Solvability of a boundary integral equation on a polyhedron

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Abstract. The boundary integral equation associated with the Dirichlet problem for the Laplace equation on a polyhedral domain is considered. Pointwise estimates for the kernel of the inverse operator are derived. As a corollary, the solvability of the integral equation in the space of continuous functions and in a weighted L_p-space is obtained.

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1 Introduction

This article is closely related to our papers [GM1], [GM3], [GM4] in which integral equations of the harmonic and elastic potential theory on surfaces with conic vertices were considered. Here we investigate the integral equation generated by the Dirichlet problem for the Laplace equation in a 3-dimensional polyhedron, which is not necessarily a Lipschitz graph domain.

We use the method proposed by one of the authors [M1] -[M4] which reduces the analysis of boundary integral equations to the study of some auxiliary boundary value problems. Different applications of the method can be found in [MZ1], [MZ2], [Z1], [Z2], [LM], [GM2], [GM7], [MS].

By estimates for the fundamental solutions of the Dirichlet and Neumann problems [MP1], [GM5] (see [M5] for detailed exposition), we arrive at the estimates for the kernel of the inverse operator of the integral equation in question. Such estimates lead to theorems on the solvability of this equation in various function spaces and, in particular, in the space C of continuous functions.

The question of the validity of the last result was stated long ago. The solvability of the boundary integral eqution in the space C over surfaces of a fairly wide class was established in the multi-dimensional case by Burago, Maz'ya [BM] and Kral [K] under the requirement that the esssential norm $|T|$ of the double layer potential T is less than 1. This condition can be formulated in geometric terms. However, it does not always hold even for sufficiently simple cones. Angell, Kleinman, Kral [AKK] and Kral, Wendland [KW] succeded in compelling the inequality $|T| < 1$ for certain 3-dimensional polyhedra to hold by replacing the usual norm in C with an equivalent weighted norm. The polyhedral surfaces considered in [AKK] are constituted by a finite number of rectangles parallel to the coordinate planes.

The soilvability in the space C for the above mentioned integral equation on surfaces in $\mathbb{R}^n$ with a finite number of conical points was proved by Grachev and Maz'ya [GM1], [GM3], [GM4] without any complementary geometric assumptions. Thus, it was shown that the use of the esssential norm had been unnecessary and dictated only by the method of proof. We, and independently Rathsfeld [R], extended this result to arbitrary polyhedra. A direct approach based on the Mellin transform was used in [R]. Some of the results of the present paper were announced in the lecture [G1] and in the preliminary publication [GM6].

Now we briefly describe our results. We assume that Γ is a polyhedron in the three-dimensional Euclidean space. By G^+ we denote the interior of this polyhedron and consider the Dirichlet problem

$$\Delta u = 0 \text{ on } G^+, \quad u = f \text{ on } \Gamma. \tag{1.1}$$

Let $O_1, \dots, O_m$ be the vertices of the polyhedron and let $\mathfrak{M}_1, \dots, \mathfrak{M}_k$ be its edges. We denote by ω_j the opening of the dihedral angle with the edge $\mathfrak{M}_j$ from the side of G^+ and we put $\lambda_j = \pi/\omega_j$. We use the notation

$$r_j(x) = \operatorname{dist}(x, \mathfrak{M}_j), \qquad \rho_i(x) = \operatorname{dist}(x, O_i),$$

$$r(x) = \min_{1\le j\le k}\{r_j(x)\}, \qquad \rho(x) = \min_{1\le i\le m}\{\rho_i(x)\}.$$

Let K_i, $k = 1, 2, \dots, m$, be the cone with vertex O_i which coincides with G^+ near the point O_i. The open set cut by the cone K_i out of the unit sphere S^2 centered at O_i is denoted by Ω_i^+ and the set $S^2\backslash\overline{\Omega_i^+}$ by Ω_i^-. Let δ_i and ν_i be positive numbers such that $\delta_i(\delta_i+1)$ and $\nu_i(\nu_i+1)$ are the first eigenvalues of the Dirichlet problem on Ω_i^+ and the Neumann problem on Ω_i^- for the Beltrami operator. Further, we denote by $\varkappa_i$ the minimum of δ_i, ν_i, and 1.

Let $W\psi$ denote the classical double layer potential with the density ψ:

$$(W\psi) = \frac{1}{4\pi}\int_\Gamma \frac{\partial}{\partial n_\xi}\Big(\frac{1}{|x-\xi|}\Big)\psi(\xi)\,ds_\xi, \quad x \in G^\pm.$$

We are looking for a solution of the equation (1.1) in the form of a double potential. It is known that the density ψ satisfies the integral equation

$$(1+T)\,\psi = 2\,f.$$

Here T is the operator on Γ defined by the equation

$$(T\psi)(x) = 2W_0\psi(x) + (1 - d(x))\,\psi(x),$$

where $d(x) = 1$ for $x \in \Omega\backslash\overline{\mathfrak{M}_j}$, $d(x) = \omega_j/\pi$ for $x \in \mathfrak{M}_j$, $d(x) = \operatorname{meas}\Omega_i^+/2\pi$ for $x \in O_i$, and $W_0\psi$ is the direct value on Γ of the double layer potential.

The following two theorems are our main results.

Theorem 1 *The operator*

$$1+T : C(\Gamma) \to C(\Gamma)$$

performs an isomorphism. The inverse operator admits the representation

$$(1+T)^{-1} f = (1+L+M) f,$$

where L and M are integral operators on Γ with kernels $L(x,y)$ and $M(x,y)$ edmitting the following estimates:

If $\mathfrak{M}_j$ is the edge nearest to the point y and O_i is the vertex nearest to y, then

$$|M(x,y)| \le c\,\rho(y)^{\varkappa_i - 1 - \varepsilon} \Big(\frac{r(y)}{\rho(y)}\Big)^{\lambda_j - 1 - \varepsilon}.$$

If the points x and y lie in a neighbourhood of a vertex O_i, $i = 1, 2, \dots, m$, this neighbourhood contains no vertices of the polyhedron O_i and if $\mathfrak{M}_j$, $\mathfrak{M}_l$ are the edges nearest to the points y and x respectively, then

$$|L(x,y)| \le c\,\rho(y)^{-2} \big(r(y)/\rho(y)\big)^{\lambda_j - 1 - \varepsilon}$$

$$+c\big(r(y) + |x-y|\big)^{-2} \Big(\frac{r(x)}{r(x)+|x-y|}\Big)^{\lambda_l - \varepsilon} \Big(\frac{r(y)}{r(y)+|x-y|}\Big)^{\lambda_j - 1 - \varepsilon}$$

for $\rho(x)/2 < \rho(y) < 2\rho(x)$ and

$$|L(x,y)| \le c\,\rho(y)^{-1} \big(\rho(x)+\rho(y)\big)^{-1} \Big(\frac{\min\{\rho(x),\rho(y)\}}{\rho(x)+\rho(y)}\Big)^{\varkappa_i - \varepsilon} \Big(\frac{r(y)}{\rho(y)}\Big)^{\lambda_j - 1 - \varepsilon}$$

in the opposite case. Here ε is an arbitrary positive number.

The next theorem concerns the operator defined by

$$T\psi = 2W_0 \psi \quad \text{a.e. on } \Gamma$$

as an operator in the weighted L_p-space $L^p_{\beta,\gamma}(\Gamma)$ endowed with the norm

$$\|u\|_{L^p_{\beta,\gamma}(\Gamma)} = \|\rho^\beta\, r^\gamma\, u\|_{L_p(\Gamma)}.$$

Theorem 2 *Let*

$$\varkappa = \min\{\varkappa_i\}, \quad \lambda = \min\{\lambda_j\}.$$

If

$$1 \le p < \infty, \quad 0 < \beta + \gamma + 2/p < 1 + \varkappa, \quad 0 < \gamma + 1/p < \lambda$$

or

$$p = \infty, \quad 0 \le \beta + \gamma < 1 + \varkappa, \quad 0 \le \gamma < \lambda,$$

then the operator

$$1 + T : L^p_{\beta,\gamma}(\Gamma) \to L^p_{\beta,\gamma}(\Gamma)$$

performs an isomorphism.

In Section 2 we collect some preliminary information on boundary value problems and find a representation for the inverse operator of the integral equation in question stated in terms of the inverse operators of boundary value problems. Estimates for $L(x,y)$ and $M(x,y)$ in Theorem 1 are obtained in Section 3. Finally, in Section 4 we prove theorems on the unique solvability of the integral equation in spaces C and $L^p_{\beta,\gamma}$.

2 Representation for the inverse operator of the boundary integral equation

2.1 Preliminary information

We shall use the notation from Introduction. Besides, let $G^- = \mathbb{R}^3 \backslash \overline{G^+}$ and $B(r,x) = \{y \in \mathbb{R}^3 : |x-y| < r\}$.

We define some weighted Hölder spaces. For simplicity we introduce the same weight r^γ for all edges and the same weight ρ^β for all vertices. We denote by $N^{l,\alpha}_{\beta,\gamma}(G^+)$ the space of functions on G^+ with the finite norm

$$\|u\|_{N^{l,\alpha}_{\beta,\gamma}(G^+)} = \sup_{x \in F^+} \rho(x)^\beta r(x)^\gamma [u]^{l+\alpha}_{B(r/2,x)\cap G^+} + \sup_{x \in G^+} \rho(x)^\beta r(x)\gamma - l - \alpha |u(x)|. \quad (2.1)$$

Here β, γ are real numbers, $\alpha \in (0,1)$, l is an integer, $l \geq 0$, and

$$[u]^\rho_E = \sup_{x,y \in E} \sum_{|\sigma| = [\rho]} |x-y|^{[\rho]-\rho} |\partial^\sigma_x u(x) - \partial^\sigma_y u(y)|,$$

where E is a subset of $\mathbb{R}^3$, ρ is a positive noninteger, $[\rho]$ is the integer part of ρ.

We also introduce the space $C^{l,\alpha}_{\beta,\gamma}(G^+)$ $(0 < \gamma < l+\alpha$, $l+\alpha-\gamma$ is not integer) of functions u in G^+ with the finite norm

$$\|u\|_{C^{l,\alpha}_{\beta,\gamma}(K)} = \sup_{x \in G^+} \rho(x)^\beta r(x)^\gamma \left[u\right]^{l+\alpha}_{G^+ \cap B(r/2,x)}$$

$$+ \sup_{x \in G^+} \rho(x)^\beta \left[u\right]^{l+\alpha-\gamma}_{G^+ \cap B(\rho/2,x)} + \sup_{x \in G^+} \rho(x)^{\beta+\gamma-l-\alpha} |u(x)|. \quad (2.2)$$

For the domain G^- we define similar spaces $N^{l,\alpha}_{\beta,\gamma}(G^-)$ and $C^{l,\alpha}_{\beta,\gamma}(G^-)$. Suppose that the ball $B(R,0)$ contains $\overline{G^+}$. We denote by χ a function from the space $C^\infty(\mathbb{R}^3)$ equal to one on $B(R,0)$ and to zero on $\mathbb{R}^3 \backslash B(R+1,0)$. A function u in G^- belongs to $N^{l,\alpha}_{\beta,\gamma}(G^-)$ and respectively to $C^{l,\alpha}_{\beta,\gamma}(G^-)$ if and only if the norm (2.1), respectively (2.2), of $u\chi$ and the norm

$$\sup_{x \in G^-} |x|^{l+\alpha+1} \left[v\right]^{l+\alpha}_{B(|x|/2,x)} + \sup_{x \in G^-} |x|\, |v(x)|$$

of the function $v = (1-\chi)u$ are finite.

Let Γ_i denote a face of the polyhedron Γ. We denote by $N^{l,\alpha}_{\beta,\gamma}(\Gamma_i)$ the space of traces on Γ_i of functions from $N^{l,\alpha}_{\beta,\gamma}(G^+)$ or from $N^{l,\alpha}_{\beta,\gamma}(G^-)$. We say that u belongs to $N^{l,\alpha}_{\beta,\gamma}(\Gamma)$ if and only if the restriction u_i on each Γ_i is in $N^{l,\alpha}_{\beta,\gamma}(\Gamma_i)$ and we introduce the norm

$$\|u\|_{N^{l,\alpha}_{\beta,\gamma}(\Gamma)} = \sum_i \|u\|_{N^{l,\alpha}_{\beta,\gamma}(\Gamma_i)}.$$

The space of traces on Γ of functions from $C^{l,\alpha}_{\beta,\gamma}(G^+)$ or from $C^{l,\alpha}_{\beta,\gamma}(G^-)$ will be denoted by $C^{l,\alpha}_{\beta,\gamma}(\Gamma)$.

Consider the interior Dirichlet problem and the exterior Neumann problem for the Laplace equation

$$\Delta u = 0 \text{ on } G^+, \qquad u = f \text{ on } \Gamma, \quad (2.3)$$

$$\Delta v = 0 \text{ on } G^-, \qquad \partial v / \partial n = g \text{ on } \Gamma \backslash \mathfrak{M}. \quad (2.4)$$

Here $\partial/\partial n$ stands for the derivative in the direction of the outer normal to

$$\Gamma \backslash \mathfrak{M} = \bigcup_{1\le i\le k} \overline{\mathfrak{M}_i}.$$

Now we formulate estimates for the fundamental solutions of the problems (2.3) and (2.4). Let K_i, $i = 1, 2, \ldots, m$, be the cone with the vertex O_i which coincides with G^+ near the point O_i. The open set that the cone K_i cuts from the unit sphere S^2 centered at O_i is denoted by Ω^+ and the set $S^2\backslash\overline{\Omega^+}$ is denoted by Ω^-. Let δ_i and ν_i be positive numbers such that $\delta_i(\delta_i + 1)$ and $\nu_i(\nu_i + 1)$ are the first positive eigenvalues of the Dirichlet problem in Ω^+ and the Neumann problem in Ω^- for the Laplace-Beltrami operator on S^2. The result formulated here is contained in [MP1].

Theorem 3 *Let*

$$\delta^+ = \min_{1\le j\le} \delta_i, \qquad \lambda^+ = \min_{1\le i\le k} \pi/\omega_i,$$

and let l be a positive integer. If

$$-\delta^+ < \beta + \gamma - \alpha < 1 + \delta^+, \qquad 0 < \alpha - \gamma < \min\{1, \lambda^+\},$$

then for any $f \in C^{l,\alpha}_{\beta,\gamma+l}(\Gamma)$ there exists a unique solution $u \in C^{l,\alpha}_{\beta,\gamma+l}(G^+)$ of the Dirichlet problem (2.3) and the solution admits the representation

$$u(x) = \int_\Gamma \mathcal{P}^+(x,\xi)\, f(\xi)\, ds_\xi \tag{2.5}$$

Suppose that the points x and ξ lie in a neighbourhood of a vertex O_i, $i = 1, 2, \ldots, m,$. If either $2\,\rho(\xi) < \rho(x)$ or $\rho(\xi) > 2\,\rho(x)$, then

$$|\partial_x^\sigma \partial_\xi^\tau \mathcal{P}^+(x,\xi)| \le c_{\sigma,\tau}\, \rho(x)^{-|\sigma|} \rho(\xi)^{-1-|\tau|} \big(\rho(x) + \rho(\xi)\big)^{-1}$$

$$\times \Big(\frac{\min\{\rho(x), \rho(\xi)\}}{\rho(x) + \rho(\xi)}\Big)^{\delta^+ - \varepsilon} \Big(\frac{r(x)}{\rho(x)}\Big)^{\lambda^+ - |\sigma| - \varepsilon} \Big(\frac{r(\xi)}{\rho(\xi)}\Big)^{\lambda^+ - |\tau| - 1 - \varepsilon}.$$

In the zone $\rho(\xi) < 2\,\rho(x) < 4\,\rho(\xi)$, the estimates have the form

$$|\partial_x^\sigma \partial_\xi^\tau \mathcal{P}^+(x,\xi)| \le c_{\sigma,\tau}\, |x - \xi|^{-2-|\sigma|-|\tau|}$$

$$\times \Big(\frac{r(x)}{r(x) + |x-\xi|}\Big)^{\lambda^+ - |\sigma| - \varepsilon} \Big(\frac{r(\xi)}{r(\xi) + |x-\xi|}\Big)^{\lambda^+ - 1 - |\tau| - \varepsilon}.$$

In the case $x \in U_i$, $\xi \in U_q$, where U_i and U_q are small neighbourhoods of the vertices O_i and O_q with $i \ne q$, the estimates take the form

$$|\partial_x^\sigma \partial_\xi^\tau \mathcal{P}^+(x,\xi)| \le c_{\sigma,\tau}\, \rho(x)^{\delta^+ - |\sigma| - \varepsilon} \rho(\xi)^{\delta^+ - |\tau| - 1 - \varepsilon} \Big(\frac{r(x)}{\rho(x)}\Big)^{\lambda^+ - |\sigma| - \varepsilon} \Big(\frac{r(\xi)}{\rho(\xi)}\Big)^{\lambda^+ - |\tau| - 1 - \varepsilon}.$$

Here σ and τ are arbitrary multi-indices, ε is a sufficiently small positive number.

The next result is essentially proved in [GM5].

Theorem 4 *Let*

$$\nu^- = \min_{1\le i\le m} \nu_i, \qquad \lambda^- = \min_{1\le j\le k} \{\pi/(2\pi - \omega_j)\}.$$

and let l be a positive integer. If

$$0 < \beta + \gamma - \alpha < 1, \qquad 0 < \alpha - \gamma < \min\{1, \lambda^-\}$$

then for any $g \in N^{l,\alpha}_{\beta,\gamma+l}(\Gamma)$ there exists a unique solution $v \in C^{l,\alpha}_{\beta,\gamma+l}(G^-)$ of the Neumann problem (2.4) and

$$v(x) = \int_\Gamma Q^-(x,\xi)\, g(\xi)\, ds_\xi. \tag{2.6}$$

Suppose that the points x and ξ lie in a neighbourhood of the vertex O_i, $i = 1, 2, \dots, m$,. If either $2\,\rho(x) < \rho(\xi)$ or $\rho(x) > 2\,\rho(\xi)$, then

$$Q^-(x,\xi) = Q^-(0,\xi) + R^-(x,\xi) \quad \text{for } 2\,\rho(x) < \rho(\xi), \tag{2.7}$$

$$Q^-(x,\xi) = Q^-(x,0) + R^-(\xi,x) \quad \text{for } 2\,\rho(\xi) < \rho(x), \tag{2.8}$$

where

$$Q^-(0,\xi) = Q^-(\xi,0) = a_i^-/\rho(\xi) + b_i^- + d_i^-(\xi) \tag{2.9}$$

and

$$a_i^- = 1/\mathrm{meas}(\Omega_i^-), \quad b_i^- = \mathrm{const}.$$

For $R^-(x,\xi)$ and $d_i^-(\xi)$ one has the estimates

$$|\partial_\xi^\sigma\, d_i^-(\xi)| \le c_\sigma\, \rho(x)^{\nu^- - |\alpha| - \varepsilon} \Big(\frac{r(\xi)}{\rho(\xi)}\Big)^{\lambda^-_{\sigma\varepsilon}},$$

$$|\partial_x^\sigma \partial_\xi^\tau \mathcal{R}^-(x,\xi)| \le c_{\sigma,\tau}\, \rho(x)^{\nu^- - |\sigma| - \varepsilon} \rho(\xi)^{-1-\nu^- - |\tau| + \varepsilon} \Big(\frac{r(x)}{\rho(x)}\Big)^{\lambda^-_{\sigma\varepsilon}} \Big(\frac{r(\xi)}{\rho(\xi)}\Big)^{\lambda^-_{\sigma\varepsilon}}.$$

In the intermediate zone $\rho(x) < 2\,\rho(\xi) < 4\,\rho(x)$, the estimate takes the form

$$|\partial_x^\sigma \partial_\xi^\tau Q^-(x,\xi)| \le \frac{c_{\sigma\tau}}{|x-\xi|^{1+|\sigma|+|\tau|}} \Big(\frac{r(x)}{r(x) + |x-\xi|}\Big)^{\lambda^-_{\sigma\varepsilon}} \Big(\frac{r(\xi)}{r(\xi) + |x-\xi|}\Big)^{\lambda^-_{\tau\varepsilon}}.$$

In the case $x \in U_i$, $\xi \in U_q$, where U_i and U_q are small neighbourhoods of the vertices O_i and O_q with $i \ne q$, we have

$$|\partial_x^\sigma \partial_\xi^\tau Q^-(x,\xi)| \le c_{\sigma,\tau}\, \rho(x)^{\nu^-_{\sigma\varepsilon}} \rho(\xi)^{\nu^-_{\tau\varepsilon}} \Big(\frac{r(x)}{\rho(x)}\Big)^{\lambda^-_{\sigma\varepsilon}} \Big(\frac{r(\xi)}{\rho(\xi)}\Big)^{\lambda^-_{\tau\varepsilon}}.$$

Here we use the notation

$$\lambda^-_{\sigma\varepsilon} = \min\{0, \lambda^- - |\sigma| - \varepsilon\}, \quad \lambda^-_{\tau\varepsilon} = \min\{0, \lambda^- - |\tau| - \varepsilon\},$$

$$\nu^-_{\sigma\varepsilon} = \min\{0, \nu^- - |\sigma| - \varepsilon\}, \quad \nu^q_{\tau\varepsilon} = \min\{0, \nu^- - |\tau| - \varepsilon\}.$$

In what follows we need some estimates for the fundamental solutions of the Dirichlet and Neumann problems in a dihedral angle. Let D^+ be the interior of the angle with opening ω and let $D^- = \mathbb{R}^3 \backslash \overline{D^+}$. We denote by F^+ and F^- the sides of D^+, by $\mathfrak{M}$ the edge and by F the boundary, i.e. $F = F^+ \cup F^- \cup \mathfrak{M}$.

We introduce the space $N^{l,\alpha}_\gamma(D^+)$ with the norm

$$\|u\|_{N^{l,\alpha}_\gamma(D^+)} = \sup_{x\in D^+} r(x)^\gamma \big[u\big]^{l+\alpha}_{D^+\cap B(r/2,x)} + \sup_{x\in D^+} r(x)^{\gamma-l-\alpha} |u(x)|.$$

and the space $C^{l,\alpha}_{\gamma}(D^+)$, $l+\alpha-\gamma>0$, with the norm

$$\|u\|_{C^{l,\alpha}_{\gamma}(D^+)} = \sup_{x\in D^+} r(x)^{\gamma}\,[u]^{l+\alpha}_{D^+\cap B(r/2,x)} + \|u\|_{C^{l+\alpha-\gamma}(\overline{D^+})}.$$

Here $C^s(\overline{D^+})$ is the Hölder space of order s and $r(x) = \mathrm{dist}(x, \mathfrak{M})$.

We denote by $N^{l,\alpha}_{\gamma}(F^{\pm})$ the space of traces on $F^{\pm}$ of functions from $N^{l,\alpha}_{\gamma}(D^+)$ or from $N^{l,\alpha}_{\beta,\gamma}(G^-)$. We say that u belongs to $N^{l,\alpha}_{\gamma}(F)$ if and only if the restriction $u^{\pm}$ to $F^{\pm}$ is in $N^{l,\alpha}_{\gamma}(F^{\pm})$ and we introduce the norm

$$\|u\|_{N^{l,\alpha}_{\gamma}(F)} = \sum_{\pm} \|u\|_{N^{l,\alpha}_{\gamma}(F^{\pm})}.$$

The space of traces on F of functions from $C^{l,\alpha}_{\gamma}(D^+)$ is denoted by $C^{l,\alpha}_{\gamma}(F)$.

Similarly, one defines spaces of functions on D^-.

Consider two boundary value problems

$$\Delta u = 0 \ \text{ in } \ D^+, \qquad u = f \ \text{ on } \ F, \tag{2.10}$$

$$\Delta v = 0 \ \text{ in } \ D^-, \qquad \partial v/\partial n = g \ \text{ on } \ F\backslash\mathfrak{M}. \tag{2.11}$$

The following theorem was proved in [MP1].

Theorem 5 *Let $0 < \alpha-\gamma < \min\{1, \pi/\omega\}$ and let l be a positive integer. Then for any $f \in C^{l,\alpha}_{\beta,\gamma+l}(F)$ there exists a unique solution $u \in C^{l,\alpha}_{\beta,\gamma+l}(D^+)$ of the Dirichlet problem (2.10). It admits the representation*

$$u(x) = \int_F P^+(x,\xi)\, f(\xi)\, ds_{\xi}, \tag{2.12}$$

where

$$|\partial^{\sigma}_x \partial^{\tau}_{\xi} P^+(x,\xi)| \le c_{\sigma,\tau}\, |x-\xi|^{-2-|\sigma|-|\tau|}$$

$$\times \Big(\frac{r(x)}{r(x)+|x-\xi|}\Big)^{\pi/\omega-|\sigma|-\varepsilon} \Big(\frac{r(\xi)}{r(\xi)+|x-\xi|}\Big)^{\pi/\omega-1-|\tau|-\varepsilon}.$$

Now we formulate an analogous result for the Neumann problem obtained in [GM5].

Theorem 6 *Let $0 < \alpha-\gamma < \lambda^-$, $\lambda^- = \min\{1, \pi/(2\pi-\omega)\}$ and let l be a positive integer. Then for any $g \in C^{l,\alpha}_{\beta,\gamma+l}(F)$ there exists a unique solution $v \in C^{l,\alpha}_{\beta,\gamma+l}(D^-)$ of the Dirichlet problem (2.11). It admits the representation*

$$v(x) = \int_F Q^-(x,\xi)\, g(\xi)\, ds_{\xi}, \tag{2.13}$$

where

$$|\partial^{\sigma}_x \partial^{\tau}_{\xi} \big(Q^-(x,\xi) - a/|x-\xi|\big)| \le c_{\sigma,\tau}\, |x-\xi|^{-1-|\sigma|-|\tau|}$$

$$\times \Big(\frac{r(x)}{r(x)+|x-\xi|}\Big)^{\lambda^- -|\sigma|-\varepsilon} \Big(\frac{r(\xi)}{r(\xi)+|x-\xi|}\Big)^{\lambda^- -|\tau|-\varepsilon}.$$

Here $a = 1/\mathrm{meas}(S^2\cap D^-)$ and S^2 is the unit sphere with center at $x\in\mathfrak{M}$.

2.2 Representations for the inverse operators

We denote by $V\psi$ and $W\psi$ the single and double layer potentials:

$$(V\psi) = \frac{1}{4\pi}\int_\Gamma \frac{1}{|x-\xi|}\varphi(\xi)\,ds_\xi, \quad x \in \mathbb{R}^3.$$

$$(W\psi) = \frac{1}{4\pi}\int_\Gamma \frac{\partial}{\partial n_\xi}\Big(\frac{1}{|x-\xi|}\Big)\psi(\xi)\,ds_\xi, \quad x \in G^\pm.$$

In what follows we denote by $(\cdot)^+$ and $(\cdot)^-$ the interior and exterior limit values with respect to G^+. By $W_0\psi$ we mean the direct values of the double layer potential $W\psi$ on Γ. Let the operator T be defined by the equality

$$(T\psi)(x) = 2W_0\psi(x) + (1-d(x))\,\psi(x),$$

where

$$d(x) = \lim_{\delta\to 0^+}(\mathrm{meas}(G^+\cap B(\delta,x))/\mathrm{meas}B(\delta,x).$$

Lemma 1 *Let $0<\beta+\gamma-\alpha<2$, $0<\alpha-\gamma<1$ and let l be a positive integer. If $\psi \in C^{l,\alpha}_{\beta,\gamma+l}(\Gamma)$, then $W_0\psi \in C^{l,\alpha}_{\beta,\gamma+l}(\Gamma)$ and*

$$(W\psi)^\pm = W_0\psi \pm \psi/2, \qquad \Big(\frac{\partial(W\psi)}{\partial n}\Big)^+ = \Big(\frac{\partial(W\psi)}{\partial n}\Big)^- \tag{2.14}$$

holds on $\Gamma\backslash\mathcal{M}$.

Proof. Let the number δ be so small that the ball $B(2\delta,O_i)$ contains no vertices except O_i. One verifies directly the estimate

$$\sup_{x\in B(\delta,O_i)} \rho(x)^{\beta+\gamma-\alpha}\,|(W\psi)(x)| \le c\sup_{x\in\Gamma}\rho(x)^{\beta+\gamma-\alpha}\,|\psi(x)|. \tag{2.15}$$

We consider the following transmission problem

$$\Delta u = 0 \ \text{ in } \ G^+\cup G^-, \quad u^+ - u^- = \psi \ \text{ on } \ \Gamma,$$
$$\Big(\frac{\partial u}{\partial n}\Big)^+ - \Big(\frac{\partial u}{\partial n}\Big)^- = 0 \ \text{ on } \ \Gamma\backslash\mathfrak{M} \tag{2.16}$$

which is satisfied by $W\psi \in C^{l,\alpha}_{\mathrm{loc}}(\overline{G^\pm}\backslash\mathfrak{M})$. We introduce the sets

$$U_k = \{\xi : 1/2 < 2^k\,|\xi| < 2\}, \qquad V_k = \{\xi : 1/4 < 2^k\,|\xi| < 4\}$$

for $k = 1,2,\ldots$.

Well-known local Schauder estimate for solutions of (2.16) leads to the inequality

$$2^{-k(l+\beta)}\sup_{U_k\cap G^\pm} r(x)^\gamma [u]^{l+\alpha}_{B(r/2,x)\cap G^\pm} + 2^{-k(l+\beta)}[u]^{l+\alpha-\gamma}_{U_k\cap G^\pm}$$
$$\le c\big(2^{-k(l+\beta)}\sup_{V_k\cap\Gamma} r(x)^\gamma[\psi]^{l+\alpha}_{B(r/2,x)\cap\Gamma} + 2^{-k(l+\beta)}[\psi]^{l+\alpha-\gamma}_{V_k\cap\Gamma}$$
$$+2^{-k(\beta+\gamma-\alpha)}\sup_{V_k\cap\Gamma}|\psi(x)| + 2^{-k(\beta+\gamma-\alpha)}\sup_{x\in V_k\cap\Gamma}|u(x)|\big).$$

From this inequality and from (2.15) we conclude that $W\psi \in C^{l,\alpha}_{\beta,\gamma+l}(G^\pm)$.

The relations (2.14) follow from similar relations for domains with smooth boundaries.

Lemma 2 *Let $0 < \beta+\gamma-\alpha < 1$, $0 < \alpha-\gamma < 1$ and let l be a positive integer. If $\varphi \in N^{l-1,\alpha}_{\beta,\gamma+l}(\Gamma)$, then $V\varphi \in N^{l,\alpha}_{\beta,\gamma+l}(\Gamma^\pm)$ and*

$$\Big(\frac{\partial(V\varphi)}{\partial n}\Big)^{\pm} = -W_0^*\varphi \pm \varphi/2, \qquad (V\varphi)^+ = (V\varphi)^-$$

hold on $\Gamma\backslash\mathcal{M}$. Here W_0^* is the operator formally adjoint of W_0.

Proof. One verifies directly the estimates

$$\sup_{x\in B(\delta,O_i)} \rho(x)^{\beta+\gamma-\alpha}\,|(V\varphi)(x)| \le c\sup_{x\in\Gamma}\rho(x)^{\beta} r(x)^{\gamma-\alpha+1}\,|\varphi(x)|,\ i=1,2,\dots,m,$$

where δ is the same as in (2.15). To get the result, it is sufficient to apply the same argument as in the proof of Lemma 1 to the transmission problem

$$\Delta v = 0 \ \text{ on } \ G^+\cup G^-, \quad v^+ - v^- = 0 \ \text{ on } \ \Gamma,$$

$$\Big(\frac{\partial v}{\partial n}\Big)^{+} - \Big(\frac{\partial v}{\partial n}\Big)^{-} = \varphi \ \text{ on } \ \Gamma\backslash\mathcal{M}.$$

Lemma 3 *Let $0 < \beta+\gamma-\alpha < 1$, $0 < \alpha-\gamma < 1$ and let l be a positive integer. Then the representation*

$$u = V\Big(\Big(\frac{\partial u}{\partial n}\Big)^{+} - \Big(\frac{\partial u}{\partial n}\Big)^{-}\Big) + W(u^+ - u^-)$$

holds on $G^+\cup G^-$ for all functions $u \in C^{l,\alpha}_{\beta,\gamma+l}(G^+\cup G^-)$ satisfying the equation $\Delta u = 0$ on $G^+\cup G^-$. Here $C^{l,\alpha}_{\beta,\gamma}(G^+\cup G^-)$ is the space of functions u in $G^+\cup G^-$ whose restrictions to $G^\pm$ belong to $C^{l,\alpha}_{\beta,\gamma}(G^\pm)$.

Proof. We use the following classic relations

$$\begin{aligned}
u(x) &= (V(\partial u/\partial n)^+)(x) + (Wu^+)(x), && x\in G^+,\\
0 &= (V(\partial u/\partial n)^+)(x) + (Wu^+)(x), && x\in G^-,\\
0 &= -(V(\partial u/\partial n)^-)(x) - (Wu^-)(x), && x\in G^+,\\
u(x) &= -(V(\partial u/\partial n)^-)(x) - (Wu^-)(x), && x\in G^-,
\end{aligned}$$

hold for all functions u such that

$$u\in C^\infty(\overline{G^\pm}), \quad u = O(|x|^{-1}) \ \text{as}\ x\to\infty \quad \text{and} \ \ \Delta u = 0 \ \text{on} \ G^+\cup G^-.$$

One shows, using Lemmas 1, 2, that these relations extend to all $u \in C^{l,\alpha}_{\beta,\gamma+l}(G^+\cup G^-)$, harmonic on $G^+\cup G^-$.

Theorem 7 *Let $0 < \alpha-\gamma < \min\{\lambda^+,\lambda^-\}$, $0 < \beta+\gamma-\alpha < \min\{\delta^+,\nu^-,1\}$, and let l be a positive integer. If $f \in C^{l,\alpha}_{\beta,\gamma+l}(\Gamma)$, then there exists a unique solution $\varphi \in C^{l,\alpha}_{\beta,\gamma+l}(\Gamma)$ of the integral equation $(1+T)\varphi = f$ and this solution can be represented in the form*

$$(1+T)^{-1}f = \frac{1}{2}\Big(1 - Q^-\frac{\partial}{\partial n}P^+\Big)f. \tag{2.17}$$

Here P^+ and Q^- are the inverse operators of the boundary value problems (2.3) and (2.4) (see Theorems 3 and 4).

Proof. By Theorems 3 and 4, the function

$$\varphi = \frac{1}{2}\Big(1 - Q^- \frac{\partial}{\partial n} P^+\Big) f$$

is in the space $C^{l,\alpha}_{\beta,\gamma+l}(\Gamma)$. We shall prove that φ is the solution of the equation $(1+T)\varphi = f$. We introduce the function $u \in C^{l,\alpha}_{\beta,\gamma+l}(G^+ \cup G^-)$ which is a solution of the boundary value problem

$$\Delta u = 0 \ \text{ on } \ G^+ \cup G^-, \quad u^+ = f \ \text{ on } \ \Gamma,$$

$$\Big(\frac{\partial u}{\partial n}\Big)^+ - \Big(\frac{\partial u}{\partial n}\Big)^- = 0 \ \text{ on } \ \Gamma\backslash\mathfrak{M}.$$

It is clear that $u^- = Q^- \frac{\partial}{\partial n} P^+ f$. Hence $\varphi = (u^+ - u^-)/2$. By this and Lemmas 1, 3 we arrive at the chain of equalities

$$\big((1+T)\varphi\big)(x) = 2(W\varphi)^+(x) = (W(u^+ - u^-))^+(x) = u^+(x) = f(x) \tag{2.18}$$

for $x \in \Gamma\backslash\mathfrak{M}$. Since $(W_0 1)(x)$ is the solid angle under which the surface Γ is seen from x, we conclude that $T\varphi \in C(\Gamma)$ for $\varphi \in C(\Gamma)$ and that the relations (2.18) hold for all $x \in \Gamma$.

It remains to verify the uniqueness of the solution. Let $\varphi_0 \in C^{l,\alpha}_{\beta,\gamma+l}(\Gamma)$ satisfy $(1+T)\varphi_0 = 0$. Consider the function $u = W\varphi_0$. By Lemma 1, u is a solution of (2.3) with $f = 0$. In view of the uniqueness of the solution of (2.3) we conclude that $W\varphi_0 = 0$ on G^+. Since

$$\Big(\frac{\partial(W\varphi_0)}{\partial n}\Big)^- = \Big(\frac{\partial(W\varphi_0)}{\partial n}\Big)^+ = 0 \qquad \text{on} \quad \Gamma\backslash\mathfrak{M}$$

and since (2.4) is uniquely solvable, we conclude that $W\varphi_0 = 0$ in G^-. Thus,

$$\varphi_0 = (W\varphi_0)^+ - (W\varphi_0)^- = 0,$$

which completes the proof.

3 Estimates for the kernel of the inverse operator

In what follows we use the notations

$$\varkappa = \min\{\delta^+, \nu^-, 1\}, \qquad \lambda = \min\{\lambda^+, \lambda^-\}.$$

The aim of this section is to prove the following assertion

Theorem 8 *Let $0 < \alpha - \gamma < \lambda$, $0 < \beta + \gamma - \alpha < 1$, and let l be a positive integer. Then*

$$(1+T)^{-1} f = (1 + L + M) f, \quad f \in C^{l,\alpha}_{\beta,\gamma+l}(\Gamma). \tag{3.1}$$

where L and M are integral operators on Γ. The kernel $M(x,y)$ of the operator M admits the estimate

$$|M(x,y)| \le c\,\rho(y)^{\varkappa-1-\varepsilon}\Big(\frac{r(y)}{\rho(y)}\Big)^{\lambda-1-\varepsilon}.$$

The kernel $L(x,y)$ of L equals zero if $\operatorname{dist}(x,y) \ge \delta$, where δ is a sufficiently small positive number.

If the points x and y lie in a neighbourhood of a vertex O_i, $i = 1, 2, \dots, m$, and this neighbourhood contains no vertices of the polyhedron other than O_i, then the kernel $\mathcal{L}(x, y)$ satisfies

$$|L(x,y)| \le c\,\rho(y)^{-2}\Big(\frac{r(y)}{\rho(y)}\Big)^{\lambda-1-\varepsilon}$$
$$+c\,\big(r(y)+|x-y|\big)^{-2}\Big(\frac{r(x)}{r(x)+|x-y|}\Big)^{\lambda-\varepsilon}\Big(\frac{r(y)}{r(y)+|x-y|}\Big)^{\lambda-1-\varepsilon}$$

provided $\rho(x)/2 < \rho(y) < 2\rho(x)$ and

$$|L(x,y)| \le c\,\rho(y)^{-1}\big(\rho(x)+\rho(y)\big)^{-1}\Big(\frac{\min\{\rho(x),\rho(y)\}}{\rho(x)+\rho(y)}\Big)^{\varkappa-\varepsilon}\Big(\frac{r(y)}{\rho(y)}\Big)^{\lambda-1-\varepsilon}$$

in the opposite case. Here ε is an arbitrary positive number.

In what follows by $\{\chi_k\}_{k=1}^{3}$, $\eta_1$and, η_2 we mean functions in $C^\infty([0,\infty))$ such that

(1) $$\sum_{1\le k\le 3}\chi_k = 1,\quad \operatorname{supp}\chi_1\subset[0,5/8),\quad \operatorname{supp}\chi_2\subset(1/2,2),\quad \chi_3\subset(8/5,\infty);$$

(2) $$\eta_1(t)=1 \quad\text{for } t<1/8 \quad\text{and}\quad \eta_1(t)=0 \quad\text{for } t\ge 1/4;$$

(3) $$\eta_2(t)=1 \quad\text{for } t<5/6 \quad\text{and}\quad \eta_2(t)=0 \quad\text{for } t\ge 6/7.$$

We assume that the points x and y lie in a neighbourhood U_i of the vertex O_i, $i = 1, 2, \dots, m$ and that U_i contains no other vertices than O_i. Let the origin coincide with O_i.

3.1 Estimates for the kernels $L(x, y)$ and $M(x, y)$ for $|y| < 5|x|/8$

Given $x \in \Gamma\backslash\mathfrak{M}$, consider the Dirichlet problem

$$\begin{aligned}\Delta_y R^+(y,x) &= 0, \quad y\in G^+\\ R^+(y,x) &= \eta_2(|y|/|x|)R^-(y,x), \quad y\in\Gamma.\end{aligned}\tag{3.2}$$

Lemma 4 *Let $0 < -\alpha - \gamma < \lambda$, $-\varkappa < \beta + \gamma - \alpha < 1 + \varkappa$ and let l be a positive integer. Then there exists a unique solution $R^+(\cdot, x) \in C^{l,\alpha}_{\beta,\gamma+l}(G^+)$ of the problem (3.2) for all $x \in U_i \cap (\Gamma\backslash O_i)$ and*

$$|\partial_y^\tau R^+(y,x)| \le c_\tau\,|x|^{-1}|y|^{-|\tau|}\Big(\frac{|y|}{|x|}\Big)^{\varkappa-\varepsilon}\Big(\frac{r(y)}{|y|}\Big)^{\lambda_{\tau\varepsilon}},\tag{3.3}$$

where $\lambda_{\tau\varepsilon} = \min\{0, \lambda - |\tau| - \varepsilon\}$ and $y \in U_i \cap G^+$.

Proof. We set $x = |x|X$, $y = |x|Y$ and let $G_{|x|}$ and $\Gamma_{|x|}$ be the images of the sets G^+ and Γ under the mapping $y \to Y$. The problem (3.2) can be written in the form

$$\begin{aligned}\Delta_Y R_{|x|}(Y,X) &= 0 \quad Y\in G_{|x|}\\ R_{|x|}(Y,X) &= H_x(y), \quad Y\in\Gamma_{|x|}\end{aligned}\tag{3.4}$$

where

$$R_{|x|}(Y,X) = |x|R^+(|x|Y,|x|X), \quad |X|=1$$

and in view of the inequalities for $\partial_x^\sigma \partial_\xi^\tau R^-(x,\xi)$ from Theorem 4

$$\|H_x\|_{C^{l,\alpha}_{\beta,\gamma+l}(G^+_{|x|})} \le c.$$

Applying Theorem 3 to the solution of the problem (3.4), we get

$$\|R_{|x|}(\cdot,X)\|_{C^{l,\alpha}_{\beta,l+\gamma}(G^+_{|x|})} \le c.$$

Setting $\gamma = \alpha - \lambda + \varepsilon$, $\beta = -\varkappa - \gamma + \alpha + \varepsilon$, we find

$$|\partial_Y^\tau R_{|x|}(Y,X)| \le c_\tau\, |Y|^{\varkappa - |\tau| - \varepsilon} \Big(\frac{r(Y)}{|Y|}\Big)^{\lambda_{\tau\varepsilon}}.$$

Returning back to the function $R^+(y,x)$, we arrive at (3.3). The lemma is proved.

Lemma 5 *Let $0 < \alpha - \gamma < \lambda$, $0 < \beta + \gamma - \alpha < 1$ and let l be a positive integer. For any $\varphi \in C^{l,\alpha}_{\beta,l+\gamma}(\Gamma)$ the representation*

$$\begin{aligned} &\int_\Gamma Q^-(x,\xi)\frac{\partial}{\partial n_\xi}\int_\Gamma P^+(\xi,y)\chi_1\Big(\frac{|y|}{|x|}\Big)\varphi(y)ds_y ds_\xi \\ &= \int_\Gamma L(x,y)\chi_1\Big(\frac{|y|}{|x|}\Big)\varphi(y)ds_y, \quad x \in \Gamma \cap U_i \end{aligned} \tag{3.5}$$

is valid, where

$$|L(x,y)| \le c\,|x|^{-1}|y|^{-1}\Big(\frac{|y|}{|x|}\Big)^{\varkappa-\varepsilon}\Big(\frac{r(y)}{|y|}\Big)^{\lambda-1-\varepsilon}. \tag{3.6}$$

Proof. Setting

$$v(\xi) = \int_\Gamma P^+(\xi,y)\chi_1\Big(\frac{|y|}{|x|}\Big)\varphi(y)ds_y$$

and using (2.8), we rewrite the left-hand side of (3.5) in the form

$$\begin{aligned} &\int_\Gamma (\eta_2\Big(\frac{|\xi|}{|x|}\Big)Q^-(0,x)\frac{\partial}{\partial n_\xi}v(\xi)ds_\xi + \int_\Gamma R^+(\xi,x)\frac{\partial}{\partial n_\xi}v(\xi)ds_\xi \\ &\quad + \int_\Gamma (1-\eta_2\Big(\frac{|\xi|}{|x|}\Big)Q^-(x,\xi)\frac{\partial}{\partial n_\xi}v(\xi)ds_\xi, \end{aligned}$$

where R^+ is the solution of (3.2)

Applying Green's formula to the first and second integrals, we arrive at the representation (3.5) with

$$L(x,y) = \sum_{1\le k\le 3} L_k(x,y), \tag{3.7}$$

where

$$\begin{aligned} L_1(x,y) &= \frac{\partial}{\partial n_y}R^+(y,x), \\ L_2(x,y) &= -\int_{G^+}\Big(\Delta_\xi \eta_2\Big(\frac{|\xi|}{|x|}\Big)Q^-(0,x)\Big)P^+(\xi,y)ds_\xi, \\ L_3(x,y) &= -\int_\Gamma \Big(1-\eta_2\Big(\frac{|\xi|}{|x|}\Big)\Big)Q^-(x,\xi)\frac{\partial}{\partial n_\xi}P^+(\xi,y)ds_\xi. \end{aligned}$$

We estimate each term in (3.7). Inequality (3.6) for $\mathcal{L}_1(x,y)$ follows directly from (3.3). Let us estimate $\mathcal{L}_2(x,y)$. It is clear that $6|\xi| > 5|x|$ on the support of the function $\xi \to \Delta_\xi \eta_2(|\xi|/|x|)$. From this and from the inequality $8|y| < 5|x|$ we conclude that $3|\xi| > 4|y|$. Hence, by Theorems 3 and 4 we have

$$
\begin{aligned}
|L_2(x,y)| &\le c \int_{\xi \in G^+ : 6|\xi|>5|x|} |\xi|^{-3}|x|^{-1}|y|^{-1}\Big(\frac{|y|}{|\xi|}\Big)^{\delta^+-\varepsilon}\Big(\frac{r(y)}{|y|}\Big)^{\lambda^+-1-\varepsilon} d\xi \\
&\le c|x|^{-1}|y|^{-1}\Big(\frac{|y|}{|x|}\Big)^{\delta^+-\varepsilon}\Big(\frac{r(y)}{|y|}\Big)^{\lambda^+-1-\varepsilon}.
\end{aligned}
$$

Finally, to obtain the required estimate for $\mathcal{L}_3(x,y)$, we write it as the sum of two integrals over the sets

$$
\Gamma_1 = \{\xi \in \Gamma : |\xi| < 2\,|x|\} \qquad \text{and} \qquad \Gamma_2 = \{\xi \in \Gamma : |\xi| > 2\,|x|\}.
$$

By Theorems 3 and 4,

$$
\begin{aligned}
|L_3(x,y)| &\le c \int_{\substack{\xi \in \Gamma : 6|\xi|>5|x| \\ |x-\xi|<3|x|}} |x-\xi|^{-1}|\xi|^{-2} K(y,\xi)\, ds_\xi \\
&+ c\int_{\xi \in \Gamma : |\xi|>2|x|} \big(|\xi|^{-3} + \rho(\xi)^{\delta^+-1-\varepsilon}\big) K(y,\xi)\, ds_\xi \\
&\le c\,|x|^{-1}|y|^{-1}\Big(\frac{|y|}{|x|}\Big)^{\delta^+-\varepsilon}\Big(\frac{r(y)}{|y|}\Big)^{\lambda-1-\varepsilon}.
\end{aligned}
$$

Here we used the notation

$$
K(y,\xi) = |y|^{-1}(|y|/|\xi|)^{\delta^+-\varepsilon}\Big(\frac{r(y)}{|y|}\Big)^{\lambda^+-1-\varepsilon}\Big(\frac{r(\xi)}{\rho(\xi)}\Big)^{\lambda^+-1-\varepsilon}.
$$

The lemma is proved.

3.2 Estimates for the kernels $L(x,y)$ and $M(x,y)$ for $5|y| > 8|x|$

Let $x \in \Gamma \cap U_i$. Consider the following boundary value problems

$$
\Delta_y R^+(y,x) = 0, \quad y \in G^+ \qquad R^+(y,x) = \eta_2\Big(\frac{|x|}{|y|}\Big) R^-(y,x), \quad y \in \Gamma, \tag{3.8}
$$

$$
\Delta d^+(y) = 0, \quad y \in G^+ \qquad d^+(y) = d^-(y), \quad y \in \Gamma. \tag{3.9}
$$

Lemma 6 *Let $0 < \alpha - \gamma < \lambda$, $-\varkappa < \beta + \gamma - \alpha < 1 + \varkappa$ and let l be a positive integer. Then problems (3.8) and (3.9) have unique solutions $R^+(\cdot, x) \in C^{l,\alpha}_{\beta,\gamma+l}(G^+)$, respectively, $d^+ \in C^{l,\alpha}_{\beta,\gamma+l}(G^+)$ for all $x \in U_i \cap (\Gamma \backslash O_i)$ and*

$$
|\partial_y^\tau R^+(y,x)| \le c_\tau\, |y|^{-1-|\tau|}\Big(\frac{|x|}{|y|}\Big)^{\varkappa-\varepsilon}\Big(\frac{r(y)}{|y|}\Big)^{\lambda_{\tau\varepsilon}}, \quad y \in G^+ \cap O_i, \tag{3.10}
$$

$$
|\partial_y^\tau d^+(y)| \le c_\tau\, \rho(y)^{\varkappa-|\tau|-\varepsilon}\Big(\frac{r(y)}{\rho(y)}\Big)^{\lambda_{\tau\varepsilon}}, \quad y \in G^+ \cap O_i. \tag{3.11}
$$

Proof. Inequality (3.11) is a direct corollary of Theorem 3 and inequality (3.11) is proved in a similar manner as Lemma 4.

Lemma 7 *Let $0 < \alpha - \gamma < \lambda$, $0 < \beta + \gamma - \alpha < 1$, and let l be a positive integer. For any $\varphi \in C^{l,\alpha}_{\beta+l}(\Gamma)$*

$$\int_\Gamma Q^-(x,\xi)\frac{\partial}{\partial n_\xi}\int_\Gamma P^+(\xi,y)\chi_3\Big(\frac{|y|}{|x|}\Big)\varphi(y)ds_y\big)ds_\xi$$
$$= \int_\Gamma \big(M(x,y)+L(x,y)\big)\chi_3\Big(\frac{|y|}{|x|}\Big)\varphi(y)ds_y, \quad x\in\Gamma\backslash\mathfrak{M}, \tag{3.12}$$

where $y \in \Gamma\cap O_i$ and

$$|M(x,y)| \le c\,|y|^{\varkappa-1-\varepsilon}\Big(\frac{r(y)}{|y|}\Big)^{\lambda-1-\varepsilon}, \tag{3.13}$$

$$|L(x,y)| \le c\,|y|^{-2}(|x|/|y|)^{\varkappa-\varepsilon})\Big(\frac{r(y)}{|y|}\Big)^{\lambda-1-\varepsilon}. \tag{3.14}$$

Proof. Setting
$$v(\xi) = P^+(\xi,y)\chi_3(|y|/|x|)\varphi(y)ds_y$$
and using (2.7), (2.9), we write the left-hand side of (3.12) in the form

$$\int_\Gamma (\eta_2(|x|/|\xi|)(\frac{a_i^-}{|\xi|}+b_i^-+d^+(\xi))\frac{\partial}{\partial n_\xi}v(\xi)ds_\xi$$
$$+\int_\Gamma R^+(\xi,x)\frac{\partial}{\partial n_\xi}v(\xi)ds_\xi + \int_\Gamma (1-\eta_2\Big(\frac{|x|}{|\xi|}\Big)Q^-(x,\xi)\frac{\partial}{\partial n_\xi}v(\xi)ds_\xi.$$

Here $R^+(\xi,x)$ and $d_i^+(\xi)$ are solutions of (3.8) and (3.9). Applying Green's formula, we arrive at (3.12), where

$$L(x,y)+M(x,y) = \frac{\partial}{\partial n_y}R^+(y,x) + \frac{\partial}{\partial n_y}\Big(\eta_2\Big(\frac{|x|}{|y|}\Big)\Big(\frac{a_i^-}{|y|}+b_i^-+d^+(y)\Big)\Big)$$
$$-\int_{G^+}\Delta_\xi\Big(\eta_2\Big(\frac{|x|}{|\xi|}\Big)\Big(\frac{a_i^-}{|\xi|}+b_i^-+d^+(\xi)\Big)\Big)P^+(\xi,y)\,ds_\xi$$
$$+\int_\Gamma\Big(1-\eta_2\Big(\frac{|x|}{|\xi|}\Big)\Big)Q^-(x,\xi)\frac{\partial}{\partial n_\xi}P^+(\xi,y)\,ds_\xi.$$

To obtain estimates (3.13), (3.14), it is sufficient to use Theorems 3, 4, and Lemma 6 (see the proof of Lemma 5).

3.3 Estimates for the kernel $L(x,y)$ for $|y|/2 < |x| < 2|y|$

The purpose of this subsection is to prove the following assertion.

Lemma 8 *Let $0 < \alpha - \gamma < \lambda$, $0 < \beta + \gamma - \alpha < 1$, and let l be a positive integer. For any $\varphi \in C^{l,\alpha}_{\beta,l+\gamma}(\Gamma)$*

$$\int_\Gamma Q^-(x,\xi)\frac{\partial}{\partial n_\xi}\int_\Gamma \mathcal{P}^+(\xi,y)\chi_2\Big(\frac{|y|}{|x|}\Big)\varphi(y)ds_y\,ds_\xi$$
$$= -\varphi(x) + \int_\Gamma \mathcal{L}(x,y)\chi_2\Big(\frac{|y|}{|x|}\Big)\varphi(y)ds_y, \tag{3.15}$$

where

$$|L(x,y)| \le c\,(r(y))^{-2} \quad \text{if} \quad |x-y| < r(x)/2,$$

and

$$|L(x,y)| \le \frac{c}{|x-y|^2}\Big(\frac{r(x)}{|x-y|}\Big)^{\lambda-\varepsilon}\Big(\frac{r(y)}{|x-y|}\Big)^{\lambda-1-\varepsilon} + \frac{c}{|y|^2}\Big(\frac{r(y)}{|y|}\Big)^{\lambda-1-\varepsilon}$$

otherwise.

First we formulate an auxiliary assertion. Suppose that the point $x \in \Gamma\backslash O_i$ lies in a neighbourhood of the edge $\mathfrak{M}_j$ together with the ball $B(\delta|x|, x)$ of radius $\delta|x|$, where δ is a sufficiently small positive number. We denote by D_j^+ and D_j^- the interior and the exterior of the dihedral angle which coincides with G^+ near the edge $\mathfrak{M}_j$. In what follows we omit the index j in $D_j^\pm$ and use the notations for the diherdral angle $D^\pm$ introduced in Subsection 1.2.

Lemma 9 *The following estimates hold on the set* $\{y \in \Gamma : |x|/2 < |y| < 2|x|\}$*:*

$$|\partial_x^\sigma \partial_y^\tau (P^+(x,y) - \mathcal{P}^+(x,y)| \le c_{\sigma\tau}|x|^{-2-|\tau|-|\sigma|},$$

$$|\partial_x^\sigma \partial_y^\tau (Q^-(x,y) - \mathcal{Q}^-(x,y)| \le c_{\sigma\tau}|x|^{-1-|\tau|-|\sigma|}\Big(\frac{r(x)}{|x|}\Big)^{\lambda^-_{\sigma\varepsilon}}\Big(\frac{r(y)}{|y|}\Big)^{\lambda^-_{\tau\varepsilon}},$$

where $P^+(x,y)$*,* $Q^-(x,y)$ *are the kernels of the operators (2.12), (2.13), and*

$$\lambda^-_{\sigma\varepsilon} = \min\{0, \lambda^- - |\sigma| - \varepsilon\}, \quad \lambda^-_{\tau\varepsilon} = \min\{0, \lambda^- - |\tau| - \varepsilon\}.$$

The proof is similar to that of Lemma 2.6 in [GM3]. The only difference is that one has to use theorems on the solvability of the Dirichlet and Neumann problems in domains with edges (see [MP1], [**?**]) instead of similar assertions for smooth boundaries.

Let $x \in \Gamma\backslash O_i$. Consider the following problem

$$\Delta\mathcal{R}^+(x,y) = 0, \quad y \in G^+,$$
$$\mathcal{R}^+(x,y) = \chi_2\Big(\frac{|x|}{|y|}\Big)\big(Q^-(x,y) - \mathcal{Q}^-(x,y)\big), \quad y \in \Gamma.$$

It follows essentially from Theorem 3 that the estimate

$$|\partial_y^\tau \mathcal{R}^+(x,y)| \le c_\tau\,|x|^{-1-|\tau|}\Big(\frac{r(y)}{|y|}\Big)^{\lambda_{\tau\varepsilon}} \tag{3.16}$$

holds for all y with $|x|/2 < |y| < 2|x|$, where $\lambda_{\tau\varepsilon} = \min\{0, \lambda - |\tau| - \varepsilon\}$.

Proof of Lemma 8. We write the left-hand side of (3.16) as

$$\begin{aligned}&\int_\Gamma \chi_2\Big(\frac{|x|}{|y|}\Big)\mathcal{Q}^-(x,\xi)\frac{\partial}{\partial n_\xi}\,v(\xi)\,ds_\xi + \int_\Gamma \mathcal{R}^+(x,\xi)\frac{\partial}{\partial n_\xi}\,v(\xi)\,ds_\xi \\ &+ \int_\Gamma (1-\chi_2\Big(\frac{|x|}{|\xi|}\Big))Q^-(x,\xi)\frac{\partial}{\partial n_\xi}\,v(\xi)\,ds_\xi,\end{aligned} \tag{3.17}$$

where

$$v(\xi) = \int_\Gamma P^+(\xi,y)\chi_2\Big(\frac{|y|}{|x|}\Big)\,\varphi(y)\,ds_y.$$

Replacing $P^+(x,\xi)$ by $\mathcal{P}^+(x,\xi)$ in the first term and applying Green's formula to the second term, we write (3.17) in the form

$$\int_\Gamma \mathcal{Q}^-(x,\xi)\frac{\partial}{\partial n_\xi}\int_\Gamma \mathcal{P}^+(\xi,y)\,\eta_1\Big(\frac{|y-x|}{\delta|x|}\Big)\varphi(y)\,ds_y\,ds_\xi \\ +\int_\Gamma L'(x,y)\chi_2\Big(\frac{|y|}{|x|}\Big)\,\varphi(y)\,ds_y, \tag{3.18}$$

where

$$L'(x,y)=\frac{\partial}{\partial n_\xi}\mathcal{R}^+(x,y) \\ -\int_\Gamma \mathcal{Q}^-(x,\xi)\frac{\partial}{\partial n_\xi}\mathcal{P}^+(\xi,y)\,\Big(\eta_1\Big(\frac{|y-\xi|}{\delta|x|}\Big)-1\Big)\eta_1\Big(\frac{|y-x|}{\delta|x|}\Big)\,\varphi(y)\,ds_\xi \\ +\int_\Gamma \chi_2\Big(\frac{|x|}{|\xi|}\Big)\,\mathcal{Q}^-(x,\xi)\frac{\partial}{\partial n_\xi}P^+(\xi,y)\Big(1-\eta_1\Big(\frac{|y-x|}{\delta|x|}\Big)\Big)ds_\xi \\ +\int_\Gamma \mathcal{Q}^-_\chi(x,\xi)\frac{\partial}{\partial n_\xi}P^+(\xi,y)\Big(1-\eta_1\Big(\frac{|y-\xi|}{\delta|x|}\Big)\Big)\eta_1\Big(\frac{|y-x|}{\delta|x|}\Big)\,ds_\xi \\ +\int_\Gamma \mathcal{Q}^-_\chi(x,\xi)\frac{\partial}{\partial n_\xi}\big(P^+(\xi,y)-\mathcal{P}^+(\xi,y)\big)\eta_1\Big(\frac{|y-\xi|}{\delta|x|}\Big)\eta_1\Big(\frac{|y-x|}{\delta|x|}\Big)ds_\xi \\ +\int_\Gamma\Big(1-\chi_2\Big(\frac{|x|}{|\xi|}\Big)\Big)Q^-(x,\xi)\frac{\partial}{\partial n_\xi}P^+(\xi,y)\,ds_\xi.$$

Here $\mathcal{Q}^- = \mathcal{Q}^-_\chi(x,\xi) = \chi_2\Big(\frac{|x|}{|\xi|}\Big)\mathcal{Q}^-(x,\xi)$. Estimating each term with the help of Theorems 3, 4 and Lemma 9, we arrive at the inequality

$$|L'(x,y)|\le c\,|x|^{-2}\Big(\frac{r(y)}{|y|}\Big)^{\lambda-1-\varepsilon}.$$

We set $x'=x/|x|$, $\xi'=\xi/|x|$, $y'=y/|x|$. Since the functions $\mathcal{Q}^-(x,y)$ and $\mathcal{P}^+(x,y)$ are homogeneous, the first term in (3.18) takes the form

$$\int_\Gamma \mathcal{Q}^-(x',\xi')\frac{\partial}{\partial n_\xi}\int_\Gamma \mathcal{P}^+(\xi',y')\,\eta_1\Big(\frac{|y'-x'|}{\delta}\Big)\varphi(y)\,ds_{y'}\,ds_{\xi'}.$$

To complete the proof, it is sufficient to refer the following assertion.

Lemma 10 *Let F be the boundary of the dihedral angle with opening ω and let $\Lambda=\pi/(\pi+|\pi-\omega|)$. If $\varphi\in C^{l,\alpha}_{\gamma+l}(F)$, $0<\alpha-\gamma<\Lambda$, $\operatorname{supp}\varphi\subset B(1,0)$, then the representation*

$$\int_\Gamma \mathcal{Q}^-(x,\xi)\frac{\partial}{\partial n_\xi}\mathcal{P}^+(\xi,y)\varphi(y)\,ds_y\,ds_\xi=-\varphi(x)+\int_F \mathcal{L}(x,y)\varphi(y)\,ds_y \tag{3.19}$$

holds for $x\in F\backslash\mathfrak{M}$. Moreover,

$$|\mathcal{L}(x,y)|\le c\,(r(y))^{-2}\qquad \text{for}\quad |x-y|<r(x)/2,$$

and

$$|\mathcal{L}(x,y)|\le c\,|x-y|^{-2}\Big(\frac{r(x)}{|x-y|}\Big)^{\Lambda-\varepsilon}\Big(\frac{r(y)}{|x-y|}\Big)^{\Lambda-1-\varepsilon}$$

otherwise.

Now we formulate an auxiliary assertion. Consider the problem

$$\Delta \mathcal{R}^+(x,y) = 0, \quad y \in D^+,$$
$$\mathcal{R}^+(x,y) = \Big(\mathcal{Q}^-(x,y) - \frac{a}{|x-y|}\Big)\Big(1-\eta_1\Big(\frac{|y-x|}{r(x)}\Big)\Big), \quad y \in F, \qquad (3.20)$$

where a, $\mathcal{Q}^-$, and r are the same as in Theorem 6.

Lemma 11 *Let $0 < \alpha - \gamma < \Lambda$, $\Lambda = \pi/(\pi + |\pi - \omega|)$, $x \in B(1,0)$. Then there exists a unique solution of (3.20) and the estimate*

$$|\partial_y^\sigma \mathcal{R}^+(x,y)| \le c\,|x-y|^{-1-|\sigma|}\Big(\frac{r(x)}{|x-y|}\Big)^{\Lambda-\varepsilon}\Big(\frac{r(y)}{|x-y|}\Big)^{\Lambda-|\sigma|-\varepsilon}$$

holds for $|x-y| > r(x)$.

Proof. Since $\mathcal{R}^+(x,y)$ is homogeneous, we can assume that $|x-y| = 1$. We introduce the function

$$v(y) = r(x)^{-\Lambda+\varepsilon}\mathcal{R}^+(x,y).$$

Clearly, v solves the problem

$$\Delta v = 0 \quad \text{on } D^+, \quad v = \psi \text{ on } F,$$

where

$$|\partial_y^\sigma \psi| \le c\, r(y)^{\Lambda-|\sigma|-\varepsilon}.$$

The required estimate follows from Theorem 5.

Proof of Lemma 10. In what follows, for definiteness, we assume that x lies on the face F^+ of the polyhedron. We represent the left-hand side of (3.19) as the sum of two terms obtained from the initial expression by replacing φ by φ_1 and by φ_2, where

$$\varphi_1 = \varphi - \varphi_2, \quad \varphi_2(x,y) = \varphi(x,y)\eta_1(|x-y|/r(x)\delta)).$$

Here δ is so small that $F^- \cap B(\delta r(x), x) = \emptyset$.

We write the first term in the form

$$\int_F \eta_1\Big(\frac{4|x-y|}{\delta r(x)}\Big)\mathcal{Q}^-(x,\xi)\frac{\partial}{\partial n_\xi}v(\xi)ds_\xi + \int_F\Big(1-\eta_1\Big(\frac{4|x-y|}{\delta r(x)}\Big)\Big)\mathcal{Q}^-(x,\xi)\frac{\partial}{\partial n_\xi}v(\xi)ds_\xi,$$

where

$$v(\xi) = \int_F \mathcal{P}^+(\xi,y)\Big(1-\eta_1\Big(\frac{|x-y|}{\delta r(x)}\Big)\Big)\varphi(y)ds_y.$$

Applying Green's formula to the second integral and using the solution of (3.20), we find that it is equal to

$$\int_F \mathcal{L}_1^+(x,y)\Big(1-\eta_1\Big(\frac{|x-y|}{\delta r(x)}\Big)\Big)\varphi(y)ds_y,$$

where

$$\mathcal{L}_1^+(x,y) = \frac{\partial}{\partial n_y}\Big(\mathcal{R}^+(x,y) + \frac{a}{|x-y|}\Big) - \int_F \eta_1\Big(\frac{4|x-\xi|}{\delta r(x)}\Big)\mathcal{Q}^-(x,\xi)\frac{\partial}{\partial n_\xi}\mathcal{P}^+(\xi,y)ds_\xi$$
$$+\int_{D+}\Delta_\xi\Big(\mathcal{R}^+(x,\xi) + \frac{a}{|x-\xi|}\Big)\Big(1-\eta_1\Big(\frac{4|x-\xi|}{\delta r(x)}\Big)\mathcal{P}^+(\xi,y)\,d\xi.$$

Since

$$\Big|\frac{\partial}{\partial n_y}\frac{1}{|x-y|}\Big| \le \frac{r(x)}{|x-y|^3} \quad \text{for } |x-y|>r(x),$$

the estimate

$$|\mathcal{L}_1(x,y)| \le c\,|x-y|^{-2}\Big(\frac{r(x)}{|x-y|}\Big)^{\Lambda-\varepsilon}\Big(\frac{r(y)}{|x-y|}\Big)^{\Lambda-1-\varepsilon}$$

follows from Theorems 5, 6 and from Lemma 11.

Next we need to show that

$$\int_F \mathcal{Q}^-(x,\xi)\frac{\partial}{\partial n_\xi}\int_F \mathcal{P}^+(\xi,y)\eta_1\Big(\frac{|x-y|}{\delta r(x)}\Big)\varphi(y)ds_y\,d\xi$$

$$= -\varphi(x) + \int_F \mathcal{L}_2(x,y)\eta_1\Big(\frac{|x-y|}{\delta r(x)}\Big)\varphi(y)ds_y, \quad x\in\Gamma\backslash\mathfrak{M}, \tag{3.21}$$

where

$$|\mathcal{L}_2(x,y)| \le c\,(r(y))^{-2}. \tag{3.22}$$

We write the left-hand side of (3.21) as the sum of two terms by setting

$$\mathcal{Q}^-(x,\xi) = Q_1(x,\xi) + Q_2(x,\xi),$$

$$Q_1(x,\xi) = \mathcal{Q}^-(x,\xi)\,\eta_1(|x-\xi|/(4\delta r(x)).$$

It is clear that the inequality $|y-\xi| > s\,r(y)$ holds for

$$|x-\xi| > \delta\, r(x)/2, \qquad |x-y| < \delta\, r(x)/4,$$

where s is a certain positive number. Hence, by Theorems 5, 6, the second term is the integral operator with a kernel satisfying (3.22).

We denote by $P_0^+(x,y)$ and $Q_0^-(x,y)$ the kernels of the inverse operators of the corresponding boundary value problems in the half-space, that is the problems obtained from (2.10) and (2.11) by replacing $D^\pm$ and F by $\mathbb{R}^3_\pm$ and $\mathbb{R}^2$, where $\mathbb{R}^2$ is the plane containing F^+, $\mathbb{R}^3_+$ is the half-space with boundary $\mathbb{R}^2$ containing points of the polyhedron near the origin and $\mathbb{R}^3_- = \mathbb{R}^3\backslash(\mathbb{R}^3_+\cup\mathbb{R}^2)$. It is well known that the estimates

$$|\partial_\xi^\sigma\partial_y^\tau(\mathcal{P}^+(\xi,y) - P_0^+(\xi,y))| \le c_{\sigma\tau}r(x)^{-2-|\tau|-|\sigma|},$$

$$|\partial_\xi^\sigma\partial_y^\tau(\mathcal{Q}^-(\xi,y) - Q_0^-(\xi,y))| \le c_{\sigma\tau}r(x)^{-1-|\tau|-|\sigma|},$$

hold for $\xi, y\in D^\pm\cap B(\delta r(x),x)$.

Therefore, to obtain the representation (3.21), it is sufficient to show that the expression

$$\int_{F^+}\eta_1\Big(\frac{|x-\xi|}{4\delta r(x)}\Big)Q_0^-(x,\xi)\frac{\partial}{\partial n_\xi}\int_{F+}P_0^+\eta_1\Big(\frac{|x-\xi|}{\delta r(x)}\Big)\varphi(y)ds_y\,ds_\xi$$

admits a similar representation. The last follows directly from the relation

$$\int_{\mathbb{R}^2}Q_0^-(x,\xi)\frac{\partial}{\partial n_\xi}\int_{\mathbb{R}^2}P_0^+(\xi,y)\varphi(y)ds_y\,ds_\xi = -\varphi(x).$$

The lemma is proved.

Proof of Theorem 8. If the point x is placed near y, then the estimates for the kernels $M(x,y)$, $L(x,y)$ follow from Lemmas 5, 7, 8. In the opposite case such estimates can be obtained similarly and even simpler.

Remark 1 *Using known results on the asymptotic behaviour of solutions to the Dirichlet and Neumann problems near boundary singularities, one can improve estimates of the kernels $M(x, y)$, $L(x, y)$. For example, if the points x, y lie in the neighbourhood of a vertex O_i which does not contain other vertices and if for a certain edge $\mathfrak{M}_j$ the estimates*

$$\operatorname{dist}(x, \mathfrak{M}_j) \le c \operatorname{dist}(x, \mathfrak{M}_s), \qquad \operatorname{dist}(y, \mathfrak{M}_j) \le c \operatorname{dist}(y, \mathfrak{M}_s)$$

hold for all $s : 1 \le s \le k$, $s \ne j$, then the numbers $\varkappa$ and λ may be replaced by $\min\{\delta_j, \nu_j\}$ and $\pi/(\pi + |\pi - \omega_j|)$, where ω_j is the opening of the dihedral angle with the edge $\mathfrak{M}_j$.

Remark 2 *The following representation holds for the inverse operator of the integral equation associated with the exterior Neumann problem:*

$$(1 + T^*)^{-1} g = (1 + L^* + M^*) g, \qquad g \in N^{l-1,\alpha}_{\beta,\gamma+l}(\Gamma),$$

where the kernels of the operators L^* and $M*$ obey the estimates which can be obtained from the estimates for the kernels L and M in (3.1) by replacing x by y and vice versa.

4 Solvability of the integral equation

In this section we use our previous notatio. Besides, we denote by $L^p_{\beta,\gamma}(\Gamma)$ the space of functions u with the norm

$$\|u\|_{L^p_{\beta,\gamma}(\Gamma)} = \|\rho^\beta r^\gamma u\|_{L_p(\Gamma)}.$$

Lemma 12 *The operators L and M satisfying the estimates given in Theorem 8 are continuous in $L^p_{\beta,\gamma}(\Gamma)$ for*

$$1 \le p < \infty, \quad 0 < \beta + \gamma + 2/p < 1 + \varkappa, \quad 0 < \gamma + 1/p < \lambda$$

and for

$$p = \infty, \quad 0 \le \beta + \gamma < 1 + \varkappa, \quad 0 \le \gamma < \lambda.$$

Proof. Let $\varphi \in L^p_{\beta,\gamma}(\Gamma)$. It is sufficient to show that $L\varphi \in L^p_{\beta,\gamma}(\Gamma \cap U)$, respectively, $M\varphi \in L^p_{\beta,\gamma}(\Gamma \cap U)$, where U is a neighbourhood of a vertex O_i. For convenience we assume that the point O_i coincides with the origin. We denote by χ a function from $C_0^\infty(\mathbb{R}^3)$ which equals one on $\overline{U}$. We also assume that $\operatorname{supp}\chi$ contains no other vertices of the polyhedron except O_i.

We shall verify the following inequality for the function $\psi = \varphi\chi$

$$\|L\psi\|_{L^p_{\beta,\gamma}(\Gamma\cap U)} \le c\, \|\psi\|_{L^p_{\beta,\gamma}(\Gamma)}. \tag{4.1}$$

The same estimate for the operator M is obvious for $1 \le 0 < \infty$, $\beta + \gamma + 2/p > 0$, $\gamma + 1/p > 0$, and $p = \infty$, $\beta + \gamma \ge 0$, $\gamma \ge 0$.

We set

$$L\psi = \sum_{1 \le i \le 3} L_i \psi = \sum_{1 \le i \le 3} \int_{\Gamma_i} L(x, y)\, \psi(y)\, ds_y,$$

where $\Gamma_1 = \{\xi \in \Gamma : 2|\xi| < |x|\}$, $\Gamma_2 = \{\xi \in \Gamma : |x|/2 < |\xi| < 2|x|\}$, $\Gamma_3 = \{\xi \in \Gamma : |\xi| > 2|x|\}$, and prove (4.1) for each integral $L_i\psi$. We shall use the Hardy inequality

$$\|\rho^\alpha F\|_{L_p(\mathbb{R}^1_+)} \le c\, \|\rho^{\alpha+1} f\|_{L_p(\mathbb{R}^1_+)}, \tag{4.2}$$

where

$$F(\rho) = \int_0^\rho f(t)dt, \quad \alpha < -1/p, \qquad \text{and} \qquad F(\rho) = \int_\rho^\infty f(t)dt, \quad \alpha > -1/p.$$

Let $\overline{\psi}$ be the function on $\mathbb{R}^1_+$ defined by

$$\overline{\psi}(\rho) = \int_\ell r(\theta)^{\lambda-1-\varepsilon} |\psi(\rho\theta)| \, d\ell_\theta,$$

where $\ell = \partial K_i \cap S^2$, S^2 is the unit sphere with center at O_i, ∂K_i is the boundary of the cone K_i which coincides with G^+ near trhe point O_i and a positive ε is so small that $\lambda - \varepsilon > \gamma + 1/p$.

We set

$$F(\rho) = \int_0^\rho \tau^{\varkappa-\varepsilon} \overline{\psi}(\tau) \, d\tau$$

and in vew of estimates for $L(x,y)$ in Theorem 8 we get

$$\|\rho^{\beta+\gamma} L_1 \psi\|_{L_p(\Gamma \cap U)} \le c \, \|\rho^{\beta+\gamma-1-\varkappa+\varepsilon+1/p} F\|_{L_p(\mathbb{R}^1_+)}.$$

Using (4.2) and taking into account thet $\lambda - \varepsilon > \gamma + 1/p$, we arrive at the desired estimate for $L_1\psi$ for $\beta + \gamma + 2/p < 1 + \varkappa - \varepsilon$.

The estimate (4.1) for $L_3\psi$ is proved similarly. It is sufficient to consider the function

$$F(\rho) = \int_\rho^\infty \tau^{-1-\varkappa+\varepsilon} \overline{\psi}(\tau) \, d\tau$$

and to apply the inequality (4.2).

In order to obtain (4.1) for $L_2\psi$, we use the following assertion.

Lemma 13 *Let F be the boundary of the dihedral angle with edge $\mathfrak{M}$ and let $\mathcal{L}$ be the integral operator on F with the kernel $\mathcal{L}(x,y)$ satisfying the estimate in Lemma 10. Then the operator $\mathcal{L}$ is continuous in $L_{p,t}(F)$ for $1 \le p \le \infty$, $-\lambda < t + 1/p < \lambda$, where $L_{p,t}(F)$ is the function space with the norm*

$$\|u\|_{L_{p,t}(F)} = \|r^t u\|_{L_p(F)}.$$

Proof. Let $1 < p < \infty$. We denote by $\mathcal{L}_i$, $i = 1, 2$, the operator with the kernel $(\mathcal{L}\zeta_i)(x,y)$, where

$$\zeta_1(x,y) = \Big(1 - \eta_1\Big(\frac{|x-y|}{r(x)}\Big)\Big), \qquad \zeta_2(x,y) = \eta_1\Big(\frac{|x-y|}{r(x)}\Big).$$

For the operator $\mathcal{L}_1$ we have

$$\|\mathcal{L}_1\varphi\|^p_{L_{p,t}(F)} \le c \int_F r(x)^{p(t+\lambda-\varepsilon)} \Big(\int_F \frac{r(y)^{\lambda-1-\varepsilon}}{|x-y|^{1+2\lambda-2\varepsilon}} \zeta_1(x,y)\varphi(y) ds_y\Big)^p ds_x. \tag{4.3}$$

By Hölder's inequality, the interior integral is majorized by

$$\Big(\int_F \frac{r(y)^{\delta q}}{|x-y|^{2+\alpha q}} \zeta_1 ds_y\Big)^{\frac{1}{q}} \Big(\int_F \frac{r(y)^{p(\lambda-1-\varepsilon-\delta)}}{|x-y|^{2+p(2\lambda-1-2\varepsilon-\alpha)}} \zeta_1 |\varphi|^p ds_y\Big)^{\frac{1}{p}},$$

where $q = p/(p-1)$, $p \ne 1$. Setting $\delta > -1/q$, $\alpha - \delta > 0$, we conclude that the first factor in the last expression is estimated by $r(x)^{\delta-\alpha}$.

Hence

$$\|\mathcal{L}_1\varphi\|^p_{L_{p,t}(F)} \le c\int_F r(y)^{p(\lambda-1-\varepsilon-\delta)}|\varphi|^p\Big(\int_F \frac{r(x)^{p(t+\lambda-\varepsilon+\delta-\alpha)}}{|x-y|^{2+2(\lambda-1-2\varepsilon-\alpha)p}}\zeta_1(x,y)ds_x\Big)^p ds_y.$$

Suppose that ε, $\alpha-\delta$, $\delta+1-1/p$ are so small that

$$\lambda+t+1/p-\varepsilon+\delta-\alpha>0, \qquad \lambda-1-\varepsilon-\delta-t>0.$$

The the last inequality leads to the estimate

$$\|\mathcal{L}_1\varphi\|_{L_{p,t}(F)} \le c\,\|\varphi\|_{L_{p,t}(F)}. \tag{4.4}$$

In order to establish (4.4) for $p=1$, it is sufficient to change the order of integration in the right-hand side of (4.3). In the case $p=\infty$, the estimate (4.4) follows directly from the estimates for the kernel of thew operator L_1.

Using the inequalities

$$c_1 r(y) < r(x) < c_2 r(y), \qquad c_1, c_2 > 9,$$

for $x, y \in \operatorname{supp}\zeta_2$, we arrive at (4.4) for the operator $\mathcal{L}_2$.

Lemma 14 *The operator T is continuous in spaces $C(\Gamma)$ and $L^p_{\beta,\gamma}(\Gamma)$ for all $1 \le p < \infty$, $0 < \beta+\gamma+2/p < 2$, $0 < \gamma+1/p < 1$, and for $p=\infty$, $0 \le \beta+\gamma < 2$, $0 \le \gamma < 1$.*

Proof. Let the points x, y be placed in a neighbourhood of a vertex O_i. One verifies directly that the kernel $T(x,y)$ of the operator T admits the estimates

$$|T(x,y)| \le c\,\frac{r(x)}{(r(x)+|x-y|)^3} + c\,\frac{1}{\rho(x)^2}$$

if $\rho(x)/2 < \rho(y) < 2\rho(x)$, and

$$|T(x,y)| \le c\,\frac{\rho(x)}{(\rho(x)+\rho(y))^3} + c$$

otherwise.

It is known that $T\varphi \in C(\Gamma)$ for $\varphi \in C(\Gamma)$ (see [BM], [K]). Hence, by the above estimates for $T(x,y)$ all assertions of this lemma follow from Lemma 12.

Using Theorems 7 and Lemmas 12, 14, we arriuve at the following assertion.

Theorem 9 *Let $1 \le p < \infty$, $0 < \beta+\gamma+2/p < 1+\varkappa$, $0 < \gamma+1/p < \lambda$, and for $p=\infty$, $0 \le \beta+\gamma < 1+\varkappa$, $0 \le \gamma < \lambda$. Then the inverse operator of the integral equation associated with the Dirichlet problem is continuous in the spaces $C(\Gamma)$ and $L^p_{\beta,\gamma}(\Gamma)$.*

This result along with Lemma 14 shows in particular that the mappings

$$1+T : L_p(\Gamma) \to L_p(\Gamma), \qquad 1+T^* : L_{p/(p-1)}(\Gamma) \to L_{p/(p-1)}(\Gamma),$$

where $p > 2/(1+\varkappa)$ and $p > 1/\lambda$, are isomorphic.

References

[AKK] Angell, T.S., Kleinman, R.E, Kral, J. *Layer potentials on boundaries with corners and edges*, Časopis Pěst. Mat., **113** (1988), no. 4, 387–402

[BM] Burago, Yu.D., Maz'ya, V. *Multi-dimensional potential theory and the solution of boundary value problems in domains with non-regular boundaries*, Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova **3** (1967), 1-152; English translation: Sem. Math. V.A. Steklov Math. Inst. Leningr. **3** (1969), 1–68.

[G1] Grachev, N.V. *Representations and estimates for inverse operators of the potential theory integral equations in a polyhedron*, Proc. Int. Conf. on the Potential Theory, Nagoya, 1990, Potential Theory, Walter de Gruye, 1992, 201–206.

[GM1] Grachev, N.V., Maz'ya, V.G. *Representations and estimates for inverse operators of the potential theory integral equations on surfaces with conic points*, Soobsch. Akad. Nauk Gruzin. SSR **32**:1 (1988), 21-24.

[GM2] Grachev, N.V., Maz'ya, V.G. *On the Fredholm radius of integral equations of the potential theory*, In *Differential Equations. Spectral Theory. Wave Propogation*, pp. 109–133, Leningrad University Press, 1990.

[GM3] Grachev, N.; Maz'ya, V. *Estimates for kernels of inverse operators of integral equations of elasticity theory on surfaces with conic points*, J. Math. Sci. (N. Y.), **186**:2 (2012), 179–208 (first published as Preprint of Linköping University LiTH-MAT-R-91-06).

[GM4] Grachev, N.; Maz'ya, V. *Invertibility of boundary integral operators of elasticity on surfaces with conic points*, J. Math. Sci. (N. Y.), **186**:2 (2012), 209–233 (first published as Preprint of Linköping University LiTH-MAT-R-91-07).

[GM5] Grachev, N.V., Maz'ya, V.G. *Estimates of the fundamental solutions of the Neumann problem in polyhedral cone*, Preprint of Linköping University LiTH-MAT-R-91-28, 1–45.

[GM6] Grachev, N.V., Maz'ya, V.G. *Solvability of a boundary integral equation on a polyhedron*, Preprint of Linköping University LiTH-MAT-R-91-50.

[GM7] Grachev, N. V., Maz'ya, V. G. *A contact problem for the Laplace equation in the exterior of the boundary of a dihedral angle.* (Russian) Math. Nachr. **151** (1991), 207–231.

[K] Kral, J. *Integral Operators in Potential Theory*, Lect. Notes Math. **823**, Springer, 1980.

[KW] Kral, J. , Wendland, W. *Some examples concerning applicability of the Fredholm-Radon method in potential theory*, Apl. Mat. **31** (1986), 293–318.

[LM] Levin, A. V., Maz'ya, V. G. *Asymptotics of the densities of harmonic potentials near the apex of a cone.* (Russian) Z. Anal. Anwendungen **8** (1989), no. 6, 501–514.

[M1] Maz'ya, V. *Integral equations of the potential theory in domains with piecewise smooth boundaries*, Uspehi Matem. Nauk, **36**: 4 (1981), 229-230.

[M2] Maz'ya, V. *The potential theory for the Lamé system in domains with piecewise smooth boundaries*, Proc. of the Conference on partial Diff. Equations in memoriam of I.N. Vekua, Tbilisi, 1982, 123–129.

[M3] Maz'ya, V. *Boundary integral equations of elasticity in domains with piecewise smooth boundaries*, Equadiff 6, Proc. Int. Conf. Brno 1985, Lect. Notes Math. **1192**, 1986, 235–242.

[M4] Maz'ya, V. *Boundary integral equations*, Encyclopaedia of Math. Sciences, **27** (1991), 127–223.

[M5] Maz'ya, V. *Representations and estimates for inverse operators in the harmonic potential theory for polyhedra*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **23** (2012), no. 3, 229–258.

[MP1] Maz'ya, V., Plamenevsky, B. *The first boundary value problem for classical equations of mathematical physics in domains with piecewise smooth boundaries*, ZAA, 1: **24** (1983), 335–339; 2: **26** (1983), 523–551.

[MS] Maz'ya, V., Shaposhnikova, T. *Higher regularity in the layer potential theory for Lipschitz domains*, Indiana Univ. Math. J. **54** (2005), no. 1, 99–142.

[MZ1] Maz'ya, V., Zargaryan, S. *Singularities of solutions of a system of equations of potential theory for Zaremba's problem* (Russian) Vestnik Leningrad. Univ. Mat. Mekh. Astronom. **1** (1983), 43–48; English transl. Vestnik Leningrad. Univ. Math. **16** (1984), 49–55.

[MZ2] Maz'ya, V., Zargaryan, S. *The asymptotic form of the solutions of integral equations of potential theory in the neighbourhood of the corner points of a contour* (Russian) Prikl. Mat. Mekh. **48** (1984), no. 1, 169–174; English transl. J. Appl. Math. Mech. **48** (1985), no. 1, 120–124.

[R] Rathsfeld, A. *The invertibility of the double layer potential operator in the space of continuous functions defined on a polyhedron: the panel method*, Appl. Anal. **45** (1992), no. 1-4, 135–177.

[Z1] Zargaryan, S. *Asymptotic behavior of solutions of a system of singular integral equations generated by Lamé equations in the neighborhood of corner points of a contour.* (Russian) Akad. Nauk Armyan. SSR Dokl. **77** (1983), no. 1, 30–35.

[Z2] Zargaryan, S. *Singularities of solutions of a system of singular integral equations of the plane theory of elasticity under stresses given on the boundary.* (Russian) Akad. Nauk Armyan. SSR Dokl. **77** (1983), no. 4, 167–172.

[ZS] Zaĭonchkovskiĭ, V., Solonnikov, V. A. *The Neumann problem for second-order elliptic equations in domains with ribs on the boundary*, Boundary value problems of mathematical physics and related questions in the theory of functions, 15, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **127** (1983), 7–48.