REGULARITY OF SOLUTIONS TO THE POLYHARMONIC EQUATION IN GENERAL DOMAINS

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ABSTRACT. The present paper establishes boundedness of $\left[m - \frac{n}{2} + \frac{1}{2}\right]$ derivatives for the solutions to the polyharmonic equation of order $2m$ in arbitrary bounded open sets of $\mathbb{R}^n$, $2 \le n \le 2m + 1$, without any restrictions on the geometry of the underlying domain. It is shown that this result is sharp and cannot be improved in general domains. Moreover, it is accompanied by sharp estimates on the polyharmonic Green function.

CONTENTS

1. INTRODUCTION

Higher order differential operators are important in physics and in engineering [34] and have been integrated in many areas of mathematics, including conformal geometry (Paneitz operator, Q-curvature [8], [9]), free boundary problems [1], non-linear elasticity [50], [10], [5], to mention just a few. Unfortunately, in spite of evident demand, the properties of higher order PDEs on *general* domains remained largely beyond reach. Their investigation brought challenging hypotheses and surprising counterexamples, and few general positive results. For instance, Hadamard's 1908 conjecture regarding positivity of the biharmonic Green function [16] was actually refuted in 1949 (see [14], [15], [45]), and later on the weak maximum principle was proved to fail as well, at least

in high dimensions [32], [44]. Another curious feature is a paradox of passage to the limit for solutions under approximation of a smooth domain by polygons [6], [28].

For a long time, almost all results for higher order PDEs pertained to piecewise smooth domains (see, e.g., [19], [20], [33]). The recent two decades have witnessed a great burst of activity in the study of boundary value problems on Lipschitz domains, which brought some powerful and delicate estimates for the bi-Laplacian and for the polyharmonic operator due to B. Dahlberg, C. Kenig, J. Pipher, G. Verchota, Z. Shen, M. Mitrea, and others ([11], [53], [40], [41], [42], [17], [44], [43], [54], [12], [2], [55], [46], [47], [48], [49], [35], [27], [56], [18], [36]). Unfortunately, none of these results could be extended to domains of more complicated geometry.

The present paper establishes sharp pointwise estimates on the solutions to the polyharmonic equation and their derivatives in arbitrary bounded open sets, without any restrictions on the geometry of the underlying domain. It is shown that these estimates are sharp and can not be improved.

Let us recall the second order case. One of the fundamental results of elliptic theory is the maximum principle for harmonic functions. It holds in *arbitrary* domains and guarantees that every solution to the Dirichlet problem for the Laplace's equation, with bounded data, is bounded. A similar statement is valid for a broad class of second order elliptic operators.

In 1960 an appropriate, weak, version of the maximum principle was obtained for higher order equations on *smooth* domains ([3], see also [37], [38]). Roughly speaking, it amounts to the estimate

$$(1.1) \qquad \|\nabla^{m-1}u\|_{L^\infty(\overline{\Omega})} \le C\|\nabla^{m-1}u\|_{L^\infty(\partial\Omega)},$$

where u is a solution of an elliptic differential equation of order $2m$ such that $|\nabla^{m-1}u|$ is continuous up to the boundary, $\nabla^{m-1}u = \{\partial^\alpha u\}_{|\alpha|=m-1}$ is a vector of all partial derivatives of u of order $m-1$ and we adopt the usual convention that the zeroth order derivative of u is u itself. The formulation in non-smooth domains is somewhat trickier (see, e.g., [39]). However, in any setting, the weak maximum principle would always guarantee that the solution with "nice" data has bounded derivatives of order $m-1$. In the early 1990s, (1.1) was extended to three-dimensional domains diffeomorphic to a polyhedron ([20], [31]) or having a Lipschitz boundary ([42], [44]). In general domains, no direct analog of the maximum principle exists (see Problem 4.3, p. 275, in Nečas's book [39]).

Moreover, it turns out that for every elliptic operator of order *greater than two* the maximum principle can be violated, in a four-dimensional cone ([32], see also [29], [41], [44]). It has been shown, in particular, that in dimensions $n \ge 4$ there are solutions to the polyharmonic equation with unbounded $(m-1)$-st derivatives (cf. (1.1)). This phenomenon raises two fundamental questions: whether the boundedness of the $(m-1)$-st derivatives remains valid in dimensions $n \le 3$, and whether there are some other, possibly lower-order, estimates that characterize the solutions when $n \ge 4$. The main result of the present paper is as follows.

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$, $2 \le n \le 2m+1$, and*

$$(1.2) \qquad (-\Delta)^m u = f \text{ in } \Omega, \quad f \in C_0^\infty(\Omega), \quad u \in \mathring{W}^{m,2}(\Omega).$$

Then the solution to the boundary value problem (1.2) *satisfies*

(1.3) $\nabla^{m-n/2+1/2}u \in L^\infty(\Omega)$ *when n is odd*   *and* $\nabla^{m-n/2}u \in L^\infty(\Omega)$ *when n is even.*

In particular,

(1.4) $\nabla^{m-1}u \in L^\infty(\Omega)$ *when n = 2, 3.*

Here the space $\mathring{W}^{m,2}(\Omega)$, is, as usually, a completion of $C_0^\infty(\Omega)$ in the norm given by $\|u\|_{\mathring{W}^{m,2}(\Omega)} = \|\nabla^m u\|_{L^2(\Omega)}$. We note that $\mathring{W}^{m,2}(\Omega)$ embeds into $C^k(\Omega)$ only when k is strictly smaller than $m - \frac{n}{2}$, $n < 2m$. Thus, whether the dimension is even or odd, Theorem 1.1 gains one derivative over the outcome of Sobolev embedding.

The results of Theorem 1.1 are sharp, in the sense that the solutions do not exhibit higher smoothness than warranted by (1.3)–(1.4) in general domains. Indeed, assume that $n \in [3, 2m + 1] \cap \mathbb{N}$ is odd and let $\Omega \subset \mathbb{R}^n$ be the punctured unit ball $B_1 \setminus \{O\}$, where $B_r = \{x \in \mathbb{R}^n : |x| < r\}$. Consider a function $\eta \in C_0^\infty(B_{1/2})$ such that $\eta = 1$ on $B_{1/4}$. Then let

(1.5) $u(x) := \eta(x)\, \partial_x^{m-\frac{n}{2}-\frac{1}{2}}(|x|^{2m-n}), \qquad x \in B_1 \setminus \{O\},$

where ∂_x stands for a derivative in the direction of x_i for some $i = 1, ..., n$. It is straightforward to check that $u \in \mathring{W}^{m,2}(\Omega)$ and $(-\Delta)^m u \in C_0^\infty(\Omega)$. While $\nabla^{m-\frac{n}{2}+\frac{1}{2}}u$ is bounded, the derivatives of the order $m - \frac{n}{2} + \frac{3}{2}$ are not, and moreover, $\nabla^{m-\frac{n}{2}+\frac{1}{2}}u$ is not continuous at the origin. Therefore, the estimates (1.3)–(1.4) are optimal in general domains.

As for the case when n is even, the results in [20, Section 10.4] demonstrate that in an exterior of a ray there is an m-harmonic function behaving as $|x|^{m-\frac{n}{2}+\frac{1}{2}}$. Thus, upon truncation by the aforementioned cut-off η, one obtains a solution to (1.2) in $B_1 \setminus \{x_1 = 0, ..., x_{n-1} = 0, 0 \le x_n < 1\}$, whose derivatives of order $m - \frac{n}{2} + 1$ are not bounded. Therefore, in even dimensions (1.3) is a sharp property as well.

Furthermore, Theorem 1.1 has several quantitative manifestations, providing specific estimates on the solutions to (1.2). Most importantly, *we establish sharp pointwise estimates on Green's function of the polyharmonic operator and its derivatives*, once again without any restrictions on the geometry of the domain. The full list of the estimates is quite extended. For the purposes of the introduction, let us just highlight the highest order case.

As customary, we denote by $G(x, y)$, $x, y \in \Omega$, Green's function for the polyharmonic equation and by Γ its fundamental solution, so that, in particular, $G(x, y) - \Gamma(x - y)$, $x, y \in \Omega$, is the regular part of the Green function. By definition, for every fixed $y \in \Omega$ the function $G(\cdot, y)$ satisfies

(1.6) $(-\Delta_x)^m G(x, y) = \delta(x - y), \qquad x \in \Omega,$

in the space $\mathring{W}^{m,2}(\Omega)$. Here Δ_x stands for the Laplacian in x variable, and similarly we use the notation $\Delta_y, \nabla_y, \nabla_x$ for the Laplacian and gradient in y, and gradient in x, respectively. By $d(x)$ we denote the distance from $x \in \Omega$ to $\partial\Omega$.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary bounded domain. If $n \in [3, 2m + 1] \cap \mathbb{N}$ is odd then*

(1.7) $\left| \nabla_x^{m-\frac{n}{2}+\frac{1}{2}} \nabla_y^{m-\frac{n}{2}+\frac{1}{2}} (G(x, y) - \Gamma(x - y)) \right| \le \dfrac{C}{\max\{d(x), d(y), |x - y|\}}, \quad$ *for every $x, y \in \Omega$,*

and, in particular,

$$(1.8) \qquad \left| \nabla_x^{m-\frac{n}{2}+\frac{1}{2}} \nabla_y^{m-\frac{n}{2}+\frac{1}{2}} G(x,y) \right| \leq \frac{C}{|x-y|}, \quad \text{for every } x, y \in \Omega.$$

If $n \in [2, 2m] \cap \mathbb{N}$ is even, then

$$(1.9) \qquad \left| \nabla_x^{m-\frac{n}{2}} \nabla_y^{m-\frac{n}{2}} (G(x,y) - \Gamma(x-y)) \right| \leq C \log \left(1 + \frac{\operatorname{diam} \Omega}{\max\{d(x), d(y), |x-y|\}} \right), \text{ for every } x, y \in \Omega,$$

and

$$(1.10) \qquad \left| \nabla_x^{m-\frac{n}{2}} \nabla_y^{m-\frac{n}{2}} G(x,y) \right| \leq C \log \left(1 + \frac{\min\{d(x), d(y)\}}{|x-y|} \right), \quad \text{for every } x, y \in \Omega.$$

The constant C in (1.7)–(1.10) *depends on m and n only. In particular, it does not depend on the size or the geometry of the domain* Ω.

We mention that the pointwise bounds on the absolute value of Green's function itself have been treated previously in dimensions $2m + 1$ and $2m + 2$ for $m > 2$ and dimensions $5, 6, 7$ for $m = 2$ in [25, Section 10] (see also [24]). Also, as for the solutions, there exist results in smooth domains [13], [21], [51], [52], in conical domains [30], [20], and in polyhedra [31]. The estimates on the derivatives of Green's function in *arbitrary* bounded domains, provided by Theorem 1.2, are new.

Furthermore, using standard techniques, the Green's function estimates can be employed to establish the bounds on the solution to (1.2) for general classes of data f, such as L^p for a certain range of p, Lorentz spaces etc. We defer the detailed discussion of those to the body of the paper.

Here we just would like to point out that until recently, the properties stated in Theorems 1.1–1.2 seemed accessible exclusively under heavy restrictions on the geometry of the domain (see the references above). The present paper rests on a new method, based on some intricate weighted integral identities. The biggest challenge, and the core idea, is the proper choice of the weight function w. It is very subtle and finely tuned to the underlying elliptic operator in such a way that the positivity, or, rather, suitable bounds from below, could be obtained for expressions akin to $\int_\Omega (-\Delta)^m u(x) u(x) w(x) \, dx$. When such integrals have been considered before (in particular, we have established the three-dimensional biharmonic version of the results of this paper earlier in [22]), the difficulties of handling general m and n seemed insurmountable. One of the main technical achievement of this paper is the novel systematic construction of the weight leading to sharp bounds for the solutions. It invokes numerous new aspects: employing induction in eigenvalues of the Laplace-Beltrami operator on the sphere, preservation of some positivity properties under a change of underlying higher order operator, exploiting delicate peculiarities of $(-\Delta)^m$ depending on the parity of $m, n, m - n/2$, and others. The construction appears for the first time in this manuscript and is likely to be applicable to general classes of elliptic equations.

2. INTEGRAL INEQUALITIES AND GLOBAL ESTIMATE: THE CASE OF ODD DIMENSION. PART I: POWER WEIGHT

Let us start with a list of notation and conventions used throughout the paper.

For any domain $\Omega \subset \mathbb{R}^n$ a function $u \in C_0^\infty(\Omega)$ can be extended by zero to $\mathbb{R}^n$ and we will write $u \in C_0^\infty(\mathbb{R}^n)$ whenever convenient. Similarly, the functions in $\mathring{W}_2^m(\Omega)$, $m \in \mathbb{N}$, will be extended by zero and treated as functions on $\mathbb{R}^n$ or other open sets containing Ω without further comments.

The symbols $B_r(Q)$ and $S_r(Q)$ denote, respectively, the ball and the sphere with radius r centered at Q and $C_{r,R}(Q) = B_R(Q) \setminus \overline{B_r(Q)}$. When the center is at the origin, we write B_r in place of $B_r(O)$, and similarly $S_r := S_r(O)$ and $C_{r,R} := C_{r,R}(O)$.

Let (r, ω) be spherical coordinates in $\mathbb{R}^n$, $n \geq 2$, i.e. $r = |x| \in (0, \infty)$ and $\omega = x/|x|$ is a point of the unit sphere S^{n-1}. In fact, it will be more convenient to use e^{-t}, $t \in \mathbb{R}$, in place of r, so that $t = \log r^{-1} = \log |x|^{-1}$. Then by $\varkappa$ we denote the mapping

$$(2.1) \qquad \mathbb{R}^n \ni x \xrightarrow{\varkappa} (t, \omega) \in \mathbb{R} \times S^{n-1}, \quad n \geq 2.$$

The symbols δ_ω and ∇_ω refer, respectively, to the Laplace-Beltrami operator and the gradient on S^{n-1}.

Finally, by C, c, C_i and c_i, $i \in \mathbb{N}$, we generally denote some constants, possibly depending on the order of operator m and the dimension n but not on any other variables and not on the domain, unless explicitly stated otherwise. Their exact values are of no importance and can change from line to line. Also, we write $A \approx B$, if $C^{-1} A \leq B \leq C A$ for some $C > 0$.

Theorem 2.1. *Assume that $m \in \mathbb{N}$ and $n \in [3, 2m + 1] \cap \mathbb{N}$ is odd. Let Ω be a bounded domain in $\mathbb{R}^n$, $O \in \mathbb{R}^n \setminus \Omega$, $u \in C_0^\infty(\Omega)$ and $v = e^{\left(m - \frac{n}{2} + \frac{1}{2}\right)t}(u \circ \varkappa^{-1})$. Then*

$$\int_{\mathbb{R}^n} (-\Delta)^m u(x)\, u(x)|x|^{-1}\, dx$$

$$(2.2) \qquad \geq C \sum_{k=1}^m \int_{\mathbb{R}} \int_{S^{n-1}} \left(\partial_t^k v\right)^2 d\omega dt + C \int_{\mathbb{R}} \int_{S^{n-1}} v \prod_{p=-\frac{n}{2}+\frac{3}{2}}^{m-\frac{n}{2}+\frac{1}{2}} (-\delta_\omega - p\,(p + n - 2))\, v\, d\omega dt,$$

where $C > 0$ is some constant depending on m and n only.

Proof. **Step I.** In the system of coordinates (t, ω) the polyharmonic operator can be written as

$$(2.3) \qquad (-\Delta)^m = (-1)^m e^{2mt} \prod_{j=0}^{m-1} \left((-\partial_t - 2j)(-\partial_t - 2j + n - 2) + \delta_\omega\right).$$

Then

$$(2.4) \qquad \int_{\mathbb{R}^n} (-\Delta)^m u(x)\, u(x)|x|^{-1}\, dx = \int_{\mathbb{R}} \int_{S^{n-1}} \mathcal{L}^{m,n}(\partial_t, \delta_\omega) v(t, \omega)\, v(t, \omega)\, d\omega dt,$$

with

$$(2.5) \qquad \mathcal{L}^{m,n}(\partial_t, \delta_\omega) = (-1)^m \prod_{j=0}^{m-1} \left(\left(-\partial_t + m - \frac{n}{2} + \frac{1}{2} - 2j\right)\left(-\partial_t + m + \frac{n}{2} - \frac{3}{2} - 2j\right) + \delta_\omega\right).$$

Denote by v_{pl} the coefficients of the expansion of v into spherical harmonics:

$$(2.6) \qquad v(t, \omega) = \sum_{p=0}^{\infty} \sum_{l=-p}^{p} v_{pl}(t) Y_l^p(\omega), \qquad t \in \mathbb{R}, \;\; \omega \in S^{n-1}.$$

Then we can write the expression on the right-hand side of (2.4) as

$$(2.7) \qquad \sum_{p=0}^{\infty} \sum_{l=-p}^{p} \int_{\mathbb{R}} \mathcal{L}^{m,n}(\partial_t, -p(p+n-2)) v_{pl}(t) \, v_{pl}(t) \, dt.$$

As usually, denote by $\widehat{v}$ the Fourier transform of v, i.e.

$$(2.8) \qquad \widehat{v}(\gamma) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\gamma t} v(t) \, dt, \qquad \gamma \in \mathbb{R}.$$

By the Plancherel's identity (2.7) is equal to

$$\sum_{p=0}^{\infty} \sum_{l=-p}^{p} \int_{\mathbb{R}} \mathcal{L}^{m,n}(i\gamma, -p(p+n-2)) \left| \widehat{v_{pl}}(\gamma) \right|^2 \, d\gamma$$

$$(2.9) \qquad\qquad = \sum_{p=0}^{\infty} \sum_{l=-p}^{p} \int_{\mathbb{R}} \mathfrak{Re}\, \mathcal{L}^{m,n}(i\gamma, -p(p+n-2)) \left| \widehat{v_{pl}}(\gamma) \right|^2 \, d\gamma.$$

Indeed, the imaginary part of the polynomial $\mathcal{L}^{m,n}(i\gamma, -p(p+n-2))$ is odd, while $\widehat{v_{pl}}$ is even for any real-valued function v_{pl}, hence the imaginary part of the first integral above is equal to zero.

From now we shall carry out all calculations for $n = 3$ (Steps II-VII) and then show that the general case can be reduced to the three-dimensional one (Step VIII). For brevity, $\mathcal{L}^{m,3}$ will be denoted $\mathcal{L}^m$.

Step II. We claim that $\mathcal{L}^m$, $m \in \mathbb{N}$, satisfy the following relations:

$$(2.10) \quad \mathcal{L}^m(i\gamma, -p(p+1)) = a_p + i\, b_p,$$

where

$$(2.11) \qquad\qquad a_0 \;\; = \;\; \prod_{k=0}^{m-1} \left(\gamma^2 + k^2 \right),$$

$$(2.12) \qquad\qquad a_1 \;\; = \;\; (\gamma^2 + m^2 + 1) \prod_{k=0}^{m-2} \left(\gamma^2 + k^2 \right),$$

$$(2.13) \qquad\qquad a_p \;\; = \;\; \frac{(2p-1)a_{p-1} + \left(\gamma^2 + (p-1+m)^2 \right) a_{p-2}}{\gamma^2 + (m-p)^2}, \qquad p \geq 2,$$

and

$$(2.14) \qquad b_0 \;=\; \gamma m \prod_{k=1}^{m-1} \left(\gamma^2 + k^2 \right),$$

$$(2.15) \qquad b_1 \;=\; \gamma m (\gamma^2 + m^2 - 1) \prod_{k=1}^{m-2} \left(\gamma^2 + k^2 \right),$$

$$(2.16) \qquad b_p \;=\; \frac{-(2p-1)b_{p-1} + \left(\gamma^2 + (p-1+m)^2 \right) b_{p-2}}{\gamma^2 + (m-p)^2}, \qquad p \geq 2.$$

When $p = m$ and $\gamma = 0$, the expressions in (2.13) and (2.16) should be understood in the sense of the corresponding limits. Formulas (2.11)–(2.16) show that each of the polynomials a_{m-1}, a_m, b_{m-1}, b_m contains the factor γ^2, hence the aforementioned limits are finite. Below we shall not consider separately the case $p = m$, $\gamma = 0$, as all arguments can be justified for this case by a simple limiting procedure.

Let us denote $\mathcal{L}^m(i\gamma, -p\,(p+1))$ by $\mathcal{L}_p$ throughout this argument. According to (2.5)

$$\mathcal{L}_p \;=\; (-1)^m \prod_{j=0}^{m-1} \left((-i\gamma + m - 1 - 2j)(-i\gamma + m - 2j) - p\,(p+1) \right)$$

$$(2.17) \qquad\qquad =\; (-1)^m \prod_{j=0}^{m-1} (-i\gamma + m - 1 - 2j - p)(-i\gamma + m - 2j + p).$$

It remains to show that the polynomials above satisfy (2.10)–(2.16). First,

$$\mathcal{L}_0 \;=\; (-1)^m \prod_{j=0}^{m-1} (-i\gamma + m - 1 - 2j)(-i\gamma + m - 2j) = (-1)^m \prod_{k=-m+1}^{m} (-i\gamma + k)$$

$$(2.18) \qquad =\; (-1)^m (-i\gamma)(m - i\gamma) \prod_{k=1}^{m-1} (-i\gamma + k)(-i\gamma - k) = (\gamma^2 + i\gamma m) \prod_{k=1}^{m-1} (\gamma^2 + k^2)$$

and

$$\mathcal{L}_1 \;=\; (-1)^m \prod_{j=0}^{m-1} (-i\gamma + m - 2 - 2j)(-i\gamma + m - 2j + 1)$$

$$=\; (-1)^m (m + 1 - i\gamma)(m - 1 - i\gamma)(-i\gamma)(-m - i\gamma) \prod_{k=1}^{m-2} (-i\gamma + k)(-i\gamma - k)$$

$$=\; (m^2 - 2mi\gamma - \gamma^2 - 1)(mi\gamma - \gamma^2) \prod_{k=1}^{m-2} (\gamma^2 + k^2)$$

$$(2.19) \qquad = \ \left(\gamma^2(\gamma^2 + m^2 + 1) + mi\gamma(\gamma^2 + m^2 - 1)\right) \prod_{k=1}^{m-2}(\gamma^2 + k^2),$$

as desired. As for $p \geq 2$, we have to show that

$$\mathcal{L}_p \ = \ \frac{1}{\gamma^2 + (m-p)^2}\left((2p-1)(a_{p-1} - ib_{p-1}) + \left(\gamma^2 + (p-1+m)^2\right)(a_{p-2} + ib_{p-2})\right)$$

$$(2.20) \qquad = \ \frac{1}{\gamma^2 + (m-p)^2}\left((2p-1)\overline{\mathcal{L}_{p-1}} + \left(\gamma^2 + (p-1+m)^2\right)\mathcal{L}_{p-2}\right).$$

It follows from (2.17) that

$$\overline{\mathcal{L}_{p-1}} \ = \ (-1)^m \prod_{j=0}^{m-1}(i\gamma + m - 2j - p)(i\gamma + m - 2j + p - 1)$$

$$= \ (-1)^m \prod_{j=0}^{m-1}(i\gamma + m - 2(m-1-j) - p)(i\gamma + m - 2(m-1-j) + p - 1)$$

$$(2.21) \qquad = \ (-1)^m \prod_{j=0}^{m-1}(m - 2j - i\gamma - p - 1)(m - 2j - i\gamma + p - 2),$$

and

$$\mathcal{L}_{p-2} \ = \ (-1)^m \prod_{j=0}^{m-1}(-i\gamma + m - 2j - p + 1)(-i\gamma + m - 2j + p - 2)$$

$$(2.22) \qquad = \ (-1)^m \prod_{j=0}^{m-1}(-i\gamma + m - 2j + p - 2) \prod_{k=-1}^{m-2}(-i\gamma + m - 2k - p - 1).$$

Hence,

$$\frac{1}{\gamma^2 + (m-p)^2}\left((2p-1)\overline{\mathcal{L}_{p-1}} + \left(\gamma^2 + (p-1+m)^2\right)\mathcal{L}_{p-2}\right)$$

$$= \ \frac{1}{\gamma^2 + (m-p)^2}(-1)^m \prod_{j=0}^{m-1}(-i\gamma + m - 2j + p - 2) \prod_{k=0}^{m-2}(-i\gamma + m - 2k - p - 1) \times$$

$$\times\left((2p-1)(-m+1-i\gamma-p) + \left(\gamma^2 + (p-1+m)^2\right)(m+1-i\gamma-p)\right)$$

$$= \ \frac{1}{\gamma^2 + (m-p)^2}(-1)^m \prod_{k=1}^{m}(-i\gamma + m - 2k + p) \prod_{k=0}^{m-2}(-i\gamma + m - 2k - p - 1) \times$$

$$(2.23) \qquad \times(-m + 1 - i\gamma - p)\left((2p-1) - (m - 1 - i\gamma + p)(m + 1 - i\gamma - p)\right).$$

The latter expression is equal to

$$\mathcal{L}_p \frac{1}{\gamma^2 + (m-p)^2} \frac{(-i\gamma - m + p)}{(-i\gamma + m + p)} \frac{1}{(-m + 1 - i\gamma - p)} \times$$

$$(2.24) \qquad \times (-m + 1 - i\gamma - p)(-1)(-i\gamma + m + p)(-i\gamma + m - p) = \mathcal{L}_p.$$

This finishes the proof of (2.10)–(2.16), in particular, we have

$$(2.25) \quad \mathfrak{Re}\, \mathcal{L}^m(i\gamma, -p\,(p+1)) = a_p \text{ for every } p \in \mathbb{N} \cup \{0\},$$

and hence

$$(2.26) \quad \int_{\mathbb{R}^3} (-\Delta)^m u(x)\, u(x)|x|^{-1}\, dx = \sum_{p=0}^{\infty} \sum_{l=-p}^{p} \int_{\mathbb{R}} a_p(\gamma)\, \left|\widehat{v_{pl}}(\gamma)\right|^2\, d\gamma.$$

Step III. In order to prove (2.2) for $n = 3$ one has to show that

$$(2.27) \quad a_q(\gamma) \geq C \sum_{k=1}^{m} \gamma^{2k} + C \prod_{p=0}^{m-1} \left(q(q+1) - p\,(p+1)\right) \text{ for every } q \in \mathbb{N} \cup \{0\},\ \gamma \in \mathbb{R},$$

with some constant $C > 0$ independent of q and γ. Indeed, if (2.27) is satisfied, then

$$\sum_{q=0}^{\infty} \sum_{l=-q}^{q} \int_{\mathbb{R}} a_q(\gamma)\, \left|\widehat{v_{ql}}(\gamma)\right|^2\, d\gamma$$

$$\geq C \sum_{q=0}^{\infty} \sum_{l=-q}^{q} \int_{\mathbb{R}} \left(\sum_{k=1}^{m} \left|(i\gamma)^k\, \widehat{v_{ql}}(\gamma)\right|^2 + \overline{\widehat{v_{ql}}(\gamma)} \prod_{p=0}^{m-1} \left(q(q+1) - p\,(p+1)\right)\widehat{v_{ql}}(\gamma) \right) dt$$

$$= C \sum_{q=0}^{\infty} \sum_{l=-q}^{q} \int_{\mathbb{R}} \left(\sum_{k=1}^{m} \left(\partial_t^k v_{ql}(t)\right)^2 + v_{ql}(t) \prod_{p=0}^{m-1} \left(q(q+1) - p\,(p+1)\right)v_{ql}(t) \right) dt$$

$$(2.28) \qquad = C \int_{\mathbb{R}} \int_{S^2} \left(\sum_{k=1}^{m} \left(\partial_t^k v(t,\omega)\right)^2 + v(t,\omega) \prod_{p=0}^{m-1} \left(-\delta_\omega - p\,(p+1)\right)v(t,\omega) \right) dt$$

which combined with (2.26) gives (2.2) for $n = 3$. Let us now concentrate on (2.27).

Step IV. The formulas (2.11)–(2.13) show that all a_q are nonnegative. Moreover, since

$$(2.29) \quad (q - 1 + m)^2 > (m - q)^2 \text{ for every } q \in \mathbb{N} \text{ and } m \in \mathbb{N},$$

we have $a_q \geq a_{q-2}$ for all $q \geq 2$, and therefore,

$$(2.30) \quad a_q(\gamma) \geq \min\{a_0(\gamma), a_1(\gamma)\} \geq \prod_{k=0}^{m-1} (\gamma^2 + k^2) \text{ for every } q \in \mathbb{N} \cup \{0\},\ \gamma \in \mathbb{R}.$$

Hence,

$$(2.31) \qquad a_q(\gamma) \geq \sum_{k=1}^{m} \gamma^{2k} \text{ for every } q \in \mathbb{N} \cup \{0\}, \gamma \in \mathbb{R}.$$

Step V. As for the second term on the right hand side of (2.27), it is clear that

$$(2.32) \qquad a_q(\gamma) \geq \prod_{p=0}^{m-1} \Big(q(q+1) - p(p+1) \Big) \text{ for } q = 0, 1, ..., m-1, \gamma \in \mathbb{R},$$

since for such q the product in (2.32) is equal to 0. Now assume that $q \geq m$. If $|\gamma| \geq q + 1$, then by (2.30)

$$(2.33) \qquad a_q(\gamma) \geq \prod_{k=0}^{m-1} \Big((q+1)^2 + k^2 \Big) \geq \prod_{p=0}^{m-1} \Big(q(q+1) - p(p+1) \Big),$$

and it remains to consider $|\gamma| \leq q + 1$, $q \geq m$.

Step VI. By definition

$$a_q(\gamma) \ = \ (-1)^m \, \mathfrak{Re} \prod_{j=0}^{m-1} (-i\gamma + m - 1 - 2j - q)(-i\gamma + m - 2j + q)$$

$$(2.34) \qquad = \ \mathfrak{Re} \prod_{j=0}^{m-1} \Big(\gamma^2 + (-m+1+2j+q)(m-2j+q) + i\gamma(2m-4j-1) \Big).$$

Since $q \geq m$, we have $(-m + 1 + 2j + q) > 0$ and $(m - 2j + q) > 0$. Therefore,

$$a_q(\gamma) \ = \ \prod_{j=0}^{m-1} \Big(\gamma^2 + (-m+1+2j+q)(m-2j+q) \Big)$$

$$(2.35) \qquad \times \mathfrak{Re} \prod_{j=0}^{m-1} \Big(1 + i\gamma \frac{(2m-4j-1)}{\gamma^2 + (-m+1+2j+q)(m-2j+q)} \Big).$$

Let us show that there exists $C(m) \geq m$ such that for every $q \geq C(m)$ and $|\gamma| \leq q + 1$

$$(2.36) \qquad \mathfrak{Re} \prod_{j=0}^{m-1} \Big(1 + i\gamma \frac{(2m-4j-1)}{\gamma^2 + (-m+1+2j+q)(m-2j+q)} \Big) \geq \frac{1}{2}.$$

A general idea is that the expression above can be written as $1 + \mathfrak{Re} R$, where $|R|$ is sufficiently small when q is large. Indeed, R is a sum of at most $2^m - 1$ terms, each of those being a product of (at least one, at most m) elements

$$(2.37) \qquad c_j := i\gamma \frac{(2m-4j-1)}{\gamma^2 + (-m+1+2j+q)(m-2j+q)}, \qquad j = 0, ..., m-1.$$

If $q \geq 2m$, then $|2j - m| \leq q/2$ and

$$(2.38) \quad |c_j| \leq (q+1)\frac{2m}{(q/2+1)(q/2)} \leq \frac{8m}{q}.$$

Moreover, if $q > 8m$, the expression above is less than 1. Hence,

$$(2.39) \quad |\Re e\, R| \leq |R| \leq (2^m - 1)\frac{8m}{q}.$$

Set now $C(m) := 2^{m+4}m$. Then $|\Re eR| \leq 1/2$ for every $q \geq C(m)$, and hence (2.36) holds for such q. Combining this with (2.35), we deduce that

$$
\begin{aligned}
a_q(\gamma) \;\geq\; & \frac{1}{2}\prod_{j=0}^{m-1}\Big(\gamma^2 + (-m+1+2j+q)(m-2j+q)\Big) \\[2mm]
\geq\; & \frac{1}{2}\prod_{j=0}^{m-1}(-m+1+2j+q)(m-2j+q) \geq \frac{1}{2}\left(\frac{(q+2)q}{4}\right)^m \\[2mm]
(2.40) \qquad\qquad \geq\; & \frac{1}{2^{2m+1}}\prod_{p=0}^{m-1}\Big(q(q+1) - p(p+1)\Big)
\end{aligned}
$$

whenever $|\gamma| \leq q+1$, $q \geq 2^{m+4}m$.

Step VII. Finally, we claim that there exists $D(m) > 0$ such that

$$(2.41) \quad a_q(\gamma) \geq D(m)\prod_{p=0}^{m-1}\Big(q(q+1) - p(p+1)\Big) \text{ if } m \leq q \leq 2^{m+4}m \text{ and } |\gamma| \leq q+1.$$

Since $a_q \geq a_{q-2}$ for $q \geq 2$, it is enough to show that for some $D > 0$

$$(2.42) \quad \min\{a_m(\gamma), a_{m+1}(\gamma)\} \geq D, \qquad |\gamma| \leq q+1,$$

as (2.42) implies (2.41) with

$$(2.43) \quad D(m) = D\left(\prod_{p=0}^{m-1}\Big(2^{m+4}m(2^{m+4}m + 1) - p(p+1)\Big)\right)^{-1}.$$

However, the formula (2.17) implies

$$(2.44) \quad a_m(0) = \prod_{j=0}^{m-1}(1+2j)(2m-2j) \geq 2 \quad\text{and}\quad a_{m+1}(0) = \prod_{j=0}^{m-1}(2+2j)(2m+1-2j) \geq 2,$$

so that for some $\varepsilon_0 = \varepsilon_0(m) > 0$ we have $\min\{a_m(\gamma), a_{m+1}(\gamma)\} \geq 1$ when $|\gamma| < \varepsilon_0$. On the other hand, by (2.30)

$$(2.45) \quad a_q(\gamma) \geq \prod_{k=0}^{m-1}(\varepsilon_0^2 + k^2) \text{ for every } q \in \mathbb{N} \cup \{0\}, |\gamma| > \varepsilon_0.$$

Hence, (2.42) holds with D given by the minimum of 1 and the right hand side of (2.45). This concludes the proof of (2.41) and the proof of the Theorem for the case $n = 3$.

Step VIII. Let us now consider any odd $n \geq 3$. Similarly to (2.27)–(2.28) in order to prove (2.2) one has to show that

$$(2.46) \quad \mathfrak{Re}\, \mathcal{L}^{m,n}(i\gamma, -q(q+n-2)) \geq C \sum_{k=1}^{m} \gamma^{2k} + C \prod_{p=-\frac{n}{2}+\frac{3}{2}}^{m-\frac{n}{2}+\frac{1}{2}} \big(q(q+n-2) - p(p+n-2)\big)$$

for every $q \in \mathbb{N} \cup \{0\}, \gamma \in \mathbb{R}$. However,

$$\mathcal{L}^{m,n}(i\gamma, -q(q+n-2))$$

$$= (-1)^m \prod_{j=0}^{m-1}\left(-i\gamma + m - \frac{n}{2} + \frac{1}{2} - 2j - q\right)\left(-i\gamma + m + \frac{n}{2} - \frac{3}{2} - 2j + q\right)$$

$$= (-1)^m \prod_{j=0}^{m-1}(-i\gamma + m - 1 - 2j - p)(-i\gamma + m - 2j + p)$$

$$(2.47) \qquad\qquad = \mathcal{L}^{m,3}(i\gamma, -p(p+1)),$$

for $p = q + \frac{n}{2} - \frac{3}{2}$. Hence,

$$\mathfrak{Re}\, \mathcal{L}^{m,n}(i\gamma, -q(q+n-2)) = a_{q+\frac{n}{2}-\frac{3}{2}}(\gamma)$$

$$\geq C \sum_{k=1}^{m} \gamma^{2k} + C \prod_{p=0}^{m-1}\left(\left(q + \frac{n}{2} - \frac{3}{2}\right)\left(q + \frac{n}{2} - \frac{1}{2}\right) - p(p+1)\right)$$

$$= C \sum_{k=1}^{m} \gamma^{2k} + C \prod_{s=-\frac{n}{2}+\frac{3}{2}}^{m-\frac{n}{2}+\frac{1}{2}}\left(\left(q + \frac{n}{2} - \frac{3}{2}\right)\left(q + \frac{n}{2} - \frac{1}{2}\right) - \left(s + \frac{n}{2} - \frac{3}{2}\right)\left(s + \frac{n}{2} - \frac{1}{2}\right)\right)$$

$$(2.48) \qquad = C \sum_{k=1}^{m} \gamma^{2k} + C \prod_{s=-\frac{n}{2}+\frac{3}{2}}^{m-\frac{n}{2}+\frac{1}{2}}\big(q(q+n-2) - s(s+n-2)\big)$$

where we used (2.27) and substitution $s := p - \frac{n}{2} + \frac{3}{2}$. As in (2.27) the constant $C > 0$ above is independent of q and γ. Now (2.48) leads to (2.46) and finishes the argument. □

3. Preservation of positivity for solutions of differential operators

Proposition 3.1. [1] *Assume that $m \in \mathbb{N}$ and $n \in [3, 2m+1] \cap \mathbb{N}$ is odd. Let us define h as the unique solution of*

$$(3.1) \qquad \mathcal{L}^{m,n}\left(-\partial_t, 0\right) h = \delta,$$

which is bounded and vanishes at $+\infty$. Then,

$$(3.2) \qquad \mathcal{L}^{m,n}\left(-\partial_t, -p(p+n-2)\right) h \geq 0,$$

for all $p \in [0, m - n/2 + 1/2]$.

Let us start with the operators of the second order.

Let a, b, c, and d four real numbers such that $b < 0 \leq a$ and $d < 0 \leq c$ and let f be a (tempered) distribution on $\mathbb{R}$. We define h as a solution of

$$(3.3) \qquad \left(\partial_t - a\right)\left(\partial_t - b\right) h = f,$$

and let the distribution g be given by

$$(3.4) \qquad g := \left(\partial_t - c\right)\left(\partial_t - d\right) h$$

We want to investigate under what condition on a, b, c, and d the positivity of f entails the positivity of g.

First, one has to pick a suitable solution (clearly, by subtracting some solution of a homogeneous equation one can always destroy positivity). Also, f has to be reasonable in order for a solution to exist.

To keep the discussion at full generality and, at the same time, ensure existence and uniqueness, in the preliminary results of this section we simply explicitly assume that the solution is obtained by convolution with the bounded fundamental solution vanishing at $+\infty$ (and, in particular, that f is such that the convolution integral converges), and later we will check that this hypothesis fits the argument proving Proposition 3.1.

Also, we record a few conventions. As usually, a distribution f has the singular support at 0 if $f \in C^\infty(\mathbb{R} \setminus \{0\})$. Furthermore, we say that such an f has an exponential decay at $+\infty$ if $|f(t)| \leq Ce^{-\alpha t}$, $t > 0$, for some $\alpha, C > 0$, and similarly, f has an exponential decay at $-\infty$ if $|f(t)| \leq Ce^{\beta t}$, $t < 0$ for some $\beta, C > 0$. Within this section all distributions of interest are bounded in a punctured neighborhood of 0 and thus, there is no loss of generality in considering all $t > 0$ (or, respectively, $t < 0$) in the definitions above.

Proposition 3.2. *Fix $a, b, c, d \in \mathbb{R}$ such that $b < 0 \leq a$ and $d < 0 \leq c$. Let f and h be two distributions on $\mathbb{R}$ such that h is a convolution of f with the bounded fundamental solution of the operator $\left(\partial_t - a\right)\left(\partial_t - b\right)$, which vanishes at $+\infty$. In particular,*

$$(3.5) \qquad \left(\partial_t - a\right)\left(\partial_t - b\right) h = f.$$

---

[1]The idea of the proof has been suggested by Marcel Filoche.

Assume that f is positive. If $[b, a] \subset [d, c]$, then

$$(3.6) \qquad g := (\partial_t - c)(\partial_t - d) h$$

is also a positive distribution.

Furthermore, if f has the singular support at 0 and exponential decay at $+\infty$ and $-\infty$, then g also has the singular support at 0 and exponential decay at $+\infty$ and $-\infty$ provided that either $a, c \neq 0$ or $a = c = 0$.

Proof. Let us start with the case $f = \delta$. Problem (3.5) therefore writes

$$(3.7) \qquad (\partial_t - a)(\partial_t - b) h = \delta.$$

The solution of (3.7) which is bounded and vanishes at $+\infty$ is given by

$$(3.8) \qquad h_0(t) = \kappa \begin{cases} e^{at} & \text{if } t \leq 0 \\ e^{bt} & \text{if } t > 0 \end{cases} \qquad \text{with} \quad \kappa = \frac{1}{b - a}.$$

This solution is unique because any other bounded solution vanishing at $+\infty$, h_1, would be such that $h_1 - h_0$ solves the homogeneous problem corresponding to (3.5). Thus, $h_1 - h_0$ would have an exponential growth at $-\infty$ and/or violate the assumption of vanishing at $+\infty$.

Applying the operator in (3.6) to this solution h_0 yields

$$(3.9) \qquad g_0(t) = \delta + \begin{cases} \kappa (a - c)(a - d) e^{at} & \text{if } t \leq 0, \\ \kappa (b - c)(b - d) e^{bt} & \text{if } t > 0, \end{cases}$$

i.e., g_0 is the sum of the Dirac delta function, coming from the second derivative appearing in (3.6), and a piecewise continuous function. The constant κ being always negative, it follows immediately that g_0 will be positive if and only if $(a - c)(a - d)$ and $(b - c)(b - d)$ are both non-positive, i.e., if both a and b belong to the interval $[d, c]$. On the contrary, due to the Dirac distribution present in g_0, if either a or b is outside the interval $[d, c]$ then g_0 has no specific sign.

We now turn back to the initial problem (3.5). As per our assumptions, we consider the solution of this problem obtained from a convolution product of f with the fundamental solution h_0 defined in (3.8), that is,

$$(3.10) \quad h(t) = \int_{\mathbb{R}} f(u) h_0(t - u) \, du.$$

Applying the operator in (3.6) to this h yields

$$(3.11) \quad g(t) = \int_{\mathbb{R}} f(u) g_0(t - u) \, du$$

We see here that the positivity of g for any positive f follows from the positivity of g_0, hence a and b belonging to the interval $[d, c]$. Given that $b < 0 \leq a$, this amounts to $[b, a] \subset [d, c]$.

Furthermore, if f has an exponential decay at both $+\infty$ and $-\infty$ and either $a, c \neq 0$ or $a = c = 0$ then, due to formula (3.9), g_0 also has an exponential decay at both $+\infty$ and $-\infty$, and hence, so does their convolution in (3.11), as desired. $\qquad \square$

Corollary 3.3. *Let P and Q be two polynomials of order $2m$, with all their roots real and their higher order coefficient equal to 1. Therefore P and Q write*

$$(3.12) \quad P(x) = \prod_{i=0}^{m-1}(x - a_i)(x - b_i) \qquad \text{and} \qquad Q(x) = \prod_{i=0}^{m-1}(x - c_i)(x - d_i).$$

Let us assume further that both P and Q have m non-negative roots, noted respectively (a_i) and (c_i) $(0 \le i \le m - 1)$, and m negative roots, noted respectively (b_j) and (d_j) $(1 \le j \le m)$, such that $\forall i \in [0, m - 1], [b_i, a_i] \subset [d_i, c_i]$. Moreover, either none of the roots of both polynomials is zero, or both P and Q have exactly one zero root $a_0 = c_0 = 0$.

Let f be a positive distribution with the singular support at 0 and exponential decay at $+\infty$ and $-\infty$, and h be the unique bounded solution of

$$(3.13) \quad P(\partial_t)\, h = f,$$

which vanishes at $+\infty$. Then

$$(3.14) \quad g := Q(\partial_t)\, h,$$

is also a positive distribution with the singular support at 0 and exponential decay at $+\infty$ and $-\infty$.

Remark. The gist of this Proposition, as well as Proposition 3.2, is preservation of the positivity under appropriate change of a differential operator, or more precisely, under the change that widens the distances between the roots. The exponential decay is not a necessary condition, but rather a convenient restriction that ensures the choice of a suitable solution at each step.

Proof. First of all, let us mention that there exists a solution h of (3.13) which is bounded and vanishes at $+\infty$. It is the solution obtained by the convolution of f with the fundamental solution of the operator $P(\partial_t)$ which is bounded and vanishes at $+\infty$.

The Corollary will be obtained by successively applying Proposition 3.2 for all pairs $[b_i, a_i]$. The function f_0 is defined as $f_0 = f$, and f_{i+1} is deduced from f_i by

$$(3.15) \quad f_{i+1} := (\partial_t - c_i)\,(\partial_t - d_i)\, h_i,$$

where h_i is the solution of

$$(3.16) \quad (\partial_t - a_i)\,(\partial_t - b_i)\, h_i = f_i,$$

obtained by convolution with the bounded fundamental solution vanishing at $+\infty$, following Proposition 3.2. Then it is immediate to see that, by Proposition 3.2, all f_i are positive distributions with the singular support at 0 and exponential decay at $+\infty$ and $-\infty$.

We claim that

$$(3.17) \quad h_i = \prod_{j=0}^{i-1}(\partial_t - c_j)(\partial_t - d_j) \prod_{j=i+1}^{m-1}(\partial_t - a_j)(\partial_t - b_j)h, \quad i = 1, ..., m - 1,$$

and

$$(3.18) \quad f_{i+1} = \prod_{j=0}^{i}(\partial_t - c_j)(\partial_t - d_j) \prod_{j=i+1}^{m-1}(\partial_t - a_j)(\partial_t - b_j)h, \quad i = 0, ..., m - 1,$$

where in the definitions of h_0, h_{m-1}, f_m one of the products is void.

First of all, letting $f_0 = f$, we see that

$$(3.19) \quad h_0' := \prod_{j=1}^{m-1} (\partial_t - a_j)(\partial_t - b_j)h, \quad i = 1, ..., m - 2,$$

satisfies equation (3.16). The only question is whether it is the same solution as h_0. However, much as for the operators of the second order in Proposition 3.2, h_0' is bounded and vanishes at $+\infty$ for a particular h that we are working with (see the beginning of the proof). Clearly, the same is true for h_0 obtained from (3.16) by convolution of f with the bounded fundamental solution of $(\partial_t - a_0)(\partial_t - b_0)$ which vanishes at $+\infty$. However, then the difference $h_0 - h_0'$ must solve the homogeneous equation for the operator $(\partial_t - a_0)(\partial_t - b_0)$ and at the same time, $h_0 - h_0'$ must be bounded and vanish at $+\infty$, hence, $h_0 - h_0' = 0$. Note, further, that f_1 can now be defined equivalently by (3.18) and (3.15) and, moreover, f_1 has an exponential decay at both $+\infty$ and $-\infty$. The latter follows from (3.15)–(3.16) and the last statement of Proposition 3.2.

Following the same lines, we verify (3.17)–(3.18) identifying them with functions in (3.15)–(3.16) as the unique solutions of the corresponding differential equations satisfying aforementioned decay properties.

As we already pointed out, by virtue of Proposition 3.2, all f_i are positive distributions with the singular support at 0 and exponential decay at $+\infty$ and $-\infty$. It is now clear from (3.18) that $f_m = g$. This finishes the proof of the Corollary.   □

Proof of Proposition 3.1. We now focus on the following differential operator

$$\mathcal{L}^{m,n}(-\partial_t, -p(p + n - 2)) :=$$

$$(3.20) \quad (-1)^m \prod_{j=0}^{m-1} \left(\partial_t + m - \frac{n}{2} + \frac{1}{2} - 2j - p \right) \left(\partial_t + m + \frac{n}{2} - \frac{3}{2} - 2j + p \right).$$

Let us introduce the numbers

$$(3.21) \quad c_j = 2j - \left(m - \frac{n-1}{2} \right), \quad 0 \le j \le m - 1.$$

Note that

$$m + \frac{n}{2} - \frac{3}{2} - 2j + p = -m + \frac{n}{2} + \frac{1}{2} + 2(m - 1 - j) + p.$$

The operator $\mathcal{L}^{m,n}$ thus writes

$$(3.22) \quad \mathcal{L}^{m,n}(-\partial_t, -p(p + n - 2)) = (-1)^m \prod_{j=0}^{m-1} \left(\partial_t - c_j - p \right) \left(\partial_t + c_j + 1 + p \right).$$

The roots of the corresponding polynomial are $c_j + p$ and $-c_j - 1 - p$, $j = 0, ..., m - 1$.

Consider an integer p such that $0 \le p \le m - \dfrac{n-1}{2} =: k$ and let us examine the following set of roots:

$$(3.23) \quad S_0 = \{c_j\} \cup \{-c_j - 1\} = \{-k, -k - 2, ..., 2(m - 1) - k\}$$

(3.24) $$\cup \{-2(m-1)+k-1, -2(m-1)+k+1, ..., k-1\}.$$

This set contains $2m$ elements. The largest of the $\{c_j\}$ is larger than the largest of the $\{-c_j - 1\}$:

(3.25) $\quad 2(m-1) - k > k - 1 \quad$ because $\quad 2(m-1) \geq 2k > 2k - 1.$

Similarly, the smallest of the $\{-c_j - 1\}$ is smaller than the smallest of the $\{c_j\}$:

(3.26) $\quad -2(m-1) + k - 1 < -k \quad$ because $\quad 2(m-1) \geq 2k > 2k - 1.$

Moreover, the $\{c_j\}$ and the $\{-c_j - 1\}$ have different parities. So when these two subsets of roots overlap, the overlap covers all integers, odd and even. Hence, one can rewrite the set S as the union of 3 sets, the center interval and two "lateral" sets, these two being respectively composed of only negative and positive numbers:

(3.27) $\quad S_0 = \{-2(m-1)+k-1, ..., -k-5, -k-3\} \cup$

(3.28) $\quad \big([-k-1, k] \cap \mathbb{Z}\big) \cup$

(3.29) $\quad \{k+2, k+4, ..., 2(m-1)-k\}.$

The first and the third set might possibly be empty.

Next, the set of roots $S_p = \{c_j + p\} \cup \{-c_j - p - 1\}$ can be described in similar terms replacing k by $k - p$:

(3.30) $\quad S_p = \{-2(m-1)+k-p-1, ..., -k+p-5, -k+p-3\} \cup$

(3.31) $\quad \big([-k+p-1, k-p] \cap \mathbb{Z}\big) \cup$

(3.32) $\quad \{k-p+2, k-p+4, ..., 2(m-1)-k+p\}.$

To compare S_0 and S_p, one rewrites S_0 in the following way

(3.33) $\quad S_0 = \{-2(m-1)+k-1, ..., -k-5, -k-3\} \cup \big([-k-1, -k+p-2] \cap \mathbb{Z}\big)$

(3.34) $\quad \big([-k+p-1, k-p] \cap \mathbb{Z}\big) \cup$

(3.35) $\quad \big([k-p+1, k] \cap \mathbb{Z}\big) \cup \{k+2, k+4, ..., 2(m-1)-k\}.$

Now the center interval is identical in both sets and composed of consecutive integers, $k - p + 1$ negative numbers and $k - p + 1$ non negative numbers. In particular, this interval always contains $\{-1, 0\}$. Also, compared one-by-one, the roots in the first subset of S_p (3.30), are smaller than the roots in the first subset of S_0 (3.33), and similarly, the roots in the last subset of S_p (3.32), are larger than the roots in the last subset of S_0 (3.35).

We now can name the sequence of roots of S_0 (in the order they appear in (3.33)–(3.35))

(3.36) $\quad b_{m-1} < b_{m-2} < ... < b_0 = -1 < a_0 = 0 < a_1 < ... < a_m,$

and, respectively, the roots of S_p (in the order of appearance in (3.30)–(3.32))

(3.37) $\quad d_{m-1} < d_{m-2} < ... < d_0 = -1 < c_0 = 0 < c_1 < ... < c_m.$

Then $[b_i, a_i] \subset [d_i, c_i]$ for all $i = 0, ..., m - 1$, and moreover, $a_0 = c_0 = 0$ are the only zero roots. Thus, we are in the setting of Corollary 3.3, which finishes the argument. (Note that when m is odd, we apply Corollary 3.3 with $-h$ in place of h). $\qquad\qquad\square$

4. Integral inequalities and global estimate: the case of odd dimension. Part II: weight g

Incorporating the results of Section 3, in this section we introduce a new weight function g. Together with the inequalities in Section 2, this yields further improvement of the key integral estimates and ultimately, improved pointwise estimates on the solution of the polyharmonic equation.

We start with the following auxiliary result which provides an explicit formula for the solution of (3.1).

Lemma 4.1. *Assume that $m \in \mathbb{N}$ and $n \in [3, 2m + 1] \cap \mathbb{N}$ is odd. Consider the equation*

$$(4.1) \qquad \mathcal{L}^{m,n}(-\partial_t, 0)\, h = \delta,$$

where δ stands for the Dirac delta function. A unique solution to (4.1) which is bounded and vanishes at $+\infty$ has a form

$$(4.2) \qquad h(t) = \begin{cases} \sum_{j=1}^m \nu_j\, e^{-\alpha_j t}, & t > 0, \\ \sum_{j=1}^m \mu_j\, e^{\beta_j t}, & t < 0. \end{cases}$$

Here $\alpha_j > 0$, $j = 1, 2, ..., m$, $\beta_j > 0$ for $j = 2, ..., m$ and $\beta_1 = 0$ are such that

$$(4.3) \qquad \{-\alpha_j\}_{j=1}^m \bigcup \{\beta_j\}_{j=1}^m = \left\{ -m + \frac{n}{2} - \frac{1}{2} + 2j \right\}_{j=0}^{m-1} \cup \left\{ -\frac{n}{2} - \frac{1}{2} + m - 2j \right\}_{j=0}^{m-1},$$

and with the notation

$$(4.4) \qquad \vec{\gamma} = (-\alpha_1, ..., -\alpha_m, \beta_1, ..., \beta_m), \qquad \vec{\kappa} = (\nu_1, ..., \nu_m, -\mu_1, ..., -\mu_m)$$

the coefficients $\nu_j, \mu_j \in \mathbb{R}$ satisfy

$$(4.5) \qquad \kappa_i = (-1)^{m+1} \Big(\prod_{j \neq i} (\gamma_j - \gamma_i) \Big)^{-1}.$$

Proof. Recall the representation formula (3.22) and let $p = 0$. Similarly to the analysis in the proof of Proposition 3.1, one can see that the characteristic polynomial of the differential equation (4.1) has $2m$ distinct roots given by

$$(4.6) \qquad c_j := -m + \frac{n}{2} - \frac{1}{2} + 2j, \quad j = 0, 1, ..., m - 1,$$

and

$$(4.7) \qquad -\frac{n}{2} - \frac{1}{2} + m - 2j = -1 - \Big(-m + \frac{n}{2} - \frac{1}{2} + 2j\Big) = -1 - c_j, \quad j = 0, 1, ..., m - 1.$$

Following (3.27)–(3.29), (3.33)–(3.35) and (3.36), we conclude that there are m distinct negative roots, 1 root equal to zero and $m - 1$ distinct positive roots for the characteristic polynomial. Hence,

a bounded solution vanishing at infinity must have the form (4.2). In addition, h should be such that that $\partial_t^k g$ are continuous for $k = 0, 1, ..., 2m - 2$ and $\lim_{t \to 0^+} \partial_t^{2m-1} h(t) - \lim_{t \to 0^-} \partial_t^{2m-1} h(t) = (-1)^m$. This gives rise to the system of equations

$$(4.8) \qquad \sum_{j=1}^{m} \nu_j (-\alpha_j)^k - \sum_{j=1}^{m} \mu_j (\beta_j)^k = 0, \quad k = 0, 1, ..., 2m - 2,$$

$$(4.9) \qquad \sum_{j=1}^{m} \nu_j (-\alpha_j)^{2m-1} - \sum_{j=1}^{m} \mu_j (\beta_j)^{2m-1} = (-1)^m.$$

With the notation (4.4) and the convention $0^0 = 1$, (4.8)–(4.9) can then be written as $A \vec{\kappa}^{\perp} = (-1)^m (0, ..., 0, 1)^{\perp}$, where A is a matrix of elements $A_{ji} = \gamma_i^{j-1}$, $i, j = 1, ..., 2m$. Now using the formula for the Vandermonde determinant, we find that the solution is $\vec{\kappa} = (\kappa_j)_{j=1}^{2m}$,

$$(4.10) \quad \kappa_j = (-1)^{j+m} \prod_{i<k,\, i \neq j,\, k \neq j} (\gamma_j - \gamma_i) \left(\prod_{i<k} (\gamma_j - \gamma_i) \right)^{-1} = (-1)^{m+1} \left(\prod_{j \neq i} (\gamma_j - \gamma_i) \right)^{-1}.$$

$$\square$$

Theorem 4.2. *Assume that $m \in \mathbb{N}$ and $n \in [3, 2m + 1] \cap \mathbb{N}$ is odd. Let Ω be a bounded domain in $\mathbb{R}^n$, $O \in \mathbb{R}^n \setminus \Omega$, $u \in C_0^{\infty}(\Omega)$ and $v = e^{\left(m - \frac{n}{2} + \frac{1}{2} \right)t} (u \circ \varkappa^{-1})$. Then for every $\xi \in \Omega$ and $\tau = \log |\xi|^{-1}$*

$$(4.11) \quad \int_{S^{n-1}} v^2(\tau, \omega) \, d\omega \leq C \int_{\mathbb{R}^n} (-\Delta)^m u(x) \, u(x) g(\log |x|^{-1}, \log |\xi|^{-1}) \, dx,$$

with

$$(4.12) \quad g(t, \tau) = e^t \left(C_1 h(t - \tau) + C_2 \right), \qquad t, \tau \in \mathbb{R},$$

and h given by Lemma 4.1. Here C, C_1, C_2 are some constants depending on m and n only.

Proof. **Step I.** First of all, let us pass to (t, ω) coordinates. We aim to show that for some positive constants C and C'

$$\int_{S^{n-1}} v^2(\tau, \omega) \, d\omega \leq C \int_{\mathbb{R}} \int_{S^{n-1}} \mathcal{L}^{m,n}(\partial_t, \delta) v(t, \omega) \, v(t, \omega) h(t - \tau) \, d\omega dt$$

$$(4.13) \qquad\qquad\qquad + C' \int_{\mathbb{R}} \int_{S^{n-1}} \mathcal{L}^{m,n}(\partial_t, \delta) v(t, \omega) \, v(t, \omega) \, d\omega dt.$$

Clearly, (4.13) implies (4.11).

For future reference, observe that due to (4.1)

$$(4.14) \quad \mathcal{L}^{m,n}(-\partial_t, 0) h(t - \tau) = \delta(t - \tau), \qquad t, \tau \in \mathbb{R}.$$

Step II. Going further, let us concentrate on the first integral on the right-hand side of (4.13):

$$\int_{\mathbb{R}} \int_{S^{n-1}} \mathcal{L}^{m,n}(\partial_t, \delta) v(t, \omega) \, v(t, \omega) h(t - \tau) \, d\omega dt$$

$$= \int_{\mathbb{R}} \int_{S^{n-1}} \mathcal{L}^{m,n}(0, \delta) v(t, \omega) \, v(t, \omega) h(t - \tau) \, d\omega dt$$

(4.15)
$$+ \int_{\mathbb{R}} \int_{S^{n-1}} (\mathcal{L}^{m,n}(\partial_t, \delta) - \mathcal{L}^{m,n}(0, \delta)) \, v(t, \omega) \, v(t, \omega) h(t - \tau) \, d\omega dt.$$

All terms of the operator $(\mathcal{L}^{m,n}(\partial_t, \delta) - \mathcal{L}^{m,n}(0, \delta))$ contain ∂_t^k for some $k \geq 1$, and therefore, we can write

(4.16) $\mathcal{L}^{m,n}(\partial_t, \delta) - \mathcal{L}^{m,n}(0, \delta) = \sum_{\substack{k \geq 1, i \geq 0 \\ 2i+k \leq 2m}} d_{ik}(-\delta)^i \partial_t^k$,   for some $d_{ik} \in \mathbb{R}$.

Hence,

$$\int_{\mathbb{R}} \int_{S^{n-1}} (\mathcal{L}^{m,n}(\partial_t, \delta) - \mathcal{L}^{m,n}(0, \delta)) v(t, \omega) \, v(t, \omega) h(t - \tau) \, d\omega dt$$

(4.17)
$$= \sum_{\substack{k \geq 1, i \geq 0 \\ 2i+k \leq 2m}} d_{ik} \int_{\mathbb{R}} \int_{S^{n-1}} \partial_t^k \nabla_\omega^i v(t, \omega) \, \nabla_\omega^i v(t, \omega) h(t - \tau) \, d\omega dt.$$

We claim that

(4.18) $\int_{\mathbb{R}} \partial_t^k v \, vh \, dt = \frac{1}{2} \int_{\mathbb{R}} v^2 \, (-\partial_t)^k h \, dt + \sum_{\substack{i \geq 1, j \geq 0 \\ 2i+j \leq k}} b_{ij}^k \int_{\mathbb{R}} (\partial_t^i v)^2 \, \partial_t^j h \, dt, \quad b_{ij}^k \in \mathbb{R},$

for any $k \geq 1$. This can be proved by induction. For $k = 1$ and $k = 2$ we have, respectively,

$$\int_{\mathbb{R}} \partial_t v \, vh \, dt = \frac{1}{2} \int_{\mathbb{R}} v^2 \, (-\partial_t) h \, dt,$$

$$\int_{\mathbb{R}} \partial_t^2 v \, vh \, dt = \frac{1}{2} \int_{\mathbb{R}} v^2 \, (-\partial_t)^2 h \, dt - \int_{\mathbb{R}} (\partial_t v)^2 \, h \, dt.$$

Let us now assume that (4.18) holds for $k = 1, 2, ..., l - 1$, and prove it for $k = l$. First,

(4.19) $\int_{\mathbb{R}} \partial_t^l v \, vh \, dt = - \int_{\mathbb{R}} \partial_t^{l-2}(\partial_t v) \, (\partial_t v) h \, dt - \int_{\mathbb{R}} \partial_t^{l-1} v \, v \, \partial_t h \, dt.$

Next, using (4.18) for $l - 1$ and $l - 2$, we deduce that the expression above is equal to

$$-\frac{1}{2} \int_{\mathbb{R}} (\partial_t v)^2 \, (-\partial_t)^{l-2} h \, dt - \sum_{\substack{i \geq 1, j \geq 0 \\ 2i+j \leq l-2}} b_{ij}^{l-2} \int_{\mathbb{R}} (\partial_t^{i+1} v)^2 \, \partial_t^j h \, dt$$

$$(4.20) \qquad + \frac{1}{2} \int_{\mathbb{R}} v^2 \, (-\partial_t)^l h \, dt + \sum_{\substack{i \geq 1, \, j \geq 0 \\ 2i+j \leq l-1}} b_{ij}^{l-1} \int_{\mathbb{R}} (\partial_t^i v)^2 \, \partial_t^{j+1} h \, dt,$$

and (4.20) can be written in the form (4.18) for $k = l$.

Then, using (4.18) for v and $\nabla_\omega^i v$, (4.17) leads to the representation

$$\int_{\mathbb{R}} \int_{S^{n-1}} (\mathcal{L}^{m,n}(\partial_t, \delta) - \mathcal{L}^{m,n}(0, \delta)) v(t, \omega) \, v(t, \omega) h(t - \tau) \, d\omega dt$$

$$= \frac{1}{2} \sum_{\substack{k \geq 1, \, i \geq 0 \\ 2i+k \leq 2m}} d_{ik} \int_{\mathbb{R}} \int_{S^{n-1}} (\nabla_\omega^i v(t, \omega))^2 (-\partial_t)^k h(t - \tau) \, d\omega dt$$

$$(4.21) \qquad + \sum_{\substack{k \geq 1, \, i \geq 0 \\ 2i+k \leq 2m}} \sum_{\substack{l \geq 1, \, j \geq 0 \\ 2l+j \leq k}} b_{lj}^{ki} \int_{\mathbb{R}} \int_{S^{n-1}} (\partial_t^l \nabla_\omega^i v)^2 \, \partial_t^j h(t - \tau) \, d\omega dt =: I_1 + I_2.$$

Step II. Consider I_1 first. Let us decompose v into spherical harmonics. Then

$$I_1 = \frac{1}{2} \sum_{\substack{k \geq 1, \, i \geq 0 \\ 2i+k \leq 2m}} d_{ik} \int_{\mathbb{R}} \int_{S^{n-1}} (\nabla_\omega^i v(t, \omega))^2 (-\partial_t)^k h(t - \tau) \, d\omega dt$$

$$= \frac{1}{2} \sum_{\substack{k \geq 1, \, i \geq 0 \\ 2i+k \leq 2m}} d_{ik} \sum_{p=0}^{\infty} \sum_{l=-p}^{p} \int_{\mathbb{R}} p^i (p + n - 2)^i v_{pl}^2(t) (-\partial_t)^k h(t - \tau) \, dt$$

$$= \frac{1}{2} \sum_{p=0}^{\infty} \sum_{l=-p}^{p} \int_{\mathbb{R}} v_{pl}^2(t) \sum_{\substack{k \geq 1, \, i \geq 0 \\ 2i+k \leq 2m}} d_{ik} p^i (p + n - 2)^i (-\partial_t)^k h(t - \tau) \, dt$$

$$(4.22) \qquad = \frac{1}{2} \sum_{p=0}^{\infty} \sum_{l=-p}^{p} \int_{\mathbb{R}} v_{pl}^2(t) \left(\mathcal{L}^{m,n}(-\partial_t, -p(p + n - 2)) - \mathcal{L}^{m,n}(0, -p(p + n - 2)) \right) h(t - \tau) \, dt,$$

where we employed (4.16).

Now,

$$\frac{1}{2} \sum_{p=0}^{\infty} \sum_{l=-p}^{p} \int_{\mathbb{R}} v_{pl}^2(t) \mathcal{L}^{m,n}(0, -p(p + n - 2)) h(t - \tau) \, dt$$

$$(4.23) \qquad = \frac{1}{2} \int_{\mathbb{R}} \int_{S^{n-1}} \mathcal{L}^{m,n}(0, \delta) v(t, \omega) v(t, \omega) h(t - \tau) \, d\omega dt.$$

Also, taking into account (4.14), we see that

$$(4.24) \qquad \frac{1}{2} \sum_{p=0}^{\infty} \sum_{l=-p}^{p} \int_{\mathbb{R}} v_{pl}^2(t) \mathcal{L}^{m,n}(-\partial_t, 0) h(t - \tau) \, d\omega dt = \frac{1}{2} \int_{S^{n-1}} v^2(\tau, \omega) \, d\omega.$$

Hence,

$$I_1 = \frac{1}{2} \sum_{p=0}^{\infty} \sum_{l=-p}^{p} \int_{\mathbb{R}} v_{pl}^2(t) \left(\mathcal{L}^{m,n}(-\partial_t, -p(p + n - 2)) - \mathcal{L}^{m,n}(-\partial_t, 0) \right) h(t - \tau) \, dt$$

$$- \frac{1}{2} \int_{\mathbb{R}} \int_{S^{n-1}} \mathcal{L}^{m,n}(0, \delta) v(t, \omega) v(t, \omega) h(t - \tau) \, d\omega dt$$

$$(4.25) \qquad + \frac{1}{2} \int_{S^{n-1}} v^2(\tau, \omega) \, d\omega.$$

Let us discuss first in rough terms the bounds on (4.25) and then present the details. The third term above isolates v^2 on the sphere, as desired (cf. (4.13)), and the second term will be combined with the first integral on the right-hand side of (4.15), and ultimately estimated by (2.2). As for the first term, we split it into the sum over $p \leq m - n/2 + 1/2$ and the complementary one. The sum over $p \geq m - n/2 + 3/2$ will be estimated by (2.2). The sum over $p \leq m - n/2 + 1/2$ must be positive by itself. This amounts to showing that

$$(4.26) \quad \left(\mathcal{L}^{m,n}(-\partial_t, -p(p + n - 2)) - \mathcal{L}^{m,n}(-\partial_t, 0) \right) h(t) \geq 0, \quad \text{for all} \quad t \neq 0, \quad 0 \leq p \leq m - \frac{n}{2} + \frac{1}{2},$$

or, equivalently (see (4.14)),

$$(4.27) \quad \mathcal{L}^{m,n}(-\partial_t, -p(p + n - 2)) h(t) \geq 0, \quad \text{for all} \quad t \neq 0 \quad \text{and} \quad 0 \leq p \leq m - \frac{n}{2} + \frac{1}{2}.$$

This, however, is exactly the result of Proposition 3.1.

Step III. Combining all of the above, we have

$$\int_{\mathbb{R}} \int_{S^{n-1}} \mathcal{L}^{m,n}(\partial_t, \delta) v(t, \omega) \, v(t, \omega) h(t - \tau) \, d\omega dt$$

$$= \frac{1}{2} \int_{S^{n-1}} v^2(\tau, \omega) \, d\omega + \frac{1}{2} \int_{\mathbb{R}} \int_{S^{n-1}} \mathcal{L}^{m,n}(0, \delta) v(t, \omega) \, v(t, \omega) h(t - \tau) \, d\omega dt$$

$$+ \frac{1}{2} \sum_{p=0}^{\infty} \sum_{l=-p}^{p} \int_{\mathbb{R}} v_{pl}^2(t) \left(\mathcal{L}^{m,n}(-\partial_t, -p(p + n - 2)) - \mathcal{L}^{m,n}(-\partial_t, 0) \right) h(t - \tau) \, dt$$

$$+ \sum_{\substack{k \geq 1, i \geq 0 \\ 2i+k \leq 2m}} \sum_{\substack{l \geq 1, j \geq 0 \\ 2l+j \leq k}} b_{lj}^{ki} \int_{\mathbb{R}} \int_{S^{n-1}} (\partial_t^l \nabla_\omega^i v)^2 \, \partial_t^j h(t - \tau) \, d\omega dt$$

$$\geq \frac{1}{2} \int_{S^{n-1}} v^2(\tau, \omega) \, d\omega + \frac{1}{2} \int_{\mathbb{R}} \int_{S^{n-1}} \mathcal{L}^{m,n}(0, \delta) v(t, \omega) \, v(t, \omega) h(t - \tau) \, d\omega dt$$

$$+ \sum_{\substack{k\geq 1,\, i\geq 0 \\ 2i+k\leq 2m}} \sum_{\substack{l\geq 1,\, j\geq 0 \\ 2l+j\leq k}} b_{lj}^{ki} \int_{\mathbb{R}} \int_{S^{n-1}} (\partial_t^l \nabla_\omega^i v)^2 \, \partial_t^j h(t-\tau) \, d\omega dt$$

(4.28) $$+ \frac{1}{2} \sum_{p=m-\frac{n}{2}+\frac{3}{2}}^{\infty} \sum_{l=-p}^{p} \int_{\mathbb{R}} v_{pl}^2(t) \left(\mathcal{L}^{m,n}(-\partial_t, -p(p+n-2)) - \mathcal{L}^{m,n}(-\partial_t, 0) \right) h(t-\tau) \, dt.$$

Note that $\mathcal{L}^{m,n}(0,\delta)$ is exactly the operator showing up in the last expression in (2.2). Indeed,

(4.29) $$\mathcal{L}^{m,n}(0,\delta) = (-1)^m \prod_{j=0}^{m-1} \left(\left(m - \frac{n}{2} + \frac{1}{2} - 2j \right)\left(m + \frac{n}{2} - \frac{3}{2} - 2j \right) + \delta_\omega \right) = \prod_{p=-\frac{n}{2}+\frac{3}{2}}^{m-\frac{n}{2}+\frac{1}{2}} \left(-\delta_\omega - p\,(p+n-2) \right).$$

Then the absolute value of the second integral on the right-hand side of (4.28) can be majorized by the right-hand side of (2.2). This is easy to see, for example, after the decomposition into spherical harmonics:

$$\left| \sum_{q=0}^{\infty} \sum_{l=-q}^{q} \int_{\mathbb{R}} \mathcal{L}^{m,n}(0, -q\,(q+n-2)) v_{ql}(t)\, v_{ql}(t) h(t-\tau)\, dt \right|$$

$$\leq \sum_{q=0}^{\infty} \sum_{l=-q}^{q} \int_{\mathbb{R}} \prod_{p=-\frac{n}{2}+\frac{3}{2}}^{m-\frac{n}{2}+\frac{1}{2}} (q(q+n-2) - p\,(p+n-2))\,(v_{ql}(t))^2\, |h(t-\tau)|\, dt$$

$$\leq C \sum_{q=0}^{\infty} \sum_{l=-q}^{q} \int_{\mathbb{R}} \prod_{p=-\frac{n}{2}+\frac{3}{2}}^{m-\frac{n}{2}+\frac{1}{2}} (q(q+n-2) - p\,(p+n-2))\,(v_{ql}(t))^2\, dt$$

(4.30) $$= C \sum_{q=0}^{\infty} \sum_{l=-q}^{q} \int_{\mathbb{R}} \mathcal{L}^{m,n}(0, -q\,(q+n-2)) v_{ql}(t)\, v_{ql}(t)\, dt,$$

since h is bounded by a constant and the polynomial

(4.31) $$\prod_{p=-\frac{n}{2}+\frac{3}{2}}^{m-\frac{n}{2}+\frac{1}{2}} (q(q+n-2) - p\,(p+n-2)) \geq 0 \quad \text{for all} \quad q \in \mathbb{N} \cup \{0\}.$$

Hence, (4.11) follows from (4.28) once we prove that

$$\sum_{\substack{k\geq 1,\, i\geq 0 \\ 2i+k\leq 2m}} \sum_{\substack{l\geq 1,\, j\geq 0 \\ 2l+j\leq k}} |b_{lj}^{ki}| \int_{\mathbb{R}} \int_{S^{n-1}} (\partial_t^l \nabla_\omega^i v)^2 \, |\partial_t^j h(t-\tau)|\, d\omega dt$$

(4.32) $$\leq C \int_{\mathbb{R}^n} (-\Delta)^m u(x)\, u(x) |x|^{-1}\, dx$$

and

$$\sum_{p=m-\frac{n}{2}+\frac{3}{2}}^{\infty} \sum_{l=-p}^{p} \int_{\mathbb{R}} v_{pl}^2(t) \left| (\mathcal{L}^{m,n}(-\partial_t, -p(p+n-2)) - \mathcal{L}^{m,n}(-\partial_t, 0)) h(t-\tau) \right| dt$$

(4.33)
$$\leq C \int_{\mathbb{R}^n} (-\Delta)^m u(x) \, u(x) |x|^{-1} \, dx$$

for some $C > 0$.

Step IV. Since all derivatives of h are bounded (see (4.2)),

$$\sum_{\substack{k\geq 1, i\geq 0 \\ 2i+k\leq 2m}} \sum_{\substack{l\geq 1, j\geq 0 \\ 2l+j\leq k}} |b_{lj}^{ki}| \int_{\mathbb{R}} \int_{S^{n-1}} (\partial_t^l \nabla_\omega^i v)^2 \, |\partial_t^j h(t-\tau)| \, dt$$

(4.34)
$$\leq C \sum_{\substack{k\geq 1, i\geq 0 \\ i+k\leq m}} \int_{\mathbb{R}} \int_{S^{n-1}} (\partial_t^k \nabla_\omega^i v(t,\omega))^2 \, d\omega dt.$$

Furthermore, for all k, i as above

(4.35)
$$\int_{\mathbb{R}} \int_{S^{n-1}} (\partial_t^k \nabla_\omega^i v(t,\omega))^2 \, d\omega dt = \sum_{p=0}^{\infty} \sum_{l=-p}^{p} \left(p(p+n-2) \right)^i \int_{\mathbb{R}} (\partial_t^k v_{pl}(t))^2 \, dt.$$

We break down the sum above into two parts, corresponding to the cases $p \leq m - \frac{n}{2} + \frac{1}{2}$ and $p \geq m - \frac{n}{2} + \frac{3}{2}$, respectively. In the first case,

$$\sum_{p=0}^{m-\frac{n}{2}+\frac{1}{2}} \sum_{l=-p}^{p} \left(p(p+n-2) \right)^i \int_{\mathbb{R}} (\partial_t^k v_{pl}(t))^2 \, dt \leq C_{m,n} \sum_{p=0}^{m-\frac{n}{2}+\frac{1}{2}} \sum_{l=-p}^{p} \int_{\mathbb{R}} (\partial_t^k v_{pl}(t))^2 \, dt$$

(4.36)
$$\leq C_{m,n} \int_{\mathbb{R}} \int_{S^{n-1}} (\partial_t^k v(t,\omega))^2 \, d\omega dt \leq C_{m,n} \int_{\mathbb{R}^n} (-\Delta)^m u(x) \, u(x) |x|^{-1} \, dx,$$

where we used (2.2) for the last inequality (note that $k \geq 1$ since we are in the range of (4.35)).

As for $p \geq m - \frac{n}{2} + \frac{3}{2}$, we claim that

(4.37) $\left(p(p+n-2) \right)^m \leq C \prod_{s=-\frac{n}{2}+\frac{3}{2}}^{m-\frac{n}{2}+\frac{1}{2}} \left(p(p+n-2) - s(s+n-2) \right),$ for every $p \geq m - \frac{n}{2} + \frac{3}{2}$,

where $C > 0$ depends on m and n only. Indeed, one can choose C such that

(4.38) $\dfrac{1}{C} \leq \left(1 - \dfrac{\left(m - \frac{n}{2} + \frac{1}{2}\right)\left(m + \frac{n}{2} - \frac{3}{2}\right)}{\left(m - \frac{n}{2} + \frac{3}{2}\right)\left(m + \frac{n}{2} - \frac{1}{2}\right)} \right)^m \leq \prod_{s=-\frac{n}{2}+\frac{3}{2}}^{m-\frac{n}{2}+\frac{1}{2}} \left(1 - \dfrac{s(s+n-2)}{p(p+n-2)} \right),$

for every $p \geq m - \frac{n}{2} + \frac{3}{2}$.

However, since $i + k \leq m$ then by Young's inequality

$$\left(p\,(p+n-2)\right)^i \int_{\mathbb{R}} (\partial_t^k v_{pl}(t))^2\, dt = \int_{\mathbb{R}} \left(p\,(p+n-2)\right)^i \gamma^{2k} |\widehat{v_{pl}}(\gamma)|^2\, d\gamma$$

$$\leq \int_{\mathbb{R}} \left(\left(p\,(p+n-2)\right)^m + \gamma^{2m}\right) |\widehat{v_{pl}}(\gamma)|^2\, d\gamma$$

$$(4.39) \qquad \leq \left(p\,(p+n-2)\right)^m \int_{\mathbb{R}} (v_{pl}(t))^2\, dt + \int_{\mathbb{R}} (\partial_t^m v_{pl}(t))^2\, dt.$$

Therefore,

$$\sum_{p=m-\frac{n}{2}+\frac{3}{2}}^{\infty} \sum_{l=-p}^{p} \left(p\,(p+n-2)\right)^i \int_{\mathbb{R}} (\partial_t^k v_{pl}(t))^2\, dt$$

$$\leq \sum_{p=m-\frac{n}{2}+\frac{3}{2}}^{\infty} \sum_{l=-p}^{p} \prod_{s=-\frac{n}{2}+\frac{3}{2}}^{m-\frac{n}{2}+\frac{1}{2}} \left(p\,(p+n-2) - s\,(s+n-2)\right) \int_{\mathbb{R}} (v_{pl}(t))^2\, dt$$

$$(4.40) \qquad + \sum_{p=m-\frac{n}{2}+\frac{3}{2}}^{\infty} \sum_{l=-p}^{p} \int_{\mathbb{R}} (\partial_t^m v_{pl}(t))^2\, dt \leq C \int_{\mathbb{R}^n} (-\Delta)^m u(x)\, u(x) |x|^{-1}\, dx,$$

by Theorem 2.1.

Analogously, the left-hand side of (4.33) will give rise to terms

$$(4.41) \qquad \sum_{p=m-\frac{n}{2}+\frac{3}{2}}^{\infty} \sum_{l=-p}^{p} \left(p\,(p+n-2)\right)^i \int_{\mathbb{R}} v_{pl}^2(t)\, dt$$

for $0 \leq i \leq m$, which are bounded by

$$C \sum_{p=m-\frac{n}{2}+\frac{3}{2}}^{\infty} \sum_{l=-p}^{p} \prod_{s=-\frac{n}{2}+\frac{3}{2}}^{m-\frac{n}{2}+\frac{1}{2}} \left(p\,(p+n-2) - s\,(s+n-2)\right) \int_{\mathbb{R}} (v_{pl}(t))^2\, dt$$

$$(4.42) \qquad \leq C \int_{\mathbb{R}^n} (-\Delta)^m u(x)\, u(x) |x|^{-1}\, dx,$$

as above. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

5. Integral identity and global estimate: the case of even dimension. Part I: power-logarithmic weight

Theorem 5.1. *Assume that $m \in \mathbb{N}$ and $n \in [2, 2m] \cap \mathbb{N}$ is even. Let Ω be a bounded domain in $\mathbb{R}^n$, $O \in \mathbb{R}^n \setminus \Omega$, $u \in C_0^\infty(\Omega)$ and $v = e^{\left(m-\frac{n}{2}\right)t}(u \circ \varkappa^{-1})$. Furthermore, let R be a positive constant such*

that the support of u is contained in B_{2R}, $C_R := \log(4R)$, and let ψ be a weight function such that either $\psi(t) = C_R + t$ for all $t \in \mathbb{R}$ or $\psi(t) = 1$ for all $t \in \mathbb{R}$. Then whenever m is even,

$$\int_{\mathbb{R}^n} (-\Delta)^m u(x)\, u(x)\, \psi(\log|x|^{-1})\, dx \geq C \sum_{k=1}^{m} \sum_{i=0}^{m-k} \int_{\mathbb{R}} \int_{S^{n-1}} \left(\partial_t^k \nabla_\omega^i v\right)^2 \psi(t)\, d\omega dt$$

(5.1)
$$+ C \int_{\mathbb{R}} \int_{S^{n-1}} v \prod_p (-\delta_\omega - p\,(p+n-2))^2\, v\, \psi(t)\, d\omega dt,$$

where the product is over $p = -n/2+2, -n/2+4, ..., m-n/2-2, m-n/2$, that is, $p = -n/2+2j$ with $j = 1, 2, ..., m/2$. If m is odd,

$$\int_{\mathbb{R}^n} (-\Delta)^m u(x)\, u(x)\, \psi(\log|x|^{-1})\, dx \geq C \sum_{k=1}^{m} \sum_{i=0}^{m-k} \int_{\mathbb{R}} \int_{S^{n-1}} \left(\partial_t^k \nabla_\omega^i v\right)^2 \psi(t)\, d\omega dt$$

(5.2)
$$+ C \int_{\mathbb{R}} \int_{S^{n-1}} v \prod_p (-\delta_\omega - p\,(p+n-2))^2 \left(-\delta_\omega + (n/2-1)^2\right) v\, \psi(t)\, d\omega dt,$$

where the product is over $p = -n/2+3, -n/2+5, ..., m-n/2-2, m-n/2$, that is, $p = -n/2+1+2j$ with $j = 1, 2, ..., (m-1)/2$. In both cases $C > 0$ is some constant depending on m and n only.

Proof. Passing to the coordinates (t, ω) and decomposing v into spherical harmonics, we see that

$$\int_{\mathbb{R}^n} (-\Delta)^m u(x)\, u(x)\, \psi(\log|x|^{-1})\, dx = \int_{\mathbb{R}} \int_{S^{n-1}} \mathcal{L}_o^{m,n}(\partial_t, \delta_\omega) v(t, \omega)\, v(t, \omega)\, \psi(t)\, d\omega dt$$

(5.3)
$$= \sum_{q=0}^{\infty} \sum_{l=-q}^{q} \int_{\mathbb{R}} \mathcal{L}_o^{m,n}(\partial_t, -q(q+n-2)) v_{ql}(t)\, v_{ql}(t)\, \psi(t)\, dt,$$

where

$$\mathcal{L}_o^{m,n}(\partial_t, \delta_\omega) = (-1)^m \prod_{j=0}^{m-1} \left(\left(-\partial_t + m - n/2 - 2j\right)\left(-\partial_t + m + n/2 - 2 - 2j\right) + \delta_\omega \right)$$

(5.4)
$$= \prod_{j=0}^{m-1} \left(-\left(\partial_t - m + 2j + 1\right)^2 + \left(n/2 - 1\right)^2 - \delta_\omega \right).$$

Let us denote $A := \left(n/2 - 1\right)^2 - \delta_\omega$. Then the expression above is equal to

$$\prod_{j=0}^{m-1} \left(\sqrt{A} - \partial_t + m - 2j - 1 \right)\left(\sqrt{A} + \partial_t - m + 2j + 1 \right)$$

(5.5)
$$= \prod_{j=0}^{m-1} \left(\sqrt{A} - \partial_t + m - 2j - 1 \right)\left(\sqrt{A} + \partial_t + m - 2j - 1 \right) = \prod_{j=0}^{m-1} \left(-\partial_t^2 + \left(\sqrt{A} + m - 2j - 1\right)^2 \right).$$

Hence,

$$(5.6) \qquad \mathcal{L}_o^{m,n}(\partial_t, -q(q+n-2)) = \prod_{j=0}^{m-1}\left(-\partial_t^2 + \mathcal{B}_j(q)^2\right),$$

with

$$(5.7) \quad \mathcal{B}_j(q)^2 = \left(\sqrt{(n/2-1)^2 + q(q+n-2)} + m - 2j - 1\right)^2 = \left(q + n/2 + m - 2j - 2\right)^2, \quad q \in \mathbb{N} \cup \{0\}.$$

We claim that

$$(5.8) \qquad \mathcal{B}_j(q)^2 \geq C\, q(q+n-2), \text{ when } q \in \mathbb{N} \cup \{0\} \text{ is such that } q \neq 2j - m - n/2 + 2,$$

for some $C > 0$ depending on m and n only, and

$$(5.9) \qquad \mathcal{B}_j(q) = 0 \quad \text{if} \quad q = 2j - m - n/2 + 2.$$

Indeed, if $q > m - n/2$ then (5.8) holds with any $C \in (0,1)$ satisfying

$$(5.10) \quad (1-C)\, q(q+n-2) > (m-n/2)(m+n/2-2) \quad \forall\, q > m - n/2, \ q \in \mathbb{N}.$$

Furthermore, one can directly check that $\mathcal{B}_j(q) = 0$ for the values of q described in (5.9) and otherwise $\mathcal{B}_j(q)^2 > 0$. Hence, if we choose C to be the minimum of $\mathcal{B}_j(q)^2 / (q(q+n-2))$ over all $j = 0, \ldots, m-1$, $0 < q \leq m - n/2$, such that $q \neq 2j - m - n/2 + 2$, we obtain (5.8) for the values of $q \leq m - n/2$. This finishes the proof of (5.8).

Now one can show that

$$\int_{\mathbb{R}} \mathcal{L}_o^{m,n}(\partial_t, -q(q+n-2)) v_{ql}(t)\, v_{ql}(t)\, \psi(t)\, dt$$

$$= \sum_{j=1}^{m} \sum_{k_1 < \ldots < k_j} \int_{\mathbb{R}} (-\partial_t^2)^{m-j} \mathcal{B}_{k_1}(q)^2 \ldots \mathcal{B}_{k_j}(q)^2 v_{ql}(t)\, v_{ql}(t)\, \psi(t)\, dt + \int_{\mathbb{R}} (-\partial_t^2)^m v_{ql}(t)\, v_{ql}(t)\, \psi(t)\, dt$$

$$= \sum_{j=1}^{m} \sum_{k_1 < \ldots < k_j} \int_{\mathbb{R}} \mathcal{B}_{k_1}(q)^2 \ldots \mathcal{B}_{k_j}(q)^2 (\partial_t^{m-j} v_{ql}(t))^2\, \psi(t)\, dt + \int_{\mathbb{R}} (-\partial_t^2)^m v_{ql}(t)\, v_{ql}(t)\, \psi(t)\, dt$$

$$\geq C \sum_{\substack{i \geq 0,\, k \geq 1 \\ i+k=m}} \int_{\mathbb{R}} (q(q+n-2))^i\, (\partial_t^k v_{ql}(t))^2\, \psi(t)\, dt + \int_{\mathbb{R}} \prod_{j=0}^{m-1} \mathcal{B}_j(q)^2 v_{ql}^2(t)\, \psi(t)\, dt$$

$$(5.11) \qquad \geq C \sum_{\substack{i \geq 0,\, k \geq 1 \\ i+k \leq m}} \int_{\mathbb{R}} (q(q+n-2))^i\, (\partial_t^k v_{ql}(t))^2\, \psi(t)\, dt + \int_{\mathbb{R}} \mathcal{L}_o^{m,n}(0, -q(q+n-2)) v_{ql}^2(t)\, \psi(t)\, dt.$$

Note that the second equality uses the particular form of the weight function. Indeed, when $\psi \equiv 1$, this is just integration by parts, and when $\psi(t) = C_R + t$, $t \in \mathbb{R}$, we prove by induction in $k \in \mathbb{N} \cup \{0\}$ that

$$(5.12) \qquad \int_{\mathbb{R}} (-\partial_t^2)^k v^2(t)\, (C_R + t)\, dt = \int_{\mathbb{R}} (\partial_t^k v)^2(t)\, (C_R + t)\, dt, \qquad k \in \mathbb{N}.$$

Finally, a straightforward calculation shows that whenever m is even,

$$(5.13) \quad \mathcal{L}_o^{m,n}(0, -q(q + n - 2)) = \prod_p (q(q + n - 2) - p(p + n - 2))^2, \quad q \in \mathbb{N} \cup \{0\},$$

where the product above is over $p = -n/2 + 2, -n/2 + 4, ..., m - n/2 - 2, m - n/2$, and if m is odd,

$$(5.14) \quad \mathcal{L}_o^{m,n}(0, -q(q + n - 2)) = \prod_p (q(q + n - 2) - p(p + n - 2))^2 \left(q(q + n - 2) + (n/2 - 1)^2 \right),$$

for every $q \in \mathbb{N} \cup \{0\}$, with the product above over $p = -n/2 + 3, -n/2 + 5, ..., m - n/2 - 2, m - n/2$. This finishes the argument. $\qquad\square$

6. Integral identity and global estimate: the case of even dimension. Part II: weight g

Lemma 6.1. *Assume that $m \in \mathbb{N}$, $n \in [2, 2m] \cap \mathbb{N}$ is even and $m - n/2$ is even. Recall that in this case*

$$(6.1) \quad \mathcal{L}_o^{m,n}(-\partial_t, 0) = \prod_{j=0}^{m-1} \left(-\partial_t^2 + \left(m - \frac{n}{2} - 2j \right)^2 \right)$$

and consider the equation

$$(6.2) \quad \prod_{j=0}^{m-1} \left(-\partial_t^2 + \left(m - \frac{n}{2} - 2j \right)^2 \right) h = \delta,$$

where δ stands for the Dirac delta function. A unique solution to (6.2) which vanishes at $+\infty$ and has at most linear growth or decay at $-\infty$ has a form

$$(6.3) \quad h(t) = \begin{cases} \sum_{i=1}^{(m-n/2)/2} v_i^{(1)} e^{-2it} + \sum_{i=1}^{(m-n/2)/2} v_i^{(2)} t e^{-2it} + \sum_{i=1}^{n/2-1} v_i^{(3)} e^{-(m-n/2+2i)t}, & t > 0, \\ \sum_{i=1}^{(m-n/2)/2} \mu_i^{(1)} e^{2it} + \sum_{i=1}^{(m-n/2)/2} \mu_i^{(2)} t e^{2it} + \sum_{i=1}^{n/2-1} \mu_i^{(3)} e^{(m-n/2+2i)t} + \mu^{(4)} t + \mu^{(5)}, & t < 0. \end{cases}$$

Here $v_i^{(1)}, v_i^{(2)}, \mu_i^{(1)}, \mu_i^{(2)}$, $i = 1, ..., (m - n/2)/2$, $v_i^{(3)}, \mu_i^{(3)}$, $i = 1, ..., n/2 - 1$, and $\mu^{(4)}, \mu^{(5)}$ are some real numbers depending on m and n only.

Proof. The characteristic polynomial of the differential equation (6.2) has $n/2 - 1$ single roots given by

$$(6.4) \quad \text{even numbers} \quad \text{from} \quad -m - n/2 + 2 \quad \text{to} \quad -m + n/2 - 2,$$

then $m - n/2 + 1$ double roots given by

$$(6.5) \quad \text{even numbers} \quad \text{from} \quad -m + n/2 \quad \text{to} \quad m - n/2,$$

(including 0), and then another $n/2 - 1$ single roots given by

$$(6.6) \quad \text{even numbers} \quad \text{from} \quad m - n/2 + 2 \quad \text{to} \quad m + n/2 - 2.$$

This determines the structure of the solution (6.3). Furthermore, the solution to (6.2) h should be such that that $\partial_t^k h$ are continuous for $k = 0, 1, ..., 2m - 2$, i.e., $\lim_{t \to 0^+} \partial_t^k h(t) - \lim_{t \to 0^-} \partial_t^k h(t) = 0$ for

$k = 0, 1, ..., 2m - 2$, and $\lim_{t \to 0^+} \partial_t^{2m-1} g(t) - \lim_{t \to 0^-} \partial_t^{2m-1} h(t) = (-1)^m$. Hence, the coefficients in (6.3) satisfy the linear system

$$(6.7) \qquad A \vec{\kappa}^{\perp} = (-1)^m (0, ..., 0, 1)^{\perp},$$

where the vector on the right-hand side is of the length $2m$ (it has first $2m - 1$ entries equal to 0 and the last entry equal to 1), the vector $\vec{\kappa}$ is also of the length $2m$ with the entries (in order) $v_i^{(1)}, v_i^{(2)}, i = 1, ..., (m - n/2)/2, v_i^{(3)}, i = 1, ..., n/2 - 1$, and $-\mu_i^{(1)}, -\mu_i^{(2)}, i = 1, ..., (m - n/2)/2, -\mu_i^{(3)},$ $i = 1, ..., n/2 - 1, -\mu^{(4)}, -\mu^{(5)}$. Finally, A is a $2m \times 2m$ matrix built as follows:

$A_{jk} = (-2i)^{j-1}, \text{ for } i = k, \ j = 1, ..., 2m, \ k = 1, ..., (m - n/2)/2,$

$A_{0k} = 0, \text{ for } k = (m - n/2)/2 + 1, ..., m - n/2,$

$A_{jk} = (j - 1)(-2i)^{j-2}, \text{ for } i = k - (m - n/2)/2, \ j = 2, ..., 2m, \ k = (m - n/2)/2 + 1, ..., m - n/2,$

$A_{jk} = (-m + n/2 - 2i)^{j-1}, \text{ for } i = k - m + n/2, \ j = 1, ..., 2m, \ k = m - n/2 + 1, ..., m - 1,$

$A_{jk} = (2i)^{j-1}, \text{ for } i = k - m + 1, \ j = 1, ..., 2m, \ k = m, ..., (3m - n/2)/2 - 1,$

$A_{0k} = 0, \text{ for } k = (3m - n/2)/2, ..., 2m - n/2 - 1,$

$A_{jk} = (j - 1)(2i)^{j-2}, \text{ for } i = k - (3m - n/2)/2 + 1, \ j = 2, ..., 2m, \ k = (3m - n/2)/2, ..., 2m - n/2 - 1,$

$A_{jk} = (m - n/2 + 2i)^{j-1}, \text{ for } i = k - 2m + n/2 + 1, \ j = 1, ..., 2m, \ k = 2m - n/2, ..., 2m - 2,$

$A_{jk} = 0, \text{ for } j = 1 \text{ and } j = 3, ..., 2m, \text{ and } A_{2k} = 1, \text{ when } k = 2m - 1,$

$A_{jk} = 0, \text{ for } j = 1, ..., 2m, \text{ and } A_{1k} = 1, \text{ when } k = 2m.$

It remains to show that the system (6.7) has a unique solution, i.e., that the determinant of A is not 0. This, however, is a standard argument in the theory of ordinary differential equations since determinant of A is Wronskian at 0 for a complete system of solutions for an ODE of order $2m$. We omit the details. $\qquad \square$

Lemma 6.2. *Assume that $m \in \mathbb{N}$, $n \in [2, 2m] \cap \mathbb{N}$ is even and $m - n/2$ is odd. Recall that in this case*

$$(6.8) \qquad \mathcal{L}_o^{m,n}(-\partial_t, 1 - n) = \mathcal{L}_o^{m,n}(-\partial_t, -1(1 + n - 2)) = \prod_{j=0}^{m-1} \left(-\partial_t^2 + \left(m + \frac{n}{2} - 2j - 1 \right)^2 \right)$$

and consider the equation

$$(6.9) \qquad \prod_{j=0}^{m-1} \left(-\partial_t^2 + \left(m + \frac{n}{2} - 2j - 1 \right)^2 \right) h = \delta,$$

where δ stands for the Dirac delta function. A unique solution to (6.9) which vanishes at $+\infty$ and has at most linear growth or decay at $-\infty$ has a form

(6.10)

$$h(t) = \begin{cases} \sum_{i=1}^{(m-n/2-1)/2} v_i^{(1)} e^{-2it} + \sum_{i=1}^{(m-n/2-1)/2} v_i^{(2)} te^{-2it} + \sum_{i=1}^{n/2} v_i^{(3)} e^{-(m-n/2-1+2i)t}, & t > 0, \\ \sum_{i=1}^{(m-n/2-1)/2} \mu_i^{(1)} e^{2it} + \sum_{i=1}^{(m-n/2-1)/2} \mu_i^{(2)} te^{2it} + \sum_{i=1}^{n/2} \mu_i^{(3)} e^{(m-n/2-1+2i)t} + \mu^{(4)}t + \mu^{(5)}, & t < 0. \end{cases}$$

Here $v_i^{(1)}, v_i^{(2)}, \mu_i^{(1)}, \mu_i^{(2)}, i = 1, ..., (m - n/2 - 1)/2, v_i^{(3)}, \mu_i^{(3)}, i = 1, ..., n/2, and \mu^{(4)}, \mu^{(5)} are some real numbers depending on m and n only.

We would like to remark a curious difference in h for three considered situations: n odd; n even and $m - n/2$ even; n even and $m - n/2$ odd. While formula (6.3) is rather different from (4.2), the underlying differential equation in (6.1)–(6.2) is a quite natural analogue of (4.1). Lemma 6.2, however, deals with a different equation (6.8). Just as in Section 4, we will show below how one can build a suitable weight from the corresponding function h.

Proof. The characteristic polynomial of the differential equation (6.9) has $n/2$ single roots given by

(6.11)   even numbers   from $-m - n/2 + 1$ to $-m + n/2 - 1,$

then $m - n/2$ double roots given by

(6.12)   even numbers   from $-m + n/2 + 1$ to $m - n/2 - 1,$

(including 0), and then another $n/2$ single roots given by

(6.13)   even numbers   from $m - n/2 + 1$ to $m + n/2 - 1.$

In fact, these are the same collections of numbers as those appearing in (6.4)–(6.6), in the following sense. Let us denote the sequence of numbers described in (6.4)–(6.6) by $S_{m,n}^e$ and the one described by (6.11)–(6.13) by $S_{m,n}^o$. Then

$$\left\{ S_{m,n}^e \right\}_{m \in \mathbb{N}, n \in [2,2m] \cap \mathbb{N}, m-n/2 \text{ even}} \supset \left\{ S_{m,n}^o \right\}_{m \in \mathbb{N}, n \in [2,2m] \cap \mathbb{N}, m-n/2 \text{ odd}},$$

via the identification $S_{m,n}^o = S_{m,n+2}^e$ for m, n as above. This identification immediately leads to the observation that the solution to (6.9) must be given by the formula (6.3) with n replaced by $n + 2$. □

Theorem 6.3. *Assume that $m \in \mathbb{N}$ and $n \in [2, 2m] \cap \mathbb{N}$ is even. Let Ω be a bounded domain in $\mathbb{R}^n$, $O \in \mathbb{R}^n \setminus \Omega$, $u \in C_0^\infty(\Omega)$ and $v = e^{(m-\frac{n}{2})t}(u \circ \varkappa^{-1})$. Let R be a positive constant such that the support of u is contained in B_{2R}. Then there exist positive constants C, C', C'', depending on m and n only, such that for every $\xi \in B_{2R}$ and $\tau = \log |\xi|^{-1}$ we have*

$$(6.14) \qquad \int_{S^{n-1}} v^2(\tau, \omega) \, d\omega \leq C \int_{\mathbb{R}^n} (-\Delta)^m u(x) \, u(x) g(\log |x|^{-1}, \log |\xi|^{-1}) \, dx$$

where $C_R = \log(4R)$ and g is defined by

$$(6.15) \quad g(t, \tau) = h(t - \tau) + \mu^{(4)}(C_R + \tau) + C' + C''(C_R + t),$$

where h and $\mu^{(4)}$ are given by (6.3) and (6.10), depending on the parity of $m - \frac{n}{2}$.

Proof. **Step I, the preliminary choice of weight**. Recall the function h defined in (6.3) and (6.10) for the cases when $m - n/2$ is even and odd, respectively. Let

$$(6.16) \quad \widetilde{g}(t, \tau) := h(t - \tau) + \mu^{(4)}(C_R + \tau). \qquad t, \tau \in \mathbb{R}.$$

In particular, when $m - n/2$ is even, we have

$$(6.17) \quad \widetilde{g}(t, \tau) = \begin{cases} \sum_{i=1}^{(m-n/2)/2} \nu_i^{(1)} e^{-2i(t-\tau)} + \sum_{i=1}^{(m-n/2)/2} \nu_i^{(2)} (t - \tau) e^{-2i(t-\tau)} \\ \qquad + \sum_{i=1}^{n/2-1} \nu_i^{(3)} e^{-(m-n/2+2i)(t-\tau)} + \mu^{(4)}(C_R + \tau), \qquad t > \tau, \\ \sum_{i=1}^{(m-n/2)/2} \mu_i^{(1)} e^{2i(t-\tau)} + \sum_{i=1}^{(m-n/2)/2} \mu_i^{(2)} (t - \tau) e^{2i(t-\tau)} \\ \qquad + \sum_{i=1}^{n/2-1} \mu_i^{(3)} e^{(m-n/2+2i)(t-\tau)} + \mu^{(4)}(C_R + t) + \mu^{(5)}, \qquad t < \tau, \end{cases}$$

and in the case when $m - n/2$ is odd, we have

$$(6.18) \quad \widetilde{g}(t, \tau) = \begin{cases} \sum_{i=1}^{(m-n/2-1)/2} \nu_i^{(1)} e^{-2i(t-\tau)} + \sum_{i=1}^{(m-n/2-1)/2} \nu_i^{(2)} (t - \tau) e^{-2i(t-\tau)} \\ \qquad + \sum_{i=1}^{n/2} \nu_i^{(3)} e^{-(m-n/2-1+2i)(t-\tau)} + \mu^{(4)}(C_R + \tau), \qquad t > \tau, \\ \sum_{i=1}^{(m-n/2-1)/2} \mu_i^{(1)} e^{2i(t-\tau)} + \sum_{i=1}^{(m-n/2-1)/2} \mu_i^{(2)} (t - \tau) e^{2i(t-\tau)} \\ \qquad + \sum_{i=1}^{n/2} \mu_i^{(3)} e^{(m-n/2-1+2i)(t-\tau)} + \mu^{(4)}(C_R + t) + \mu^{(5)}, \qquad t < \tau. \end{cases}$$

For future reference, we record a few estimates. First, by our assumptions on ξ and the support of u, the discussion will be naturally restricted to the case $t, \tau \geq \log(2R)^{-1}$. Hence, both $C_R + t$ and $C_R + \tau$ are positive. We remark that we do not claim positivity of h or that of $\widetilde{g}$. However,

$$(6.19) \quad |\widetilde{g}(t, \tau)| \leq C_0(m, n) + |\mu^{(4)}|(C_R + t), \quad t, \tau \geq \log(2R)^{-1},$$

and

$$(6.20) \quad |\partial_t^l \widetilde{g}(t, \tau)| \leq C_1(m, n), \quad t, \tau \geq \log(2R)^{-1}, \quad 1 \leq l \leq 2m,$$

for some constants $C_0(m, n), C_1(m, n) > 0$ depending on m, n only. We note that $\partial_t^l \widetilde{g}$ can be defined at $t = \tau$ for all $l < 2m$ by continuity, and for $l = 2m$ one assumes $t \neq \tau$ in (6.20).

Step II, the set-up. We commence similarly to (4.15):

$$\int_{\mathbb{R}^n} (-\Delta)^m u(x) \, u(x) \widetilde{g}(\log |x|^{-1}, \log |\xi|^{-1}) \, dx = \int_{\mathbb{R}} \int_{S^{n-1}} \mathcal{L}_o^{m,n}(\partial_t, \delta) v(t, \omega) \, v(t, \omega) \widetilde{g}(t, \tau) \, d\omega dt$$

$$= \int_{\mathbb{R}} \int_{S^{n-1}} \mathcal{L}_o^{m,n}(0, \delta) v(t, \omega) \, v(t, \omega) \widetilde{g}(t, \tau) \, d\omega dt$$

$$(6.21) \qquad + \int_{\mathbb{R}} \int_{S^{n-1}} (\mathcal{L}_o^{m,n}(\partial_t, \delta) - \mathcal{L}_o^{m,n}(0, \delta)) \, v(t, \omega) \, v(t, \omega) \widetilde{g}(t, \tau) \, d\omega dt.$$

Furthermore, as in (4.21)–(4.22),

$$\int_{\mathbb{R}} \int_{S^{n-1}} (\mathcal{L}_o^{m,n}(\partial_t, \delta) - \mathcal{L}_o^{m,n}(0, \delta)) v(t, \omega) \, v(t, \omega) \widetilde{g}(t, \tau) \, d\omega dt$$

$$= \frac{1}{2} \sum_{p=0}^{\infty} \sum_{l=-p}^{p} \int_{\mathbb{R}} v_{pl}^2(t) \left(\mathcal{L}^{m,n}(-\partial_t, -p(p+n-2)) - \mathcal{L}^{m,n}(0, -p(p+n-2)) \right) \widetilde{g}(t,\tau) \, dt$$

(6.22)
$$+ \sum_{\substack{k \geq 1, i \geq 0 \\ 2i+k \leq 2m}} \sum_{\substack{l \geq 1, j \geq 0 \\ 2l+j \leq k}} c_{lj}^{ki} \int_{\mathbb{R}} \int_{S^{n-1}} (\partial_t^l \nabla_\omega^i v)^2 \, \partial_t^j \widetilde{g}(t,\tau) \, d\omega dt =: I_1^o + I_2^o,$$

for some constants c_{lj}^{ki} depending on m, n only.

Let us now denote

(6.23) $\quad p_0 := \begin{cases} 0, & \text{when } m - n/2 \text{ is even,} \\ 1, & \text{when } m - n/2 \text{ is odd.} \end{cases}$

Then

$$I_1^o = \frac{1}{2} \sum_{p=0}^{\infty} \sum_{l=-p}^{p} \int_{\mathbb{R}} v_{pl}^2(t) \bigg(\mathcal{L}_o^{m,n}(-\partial_t, -p(p+n-2)) - \mathcal{L}_o^{m,n}(0, -p(p+n-2))$$

$$-\mathcal{L}_o^{m,n}(-\partial_t, -p_0(p_0+n-2)) \bigg) \widetilde{g}(t,\tau) \, dt$$

(6.24)
$$+ \frac{1}{2} \int_{S^{n-1}} v^2(\tau, \omega) \, d\omega,$$

where we used (6.1)–(6.2), (6.8)–(6.9), as well as the fact that the operator $\mathcal{L}_o^{m,n}(-\partial_t, -p_0(p_0+n-2))$ kills constants and thus,

(6.25) $\quad \mathcal{L}_o^{m,n}(-\partial_t, -p_0(p_0+n-2))\widetilde{g}(t,\tau) = \mathcal{L}_o^{m,n}(-\partial_t, -p_0(p_0+n-2))h(t-\tau) = \delta(t-\tau).$

Step III. The term in the representation of I_1^o in (6.24) associated to $\mathcal{L}_o^{m,n}(0, -p(p+n-2))$ and the first term on the right-hand side of (6.21) are, modulo a multiplicative constant, the same, and can be estimated as follows:

$$\left| \int_{\mathbb{R}} \int_{S^{n-1}} \mathcal{L}_o^{m,n}(0, \delta) v(t, \omega) v(t, \omega) \widetilde{g}(t, \tau) \, d\omega dt \right|$$

$$= \left| \sum_{p=0}^{\infty} \sum_{l=-p}^{p} \int_{\mathbb{R}} v_{pl}^2(t) \mathcal{L}_o^{m,n}(0, -p(p+n-2)) \widetilde{g}(t, \tau) \, dt \right|$$

$$\leq \sum_{p=0}^{\infty} \sum_{l=-p}^{p} \int_{\mathbb{R}} v_{pl}^2(t) \mathcal{L}_o^{m,n}(0, -p(p+n-2)) |\widetilde{g}(t, \tau)| \, dt$$

$$\leq \sum_{p=0}^{\infty} \sum_{l=-p}^{p} \int_{\mathbb{R}} v_{pl}^2(t) \mathcal{L}_o^{m,n}(0, -p(p+n-2)) \left(C_0(m, n) + |\mu^{(4)}| (C_R + t) \right) dt$$

$$= \int_{\mathbb{R}} \int_{S^{n-1}} \mathcal{L}_o^{m,n}(0, \delta) v(t, \omega) v(t, \omega) \left(C_0(m, n) + |\mu^{(4)}| (C_R + t) \right) \, d\omega dt$$

(6.26)
$$\leq C \int_{\mathbb{R}^n} (-\Delta)^m u(x) \, u(x) \left(C_0(m, n) + |\mu^{(4)}| (C_R + \log |x|^{-1}) \right) \, dx,$$

where we used the fact that the polynomial $\mathcal{L}^{m,n}(0, -p(p+n-2)) \geq 0$ (see (5.13)–(5.14)) for the first inequality, (6.19) for the second one, and (5.1)–(5.2) with (5.13)–(5.14) for the last inequality above.

It remains to estimate I_2^o (see (6.22) for definition) and

(6.27)
$$\sum_{p=0}^{\infty} \sum_{l=-p}^{p} \int_{\mathbb{R}} v_{pl}^2(t) \left(\mathcal{L}_o^{m,n}(-\partial_t, -p(p+n-2)) - \mathcal{L}_o^{m,n}(-\partial_t, -p_0(p_0+n-2)) \right) \widetilde{g}(t, \tau) \, dt.$$

For the terms in the expression above (and for I_2^o), it is sufficient to either prove positivity or, for those which are not necessarily positive, a bound on (the sum of) their absolute values in terms of the right-hand side of (6.14).

Step IV, $0 \leq p \leq m - n/2$, $m - n/2$ **even**. At this stage, the discussion splits according to whether $m - n/2$ is even or odd. In fact, the two cases are, in some sense, symmetric, but tracking both at the same time would make the discussion too cumbersome. We start with the situation when $m - n/2$ is even, that is, we are in the regimen of Lemma 6.1, and then we will list the necessary modifications for $m - n/2$ odd.

To outline in a few words the strategy, we are going to show that the terms in the expression above corresponding to $0 \leq p \leq m - n/2$ even are positive by virtue of the theory developed in Section 3, while the terms corresponding to $0 \leq p \leq m - n/2$ odd, as well as the sum of those corresponding to $p > m - n/2$, are bounded by the right-hand side of (6.14). Essentially, for large p we use the same considerations as in Section 4, while for $0 \leq p \leq m - n/2$ odd we observe that the last integrals in (5.1), (5.2) do not vanish and provide the desired bounds. Finally, I_2^o can be analyzed much as in Section 4. Now we turn to the details.

Step IV(a), $m - n/2$ **even,** $0 \leq p \leq m - n/2$ **even**. In this case, we shall employ the strategy suggested by the results of Section 3. One would like to show that

(6.28) $\quad \mathcal{L}_o^{m,n}(-\partial_t, 0)h = \delta \quad \Longrightarrow$

$$\mathcal{L}_o^{m,n}(-\partial_t, -p(p+n-2))h(t) \geq 0, \quad t \neq 0, \quad \text{for all } 0 \leq p \leq m - n/2 \text{ even.}$$

As we shall demonstrate below, $\mathcal{L}_o^{m,n}(-\partial_t, -p(p+n-2))$, viewed as a polynomial in ∂_t, has a double root at zero for all $0 \leq p \leq m - n/2$ even and hence, the result of its action (as an operator) on $\widetilde{g}$ is the same as the result of its action on h. Thus, it is indeed enough to consider h in place of $\widetilde{g}$ in the terms corresponding to such values of p.

Next, due to (5.6)–(5.7), the roots of $\mathcal{L}_o^{m,n}(-\partial_t, -p(p+n-2))$ as a polynomial in ∂_t are given by

$$\pm \left(p + n/2 + m - 2j - 2 \right), \quad j = 0, ..., m - 1.$$

Clearly, the sequence of the roots is symmetric with respect to zero and moreover, the roots can be ordered in non-decreasing order as single roots given by

$$(6.29) \quad \text{even numbers} \quad \text{from} \quad -p-m-n/2+2 \quad \text{to} \quad p-m+n/2-2,$$

then double roots given by

$$(6.30) \quad \text{even numbers} \quad \text{from} \quad p-m+n/2 \quad \text{to} \quad -p+m-n/2,$$

(including 0), and then single roots given by

$$(6.31) \quad \text{even numbers} \quad \text{from} \quad -p+m-n/2+2 \quad \text{to} \quad p+m+n/2-2.$$

Now, in the spirit of the Proof of Proposition 3.1, let us denotes the non-decreasing sequence of the roots of $\mathcal{L}_o^{m,n}(-\partial_t, -p(p+n-2))$ for $p = 0$ by

$$(6.32) \quad b_{m-1} \le b_{m-2} \le \dots \le b_1 < b_0 = 0 = a_0 < a_1 \le \dots \le a_m,$$

and, respectively, the roots of $\mathcal{L}_o^{m,n}(-\partial_t, -p(p+n-2))$ for a given even $0 < p \le m - n/2$ by

$$(6.33) \quad d_{m-1} \le d_{m-2} \le \dots \le d_1 < d_0 = 0 = c_0 < c_1 \le \dots \le c_m.$$

It is not hard to see that for all even $0 < p \le m - n/2$ there holds $[b_i, a_i] \subset [d_i, c_i]$ for all $i = 0, \dots, m-1$, and moreover, $a_0 = c_0 = 0 = b_0 = d_0$ are the only zero roots. We are now in a position to apply Corollary 3.3. The only problem lies in extra zero roots which appear in the current scenario. To avoid it, we do the following.

The goal is to establish (6.28). In the framework of Corollary 3.3, let us denote by P the differential operator of the order $2(m-1)$ such that $\partial_t^2 P(\partial_t) = (-1)^m \mathcal{L}_o^{m,n}(-\partial_t, 0)$, $f := \delta$, and let Q be the differential operator of the order $2(m-1)$ such that $\partial_t^2 Q(\partial_t) = (-1)^m \mathcal{L}_o^{m,n}(-\partial_t, -p(p+n-2))$ with $0 < p \le m - n/2$ even. Then the roots of P and Q are, respectively, $b_i, a_i, i = 1, \dots, m-1$, and $d_i, c_i, i = 1, \dots, m-1$, as above. They satisfy $[b_i, a_i] \subset [d_i, c_i], i = 1, \dots, m-1$, and they are all non-zero. Hence, applying Corollary 3.3 with $m - 1$ in place of m we have

$$(6.34) \quad P(\partial_t) H = \delta \quad \Longrightarrow \quad Q(\partial_t) H \text{ is a positive distribution for all } 0 \le p \le m - n/2 \text{ even,}$$

assuming that H is the (unique) solution to $P(\partial_t) H = \delta$ which is bounded and vanishes at $+\infty$. It remains to show that $H = (-1)^m \partial_t^2 h$, where h is the unique solution to $\mathcal{L}_o^{m,n}(-\partial_t, 0)h = \delta$ which vanishes at $+\infty$ and has at most linear growth or decay at $-\infty$ (that is, h is given by Lemma 6.1). However, evidently, $\partial_t^2[(-1)^m h]$ is bounded and vanishes at $+\infty$. Since a solution with such decay properties is unique, (6.34) indeed proves (6.28).

Step IV(b), $m - n/2$ even, $0 \le p \le m - n/2$ odd. We first note that

$$(6.35) \quad \mathcal{L}_o^{m,n}(0, -p(p+n-2)) \ge C, \quad \text{for } m - n/2 \text{ even and } 0 \le p \le m - n/2 \text{ odd,}$$

where $C > 0$ is a *strictly* positive constant depending on m and n only. This can be seen immediately from (5.13)–(5.14), or, alternatively, from (5.6)–(5.7). Indeed, the polynomial $\mathcal{L}_o^{m,n}(0, -p(p+n-2))$ is a product of squares. Thus, it is non-negative. Moreover, if $m - n/2$ even, $\mathcal{L}_o^{m,n}(0, -p(p+n-2))$ can only be zero when p is even too. Thus, one can take C as a (strictly positive) minimum of $\mathcal{L}_o^{m,n}(0, -p(p+n-2))$ over $0 \le p \le m - n/2$ odd, justifying (6.35).

Thus, for any $0 \le p \le m - n/2$ odd

$$(6.36) \quad \left| \left(\mathcal{L}_o^{m,n}(-\partial_t, -p(p+n-2)) - \mathcal{L}_o^{m,n}(-\partial_t, 0) \right) \widetilde{g}(t, \tau) \right| \le C_1 + C_2(C_R + t)$$

$$\le \mathcal{L}_o^{m,n}(0, -p(p+n-2)) \left(C_3 + C_4(C_R + t) \right), \qquad t, \tau \ge \log(2R)^{-1},$$

where we used (6.19)–(6.20) for the first inequality above and (6.35) for the second one. The constants C_i, $i = 1, 2, 3, 4$, depend on m and n only. Note that the operator $\mathcal{L}_o^{m,n}(-\partial_t, -p(p+n-2)) - \mathcal{L}_o^{m,n}(-\partial_t, 0)$ has order strictly smaller than $2m$, thus, the corresponding derivatives of $\widetilde{g}$ are continuous at $t = \tau$.

Therefore, for any $0 \le p \le m - n/2$ odd we have

$$(6.37) \quad \left| \int_{\mathbb{R}} v_{pl}^2(t) \left(\mathcal{L}_o^{m,n}(-\partial_t, -p(p+n-2)) - \mathcal{L}_o^{m,n}(-\partial_t, 0) \right) \widetilde{g}(t, \tau) \, dt \right|$$

$$\le C \int_{\mathbb{R}} v_{pl}^2(t) \mathcal{L}_o^{m,n}(0, -p(p+n-2)) \left(C_3 + C_4(C_R + t) \right) \, dt$$

$$\le C \int_{\mathbb{R}^n} (-\Delta)^m u(x) \, u(x) \left(C_3 + C_4 \left(C_R + \log |x|^{-1} \right) \right) \, dx,$$

as desired.

Step V, $0 \le p \le m - n/2$, $m - n/2$ **odd**. The argument in this case is very similar to that in Step IV, except that the approach to even p for $m - n/2$ odd resembles the approach to odd p for $m - n/2$ even and vice versa.

Indeed, when $m - n/2$ is odd, the case of $0 \le p \le m - n/2$ even can be handled following verbatim the argument in Step IV(b), observing that

$$(6.38) \quad \mathcal{L}_o^{m,n}(0, -p(p+n-2)) \ge C, \quad \text{for } m - n/2 \text{ odd and } 0 \le p \le m - n/2 \text{ even,}$$

where $C > 0$ is a *strictly* positive constant depending on m and n only. Then, as in Step IV(b), when $m - n/2$ is odd, for any $0 \le p \le m - n/2$ even we have with $p_0 = 1$

$$(6.39) \quad \left| \int_{\mathbb{R}} v_{pl}^2(t) \left(\mathcal{L}_o^{m,n}(-\partial_t, -p(p+n-2)) - \mathcal{L}_o^{m,n}(-\partial_t, -p_0(p_0+n-2)) \right) \widetilde{g}(t, \tau) \, dt \right|$$

$$\le C \int_{\mathbb{R}} v_{pl}^2(t) \mathcal{L}_o^{m,n}(0, -p(p+n-2)) \left(C_3 + C_4(C_R + t) \right) \, dt$$

$$\le C \int_{\mathbb{R}^n} (-\Delta)^m u(x) \, u(x) \left(C_3 + C_4 \left(C_R + \log |x|^{-1} \right) \right) \, dx.$$

Thus, it remains to treat $0 \le p \le m - n/2$ odd. The argument essentially follows Step IV(a). First of all, the roots of $\mathcal{L}_o^{m,n}(-\partial_t, -p(p+n-2))$ for any $0 \le p \le m - n/2$ odd, $m - n/2$ odd, are exactly described by (6.29)–(6.31), including the parity. Hence, for all such p the operator $\mathcal{L}_o^{m,n}(-\partial_t, -p(p+n-2))$ kills constants and it is enough to show that with $m - n/2$ odd

$$(6.40) \quad \mathcal{L}_o^{m,n}(-\partial_t, -1(1+n-2))h = \delta \quad \Longrightarrow$$

$$\mathcal{L}_o^{m,n}(-\partial_t, -p(p + n - 2))h(t) \geq 0, \quad t \neq 0, \quad \text{for all } 0 \leq p \leq m - n/2 \text{ odd.}$$

As before, the crux of the matter is the structure of the roots of $\mathcal{L}_o^{m,n}(-\partial_t, -p(p + n - 2))$ and $\mathcal{L}_o^{m,n}(-\partial_t, -1(1 + n - 2))$ in the context of the results of Section 3. However, if we denote the set of the numbers described by (6.29)–(6.31) as $S_{m,n,p}^e$ when $m - n/2$ is even and $0 \leq p \leq m - n/2$ is even and as $S_{m,n,p}^o$ when $m - n/2$ is odd and $0 \leq p \leq m - n/2$ is odd, then they can be identified via $S_{m,n,p}^o = S_{m,n+2,p-1}^e$. Hence, the results of Step IV(a) translate into the present setting and give exactly (6.40), as desired.

Step VI, $p > m - n/2$. Our treatment of $p > m - n/2$ does not depend on whether $m - n/2$ is even or odd, and we record it here in full generality.

First of all, by (5.8) we have $\mathcal{B}_j(q)^2 \geq C q(q + n - 2)$ for all $q > m - n/2$ and $j = 0, ..., m - 1$, with $C > 0$ depending on m and n only. Thus,

$$(6.41) \quad \mathcal{L}_o^{m,n}(0, -p(p + n - 2)) \geq C p^m (p + n - 2)^m, \quad \text{for } p > m - n/2,$$

with $C > 0$ depending on m and n only. Secondly, recalling the definition of p_0 in (6.23), we have

$$(6.42) \quad \left| \left(\mathcal{L}_o^{m,n}(-\partial_t, -p(p + n - 2)) - \mathcal{L}_o^{m,n}(-\partial_t, -p_0(p_0 + n - 2)) \right) \widetilde{g}(t, \tau) \right|$$

$$\leq (C_1 + C_2(C_R + t)) \, p^m (p + n - 2)^m,$$

by (6.19)–(6.20). Combining (6.41) with (6.42) yields then for any $p > m - n/2$

$$(6.43) \quad \sum_{p=m-n/2+1}^{\infty} \sum_{l=-p}^{p} \left| \int_{\mathbb{R}} v_{pl}^2(t) \left(\mathcal{L}_o^{m,n}(-\partial_t, -p(p + n - 2)) - \mathcal{L}_o^{m,n}(-\partial_t, -p_0(p_0 + n - 2)) \right) \widetilde{g}(t, \tau) \, dt \right|$$

$$\leq C \sum_{p=m-n/2+1}^{\infty} \sum_{l=-p}^{p} \int_{\mathbb{R}} v_{pl}^2(t) p^m (p + n - 2)^m (C_1 + C_2(C_R + t)) \, dt$$

$$\leq \sum_{p=m-n/2+1}^{\infty} \sum_{l=-p}^{p} \int_{\mathbb{R}} v_{pl}^2(t) \mathcal{L}_o^{m,n}(0, -p(p + n - 2)) (C_3 + C_4(C_R + t)) \, dt$$

$$\leq C \int_{\mathbb{R}^n} (-\Delta)^m u(x) \, u(x) \left(C_3 + C_4 \left(C_R + \log |x|^{-1} \right) \right) \, dx.$$

This finishes the argument for I_1^o.

Step VII, the bound on I_2^o. The estimate on I_2^o is essentially already incorporated in (5.1), (5.2). Indeed,

$$I_2^o \leq \sum_{\substack{k \geq 1, i \geq 0 \\ 2i+k \leq 2m}} \sum_{\substack{l \geq 1, j \geq 0 \\ 2l+j \leq k}} |c_{lj}^{ki}| \int_{\mathbb{R}} \int_{S^{n-1}} (\partial_t^l \nabla_\omega^i v)^2 |\partial_t^j \widetilde{g}(t, \tau)| \, dt$$

$$\leq C \sum_{\substack{k \geq 1, i \geq 0 \\ i+k \leq m}} \int_{\mathbb{R}} \int_{S^{n-1}} (\partial_t^k \nabla_\omega^i v(t, \omega))^2 (C_1 + C_2(C_R + t)) \, d\omega dt$$

$$(6.44) \qquad \leq \int_{\mathbb{R}^n} (-\Delta)^m u(x) \, u(x) \left(C_3 + C_4 (C_R + \log |x|^{-1}) \right) dx,$$

employing (6.19)–(6.20) for the first inequality and (5.1), (5.2) for the second one.

This finishes the argument. $\qquad \square$

7. Pointwise and local L^2 estimates for solutions to the polyharmonic equation

This section is devoted to the proof of Theorem 1.1. In addition, we will establish sharp local estimates for solutions in a neighborhood of a boundary point and "at infinity", that is, when moving away from a given $Q \in \partial\Omega$.

To start, we record for future reference a well-known result that follows from the energy estimates for solutions of elliptic equations.

Lemma 7.1. *Let Ω be an arbitrary domain in $\mathbb{R}^n$, $n \geq 2$, $Q \in \mathbb{R}^n \setminus \Omega$ and $R > 0$. Suppose*

$$(7.1) \qquad (-\Delta)^m u = f \text{ in } \Omega, \quad f \in C_0^\infty(\Omega \setminus B_{4R}(Q)), \quad u \in \mathring{W}^{m,2}(\Omega).$$

Then

$$(7.2) \qquad \sum_{i=1}^m \frac{1}{\rho^{2m-2i}} \int_{B_\rho(Q) \cap \Omega} |\nabla^i u|^2 \, dx \leq \frac{C}{\rho^{2m}} \int_{C_{\rho,2\rho}(Q) \cap \Omega} |u|^2 \, dx$$

for every $\rho < 2R$.

The following Proposition reflects the rate of growth of solutions near a boundary point encoded in Theorems 4.2, 6.3, and ultimately provides a passage to Theorem 1.1.

Proposition 7.2. *Let Ω be a bounded domain in $\mathbb{R}^n$, $2 \leq n \leq 2m + 1$, $Q \in \mathbb{R}^n \setminus \Omega$, and $R > 0$. Suppose*

$$(7.3) \qquad (-\Delta)^m u = f \text{ in } \Omega, \quad f \in C_0^\infty(\Omega \setminus B_{4R}(Q)), \quad u \in \mathring{W}^{m,2}(\Omega).$$

Then

$$(7.4) \qquad \frac{1}{\rho^{2\lambda+n-1}} \int_{S_\rho(Q) \cap \Omega} |u(x)|^2 \, d\sigma_x \leq \frac{C}{R^{2\lambda+n}} \int_{C_{R,4R}(Q) \cap \Omega} |u(x)|^2 \, dx \quad \text{for every} \quad \rho < R,$$

where C is a constant depending on m and n only, and $\lambda = \left[m - \frac{n}{2} + \frac{1}{2} \right]$, that is,

$$(7.5) \qquad \lambda = \begin{cases} m - n/2 + 1/2 & \text{when } n \text{ is odd,} \\ m - n/2 & \text{when } n \text{ is even.} \end{cases}$$

Moreover, for every $x \in B_{R/4}(Q) \cap \Omega$

$$(7.6) \qquad |\nabla^i u(x)|^2 \leq C \frac{|x - Q|^{2\lambda-2i}}{R^{n+2\lambda}} \int_{C_{R/4,4R}(Q) \cap \Omega} |u(y)|^2 \, dy, \qquad 0 \leq i \leq \lambda.$$

In particular, for every bounded domain $\Omega \subset \mathbb{R}^n$ the solution to the boundary value problem (7.3) *satisfies*

(7.7) $|\nabla^{m-n/2+1/2} u| \in L^\infty(\Omega)$ *when n is odd   and* $|\nabla^{m-n/2} u| \in L^\infty(\Omega)$ *when n is even.*

This Proposition will conclude the proof of Theorem 1.1.

Proof of Proposition 7.2. Without loss of generality we can assume that $Q = O$. Let us approximate Ω by a sequence of domains with smooth boundaries $\{\Omega_n\}_{n=1}^\infty$ satisfying

(7.8) $\displaystyle\bigcup_{n=1}^\infty \Omega_n = \Omega$ and $\overline{\Omega}_n \subset \Omega_{n+1}$ for every $n \in \mathbb{N}.$

Choose $n_0 \in \mathbb{N}$ such that $\operatorname{supp} f \subset \Omega_n$ for every $n \geq n_0$ and denote by u_n a unique solution of the Dirichlet problem

(7.9) $(-\Delta)^m u_n = f$ in $\Omega_n,$ $u_n \in \mathring{W}^{m,2}(\Omega_n),$ $n \geq n_0.$

The sequence $\{u_n\}_{n=n_0}^\infty$ converges to u in $\mathring{W}^{m,2}(\Omega)$ (see, e.g., [39], §6.6).

Next, take some $\eta \in C_0^\infty(B_{2R})$ such that

(7.10) $0 \leq \eta \leq 1$ in B_{2R}, $\eta = 1$ in B_R and $|\nabla^k \eta| \leq C R^{-k}$, $k \leq 2m.$

Also, fix $\tau = \log \rho^{-1}$ and let g be the function defined in (4.12) when n is odd and by (6.15) when n is even.

We observe that when n is odd, the formula (4.12) yields

(7.11) $\left|\nabla_x^k g(\log |x|^{-1}, \log \rho^{-1})\right| \leq C |x|^{-k-1},$ $0 \leq k \leq 2m,$ $x \in \mathbb{R}^n \setminus \{0\},$ $\rho \in (0, \infty),$

while for n even by (6.15) and (6.19)–(6.20)

(7.12) $\left|g(\log |x|^{-1}, \log \rho^{-1})\right| \leq C_1 + C_2(C_R + \log |x|^{-1}),$ $0 < |x|, \rho < 2R,$

and

(7.13) $\left|\nabla_x^k g(\log |x|^{-1}, \log \rho^{-1})\right| \leq C |x|^{-k},$ $1 \leq k \leq 2m,$ $0 < |x|, \rho < 2R.$

Here, as usually, we assume $|x| \neq \rho$ when $k = 2m$, and lower derivatives of g as well as g itself are defined at x such that $|x| \neq \rho$ by continuity. Hence, in particular, when n is odd,

(7.14) $\left|\nabla_x^k g(\log |x|^{-1}, \log \rho^{-1})\right| \leq C R^{-k-1},$ $0 \leq k \leq 2m,$ $x \in C_{R,2R},$ $\rho < R,$

and when n is even,

(7.15) $\left|\nabla_x^k g(\log |x|^{-1}, \log \rho^{-1})\right| \leq C R^{-k},$ $0 \leq k \leq 2m,$ $x \in C_{R,2R},$ $\rho < R,$

since for $x \in C_{R,2R}$ we have

$$C_R + \log |x|^{-1} = \log(4R) + \log |x|^{-1} = \log \frac{4R}{|x|} \approx C.$$

Consider now

(7.16) $\displaystyle\int_{\mathbb{R}^n} \left([(-\Delta)^m, \eta] u_n(x) \right) \left(\eta(x) u_n(x) g(\log |x|^{-1}, \log \rho^{-1}) \right) dx,$

where the brackets denote the commutator

$$[(-\Delta)^m, \eta] u_n(x) = (-\Delta)^m (\eta(x) u_n(x)) - \eta(x)(-\Delta)^m u_n(x),$$

the integral in (7.16) is understood in the sense of pairing between $\mathring{W}^{m,2}(\Omega_n)$ and its dual. Evidently, the support of the integrand is a subset of supp $\nabla \eta \subset C_{R,2R}$, and therefore, the expression in (7.16) is equal to

$$(7.17) \quad \int_{C_{R,2R}} \Big([(-\Delta)^m, \eta] u_n(x) \Big) \Big(\eta(x) u_n(x) g(\log |x|^{-1}, \log \rho^{-1}) \Big) dx$$

$$\leq C \sum_{i=0}^m \frac{1}{R^{2\lambda+n-2i}} \int_{C_{R,2R}} |\nabla^i u_n(x)|^2 \, dx \leq \frac{C}{R^{2\lambda+n}} \int_{C_{R,4R}} |u_n(x)|^2 \, dx,$$

using Cauchy-Schwarz inequality, (7.14)–(7.15), and (7.10) for the first inequality and Lemma 7.1 for the second one.

On the other hand, since u_n is biharmonic in $\Omega_n \cap B_{4R}$ and η is supported in B_{2R}, one can see that $\eta(-\Delta)^m u_n = 0$ and hence the integral in (7.16) is equal to

$$(7.18) \quad \int_{\mathbb{R}^n} (-\Delta)^m (\eta(x) u_n(x)) \, \eta(x) u_n(x) g(\log |x|^{-1}, \log \rho^{-1}) \, dx.$$

To estimate it we employ Theorems 4.2 and 6.3 with $u = \eta u_n$. The results (4.11) and (6.14) hold for such a choice of u. This can be seen directly by inspection of the argument or one can approximate each u_n by a sequence of $C_0^\infty(\Omega_n)$ functions in $\mathring{W}^{m,2}(\Omega_n)$ and then take a limit using that $O \notin \overline{\Omega}_n$. Then (7.18) is bounded from below by

$$(7.19) \quad \frac{C}{\rho^{2\lambda+n-1}} \int_{S_\rho} |\eta(x) u_n(x)|^2 \, d\sigma_x.$$

Hence, for every $\rho < R$

$$(7.20) \quad \frac{1}{\rho^{2\lambda+n-1}} \int_{S_\rho} |u_n(x)|^2 \, d\sigma_x \leq \frac{C}{R^{2\lambda+n}} \int_{C_{R,4R}} |u_n(x)|^2 \, dx.$$

Now the proof of (7.4) can be finished by taking the limit as $n \to \infty$.

Going further, by virtue of the local estimates for solutions of elliptic equations, the bound in (7.4) transforms into the uniform pointwise estimates for $\nabla^\lambda u$ (observe that the exponent λ, as defined in (7.5), is an integer number both when n is odd and when n is even).

Indeed, by interior estimates for solutions of the elliptic equations (see [4])

$$(7.21) \quad |\nabla^i u(x)|^2 \leq \frac{C}{d(x)^n} \int_{B_{d(x)/2}(x)} |\nabla^i u(y)|^2 \, dy, \qquad 0 \leq i \leq m,$$

where $d(x)$ denotes the distance from x to $\partial\Omega$. Let x_0 be a point on the boundary of Ω such that $d(x) = |x - x_0|$. Since $x \in B_{R/4}(Q) \cap \Omega$ and $Q \in \mathbb{R}^n \setminus \Omega$, we have $x \in B_{R/4}(x_0)$, and therefore

$$(7.22) \quad \frac{1}{d(x)^n} \int_{B_{d(x)/2}(x)} |\nabla^i u(y)|^2 \, dy \leq \frac{C}{d(x)^{n+2i}} \int_{B_{2d(x)}(x_0)} |u(y)|^2 \, dy \leq C \frac{d(x)^{-2i+2\lambda}}{R^{n+2\lambda}} \int_{C_{3R/4,3R}(x_0)} |u(y)|^2 \, dy,$$

as soon as $i \leq \lambda$. Indeed, the first inequality in (7.22) follows from Lemma 7.1. Turning to the second one, we observe that $d(x) \leq R/4$ and therefore, $2d(x) < 3R/4$. On the other hand, u is polyharmonic in $B_{4R}(Q) \cap \Omega$ and $|Q - x_0| \leq |Q - x| + |x - x_0| \leq R/2$. Hence, u is polyharmonic in $B_{3R}(x_0) \cap \Omega$ and Proposition 7.2 holds with x_0 in place of Q, $3R/4$ in place of R and $\rho = 2d(x)$. Furthermore, $C_{3R/4,3R}(x_0) \subset C_{R/4,4R}(Q)$, and that finishes the argument for the second inequality in (7.22). Clearly, $d(x) \leq |x - Q|$, so that (7.21)–(7.22) entail (7.6). $\square$

At this point, we are ready to address the behavior of solutions "at infinity".

Proposition 7.3. *Let Ω be a bounded domain in $\mathbb{R}^n$, $2 \leq n \leq 2m+1$, $Q \in \mathbb{R}^n \setminus \Omega$, $r > 0$ and assume that*

$$(7.23) \quad (-\Delta)^m u = f \text{ in } \Omega, \quad f \in C_0^\infty(B_{r/4}(Q) \cap \Omega), \quad u \in \mathring{W}^{m,2}(\Omega).$$

Then

$$(7.24) \quad \rho^{2\lambda+n+1-4m} \int_{S_\rho(Q)\cap\Omega} |u(x)|^2 \, d\sigma_x \leq C \, r^{2\lambda+n-4m} \int_{C_{r/4,r}(Q)\cap\Omega} |u(x)|^2 \, dx,$$

for any $\rho > r$ and λ given by (7.5).

Furthermore, for any $x \in \Omega \setminus B_{4r}(Q)$

$$(7.25) \quad |\nabla^i u(x)|^2 \leq C \, \frac{r^{2\lambda+n-4m}}{|x - Q|^{2\lambda+2n-4m+2i}} \int_{C_{r/4,4r}(Q)\cap\Omega} |u(y)|^2 \, dy, \qquad 0 \leq i \leq \lambda.$$

Proof. Without loss of generality, one can consider $Q = O$. Retain the approximation of Ω with the sequence of smooth domains Ω_n satisfying (7.8) and define u_n according to (7.9). We denote by $\mathcal{I}$ the inversion $x \mapsto y = x/|x|^2$ and by U_n the Kelvin transform of u_n,

$$(7.26) \quad U_n(y) := |y|^{2m-n} u_n(y/|y|^2), \quad y \in \mathcal{I}(\Omega_n).$$

Then

$$(7.27) \quad (-\Delta)^m U_n(y) = |y|^{-n-2m}((-\Delta)^m u_n)(y/|y|^2),$$

and therefore, U_n is polyharmonic in $\mathcal{I}(\Omega_n) \cap B_{4/r}$. Moreover, $U_n \in \mathring{W}^{m,2}(\mathcal{I}(\Omega_n))$ if and only if $u_n \in \mathring{W}^{m,2}(\Omega_n)$. Note also that Ω_n is a bounded domain with $O \notin \overline{\Omega_n}$, hence, so is $\mathcal{I}(\Omega_n)$ and $O \notin \overline{\mathcal{I}(\Omega_n)}$.

At this point we can invoke Proposition 7.2 to show that

$$(7.28) \quad \rho^{2\lambda+n-1} \int_{S_{1/\rho}} |U_n(y)|^2 \, d\sigma_y \leq C \, r^{2\lambda+n} \int_{C_{1/r,4/r}} |U_n(y)|^2 \, dy,$$

so that

$$(7.29) \quad \rho^{2\lambda+n+1-4m} \int_{S_\rho} |u_n(x)|^2 \, d\sigma_x \leq C \, r^{2\lambda+n-4m} \int_{C_{r/4,r}} |u_n(x)|^2 \, dx.$$

using the substitution (7.26) and the change of coordinates.

This furnishes the desired L^2 estimates. It remains to establish pointwise control. To this end, let us fix some $x \in \Omega \setminus B_{4r}(Q)$ and recall that

$$u_n(x) = |x|^{2m-n} U_n(x/|x|^2),$$

hence,

$$(7.30) \quad |\nabla^i u_n(x)| \le C \sum_{k=0}^{i} |x|^{2m-n-i-k} (\nabla^k U_n)(x/|x|^2).$$

Therefore, combining (7.30) and Proposition 7.2 applied to the function U_n, we deduce that

$$(7.31) \quad |\nabla^i u_n(x)|^2 \le C \frac{r^{n+2\lambda}}{|x|^{2\lambda+2n-4m+2i}} \int_{C_{1/(4r),4/r}} |U_n(z)|^2 \, dz = C \frac{r^{n+2\lambda-4m}}{|x|^{2\lambda+2n-4m+2i}} \int_{C_{r/4,4r}} |u_n(z)|^2 \, dz,$$

for all $0 \le i \le \lambda$. Now one can use the limiting procedure to complete the argument. Indeed, since u_n converges to u in $\mathring{W}^{m,2}(\Omega)$ and $i \le \lambda \le m$, the integrals on the right-hand side of (7.29) and (7.31) converge to the corresponding integrals with u_n replaced by u. Turning to $|\nabla^i u_n(x)|$, we observe that both u_n and u are m-harmonic in a neighborhood of x, in particular, for sufficiently small d

$$(7.32) \quad |\nabla^i (u_n(x) - u(x))|^2 \le \frac{C}{d^{n+2i}} \int_{B_{d/2}(x)} |u_n(z) - u(z)|^2 \, dz.$$

As $n \to \infty$, the integral on the right-hand side of (7.32) vanishes and therefore, $|\nabla^i u_n(x)| \to |\nabla^i u(x)|$. Similar considerations apply to $u_n(x)$ and the integral on the left-hand side of (7.29). □

8. Green's function estimates

The present section addresses sharp pointwise estimates on Green's function of the polyharmonic equation and the regular part of Green's function, that is, the difference between Green's function and the fundamental solution. We have discussed some of these bounds, as well as their applications for the solutions of the Dirichlet problem, in [23]. The estimates listed below are more refined compared to the statements in [23] and we present them here with full proofs.

To start, let us recall the definition of the fundamental solution for the polyharmonic equation (see. e.g., [7]). A fundamental solution for the m-Laplacian is a linear combination of the characteristic singular solution (defined below) and any m-harmonic function in $\mathbb{R}^n$. The characteristic singular solution is

$$(8.1) \qquad\qquad C_{m,n}|x|^{2m-n}, \qquad \text{if } n \text{ is odd, or if } n \text{ is even with } n \ge 2m + 2,$$

$$(8.2) \qquad\qquad C_{m,n}|x|^{2m-n} \log |x|, \qquad \text{if } n \text{ is even with } n \le 2m.$$

The exact expressions for constants $C_{m,n}$ can be found in [7], p.8. Hereafter we will use the fundamental solution given by

$$(8.3) \qquad \Gamma(x) = C_{m,n} \begin{cases} |x|^{2m-n}, & \text{if } n \text{ is odd}, \\ |x|^{2m-n} \log \frac{\operatorname{diam}\Omega}{|x|}, & \text{if } n \text{ is even and } n \le 2m, \\ |x|^{2m-n}, & \text{if } n \text{ is even and } n \ge 2m + 2. \end{cases}$$

As in the Introduction, by G we denote the Green function of the operator $(-\Delta)^m$ (see (1.6)) and S stands for its regular part, that is $S(x, y) = G(x, y) - \Gamma(x - y)$, $x, y \in \Omega$.

Theorem 8.1. *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary bounded domain, $m \in \mathbb{N}$, $n \in [2, 2m + 1] \cap \mathbb{N}$, and let λ retain the significance of (7.5). Fix any number $N \geq 25$. Then there exists a constant C depending on m, n, N only such that for every $x, y \in \Omega$ the following estimates hold.*

If $n \in [3, 2m + 1] \cap \mathbb{N}$ is odd then

$$(8.4) \qquad |\nabla_x^i \nabla_y^j G(x, y)| \leq C \frac{d(y)^{\lambda - j}}{|x - y|^{\lambda + n - 2m + i}}, \quad \text{when} \quad |x - y| \geq N d(y), \quad 0 \leq i, j \leq \lambda,$$

and

$$(8.5) \qquad |\nabla_x^i \nabla_y^j G(x, y)| \leq C \frac{d(x)^{\lambda - i}}{|x - y|^{\lambda + n - 2m + j}}, \quad \text{when} \quad |x - y| \geq N d(x), \quad 0 \leq i, j \leq \lambda.$$

Next,

$$(8.6) \qquad |\nabla_x^i \nabla_y^j G(x, y)| \leq \frac{C}{|x - y|^{n - 2m + i + j}},$$

$$\text{when } |x - y| \leq N^{-1} \max\{d(x), d(y)\}, \text{ and } i + j \geq 2m - n, \ 0 \leq i, j \leq m - n/2 + 1/2,$$

and

$$(8.7) \qquad |\nabla_x^i \nabla_y^j G(x, y)| \leq C \min\{d(x), d(y)\}^{2m - n - i - j},$$

$$\text{when } |x - y| \leq N^{-1} \max\{d(x), d(y)\}, \text{ and } i + j \leq 2m - n, \ 0 \leq i, j \leq m - n/2 + 1/2.$$

Finally,

$$(8.8) \qquad |\nabla_x^i \nabla_y^j G(x, y)| \leq \frac{C}{\min\{d(x), d(y), |x - y|\}^{n - 2m + i + j}} \approx \frac{C}{\max\{d(x), d(y), |x - y|\}^{n - 2m + i + j}},$$

$$\text{when } N^{-1} d(x) \leq |x - y| \leq N d(x) \text{ and } N^{-1} d(y) \leq |x - y| \leq N d(y), \quad 0 \leq i, j \leq \lambda.$$

Furthermore, if $n \in [3, 2m + 1] \cap \mathbb{N}$ is odd, the estimates on the regular part of the Green function S are as follows:

$$(8.9) \qquad |\nabla_x^i \nabla_y^j S(x - y)| \leq \frac{C}{|x - y|^{n - 2m + i + j}} \quad \text{when} \quad |x - y| \geq N \min\{d(x), d(y)\}, \quad 0 \leq i, j \leq \lambda.$$

Next,

$$(8.10) \qquad |\nabla_x^i \nabla_y^j S(x, y)| \leq \frac{C}{\max\{d(x), d(y)\}^{n - 2m + i + j}},$$

$$\text{when } |x - y| \leq N^{-1} \max\{d(x), d(y)\}, \text{ and } i + j \geq 2m - n, \ 0 \leq i, j \leq m - n/2 + 1/2,$$

and

$$(8.11) \qquad |\nabla_x^i \nabla_y^j S(x, y)| \leq C \min\{d(x), d(y)\}^{2m - n - i - j},$$

$$\text{when } |x - y| \leq N^{-1} \max\{d(x), d(y)\}, \text{ and } i + j \leq 2m - n, \ 0 \leq i, j \leq m - n/2 + 1/2.$$

Finally,

$$(8.12) \quad |\nabla_x^i \nabla_y^j S(x, y)| \leq \frac{C}{\min\{d(x), d(y), |x - y|\}^{n-2m+i+j}} \approx \frac{C}{\max\{d(x), d(y), |x - y|\}^{n-2m+i+j}},$$

$$\text{when } N^{-1} d(x) \leq |x-y| \leq Nd(x) \text{ and } N^{-1} d(y) \leq |x-y| \leq Nd(y), \quad 0 \leq i, j \leq m-n/2+1/2.$$

If $n \in [2, 2m] \cap \mathbb{N}$ is even, then (8.4)–(8.5) and (8.8) are valid with $\lambda = m - \frac{n}{2}$, and

$$(8.13) \quad |\nabla_x^i \nabla_y^j G(x, y)| \leq C \min\{d(x), d(y)\}^{2m-n-i-j} \left(C' + \log \frac{\min\{d(x), d(y)\}}{|x - y|} \right),$$

$$\text{when } |x - y| \leq N^{-1} \max\{d(x), d(y)\} \text{ and } 0 \leq i, j \leq m - n/2.$$

Furthermore, if $n \in [2, 2m] \cap \mathbb{N}$ is even, the estimates on the regular part of the Green function S are as follows:

$$(8.14) \quad |\nabla_x^i \nabla_y^j S(x - y)| \leq C |x - y|^{-n+2m-i-j} \left(C' + \log \frac{\operatorname{diam}(\Omega)}{|x - y|} \right)$$

$$\text{when} \quad |x - y| \geq N \min\{d(x), d(y)\}, \quad 0 \leq i, j \leq m - n/2.$$

Next,

$$(8.15) \quad |\nabla_x^i \nabla_y^j S(x, y)| \leq C \min\{d(x), d(y)\}^{2m-n-i-j} \left(C' + \log \frac{\operatorname{diam} \Omega}{\max\{d(x), d(y)\}} \right),$$

$$\text{when} \quad |x - y| \leq N^{-1} \max\{d(x), d(y)\}, \quad 0 \leq i, j \leq m - n/2.$$

Finally,

$$(8.16) \quad |\nabla_x^i \nabla_y^j S(x, y)| \leq C \min\{d(x), d(y), |x - y|\}^{2m-n-i-j} \left(C' + \log \frac{\operatorname{diam} \Omega}{\max\{d(x), d(y), |x - y|\}^{n-2m+i+j}} \right)$$

$$\text{when } N^{-1} d(x) \leq |x - y| \leq Nd(x) \text{ and } N^{-1} d(y) \leq |x - y| \leq Nd(y), \quad 0 \leq i, j \leq m - n/2.$$

Before passing to the proof of the Theorem, we would like to point out that the bounds on the highest order derivatives highlighted in Theorem 1.2 is a particular case of Theorem 8.1.

Proof of Theorem 8.1. Recall the definition of the fundamental solution Γ in (8.3). For any α, a multi-index of length less than or equal to λ, the function $\partial_y^\alpha \Gamma(x - y)$ can be written as

$$(8.17) \quad \partial_y^\alpha \Gamma(x - y) = P^\alpha(x - y) \log \frac{\operatorname{diam} \Omega}{|x - y|} + Q^\alpha(x - y).$$

Here, when the dimension is odd, $P^\alpha \equiv 0$ and Q^α is a homogeneous function of order $2m - n - |\alpha|$. If the dimension is even (and less than or equal to $2m$ by the assumptions of the theorem) then P^α and Q^α are homogeneous polynomials of order $2m - n - |\alpha|$ as long as $|\alpha| \leq 2m - n$. In both cases, P^α and Q^α do not depend in any way on the domain Ω.

Consider a function η such that

(8.18) $\eta \in C_0^\infty(B_{1/2})$ and $\eta = 1$ in $B_{1/4}$,

and define

(8.19) $\mathcal{R}_\alpha(x, y) := \partial_y^\alpha G(x, y) - \eta\left(\dfrac{x-y}{d(y)}\right)\left(-P^\alpha(x-y)\log\dfrac{|x-y|}{d(y)} + Q^\alpha(x-y)\right),$ $x, y \in \Omega.$

For every fixed $y \in \Omega$ the function $x \mapsto \mathcal{R}_\alpha(x, y)$ is a solution of the boundary value problem

(8.20) $(-\Delta_x)^m \mathcal{R}_\alpha(x, y) = f_\alpha(x, y)$ in Ω, $f_\alpha(\cdot, y) \in C_0^\infty(\Omega)$, $\mathcal{R}_\alpha(\cdot, y) \in \mathring{W}^{m,2}(\Omega),$

where

(8.21) $f_\alpha(x, y) := (-\Delta_x)^m \mathcal{R}_\alpha(x, y) = -\left[(-\Delta_x)^m, \eta\left(\dfrac{x-y}{d(y)}\right)\right]\left(P^\alpha(x-y)\log\dfrac{|x-y|}{d(y)} + Q^\alpha(x-y)\right).$

Indeed, it is not hard to see that for every α

(8.22) $f_\alpha(\cdot, y) \in C_0^\infty(C_{d(y)/4, d(y)/2}(y))$ and $|f_\alpha(x, y)| \le Cd(y)^{-n-|\alpha|}$, $x, y \in \Omega,$

so that, in particular, $f_\alpha(\cdot, y) \in C_0^\infty(\Omega)$ as stated in (8.20). Furthermore, by (8.21),

(8.23) $\left\|\nabla_x^m \mathcal{R}_\alpha(\cdot, y)\right\|_{L^2(\Omega)} = \|\mathcal{R}_\alpha(\cdot, y)\|_{W^{m,2}(\Omega)} \le C\|f_\alpha(\cdot, y)\|_{W^{-m,2}(\Omega)},$

where $W^{-m,2}(\Omega)$ stands for the Banach space dual of $\mathring{W}^{m,2}(\Omega)$, i.e.

(8.24) $\|f_\alpha(\cdot, y)\|_{W^{-m,2}(\Omega)} = \sup_{v \in \mathring{W}^{m,2}(\Omega): \|v\|_{\mathring{W}^{m,2}(\Omega)} = 1} \int_\Omega f_\alpha(x, y)v(x)\,dx.$

Recall that by Hardy's inequality

(8.25) $\left\|\dfrac{v}{|\cdot - Q|^m}\right\|_{L^2(\Omega)} \le C\,\|\nabla^m v\|_{L^2(\Omega)}$ for every $v \in \mathring{W}^{m,2}(\Omega)$, $Q \in \partial\Omega.$

Then by (8.24)

(8.26) $\|f_\alpha(\cdot, y)\|_{W^{-m,2}(\Omega)} = \sup_{v \in \mathring{W}^{m,2}(\Omega): \|v\|_{\mathring{W}^{m,2}(\Omega)} = 1} \int_\Omega f_\alpha(x, y)v(x)\,dx$

$\le C \sup_{v \in \mathring{W}^{m,2}(\Omega): \|v\|_{\mathring{W}^{m,2}(\Omega)} = 1} \left\|\dfrac{v}{|\cdot - y_0|^m}\right\|_{L^2(\Omega)} \|f_\alpha(\cdot, y)\,|\cdot - y_0|^m\|_{L^2(\Omega)}$

$\le C \sup_{v \in \mathring{W}^{m,2}(\Omega): \|v\|_{\mathring{W}^{m,2}(\Omega)} = 1} d(y)^m \|\nabla^m v\|_{L^2(\Omega)} \|f_\alpha(\cdot, y)\|_{L^2(C_{d(y)/4, d(y)/2}(y))}$

$\le Cd(y)^m\|f_\alpha(\cdot, y)\|_{L^2(C_{d(y)/4, d(y)/2}(y))},$

where $y_0 \in \partial\Omega$ is such that $|y - y_0| = d(y)$. Therefore, by (8.22)

(8.27) $\left\|\nabla_x^m \mathcal{R}_\alpha(\cdot, y)\right\|_{L^2(\Omega)} \le Cd(y)^{m-|\alpha|-n/2}.$

Now we split the discussion into a few cases.

Case I: either $|x - y| \geq N d(y)$ or $|x - y| \geq N d(x)$ for some $N \geq 25$.

Let us first assume that $|x - y| \geq Nd(y)$, $N \geq 25$. In this case we have $\eta\left(\frac{x-y}{d(y)}\right) = 0$ and hence,

$$(8.28) \quad \nabla_x^i \mathcal{R}_\alpha(x, y) = \nabla_x^i \partial_y^\alpha G(x, y), \quad \text{for } x, y \in \Omega \text{ such that } |x - y| \geq N d(y), \quad 0 \leq i, |\alpha| \leq \lambda.$$

As before, we denote by y_0 some point on the boundary such that $|y - y_0| = d(y)$. Then by (8.20), (8.22) the function $x \mapsto \mathcal{R}_\alpha(x, y)$ is m-harmonic in $\Omega \setminus B_{3d(y)/2}(y_0)$. Hence, by Proposition 7.3 with $r = 6d(y)$

$$(8.29) \quad |\nabla_x^i \mathcal{R}_\alpha(x, y)|^2 \leq C \frac{d(y)^{2\lambda+n-4m}}{|x - y_0|^{2\lambda+2n-4m+2i}} \int_{C_{3d(y)/2, 24d(y)}(y_0)} |\mathcal{R}_\alpha(z, y)|^2 \, dz, \quad \text{for } \quad 0 \leq i \leq \lambda,$$

provided $|x - y_0| \geq 4r = 24 \, d(y)$. The latter condition is automatically satisfied for x such that $|x - y| \geq N \, d(y)$ with $N \geq 25$, which is the current assumption on x, y. The right-hand side of (8.29) is bounded by

$$C \frac{d(y)^{2\lambda+n-2m}}{|x - y_0|^{2\lambda+2n-4m+2i}} \int_{C_{3d(y)/2, 24d(y)}(y_0)} \frac{|\mathcal{R}_\alpha(z, y)|^2}{|z - y_0|^{2m}} \, dz$$

$$(8.30) \qquad \leq C \frac{d(y)^{2\lambda+n-2m}}{|x - y_0|^{2\lambda+2n-4m+2i}} \int_\Omega |\nabla_z^m \mathcal{R}_\alpha(z, y)|^2 \, dz \leq C \frac{d(y)^{2\lambda-2|\alpha|}}{|x - y|^{2\lambda+2n-4m+2i}},$$

by Hardy's inequality and (8.27). Therefore,

$$(8.31) \quad |\nabla_x^i \mathcal{R}_\alpha(x, y)|^2 \leq C \frac{d(y)^{2\lambda-2|\alpha|}}{|x - y|^{2\lambda+2n-4m+2i}}, \quad \text{when} \quad |x - y| \geq N \, d(y), \quad 0 \leq i, |\alpha| \leq \lambda.$$

By (8.28), the estimate (8.31) with $j := |\alpha|$ implies (8.4). Also, by the symmetry of Green's function we automatically deduce (8.5). This furnishes the desired estimates on Green's function when either $|x - y| \geq N \, d(y)$ or $|x - y| \geq N \, d(x)$, both in the case when n is odd and when n is even.

Turning to the estimates on the regular part of Green's function, we observe that, in particular, (8.4) and (8.5) combined give the estimate

$$(8.32) \quad |\nabla_x^i \nabla_y^j G(x, y)|^2 \leq \frac{C}{|x - y|^{2n-4m+2i+2j}}, \quad \text{when} \quad |x - y| \geq N \min\{d(x), d(y)\}, \quad 0 \leq i, j \leq \lambda.$$

Furthermore, if n is odd, then

$$(8.33) \quad |\nabla_x^i \nabla_y^j \Gamma(x - y)| \leq \frac{C}{|x - y|^{n-2m+i+j}} \qquad \text{for all} \qquad x, y \in \Omega, \quad i, j \geq 0,$$

while if n is even, then

$$(8.34) \quad |\nabla_x^i \nabla_y^j \Gamma(x - y)| \leq C_1 |x - y|^{-n+2m-i-j} \log \frac{\text{diam}(\Omega)}{|x - y|} + C_2 |x - y|^{-n+2m-i-j} \quad \text{if } 0 \leq i + j \leq 2m - n,$$

for all $x, y \in \Omega$.

Combining this with (8.32) we deduce (8.9) and (8.14) in the cases when the dimension is odd and even, respectively.

Case II: either $|x - y| \leq N^{-1}d(y)$ or $|x - y| \leq N^{-1}d(x)$ for some $N \geq 25$.

Assume that $|x - y| \le N^{-1} d(y)$. For such x we have $\eta\left(\frac{x-y}{d(y)}\right) = 1$ and therefore

$$(8.35) \quad \mathcal{R}_\alpha(x, y) = \partial_y^\alpha G(x, y) + P^\alpha(x - y) \log \frac{|x - y|}{d(y)} - Q^\alpha(x - y).$$

Hence, if n is odd,

$$(8.36) \quad \mathcal{R}_\alpha(x, y) = \partial_y^\alpha (G(x, y) - \Gamma(x - y)), \quad \text{when} \quad |x - y| \le N^{-1} d(y),$$

and if n is even,

$$(8.37) \quad \mathcal{R}_\alpha(x, y) = \partial_y^\alpha (G(x, y) - \Gamma(x - y)) + P^\alpha(x - y) \log \frac{\operatorname{diam}\Omega}{d(y)}, \quad \text{when} \quad |x - y| \le N^{-1} d(y).$$

By the interior estimates for solutions of elliptic equations for any $i \le m$

$$(8.38) \quad |\nabla_x^i \mathcal{R}_\alpha(x, y)|^2 \le \frac{C}{d(y)^{n+2i}} \int_{B_{d(y)/8}(x)} |\mathcal{R}(z, y)|^2 \, dz,$$

since by (8.22) for any fixed y the function $\mathcal{R}_\alpha(\cdot, y)$ is m-harmonic in $B_{d(y)/4}(y) \supset B_{d(y)/8}(x)$. Next, since for every $z \in B_{d(y)/8}(x) \subset B_{d(y)/4}(y)$ and any $y_0 \in \partial\Omega$ such that $|y - y_0| = d(y)$ we have $|z - y_0| \approx d(y)$, one can bound the expression above by

$$(8.39) \quad \frac{C}{d(y)^{n+2i-2m}} \int_{B_{d(y)/4}(y)} \frac{|\mathcal{R}(z, y)|^2}{|z - y_0|^{2m}} \, dz \le \frac{C}{d(y)^{n+2i-2m}} \left\| \nabla_x^m \mathcal{R}(\cdot, y) \right\|_{L^2(\Omega)}^2 \le \frac{C}{d(y)^{2n-4m+2i+2|\alpha|}},$$

with $0 \le |\alpha| \le \lambda$. Then, overall,

$$(8.40) \quad |\nabla_x^i \mathcal{R}_\alpha(x, y)|^2 \le \frac{C}{d(y)^{2n-4m+2i+2|\alpha|}}, \quad |x - y| \le N^{-1} d(y), \quad 0 \le |\alpha| \le \lambda.$$

Having this at hand, we turn to the estimates on the regular part of Green's function, starting with the case when n is odd. It follows from (8.36) and (8.40) that

$$(8.41) \quad |\nabla_x^i \nabla_y^j S(x, y)| \le \frac{C}{d(y)^{n-2m+i+j}}, \quad 0 \le i \le m, \quad 0 \le j \le \lambda, \quad |x - y| \le N^{-1} d(y).$$

and hence, by symmetry,

$$(8.42) \quad |\nabla_x^i \nabla_y^j S(x, y)| \le \frac{C}{d(x)^{n-2m+i+j}}, \quad 0 \le i \le \lambda, \quad 0 \le j \le m, \quad |x - y| \le N^{-1} d(x).$$

However, we have

$$(8.43) \quad |x - y| \le N^{-1} d(y) \quad \Longrightarrow \quad (N - 1) \, d(y) \le N d(x) \le (N + 1) \, d(y),$$

i.e. $d(y) \approx d(x) \approx \min\{d(y), d(x)\} \approx \max\{d(y), d(x)\}$ whenever $|x-y|$ is less than or equal to $N^{-1} d(y)$ or $N^{-1} d(x)$. Therefore, when the dimension is odd, we obtain (8.10) and (8.11) depending on the range of i, j.

Moreover, by (8.33), for the Green function itself, we then arrive at (8.6), (8.7), once again depending on the range of i, j.

Similar considerations apply to the case when the dimension is even, leading by (8.37) to the following results:

$$(8.44) \quad |\nabla_x^i \nabla_y^j S(x,y)| \leq C \, d(y)^{2m-n-i-j} \left(C' + \log \frac{\operatorname{diam} \Omega}{d(y)} \right),$$

$$\text{for } 0 \leq i \leq m, \ 0 \leq j \leq m - \frac{n}{2}, \ |x-y| \leq N^{-1} d(y),$$

and

$$(8.45) \quad |\nabla_x^i \nabla_y^j S(x,y)| \leq C \, d(x)^{2m-n-i-j} \left(C' + \log \frac{\operatorname{diam} \Omega}{d(x)} \right),$$

$$\text{for } 0 \leq i \leq m - \frac{n}{2}, \ 0 \leq j \leq m, \ |x-y| \leq N^{-1} d(x).$$

Overall, in view of (8.43), and the fact that $2m - n - i - j \geq 0$ whenever $0 \leq i,j \leq m - \frac{n}{2}$, (8.44)–(8.45) yield (8.15).

As for the Green's function estimates, when n is even, (8.35) and (8.40) lead to the bounds

$$(8.46) \quad |\nabla_x^i \nabla_y^j G(x,y)| \leq C \, d(y)^{2m-n-i-j} \left(C' + \log \frac{d(y)}{|x-y|} \right), \ 0 \leq i,j \leq m - \frac{n}{2}, \ |x-y| \leq N^{-1} d(y),$$

$$(8.47) \quad |\nabla_x^i \nabla_y^j G(x,y)| \leq C \, d(x)^{2m-n-i-j} \left(C' + \log \frac{d(x)}{|x-y|} \right), \ 0 \leq i,j \leq m - \frac{n}{2}, \ |x-y| \leq N^{-1} d(x),$$

and therefore, one obtains (8.13).

Finally, it remains to consider

Case III: $|x-y| \approx d(y) \approx d(x)$, or more precisely, the situation when

$$(8.48) \quad N^{-1} d(x) \leq |x-y| \leq N d(x) \quad \text{and} \quad N^{-1} d(y) \leq |x-y| \leq N d(y) \quad \text{for some} \quad N \geq 25.$$

Fix any x, y as in (8.48). Then the mapping $z \mapsto G(z,y)$ is m-harmonic in $B_{d(x)/(2N)}(x)$. Assume first that n is odd. Then by the interior estimates, with $x_0 \in \partial\Omega$ such that $|x - x_0| = d(x)$, we have

$$\begin{aligned}
|\nabla_x^i \nabla_y^{|\alpha|} G(x,y)|^2 \ &\leq \ \frac{C}{d(x)^{n+2i}} \int_{B_{d(x)/(2N)}(x)} |\nabla_y^{|\alpha|} G(z,y)|^2 \, dz \\
(8.49) \qquad &\leq \ \frac{C}{d(x)^{n+2i}} \int_{B_{d(x)/(2N)}(x)} |\nabla_y^{|\alpha|} \Gamma(z-y)|^2 \, dz + \frac{C}{d(x)^{n+2i-2m}} \int_{B_{2d(x)}(x_0)} \frac{|\mathcal{R}_\alpha(z,y)|^2}{|z-x_0|^{2m}} \, dz
\end{aligned}$$

by definition (8.19). Then the expression above is bounded by

$$\frac{C}{d(x)^{n+2i}} \int_{B_{d(x)/(2N)}(x)} |\nabla_y^{|\alpha|} \Gamma(z-y)|^2 \, dz + \frac{C}{d(x)^{n+2i-2m}} \int_\Omega |\nabla_z^m \mathcal{R}_\alpha(z,y)|^2 \, dz$$

$$(8.50) \qquad \leq \frac{C}{d(x)^{2n-4m+2i+2|\alpha|}} + \frac{C}{d(x)^{n-2m+2i} d(y)^{n-2m+2|\alpha|}},$$

provided that $0 \leq i \leq m$, $0 \leq |\alpha| \leq m - \frac{n}{2} + \frac{1}{2}$.

The same estimate on derivatives of Green's function holds when n is even, since

$$|\nabla_x^i \nabla_y^{|\alpha|} G(x,y)|^2 \leq \frac{C}{d(x)^{n+2i}} \int_{B_{d(x)/(2N)}(x)} |\nabla_y^{|\alpha|} G(z,y)|^2 \, dz \leq \frac{C}{d(x)^{n+2i}} \int_{B_{d(x)/(2N)}(x)} |P^\alpha(z-y)|^2 \, dz$$

$$(8.51) \qquad + \frac{C}{d(x)^{n+2i}} \int_{B_{d(x)/(2N)}(x)} |Q^\alpha(z-y)|^2 \, dz + \frac{C}{d(x)^{n+2i-2m}} \int_{B_{2d(x)}(x_0)} \frac{|\mathcal{R}_\alpha(z,y)|^2}{|z-x_0|^{2m}} \, dz,$$

since the absolute value of $\log \frac{|z-y|}{d(y)}$ is bounded by a constant for z, x, y as in (8.51), (8.48). This yields (8.8), both in the case when n is even and odd.

Furthermore, if n is odd, the same argument (or simply an estimate on the difference between the Green function and S, that is, the fundamental solution) can be used to deduce

$$(8.52) \quad |\nabla_x^i \nabla_y^j S(x,y)| \leq \frac{C}{\min\{d(x), d(y), |x-y|\}^{n-2m+i+j}} \approx \frac{C}{\max\{d(x), d(y), |x-y|\}^{n-2m+i+j}},$$

$$\text{when } x, y \text{ satisfy } (8.48), \quad 0 \leq i, j \leq m - n/2 + 1/2,$$

which gives (8.12). However, if n is even, using (8.48), we are led to the bound

$$|\nabla_x^i \nabla_y^j S(x,y)| \leq \frac{C}{\min\{d(x), d(y), |x-y|\}^{n-2m+i+j}} \left(C' + \log \frac{\operatorname{diam} \Omega}{\max\{d(x), d(y), |x-y|\}^{n-2m+i+j}} \right)$$

$$(8.53) \qquad \approx \frac{C}{\max\{d(x), d(y), |x-y|\}^{n-2m+i+j}} \left(C' + \log \frac{\operatorname{diam} \Omega}{\max\{d(x), d(y), |x-y|\}^{n-2m+i+j}} \right),$$

for $0 \leq i, j \leq m - \frac{n}{2}$. This establishes (8.16) and finishes the argument. $\square$

9. Estimates for solutions of the Dirichlet problem

The results of Section 7 provide certain local monotonicity estimates for solutions of the Dirichlet problem. They can, in principle, be translated into the bounds on solution u in terms of data f when f is a suitably supported C_0^∞ function. However, the pointwise control of the Green function that we obtained in Section 8 allows us to estimate the solutions of the Dirichlet problem for a wide, in some sense, optimal, class of data.

Proposition 9.1. *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary bounded domain, $m \in \mathbb{N}$, $n \in [2, 2m+1] \cap \mathbb{N}$, and let λ retain the significance of (7.5). Consider the boundary value problem*

$$(9.1) \qquad (-\Delta)^m u = \sum_{|\alpha| \leq \lambda} c_\alpha \partial^\alpha f_\alpha, \quad u \in \mathring{W}^{m,2}(\Omega).$$

Then the solution satisfies the following estimates.

If $n \in [3, 2m+1] \cap \mathbb{N}$ is odd, then

$$(9.2) \qquad |\nabla^{m-\frac{n}{2}+\frac{1}{2}} u(x)| \leq C_{m,n} \sum_{|\alpha| \leq m-\frac{n}{2}+\frac{1}{2}} \int_\Omega \frac{d(y)^{m-\frac{n}{2}+\frac{1}{2}-|\alpha|}}{|x-y|} |f_\alpha(y)| \, dy, \qquad x \in \Omega,$$

whenever the integrals on the right-hand side of (9.2) *are finite. In particular,*

$$(9.3) \qquad \|\nabla^{m-\frac{n}{2}+\frac{1}{2}}u\|_{L^\infty(\Omega)} \ \leq \ C_{m,n,\Omega} \sum_{|\alpha|\leq m-\frac{n}{2}+\frac{1}{2}} \|d(\cdot)^{m-\frac{n}{2}-\frac{1}{2}-|\alpha|}f_\alpha\|_{L^p(\Omega)}, \ \ p > \frac{n}{n-1},$$

provided that the norms on the right-hand side of (9.3) *are finite.*

If $n \in [2, 2m] \cap \mathbb{N}$ is even, then

$$(9.4) \qquad |\nabla^{m-\frac{n}{2}}u(x)| \ \leq \ C_{m,n} \sum_{|\alpha|\leq m-\frac{n}{2}} \int_\Omega d(y)^{m-\frac{n}{2}-|\alpha|} \log\left(1 + \frac{d(y)}{|x-y|}\right) |f_\alpha(y)| \, dy, \qquad x \in \Omega,$$

whenever the integrals on the right-hand side of (9.4) *are finite. In particular,*

$$(9.5) \qquad \|\nabla^{m-\frac{n}{2}}u\|_{L^\infty(\Omega)} \ \leq \ C_{m,n,\Omega} \sum_{|\alpha|\leq m-\frac{n}{2}} \|d(\cdot)^{m-\frac{n}{2}-|\alpha|}f_\alpha\|_{L^p(\Omega)}, \ \ p > 1,$$

provided that the norms on the right-hand side of (9.5) *are finite.*

The constants $C_{m,n}$ above depend on m and n only, while the constants denoted by $C_{m,n,\Omega}$ depend on m, n, and the diameter of the domain Ω.

Proof. First of all, the integral representation formula

$$(9.6) \qquad u(x) = \int_\Omega G(x,y) \sum_{|\alpha|\leq\lambda} c_\alpha \partial^\alpha f_\alpha(y) \, dy, \qquad x \in \Omega,$$

follows directly from the definition of Green's function. It implies that

$$(9.7) \qquad \nabla^\lambda u(x) \ = \ \sum_{|\alpha|\leq\lambda} c_\alpha(-1)^{|\alpha|} \int_\Omega \nabla_x^\lambda \partial_y^\alpha G(x,y) \, f_\alpha(y) \, dy \qquad x \in \Omega.$$

Let us now assume that $n \in [3, 2m+1] \cap \mathbb{N}$ is odd. We claim that in this case

$$(9.8) \qquad \left|\nabla_x^{m-\frac{n}{2}+\frac{1}{2}}\partial_y^\alpha G(x,y)\right| \leq C \frac{d(y)^{m-\frac{n}{2}+\frac{1}{2}-|\alpha|}}{|x-y|}, \qquad \text{for all } x, y \in \Omega, \quad 0 \leq |\alpha| \leq m - \frac{n}{2} + \frac{1}{2}.$$

Indeed, when $|\alpha| = m-\frac{n}{2}+\frac{1}{2}$, (9.8) is equivalent to (1.8). Thus, it remains consider $|\alpha| \leq m-\frac{n}{2}-\frac{1}{2}$. Now we split the cases, essentially according to Theorem 8.1. When $|x - y| \geq Nd(y)$, for some $N \geq 25$, (8.4) directly yields (9.8). If $x, y \in \Omega$ such that $|x - y| \leq N^{-1}d(y)$ then (9.8) follows from (8.7). Finally, if $N^{-1}d(y) \leq |x - y| \leq Nd(y)$, we observe that $d(x) \leq |x - y| + d(y) \leq (1 + N)|x - y|$. The latter estimate shows that (8.5)–(8.8) imply the bound on the left-hand side of (9.8) by $C |x - y|^{m-\frac{n}{2}-\frac{1}{2}-|\alpha|}$, and hence, (9.8), since here $d(y) \approx |x - y|$.

Now (9.8) allows us to deduce (9.2), and (9.3) follows from it via the mapping properties of the Riesz potential.

Let us now turn to the case when $n \in [2, 2m] \cap \mathbb{N}$ is even. Then we have to show that

$$(9.9) \quad \left|\nabla_x^{m-\frac{n}{2}}\partial_y^\alpha G(x,y)\right| \leq C \, d(y)^{m-\frac{n}{2}-|\alpha|} \log\left(1 + \frac{d(y)}{|x-y|}\right), \ \ \text{for all } x, y \in \Omega, \ 0 \leq |\alpha| \leq m - \frac{n}{2}.$$

Once again, we split to cases according to Theorem 8.1. When $|x-y| \geq Nd(y)$, for some $N \geq 25$, then $1 + \frac{d(y)}{|x-y|}$ is bounded from below and above by a positive constant, so that (8.4) is the same as

(9.9). If $x, y \in \Omega$ are such that $|x - y| \leq N^{-1}d(y)$ then (9.9) follows directly from (8.13). Finally, if $N^{-1}d(y) \leq |x - y| \leq Nd(y)$, then, as before, $d(x) \leq (1 + N)|x - y|$ and $1 + \frac{d(y)}{|x-y|}$ is bounded from below and above by a positive constant. Then (8.5), (8.13), and (8.8) imply the bound on the left-hand side of (9.9) by $C|x - y|^{m-\frac{n}{2}-|\alpha|}$, and hence, (9.9), using that $d(y) \approx |x - y|$. This finishes the proof of (9.4).

Finally, we obtain from (9.4) the bound

$$(9.10) \quad |\nabla^{m-\frac{n}{2}}u(x)| \;\leq\; C \sum_{|\alpha|\leq m-\frac{n}{2}} \int_\Omega d(y)^{m-\frac{n}{2}-|\alpha|} \left(\frac{d(y)}{|x-y|}\right)^\varepsilon |f_\alpha(y)|\,dy, \qquad x \in \Omega. \quad \varepsilon > 0,$$

Then, by the mapping properties of the Riesz potential we recover an estimate

$$(9.11) \|\nabla^{m-\frac{n}{2}}u\|_{L^\infty(\Omega)} \;\leq\; C_\Omega \sum_{|\alpha|\leq m-\frac{n}{2}} \left\|d(y)^{m-\frac{n}{2}-|\alpha|+\varepsilon}f_\alpha\right\|_{L^p(\Omega)}, \quad \text{provided that } \varepsilon > 0, \; p > \frac{n}{n-\varepsilon},$$

which, in turn, leads to (9.5). $\qquad\qquad\square$

REFERENCES

[1] D.R. Adams, *L^p potential theory techniques and nonlinear PDE*, Potential theory (Nagoya, 1990), 1–15, de Gruyter, Berlin, 1992. 1

[2] V. Adolfsson, J. Pipher, *The inhomogeneous Dirichlet problem for Δ^2 in Lipschitz domains*, J. Funct. Anal. 159 (1998), no. 1, 137–190. 2

[3] S. Agmon, *Maximum theorems for solutions of higher order elliptic equations*, Bull. Amer. Math. Soc. 66 (1960), 77–80. 2

[4] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I*, Comm. Pure Appl. Math., 12 (1959), 623–727. 39

[5] S.S. Antman, *Nonlinear problems of elasticity.* Second edition. Applied Mathematical Sciences, 107. Springer, New York, 2005. 1

[6] I. Babuška, *The theory of small changes in the domain of existence in the theory of partial differential equations and its applications.* 1963 Differential Equations and Their Applications (Proc. Conf., Prague, 1962) pp. 13–26. 2

[7] N. Aronszajn, T. Creese, L. Lipkin, *Polyharmonic functions*, Notes taken by Eberhard Gerlach. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1983. 41

[8] S.-Y. A. Chang, *Conformal invariants and partial differential equations*, Bull. Amer. Math. Soc. (N.S.) 42 (2005), no. 3, 365–393 (electronic). 1

[9] S.-Y. A. Chang, P.C. Yang, *Non-linear partial differential equations in conformal geometry,* Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002), 189–207, Higher Ed. Press, Beijing, 2002. 1

[10] Ph. Ciarlet, *Mathematical elasticity.* Vol. II. Theory of plates. Studies in Mathematics and its Applications, 27. North-Holland Publishing Co., Amsterdam, 1997. 1

[11] B. Dahlberg, C. Kenig, G. Verchota, *The Dirichlet problem for the biharmonic equation in a Lipschitz domain*, Ann. Inst. Fourier (Grenoble) 36 (1986), no. 3, 109–135. 2

[12] B. E. J. Dahlberg, C. E. Kenig, J. Pipher, and G. C. Verchota, *Area integral estimates for higher order elliptic equations and systems*, Ann. Inst. Fourier (Grenoble) 47 (1997), no. 5, 1425–1461. 2

[13] A. Dall'Acqua, G. Sweers, *Estimates for Green function and Poisson kernels of higher-order Dirichlet boundary value problems*, J. Differential Equations 205 (2004), no. 2, 466–487. 4

[14] R.J. Duffin, *On a question of Hadamard concerning super-biharmonic functions*, J. Math. Physics 27, (1949), 253–258. 1

[15] P.R. Garabedian, *A partial differential equation arising in conformal mapping*, Pacific J. Math. 1, (1951), 485–524. 1

[16] J. Hadamard, *Mémoire sur le problème d'analyse relatif à l'équilibre des plaques élastiques encastrées*, Mémoires présentés par divers savants à l'Académie des Sciences, (2) 33 (1908). 1

[17] C. Kenig, *Harmonic analysis techniques for second order elliptic boundary value problems,* CBMS Regional Conference Series in Mathematics, 83. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1994. 2

[18] J. Kilty and Z. Shen, *A bilinear estimate for biharmonic functions in Lipschitz domains*, Math. Ann. 349 (2011), no. 2, 367–394. 2

[19] V. Kozlov, V. Maz'ya, J. Rossmann, *Elliptic boundary value problems in domains with point singularities.* Mathematical Surveys and Monographs, 52. American Mathematical Society, Providence, RI, 1997. 2

[20] V. Kozlov, V. Maz'ya, J. Rossmann, *Spectral problems associated with corner singularities of solutions to elliptic equations*, Mathematical Surveys and Monographs, 85. American Mathematical Society, Providence, RI, 2001. 2, 3, 4

[21] Ju.P. Krasovskiĭ, *Isolation of the singularity in Green's function*, Izv. Akad. Nauk SSSR Ser. Mat. 31 (1967), 977–1010. 4

[22] S. Mayboroda, V. Maz'ya, *Boundedness of the gradient of a solution and Wiener test of order one for the biharmonic equation*, Invent. Math. 175 (2009), no. 2, 287–334. 4

[23] S. Mayboroda, V. Maz'ya, *Pointwise estimates for the polyharmonic Green function in general domains*, Cialdea, Alberto (ed.) et al., Analysis, partial differential equations and applications. The Vladimir Maz'ya anniversary volume. Selected papers of the international workshop, Rome, Italy, June 30–July 3, 2008. Basel: BirkhŁuser. Operator Theory: Advances and Applications 193, 143-158 (2009). 41

[24] V. Maz'ya, *Behaviour of solutions to the Dirichlet problem for the biharmonic operator at a boundary point*, Equadiff IV (Proc. Czechoslovak Conf. Differential Equations and their Applications, Prague, 1977), pp. 250–262, Lecture Notes in Math., 703, Springer, Berlin, 1979. 4

[25] V. Maz'ya, *On Wiener's type regularity of a boundary point for higher order elliptic equations*, Nonlinear analysis, function spaces and applications, Vol. 6 (Prague, 1998), Acad. Sci. Czech Repub., Prague, 1999, pp. 119–155. 4

[26] V. Maz'ya, *The Wiener test for higher order elliptic equations.*, Duke Math. J. 115 (2002), no. 3, 479–512.

[27] V. Maz'ya, M. Mitrea, and T. Shaposhnikova, *The Dirichlet problem in Lipschitz domains for higher order elliptic systems with rough coefficients*, J. Anal. Math. 110 (2010), 167–239. 2

[28] V. Maz'ya, S.A. Nazarov, *Paradoxes of the passage to the limit in solutions of boundary value problems for the approximation of smooth domains by polygons.* Izv. Akad. Nauk SSSR Ser. Mat. 50 (1986), no. 6, 1156–1177, 1343. 2

[29] V. Maz'ya, S.A. Nazarov, and B.A. Plamenevskiĭ, *Singularities of solutions of the Dirichlet problem in the exterior of a thin cone*, Mat. Sb. (N.S.) 122 (164) (1983), no. 4, 435–457. English translation: Math. USSR-Sb. 50 (1985), no. 2, 415–437. 2

[30] V. Maz'ya, B.A. Plamenevskiĭ, *Asymptotic behavior of the fundamental solutions of elliptic boundary value problems in domains with conical points*, (Russian) Boundary value problems. Spectral theory (Russian), pp. 100–145, 243, Probl. Mat. Anal., 7, Leningrad. Univ., Leningrad, 1979. 4

[31] V. Maz'ya, J. Rossmann, *On the Agmon-Miranda maximum principle for solutions of elliptic equations in polyhedral and polygonal domains*, Ann. Global Anal. Geom. 9 (1991), no. 3, 253–303. 2, 4

[32] V. Maz'ya, J. Rossmann, *On the Agmon-Miranda maximum principle for solutions of strongly elliptic equations in domains of $\mathbb{R}^n$ with conical points*, Ann. Global Anal. Geom. 10 (1992), no. 2, 125–150. 2

[33] V. Maz'ya, J. Rossmann, *Elliptic equations in polyhedral domains.* Mathematical Surveys and Monographs, 162. American Mathematical Society, Providence, RI, 2010. viii+608 pp. 2

[34] V.V. Meleshko, *Selected topics in the history of the two-dimensional biharmonic problem*, Appl. Mech. Rev. 56 (2003), 33–85. 1

[35] I. Mitrea, *Mapping properties of layer potentials associated with higher-order elliptic operators in Lipschitz domains*, Topics in operator theory. Volume 2. Systems and mathematical physics, Oper. Theory Adv. Appl., vol. 203, Birkhäuser Verlag, Basel, 2010, pp. 363–407. 2

[36] I. Mitrea, M. Mitrea, and M. Wright, *Optimal estimates for the inhomogeneous problem for the bi-laplacian in three-dimensional Lipschitz domains*, J. Math. Sci. 172 (2011), no. 1, 24–134. 2

[37] C. Miranda, *Formule di maggiorazione e teorema di esistenza per le funzioni biarmoniche de due variabili*, Giorn. Mat. Battaglini (4) 2(78) (1948), 97–118. 2

[38] C. Miranda, *Teorema del massimo modulo e teorema di esistenza e di unicità per il problema di Dirichlet relativo alle equazioni ellittiche in due variabili*, Ann. Mat. Pura Appl. (4) 46 (1958), 265–311. 2

[39] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Masson et Cie, Éditeurs, Paris; Academia, Éditeurs, Prague 1967. 2, 38

[40] J. Pipher and G. C. Verchota, *Area integral estimates for the biharmonic operator in Lipschitz domains*, Trans. Amer. Math. Soc. 327 (1991), no. 2, 903–917. 2

[41] J. Pipher, G. Verchota, *The Dirichlet problem in L^p for the biharmonic equation on Lipschitz domains*, Amer. J. Math. 114 (1992), no. 5, 923–972. 2

[42] J. Pipher, G. Verchota, *A maximum principle for biharmonic functions in Lipschitz and C^1 domains*, Comment. Math. Helv. 68 (1993), no. 3, 385–414. 2

[43] J. Pipher, G. Verchota, *Dilation invariant estimates and the boundary Gårding inequality for higher order elliptic operators*, Ann. of Math. (2) 142 (1995), no. 1, 1–38. 2

[44] J. Pipher, G. Verchota, *Maximum principles for the polyharmonic equation on Lipschitz domains*, Potential Anal. 4 (1995), no. 6, 615–636. 2

[45] H.S. Shapiro, M. Tegmark, *An elementary proof that the biharmonic Green function of an eccentric ellipse changes sign*, SIAM Rev. 36 (1994), no. 1, 99–101. 1

[46] Z. Shen, *The L^p Dirichlet problem for elliptic systems on Lipschitz domains*, Math. Res. Lett. 13 (2006), no. 1, 143–159. 2

[47] Z. Shen, *Necessary and sufficient conditions for the solvability of the L^p Dirichlet problem on Lipschitz domains*, Math. Ann. 336 (2006), no. 3, 697–725. 2

[48] Z. Shen, *On estimates of biharmonic functions on Lipschitz and convex domains*, J. Geom. Anal. 16 (2006), no. 4, 721–734. 2

[49] Z. Shen, *The L^p boundary value problems on Lipschitz domains*, Adv. Math. 216 (2007), no. 1, 212–254. 2

[50] I.V. Skrypnik, *Methods for analysis of nonlinear elliptic boundary value problems.* Translated from the 1990 Russian original by Dan D. Pascali. Translations of Mathematical Monographs, 139. American Mathematical Society, Providence, RI, 1994. 1

[51] V.A. Solonnikov, *The Green's matrices for elliptic boundary value problems. I*, (Russian) Trudy Mat. Inst. Steklov. 110 (1970), 107–145. 4

[52] V.A. Solonnikov, *The Green's matrices for elliptic boundary value problems. II*, (Russian) Boundary value problems of mathematical physics, 7. Trudy Mat. Inst. Steklov. 116 (1971), 181–216, 237. 4

[53] G. Verchota, *The Dirichlet problem for the polyharmonic equation in Lipschitz domains*, Indiana Univ. Math. J. 39 (1990), no. 3, 671–702. 2

[54] G. Verchota, *Potentials for the Dirichlet problem in Lipschitz domains*, Potential theory—ICPT 94 (Kouty, 1994), de Gruyter, Berlin, 1996, pp. 167–187. 2

[55] G. Verchota, *The biharmonic Neumann problem in Lipschitz domains*, Acta Math. 194 (2005), no. 2, 217–279. 2

[56] G. Verchota, *Boundary coerciveness and the Neumann problem for 4th order linear partial differential operators*, Around the research of Vladimir Maz'ya. II, Int. Math. Ser. (N. Y.), vol. 12, Springer, New York, 2010, pp. 365–378. 2

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