\tau)|\,d\tau$$

and using (2.8) and (2.10), we arrive at (2.9) for $k=3$, $t > -2/p$.

Since $|x| < 2\,|\xi| < 4\,|x|$ on Γ_2, the operator with the kernel $|x|^{-t}\mathcal{T}_2(x,\xi)\,|\xi|^t$ is continuous in $L_p(\Gamma)$ for all t, where $\mathcal{T}_2(x,\xi)$ is the kernel of T_2. Hence the operator T is continuous in $L_{p,t}(\Gamma)$ for $1 < p \le \infty$, $0 < t + 2/p < 2$.

Replacing T by T^* in the above argument and using (2.8) for the kernel $\mathcal{T}^*(x,\xi)$, we conclude that T^* is continuous in $L_{p,t}(\Gamma)$ for $0 < t + 2/p < 2$.

We showed that for $p=\infty$, $t=0$ the estimate (2.9) holds for $k=1$ and $k=2$. By (2.6) this estimate is valid for $k=3$. Thus, the operator T is continuous in $L_\infty(\Gamma)$. The proof is complete.

Now we present estimates for the kernels of $(1+T)^{-1}$ and $(1+T^*)^{-1}$ obtained in [MG] to be used henceforth.

Theorem 1 (see [MG]) *Let $0 < \beta < 1$ and let l be a positive integer. It $f \in N^{l,\alpha}_{\beta+l}(\Gamma)$ and $g \in N^{l-1,\alpha}_{\beta+l}(\Gamma)$, then*

$$(1+T)^{-1} f = (1+L) f, \qquad (1+T^*)^{-1} g = (1+M) g. \tag{2.11}$$

Here L and M are integral operators on $\Gamma\backslash 0$ with the kernels $\mathcal{L}(x,y)$ and $\mathcal{M}(x,y)$ obeying the estimates

$$|\mathcal{L}(x,y)| \le \begin{cases} c\,|y|^{-2}(|x|/|y|)^{\varkappa-\varepsilon} + c\,|y|^{\varkappa-1-\varepsilon}, & 2|x| < |y| \\ c\,|y|^{-1}|x-y|^{-1}, & |y| < 2|x| < 4|y|, \\ c\,|x|^{-1}|y|^{-1}(|y|/|x|)^{\varkappa-\varepsilon}, & |x| > 2|y|, \end{cases} \tag{2.12}$$

and

$$|\mathcal{M}(x,y)| \le \begin{cases} c\,|y|^{-2}(|x|/|y|)^{\varkappa-\varepsilon}, & 2|x| < |y| \\ c\,|y|^{-1}|x-y|^{-1}, & |y| < 2|x| < 4|y|, \\ c\,|x|^{-1}|y|^{-1}(|y|/|x|)^{\varkappa-\varepsilon} + c\,|y|^{\varkappa-1-\varepsilon}, & |x| > 2|y|, \end{cases} \tag{2.13}$$

where $\varkappa$ is a number in $(0,1]$ depending on the shape of the cone K^+ and ε is a sufficiently small positive number.

Theorem 2 *Let $1 < p \le \infty$, $0 < t + 2/p < 1 + \varkappa$, $1 - \varkappa < \beta + 2/p < 2$. Then the operators $(1+T)^{-1}$ and $(1+T^*)^{-1}$ are continuous in the spaces $L_{p,t}(\Gamma)$ and $L_{p,\beta}(\Gamma)$ respectively. The representations (2.11) can be extended to all functions $f \in L_{p,t}(\Gamma)$ and $g \in L_{p,\beta}(\Omega)$. Moreover, the operator $(1+T)^{-1}$ is continuous in $L_\infty(\Gamma)$ and $C(\Gamma)$.*

Proof. By (2.12), the arguments used in the proof of Lemma 5 show that the operator L in (2.11) is continuous in $L_\infty(\Gamma)$ and $L_{p,t}(\Gamma)$ for $0 < t + 2/p < 1 + \varkappa$.

Since the space $C_0^\infty(\Gamma\backslash 0)$ is dense in $L_{p,t}(\Gamma)$ for $p < \infty$, the first representation in (2.11) extends to all $f \in L_{p,t}(\Gamma)$ for $0 < t + 2/p < 1 + \varkappa$. For $f \in L_{\infty,t}(\Gamma)$ with $0 < t < 1 + \varkappa$ the same representation is valid by the embedding

$$L_{\infty,t}(\Gamma) \subset L_{2,t-1+\varepsilon}(\Gamma), \qquad \varepsilon = (1 + \varkappa - t)/2.$$

Thus, $(1+T)^{-1}$ is continuous in $L_{p,t}(\Gamma)$ and $L_\infty(\Gamma)$.

Let f be the restriction of an arbitrary function from $C^\infty(\mathbb{R}^3)$ to Γ and let $\varphi = (1+T)^{-1} f$. Using the identity $T(f(0)) = f(0)$, we obtain

$$\varphi = (1+T)^{-1}(f - f(0)) + \frac{1}{2} f(0).$$

By (2.11),

$$\varphi = f + L(f - f(0)) - \frac{1}{2} f(0)$$

which along with (2.12) and $f(y) - f(0) = O(|y|)$ shows that φ is continuous at the point 0. Since $(1+T)^{-1}$ is a bounded operator in $L_\infty(\Gamma)$ and since it maps a dense subset of $C(\Gamma)$ into $C(\Gamma)$, it follows that $(1+T)^{-1}$ is a bounded operator in $C(\Gamma)$.

By Theorem 1 we see that the second formula in (2.11) extends to $g \in L_{p,\beta}(\Gamma)$ and the operator $(1+T^*)^{-1}$ is continuous in $L_{p,\beta}(\Gamma)$. The proof is complete.

Lemma 6 *The operators T and T^* are continuous in the space $V^l_{p,\beta+l}(\Gamma)$ for $1 < p < \infty$, $0 < \beta + 2/p < 2$, $l = 1, 2, \ldots$ and in the space $N^{l,\alpha}_{\delta+l+\alpha}(\Gamma)$ for $0 \le \delta < 2$, $\alpha \in (0,1)$, $l = 1, 2, \ldots$.*

Proof. Let $\{\zeta_j\}_{-\infty<j<\infty}$ be a partition of unity on $\mathbb{R}^3\backslash 0$ subbordinate to the covering U_j, where $U_j = \{x : 2^{j-1} < |x| < 2^{j+1}\}$, and let χ_j be a function supported by $\{x : 2^{j-2} < |x| < 2^{j+2}\}$ such that $\zeta_j\, \chi_j = \zeta_j$. Suppose that

$$|\partial^\sigma \zeta_j| + |\partial^\sigma \chi_j| \le C_\sigma\, 2^{-j|\sigma|}$$

for all multiindices σ. Moreover, we assume that one of two inequalities $|x| < d\,|\xi|$ or $|\xi| < d\,|x|$, where d is a number in the interval $(0, 1)$, is valid on the support of the function

$$(x, \xi) \;\to\; \zeta_j(x)(1 - \chi_j(\xi)).$$

We write the function $\zeta_j T\varphi$ as

$$\zeta_j T\varphi = \zeta_j\, T\chi_j\, \varphi + \zeta_j T(1 - \chi_j)\, \varphi.$$

Since the operator of the direct value of the double layer potential on smooth surface is continuous in Sobolev spaces, we have

$$\begin{aligned} &\|\zeta_j T\varphi\|^p_{V^l_{p,l}(\Gamma)} \\ &\le C\Big(\|\chi_j\, \varphi\|^p_{V^l_{p,l}(\Gamma)} + \int_\Gamma \Big(\int_\Gamma |\chi_j(x)|(|x| + |\xi|)^{-2}|u(\xi)|\, ds_\xi\Big)^p ds_x\Big). \end{aligned} \tag{2.14}$$

Let N be so large that the ball of radius 2^N centered at 0 contains Γ. Multiplying inequalities (2.14) by $2^{jp\beta}$ and adding them for all $j = -\infty, \dots, N$, we obtain

$$\begin{aligned} &\|u\|^p_{V^l_{p,\beta+l}(\Gamma)} \\ &\le C\Big(\|\varphi\|^p_{V^l_{p,\beta+l}(\Gamma)} + \int_\Gamma \Big(|x|^{p\beta} \int_\Gamma (|x| + |\xi|)^{-2}|\varphi(\xi)|\, ds_\xi\Big)^p ds_x\Big). \end{aligned} \tag{2.15}$$

To obtain the continuity of T in $V^l_{p,\beta+l}(\Gamma)$ with $0 < \beta + 2/p < 2$, it remains to show that the right-hand side of (2.15) is estimated by $C\, \|r^\beta \varphi\|^p_{L_l(\Gamma)}$.

Clearly, it suffices to prove that

$$\int_{\Gamma_0} \Big(|x|^{p\beta} \int_{\Gamma_0} (|x| + |\xi|)^{-2}|\varphi(\xi)|\, ds_\xi\Big)^p ds_x \le C\, \|\varphi\|_{L_{p,\beta}(\Gamma)} \tag{2.16}$$

holds for functions supported by $\Gamma_0 = \Gamma \cap \partial K^+$. Let

$$\overline{\varphi}(r) = \int_\gamma \varphi(ry)\, d\gamma_y,$$

where $\gamma = \partial K^+ \cap S^2$. The left-hand side of (2.16) is majorized by

$$\begin{aligned} &C \int_0^\infty r^{p\beta+1} \Big(\int_0^\infty \tau(r^2 + \tau^2)^{-1}|\overline{\varphi}(\tau)| d\tau\Big)^p dr \\ \le C &\int_0^\infty r^{p(\beta-2)+1} \Big(\int_0^r \tau|\overline{\varphi}(\tau)| d\tau\Big)^p dr + C \int_0^\infty r^{p\beta+1} \Big(\int_r^\infty \tau^{-1}|\overline{\varphi}(\tau)| d\tau\Big)^p dr. \end{aligned}$$

Using Hardy's inequality (2.10), we arrive at (2.16).

In the case of Hölder spaces, inequality (2.14) takes the form

$$\begin{aligned} &\|\zeta_j T\varphi\|_{N^{l,\alpha}_{l+\alpha}(\Gamma)} \\ &\leq C\Big(\|\chi_j\, \varphi\|_{N^{l,\alpha}_{l+\alpha}(\Gamma)} + \int_\Gamma \Big(1+2^j(|\xi|+2^j)^{-3}\Big)|\varphi(\xi)|\, ds_\xi\Big). \end{aligned} \tag{2.17}$$

Multiplying both sides by $2^{j\delta}$ and estimating the last integral in (2.17), we arrive at

$$\|\zeta_j T\varphi\|_{N^{l,\alpha}_{l+\alpha+\delta}(\Gamma)} \leq C\Big(\|\chi_j\, \varphi\|_{N^{l,\alpha}_{l+\alpha+\delta}(\Gamma)} + \|\varphi\|_{L_{\infty,\delta}(\Gamma)}\Big)$$

for $0 \leq \delta < 2$. Combining this with Lemma 5, we conclude that T is continuous in the spaces $N^{l,\alpha}_{l+\alpha+\delta}(\Gamma)$ for $0 \leq \delta < 2$.

The continuity of T^* is proved in a similar way.

Theorem 3 *Let $1 < p \leq \infty$. Then the operators $(1+T)^{-1}$ and $(1+T^*)^{-1}$ are continuous in the space $V^l_{p,\beta+l}(\Gamma)$ with $0 < \beta + 2/p < 1 + \varkappa$ and $1 - \varkappa < \beta + 2/p < 2$, respectively, as well as in $N^{l,\alpha}_{l+\alpha+\delta}(\Gamma)$ with $0 < \delta < 2$ and $1 - \varkappa < \delta < 2$, respectively.*

Proof. The arguments used in the proof of Lemma 6 show that the assertions of the theorem follow from quasilocal estimates (see Lemmas 1 -4) and the continuity of $(1+T)^{-1}$ and $(1+T^*)^{-1}$ in the space $L_{p,t}(\Gamma)$ (see Theorem 2).

2.3 Invertibility theorem for the operator $1+T$ in weighted Hölder spaces with nonhomogeneous norms

Let G^+ be a domain with compact closure $\overline{G^+}$ bounded by Γ. We denote by $C^{l,\alpha}_\beta(G^+)$, $0 < l + \alpha - \beta < 1$, $l = 1, 2, \ldots$, the space of functions with continuous derivatives up to order l in $G^+\backslash 0$ endowed with the norm

$$\begin{aligned} \|u\|_{C^{l,\alpha}_\beta(G^+)} &= \sup_{x\in G^+} |x|^\beta [u]^{l+\alpha}_{B(|x|/2,x)\cap G^+} \\ &+ \sup_{x\in G^+} |x|^{\beta-l-\alpha+1} \sum_{|\sigma|=1} |\partial^\sigma_x u(x)| + \|u\|_{C^{l+\alpha-\beta}(\overline{G^+})}. \end{aligned} \tag{2.18}$$

Here $B(r,x)$ is the open ball in $\mathbb{R}^3$ of radius r centered at x,

$$[u]^\rho_\Omega = \sup_{x,y\in\Omega} \sum_{|\sigma|=[\rho]} |x-y|^{[\rho]-\rho} |\partial^\sigma_x u(x) - \partial^\sigma_y u(y)|,$$

$[\rho]$ is the integer part of ρ, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is a multiindex of order $|\sigma| = \sigma_1 + \sigma_2 + \sigma_3$, and $\partial^\sigma_x = \partial^{|\sigma|}/\partial x_1^{\sigma_1} \partial x_2^{\sigma_2} \partial x_3^{\sigma_3}$.

Similarly, for $G^- = \mathbb{R}^3 \backslash \overline{G^+}$ we introduce the space $C^{l,\alpha}_\beta(G^-)$. We say that a function supported near 0 belongs to this space if the norm obtained from (2.18) after replacement of G^+0 by G^- is finite. If $\mathrm{dist}(\mathrm{supp}u, 0) \geq 1$, the function u belongs to $C^{l,\alpha}_\beta(G^-)$ if and only if the norm

$$\sup_{x\in G^-} |x|^{l+1+\alpha} [u]^{l+\alpha}_{B(|x|/2,x)} + \sup_{x\in G^-} |x|\, |u(x)|$$

is finite. For an arbitrary function. the norm is obtained with the help of a partition of unity.

The space of traces on $\Gamma\backslash 0$ for functions in $C^{l,\alpha}_{\beta}(G^{\pm})$ is denoted by $C^{l,\alpha}_{\beta}(\Gamma)$.

Further, let $C^{l,\alpha}_{\beta}(G)$ be the space of functions in $G = G^{+} \cap G^{-}$ whose restrictions $u^{\pm}$ to $G^{\pm}$ belong to $C^{l,\alpha}_{\beta}(G^{\pm})$ and

$$\|u\|_{C^{l,\alpha}_{\beta}(G)} = \sum_{\pm} \|u^{\pm}\|_{C^{l,\alpha}_{\beta}(G^{\pm})}.$$

Consider the transmission problem

$$\begin{aligned} &\Delta^{*} u = 0 \qquad \text{in } G, \\ &u^{+} - u^{-} = \varphi \quad \text{on } \Gamma, \qquad (\mathcal{N}u)^{+} - (\mathcal{N}u)^{-} = \psi \ \text{ on } \Gamma\backslash 0. \end{aligned} \tag{2.19}$$

Here

$$\Delta^{*} = \mu\,\Delta + (\lambda + \mu)\nabla \mathrm{div},$$

$\mathcal{N} = \mathcal{N}(\partial_x, n_x)$ is the pseudostress operator, $u^{\pm}$ is the limit value of u on Γ as the point $x \in G^{\pm}$ approaches $x_0 \in \Gamma$.

Lemma 7 *Let $0 < \alpha - \beta < 1$ and let l be a positive integer. If $\varphi \in C^{l,\alpha}_{\beta+l}(\Gamma)$ and $\psi \in N^{l-1,\alpha}_{\beta+l}(\Gamma)$, then there exists only one solution $u \in C^{l,\alpha}_{\beta+l}(G)$ of (2.19) and*

$$\|u\|_{C^{l,\alpha}_{\beta+l}(G)} \leq C\Big(\|\varphi\|_{C^{l,\alpha}_{\beta+l}(\Gamma)} + \|\psi\|_{N^{l-1,\alpha}_{\beta+l}(\Gamma)}\Big).$$

Proof. The homogeneous problem (2.19) (i.e. $\varphi = 0$, $\psi = 0$) is equivalent to the equation

$$\Delta^{*} u = 0 \qquad \text{in } \mathbb{R}^3\backslash 0. \tag{2.20}$$

Consider the operator pencil on the unit sphere S^2 defined by

$$(s_\gamma v)(\theta) = r^{-\gamma+2}\Delta^{*} r^{\gamma}\, v(\theta),$$

where v is a vector-valued function on S^2. It is known that eigenvalues of s_γ are integer. Moreover, the multiplicity of the eigenvalue $\gamma = 0$ is equal to 3 and the Jordan chains corresponding to $\gamma = 0$ consist of the only eigenfunction $v = \text{const}$. Using the fact that problem (2.19) is solvable in weighted spaces with homogeneous norms together with the asymptotic representation of the solution of (2.19) near the conic point (see [MP]), we complete the proof of the lemma.

Theorem 4 *Let $\alpha \in (0, 1)$ and let l be a positive integer. The operators T and $(1+T)^{-1}$ are continuous in the space $C^{l,\alpha}_{\beta+l}(\Gamma)$ with $0 < \alpha - \beta < 1$ and $0 < \alpha - \beta < \varkappa$, respectively.*

Proof. Let $u \in C^{l,\alpha}_{\beta+l}(G)$ be a solution of (2.19) foe $\psi = 0$ and $\varphi \in C^{l,\alpha}_{\beta+l}(\Gamma)$. By Lemma 7,

$$\|u\|_{C^{l,\alpha}_{\beta+l}(G)} \leq C\,\|\varphi\|_{C^{l,\alpha}_{\beta+l}(\Gamma)}. \tag{2.21}$$

By Lemma 7 from [MG] we can represent u in the form

$$u = W(u^{+} - u^{-}) = W\,\varphi,$$

where $W\varphi$ is the double layer potential. Using this together with (2.21) and the relation $T\varphi = 2(W\varphi)^+ - \varphi$ (see Lemma 5 of [MG]), we conclude that the operator T is continuous in $C^{l,\alpha}_{\beta+l}(\Gamma)$ for $0 < \alpha - \beta < 1$.

The continuity of the operator $(1+T)^{-1}$ is an immediate corollary of the representation of $(1+T)^{-1}$ in terms of inverse operators of boundary value problems (see Theorem 4 in [MG]) and the continuity of these operators (see Lemmas 1 and 4 in [MG]).

3 Pointwise estimates for derivatives of the kernels of the operators $(1+T)^{-1}$ and $(1+T^*)^{-1}$. Continuity of $(1+T)^{-1}$ in the Hölder space $C^{0,\alpha}(\Gamma)$

3.1 Pointwise estimates for derivatives of the kernels $\mathcal{L}(x,y)$ and $\mathcal{M}(x,y)$ of operators L and M in (2.11)

Lemma 8 *If $x \neq y$, then*

$$\mathcal{L}(x,y) = \big(\mathcal{M}(y,x)\big)^\star, \tag{3.1}$$

where $\mathcal{M}^\star$ is the ajoint matrix of $\mathcal{M}$.

Proof. Let

$$\varphi \in L_{2,t}(\Gamma), \qquad \psi \in L_{2,-t}(\Gamma), \quad -1 < t < \varkappa.$$

We substitute the functions

$$u = (1+L)\,\varphi, \qquad v = (1+M)\,\psi$$

into the equality

$$\big((1+T)u, v\big)_\Gamma = \big(u, (1+T^*)v\big)_\Gamma,$$

where $(\cdot,\cdot)_\Gamma$ is the scalar product in $L_2(\Gamma)$.

Combining this with the identities

$$(1+T)(1+L)\varphi = \varphi, \qquad (1+T^*)(1+M)\psi = \psi,$$

which follow from Theorem 2, we arrive at

$$(\varphi, M\psi)_\Gamma = (L\varphi, \psi)_\Gamma.$$

By Fubini's theorem, one can write the last equality as

$$(\varphi, K\psi)_\Gamma = 0, \qquad \varphi \in L_{2,t}(\Gamma), \qquad \psi \in L_{2,-t}(\Gamma), \tag{3.2}$$

where K is the integral operator with the kernel

$$\mathcal{K}(x,y) = \mathcal{M}(x,y) - \mathcal{L}^\star(y,x).$$

By Theorem 2, the kernel $|x|^{-t}\mathcal{K}(x,y)|y|^t$ generates a linear functional on $L_2(\Gamma) \times L_2(\Gamma)$. Therefore, it follows from (3.2) that $\mathcal{K}(x,y) = 0$ almost everywhere on $\Gamma \times \Gamma$.

The representations for the kernels $\mathcal{L}(x,y)$ and $\mathcal{M}(x,y)$ obtained in [MG] show that the functions $(x,y) \to \mathcal{L}(x,y)$ and $(x,y) \to \mathcal{M}(x,y)$ are continuous on the set

$$\{(x,y) \in (\Gamma\backslash 0)\ \times\ (\Gamma\backslash 0) : x \neq y\}.$$

Thus, $\mathcal{K}(x,y) = 0$ for $x \neq y$. The lemma is proved.

Corollary 1 *Let y be a fixed point in $\Gamma\backslash 0$. Then the functions $x \to \mathcal{L}(x,y)$ and $x \to \mathcal{M}(x,y)$ are solutions of the equations*

$$(1+T)\,\mathcal{L}(\cdot,y) = -\mathcal{T}(\cdot,y), \tag{3.3}$$

$$(1+T^*)\,\mathcal{M}(\cdot,y) = -\mathcal{T}^*(\cdot,y), \tag{3.4}$$

which are equivalent to the relations

$$\int_\Gamma \mathcal{L}^\star(x,y)((1+T^*)\psi)(x)\,ds_x = -\int_\Gamma \mathcal{T}^\star(x,y)\,\psi(x)\,ds_x,$$
$$\int_\Gamma \mathcal{M}^\star(x,y)((1+T)\varphi)(x)\,ds_x = -\int_\Gamma \mathcal{T}(x,y)\,\varphi(x)\,ds_x,$$

for $\varphi \in L_{2,t}(\Gamma)$, $\psi \in L_{2,-t}(\Gamma)$, $-1 < t < \varkappa$.

Proof. By Theorem 2 we have the equality

$$(1+M)(1+T^*)\psi = \psi$$

which can be written as

$$\int_\Gamma \mathcal{M}(y,x)((1+T^*)\psi)(x)\,ds_x = -\int_\Gamma \mathcal{T}^*(y,x)\,\psi\,ds_x.$$

Using (3.1) and the identity $\mathcal{T}^*(y,x) = \mathcal{T}(x,y)$, we arrive at (3.3).

Analogously, (3.4) is a corollary of the identity

$$(1+L)(1+T)\,\varphi = \varphi.$$

Theorem 5 *Let x and y lie in the same coordinate neighbouthood. Then*

$$|\partial^\sigma_{x'}\partial^\tau_{y'}\mathcal{L}(x,y)| \le \begin{cases} c\,|x|^{-|\sigma|+\kappa-\varepsilon}|y|^{-2-|\tau|-\varkappa-\varepsilon} + \delta^0_{|\sigma|}\,|y|^{\varkappa-1-|\tau|-\varepsilon}, & 2|x|<|y|,\\ c\,|x|^{-1}|x-y|^{-1-|\sigma|-|\tau|}, & |y|<2|x|<4|y|,\\ c\,|x|^{-1-|\sigma|-\varkappa+\varepsilon}|y|^{-1-|\tau|+\varkappa+\varepsilon}, & |x|>2|y|,\end{cases} \tag{3.5}$$

where ε is an arbitrary small positive number, x' and y' are local coordinates of x and y.

First we prove an auxiliary assertion, where the prime denotes the passage to local coordinates.

Lemma 9 *Let Γ_1 be a smooth surface and let $\frac{1}{2}T_1$ be the operator of the direct value of the double layer potential on Γ_1. If x and y lie in the same coordinate neighbouthood, then*

$$|\partial^\sigma_{x'}\partial^\tau_{y'}\mathcal{H}_1(x,y)| \le C\,|x-y|^{-1-|\sigma|-|\tau|}, \tag{3.6}$$

where $\mathcal{H}_1(x,y)$ is the kernel of the operator H_1 in the representation

$$(1+T_1)^{-1} = 1 + H_1.$$

Proof. We estimate

$$\partial^\tau_{y'}\mathcal{H}_1(x,y), \qquad |\tau| = l.$$

Let $B(\rho,y)$ be the open ball in $\mathbb{R}^3$ with radius ρ and center at y. Further, let $U_{y,\rho}$ and $V_{y,\rho}$ stand for the sets

$$\Gamma_1\cap B(\rho/2,y) \qquad \text{and} \qquad \Gamma_1\cap B(\rho,y),$$

where $\rho = |x-y|/2$.

We introduce the functions $\eta^{(j)}_{y,\rho}$, $j = 0,1,\dots,l+1$ of the class $C^\infty(\mathbb{R}^3)$ which satisfy

(*a*) $\eta^{(j)}_{y,\rho} = 1$ on $B(\rho/2, y)$, $\quad \eta^{(j)}_{y,\rho} = 0$ outside $B(\rho, y)$,

(*b*) $|\partial^\sigma_z \eta^{(j)}_{y,\rho}(z)| \le C_\sigma\, \rho^{-|\sigma|}$,

(*c*) the inequality $|z| < d\,|\xi|$, $d \in (0,1)$, is valid on the support of the function

$$(z,\xi) \to \eta^{(j)}_{y,\rho}(z)\,(1 - \eta^{(j+1)}_{y,\rho}(\xi)).$$

The same argument as in the proof of inequality (2.5) leads to the estimate

$$\rho^{l+\alpha}[v]^{l+\alpha}_{U_{y,\rho}} + \sum_{|\sigma\le l} \rho^{[\sigma|} \sup_{z\in U_{y,\rho}} |\partial^\sigma_{x'} v(z)| \tag{3.7}$$
$$\le C\Big(\rho^{l+\alpha}[\psi]^{l+\alpha}_{V_{y,\rho}} + \sup_{z\in V_{y,\rho}} |\psi(z)| + \sum_{j=0} l\Big(\rho^{l+\alpha-j}[\psi^{(j)}]^{l+\alpha-j}_{V_{y,\rho}} + \sup_{z\in V_{y,\rho}} |\psi^{(j)}(z)|\Big)\Big)$$

for solutions of the equation $(1+T_1^*)v = \psi$, where

$$\psi^{(j)} = \eta^{(j)}_{y,\rho} T_1\,(1 - \eta^{(j+1)}_{y,\rho})\,v.$$

By Lemma 7 in [MG], the kernel $\mathcal{H}_1(x,y)$ obeys the estimate

$$|\mathcal{H}_1(x,y)| \le C\,|x-y|^{-1}.$$

Unifying this with (3.7) and the fact that the function

$$z \to \mathcal{H}_1^*(z,x) = \big(\mathcal{H}_1(x,z)\big)^\star$$

is a solution of the problem

$$(1+T_1^*)\,\mathcal{H}_1^*(\cdot,x) = -\mathcal{T}_1^*(\cdot,x),$$

we conclude that

$$|\partial^\tau_{y'} \mathcal{H}_1(x,y)| \le C\,|x-y|^{-1-|\tau|}. \tag{3.8}$$

Now we estimate

$$\partial^\sigma_{x'}\partial^\tau_{y'}\mathcal{H}_1(x,y), \quad |\sigma| = k, \quad |\tau| = l.$$

Let $\rho = |x-y|/4$. Consider the equation $(1+T_1)\,u = \varphi$ instead of $(1+T_1^*)\,v = \psi$. From (3.7) we obtain

$$\rho^{l+\alpha}[u]^{k+\alpha}_{U_{y,\rho}} + \sum_{|\sigma|\le k} \rho^{[\sigma|} \sup_{\xi\in U_{x,\rho}} |\partial^\sigma_\xi u(\xi)|$$
$$\le C\Big(\sup_{\xi\in V_{x,\rho}} |\varphi(\xi)| + \rho^{k+\alpha}[\varphi]^{k+\alpha}_{V_{x,\rho}} \tag{3.9}$$
$$+ \sum_{j=0}^{k}\Big(\rho^{k+\alpha-j}[\varphi^{(j)}]^{k+\alpha-j}_{V_{x,\rho}} + \sup_{\xi\in V_{x,\rho}} |\varphi^{(j)}(\xi)|\Big)\Big).$$

Here

$$\varphi^{(j)} = \eta^{(j)}_{x,\rho} T_1\,(1 - \eta^{(j+1)}_{x,\rho})\,u.$$

Let $\Delta^l_{y'}$ be the finite difference operator of order l. It is clear that the vecto-valued function

$$\xi \to \Delta^l_{y'} \mathcal{H}_1(\xi, y)$$

satisfies the equation

$$(1+T_1)\, \Delta^l_{y'} \mathcal{H}_1(\cdot, y) = -\Delta^l_{y'} \mathcal{T}_1(\cdot, y).$$

If the shift in the finite difference $\Delta^l_{y'} \mathcal{H}_1(z,y)$ is sufficiently small, then inequalities (3.8) and (3.9) lead to (3.6). The proof is complete.

Proof of Theorem 5. We put $x = |y|\, X$, $y = |y|\, Y$ and denote by $\Gamma_{|y|}$ the image of Γ under the mapping $x \to X$. By (3.3), the function

$$X \to \mathcal{L}_{|y|}(X,Y) = |y|^2 \mathcal{L}(|y|X, |y|Y)$$

satisfies the equation

$$\big(1 + T_{|y|}\big)\, \Delta^l_{Y'} \mathcal{L}_{|y|}(\cdot, Y) = -\Delta^l_{Y'} \mathcal{T}_{|y|}(\cdot, Y) \tag{3.10}$$

on $\Gamma_{|y|}$ for all $y \in \Gamma\backslash 0$. Here $\frac{1}{2}T_{|y|}$ is the operator of the direct value of the double layer potential on $\Gamma_{|y|}$, $\frac{1}{2}\mathcal{T}_{|y|}(X,Y)$ is its kernel and $\Delta^l_{y'}$ is the finite difference of order $l = |\tau|$.

It suffices to consider the case when the point $y \in \Gamma$ lies in a neighbourhood of the vertex of the cone. First we obtain estimates for

$$\partial^\sigma_{x'} \partial^\tau_{y'} \mathcal{L}(x,y) \qquad \text{for } \ |y| < 2|x| \le 4|y|.$$

Let ζ_i, $i = 0, 1, \ldots, k+1$, $k = |\sigma| + 2$ be functions in C^∞ such that

(a) $\zeta_i = 1$ in $1 < 2|\xi| < 4$, $\zeta_i = 0$ outside of the set $1 < 4|\xi| < 16$,

(b) one of the two inequalities $|z| < d\,|\xi|$ and $[\xi| < d\,|z|$, $d \in (0,1)$, is valid on the support of the function $(z,\xi) \to \zeta_i(z)\,(1 - \zeta_{j+1}(\xi))$.

Multiplying (3.10) by ζ_0, we write it as

$$\zeta_0\, \Delta^l_{Y'} \mathcal{L}_{|y|} + \zeta_0\, T_{|y|}\, \zeta_1\, \Delta^l_{Y'} \mathcal{L}_{|y|} = -\zeta_0\, T_{|y|}(1-\zeta_1)\, \Delta^l_{Y'} \mathcal{L}_{|y|}. \tag{3.11}$$

Let Γ_1 be a closed surface coinciding with ∂K^+ on the set $1 < 4\,|\xi| < 16$ (see Section 2 in [MG] and let $\frac{1}{2}T_1$ be the operator of the direct value of the double layer potential on Γ_1. It is clear that a representation similar to (3.11) holds for the kernel $\mathcal{H}_1(X,Y)$ of the operator $(1+T_1)^{-1} - 1$:

$$\zeta_0\, \Delta^l_{Y'} \mathcal{H}_1 + \zeta_0\, T_1\, \zeta_1\, \Delta^l_{Y'} \mathcal{H}_1 = -\zeta_0\, \Delta^l_{Y'} \mathcal{T}_1 - \zeta_0 T_1 (1-\zeta_1)\, \Delta^l_{Y'} \mathcal{H}_1. \tag{3.12}$$

We set

$$\Delta^l_{Y'} \mathcal{L}_{|y|} \; + \; \zeta_0\, \Delta^l_{Y'} \mathcal{H}_1 \; + \; \Delta^l_{Y'} \mathcal{L}'_{|y|}.$$

From (3.11) and (3.12) we obtain that the vector-valued fuction

$$X \to v(X) = \zeta_0\, \Delta^l_{Y'} \mathcal{L}'_{|y|}(X,Y)$$

solves the equation

$$v + T_1\, v = \psi_1 + (T_1\, \zeta_0 - \zeta_0\, T_1)\, \zeta_1\, \Delta^l_{Y'} \mathcal{L}'_{|y|} \tag{3.13}$$

on Γ_1. Here

$$\psi_1 = \zeta_0\, T_1 (1-\zeta_1)\, \Delta^l_{Y'} \mathcal{H}_1 - \zeta_0\, T_{|y|}(1-\zeta_1) \Delta^l_{Y'} \mathcal{L}'_{|y|}.$$

The representations obtained in Section 2 of [MG] and Lemma 9 lead to the estimates

$$|\partial^\tau_{Y'}\mathcal{L}(x,y)| \le \begin{cases} c\,|y|^{-2-|\tau|}(|x|/|y|)^{\varkappa-\varepsilon} + c\,|y|^{\varkappa-1-|\tau|-\varepsilon}, & 2|x|<|y|,\\ c\,|y|^{-1}|x-y|^{-1-|\tau|}, & |y|<2|x|<4|y|,\\ c\,|x|^{-1}|y|^{-1-|\tau|}(|y|/|x|)^{\varkappa-\varepsilon}, & |x|>2|y|, \end{cases} \tag{3.14}$$

and

$$|\partial^\tau_{Y'}(\mathcal{L}_{|y|}(X,Y)-\mathcal{H}_1(X,Y))| \le C, \quad 1<2\,|Y|<4, \quad 1<4\,|X|<16. \tag{3.15}$$

By continuity of the operators

$$\begin{aligned} &(1+T_1)^{-1} : W^k_2(\Gamma_1) \;\to\; W^k_2(\Gamma_1),\\ &[T_1,\zeta_0]: \quad W^k_2(\Gamma_1) \;\to\; W^{k+1}_2(\Gamma_1) \end{aligned}$$

equation (3.13) implies

$$\|v\|_{W^k_2(\Gamma_1)} \le C\Big(\|\psi_1\|_{W^k_2(\Gamma_1)} + \|\zeta_0\,\Delta^l_{Y'}\,\mathcal{L}'_{|y|}\|_{W^{k-1}_2(\Gamma_1)}\Big).$$

It is clear that (3.11) - (3.13) are valid if ζ_i is replaced by ζ_{i+1}. Thus,

$$\|v\|_{W^k_2(\Gamma_1)} \le C\Big(\sum_{i=1}^{k}\|\psi_i\|_{W^k_2(\Gamma_1)} + \|\zeta_k\,\Delta^l_{Y'}\,\mathcal{L}'_{|y|}\|_{L_2(\Gamma_1)}\Big).$$

where

$$\psi_i = \zeta_i\,T_1(1-\zeta_{i+1})\,\Delta^l_{Y'}\mathcal{H}_1 - \zeta_i\,T_{|y|}(1-\zeta_{i+1})\Delta^l_{Y'}\mathcal{L}'_{|y|}.$$

Combining this with (3.14) , (3.15), we conclude that

$$\|v\|_{W^k_2(\Gamma_1)} \le C.$$

Hence, by Sobolev's embedding theorem and Lemma 9, we arrive at the inequality

$$|\partial^\sigma_{X'}\partial^\tau_{Y'}\mathcal{L}_{|y|}(X,Y)| \le C\Big(1+|X-Y|^{-1-|\sigma|-|\tau|}\Big)$$

for $|\tau|=l$, $|\sigma|<\varkappa-3/2$, $1<2\,|X|<4$, $|Y|=1$. Returning back to variables x and y, we write the last estimate as

$$|\partial^\sigma_{x'}\partial^\tau_{y'}\mathcal{L}_(x,y)| \le C\,|x|^{-1}|x-y|^{-1-|\sigma|-|\tau|}$$

for $|y|<2\,|x|<4\,|y|$.

Now we turn to the estimate of $\partial^\sigma_{x'}\partial^\tau_{y'}\mathcal{L}_(x,y)$ on the set $2\,|x|<|y|$. We multiply (3.10) by a function $\eta\in C^\infty(\mathbb{R}^3)$ which is equal to 1 in the ball $B_{1/2}=B(1/2,0)$ and vanishing in $\mathbb{R}^3\backslash B_{2/3}$.

The function

$$X \;\to\; v(X) = \eta(X)\,\Delta^l_{Y'}\mathcal{L}_{|y|}(X,Y)$$

solves the equation $(1+T_{|y|})v=\psi$, where

$$\psi = -\eta\Delta^l_{Y'}\mathcal{T}_{|y|} + (T_{|y|}\eta - \eta\,T_{|y|})\,\Delta^l_{Y'}\mathcal{L}_{|y|}.$$

By Theorem 4,

$$\|v\|_{C^{k,\alpha}_{k+\alpha}}(\Gamma_{|y|}) \le C\,\|\psi\|_{C^{k,\alpha}_{k+\alpha}(\Gamma_{|y|})} \tag{3.16}$$

with $0 < \alpha - \delta < \varkappa$, $k = 1, 2, \dots$. We show that the right-hand side on (3.16) is bounded by a constant independent of y.

Let θ_0 be a function of the class $C^\infty(\mathbb{R}^3)$ such that $\theta_0 = 1$ for $1 < 4\,|X| < 3$ and $\theta = 0$ outside of $\{X : 1 < 5\,|X| < 4\}$. We write the vector-valued function ψ as

$$\psi = \sum_{1 \le k \le 4} \psi_k, \tag{3.17}$$

with

$$\psi_1 = -\eta\, \Delta^l_{Y'} \mathcal{T}_{|y|}, \qquad \psi_2 = -(1 - \theta_0)(\eta T_{|y|} - T_{|y|}\eta)\, \Delta^l_{Y'} \mathcal{L}_{|y|},$$

$$\psi_3 = -\theta_0(\eta T_{|y|} - T_{|y|}\eta)(1 - \theta_0)\Delta^l_{Y'} \mathcal{L}_{|y|},$$

$$\psi_4 = -\theta_0(\eta T_{|y|} - T_{|y|}\eta)\theta_0\, \Delta^l_{Y'} \mathcal{L}_{|y|}.$$

It is clear that

$$\|\psi_1\|_{C^{k,\alpha}_{k+\delta}}(\Gamma_{|y|}) \le C.$$

To obtain the same estimate for ψ_2, it suffices to show that

$$\sup_{X \in \Gamma_{|y|}} \Big(|\psi_2(X)| + \sum_{|\sigma|=1} |D^\sigma_{X'} \psi_2(X)| + \sum_{|\sigma|=1}^{l+1} |X|^{\delta + |\sigma| - \alpha} |D^\sigma_{X'} \psi_2(X)| \Big) \le C. \tag{3.18}$$

We can write ψ_2 in the form

$$\psi_2(X) = (\theta_0(X) - 1) \int_{\Gamma_{|y|}} \mathcal{T}_{|y|}(X, Z)(\eta(X) - \eta(Z))|y|^2\, \Delta^l_{Y'} \mathcal{L}(|y|Z, |y|Y) ds_Z. \tag{3.19}$$

First consider the case $|X| < 1/4$. It is clear that the integrand does not vanish for $|Z| > 2\,|X|$. Hence

$$|\partial^\sigma_X \mathcal{T}_{|y|}(X, Z)| \le C_\sigma |Z|^{-2-|\sigma|}.$$

We write ψ_2 as the sum of two functions $\psi_2^{(1)}$ and $\psi_2^{(2)}$ with integration taken over

$$\Gamma^{(1)}_{|y|} = \{z \in \Gamma_{|y|} : 1 < 2\,|z| < 4\}$$

and

$$\Gamma^{(2)}_{|y|} = \{z \in \Gamma_{|y|} : |z| > 2\}$$

instead of $\Gamma_{|y|}$ in(3.19). Since $|Y| = 1$, estimates (3.14) lead to the inequality

$$|\partial^\sigma_{X'} \psi_2^{(2)}(X)| \le C \int_{\Gamma^{(2)}_{|y|}} |Z|^{-3-|\sigma|-\varkappa+\varepsilon} ds_Z \le C$$

for sufficiently small increment of the argument in $\Delta^l_{Y'} \mathcal{L}_{|y|}$.

The boundedness of $|\partial^\sigma \psi_2^{(1)}|$ follows from the inequality

$$|\Delta^l_{Y'} \int_{\Gamma^{(1)}_{|y|}} |y|^2 \mathcal{L}^*(|y|Y, |y|Z)\, \varphi(Z)\, ds_Z| \le C,$$

which is valid for smooth functions on $\Gamma_{|y|}$. The last assertion results from (3.15) and the continuity of the operator

$$(1+T_1^*)^{-1} : W_2^k(\Gamma_1) \to W_2^k(\Gamma_1).$$

Now let $4\,|X| > 3$. Then the integrand does not vanish for $3\,|Z| < 2$. Hence

$$|\partial_{X'}^\sigma \mathcal{T}_{|y|}(X,Z)| \le C_\sigma |X|^{-2-|\sigma|}.$$

Since $5\,|X| < 4$ on the support of the function $X \to \nabla(1-\theta_2(X))$, by (3.14) we have

$$|\partial_X^\sigma \psi_2(X)| \le C \int_{Z\in\Gamma_{|y|}:3|Z|<2} |X|^{-2-|\sigma|} ds_Z \le C\,|X|^{-2-|\sigma|}$$

for $|X| > 1$. Therefore, (3.18) is valid for $|X| > 1$. It is clear that the same estimate holds for $3 < 4\,|X| < 4$.

The estimate (3.18) for ψ_3 is obtained in a similar manner.

Using a property of the commutator $[\eta, T_{|y|}]$, we obtain

$$\|\psi_4\|_{C^{k,\alpha}_{\delta+k}(\Gamma_{|y|})} \le C\,\|\theta_0 \Delta^l_{Y'} \mathcal{L}_{|y|}\|_{C^{k-1,\alpha}(\Gamma_{|y|}\cap \operatorname{supp}\theta_0)}.$$

Consider a sequence of functions $\theta_j(x)$, $j = 1, 2, \ldots, k$ such that

$$\operatorname{supp}\theta_j \subset (0,1), \qquad \theta_j\,\theta_{i+1} = \theta_j, \quad j = 1, 2, \ldots, k.$$

Multiplying (3.10) by θ_j and using the same arguments as in the proof of Lemma 2, we obtain after l steps that

$$\|\psi_4\|_{C^{k,\alpha}_{\delta+k}(\Gamma_{|y|})} \le C.$$

Thus,

$$\|\eta\,\Delta^l_{Y'} \mathcal{L}_{|y|}\|_{C^{k,\alpha}_{\delta+k}(\Gamma_{|y|})} \le C.$$

Setting $\delta = \alpha - \varkappa + \varepsilon$, $k = |\sigma|$, $l = |\tau|$, we arrive at

$$|\partial_X^\sigma \partial_Y^\tau \mathcal{L}_{|y|}(X,Y)| \le C\,|X|^{\varkappa-|\sigma|-\varepsilon}, \quad 2\,|X| < 1, \ \ |\sigma| > 0.$$

The last estimate is equivalent to

$$|\partial_x^\sigma \partial_y^\tau \mathcal{L}(x,y)| \le C\,|x|^{-|\sigma|}\,|y|^{-2-|\tau|}(|x|/|y|)^{\varkappa-\varepsilon}, \quad 2\,|x| < |y|.$$

It remains to consider the case $|x| > 2\,|y|$. Introducing new cut-off functions η' and θ'_k, obtained from η and θ_k by replacing X by $X/|X|^2$ and using the inequality

$$\|v\|_{N^{k,\alpha}_{\delta+k}(\Gamma_{|y|})} \le C\,\|\psi\|_{N^{k,\alpha}_{\delta+k}(\Gamma_{|y|})}, \qquad 0 \le \delta - \alpha < 1 + \varkappa,$$

instead of (3.16), we get

$$\|\Delta^l_{Y'} \mathcal{L}_{|y|}\|_{N^{k,\alpha}_{\delta+k}(\Gamma_{|y|}\setminus B(2,0))} \le C.$$

Setting $\delta = \alpha + 1 + \varkappa - \varepsilon$, $k = |\sigma|$, $l = |\tau|$, we arrive at (3.5) for $|x| > 2\,|y|$. The theorem is proved.

3.2 Continuity of $(1+T)^{-1}$ in $C^{0,\alpha}(\Gamma)$

Lemma 10 *The operator T is continuous in $C^{0,\alpha}$ for all $\alpha \in (0,1)$.*

Proof. Let $u = T\,\varphi$. By Lemma 5,

$$\sup_{x\in\Gamma} |u(x)| \le \sup_{x\in\Gamma} |\varphi(x)|.$$

Next we prove the inequality

$$[u]^{\alpha}_{\Gamma} \le C\,[\varphi]^{\alpha}_{\Gamma}. \tag{3.20}$$

Let $x, z \in \Gamma$ and let $r = 2\,|x-z|$. We use the equality

$$\int_{\Gamma} \mathcal{T}(x,\xi)\,ds_\xi = 1$$

valid on Γ, where 1 is the 3×3 identity matrix, to write

$$u(x) - u(z) = \sum_{1\le k\le 4} I_k,$$

with

$$I_1 = \int_{\Gamma\cap B(r,x)} \mathcal{T}(x,\xi)\,(\varphi(\xi)-\varphi(x))\,ds_\xi,$$
$$I_2 = -\int_{\Gamma\cap B(r,x)} \mathcal{T}(x,\xi)\,(\varphi(\xi)-\varphi(z))\,ds_\xi,$$
$$I_3 = \int_{\Gamma\cap B(r,x)} \mathcal{T}(z,\xi)\,ds_\xi\,(\varphi(x)-\varphi(z)),$$
$$I_4 = \int_{\Gamma\cap B(r,x)} \big(\mathcal{T}(x,\xi)-\mathcal{T}(z,\xi)\big)\,(\varphi(\xi)-\varphi(x))\,ds_\xi,$$

Since the integral

$$\int_{\Gamma\cap B(r,x)} \mathcal{T}(z,\xi)\,ds_\xi$$

is bounded, (3.20) follows from the inequalities:

$$|I_1| + |I_2| \le C\,[\varphi]^{\alpha}_{\Gamma} \int_{\Gamma\cap B(r,x)} |x-\xi|^{-2+\alpha} ds_\xi \le C\,r^{\alpha}[\varphi]^{\alpha}_{\Gamma}$$

and

$$|I_4| \le C\,[\varphi]^{\alpha}_{\Gamma} \int_{\Gamma\cap B(r,x)} |x-\xi|^{-3+\alpha} ds_\xi \le C\,r^{\alpha}[\varphi]^{\alpha}_{\Gamma},$$

The lemma is proved.

Theorem 6 *The operator $(1+T)^{-1}$ is continuous in $C^{0,\alpha}$ for all $\alpha \in (0,\varkappa)$.*

First we prove the following auxiliary assertion.

Lemma 11 *The inequalities hold:*

$$\int_{\Gamma\setminus B(r,x)} \frac{ds_\xi}{|x-\xi|^s|\xi|^t} \le C\,r^{2-s-t}, \qquad 0\le t<2,\;\; s+t>2, \tag{3.21}$$

and

$$\int_{\Gamma\cap B(r,x)} \frac{ds_\xi}{|x-\xi|^s|\xi|^t} \le C\,r^{2-s-t}, \qquad 0\le t<2,\;\; s+t<2, \tag{3.22}$$

Proof. Let

$$E_x = \{\xi \in \Gamma : 2\,|x-\xi| < |\xi|\}.$$

We write the left-hand side of (3.21) as the sum of two integrals J_1 and J_2 over the sets

$$\Gamma_1 = (\Gamma \cap E_x)\backslash B(r,x), \qquad \Gamma_2 = \Gamma\backslash(E_x \cup B(r,x)).$$

Estimate (3.21) for J_1 is obvious. Since $3\,|x-\xi| > r + |\xi|$ on Γ_2, it follows, after passing to the variable $\rho = |\xi|/r$, that

$$|J_2| \le C \int_\Gamma (r+|\xi|)^{-s}|\xi|^{-t} ds_\xi$$

$$\le C \int_0^\infty |\xi|^{1-t}(r+|\xi|)^{-s} ds_\xi = C\, r^{2-t-s} \int_0^\infty \rho^{1-t}(1+\rho)^{-s} d\rho.$$

Estimate (3.22) is proved in a similar manner.

Proof of Theorem 6. Representation (2.11) shows that it suffices to prove the continuity of L in (2.11) in the space $C^{0,\alpha}(\Gamma)$. Let $v = L\,\psi$. By Theorem 2,

$$\sup_{x\in\Gamma} |v(x)| \le \sup_{x\in\Gamma} |\psi(x)|.$$

It remains to show that

$$[v]^\alpha_\Gamma \le C\,[\psi]^\alpha_\Gamma. \tag{3.23}$$

Equality $\frac{1}{2}(1+T)1 = 1$, where 1 is the 3×3 identity matrix, and relation (2.11) imply

$$-2\int_\Gamma \mathcal{L}(x,\xi)\, ds_\xi = 1 \qquad \text{on } \Gamma.$$

Let $x, z \in \Gamma$ and let $r = 4\,|x-z|$. We write

$$v(x) - v(z) = \sum_{1\le k\le 4} I_k,$$

with

$$I_1 = \int_{\Gamma\cap B(r,x)} \mathcal{L}(x,\xi)\,(\psi(\xi)-\psi(x))\,ds_\xi,$$
$$I_2 = -\int_{\Gamma\cap B(r,x)} \mathcal{L}(z,\xi)\,(\psi(\xi)-\psi(z))\,ds_\xi,$$
$$I_3 = \int_{\Gamma\cap B(r,x)} \mathcal{L}(z,\xi)\,ds_\xi\,(\psi(x)-\psi(z)),$$
$$I_4 = \int_{\Gamma\cap B(r,x)} \big(\mathcal{L}(x,\xi)-\mathcal{L}(z,\xi)\big)\,(\psi(\xi)-\psi(z))\,ds_\xi.$$

Making estimates (2.12) rougher, we obtain

$$|\mathcal{L}(x,y)| \le \begin{cases} c\,|x-\xi|^{-2}, & |x| < 2\,|\xi|, \\ c\,|\xi|^{-1}|x-\xi|^{-1}, & |x| > 2\,|\xi|. \end{cases}$$

Combining this with (3.22), we conclude that

$$|I_1| \le C\,[\psi]^\alpha_\Gamma \int_{\Gamma\cap B(r,x)} \Big(|x-\xi|^{-2+\alpha} + |\xi|^{-1}|x-\xi|^{-1+\alpha}\Big) ds_\xi \le C\,r^\alpha [\psi]^\alpha_\Gamma. \tag{3.24}$$

Estimate (3.24) for I_2 follows in a similar manner. The corresponding inequality for I_3 is a corollary of the boundedness of the integral

$$\int_{\Gamma\cap B(r,x)} \mathcal{L}(x,\xi)\, ds_\xi.$$

It remains to prove the inequality

$$\int_{\Gamma\cap B(r,x)} |\mathcal{L}(x,\xi) - \mathcal{L}(z,\xi)|\, |x-\xi|^\alpha\, ds_\xi \le C\, r^\alpha. \tag{3.25}$$

The case of points x and z lying near the vertex of the cone is the most significant. Without loss of generality we can assume that x and z lie in the same coordinate neighbourhood U. We denote by γ the coordinate diffeomorphism which naps U onto a bounded subset of the plane angle in $\mathbb{R}^2$. Let $\xi \in U$ and let (ρ,θ) be polar coordinates of the point $\gamma\,\xi$. Suppose that $\rho = |\xi|$ for all $\xi \in U$. It is clear that the proof of (3.25) is reduced to the proof of this estimate for two special cases:

$$(a)\quad \theta_1 = \theta_2, \qquad\qquad (b)\quad \rho_1 = \rho_2,$$

where (ρ_1,θ_1) and (ρ_2,θ_2) are polar coordinates of the points $\gamma\, x$ and $\gamma\, z$.

Let $\theta_1 = \theta_2$, $\rho_1 < \rho_2$. We write the left-hand side of (3.25) as the sum of three integrals u_k over Γ_k, where

$$\Gamma_1 = \{\xi\in\Gamma : 2\,|\xi| < |x|\},\ \Gamma_2 = \{\xi\in\Gamma : |x| < 2\,|\xi| < 4\,|x|\},\ \Gamma_3 = \{\xi\in\Gamma : |\xi| > 2\,|x|\}.$$

Consider u_1. Let $y' = (\rho,\theta)$, $\rho_1 < \rho < \rho_2$, and $y = \gamma^{-1}y'$. Since $|x| = \rho_1 \le |y|$, the inequality

$$2\,|y-\xi| < 3\,|y| \tag{3.26}$$

holds for all $\xi \in \Gamma_1$. It is easily seen that

$$3\,|x-\xi| < 4\,|y-\xi| < 5\,|x-\xi| \tag{3.27}$$

for $\xi \in \Gamma\backslash B(r,x)$. In fact, (3.27) obviously follows from the inequalities

$$4\,|y-\xi| \le 4\,|x-\xi| + 4\,|y-x| \le 4\,|x-\xi| + 4\,|z-x| < 5\,|x-\xi|,$$

and

$$4\,|y-\xi| \ge 4\,|x-\xi| - 4\,|y-x| \ge 4\,|x-\xi| - 4\,|z-x| > 3\,|x-\xi|.$$

Using the estimates for the kernel $\mathcal{L}(y,\xi)$ obtained in Theorem 5, then estimating $|y|$ by (3.26), (3.27), and applying Lemma 11, we obtain

$$\begin{aligned}
|u_1| &\le \int_{\Gamma\cap B(r,x)} \int_{\rho_1}^{\rho_2} \Big|\frac{\partial}{\partial\rho}\mathcal{L}(y,\xi)\Big|\, d\rho\, |x-\xi|^\alpha ds_\xi \\
&\le C \int_{\Gamma\cap B(r,x)} \int_{\rho_1}^{\rho_2} |y|^{-2-\varkappa+\varepsilon} |\xi|^{\varkappa-1-\varepsilon} d\rho\, |x-\xi|^\alpha ds_\xi \\
&\le C(\rho_2-\rho_1) \int_{\Gamma\cap B(r,x)} |x-\xi|^{-2-\varkappa+\varepsilon+\alpha} |\xi|^{\varkappa-1-\varepsilon}\, ds_\xi \le C\, r^\alpha.
\end{aligned}$$

Here we used the existence of positive constants c_1 and c_2 such that

$$c_1\,|\xi-\eta| < |\gamma\xi - \gamma\eta| < c_2\,|\xi-\eta| \tag{3.28}$$

for all $\xi, \eta \in U$, where $|\xi - \eta|$ stands for the distance between ξ and η in $\mathbb{R}^3$ and $|\gamma\xi - \gamma\eta|$ stands for the distance between $\gamma\xi$ and $\gamma\eta$ in $\mathbb{R}^2$.

We turn to the estimate of u_2. Let $\xi \in \Gamma \backslash B(r,x)$. Then $|\xi| < 2\,|x| < 2\,|z|$ and

$$4\,|z| < 4\,|z-x| + 4\,|x| < |x-\xi| + 4\,|x| < 5\,|x| + |\xi| < 11\,|\xi|.$$

Thus, $2\,|\xi| < 4\,|y| < 11\,|\xi|$ for $\xi \in \Gamma \backslash B(r,x)$ and $y = \gamma^{-1}y'$, $y' = (\rho, \theta)$, $\rho_1 < \rho < \rho_2$. Hence, using the estimates obtained in Theorem 5 and (3.27), (3.28), we arrive at

$$|u_2| \le \int_{\Gamma \cap B(r,x)} \int_{\rho_1}^{\rho_2} |x-\xi|^{-3+\alpha} d\rho\, ds_\xi \le C\, r^\alpha.$$

Finally, consider u_3. If $\xi \in \Gamma \backslash B(r,x)$, we have

$$8\,|z| < 8\,|x| + 8\,|x-z| < 4\,|\xi| + 2\,|x-\xi| \le 6\,|\xi| + 2\,|x| \le 7\,|\xi|.$$

Hence $7\,|\xi| > 8\,|y|$ for all $\xi \in \Gamma \backslash B(r,x)$, $y = \gamma^{-1}y'$, $y' = (\rho, \theta)$, $\rho_1 < \rho < \rho_2$.

Putting $\varepsilon = (\varkappa - \alpha)/2$, applying the estimates in Theorem 5 and then estimating $|\xi|$ using $2\,|x-\xi| < 3\,|\xi|$, we get

$$|u_3| \le C \int_{\Gamma \cap B(r,x)} \int_{\rho_1}^{\rho_2} \rho^{\varkappa-1-\varepsilon}\, d\rho\, |\xi|^{-2-\varkappa+\varepsilon}\, |x-\xi|^\alpha ds_\xi$$

$$\le C(\rho_2^{\varkappa-\varepsilon} - \rho_1^{\varkappa-\varepsilon}) \int_{\Gamma \cap B(r,x)} |x-\xi|^{-2-\varkappa+\varepsilon+\alpha}\, ds_\xi \le C\, r^\alpha.$$

Here we also used the elementary estimate

$$\rho_2^s - \rho_1^s \le (\rho_2 - \rho_1)^s, \qquad s \in (0,1).$$

It remains to consider the case $\rho_1 = \rho_2 = \rho$, $\theta_2 > \theta_1$. As before, we write the left-hand side of (3.25) as the sum of three integrals u_k. Let $y = \gamma^{-1}y'$, $y' = (\rho, \theta)$, $\theta_1 < \theta < \theta_2$. Using the identity

$$\mathcal{L}(z,\xi) - \mathcal{L}(x,\xi) = \int_{\theta_1}^{\theta_2} \frac{\partial}{\partial\theta} \mathcal{L}(y,\xi)\, d\theta$$

and estimating the derivatives of the kernel $\mathcal{L}(y,\xi)$ by Theorem 5, we obtain

$$|u_1| \le C\,\rho\,(\theta_2 - \theta_1) \int_{\Gamma \cap B(r,x)} |x-\xi|^{-2-\varkappa+\varepsilon+\alpha} |\xi|^{\varkappa-1-\varepsilon}\, ds_\xi \le C\, r^\alpha,$$

$$|u_2| \le C\,\rho\,(\theta_2 - \theta_1) \int_{\Gamma \cap B(r,x)} |x-\xi|^{-3+\alpha} ds_\xi \le C\, r^\alpha,$$

$$|u_3| \le C\,\rho^{\varkappa-\varepsilon}\,(\theta_2 - \theta_1) \int_{\Gamma \cap B(r,x)} |x-\xi|^{-2-\varkappa+\varepsilon+\alpha} ds_\xi \le C\, r^\alpha.$$

The theorem is proved.

Remark If the cone K^+ cannot be prescribed in a Cartesian coordinate system, the estimates of the kernel $\mathcal{L}(x,y)$ in Theorem 6 hold if $\varkappa$ is replaced by $\varkappa' = \min\{0, \varkappa\}$. Theorems 2 ans 3 take the following form.

Theorem 7 *(i) Let*

$$1 < p \le \infty, \qquad -\varkappa' < \beta + 2/p < 1 + \varkappa'.$$

Then the operators $(1+T)^{-1}$ *and* $(1+T^*)^{-1}$ *are continuous in the spaces* $L_{p,\beta}(\Gamma)$ *and* $L_{p,\beta+1}(\Gamma)$*, respectively.*
(ii) Let

$$1 < p \le \infty, \qquad -\varkappa' < \beta + 2/p < 1 + \varkappa', \qquad \alpha \in (0,1),$$
$$-\varkappa' < \delta < 1 + \varkappa', \qquad l = 1, 2, \ldots .$$

Then the operators $(1+T)^{-1}$ *and* $(1+T^*)^{-1}$ *are continuous in the spaces* $V^l_{p,\beta+l}(\Gamma)$, $N^{l,\alpha}_{l+\alpha+\delta}(\Gamma)$*, and* $V^l_{p,\beta+l+1}(\Gamma)$, $N^{l,\alpha}_{l+\alpha+\delta+1}(\Gamma)$*, respectively.*

3.3 Integral equations of harmonic potential theory

Since the operators Δ^* and $\mathcal{N}(\partial_x, n_x)$ coincide with operators $1\,\Delta$ and $1\,\frac{\partial}{\partial n}$ for

$$\mu = 1, \quad \lambda + \mu = 0,$$

where Δ is the Laplace operator and 1 is the 3×3 identity matrix, it follows that we can carry over all the results obtained here to boundary integral equations for the Laplace operator. Note that Lemma 2 from [MG] is valid without any assumption on the shape of the cone.

Here is a summary of results.

Theorem 8 *Let* δ^+ *and* ν^- *be positive numbers such that* $\delta^+(\delta^+ + 1)$ *and* $\nu^-(\nu^- + 1)$ *are the first eigenvalues of the Dirichlet problem in* $\Omega^+ = K^+ \cap S^2$ *and the Neumann problem in* $\Omega^- = S^2 \backslash \overline{\Omega^+}$ *for the Beltrami operator.*

Further, let $\varkappa = \min\{\delta^+, \nu^-, 1\}$*. If* $\varphi, \psi \in L_2(\Gamma)$*, then the inverse operators* $(1+T)^{-1}$ *and* $(1+T^*)^{-1}$ *of the boundary integral equations associated with the interior Dirichlet and the exterior Neumann problem for the Laplace operator admit the representation*

$$(1+T)^{-1}\varphi = (1+L)\varphi, \qquad (1+T^*)^{-1}\psi = (1+L^*)\psi.$$

Here L *is the integral operator on* Γ *with the kernel* $\mathcal{L}(x,y)$ *obeying the estimates*

$$|\mathcal{L}(x,y)| \le \begin{cases} c\,|x|^{-|\sigma|}|y|^{-2-|\tau|}(|x|/|y|)^{\varkappa-\varepsilon} + c\,|y|^{\varkappa-1-\varepsilon}\delta^0_{|\sigma|}, & 2|x| < |y|, \\ c\,|y|^{-1}|x-y|^{-1-|\sigma|-|\tau|}, & |y| < 2|x| < 4|y|, \\ c\,|x|^{-1-|\sigma|}|y|^{-1-|\tau|}(|y|/|x|)^{\varkappa-\varepsilon}, & |x| > 2|y|, \end{cases}$$

where ε *is a sufficiently small positive number.*

Theorem 9 (i) *The operator* $(1+T)^{-1}$ *is continuous in the spaces*

$$V^l_{p,\beta+l+1}(\Gamma) \qquad \text{for } 1<p<\infty,\ 0<\beta+2/p<1+\varkappa,\ l=0,1,\dots,$$

$$L_{1,\beta}(\Gamma) \qquad \text{for } 0<\beta+2<1+\varkappa$$

$$L_{\infty,\beta}(\Gamma) \qquad \text{for } 0<\beta<1+\varkappa$$

$$C(\Gamma) \text{ and } C^{0,\alpha}(\Gamma) \qquad \text{for } 0<\alpha<\varkappa,$$

$$C^{l,\alpha}_{l+\alpha+\delta}(\Gamma) \qquad \text{for } -\varkappa<\delta<0,\ l=1,2,\dots$$

$$N^{l,\alpha}_{l+\alpha+\delta}(\Gamma) \qquad \text{for } 0<\delta<1+\varkappa,\ l=1,2,\dots$$

(ii) *The operator* $(1+T^*)^{-1}$ *is continuous in the spaces*

$$L_{p,\beta}(\Gamma) \qquad \text{for } 1<p\le\infty,\ 1-\varkappa<\beta+2/p<2$$

$$V^l_{p,\beta+l+1}(\Gamma) \qquad \text{for } 1<p<\infty,\ 1-\varkappa<\beta+2/p<2,\ l=1,\dots,$$

$$N^{l,\alpha}_{l+\alpha+\delta}(\Gamma) \qquad \text{for } 1-\varkappa<\delta<2,\ l=0,1,2,\dots$$

References

[DKV] Dahlberg, B.E.J.; Kenig, C.E.; Verchota, G.C. *Boundary value problems for systems of elastostatics in Lipschitz domains*, Duke Math. J. **57**:3 (1988), 795–818.

[MG] Grachev, N.; Maz'ya, V. *Estimates for kernels of inverse operators of integral equations of elasticity theory on surfaces with conic points*, J. Math. Sci. (N. Y.), 2012.

[Ko] Kondratiev, V. A. *Boundary problems for elliptic equations in domains with conical or angular points*, Tr. Mosk. Mat. O-va., 16 (1967) 209−292. English translation: Trans. Moscow Math. Soc. **16** (1967) 227–313.

[KM] Kozlov, V.; Maz'ya, V. *Spectral properties of operator pencils generated by elliptic boundary value problems in a cone.* Funktsional. Anal. i Prilozhen. **22**:2 (1988), 38–46, 96. English translation: Functional Anal. Appl. **22**: 2 (1988), 114–121.

[M1] Maz'ya, V. *Boundary integral equations of elasticity in domains with piecewise smooth boundaries*, Equadiff 6, Proc. Int. Conf. Brno 1985, Lect. Notes Math. **1192**, 1986, 235–242.

[M2] Maz'ya, V. *Boundary integral equations*, Analysis IV, 127–222, Encyclopaedia Math. Sci. **27**, Springer, 1991.

[MP] Maz'ya, V.; Plamenevskiĭ, B. *The coefficients in the asymptotics of solutions of elliptic boundary value problems with conical points.* Math. Nachr. **76** (1977), 29–60. English translation: Transl. Amer. Math. Soc. Ser.2, **123** (1984), 57–88.