Uniform asymptotics of Green's kernels in perforated domains and meso-scale approximations

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In honour of Robert Gilbert on the occasion of his 80th Birthday

Abstract

The paper is a review of the authors' results on asymptotic approximations of Green's kernels for elliptic boundary value problems in perforated domains. A new feature is the uniformity of the asymptotics with respect to the independent variables. Formal asymptotic approximations are supplied with estimates of the remainder terms. For the case when the number of perforations or inclusions becomes large, a novel method of meso-scale asymptotic approximations is discussed, and uniform asymptotic approximations of Green's kernels as well as solutions of boundary value problems in multiply-perforated domains are presented. Such approximations do not require periodicity or other typical constrains attributed to homogenization approximations.
Keywords: Uniform asymptotic approximations, Green's functions, singularly perturbed domains.

1 Introduction

The interest to asymptotic approximations of Green's functions for boundary value problems in singularly perturbed domains is generated by a wide range of applications in physics and mechanics. In particular, this includes domains with complicated geometries such as perforated domains and bodies with defects of different types. Although some types of asymptotic approximations for Green's functions is domains with small obstacles can be found in the literature (see, for example, [1]), the question of uniformity of such approximations with respect to the independent variables was open until recently.

A comprehensive asymptotic theory of boundary value problems in singularly perturbed domains is summarized in the monographs [2], [3] and [4].

We emphasize on two significant directions outlined in the present paper:

(a) *Uniform* asymptotic approximations of Green's functions in domains containing a hole or a finite number of small holes (see [5, 6, 7]);
(b) New uniform asymptotic formulae for boundary value problems in *non-periodic* perforated structures with many holes (see [8, 9]).

Schematically, the meso-scale configuration involving a large number of small perforations, and the perforated body with finite number of holes are shown in Figs. 1 and 2.

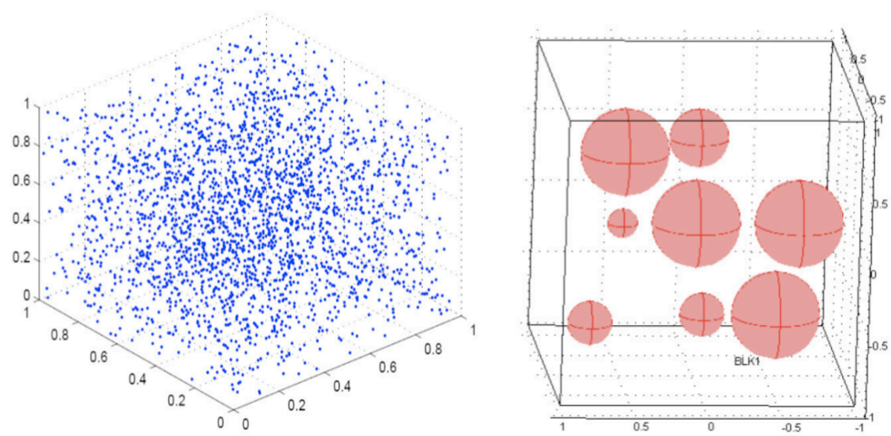

Fig. 1 A body containing perforations: meso-scale level for a body containing a large number of small holes (left) and finite number of perforations of different sizes (right).

The issue of *uniform* asymptotic approximations for Green's kernels in singularly perturbed domains has been addressed for the first time in our recent papers [5, 6, 10]. The main challenge in this work is the presence of singularities of the Green's kernels, which make the known asymptotic methods inefficient.

The history of part (b) goes back to the work by Marchenko and Khruslov [12] on boundary value problems for domains with fine grained boundary under very weak assumptions on the geometry. In particular, they discovered that the homogenized solution may satisfy equations including additional terms that were not present in the original equation. After the subsequent work by Murat and Cioranescu for the periodic case [13] where such terms are referred to as "strange terms coming from nowhere", the analysis of homogenized solutions to problems of this type has received a substantial attention of the homogenization community. The paper [8] presents a uniform approximation of Green's kernel in a multiply perforated body with many holes of a non-periodic arrangement, and both the governing equation and the boundary conditions are satisfied to the required accuracy.

The novelty of the approach [5, 6, 7, 8] to the construction of uniform asymptotic representations of Green's kernels, both in parts (a) and (b), is in the choice of a number of canonical fields defined in model domains independent of the small parameter (i.e. regular parts of Green's functions in the unperturbed domain and in the exterior of scaled holes, equilibrium potentials, capacity and dipole tensors); these fields are incorporated into the anzats of an asymptotic approximation. This work nvolves construction and analysis of global uniform asymptotic representations of Green's kernels, which are multi-scaled for singularly perturbed domains. In other words, their terms should depend on both original and stretched variables. In order to find these terms one has to study auxiliary boundary value problems independent of ε (called model problems), which are posed in appropriately chosen model domains. Certain canonical solutions of the model problems can be used as building blocks of the asymptotic formulae. The simplest physical situations to consider are in electrostatics, steady-state heat transfer and flow of an ideal fluid, described by boundary value problems for the Laplace operator. For an example involving the Dirichlet problem in a two-dimensional domain Ω_ε with a hole $F_\varepsilon = \{\mathbf{x} : \varepsilon^{-1}\mathbf{x} \in F\}$, a rigorous analysis leads to the uniform asymptotic representation of the Green's function:

$$
\begin{aligned}
G_\varepsilon(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}) + g(\boldsymbol{\xi},\boldsymbol{\eta}) + g(\boldsymbol{\xi},\infty) + g(\infty,\boldsymbol{\eta}) + \frac{1}{2\pi}\log\frac{|\mathbf{x}-\mathbf{y}|}{\varepsilon r_F} \\
-\frac{2\pi}{\log(\varepsilon r_F {R_\Omega}^{-1})}\left(G(\mathbf{x},0) + \frac{1}{2\pi}\log\frac{|\boldsymbol{\xi}|}{r_F} - g(\boldsymbol{\xi},\infty)\right) \\
\times\left(G(0,\mathbf{y}) + \frac{1}{2\pi}\log\frac{|\boldsymbol{\eta}|}{r_F} - g(\infty,\boldsymbol{\eta})\right) + O(\varepsilon),
\end{aligned}
$$

where $\boldsymbol{\xi} = \varepsilon^{-1}\mathbf{x}$, $\boldsymbol{\eta} = \varepsilon^{-1}\mathbf{y}$, G and g are Green's functions of Ω and $\mathbb{R}^2 \setminus F$, R_Ω and r_F are the inner (with respect to $\mathbf{O}$) and outer conformal radii of Ω and F, respectively (see [11]). Moreover, such asymptotic approximations are readily applicable to a range of challenging numerical problems: for example, Fig. 3 shows the results of computations based on the analytical formula of the regular part of Green's function for the Dirichlet problem in a domain with a finite number of holes, which is also compared with the finite element computations produced in COMSOL. Bearing in mind that the agreement is excellent, we note that the numerical algorithms based on asymptotic approximations are very robust and are still efficient even in situations when the finite element schemes fail (examples include Dirichlet problems in 2D and 3D bodies with a large number of small holes that require a very fine mesh and analysis of rapidly varying fields).

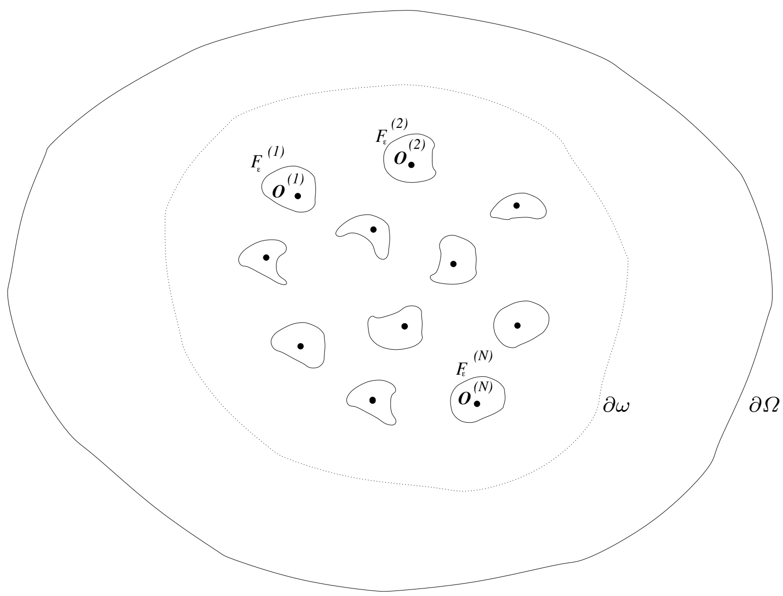

Fig. 2 Perforated domain containing N holes.

The structure of the paper can be outlined as follows. First, we describe the singular perturbation algorithm, which leads to a uniform asymptotic approximation of Green's functions in a domain with a small hole. It is noted that the two-dimensional case is characterised by the logarithmic asymptotics. The procedure is then extended to the situation when several perforations of different shapes have been introduced. Furthermore, the special class of meso-scale approximations is used when the number of perforations becomes a large parameter. A semi-analytic procedure leads to a uniform asymptotic approximation of Green's function, and the coefficients of this approximation are defined as solutions of an algebraic system, which contains the information about the shape, size and position of perforations within the body.

2 Green's functions in domains with small holes

First, we outline the results of the papers [5, 6], where uniform asymptotic approximations of Green's kernels have been constructed and rigorously justified for domains in $\mathbb{R}^n,\ n \geq 2$, containing small perforations. We note that the two-dimensional case involves logarithmic asymptotics, supplied with an additional asymptotic treatment of capacitary potentials describing boundary layers near small perforations.

Let Ω be a domain with compact closure $\overline{\Omega}$ and boundary $\partial\Omega$. The notation F will be used for a compact set of positive harmonic capacity. For convenience, we assume that Ω and F contain the origin $\mathbf{O}$ as an interior point. Without loss of generality, it is also assumed that the distance between $\mathbf{O}$ and $\partial\Omega$, as well as the diameter of F, is equal to 1. A small positive parameter ε is introduced, together with the scaled set $F_\varepsilon = \{\mathbf{x} : \varepsilon^{-1}\mathbf{x} \in F\}$, and then the perforated domain is $\Omega_\varepsilon = \Omega \setminus F_\varepsilon$. Green's function for the Dirichlet problem for the operator $-\Delta$ in Ω_ε is denoted by G_ε.

The notations G and g are used here for Green's functions of the Dirichlet problem for the operator $-\Delta$ in the unperturbed domain Ω and in the exterior of F, respectively.

2.1 Three-dimensional case

Referring to [6], which gives the derivation of uniform asymptotics of Green's kernels in $\Omega_\varepsilon \subset \mathbb{R}^n,\ n \geq 2$, we first consider an illustration for a three-dimensional domain containing a small void (similar procedure applies to the cases of $n > 3$).

It is convenient to make use of the equilibrium potential, denoted by $P(\boldsymbol{\xi})$, for $\boldsymbol{\xi} = \varepsilon^{-1}\mathbf{x} \in \mathbb{R}^3 \setminus F$, which is defined as a unique solution of the model Dirichlet problem in $F^c = \mathbb{R}^3 \setminus F$:

$$\Delta_\xi P(\boldsymbol{\xi}) = 0 \ \text{ in } F^c, \tag{1}$$

$$P(\boldsymbol{\xi}) = 1 \ \text{ on } \partial F^c, \tag{2}$$

$$P(\boldsymbol{\xi}) \to 0 \ \text{ as } \ |\boldsymbol{\xi}| \to \infty, \tag{3}$$

where the boundary condition (2) is interpreted in the sense of the Sobolev space H^1.

The boundary layer around the perforation will also involve the regular part of Green's function

$$h(\boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{1}{4\pi|\boldsymbol{\xi} - \boldsymbol{\eta}|} - g(\boldsymbol{\xi}, \boldsymbol{\eta}), \tag{4}$$

defined for $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^3 \setminus F, \quad \boldsymbol{\xi} \neq \boldsymbol{\eta}$.

The regular part of Green's function G in the unperturbed domain Ω is also used here

$$\mathcal{H}(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} - G(\mathbf{x}, \mathbf{y}). \tag{5}$$

The following assertions on asymptotics of the capacitary potential and the regular part of Green's function hold.

Lemma 1. *The potential P satisfies the estimate*

$$0 < P(\boldsymbol{\xi}) \leq \min \Big\{ 1, |\boldsymbol{\xi}|^{-1} \Big\}. \tag{6}$$

If $|\boldsymbol{\xi}| \geq 2$, then

$$\Big| P(\boldsymbol{\xi}) - \frac{\text{cap}(F)}{4\pi|\boldsymbol{\xi}|} \Big| \leq \text{Const } |\boldsymbol{\xi}|^{-2}. \tag{7}$$

and

Lemma 2. *For all $\boldsymbol{\eta} \in F^c$ and for $\boldsymbol{\xi}$ with $|\boldsymbol{\xi}| > 2$ the estimate holds:*

$$|h(\boldsymbol{\xi}, \boldsymbol{\eta}) - \frac{P(\boldsymbol{\eta})}{4\pi|\boldsymbol{\xi}|}| \leq \text{Const } \frac{P(\boldsymbol{\eta})}{|\boldsymbol{\xi}|^2}. \tag{8}$$

uniformly with respect to $\boldsymbol{\eta} \in \mathbb{R}^3 \setminus F$.

Lemma 1 is a classical result, and the inequalities (6) follow from the maximum principle for variational solutions of Laplace's equation. In turn, the inequality (7) results from the expansion of P in spherical harmonics.

Lemma 2 has been proved in [6], and the coefficient in the asymptotics of $h(\boldsymbol{\xi}, \boldsymbol{\eta})$ at infinity is evaluated by applying Green's formula to the functions $g(\boldsymbol{\xi}, \boldsymbol{\eta})$ and $1 - P(\boldsymbol{\xi})$ restricted to the domain $B_R \setminus F$, where $B_R = \{\boldsymbol{\xi} : |\boldsymbol{\xi}| < R\}$ for a sufficiently large R, and then taking the limit as $R \to \infty$.

The result on the uniform approximation of Green's function G_ε in the perforated three-dimensional domain is given by

Theorem 1. *Green's function $G_\varepsilon(\mathbf{x}, \mathbf{y})$ admits the representation*

$$\begin{aligned} G_\varepsilon(\mathbf{x}, \mathbf{y}) \ &= G(\mathbf{x}, \mathbf{y}) + \varepsilon^{-1} g(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) - \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \\ +\mathcal{H}(0, \ &\mathbf{y}) P(\varepsilon^{-1}\mathbf{x}) + \mathcal{H}(\mathbf{x}, 0) P(\varepsilon^{-1}\mathbf{y}) - \mathcal{H}(0, 0) P(\varepsilon^{-1}\mathbf{x}) P(\varepsilon^{-1}\mathbf{y}) \\ -\varepsilon \ &\text{cap}(F) \ \mathcal{H}(\mathbf{x}, 0) \mathcal{H}(0, \mathbf{y}) + O\Big(\varepsilon^2 (\min\{|\mathbf{x}|, |\mathbf{y}|\} + \varepsilon)^{-1}\Big), \end{aligned} \tag{9}$$

which is uniform with respect to $\mathbf{x}, \mathbf{y} \in \Omega_\varepsilon$. Here, $\mathcal{H}$ and h are regular parts of Green's functions G and g, respectively (see (5), (4)*), and P is the equilibrium potential of F.*

The detailed proof of the theorem is presented in [6], and a plausible formal argument leading to (9) can be described as follows.

Let G_ε be represented in the form

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} - \mathcal{H}_\varepsilon(\mathbf{x},\mathbf{y}) - h_\varepsilon(\mathbf{x},\mathbf{y}), \tag{10}$$

where $\mathcal{H}_\varepsilon$ and h_ε are solutions of the Dirichlet problems

$$\begin{aligned} \Delta_x \mathcal{H}_\varepsilon(\mathbf{x},\mathbf{y}) &= 0, \quad \mathbf{x},\mathbf{y} \in \Omega_\varepsilon, \\ \mathcal{H}_\varepsilon(\mathbf{x},\mathbf{y}) &= \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}, \quad \mathbf{x} \in \partial\Omega,\ \mathbf{y} \in \Omega_\varepsilon, \\ \mathcal{H}_\varepsilon(\mathbf{x},\mathbf{y}) &= 0, \quad \mathbf{x} \in \partial F_\varepsilon^c,\ \mathbf{y} \in \Omega_\varepsilon. \end{aligned}$$

and

$$\begin{aligned} \Delta_x h_\varepsilon(\mathbf{x},\mathbf{y}) &= 0, \quad \mathbf{x},\mathbf{y} \in \Omega_\varepsilon, \\ h_\varepsilon(\mathbf{x},\mathbf{y}) &= \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}, \quad \mathbf{x} \in \partial F_\varepsilon^c,\ \mathbf{y} \in \Omega_\varepsilon, \\ h_\varepsilon(\mathbf{x},\mathbf{y}) &= 0, \quad \mathbf{x} \in \partial\Omega,\ \mathbf{y} \in \Omega_\varepsilon. \end{aligned} \tag{11}$$

It is noted that $\mathcal{H}_\varepsilon(\mathbf{x},\mathbf{y}) - \mathcal{H}(\mathbf{x},\mathbf{y})$ is harmonic in Ω_ε, and $\mathcal{H}_\varepsilon(\mathbf{x},\mathbf{y}) - \mathcal{H}(\mathbf{x},\mathbf{y}) = 0$ for $\mathbf{x} \in \partial\Omega$. For $\mathbf{x} \in \partial F_\varepsilon^c$ the leading part of $\mathcal{H}_\varepsilon(\mathbf{x},\mathbf{y}) - \mathcal{H}(\mathbf{x},\mathbf{y})$ is equal to $-\mathcal{H}(0,\mathbf{y})$. This function can be extended onto F_ε^c, harmonically in $\mathbf{x}$, as $-\mathcal{H}(0,\mathbf{y})P(\varepsilon^{-1}\mathbf{x})$, and on the exterior boundary, as $\mathbf{x} \in \partial\Omega$, its leading-order part is equal to $-\varepsilon \ \mathrm{cap}(F)\ \mathcal{H}(\mathbf{x},0)\mathcal{H}(0,\mathbf{y})$. Hence,

$$\begin{aligned} \mathcal{H}_\varepsilon(\mathbf{x},\mathbf{y}) - \mathcal{H}(\mathbf{x},\mathbf{y}) &\sim -\mathcal{H}(0,\mathbf{y})P(\varepsilon^{-1}\mathbf{x}) \\ &+ \varepsilon \ \mathrm{cap}(F)\ \mathcal{H}(\mathbf{x},0)\mathcal{H}(0,\mathbf{y}) \quad \text{for all } \mathbf{x},\mathbf{y} \in \Omega_\varepsilon. \end{aligned} \tag{12}$$

According to the formulae (4) and (11), on the boundary of the small void we have

$$h_\varepsilon(\mathbf{x},\mathbf{y}) - \varepsilon^{-1} h(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) = 0 \text{ for } \mathbf{x} \in \partial F_\varepsilon^c.$$

Furthermore, Lemma 2 can be used to deduce

$$h_\varepsilon(\mathbf{x},\mathbf{y}) - \varepsilon^{-1} h(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) \sim -\frac{P(\varepsilon^{-1}\mathbf{y})}{4\pi|\mathbf{x}|} \quad \text{for } \mathbf{x} \in \partial\Omega.$$

The harmonic function $-\mathcal{H}(\mathbf{x},0)P(\varepsilon^{-1}\mathbf{y})$ has the Dirichlet data $-P(\varepsilon^{-1}\mathbf{y})/(4\pi|\mathbf{x}|)$ on $\partial\Omega$, and it is asymptotically equal to $-\mathcal{H}(0,0)P(\varepsilon^{-1}\mathbf{y})$ on ∂F_ε^c. In turn, the harmonic in $\mathbf{x}$ extension of $\mathcal{H}(0,0)P(\varepsilon^{-1}\mathbf{y})$ onto F_ε^c is given by $\mathcal{H}(0,0)P(\varepsilon^{-1}\mathbf{y})P(\varepsilon^{-1}\mathbf{x})$, and this function is small for $\mathbf{x} \in \partial\Omega$. Hence, we use the asymptotic representation

$$\begin{aligned} h_\varepsilon(\mathbf{x},\mathbf{y}) \ & -\varepsilon^{-1} h(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) + \mathcal{H}(\mathbf{x},0)P(\varepsilon^{-1}\mathbf{y}) \\ & \sim \mathcal{H}(0,0)P(\varepsilon^{-1}\mathbf{x})P(\varepsilon^{-1}\mathbf{y}) \qquad \text{for all } \mathbf{x},\mathbf{y} \in \Omega_\varepsilon. \end{aligned} \tag{13}$$

Substituting (12) and (13) into (10), we deduce

$$\begin{aligned} G_\varepsilon(\mathbf{x},\mathbf{y}) \ &\sim \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} - \mathcal{H}(\mathbf{x},\mathbf{y}) - \varepsilon^{-1} h(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) \\ &+\mathcal{H}(0,\mathbf{y})P(\varepsilon^{-1}\mathbf{x}) + \mathcal{H}(\mathbf{x},0)P(\varepsilon^{-1}\mathbf{y}) - \mathcal{H}(0,0)P(\varepsilon^{-1}\mathbf{x})P(\varepsilon^{-1}\mathbf{y}) \\ &-\varepsilon \mathrm{cap}(F)\ \mathcal{H}(\mathbf{x},0)\mathcal{H}(0,\mathbf{y}), \end{aligned}$$

which leads to the asymptotic approximation (9).

The rigorous proof of (9) including the remainder estimate is given in [6].

2.2 A perforated planar domain

We assume that $\Omega \subset \mathbb{R}^2$ and use the notations $\Omega_\varepsilon, \Omega, F_\varepsilon, F$, as well as the scaled coordinates $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, similar to Section 2.1. In the two-dimensional case, an asymptotic approximation of G_ε will possess the $\log \varepsilon$ dependence, which is a new feature in comparison with the formula included in Theorem 1.

Green's function $G(\mathbf{x},\mathbf{y})$ for the unperturbed domain Ω has the form

$$G(\mathbf{x},\mathbf{y}) = (2\pi)^{-1} \log|\mathbf{x}-\mathbf{y}|^{-1} - H(\mathbf{x},\mathbf{y}), \tag{14}$$

where H is its regular part satisfying

$$\Delta_x H(\mathbf{x},\mathbf{y}) = 0, \quad \mathbf{x},\mathbf{y} \in \Omega, \tag{15}$$

$$H(\mathbf{x},\mathbf{y}) = (2\pi)^{-1} \log|\mathbf{x}-\mathbf{y}|^{-1}, \quad \mathbf{x} \in \partial\Omega,\ \mathbf{y} \in \Omega. \tag{16}$$

The notations $g(\boldsymbol{\xi},\boldsymbol{\eta})$ and $h(\boldsymbol{\xi},\boldsymbol{\eta})$ are used for Green's function and its regular part in $\mathbb{R}^2 \setminus F$:

$$\Delta_\xi g(\boldsymbol{\xi},\boldsymbol{\eta}) + \delta(\boldsymbol{\xi}-\boldsymbol{\eta}) = 0, \quad \boldsymbol{\xi},\boldsymbol{\eta} \in F^c, \tag{17}$$

$$g(\boldsymbol{\xi},\boldsymbol{\eta}) = 0, \quad \boldsymbol{\xi} \in \partial F^c,\ \boldsymbol{\eta} \in F^c, \tag{18}$$

$$g(\boldsymbol{\xi},\boldsymbol{\eta}) \ \text{ is bounded } \ \text{as } |\boldsymbol{\xi}| \to \infty \text{ and } \boldsymbol{\eta} \in F^c, \tag{19}$$

and

$$h(\boldsymbol{\xi},\boldsymbol{\eta}) = (2\pi)^{-1} \log|\boldsymbol{\xi}-\boldsymbol{\eta}|^{-1} - g(\boldsymbol{\xi},\boldsymbol{\eta}). \tag{20}$$

Given $\boldsymbol{\eta}$ and taking the limit $|\boldsymbol{\xi}| \to \infty$ we introduce a function ζ:

$$\zeta(\boldsymbol{\eta}) = \lim_{|\boldsymbol{\xi}|\to\infty} g(\boldsymbol{\xi},\boldsymbol{\eta}), \tag{21}$$

and furthermore use the notation $\zeta_\infty = \lim_{|\boldsymbol{\eta}|\to\infty}\{\zeta(\boldsymbol{\eta}) - (2\pi)^{-1}\log|\boldsymbol{\eta}|\}$.

Contrary to Lemma 2, the asymptotics of the regular part of Green's functions for large $|\boldsymbol{\xi}|$ is characterised by the logarithmic growth.

Lemma 3. *Let* $|\boldsymbol{\xi}| > 2$. *Then the regular part* $h(\boldsymbol{\xi}, \boldsymbol{\eta})$ *of Green's function* g *in* F^c *admits the asymptotic representation*

$$h(\boldsymbol{\xi}, \boldsymbol{\eta}) = -(2\pi)^{-1} \log |\boldsymbol{\xi}| - \zeta(\boldsymbol{\eta}) + O(|\boldsymbol{\xi}|^{-1}), \tag{22}$$

which is uniform with respect to $\boldsymbol{\eta} \in F^c$.

The proof, which employs the inversion transformation $\boldsymbol{\xi}' = |\boldsymbol{\xi}|^{-2}\boldsymbol{\xi}$, $\boldsymbol{\eta}' = |\boldsymbol{\eta}|^{-2}\boldsymbol{\eta}$, together with the identity $|\boldsymbol{\xi} - \boldsymbol{\eta}|^{-1}|\boldsymbol{\xi}||\boldsymbol{\eta}| = |\boldsymbol{\xi}' - \boldsymbol{\eta}'|^{-1}$, is given in [6].

Instead of looking for asymptotics at infinity for an equilibrium potential, as in Lemma 1, for the two-dimensional case the *equilibrium potential* $\mathcal{P}_\varepsilon(\mathbf{x})$ is introduced as a solution of the following Dirichlet problem in Ω_ε

$$\Delta \mathcal{P}_\varepsilon(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_\varepsilon, \tag{23}$$

$$\mathcal{P}_\varepsilon(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega, \tag{24}$$

$$\mathcal{P}_\varepsilon(\mathbf{x}) = 1, \quad \mathbf{x} \in \partial F_\varepsilon^c. \tag{25}$$

Lemma 4. *The asymptotic approximation of* $\mathcal{P}_\varepsilon(\mathbf{x})$ *is given by the formula*

$$\mathcal{P}_\varepsilon(\mathbf{x}) = \frac{-G(\mathbf{x}, 0) + \zeta(\frac{\mathbf{x}}{\varepsilon}) - \frac{1}{2\pi} \log \frac{|\mathbf{x}|}{\varepsilon} - \zeta_\infty}{\frac{1}{2\pi} \log \varepsilon + H(0,0) - \zeta_\infty} + p_\varepsilon(\mathbf{x}), \tag{26}$$

where the remainder p_ε *satisfies the estimate* $|p_\varepsilon(\mathbf{x})| \leq$ *Const* $\varepsilon(\log \varepsilon)^{-1}$, *uniformly with respect to* $\mathbf{x} \in \Omega_\varepsilon$.

The proof of this lemma is straightforward. First, we note that the Dirichlet problem for the remainder p_ε is

$$\Delta p_\varepsilon(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_\varepsilon, \tag{27}$$

$$p_\varepsilon(\mathbf{x}) = -\frac{\zeta(\varepsilon^{-1}\mathbf{x}) - \frac{1}{2\pi} \log(\varepsilon^{-1}|\mathbf{x}|) - \zeta_\infty}{\frac{1}{2\pi} \log \varepsilon + H(0,0) - \zeta_\infty}, \quad \mathbf{x} \in \partial\Omega, \tag{28}$$

$$p_\varepsilon(\mathbf{x}) = 1 - \frac{H(\mathbf{x}, 0) + \frac{1}{2\pi} \log \varepsilon - \zeta_\infty}{\frac{1}{2\pi} \log \varepsilon + H(0,0) - \zeta_\infty}, \quad \mathbf{x} \in \partial F_\varepsilon^c. \tag{29}$$

On the exterior boundary, as $\mathbf{x} \in \partial\Omega$, using the expansion of $\zeta(\boldsymbol{\xi})$ in spherical harmonics, we derive $\zeta(\varepsilon^{-1}\mathbf{x}) - (2\pi)^{-1} \log(\varepsilon^{-1}|\mathbf{x}|) - \zeta_\infty = O(\varepsilon)$, and hence deduce that the right-hand side in (28) is $O(\varepsilon(\log \varepsilon)^{-1})$. Furthermore, since $H(\mathbf{x}, 0)$ is smooth in Ω, we have $H(\mathbf{x}, 0) - H(0,0) = O(\varepsilon)$, as $\mathbf{x} \in \partial F_\varepsilon^c$, and hence the right-hand side in (29) is also $O(\varepsilon(\log \varepsilon)^{-1})$. Applying the maximum principle, we arrive at the required result.

The capacitary potential $\mathcal{P}_\varepsilon$ is used in the asymptotic approximation of Green's function in a planar perforated domain defined as follows.

Theorem 2. *Green's function* G_ε *for the operator* $-\Delta$ *in* $\Omega_\varepsilon \subset \mathbb{R}^2$ *admits the representation*

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}) + g(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) + (2\pi)^{-1}\log(\varepsilon^{-1}|\mathbf{x}-\mathbf{y}|)$$
$$+\frac{\Big((2\pi)^{-1}\log\varepsilon + \zeta(\frac{\mathbf{x}}{\varepsilon}) - \zeta_\infty + H(\mathbf{x},0)\Big)\Big((2\pi)^{-1}\log\varepsilon + \zeta(\frac{\mathbf{y}}{\varepsilon}) - \zeta_\infty + H(0,\mathbf{y})\Big)}{(2\pi)^{-1}\log\varepsilon + H(0,0) - \zeta_\infty}$$
$$-\zeta(\varepsilon^{-1}\mathbf{x}) - \zeta(\varepsilon^{-1}\mathbf{y}) + \zeta_\infty + O(\varepsilon), \tag{30}$$

which is uniform with respect to $(\mathbf{x},\mathbf{y}) \in \Omega_\varepsilon \times \Omega_\varepsilon$.

For the detailed proof of the theorem we refer to [6].

2.3 Simplified asymptotic formulae

We note that some of the asymptotic formulae of this section can be simplified, and subject to certain constraints can be written in a concise elegant form.

First, for the planar case of the capacitary potential, it may convenient to use the notion of the inner conformal radius r_F of F, with respect to $\mathbf{O}$, and the outer conformal radius R_Ω of Ω, with respect to $\mathbf{O}$, defined (see [11]) as

$$r_F = \exp(-2\pi\zeta_\infty),\ R_\Omega = \exp(-2\pi H(0,0)),$$

respectively.

In this case, the function $\mathcal{P}_\varepsilon(\mathbf{x})$ is defined by the formula

$$\mathcal{P}_\varepsilon(\mathbf{x}) = \frac{-G(\mathbf{x},0) + \zeta(\frac{\mathbf{x}}{\varepsilon}) - \frac{1}{2\pi}\log\frac{|\mathbf{x}|}{\varepsilon r_F}}{\frac{1}{2\pi}\log\frac{\varepsilon r_F}{R_\Omega}} + p_\varepsilon(\mathbf{x}),$$

where the remainder $p_\varepsilon(\mathbf{x})$ is the same as in (26).

Assume that some constraints are imposed on positions of the points $\mathbf{x},\mathbf{y}$ within Ω_ε. Namely, we would like to consider the situation when both $\mathbf{x}$ and $\mathbf{y}$ are either close to F_ε or when both $\mathbf{x}$ and $\mathbf{y}$ stay outside a finite neighbourhood of F_ε.

For the three dimensional case discussed in Section 2.1 the simplified representations are given by

Corollary 1.

(a) Let $\mathbf{x}$ *and* $\mathbf{y}$ *be points of* $\Omega_\varepsilon \subset \mathbb{R}^3$, *such that*

$$\min\{|\mathbf{x}|,|\mathbf{y}|\} > 2\varepsilon. \tag{31}$$

Then

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}) \quad -\varepsilon\,\mathrm{cap}(F)\ G(\mathbf{x},0)G(0,\mathbf{y})$$
$$+O\Big(\frac{\varepsilon^2}{(|\mathbf{x}||\mathbf{y}|)\min\{|\mathbf{x}|,|\mathbf{y}|\}}\Big). \tag{32}$$

(b) If $\max\{|\mathbf{x}|,|\mathbf{y}|\} < 1/2$*, then*

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = \varepsilon^{-1} g(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y})$$
$$-\mathcal{H}(0,0)(P(\varepsilon^{-1}\mathbf{x})-1)(P(\varepsilon^{-1}\mathbf{y})-1) + O(\max\{|\mathbf{x}|,|\mathbf{y}|\}). \qquad (33)$$

Asymptotic formulae, similar to (32), are also presented in [1].

As expected the formulae (32) and (33) are symmetric with respect to interchange of $\mathbf{x}$ and $\mathbf{y}$. In the formula (32), the domain Ω is characterised by Green's function G, whereas the perforation is characterised by $\mathrm{cap}(F)$. In turn, in formula (33) the exterior of F_ε is characterised by the functions g and P, whereas the domain Ω is represented via the regular part $\mathcal{H}$ of Green's function.

For the planar case of a perforate domain, an analogue of Corollary 1 is

Corollary 2. *(a) Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be points of* $\Omega_\varepsilon \subset \mathbb{R}^2$ *subject to* (31)*. Then*

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}) + \frac{G(\mathbf{x},0)G(0,\mathbf{y})}{\frac{1}{2\pi}\log\varepsilon + H(0,0) - \zeta_\infty} + O\Big(\frac{\varepsilon}{\min\{|\mathbf{x}|,|\mathbf{y}|\}}\Big). \qquad (34)$$

(b) If $\max\{|\mathbf{x}|,|\mathbf{y}|\} < 1/2$*, then*

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = g(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) + \frac{\zeta(\varepsilon^{-1}\mathbf{x})\zeta(\varepsilon^{-1}\mathbf{y})}{\frac{1}{2\pi}\log\varepsilon + H(0,0) - \zeta_\infty} + O(\max\{|\mathbf{x}|,|\mathbf{y}|\}), \qquad (35)$$

Both formulae (34) and (35) are symmetric with respect to the interchange of $\mathbf{x}$ and $\mathbf{y}$. In (34), the domain Ω is characterised by Green's function G and its regular part H, whereas the perforation is characterised by ζ_∞, and $r_F = \exp(-2\pi\zeta_\infty)$, which is the inner conformal radius of the scaled set F.

The proofs of the above corollaries are given in [6], and they employ Lemmas 1, 2 and 3, together with the asymptotic formulae (9) and (30).

2.4 Domain with several small perforations

The results described above can be extended to the case when Ω contains a finite number N of small holes. Such a problem was addressed in [7] both for the case of Laplacian as well as the Lamé system.

We give an outline for the two-dimensional situation, and also include a numerical comparison between the asymptotic approximation and an independent finite element computation. The following theorem, presenting a uniform approximatiohas been proved in [7]:

Theorem 3. *Green's function for the operator* $-\Delta$ *in* $\Omega_\varepsilon \subset \mathbb{R}^2$ *admits the representation*

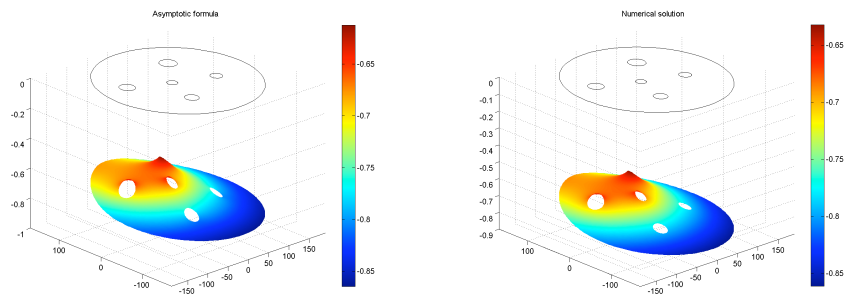

Fig. 3 An asymptotic approximation (left) of the regular part of Green's function compared to a finite element numerical solutions (right) in a domain with several holes.

$$
\begin{aligned}
G_\varepsilon(\mathbf{x},\mathbf{y}) \;=\; & G(\mathbf{x},\mathbf{y}) + \sum_{j=1}^{N} g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) + N(2\pi)^{-1}\log(\varepsilon^{-1}|\mathbf{x}-\mathbf{y}|) \\
& + \sum_{j=1}^{N}\Big\{\alpha_\varepsilon^{(j)} P_\varepsilon^{(j)}(\mathbf{y}) P_\varepsilon^{(j)}(\mathbf{x}) - \zeta^{(j)}(\boldsymbol{\xi}_j) - \zeta^{(j)}(\boldsymbol{\eta}_j) + \zeta_\infty^{(j)}\Big\} \\
& - \sum_{j=1}^{N}\sum_{k\neq j, 1\le k\le N} G(\mathbf{O}^{(k)},\mathbf{O}^{(j)}) P_\varepsilon^{(k)}(\mathbf{y}) P_\varepsilon^{(j)}(\mathbf{x}) + O(\varepsilon)\;, \quad (36)
\end{aligned}
$$

uniformly with respect to $(\mathbf{x},\mathbf{y}) \in \Omega_\varepsilon \times \Omega_\varepsilon$.

In the above formula, $\alpha_\varepsilon^{(j)} = (2\pi)^{-1}\log\varepsilon + H(\mathbf{O}^{(j)},\mathbf{O}^{(j)}) - \zeta_\infty^{(j)}$, and $P_\varepsilon^{(j)}(\mathbf{x})$ stands for the capacitary potential for the perforation $F_\varepsilon^{(j)}$, which is defined as a solution of the boundary value problem:

$$\Delta P_\varepsilon^{(j)}(\mathbf{x}) = 0\;, \quad \mathbf{x} \in \Omega_\varepsilon\;, \tag{37}$$

$$P_\varepsilon^{(j)}(\mathbf{x}) = 0\;, \quad \mathbf{x} \in \partial\Omega\;, \tag{38}$$

$$P_\varepsilon^{(j)}(\mathbf{x}) = \delta_{ij}\;, \quad \mathbf{x} \in \partial F_\varepsilon^{(i)}, i = 1,\ldots,N\;. \tag{39}$$

where δ_{ij} is the Kronecker delta.

The definitions $\boldsymbol{\xi}^{(j)} = \varepsilon^{-1}(\mathbf{x} - \mathbf{O}^{(j)})$ and $\boldsymbol{\eta}^{(j)} = \varepsilon^{-1}(\mathbf{y} - \mathbf{O}^{(j)})$ are used in the formula (36), whereas the quantities $g^{(j)}, \zeta^{(j)}$ and $\zeta_\infty^{(j)}$ have the same meaning as in Section 2.2, with F being replaced by $F^{(j)}$.

The asymptotic approximations (36) has been fully explored in [7]. In Fig. 3 we show a numerical comparison[1], made for a regular part of Green's function, involving the asymptotic formula (36) and the finite element computation made in COMSOL. The domain contains five perforations, and $\varepsilon = 0.2974$. As expected, two independent methods show excellent agreement. The computation demonstrates a high efficiency of the proposed asymptotic algorithm, which is robust and accurate.

We note that the asymptotic approximations discussed above do not cover the case of clouds of defects where the number N of perforations is considered to be a large parameter. However, this situation is well suited for the method of meso-scale asymptotic approximations outlined below.

3 Meso-scale approximations in multiply perforated domains

Assume that Ω is an arbitrary three-dimensional domain, containing a "cloud" of small perforations $F_\varepsilon^{(j)}$ of diameters ε_j, $j = 1, \dots, N$. The perforated domain is denoted by $\Omega_N = \Omega \setminus \cup_{j=1}^N F_\varepsilon^{(j)}$, and it is shown in Fig. 2. A collection of points $\{\mathbf{O}^{(j)}\}_{j=1}^N$ is chosen so that $\mathbf{O}^{(j)} \in F_\varepsilon^{(j)}$, all $F_\varepsilon^{(j)}$ are assumed to be compact subsets of Ω, and $F_\varepsilon^{(j)} \cap F_\varepsilon^{(m)} = \emptyset$, for all $m \neq j$. The "cloud" of perforations is embedded into an open set ω, i.e. $F^{(j)} \subset \omega$, for all $1 \leq j \leq N$, and $\overline{\omega} \subset \Omega$. The following notations are in use $d = 2^{-1} \min_{i \neq j, 1 \leq i,j \leq N} |\mathbf{O}^{(j)} - \mathbf{O}^{(i)}|$, $\varepsilon = \max_{1 \leq j \leq N} \varepsilon_j$. Here, it is assumed that $\varepsilon < c\, d$, and c is a sufficiently small constant. The set ω containing the cloud of perforations, as well as $F_\varepsilon^{(j)}, j = 1, \dots, N$, are required to satisfy the constraints: $\bigcup_{j=1}^N F_\varepsilon^{(j)} \subset \omega$, $\text{diam}(\omega) = 1$, $\text{dist}\ (\partial\omega, \partial\Omega) \geq 2d$, and $\text{dist}\left\{ \bigcup_{j=1}^N F_\varepsilon^{(j)}, \partial\omega \right\} \geq 2d$.

For the case when the number N of perforations becomes large, the asymptotic algorithm described in Section 2 is no longer applicable. A new method of meso-scale asymptotic approximations has been proposed in [8], and it is outlined below.

[1] The computational examples in Figs 3 and 4 have been produced by Dr. M. Nieves.

3.1 Dirichlet problem

Let u denote the variational solution of the Dirichlet problem

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega_N, \tag{40}$$

$$u(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega_N, \tag{41}$$

where f is assumed to be a smooth function with a compact support in Ω, such that $\mathrm{diam}(\mathrm{supp}\ f) \le C$ with C being an absolute constant.

The notation $P^{(j)}$ is used for the capacitary potential of a perforation $F_\varepsilon^{(j)}$.

A formal argument. Let the solution u of (40), (41) be written as

$$u(\mathbf{x}) = v(\mathbf{x}) + R^{(1)}(\mathbf{x}), \tag{42}$$

where v solves the Dirichlet problem in the unperturbed domain Ω, whereas the function $R^{(1)}$ is harmonic in Ω_N and satisfies the boundary conditions

$$R^{(1)}(\mathbf{x}) = 0 \quad \text{when} \quad \mathbf{x} \in \partial\Omega, \tag{43}$$

and

$$R^{(1)}(\mathbf{x}) = -v(\mathbf{x}) = -v(\mathbf{O}^{(k)}) + O(\varepsilon) \quad \text{when} \quad \mathbf{x} \in \partial(\mathbb{R}^3 \setminus F_\varepsilon^{(k)}). \tag{44}$$

Let us approximate the function $R^{(1)}$ in the form

$$R^{(1)}(\mathbf{x}) \sim \sum_{j=1}^{N} C_j \Big(P^{(j)}(\mathbf{x}) - \ \mathrm{cap}(F_\varepsilon^{(j)})\ H(\mathbf{x}, \mathbf{O}^{(j)}) \Big), \tag{45}$$

where C_j are unknown constant coefficients, and H is the regular part of Green's function in Ω.

For all $\mathbf{x} \in \partial\Omega,\ j = 1, \dots, N$, we deduce

$$P^{(j)}(\mathbf{x}) - \ \mathrm{cap}(F_\varepsilon^{(j)})\ H(\mathbf{x}, \mathbf{O}^{(j)}) = O(\varepsilon\ \mathrm{cap}(F_\varepsilon^{(j)}) |\mathbf{x} - \mathbf{O}^{(j)}|^{-2}). \tag{46}$$

On the boundary of a small inclusion $F_\varepsilon^{(k)} \quad (k = 1, \dots, N)$ we have

$$v(\mathbf{O}^{(k)}) + O(\varepsilon) + C_k(1 + O(\varepsilon)) \tag{47}$$

$$+ \sum_{1 \le j \le N,\ j \ne k} C_j \Big(\mathrm{cap}(F_\varepsilon^{(j)})\ G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}) + O(\varepsilon\ \mathrm{cap}(F_\varepsilon^{(j)}) |\mathbf{x} - \mathbf{O}^{(j)}|^{-2}) \Big) = 0,$$

for all $\mathbf{x} \in \partial(\mathbb{R}^3 \setminus F_\varepsilon^{(k)})$.

Equation (47) suggests that the constant coefficients $C_j,\ j = 1, \dots, N$, should be chosen to satisfy the system of linear algebraic equations

$$v(\mathbf{O}^{(k)}) + C_k + \sum_{1\le j\le N,\ j\ne k} C_j\ \mathrm{cap}(F_\varepsilon^{(j)})\ G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}) = 0, \tag{48}$$

where $k = 1, \dots, N$.

Then within certain constraints on the small parameters ε and d (see Lemma 5 below), it is shown that the above system of algebraic equations is solvable and that the harmonic function

$$R^{(2)}(\mathbf{x}) = R^{(1)}(\mathbf{x}) - \sum_{j=1}^{N} C_j \Big(P^{(j)}(\mathbf{x}) - \ \mathrm{cap}(F_\varepsilon^{(j)})\ H(\mathbf{x}, \mathbf{O}^{(j)}) \Big)$$

is small on $\partial\Omega_N$. Further application of the maximum principle for harmonic functions leads to an estimate of the remainder $R^{(2)}$ in Ω_N.

Hence, the solution (42) takes the form

$$u(\mathbf{x}) = v(\mathbf{x}) + \sum_{j=1}^{N} C_j \Big(P^{(j)}(\mathbf{x}) - \ \mathrm{cap}(F_\varepsilon^{(j)})\ H(\mathbf{x}, \mathbf{O}^{(j)}) \Big) + R^{(2)}(\mathbf{x}), \tag{49}$$

where C_j are obtained from the algebraic system (48).

The coefficients in the asymptotic approximation. It is convenient to define the matrices $\mathbf{S}$ and $\mathbf{D}$ as follows:

$$\mathbf{S} = \Big\{ (1 - \delta_{ik}) G(\mathbf{O}^{(k)}, \mathbf{O}^{(i)}) \Big\}_{i,k=1}^{N}, \tag{50}$$

and

$$\mathbf{D} = \mathrm{diag}\ \{\mathrm{cap}(F_\varepsilon^{(1)}), \dots, \mathrm{cap}(F_\varepsilon^{(N)})\}, \tag{51}$$

where G is Green's function of the unperturbed domain, without any perforations. Then the coefficients C_j in the formula (49) can be placed as components of the vector $\mathbf{C} = (C_1, \dots, C_N)^T$ and evaluated as

$$\mathbf{C} = -(\mathbf{I} + \mathbf{SD})^{-1}\mathbf{V}, \tag{52}$$

where

$$\mathbf{V} = (v(\mathbf{O}^{(1)}), \dots, v(\mathbf{O}^{(N)}))^T. \tag{53}$$

The solvability of the algebraic system (48) and the estimate on the coefficients C_j are given by the lemma proved in [7]:

Lemma 5. *If* $\max_{1\le j\le N} \mathrm{cap}(F_\varepsilon^{(j)}) < 5d/6$, *then the matrix* $\mathbf{I} + \mathbf{SD}$ *is invertible and the column vector* $\mathbf{C}$ *in* (52) *satisfies the estimate*

$$\sum_{j=1}^{N} \mathrm{cap}(F_\varepsilon^{(j)})\ C_j^2 \le (1 - \frac{6}{5d} \max_{1\le j\le N} \mathrm{cap}(F_\varepsilon^{(j)}))^{-2} \sum_{j=1}^{N} \mathrm{cap}(F_\varepsilon^{(j)})\ (v_f(\mathbf{O}^{(j)}))^2. \tag{54}$$

Furthermore, with a stronger restriction on the size of perforations it has been shown that the coefficients C_j can be estimated individually as follows.

Lemma 6. *Let the small parameters ε and d satisfy*

$$\varepsilon < cd^2, \tag{55}$$

where c is a sufficiently small absolute constant. Then the components C_j of vector $\mathbf{C}$ in (52) *allow for the estimate*

$$|C_k| \le c \max_{1\le j\le N} |v(\mathbf{O}^{(j)})|. \tag{56}$$

Meso-scale uniform approximation of u. As proved in [8], the following uniform asymptotic approximation of the solution u holds:

Theorem 4. *Let the parameters ε and d satisfy the inequality*

$$\varepsilon < c\ d^{7/4}, \tag{57}$$

where c is a sufficiently small absolute constant.

Then the matrix $\mathbf{I}+\mathbf{SD}$, defined according to (50), (51)*, is invertible, and the solution $u(\mathbf{x})$ to the boundary value problem* (40)–(41) *is defined by the asymptotic formula*

$$u(\mathbf{x}) = v(\mathbf{x}) + \sum_{j=1}^{N} C_j \Big(P^{(j)}(\mathbf{x}) - \ \mathrm{cap}(F_\varepsilon^{(j)})\ H(\mathbf{x}, \mathbf{O}^{(j)}) \Big) + R(\mathbf{x}), \tag{58}$$

where the column vector $\mathbf{C} = (C_1, \dots, C_N)^T$ is given by (52) *and the remainder $R(\mathbf{x})$ is a function harmonic in Ω_N, which satisfies the estimate*

$$|R(\mathbf{x})| \ \le C\Big\{\varepsilon\|\nabla v\|_{L_\infty(\omega)} + \varepsilon^2 d^{-7/2}\|v\|_{L_\infty(\omega)}\Big\}. \tag{59}$$

3.2 Meso-scale approximation of Green's function

Let $G_N(\mathbf{x},\mathbf{y})$ be Green's function of the Dirichlet problem for the operator $-\Delta$ in Ω_N. To derive an asymptotic approximation of $G_N(\mathbf{x},\mathbf{y})$ one would need to solve an algebraic system similar to (48). We also need here Green's functions $g^{(j)}(\mathbf{x},\mathbf{y})$ of the Dirichlet problem for the operator $-\Delta$ in $\mathbb{R}^3 \setminus F^{(j)}$, $j = 1, \dots, N$, and its regular part $h^{(j)}$. The asymptotic formula for Green's function in a multiply perforated domain is given by the theorem.

Theorem 5. *Let the small parameters ε and d satisfy the inequality $\varepsilon < c\ d^2$, where c is a sufficiently small absolute constant. Then*

$$G_N(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}) - \sum_{j=1}^{N} \Big\{ h^{(j)}(\mathbf{x},\mathbf{y}) \tag{60}$$

$$-P^{(j)}(\mathbf{y})H(\mathbf{x},\mathbf{O}^{(j)}) - P^{(j)}(\mathbf{x})H(\mathbf{O}^{(j)},\mathbf{y}) + \ \mathrm{cap}(F_\varepsilon^{(j)})H(\mathbf{x},\mathbf{O}^{(j)})H(\mathbf{O}^{(j)},\mathbf{y})$$

$$+H(\mathbf{O}^{(j)},\mathbf{O}^{(j)})\ T^{(j)}(\mathbf{x})T^{(j)}(\mathbf{y}) - \sum_{i=1}^{N} \mathcal{C}_{ij} T^{(i)}(\mathbf{x})T^{(j)}(\mathbf{y}) \Big\} + \mathcal{R}(\mathbf{x},\mathbf{y}),$$

where

$$T^{(j)}(\mathbf{y}) = P^{(j)}(\mathbf{y}) - \ \mathrm{cap}(F_\varepsilon^{(j)})H(\mathbf{O}^{(j)},\mathbf{y}), \tag{61}$$

with the capacitary potentials $P^{(j)}$ *and the regular part* H *of Green's function* G *of* Ω *being the same as in Section* 3.1*. The matrix* $\boldsymbol{\mathcal{C}} = (\mathcal{C}_{ij})_{i,j=1}^N$ *is defined by*

$$\boldsymbol{\mathcal{C}} = (\mathbf{I} + \mathbf{SD})^{-1}\mathbf{S}, \tag{62}$$

where $\mathbf{S}$ *and* $\mathbf{D}$ *are the same as in* (50), (51)*. The remainder* $\mathcal{R}(\mathbf{x},\mathbf{y})$ *is a harmonic function, both in* $\mathbf{x}$ *and* $\mathbf{y}$*, and satisfies the estimate*

$$|\mathcal{R}(\mathbf{x},\mathbf{y})| \le \mathrm{const}\ \varepsilon d^{-2} \tag{63}$$

uniformly with respect to $\mathbf{x}$ *and* $\mathbf{y}$ *in* Ω_N*.*

It is noted that the coefficients, represented as components of the matrix $\boldsymbol{\mathcal{C}}$ can be estimated as follows.

Lemma 7. *Let the small parameters* ε *and* d *obey the inequality* $\varepsilon < c\ d^2$*, where* c *is a sufficiently small absolute constant. Then the matrix* $\boldsymbol{\mathcal{C}}$ *in* (62) *satisfies the estimate*

$$\|\boldsymbol{\mathcal{C}}\|_{\mathbb{R}^N \to \mathbb{R}^N} \le c d^{-3}. \tag{64}$$

3.3 Meso-scale approximation versus homogenization

Let us consider a bounded domain Ω in $\mathbb{R}^3$, which contains a large number of points $\mathbf{O}_1, \mathbf{O}_2, ..., \mathbf{O}_N$ arranged in a cubic array. By F we denote a compact set containing the origin, the same as in Section 2.1. The domain Ω is perforated by small holes $F_\varepsilon^{(j)} = \{\mathbf{x} \in \Omega : \varepsilon^{-1}(\mathbf{x} - \mathbf{O}_j) \in F\}$. Assuming that d is the minimum of the distance between neighbouring holes and the distance between the holes and the exterior boundary $\partial\Omega$, we consider the case when $\varepsilon\ \mathrm{cap}(F)/d^3 = \mu = \mathrm{const}$.

In this case, Green's function of the Dirichlet problem in the perforated domain is approximated by the asymptotic formula:

$$G_N(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}) - \sum_{j=1}^{N} h^{(j)}(\mathbf{x},\mathbf{y}) + \sum_{j=1}^{N} \{P^{(j)}(\mathbf{y})H(\mathbf{x},\mathbf{O}_j) + P^{(j)}(\mathbf{x})H(\mathbf{O}_j,\mathbf{y})\}$$

$$-\varepsilon \ \mathrm{cap}(F) \ \sum_{j=1}^{N} H(\mathbf{x},\mathbf{O}_j)H(\mathbf{O}_j,\mathbf{y}) - \sum_{j=1}^{N} H(\mathbf{O}_j,\mathbf{O}_j)T^{(j)}(\mathbf{x})T^{(j)}(\mathbf{y})$$

$$+\sum_{j=1}^{N}\sum_{i\neq j} G_{hom}(\mathbf{O}_j,\mathbf{O}_i)T^{(i)}(\mathbf{y})T^{(j)}(\mathbf{x}) + O(d^{-1}\varepsilon), \tag{65}$$

where $T^{(j)}$ are the same as in (61), and $G_{hom}(\mathbf{x},\mathbf{y})$ is the Green's function of the "homogenized" operator $\Delta - \mu$ in Ω.

It is worthwhile to emphasize that the nature of formula (65) is qualitatively different from that of the classical homogenization theory. The principal difference is that this formula includes the discrete values of the homogenized Green's function, and in comparison with the classical homogenization theory the right-hand side in (65) satisfies the same equation as the perturbed function G_N. An important feature of this approximation is its uniformity with respect to $\mathbf{x}$ and $\mathbf{y}$ in Ω_N.

3.4 Mixed boundary value problem for a multiply perforated body

Finally, we show an illustration involving an asymptotic approximation, constructed in [9], for the case of a large number of perforations subjected to the Neumann boundary conditions. Namely, the paper [9] deals with a boundary value problem

$$-\Delta u_N(\mathbf{x}) = f(\mathbf{x}) \ , \quad \mathbf{x} \in \Omega_N \ , \tag{66}$$

$$u_N(\mathbf{x}) = \phi(\mathbf{x}) \ , \quad \mathbf{x} \in \partial\Omega \ , \tag{67}$$

$$\frac{\partial u_N}{\partial n}(\mathbf{x}) = 0 \ , \quad \mathbf{x} \in \partial F_\varepsilon^{(j)} \ , j = 1, \dots, N \ , \tag{68}$$

where $\phi \in L^{1/2,2}(\partial\Omega)$ and $f(\mathbf{x})$ is a function in $L^\infty(\Omega)$ with compact support at a positive distance from the cloud ω of small perforations.

Auxiliary solutions to several model problems include

1. v as the solution of the unperturbed problem in Ω (without voids),
2. $\boldsymbol{\mathcal{D}}^{(k)}$ as the vector function whose components are the dipole fields for the void $F^{(k)}$,
3. H as the regular part of Green's function G in Ω.

It is convenient to define

$$\boldsymbol{\Theta} = \Big(\frac{\partial v}{\partial x_1}(\mathbf{O}^{(1)}), \frac{\partial v}{\partial x_2}(\mathbf{O}^{(1)}), \frac{\partial v}{\partial x_3}(\mathbf{O}^{(1)}), \dots, \frac{\partial v}{\partial x_1}(\mathbf{O}^{(N)}), \frac{\partial v}{\partial x_2}(\mathbf{O}^{(N)}), \frac{\partial v}{\partial x_3}(\mathbf{O}^{(N)}) \Big)^T ,$$

and $\mathfrak{S} = [\mathfrak{S}_{ij}]_{i,j=1}^N$ which is a $3N \times 3N$ matrix with 3×3 block entries

$$\mathfrak{S}_{ij} = \begin{cases} (\nabla_{\mathbf{z}} \otimes \nabla_{\mathbf{w}}) \, (G(\mathbf{z}, \mathbf{w})) \Big|_{\substack{\mathbf{z}=\mathbf{O}^{(i)} \\ \mathbf{w}=\mathbf{O}^{(j)}}} & \text{if } i \neq j \\ 0 I_3 & \text{otherwise} \end{cases} ,$$

where I_3 is the 3×3 identity matrix. We also use the block-diagonal matrix

$$\mathbf{Q} = \text{diag}\{\boldsymbol{\mathcal{Q}}^{(1)}, \dots, \boldsymbol{\mathcal{Q}}^{(N)}\}, \tag{69}$$

where $\boldsymbol{\mathcal{Q}}^{(k)}$ is the so-called 3×3 polarization matrix for the small void $F_\varepsilon^{(k)}$ (see, for example, Appendix G of [11]). The shapes of the voids $F_\varepsilon^{(j)}, j = 1, \dots, N$, are constrained in such a way that the maximal and minimal eigenvalues $\lambda_{max}^{(j)}, \ \lambda_{min}^{(j)}$ of the matrices $-\boldsymbol{\mathcal{Q}}^{(j)}$ satisfy the inequalities

$$A_1 \varepsilon^3 > \max_{1 \leq j \leq N} \lambda_{max}^{(j)}, \quad \min_{1 \leq j \leq N} \lambda_{min}^{(j)} > A_2 \varepsilon^3, \tag{70}$$

where A_1 and A_2 are positive and independent of ε.

One of the results, for the case when $\Omega = \mathbb{R}^3$, and when (67) is replaced by the condition of decay of u_N at infinity, can be formulated as follows

Theorem 6. *Let*

$$\varepsilon < c\, d \, ,$$

where c is a sufficiently small absolute constant. Then the solution $u_N(\mathbf{x})$ admits the asymptotic representation

$$u_N(\mathbf{x}) = v(\mathbf{x}) + \sum_{k=1}^N \boldsymbol{C}^{(k)} \cdot \boldsymbol{\mathcal{D}}^{(k)}(\mathbf{x}) + \mathcal{R}_N(\mathbf{x}) \, , \tag{71}$$

where $\boldsymbol{C}^{(k)} = (C_1^{(k)}, C_2^{(k)}, C_3^{(k)})^T$ and the column vector $\mathbf{C} = (C_1^{(1)}, C_2^{(1)}, C_3^{(1)}, \dots, C_1^{(N)}, C_2^{(N)}, C_3^{(N)})^T$ satisfies the linear algebraic system

$$(\mathbf{I} + \mathfrak{S}\mathbf{Q})\mathbf{C} = -\boldsymbol{\Theta} \, . \tag{72}$$

The remainder $\mathcal{R}_N$ satisfies the energy estimate

$$\|\nabla \mathcal{R}_N\|^2_{L_2(\Omega_N)} \leq \text{const} \, \Big\{ \varepsilon^{11} d^{-11} + \varepsilon^5 d^{-3} \Big\} \|\nabla v\|^2_{L^2(\Omega)}. \tag{73}$$

Small parameters ε and d have the same meaning as in Section 3. The proof of the theorem as well as analysis of solvability of (72) are included in [9].

Illustrative example for a non-uniform cloud containing a large number of spherical voids. A cloud of voids is chosen in such a way that the number N

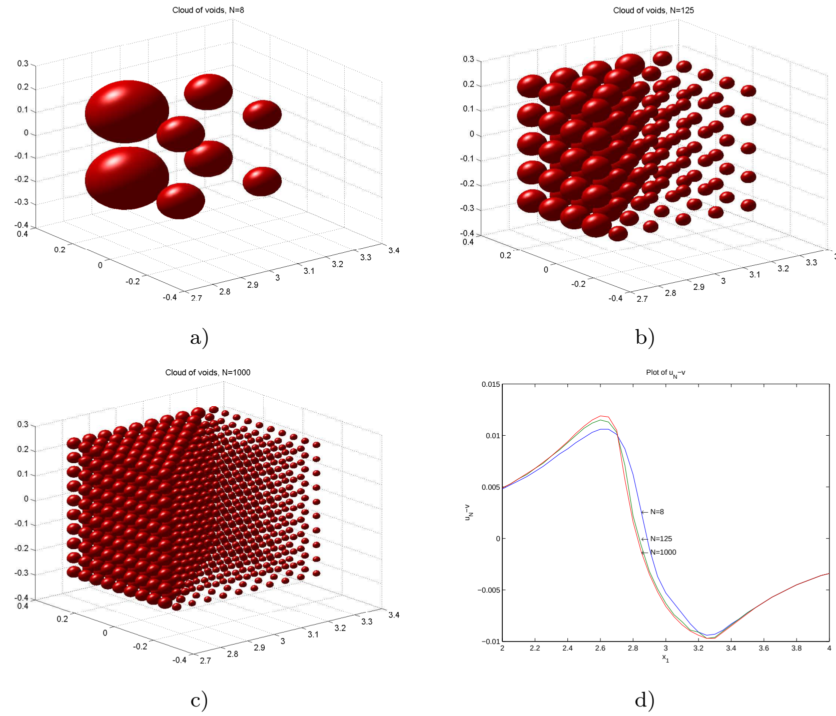

Fig. 4 The cloud of voids for the cases when a) $N = 8$, b) $N = 125$ and c) $N = 1000$. d) The graph of $u_N - v$ for $2 \leq x_1 \leq 4$ plotted along the straight line γ adjacent to the cloud of small voids.

may be large and voids may have different radii. For different N, the overall volume of voids is preserved; several examples of the clouds chosen for the illustrative example are shown in Figs. 4(a,b,c).

A cloud ω is placed in the cube ω with side length $\frac{1}{\sqrt{3}}$ and the centre at $(3, 0, 0)$. Fig. 4 shows the clouds of voids for a) $N = 8$, b) $N = 125$, and c) $N = 1000$. The plot of $u_N - v$ is shown in Fig. 4d. The asymptotic correction has been computed along the straight line $\gamma = \{x_1 \in \mathbb{R}, x_2 = -1/(2\sqrt{3}), x_3 = -1/(2\sqrt{3})\}$. Dipole type fluctuations are clearly visible on the diagram. Beyond $N = 1000$ the graphs are visually indistinguishable and hence the values $N = 8, 125, 1000$, have been chosen for this illustrative computation. The algorithm is fast and does not impose periodicity constraints on the array of small voids.

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