ACCRETIVITY AND FORM BOUNDEDNESS OF SECOND ORDER DIFFERENTIAL OPERATORS

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Abstract. Let $\mathcal{L}$ be the general second order differential operator with complex-valued distributional coefficients $A = (a_{jk})_{j,k=1}^n$, $\vec{b} = (b_j)_{j=1}^n$, and c in an open set $\Omega \subseteq \mathbb{R}^n$ $(n \geq 1)$, with principal part either in the divergence form, $\mathcal{L}u = \operatorname{div}(A\nabla u) + \vec{b} \cdot \nabla u + c\,u$, or non-divergence form, $\mathcal{L}u = \sum_{j,\,k=1}^n a_{jk}\,\partial_j \partial_k u + \vec{b} \cdot \nabla u + c\,u$.

We give a survey of the results by the authors which characterize the following two properties of $\mathcal{L}$:

(1) $-\mathcal{L}$ is accretive, i.e., $\operatorname{Re}\langle -\mathcal{L}u,\, u\rangle \geq 0$;

(2) $\mathcal{L}$ is form bounded, i.e., $|\langle \mathcal{L}u, u\rangle| \leq C\, \|\nabla u\|^2_{L^2(\Omega)}$,

for all complex-valued $u \in C_0^\infty(\Omega)$.

Contents

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1. Introduction

We consider the general second order differential operator

$$\mathcal{L}_0 u = \sum_{j,\,k=1}^{n} a_{jk}\,\partial_j\partial_k u + \sum_{j=1}^{n} b_j\,\partial_j u + c\,u, \tag{1.1}$$

in an open set $\Omega \subseteq \mathbb{R}^n$, with $A = (a_{jk}) \in D'(\Omega)^{n\times n}$, $\vec{b} = (b_j) \in D'(\Omega)^n$, and $c \in D'(\Omega)$, where $D'(\Omega) = C_0^\infty(\Omega)^*$ is the space of complex-valued distributions in Ω.

We discuss the *accretivity* property of $-\mathcal{L}_0$ (or, equivalently, *dissipativity* of $\mathcal{L}_0$), i.e.,

$$\mathrm{Re}\,\langle -\mathcal{L}_0\, u,\, u\rangle \ge 0, \tag{1.2}$$

for all complex-valued functions $u \in C_0^\infty(\Omega)$.

More general differential operators

$$\mathcal{L}_1 u = \sum_{j,\,k=1}^{n} a_{jk}\,\partial_j\partial_k u + \vec{b_1}\cdot\nabla u + \mathrm{div}\,(\vec{b_2}\,u) + c_1\,u, \tag{1.3}$$

with $\vec{b_1}, \vec{b_2} \in D'(\Omega)^n$ and $c_1 \in D'(\Omega)$ can be treated as well, since $\mathcal{L}_1$ is immediately reduced to $\mathcal{L}_0$ with $\vec{b} = \vec{b_1} + \vec{b_2}$ and $c = c_1 + \mathrm{div}\,\vec{b_2}$.

Our main results on the accretivity problem for general differential operators are discussed in Sec. 4 below. (See Propositions 4.1 and 4.2, as well as Theorem V for $n = 1$, and Theorem VI for $n \ge 2$.)

For the sake of simplicity, we will focus in the Introduction on the operator

$$\tilde{\mathcal{L}} = \Delta + \vec{b}\cdot\nabla + c,$$

whose principal part is the Laplacian Δ, and the coefficients $\vec{b} = (b_j)$ and c are locally integrable functions in $\mathbb{R}^n$. Then the sesquilinear form of $-\tilde{\mathcal{L}}$ is given by

$$\langle -\tilde{\mathcal{L}}u, v\rangle = \int_{\mathbb{R}^n} (\nabla u\cdot\nabla\overline{v} - \vec{b}\cdot\nabla u\;\overline{v} - c\,u\,\overline{v})\,dx, \tag{1.4}$$

where $u,\, v \in C_0^\infty(\mathbb{R}^n)$.

In this special case, let

$$q = \mathrm{Re}\,c - \tfrac{1}{2}\mathrm{div}\,(\mathrm{Re}\,\vec{b}), \qquad \vec{d} = \tfrac{1}{2}(\mathrm{Im}\,\vec{b}). \tag{1.5}$$

We denote by $\mathcal{H} = \Delta + q$ the corresponding Schrödinger operator. The quadratic form associated with $-\mathcal{H}$ in the case $q \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is given by

$$[h]^2_{\mathcal{H}} := \langle -\mathcal{H}h, h\rangle = \int_{\mathbb{R}^n} (|\nabla h|^2 dx - q\,|h|^2)\,dx, \quad h \in C_0^\infty(\mathbb{R}^n). \tag{1.6}$$

Theorem I. *Let $\tilde{\mathcal{L}} = \Delta + \vec{b}\cdot\nabla + c$, where $\mathrm{Re}\,\vec{b} \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, and $\mathrm{Im}\,\vec{b}$, $c \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Let q, $\vec{d}$ be given by (1.5). Then the operator $-\tilde{\mathcal{L}}$ is accretive if and only if the following two conditions hold:*

(i) The operator $-\mathcal{H}$ is nonnegative definite, i.e.,

$$[h]^2_{\mathcal{H}} = \int_{\mathbb{R}^n} (|\nabla h|^2 dx - q\, |h|^2)\, dx \geq 0, \tag{1.7}$$

for all real (or complex-valued) $h \in C_0^\infty(\mathbb{R}^n)$.

(ii) The commutator inequality

$$\left| \int_{\mathbb{R}^n} \vec{d} \cdot (u \nabla v - v \nabla u)\, dx \right| \leq [u]_{\mathcal{H}}\, [v]_{\mathcal{H}} \tag{1.8}$$

holds for all real-valued $u, v \in C_0^\infty(\mathbb{R}^n)$.

A necessary and sufficient condition for property (1.7) was obtained in [13, Proposition 5.1] (see Sec. 4.3 below). Concerning condition (1.8), we observe that, under the upper and lower bounds on the quadratic form (1.7) discussed in Sec. 4.5, the expressions $[u]_{\mathcal{H}}$ and $[v]_{\mathcal{H}}$ on the right-hand side of (2.12) can be replaced, up to a constant multiple, with the corresponding Dirichlet norms $||\nabla u||_{L^2(\mathbb{R}^n)}$ and $||\nabla v||_{L^2(\mathbb{R}^n)}$, respectively. Then the corresponding commutator inequality

$$\left| \int_{\mathbb{R}^n} \vec{d} \cdot (u \nabla v - v \nabla u)\, dx \right| \leq C\, ||\nabla u||_{L^2(\mathbb{R}^n)}\, ||\nabla v||_{L^2(\mathbb{R}^n)}, \tag{1.9}$$

for all (real-valued or complex-valued) $u, v \in C_0^\infty(\mathbb{R}^n)$, can be characterized completely as follows (see [28, Lemma 4.8]).

Theorem II. *Let $\vec{d} \in L^1_{\text{loc}}(\mathbb{R}^n)$, $n \geq 2$. Then inequality (1.9) holds if and only if*

$$\vec{d} = \vec{c} + \operatorname{Div} F, \tag{1.10}$$

where $F \in \mathrm{BMO}(\mathbb{R}^n)^{n \times n}$ is a skew-symmetric matrix field, and $\vec{c}$ satisfies the condition

$$\int_{\mathbb{R}^n} |\vec{c}|^2\, |u|^2\, dx \leq\, C\, ||\nabla u||^2_{L^2(\mathbb{R}^n)}, \tag{1.11}$$

where the constant C does not depend on $u \in C_0^\infty(\mathbb{R}^n)$

Moreover, if (1.9) holds, then (1.10) is valid with $\vec{c} = \nabla \Delta^{-1}(\operatorname{div} \vec{d})$ satisfying (1.11), and $F = \Delta^{-1}(\operatorname{Curl} \vec{d}) \in \mathrm{BMO}(\mathbb{R}^n)^{n \times n}$.

In the case $n = 2$, necessarily $\vec{c} = 0$, and $\vec{d} = (-\partial_2 f, \partial_1 f)$ with $f \in \mathrm{BMO}(\mathbb{R}^2)$ in the above statements.

Here the gradient ∇, and the matrix operators Div, Curl are understood in the sense of distributions (see Sec. 2). Expressions $\Delta^{-1}(\operatorname{div} \vec{d})$, $\Delta^{-1}(\operatorname{Curl} \vec{d})$, etc., are defined in terms of the weak-$*$ BMO convergence (details can be found in [28], [29]). Theorems I & II yield an explicit criterion of accretivity for $-\tilde{\mathcal{L}}$ (see Theorem VI below in the general case).

More general commutator inequalities related to compensated compactness theory [4] were studied earlier by the authors [28] in the framework of the *form boundedness* problem,

$$|\langle \mathcal{L}_0\, u,\, v\rangle| \le C\, ||\nabla u||_{L^2(\mathbb{R}^n)}\, ||\nabla v||_{L^2(\mathbb{R}^n)}, \tag{1.12}$$

where the constant C does not depend on $u,\, v \in C_0^\infty(\mathbb{R}^n)$.

If (1.12) holds, then $\langle \mathcal{L}_0\, u,\, v\rangle$ can be extended by continuity to $u, v \in L^{1,\,2}(\mathbb{R}^n)$ $(n \ge 3)$. Here $L^{1,\,2}(\mathbb{R}^n)$ is the completion of (complex-valued) $C_0^\infty(\mathbb{R}^n)$ functions with respect to the norm $||u||_{L^{1,\,2}(\mathbb{R}^n)} = ||\nabla u||_{L^2(\mathbb{R}^n)}$. Equivalently,

$$\mathcal{L}_0 \colon L^{1,\,2}(\mathbb{R}^n) \to L^{-1,\,2}(\mathbb{R}^n) \tag{1.13}$$

is a bounded operator, where $L^{-1,\,2}(\mathbb{R}^n) = L^{1,\,2}(\mathbb{R}^n)^*$ is a dual Sobolev space. Analogous problems have been studied in [25]–[27] for the inhomogeneous Sobolev space $W^{1,\,2}(\mathbb{R}^n)$, fractional Sobolev spaces, infinitesimal form boundedness, and other related questions (see Sec. 3 below).

In the special case of the operator $\tilde{\mathcal{L}}$, we have the following characterization of form boundedness.

Theorem III. *Let $\tilde{\mathcal{L}} = \Delta + \vec{b}\cdot\nabla + q$, where $\vec{b} \in L^1_{\rm loc}(\mathbb{R}^n)^n$ and $q \in L^1_{\rm loc}(\mathbb{R}^n)$, $n \ge 2$. Then the following statements hold.*

(i) *The sesquilinear form of $\tilde{\mathcal{L}}$ given by (1.4) is bounded if and only if $\vec{b}$ and q can be represented respectively in the form*

$$\vec{b} = \vec{c} + \operatorname{Div} F, \qquad q = \operatorname{div} \vec{h}, \tag{1.14}$$

where F is a skew-symmetric matrix field such that

$$F \in \mathrm{BMO}(\mathrm{R^n})^{n\times n}, \tag{1.15}$$

whereas $\vec{c}$ and $\vec{h}$ satisfy the condition

$$\int_{\mathbb{R}^n} (|\vec{c}|^2 + |\vec{h}|^2)\, |u|^2\, dx \le C\, ||\nabla u||^2_{L^2(\mathbb{R}^n)}, \tag{1.16}$$

where the constant C does not depend on $u \in C_0^\infty(\mathbb{R}^n)$.

(ii) *If the sesquilinear form of $\tilde{\mathcal{L}}$ is bounded, then $\vec{c}$, F, and $\vec{h}$ in decomposition* (1.14) *can be determined explicitly by*

$$\vec{c} = \nabla \Delta^{-1}(\operatorname{div} \vec{b}), \qquad \vec{h} = \nabla(\Delta^{-1}\, q), \tag{1.17}$$

$$F = \Delta^{-1}(\operatorname{Curl} \vec{b}), \tag{1.18}$$

so that conditions (1.15), (1.16) hold.

If $n = 2$, then (1.16) yields that $\vec{c} = 0$ and $\vec{h} = 0$, so that $q = 0$ and $\vec{b} = (-\partial_2 f, \partial_1 f)$ with $f \in \mathrm{BMO}(\mathbb{R}^2)$.

The form boundedness problem (1.12) for the general second order differential operator $\mathcal{L}_0$ in the case $\Omega = \mathbb{R}^n$ was characterized by the authors in [28] using harmonic analysis and potential theory methods. These results are discussed in Sec. 3 below. We observe that no ellipticity assumptions are imposed on the principal part $\mathcal{A}$ of $\mathcal{L}_0$ in this context.

For the Schrödinger operator $\mathcal{H} = \Delta + q$ with $q \in D'(\Omega)$, where either $\Omega = \mathbb{R}^n$, or Ω is a bounded domain that supports Hardy's inequality (see [2]), a characterization of form boundedness was obtained earlier in [24]. A different approach for $\mathcal{H} = \operatorname{div}(P\nabla\cdot) + q$ in general open sets $\Omega \subseteq \mathbb{R}^n$, under the uniform ellipticity assumptions on P, was developed in [13]. (We remark that these assumptions on P can be relaxed in a substantial way.) There is also a quasilinear version for operators of the p-Laplace type (see [14]).

Both the accretivity and form boundedness properties have numerous applications. They include problems in mathematical quantum mechanics ([33], [34]), PDE theory ([6], [8], [15], [16], [23], [30], [10], [31]), fluid mechanics and Navier-Stokes equations ([9], [18], [35], [37]), semigroups and Markov processes ([20]), homogenization theory ([39]), harmonic analysis ([4], [7]), etc.

We conclude the Introduction with the observation that, for the form boundedness property, the case of complex-valued coefficients is easily reduced to the real-valued case. In contrast, for the accretivity property, complex-valued coefficients lead to additional difficulties that appear when the matrix $\operatorname{Im} A$ is not symmetric, or the imaginary part of $\vec{b}$ is nontrivial.

2. Preliminaries

Let $\Omega \subseteq \mathbb{R}^n$ $(n \geq 1)$ be an open set. The matrix row divergence operator $\operatorname{Div}\colon D'(\Omega)^{n\times n} \to D'(\Omega)^n$ is defined on matrix fields $F = (f_{jk})_{j,k=1}^n \in D'(\Omega)^{n\times n}$ by $\operatorname{Div} F = (\sum_{k=1}^n \partial_k f_{jk})_{j=1}^n \in D'(\Omega)^n$. If F is skew-symmetric, i.e., $f_{jk} = -f_{kj}$, then we obviously have $\operatorname{div}(\operatorname{Div} F) = 0$.

The matrix Curl operator Curl: $D'(\Omega)^n \to D'(\Omega)^{n\times n}$ is defined on vector fields $\vec{f} = (f_k)_{k=1}^n$ by $\operatorname{Curl} \vec{f} = (\partial_j f_k - \partial_k f_j)_{j,k=1}^n$. Clearly, $\operatorname{Curl} \vec{f}$ is always a skew-symmetric matrix field.

It will be convenient to use the notion of *admissible measures* $\mathfrak{M}_+^{1,2}(\Omega)$, i.e., nonnegative locally finite Borel measures μ in Ω which obey the trace inequality

$$\Big(\int_\Omega |u|^2 \, d\mu \Big)^{\frac{1}{2}} \leq C \, ||\nabla u||_{L^2(\Omega)}, \qquad \text{for all } u \in C_0^\infty(\Omega), \tag{2.1}$$

where the constant C does not depend on u. The least embedding constant C in (2.1) will be denoted by $||\mu||_{\mathfrak{M}_+^{1,2}(\Omega)}$. For admissible measures $q(x)\,dx$ with nonnegative density $q \in L^1_{\mathrm{loc}}(\Omega)$, we write $q \in \mathfrak{M}_+^{1,2}(\Omega)$.

Several characterizations of $\mathfrak{M}_+^{1,2}(\Omega)$ are known. They can be formulated in terms of capacities [23] or Green energies [7], [32], and, in the case $\Omega = \mathbb{R}^n$, in terms of local maximal estimates [17], pointwise potential inequalities [24], or dyadic Carleson measures [38] (see also [28], [29]).

Suppose that the principal part $\mathcal{A}u$ of the general differential operator is given in the divergence form,

$$\mathcal{A}u = \mathrm{div}\,(A\nabla u), \quad u \in C_0^\infty(\Omega). \tag{2.2}$$

Then we consider the operator

$$\mathcal{L}u = \mathrm{div}\,(A\nabla u) + \vec{b}\cdot\nabla u + c\,u, \tag{2.3}$$

with distributional coefficients $A = (a_{jk})$, $\vec{b} = (b_j)$, and c. The corresponding sesquilinear form $\langle \mathcal{L}u, v\rangle$ is given by

$$\langle \mathcal{L}u, v\rangle = -\langle A\nabla u, \nabla v\rangle + \langle \vec{b}\cdot\nabla u, v\rangle + \langle c\,u, v\rangle, \tag{2.4}$$

where $u, v \in C_0^\infty(\Omega)$ are complex-valued.

We observe that if $\mathcal{L}_0$ is given in the non-divergence form (1.1), then

$$\mathcal{L}_0 = \mathcal{L} - \mathrm{Div}\,A\cdot\nabla.$$

(See, for instance, [16], [29].) Hence, we can express $\langle \mathcal{L}_0 u, v\rangle$ in the form (2.4), with $\vec{b} - \mathrm{Div}\,A$ in place of $\vec{b}$, for distributional coefficients A and $\vec{b}$.

This means that, without loss of generality, we may treat the accretivity property

$$\mathrm{Re}\,\langle -\mathcal{L}\,u,\, u\rangle \ge 0, \quad \text{for all } u \in C_0^\infty(\Omega), \tag{2.5}$$

for the divergence form operator $\mathcal{L}$ given by (2.3).

This problem is of substantial interest even in the real-variable case, where the goal is to characterize operators $-\mathcal{L}$ with real-valued coefficients whose quadratic form is nonnegative definite,

$$\langle -\mathcal{L}h, h\rangle \ge 0, \quad \text{for all real-valued } h \in C_0^\infty(\Omega). \tag{2.6}$$

In this case the operator $-\mathcal{L}$ is called *nonnegative definite*.

In the special case of Schrödinger operators

$$\mathcal{H}u = \mathrm{div}\,(P\nabla u) + \sigma\,u, \tag{2.7}$$

with real-valued $P \in D'(\Omega)^{n\times n}$ and $\sigma \in D'(\Omega)$, a characterization of this property was obtained earlier in [13, Proposition 5.1] under the assumption that P is uniformly elliptic, i.e.,

$$m\,||\xi||^2 \le P(x)\xi\cdot\xi \le M\,||\xi||^2, \quad \text{for all } \xi \in \mathbb{R}^n,\ \text{a.e. } x \in \Omega, \tag{2.8}$$

with the ellipticity constants $m > 0$ and $M < \infty$.

An analogous characterization of (2.6) for more general operators which include drift terms, $\mathcal{L} = \mathrm{div}(P\nabla\cdot) + \vec{b}\cdot\nabla + c$, with real-valued coefficients and P satisfying (2.8), is given in Proposition 4.2 below.

For the general differential operator in the form (2.2), we define the symmetric part A^s, and skew-symmetric part A^c, respectively, by

$$A^s = \frac{1}{2}(A + A^\perp), \quad A^c = \frac{1}{2}(A - A^\perp). \tag{2.9}$$

Here $A = (a_{jk}) \in D'(\Omega)^{n\times n}$, and $A^\perp = (a_{kj})$ is the transposed matrix.

For $-\mathcal{L}$ to be accretive, the matrix A^s must have a nonnegative definite real part: $P = \operatorname{Re} A^s$ should satisfy

$$P\xi \cdot \xi \geq 0 \quad \text{for all } \xi \in \mathbb{R}^n, \quad \text{in } D'(\Omega). \tag{2.10}$$

Moreover, if the corresponding Schrödinger operator $\mathcal{H}$ is defined by (2.7) with

$$P = \operatorname{Re} A^s, \quad \sigma = \operatorname{Re} c - \frac{1}{2}\operatorname{div}(\operatorname{Re} \vec{b}),$$

then $-\mathcal{H}$ must be nonnegative definite:

$$[h]^2_{\mathcal{H}} = \langle -\mathcal{H}h, h\rangle = \langle P\nabla h, \nabla h\rangle - \langle \sigma h, h\rangle \geq 0, \tag{2.11}$$

for all real-valued (or complex-valued) $h \in C_0^\infty(\Omega)$.

The rest of the accretivity problem for $\mathcal{L}$ (see Sec. 4.1) is reduced to the commutator inequality

$$\Big|\langle \vec{d}, u\nabla v - v\nabla u\rangle\Big| \leq [u]_{\mathcal{H}}\, [v]_{\mathcal{H}}, \tag{2.12}$$

for all real-valued $u, v \in C_0^\infty(\Omega)$, where the real-valued vector field $\vec{d}$ is given by

$$\vec{d} = \tfrac{1}{2}\, [\operatorname{Im} \vec{b} - \operatorname{Div}(\operatorname{Im} A^c)]. \tag{2.13}$$

As mentioned in the Introduction, under some mild restrictions on $\mathcal{H}$, the "norms" $[u]_{\mathcal{H}}$ and $[v]_{\mathcal{H}}$ on the right-hand side of (2.12) can be replaced, up to a constant multiple, with the corresponding Dirichlet norms $||\nabla \cdot||_{L^2(\Omega)}$. This leads to explicit criteria of accretivity, such as Theorem VI below in the case $\Omega = \mathbb{R}^n$.

3. Form boundedness

We start with a discussion of form boundedness for the general second order differential operator $\mathcal{L}$ in the form (2.3), where a_{ij}, b_i, and c are real- or complex-valued distributions, on the homogeneous Sobolev space $L^{1,2}(\mathbb{R}^n)$, and its inhomogeneous counterpart $W^{1,2}(\mathbb{R}^n)$, obtained in [28].

In particular, this leads to criteria of the *relative form boundedness* of the operator $\vec{b} \cdot \nabla + q$ with distributional coefficients $\vec{b}$ and q with respect to the Laplacian Δ on $L^2(\mathbb{R}^n)$. Invoking the so-called KLMN Theorem (see [6, Theorem IV.4.2]; [33, Theorem X.17]), we can then demonstrate that $\tilde{\mathcal{L}} = \Delta + \vec{b} \cdot \nabla + q$ is well defined, under appropriate smallness assumptions on $\vec{b}$ and q, as an m-sectorial operator on $L^2(\mathbb{R}^n)$. In this case, the quadratic form domain of $\tilde{\mathcal{L}}$ coincides with $W^{1,2}(\mathbb{R}^n)$.

This yields a characterization of the relative form boundedness for the magnetic Schrödinger operator

$$\mathcal{M} = (i\,\nabla + \vec{a})^2 + q, \tag{3.1}$$

with arbitrary vector potential $\vec{a} \in L^2_{\text{loc}}(\mathbb{R}^n)^n$, and $q \in D'(\mathbb{R}^n)$ on $L^2(\mathbb{R}^n)$ with respect to Δ (see [28]).

Our approach is based on factorization of functions in Sobolev spaces and integral estimates of potentials of equilibrium measures, combined with compensated compactness arguments, commutator estimates, and the idea of gauge invariance. Moreover, an explicit Hodge decomposition is established for form bounded vector fields in $\mathbb{R}^n$. In this decomposition, the irrotational part of the vector field is subject to a stronger restriction than its divergence-free counterpart.

3.1. **Form boundedness in the homogeneous Sobolev space.** As was mentioned above, without loss of generality we may assume that the principal part of the differential operator is in the divergence form, i.e., $\mathcal{L}$ is given by (2.3).

We present necessary and sufficient conditions on A, $\vec{b}$, and q which ensure the boundedness in the homogeneous Sobolev space $L^{1,\,2}(\mathbb{R}^n)$ of the sesquilinear form associated with $\mathcal{L}$:

$$|\langle \mathcal{L}\, u,\, v\rangle| \le C\, ||u||_{L^{1,\,2}(\mathbb{R}^n)}\, ||v||_{L^{1,\,2}(\mathbb{R}^n)}, \tag{3.2}$$

where C does not depend on $u,\, v \in C_0^\infty(\mathbb{R}^n)$, and $||u||_{L^{1,\,2}(\mathbb{R}^n)} = ||\nabla u||_{L^2(\mathbb{R}^n)}$.

Theorem IV. *Let $\mathcal{L} = \operatorname{div}(A\,\nabla\cdot) + \vec{b}\cdot\nabla + q$, where $A \in D'(\mathbb{R}^n)^{n\times n}$, $\vec{b} \in D'(\mathbb{R}^n)^n$ and $q \in D'(\mathbb{R}^n)$, $n \ge 2$. Then the following statements hold.*

(i) *The sesquilinear form of $\mathcal{L}$ is bounded, i.e.,* (3.2) *holds if and only if $A^s \in L^\infty(\mathbb{R}^n)^{n\times n}$, and $\vec{b}$ and q can be represented respectively in the form*

$$\vec{b} = \vec{c} + \operatorname{Div} F, \qquad q = \operatorname{div} \vec{h}, \tag{3.3}$$

where F is a skew-symmetric matrix field such that

$$F - A^c \in \mathrm{BMO}(\mathbb{R}^{\mathrm{n}})^{n\times n}, \tag{3.4}$$

whereas $\vec{c}$ and $\vec{h}$ belong to $L^2_{\mathrm{loc}}(\mathbb{R}^n)^n$, and obey the condition

$$|\vec{c}|^2 + |\vec{h}|^2 \in \mathfrak{M}_+^{1,\,2}(\mathbb{R}^n). \tag{3.5}$$

(ii) *If the sesquilinear form of $\mathcal{L}$ is bounded, then $\vec{c}$, F, and $\vec{h}$ in decomposition* (3.3) *can be determined explicitly by*

$$\vec{c} = \nabla(\Delta^{-1}\operatorname{div}\vec{b}), \qquad \vec{h} = \nabla(\Delta^{-1}\, q), \tag{3.6}$$

$$F = \Delta^{-1}\operatorname{Curl}[\vec{b} - \operatorname{Div}(A^c)] + A^c, \tag{3.7}$$

where

$$\Delta^{-1}\operatorname{Curl}[\vec{b} - \operatorname{Div}(A^c)] \in \mathrm{BMO}(\mathbb{R}^n)^{n\times n}, \tag{3.8}$$

and

$$|\nabla(\Delta^{-1}\operatorname{div}\vec{b})|^2 + |\nabla(\Delta^{-1}\, q)|^2 \in \mathfrak{M}_+^{1,\,2}(\mathbb{R}^n). \tag{3.9}$$

We remark that condition (3.8) in statement (ii) of Theorem III may be replaced with

$$\vec{b} - \operatorname{Div}(A^c) \in \mathrm{BMO}^{-1}(\mathbb{R}^n)^n, \tag{3.10}$$

which ensures that decomposition (3.3) holds. Here $\mathrm{BMO}^{-1}(\mathbb{R}^n)$ stands for the space of distributions that can be represented in the form $f = \operatorname{div} \vec{g}$ where $\vec{g} \in \mathrm{BMO}(\mathbb{R}^n)^n$ (see [18]).

In the special case $n = 2$, it is easy to see that (3.2) holds if and only if $A^s \in L^\infty(\mathbb{R}^2)^{2\times 2}$, $\vec{b} -$ Div $(A^c) \in \mathrm{BMO}^{-1}(\mathbb{R}^2)^2$, and $q = \operatorname{div} \vec{b} = 0$.

As mentioned in the Introduction, expressions $\nabla(\Delta^{-1} q)$, $\nabla(\Delta^{-1}\operatorname{div} \vec{b})$, $\operatorname{Div}(\Delta^{-1}\operatorname{Curl} \vec{b})$, which involve nonlocal operators, are defined in the sense of distributions. This is possible, since $\Delta^{-1}q$, $\Delta^{-1}\operatorname{div} \vec{b}$, and $\Delta^{-1}\operatorname{Curl} \vec{b}$ can be understood in terms of the convergence in the weak-$*$ topology of $\mathrm{BMO}(\mathbb{R}^n)$ of $\Delta^{-1} \operatorname{div} (\psi_N \vec{b})$, $\Delta^{-1} \operatorname{Curl} (\psi_N \vec{b})$, and $\Delta^{-1} (\psi_N q)$, respectively, as $N \to +\infty$. Here ψ_N is a smooth cut-off function with support in the ball $\{x : |x| < N\}$, and the limits above do not depend on the choice of ψ_N.

It follows from Theorem IV that $\mathcal{L}$ is form bounded on $L^{1,2}(\mathbb{R}^n)\times L^{1,2}(\mathbb{R}^n)$ if and only if $A^s \in L^\infty(\mathbb{R}^n)^{n\times n}$, and $\vec{b}_1 \cdot \nabla + q$ is form bounded, where

$$\vec{b_1} = \vec{b} - \operatorname{Div}(A^c). \tag{3.11}$$

In particular, the principal part $\mathcal{P}u = \operatorname{div}(A \nabla u)$ is form bounded if and only if

$$A^s \in L^\infty(\mathbb{R}^n)^{n\times n}, \tag{3.12}$$

$$\operatorname{Div} (A^c) \in \mathrm{BMO}^{-1}(\mathbb{R}^n)^n. \tag{3.13}$$

A simpler condition with $A^c \in \mathrm{BMO}(\mathbb{R}^n)^{n\times n}$ in place of (3.13) is sufficient, but generally is necessary only if $n = 1, 2$.

Thus, the form boundedness problem for the general second order differential operator is reduced to the special case

$$\mathcal{L} = \vec{b} \cdot \nabla + q, \qquad \vec{b} \in D'(\mathbb{R}^n)^n, \quad q \in D'(\mathbb{R}^n). \tag{3.14}$$

As a corollary of Theorem IV, we deduce that, if $\vec{b}\cdot \nabla + q$ is form bounded, then the Hodge decomposition

$$\vec{b} = \nabla(\Delta^{-1}\operatorname{div} \vec{b}) + \operatorname{Div} (\Delta^{-1}\operatorname{Curl} \vec{b}) \tag{3.15}$$

holds, where $\Delta^{-1}(\operatorname{Curl} \vec{b}) \in \mathrm{BMO}(\mathbb{R}^n)^{n\times n}$, and

$$\int_{|x-y|<r} [\, |\nabla\Delta^{-1}(\operatorname{div} \vec{b})|^2 + |\nabla(\Delta^{-1} q)|^2 \,]\, dy \le \text{const}\ r^{n-2}, \tag{3.16}$$

for all $r > 0$, $x \in \mathbb{R}^n$, in the case $n \ge 3$; in two dimensions, it follows that $\operatorname{div} \vec{b} = q = 0$.

We observe that condition (3.16) is generally stronger than $\Delta^{-1}\operatorname{div} \vec{b} \in$ BMO and $\Delta^{-1} q \in$ BMO, while the divergence-free part of $\vec{b}$ is characterized by $\Delta^{-1}\operatorname{Curl} \vec{b} \in$ BMO, for all $n \ge 2$.

The main difficulty in the proof of Theorem IV is the interaction between the quadratic forms associated with $q - \frac{1}{2} \operatorname{div} \vec{b}$ and the divergence free part of $\vec{b}$. To this effect, we use Theorem II, which characterizes vector fields $\vec{d}$ such that the commutator inequality (1.9) holds. Theorem II is proved in

[28, Lemma 4.8] using the idea of the gauge transformation ([19, Sec. 7.19]; [33, Sec. X.4]):

$$\nabla \to e^{-i\lambda}\, \nabla\, e^{+i\lambda},$$

where the gauge λ is a real-valued function in $L^{1,\,2}_{\text{loc}}(\mathbb{R}^n)$.

The nontrivial problem of choosing an appropriate gauge is solved in [28] as follows:

$$\lambda = \tau\, \log\,(N\mu), \qquad 1 < 2\tau < \tfrac{n}{n-2},$$

where $N\mu = (-\Delta)^{-1}\mu$ is the Newtonian potential of the equilibrium measure μ associated with an arbitrary compact set e of positive capacity.

With this choice of λ, the energy space $L^{1,\,2}(\mathbb{R}^n)$ is gauge invariant, and for the irrotational part $\vec{c} = \nabla(\Delta^{-1}\text{div}\, \vec{d})$ we have $|\vec{c}|^2 \in \mathfrak{M}^{1,\,2}_{+}(\mathbb{R}^n)$. In addition, we have $F = \Delta^{-1}\text{Curl}\, \vec{d}$ belongs to $\text{BMO}(\mathbb{R}^n)^{n\times n}$, and $\vec{d} = \vec{c} + \text{Div}\, F$. These conditions are necessary and sufficient for (1.9).

Applications of Theorem IV to the magnetic Schrödinger operator $\mathcal{M}$ defined by (3.1) are given in [28, Theorem 3.4], where it is shown that $\mathcal{M}$ is form bounded if and only if both $q + |\vec{a}|^2$ and $\vec{a} \cdot \nabla$ are form bounded.

3.2. **Form boundedness in $W^{1,\,2}(\mathbb{R}^n)$.** The above results are easily extended to the Sobolev space $W^{1,\,2}(\mathbb{R}^n)$ $(n \ge 1)$ with norm $||u||_{W^{1,\,2}(\mathbb{R}^n)} = ||\nabla u||_{L^2(\mathbb{R}^n)} + ||u||_{L^2(\mathbb{R}^n)}$.

In particular, necessary and sufficient conditions are given in [28, Theorem 5.1] for the boundedness of the general second order operator

$$\mathcal{L} \colon W^{1,\,2}(\mathbb{R}^n) \to W^{-1,\,2}(\mathbb{R}^n).$$

This solves the *relative form boundedness* problem for $\mathcal{L}$, and consequently for the magnetic Schrödinger operator $\mathcal{M}$, with respect to the Laplacian on $L^2(\mathbb{R}^n)$ (see [33, Sec. X.2]). The proofs are based on an inhomogeneous version of the div-curl lemma ([28, Lemma 5.2]).

3.3. **Infinitesimal form boundedness.** Other fundamental properties of quadratic forms associated with differential operators can be characterized using our methods. In particular, for the Schrödinger operator $\mathcal{H} = \Delta + q$, criteria of relative compactness were obtained in [24], while the infinitesimal form boundedness expressed by the inequality

$$|\langle \mathcal{H}\, u,\, u\rangle| \le \epsilon\, ||\nabla u||^2_{L^2(\mathbb{R}^n)} + C(\epsilon)\, ||u||^2_{L^2(\mathbb{R}^n)}, \quad u \in C^\infty_0(\mathbb{R}^n), \tag{3.17}$$

for every $\epsilon \in (0,1)$, where $C(\epsilon)$ is a positive constant, along with Trudinger's subordination where $C(\epsilon) = C\, \epsilon^{-\beta}$ $(\beta > 0)$, was characterized in [26]. Necessary and sufficient conditions for such properties in the case of the general second order differential operator are discussed in [28].

3.4. **Form boundedness in $W^{\frac{1}{2},2}(\mathbb{R}^n)$.** Similar problems were solved for the fractional (modified relativistic) Schrödinger operator $\mathcal{L} = -(-\Delta)^{\frac{1}{2}} + q$. In particular, the boundedness of the operator

$$\mathcal{L}\colon W^{\frac{1}{2},2}(\mathbb{R}^n) \to W^{-\frac{1}{2},2}(\mathbb{R}^n)$$

has been characterized in [25] using certain extensions to higher dimensions for multipliers acting from $W^{1,2}(\mathbb{R}^{n+1})$ to $W^{-1,2}(\mathbb{R}^{n+1})$.

4. Accretivity

We now turn to the accretivity problem for $-\mathcal{L}$, where $\mathcal{L}$ is a second order linear differential operator with complex-valued distributional coefficients defined by (2.3) in an open set $\Omega \subseteq \mathbb{R}^n$ $(n \geq 1)$.

4.1. **General accretivity criterion.** Given $A = (a_{jk}) \in D'(\Omega)^{n\times n}$, we define its symmetric part A^s and skew-symmetric part A^c respectively by (2.9). The accretivity property for $-\mathcal{L}$ can be characterized in terms of the following real-valued expressions:

$$P = \operatorname{Re} A^s, \quad \vec{d} = \tfrac{1}{2}\,[\operatorname{Im}\vec{b} - \operatorname{Div}(\operatorname{Im} A^c)], \quad \sigma = \operatorname{Re} c - \tfrac{1}{2}\operatorname{div}(\operatorname{Re}\vec{b}), \tag{4.1}$$

where $P = (p_{jk}) \in D'(\Omega)^{n\times n}$, $\vec{d} = (d_j) \in D'(\Omega)^n$, and $\sigma \in D'(\Omega)$. This is a consequence of the relation (see [29, Sec.4])

$$\operatorname{Re}\langle -\mathcal{L}u, u\rangle = \operatorname{Re}\langle -\mathcal{L}_2 u, u\rangle, \quad u \in C_0^\infty(\Omega), \tag{4.2}$$

where

$$\mathcal{L}_2 = \operatorname{div}(P\nabla\cdot) + 2i\,\vec{d}\cdot\nabla + \sigma. \tag{4.3}$$

Moreover, in order that $-\mathcal{L}$ be accretive, the matrix P must be nonnegative definite, i.e., $P\xi\cdot\xi \geq 0$ in $D'(\Omega)$ for all $\xi \in \mathbb{R}^n$. In particular, each p_{jj} $(j = 1, \ldots, n)$ is a nonnegative Radon measure.

A characterization of accretive operators $-\mathcal{L}$ is given in the following proposition.

Proposition 4.1. *Let $\mathcal{L} = \operatorname{div}(A\nabla\cdot) + \vec{b}\cdot\nabla + c$, where $A \in D'(\Omega)^{n\times n}$, $\vec{b} \in D'(\Omega)^n$ and $c \in D'(\Omega)$ are complex-valued. Suppose that P, $\vec{d}$, and σ are defined by (4.1).*

The operator $-\mathcal{L}$ is accretive if and only if P is a nonnegative definite matrix, and the following two conditions hold:

$$[h]^2_{\mathcal{H}} = \langle P\nabla h, \nabla h\rangle - \langle \sigma\, h, h\rangle \geq 0, \tag{4.4}$$

for all real-valued $h \in C_0^\infty(\Omega)$, and

$$\Big|\langle \vec{d}, u\nabla v - v\nabla u\rangle\Big| \leq [u]_{\mathcal{H}}\,[v]_{\mathcal{H}}, \tag{4.5}$$

for all real-valued $u, v \in C_0^\infty(\Omega)$.

4.2. **Real-valued coefficients.** It follows from Proposition 4.1 that, for operators with real-valued coefficients, condition (4.4) alone characterizes nonnegative definite operators $-\mathcal{L}$ in an open set $\Omega \subseteq \mathbb{R}^n$ ($n \geq 1$). A more explicit characterization of this property, under the assumption that $P = A^s \in L^1_{\rm loc}(\Omega)^{n\times n}$ in the sufficiency part, and that P is uniformly elliptic in the necessity part, is given in the next theorem.

Proposition 4.2. *Let $\mathcal{L} = \mathrm{div}(A\nabla \cdot) + \vec{b}\cdot\nabla + c$, where $A \in D'(\Omega)^{n\times n}$, $\vec{b} \in D'(\Omega)^n$ and $c \in D'(\Omega)$ are real-valued. Suppose that $P = A^s \in L^1_{\rm loc}(\Omega)^{n\times n}$ is a nonnegative definite matrix a.e.*

(i) *If there exists a measurable vector field $\vec{g}$ in Ω such that $(P\vec{g})\cdot\vec{g} \in L^1_{\rm loc}(\Omega)$, and*

$$\sigma = c - \tfrac{1}{2}\mathrm{div}\,(\vec{b}) \leq \mathrm{div}\,(P\vec{g}) - (P\vec{g})\cdot\vec{g} \quad \text{in } D'(\Omega), \tag{4.6}$$

then the operator $-\mathcal{L}$ is nonnegative definite.

(ii) *Conversely, if $-\mathcal{L}$ is nonnegative definite, then there exists a vector field $\vec{g} \in L^2_{\rm loc}(\Omega)^n$ so that $(P\vec{g})\cdot\vec{g} \in L^1_{\rm loc}(\Omega)$, and (4.6) holds, provided P is uniformly elliptic.*

The uniform ellipticity condition on P in statement (ii) of Proposition 4.2 can be relaxed. This question will be treated elsewhere.

Results similar to Proposition 4.2 are well known in ordinary differential equations [11, Sec. XI.7], in relation to disconjugate Sturm-Liouville equations and Riccati equations with continuous coefficients (see also [10], [25], [29]).

4.3. **Nonnegative definite Schrödinger operators.** As was mentioned above, in the special case of Schrödinger operators $\mathcal{H} = \mathrm{div}\,(P\nabla h) + \sigma$, with real-valued $\sigma \in D'(\Omega)$ and uniformly elliptic P, Proposition 4.2 was obtained originally in [13, Proposition 5.1]. Under these assumptions, $-\mathcal{H}$ is nonnegative definite, i.e.,

$$[h]^2_{\mathcal{H}} = \langle -\mathcal{H}h, h\rangle \geq 0, \quad \text{for all } h \in C^\infty_0(\Omega),$$

if and only if there exists a vector field $\vec{g} \in L^2_{\rm loc}(\Omega)^n$ such that

$$\sigma \leq \mathrm{div}\,(P\vec{g}) - P\vec{g}\cdot\vec{g} \quad \text{in } D'(\Omega). \tag{4.7}$$

A simpler sufficient condition for $-\mathcal{H}$ to be nonnegative definite is given by $\sigma \leq \mathrm{div}\,(P\vec{g})$, where $\vec{g} \in L^2_{\rm loc}(\Omega)^n$ satisfies the inequality

$$\int_\Omega (P\vec{g}\cdot\vec{g})\, h^2\, dx \leq \frac{1}{4}\int_\Omega |P\nabla h|^2\, dx, \quad \text{for all } h \in C^\infty_0(\Omega).$$

Here $P\vec{g}\cdot\vec{g} \in \mathfrak{M}^{1,\,2}_+(\Omega)$ is admissible if P is uniformly elliptic. However, such conditions are not necessary, with any constant in place of $\frac{1}{4}$, even when $P = I$; see [13].

We observe that in Proposition 4.1 above, the nonnegative definite quadratic form $[h]^2_{\mathcal{H}}$ is associated with the Schrödinger operator $-\mathcal{H}$, where $\mathcal{H}$ has real-valued coefficients $P = \mathrm{Re}\,A^s$ and $\sigma = \mathrm{Re}\,c - \frac{1}{2}\mathrm{div}\,(\mathrm{Re}\,\vec{b})$. Hence, (4.7) characterizes the first condition of Proposition 4.1 given by (4.4). The second one, namely, the commutator condition (4.5), will be discussed further in Sections 4.5 and 4.6.

4.4. **The one-dimensional case.** In this section, the differential operator $\mathcal{L}u = (a\,u')' + bu' + c$ is defined on an open interval $I \subseteq \mathbb{R}$ (possibly unbounded). In this case, one can avoid commutator estimates using methods of ordinary differential equations ([11], [12]). In particular, the following theorem gives a generalization of Proposition 4.2 for complex-valued coefficients in the one-dimensional case. In the statements below we will make use of the standard convention $\frac{0}{0} = 0$.

Theorem V. *Let $a, b, c \in D'(I)$. Suppose that $p = \operatorname{Re} a \in L^1_{\rm loc}(I)$, and $\operatorname{Im} b \in L^1_{\rm loc}(I)$.*

(i) *The operator $-\mathcal{L}$ is accretive if and only if $\frac{(\operatorname{Im} b)^2}{p} \in L^1_{\rm loc}(I)$, where $p \geq 0$ a.e., and the following quadratic form inequality holds:*

$$\int_I p(h')^2 dx - \langle \operatorname{Re} c - \frac{1}{2}(\operatorname{Re} b)', h^2\rangle - \int_I \frac{(\operatorname{Im} b)^2}{4p}\, h^2\, dx \geq 0, \tag{4.8}$$

for all real-valued $h \in C_0^\infty(I)$.

(ii) *If there exists a function $f \in L^1_{\rm loc}(I)$ such that $\frac{f^2}{p} \in L^1_{\rm loc}(I)$, and*

$$\operatorname{Re} c - \frac{1}{2}(\operatorname{Re} b)' - \frac{(\operatorname{Im} b)^2}{4p} \leq f' - \frac{f^2}{p} \quad \text{in } D'(I), \tag{4.9}$$

then the operator $-\mathcal{L}$ is accretive.

Conversely, if $-\mathcal{L}$ is accretive, and $m \leq p(x) \leq M$ a.e. for some constants $M, m > 0$, then there exists a function $f \in L^2_{\rm loc}(I)$ such that (4.9) holds.

We remark that in Theorem V, the terms $\operatorname{Im} a$ and $\operatorname{Im} c$ play no role, but the behavior of $\operatorname{Im} b$ is essential. In higher dimensions, the situation is even more complicated. The term $\operatorname{Im} b$ may contain both the irrotational and divergence-free components, and the latter may interact with $\operatorname{Im} A^c$.

4.5. **Upper and lower bounds of quadratic forms.** For general operators with complex-valued coefficients in the case $n \geq 2$, we recall that the first condition of Proposition 4.1 is necessary for the accretivity of $-\mathcal{L}$, namely,

$$\langle \sigma\, h, h\rangle \leq \int_\Omega (P\nabla h \cdot \nabla h)\, dx, \tag{4.10}$$

for all real-valued $h \in C_0^\infty(\Omega)$, where $\sigma = \operatorname{Re} c - \frac{1}{2}\operatorname{div}(\operatorname{Re} \vec{b}) \in D'(\Omega)$, and $\operatorname{Re} A^s = P \in D'(\Omega)^{n\times n}$ is a nonnegative definite matrix.

Suppose now that σ has a slightly smaller upper form bound, that is,

$$\langle \sigma\, h, h\rangle \leq (1-\epsilon^2) \int_\Omega (P\nabla h \cdot \nabla h)\, dx, \quad h \in C_0^\infty(\Omega), \tag{4.11}$$

for some $\epsilon \in (0, 1]$. We also consider the corresponding lower bound,

$$\langle \sigma\, h, h\rangle \geq -K \int_\Omega (P\nabla h \cdot \nabla h)\, dx, \quad h \in C_0^\infty(\Omega), \tag{4.12}$$

for some constant $K \geq 0$.

Such restrictions on real-valued $\sigma \in D'(\Omega)$ were invoked in [13], for uniformly elliptic P.

We observe that (4.11) is satisfied for any $\epsilon \in (0, 1)$, up to an extra term $C\, ||h||^2_{L^2(\Omega)}$, if σ is *infinitesimally form bounded* (see Sec. 3.3). The second term on the right is sometimes included in the definition of accretivity of the operator $-\mathcal{L}$. We can always incorporate it as a constant term in $\sigma - C(\epsilon)$. The same is true with regards to the lower bound where we can use $\sigma + C(\epsilon)$.

Assuming that both bounds (4.11) and (4.12) hold for some $\epsilon \in (0, 1]$ and $K \geq 0$, we obviously have, for all $h \in C_0^\infty(\Omega)$,

$$\epsilon \int_\Omega (P\nabla h \cdot \nabla h)\, dx \leq [h]^2_{\mathcal{H}} \leq (K+1)^{\frac{1}{2}} \int_\Omega (P\nabla h \cdot \nabla h)\, dx. \tag{4.13}$$

If P satisfies the uniform ellipticity assumptions (2.8), then from (4.13) it follows that condition (4.5) equivalent, up to a constant multiple, to

$$\left| \langle \vec{d}, u\nabla v - v\nabla u\rangle \right| \leq C\, ||\nabla u||_{L^2(\Omega)}\, ||\nabla v||_{L^2(\Omega)} \tag{4.14}$$

where $C > 0$ is a constant which does not depend on real-valued $u, v \in C_0^\infty(\Omega)$. For $\Omega = \mathbb{R}^n$ and $\vec{d} \in L^1_{\text{loc}}(\mathbb{R}^n)$, see Theorem II above.

In the case $\Omega = \mathbb{R}^n$, inequality (4.14) was characterized completely in [28, Lemma 4.8] for complex-valued u, v. However, that characterization obviously works in the case of real-valued u, v as well (one only needs to change the constant C up to a factor of $\sqrt{2}$).

4.6. **Accretivity criterion in $\mathbb{R}^n$.** Combining the characterization of the commutator inequality (4.14) with Proposition 4.1 yields the following accretivity criterion, where the lower bound (4.12) in used the necessity part, whereas the upper bound (4.11) is invoked in the sufficiency part.

Theorem VI. *Let $\mathcal{L}$ be the second order differential operator (2.3) on $\mathbb{R}^n$ $(n \geq 2)$ with complex-valued coefficients $A \in D'(\mathbb{R}^n)^{n\times n}$, $\vec{b} \in D'(\mathbb{R}^n)^n$ and $c \in D'(\mathbb{R}^n)$. Let P, $\vec{d}$ and σ be defined by (4.1), where P is uniformly elliptic.*

(i) *Suppose that $-\mathcal{L}$ is accretive, i.e., (2.5) holds, and σ satisfies (4.12) for some $K \geq 0$. Then $\vec{d}$ can be represented in the form*

$$\vec{d} = \nabla f + \text{Div}\, G, \tag{4.15}$$

where $f \in D'(\mathbb{R}^n)$ is real-valued, $|\nabla f|^2 \in \mathfrak{M}^{1,\,2}_+(\mathbb{R}^n)$, and $G \in \text{BMO}(\mathbb{R}^n)^{n\times n}$ is a real-valued skew-symmetric matrix field.

Moreover, f and G above can be defined explicitly as

$$f = \Delta^{-1}(\text{div}\, \vec{d}), \qquad G = \Delta^{-1}(\text{Curl}\, \vec{d}). \tag{4.16}$$

(ii) *Conversely, suppose that σ satisfies (4.11) with some $\epsilon \in (0,1]$. Then $-\mathcal{L}$ is accretive if representation (4.15) holds, where $|\nabla f|^2 \in \mathfrak{M}^{1,2}_+(\mathbb{R}^n)$, and $G \in \mathrm{BMO}(\mathbb{R}^n)^{n\times n}$ is a real-valued skew-symmetric matrix field, provided both $\||\nabla f|^2\|_{\mathfrak{M}^{1,2}_+(\mathbb{R}^n)}$ and the* BMO*-norm of G are small enough, depending only on ϵ.*

If $n = 2$, then in Theorem VI, we have $f = 0$, and $\vec{d} = (-\partial_2 g, \partial_1 g)$ with $g \in \mathrm{BMO}(\mathbb{R}^2)$. In statement (ii), the BMO-norm of g is supposed to be small enough (depending only on ϵ).

If $n = 3$, one can use the usual vector-valued $\mathrm{curl}(\vec{g}) \in D'(\mathbb{R}^3)^3$ in place of $\mathrm{Div}\, G$ in decomposition (4.15), with $\vec{g} = \Delta^{-1}(\mathrm{curl}\, \vec{d})$ in place of G in (4.16).

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