ASYMPTOTICS FOR SOLUTIONS OF ELLIPTIC EQUATIONS IN DOUBLE DIVERGENCE FORM

VLADIMIR MAZ'YA AND ROBERT MCOWEN

Abstract. We consider weak solutions of an elliptic equation of the form $\partial_i\partial_i(a_{ij}u) = 0$ and their asymptotic properties at an interior point. We assume that the coefficients are bounded, measurable, complex-valued functions that stabilize as $x \to 0$ in that the difference $a_{ij}(x) - \delta_{ij}$ on the annulus $B_{2r}\backslash B_r$ is bounded by a function $\Omega(r)$, where $\Omega^2(r)$ satisfies the Dini condition at $r = 0$, as well as some technical monotonicity conditions. We construct a particular weak solution Z with explicit leading order term and we show that all weak solutions (subject to a natural growth restriction) are asymptotic to cZ, where c is a constant. One of the features of this work is that solutions need not be continuous, so measurements of growth or decay are made in the sense of L^p.

0. Introduction

We are interested in the local behavior of weak solutions to the elliptic equation in "double divergence form"

$$\mathcal{A}u := \partial_i\partial_j(a_{ij}(x)u(x)) = 0, \tag{1}$$

where we have used $\partial_i = \partial/\partial x_i$ and the summation convention; the coefficients $a_{ij} = a_{ji}$ are bounded, measurable, complex-valued functions in a domain to be specified. The operator $\mathcal{A}$ arises naturally as the the formal adjoint $\mathcal{L}^*$ of the operator in "non-divergent form,"

$$\mathcal{L} = \overline{a}_{ij}(x)\partial_i\partial_j. \tag{2}$$

Solutions of (1) are not only important for the solvability of $\mathcal{L}u = f$, but for properties of the Green's function for $\mathcal{L}$. When the coefficients a_{ij} are real-valued functions, the operators $\mathcal{L}$ and $\mathcal{A}$ have been studied by many authors, including Sjögren ([24]), Bauman ([3], [4], [5]), Fabes and Stroock ([12]), Fabes, Garofalo, Marín-Malavé, and Salsa ([11]), Escauriaza and Kenig ([10]), and Escauriaza ([8], [9]), using techniques derived from the maximum principle. We shall have more to say about how our results compare with theirs at the end of this Introduction, but let us here simply observe that in our case of complex coefficients, the maximum principle no longer applies.

We want to study weak solutions of (1) in the neighborhood of an interior point of the domain, say $x = 0$, where the coefficients stabilize in the following sense:

$$\sup_{r<|x|<2r} \sum_{i,j=1}^{n} |a_{ij}(x) - \delta_{ij}| \leq \Omega(r), \tag{3}$$

and $\Omega(r) \to 0$ as $r \to 0$ in a manner that we shall describe. We remark that, when the coefficients are real-valued, the more general case obtained by replacing δ_{ij} by elliptic constants α_{ij} can be reduced to (3) by means of an affine change of the x variables. Of course, this reduction of the more general case to (3) is not available when the constants α_{ij} are complex-valued, but we have chosen to treat the special case $\alpha_{ij} = \delta_{ij}$ in order to take advantage of technical simplifications in the formulations and proofs of our results.

Date: February 22, 2006.

The specific hypotheses that we impose on the function $\Omega(r)$ in (3) are as follows:

$$\int_0^1 \frac{\Omega^2(t)}{t} dt < \infty, \tag{4}$$

$$\Omega(r)\, r^{-1+\varepsilon} \quad \text{is nonincreasing for } 0 < r < 1 \text{ and,} \tag{5}$$

$$\Omega(r)\, r^{n-\varepsilon} \quad \text{is nondecreasing for } 0 < r < 1; \tag{6}$$

here $\varepsilon > 0$. Clearly, (4) together with (5) or (6) implies that $\Omega(r) \to 0$ as $r \to 0$, so the coefficients a_{ij} are approaching δ_{ij} as $x \to 0$, although perhaps at a slow rate.

A weak solution of (1) in a domain $U \subset \mathbb{R}$ is a function $u \in L^1_{loc}(U)$ that satisfies

$$\int_U a_{ij}(x)\, u(x)\, \partial_j \partial_i \eta(x)\, dx = 0 \quad \text{for all } \eta \in C_0^\infty(U). \tag{7}$$

Weak solutions of (1) need not be continuous under our assumptions on the coefficients, so to measure growth or decay as $x \to 0$, we will use the mean in L^p for some $p \in (1, \infty)$:

$$M_p(w, r) := \left(\fint_{A_r} |w|^p\, dx \right)^{1/p}, \tag{8}$$

where A_r is the annulus $B_{2r} \backslash B_r$ with $B_r = \{x \in \mathbb{R} : |x| < r\}$; here (and elsewhere in this paper) the slashed integral denotes the mean value. (We will also use the notation $M_p(w, r)$ when w is vector or matrix valued; in this case, $|w|$ denotes the norm of w.)

For our local results, we will consider (1) in the unit ball B_1 and we will assume

$$\int_0^1 \frac{\Omega^2(t)}{t} dt < \delta. \tag{9}$$

where δ is sufficiently small. In fact, this represents no additional assumption on $\Omega(r)$ since we could replace B_1 in what follows by a very small ball B_γ in order to make the integral $\int_0^\gamma \Omega^2(t) t^{-1} dt$ as small as necessary.

At times it will be useful to consider solutions of (1) in all of $\mathbb{R}^n$; in that case, we assume that $a_{ij} = \delta_{ij}$ outside of B_1. Our first result concerns such a solution.

Theorem 1. *Let $n \geq 2$, $p \in (1, \infty)$, and $\Omega(r)$ satisfy (5), (6), and (9). There exists a weak solution $Z \in L^p_{\text{loc}}(\mathbb{R}^n)$ of equation (1) in $\mathbb{R}^n$ satisfying*

$$Z(x) = \exp\left[-c_0 \int_{B_1 \backslash B_{|x|}} (a_{ii}(y) - n a_{ij}(y) y_i y_j |y|^{-2}) \frac{dy}{|y|^n} \right] (1 + \zeta(x)), \tag{10}$$

where $c_0 = |\partial B_1|^{-1}$ and $M_p(\zeta, r) \leq c \max(\Omega(r), \int_0^r \Omega^2(t) t^{-1}\, dt)$ for $0 < r < 1$.

Remark 1. *The solution Z obtained in Theorem 1 has at most a mild singularity at the origin, and has a limit at infinity, $Z(\infty)$, satisfying*

$$|Z(x) - Z(\infty)| \leq c \sqrt{\delta}\, |x|^{-n}.$$

Our second theorem uses the solution Z from Theorem 1 to characterize the asymptotics (as $x \to 0$) of weak solutions of (1); because this is a local result, we consider a solution in B_1.

Theorem 2. *Let $n > 2$, $p \in (1, \infty)$, and $\Omega(r)$ satisfy (5), (6), and (9). Suppose that $u \in L^p_{\text{loc}}(\overline{B}_1 \backslash \{0\})$ is a weak solution of (1) in B_1 subject to the growth condition*

$$M_p(u, r) \leq c\, r^{2-n+\varepsilon_0}, \tag{11}$$

where $\varepsilon_0 > 0$. Then there exists a constant C (depending on u) such that

$$u(x) = C Z(x) + w(x), \tag{12}$$

where the remainder term w satisfies

$$M_p(w,r) \le c\, r^{1-\varepsilon_1} \tag{13}$$

for $0 < r < 1$ and any $\varepsilon_1 > 0$.

Remark 2. *The restriction $n > 2$ in Theorem 2 is caused by the existence of solutions for the Laplacian with logarithmic growth at $x = 0$ when $n = 2$. A refinement of the techniques used in proving Theorem 2 would be required to cover the case $n = 2$.*

The following two results are immediate consequences of Theorems 1 and 2.

Corollary 1. *Under the hypotheses of Theorem 2, the condition*

$$\liminf_{r\to 0} \int_{B_1\setminus B_r} (a_{ii}(y) - n a_{ij}(y) y_i y_j |y|^{-2}) \frac{dy}{|y|^n} > -\infty \tag{14}$$

is necessary and sufficient for an arbitrary weak solution $u \in L^p_{\rm loc}(\overline{B}_1\setminus\{0\})$ of (1) subject to (11) to satisfy the condition that $M_p(u,r)$ is bounded for $r \to 0$.

Corollary 2. *Under the hypotheses of Theorem 2, the condition*

$$\lim_{r\to 0} \int_{B_1\setminus B_r} (a_{ii}(y) - n a_{ij}(y) y_i y_j |y|^{-2}) \frac{dy}{|y|^n} = +\infty \tag{15}$$

is necessary and sufficient for an arbitrary weak solution $u \in L^p_{\rm loc}(\overline{B}_1\setminus\{0\})$ of (1) subject to (11) to satisfy $M_p(u,r) \to 0$ as $r \to 0$.

Remark 3. *It follows immediately from (10) and (12) that the solution u in Theorem 2 satisfies*

$$M_p(u,r) \le c_1 \exp\left(c_2 \int_r^1 \Omega(s) \frac{ds}{s}\right)$$

for $r \in (0,1)$. Even this rough estimate seems to be new.

The principal analytic content of our results is contained in Theorem 1. The method of its proof is independent of, but related to, the asymptotic theory developed in [17]. In particular, L_p-means of type (8) were extensively used in [15] and [16]. The asymptotic formula that we obtain is analogous to that of [18], where an asymptotic representation near the boundary was obtained for solutions to the Dirichlet problem for elliptic equations in divergence form with discontinuous coefficients.

Now let us return to the comparison of our results with the extensive work of previous authors; we refer to the excellent exposition in Escauriaza [9] for a more detailed description of these results as well as references to the literature. When the a_{ij} are real-valued, measurable, and uniformly elliptic (although not necessarily continuous) on $\mathbb{R}^n$, these authors show i) the existence of a unique nonnegative weak solution Z of (1) in $\mathbb{R}^n$ that satisfies $\int_{B_1} Z\, dx = |B_1|$, and ii) for every weak solution $u \in L^1_{\rm loc}(B_1)$ of (1) in B_1 the function u/Z is Hölder continuous in B_1; moreover, they use Z to characterize the behavior of the Green's function for the operator $\mathcal{L}$ on $\mathbb{R}^n$ in terms of integrability properties of Z. These results are quite general, but do not apply to the case of complex coefficients that we consider because they depend upon the maximum principle. Moreover, even when the coefficients are real-valued, our results are somewhat different in nature than the previous ones: we have obtained an asymptotic description of Z near a point where the a_{ij} are continuous in the sense of (3). For a more direct comparison, when the a_{ij} are real-valued and continuous at 0, Escauriaza [9] has obtained upper and lower estimates for the L^1-norm of Z: for any $\varepsilon > 0$ there exists a constant N_ε such that

$$N_\varepsilon^{-1} r^\varepsilon \le \fint_{B_r} Z\, dx \le N_\varepsilon r^{-\varepsilon}. \tag{16}$$

When the a_{ij} satisfy our stronger sense of continuity (3), our formula (10) not only implies (16), but gives an explicit formula for the leading asymptotic term and shows that the remainder term may be bounded in terms of L^p for $1 < p < \infty$.

1. Preliminary Estimates

In this and the next section, we will use the spherical mean of a function w. For notational convenience, we denote the spherical mean using an "overbar":

$$\overline{w}(r) = ⨍_{\partial B_1} w(r\theta)\, ds. \tag{17}$$

This should cause no confusion with complex conjugation since we will not have occasion to use the latter in these sections. In particular, in this section we are concerned with solving an equation of the form

$$-\Delta v = \partial_i \partial_j (F_{ij}) - \overline{\partial_i \partial_j (F_{ij})} \quad \text{in } \mathbb{R}^n. \tag{18}$$

Here $F_{ij} \in L^1_{loc}(\mathbb{R}^n)$ and derivatives are interpreted in the sense of distributions. The norm of the matrix $\mathcal{F} = (F_{ij})$ will be denoted by $|\mathcal{F}|$.

Proposition 1. *Suppose that $F_{ij} \in L^p_{loc}(\mathbb{R}^n \backslash \{0\})$ satisfies*

$$\int_{|x|<1} |\mathcal{F}(x)|\, dx + \int_{|x|>1} |\mathcal{F}(x)||x|^{-n-1} dx < \infty. \tag{19}$$

Then there exists a weak solution $v \in L^p_{loc}(\mathbb{R}^n \backslash \{0\})$ of (18) that satisfies

$$M_p(v,r) \leq c \left(\tilde{M}_p(\mathcal{F},r) + r \int_{|x|>r} |\mathcal{F}(x)||x|^{-n-1} dx + r^{-n} \int_{|x|<r} |\mathcal{F}(x)| dx \right), \tag{20}$$

where c is independent of r and we have introduced

$$\tilde{M}_p(w,r) := \left(⨍_{r/2<|x|<4r} |w(x)|^p\, dx \right)^{1/p}.$$

Proof: It suffices to prove the result for $F_{ij} \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$ since the general case can be handled by a standard approximation argument. The function v is defined by convolution with Γ, the fundamental solution for the Laplacian:

$$v = \Gamma \star (\partial_i \partial_j F_{ij} - \overline{\partial_i \partial_j F_{ij}}).$$

Using $\int_{\mathbb{R}^n} f(y) \overline{g}(|y|)\, dy = \int_{\mathbf{R^n}} \overline{f}(|y|) g(y)\, dy$, we can write this as

$$v(x) = \int_{\mathbb{R}^n} \left(\Gamma(|x-y|) - \overline{\Gamma(|x-\cdot|)}(|y|) \right) \partial_i \partial_j F_{ij}(y)\, dy.$$

Now to compute the spherical mean of the fundamental solution, we can use the mean value theorem for harmonic functions to conclude that

$$\overline{\Gamma(|x-\cdot|)}(|y|) = \Gamma(\max\{|x|,|y|\}).$$

This enables us to express v as

$$v(x) = \frac{\partial^2}{\partial x_i \partial x_j} \int_{\mathbb{R}^n} \Gamma(|x-y|) F_{ij}(y)\, dy - \Gamma(|x|) \int_{|y|<|x|} \frac{\partial^2 F_{ij}}{\partial y_i \partial y_j}\, dy - \int_{|y|>|x|} \Gamma(|y|) \frac{\partial^2 F_{ij}}{\partial y_i \partial y_j}\, dy.$$

Now, integration by parts yields

$$\int_{|y|<|x|} \frac{\partial^2 F_{ij}}{\partial y_i \partial y_j}\, dy = \int_{|y|=|x|} \frac{\partial F_{ij}}{\partial y_j} \frac{y_i}{|y|}\, dS_y,$$

and

$$\int_{|y|>|x|} \Gamma(|y|)\frac{\partial^2 F_{ij}}{\partial y_i \partial y_j}\,dy = -\int_{|y|>|x|} \Gamma'(|y|)\frac{y_i}{|y|}\frac{\partial F_{ij}}{\partial y_j}\,dy - \int_{|y|=|x|} \Gamma(|y|)\frac{\partial F_{ij}}{\partial y_j}\frac{y_i}{|y|}\,dS_y$$

$$= \int_{|y|>|x|} \frac{\partial}{\partial y_j}\left(\Gamma'(|y|)\frac{y_i}{|y|}\right) F_{ij}(y)\,dy + \int_{|y|=|x|}\left(\Gamma'(|y|)\frac{y_j}{|y|}F_{ij}(y) - \Gamma(|y|)\frac{\partial F_{ij}}{\partial y_j}\right)\frac{y_i}{|y|}\,dS_y.$$

Consequently,

$$v(x) = \frac{\partial^2}{\partial x_i \partial x_j}\int_{\mathbb{R}^n} \Gamma(|x-y|)F_{ij}(y)\,dy - \int_{|y|>|x|}\left(\frac{\partial^2}{\partial y_j \partial y_i}\Gamma(|y|)\right)F_{ij}(y)\,dy$$

$$-\Gamma'(|x|)\int_{|y|=|x|}\frac{y_i y_j}{|y|^2}F_{ij}(y)\,dS_y.$$

Now introduce χ_0 and χ_∞ as the characteristic functions of $B_{r/2}$ and B_{4r}^c, and let $\chi_1 = 1-\chi_0-\chi_\infty$ be the characteristic function of the annulus $B_{4r}\backslash B_{r/2}$. Then

$$v(x) - \frac{\partial^2}{\partial x_i \partial x_j}\int_{\mathbb{R}^n}\Gamma(|x-y|)(\chi_1 F_{ij})(y)\,dy =$$

$$\int_{\mathbb{R}^n}\frac{\partial^2}{\partial x_i \partial x_j}\Gamma(|x-y|)(\chi_0 F_{ij})(y)\,dy + \int_{\mathbb{R}^n}\frac{\partial^2}{\partial y_i \partial y_j}(\Gamma(|x-y|)-\Gamma(|y|))(\chi_\infty F_{ij})(y)\,dy$$

$$-\int_{B_{4r}\backslash B_{|x|}}\left(\frac{\partial^2}{\partial y_i \partial y_j}\Gamma(|y|)\right)F_{ij}(y)\,dy - \Gamma'(|x|)\int_{|y|=|x|}\frac{y_i y_j}{|y|^2}F_{ij}(y)\,dS_y.$$

We can estimate the four integral kernels and obtain that the right hand side is bounded by

$$c\left(\frac{1}{|x|^n}\int_{B_{r/2}}|F_{ij}(y)|dy + |x|\int_{B_{4r}^c}\frac{|F_{ij}(y)|dy}{|y|^{n+1}} + |x|\int_{B_{4r}\backslash B_{|x|}}\frac{|F_{ij}(y)|dy}{|y|^{n+1}} + \overline{|F_{ij}|}(|x|)\right)$$

This provides us with the following pointwise bound:

$$\left|v(x) - \frac{\partial^2}{\partial x_i \partial x_j}\int_{\mathbb{R}^n}\Gamma(|x-y|)(\chi_1 F_{ij})(y)\,dy\right|$$

$$\le c\left(\overline{|F_{ij}|}(|x|) + |x|\int_{B_r^c}|F_{ij}(y)|\frac{dy}{|y|^{n+1}} + |x|^{-n}\int_{B_r}|F_{ij}(y)|dy\right). \tag{21}$$

Using the L^p-boundedness of singular integral operators on $\mathbb{R}^n$ (see [27]), we have

$$\left\|\frac{\partial^2}{\partial x_i \partial x_j}\int_{\mathbf{R}^n}\Gamma(|x-y|)(\chi_1 F_{ij})(y)dy\right\|_{L^p(A_r)} \le \|\chi_1\mathcal{F}\|_{L^p(\mathbf{R}^n)} = \|\mathcal{F}\|_{L^p(\tilde{A}_r)}.$$

Elementary estimates may be applied to the remaining terms in (21) to obtain (20), completing the proof. □

The integrals in (20) can be estimated in terms of M_p. For example, we substitute

$$|x|^{-n-1} = c_n\int_{\frac{|x|}{2}<|y|<|x|}|y|^{-2n-1}\,dy,$$

into the first integral and change order of integration to obtain the estimate

$$r\int_{|x|>r}|\mathcal{F}(x)||x|^{-n-1}dx \le c\,r\int_{|y|>r/2}\int_{|y|<|x|<2|y|}|\mathcal{F}(x)|\,dx\,\frac{dy}{|y|^{2n+1}} \le c\,r\int_{r/2}^{\infty}M_p(\mathcal{F},\rho)\,\frac{d\rho}{\rho^2}.$$

Similarly, we can show

$$r^{-n}\int_{|x|<r}|\mathcal{F}(x)|\,dx \le c\,r^{-n}\int_0^r M_p(\mathcal{F},\rho)\,\rho^{n-1}\,d\rho$$

and

$$\tilde{M}_p(\mathcal{F},r)^p \le cr^{-n}\int_{r/2}^{4r} M_p(\mathcal{F},\rho)^p\,\rho^{n-1}\,d\rho.$$

Elementary estimates show that terms involving integration over $r/2<\rho<r$ and $2r<\rho<4r$ can be respectively dominated by the terms involving integration over $0<\rho<r$ and $\rho>r$, so we obtain the following.

Corollary 3. *Under the hypotheses of Proposition 1, the weak solution v obtained there satisfies*

$$M_p(v,r)\le c\left(r\int_r^\infty M_p(\mathcal{F},\rho)\rho^{-2}\,d\rho + r^{-n}\int_0^r M_p(\mathcal{F},\rho)\rho^{n-1}\,d\rho\right). \tag{22}$$

2. Proof of Theorem 1

We shall prove Theorem 1 by reducing the problem of finding Z to solving an operator equation of the form $(I+\tilde{T})V=f$, where V and f are elements of a Banach space X of functions on $\mathbb{R}^n\backslash\{0\})$, and $\tilde{T}$ is an integral operator of small norm on X. However, this reduction will take a few steps. To begin, let $r=|x|$, $\theta=x/|x|$, and $\eta\in C_0^\infty((0,\infty))$ be arbitrary. For $Z\in L^p_{\rm loc}(\mathbb{R}^n)$ to be a weak solution of (1), we must have

$$0=\int_{\mathbb{R}^n}\partial_i\partial_j\eta(|x|)a_{ij}(x)Z(x)\,dx$$

$$=\int_0^\infty\left(\eta''(r)\int_{\partial B_1}Z(r\theta)a_{ij}(r\theta)\theta_i\theta_j ds+\frac{\eta'(r)}{r}\int_{\partial B_1}Z(r\theta)(a_{ii}(r\theta)-a_{ij}(r\theta)\theta_i\theta_j)\,ds\right)r^{n-1}dr,$$

where ds denotes surface measure on the unit sphere, ∂B_1. Hence,

$$0=\int_0^\infty\eta'(r)\left(-\frac{d}{dr}\left[r^{n-1}\int_{\partial B_1}Z(r\theta)a_{ij}(r\theta)\theta_i\theta_j\,ds\right]\right.$$
$$\left.+r^{n-2}\int_{\partial B_1}Z(r\theta)(a_{ii}(r\theta)-a_{ij}(r\theta)\theta_i\theta_j)ds\right)dr,$$

where the derivative is understood in the distributional sense. This implies

$$-r^{n-1}\frac{d}{dr}\int_{\partial B_1}Z(r\theta)a_{ij}(r\theta)\theta_i\theta_j ds+r^{n-2}\int_{\partial B_1}Z(r\theta)(a_{ii}(r\theta)-na_{ij}(r\theta)\theta_i\theta_j)ds=C, \tag{23}$$

where C is an arbitrary constant. In what follows, we will take $C=0$; as we shall see, the solution that we construct will in fact be a weak solution of (1) on all of $\mathbb{R}^n$, not just $\mathbb{R}^n\backslash\{0\}$. (See also the Remark at the end of this section.)

Let us introduce

$$v(r\theta):=Z(r\theta)-\overline{Z}(r), \tag{24}$$

where $\overline{Z}$ is the spherical mean as in (17). We may now express (23) as

$$y'(r)+\frac{Q(r)}{r}y(r)=\frac{1}{r}Kv(r), \tag{25}$$

where

$$y(r):=⨍_{\partial B_1}Z(r\theta)a_{ij}(r\theta)\theta_i\theta_j\,ds, \tag{26}$$

and

$$Q(r):=n-\frac{\alpha_0(r)}{\alpha(r)}, \tag{27}$$

with

$$\alpha_0(r) := \fint_{\partial B_1} a_{ii}(r\theta)\, ds \quad \text{and} \quad \alpha(r) := \fint_{\partial B_1} a_{ij}(r\theta)\theta_i\theta_j\, ds; \tag{28}$$

in (25) we also have used

$$Kv(r) := \fint_{\partial B_1} v(r\theta)a_{ii}(r\theta)\, ds - \frac{\alpha_0(r)}{\alpha(r)}\fint_{\partial B_1} v(r\theta)a_{ij}(r\theta)\theta_i\theta_j\, ds. \tag{29}$$

It follows from (3) that $|\alpha_0(r) - n| \leq c\,\Omega(r)$, $|\alpha(r) - 1| \leq c\,\Omega(r)$, and

$$|Q(r)| \leq c\,\Omega(r), \tag{30}$$

so $Q(r) \to 0$ as $r \to 0$, although we do not know the sign of Q. Since $\overline{v}(r) = 0$, we can also write (27) as

$$Kv(r) = \fint_{\partial B_1} v(r\theta)(a_{ii}(r\theta) - n)\, ds - \frac{\alpha_0(r)}{\alpha(r)}\fint_{\partial B_1} v(r\theta)(a_{ij}(r\theta) - \delta_{ij})\theta_i\theta_j\, ds.$$

In this last form it is evident that K satisfies

$$M_p(Kv, r) \leq c\,\Omega(r)\, M_p(v, r). \tag{31}$$

To obtain another equation involving y and v, we start from the identity

$$\Delta v = \overline{\partial_i\partial_j((a_{ij} - \delta_{ij})v)} - \partial_i\partial_j((a_{ij} - \delta_{ij})v) + \overline{\partial_i\partial_j((a_{ij} - \delta_{ij})\overline{Z})} - \partial_i\partial_j((a_{ij} - \delta_{ij})\overline{Z}). \tag{32}$$

Noting that

$$y(r) = \alpha(r)\overline{Z}(r) + \fint_{\partial B_1} v(r\theta)(a_{ij}(r\theta)\theta_i\theta_j - 1)\, ds,$$

we can rewrite (32) as

$$\Delta v = \overline{\partial_i\partial_j(B_{ij}(v))} - \partial_i\partial_j(B_{ij}(v)) + \overline{\partial_i\partial_j(\phi_{ij}y)} - \partial_i\partial_j(\phi_{ij}y), \tag{33}$$

where

$$B_{ij}(v)(x) := (a_{ij}(x) - \delta_{ij})\left(v(x) - \frac{1}{\alpha(r)}\fint_{\partial B_1} v(r\theta)(a_{ij}(r\theta)\theta_i\theta_j - 1)\, ds\right) \tag{34}$$

and

$$\phi_{ij}(x) = \frac{a_{ij}(x) - \delta_{ij}}{\alpha(r)}. \tag{35}$$

Using (3), it is clear that

$$M_p(B_{ij}(v), r) \leq c\,\Omega(r)\, M_p(v, r) \tag{36}$$

and

$$|\phi_{ij}(x)| \leq c\,\Omega(r). \tag{37}$$

These estimates indeed hold for $0 < r < \infty$, where we extend Ω to be zero for $r > 1$. Inverting the Laplacian by means of the fundamental solution, (33) becomes

$$v + Sv + Ty = 0, \tag{38}$$

where we may use Corollary 3 with (36) to obtain

$$M_p(Sv, r) \leq c\left(r\int_r^\infty \Omega(\rho)M_p(v,\rho)\rho^{-2}\, d\rho + r^{-n}\int_0^r \Omega(\rho)M_p(v,\rho)\rho^{n-1}\, d\rho\right) \tag{39}$$

and with (37) to obtain

$$M_p(Ty, r) \leq c\left(r\int_r^\infty \Omega(\rho)M_p(y,\rho)\rho^{-2}\, d\rho + r^{-n}\int_0^r \Omega(\rho)M_p(y,\rho)\rho^{n-1}\, d\rho\right). \tag{40}$$

To simplify the equations, let us introduce the function

$$E(r) = \exp\left(\int_r^\infty \frac{Q(t)}{t}\,dt\right), \tag{41}$$

where there is no problem with convergence of the integral since $Q(t) = 0$ for $t > 1$. Notice that $E(r)$ is continuous for $r \in (0,\infty)$, $E(r) \equiv 1$ for $r \geq 1$, and for any $r, \rho \in (0,\infty)$ we have

$$E^{-1}(r)E(\rho) = \exp\left(\int_\rho^r \frac{Q(t)}{t}\,dt\right). \tag{42}$$

To derive an estimate for this expression, let us use (5) with $\varepsilon \in (0, 1/2)$ and (9) to conclude

$$\delta \geq \int_{r/2}^r \frac{\Omega^2(t)}{t} dt \geq \Omega^2(r) r^{-2+2\varepsilon} \int_{r/2}^r t^{1-2\varepsilon} dt \geq c\,\Omega^2(r). \tag{43}$$

As a consequence of (30), we have

$$\left(\frac{\rho}{r}\right)^{c\sqrt{\delta}} \leq \exp\left(\pm \int_\rho^r \frac{Q(t)}{t}\,dt\right) \leq \left(\frac{r}{\rho}\right)^{c\sqrt{\delta}} \quad \text{for } 0 < \rho \leq r \leq 1. \tag{44}$$

Now let us express equations (25) and (38) in terms of the new dependent variables

$$Y(r) = E^{-1}(r)y(r) \quad \text{and} \quad V(x) = E^{-1}(|x|)v(x). \tag{45}$$

Since the operator K only involves integration in θ, it is clear that

$$Kv(r) = E(r)KV(r), \tag{46}$$

and so the equation (25) can be expressed as

$$Y'(r) = \frac{1}{r} KV(r). \tag{47}$$

We will be assuming below that $M_p(V, r) \leq c\,\Omega(r)$ for $0 < r < 1$, so $M_p(KV, r) \leq c\,\Omega^2(r)$, enabling us to integrate (47) to obtain

$$Y(r) = Y(0) + \int_0^r \frac{KV(\rho)}{\rho}\,d\rho. \tag{48}$$

The equation (38), on the other hand, is replaced by

$$V + S_1 V + T_1 Y = 0, \tag{49}$$

where

$$S_1 := E^{-1} S E \quad \text{and} \quad T_1 := E^{-1} T E, \tag{50}$$

with E representing the multiplication operator defined by the function (41).

Now let us substitute (48) into (49) to finally obtain the operator equation that we want to solve:

$$V + S_1 V + T_2 V = -Y(0) T_1(1), \tag{51}$$

where

$$T_2 V(x) = E^{-1}(|x|) T_{\rho\to x}\left[E(\rho)\int_0^\rho KV(t)\,dt\right]. \tag{52}$$

(In (51), notice that T_1 operates on functions of a single variable, say ρ, and $T_1(1)$ represents the action of T_1 on the function that is identically 1.) For a given choice of $Y(0)$, we can solve (51) uniquely for V provided we can show that the two integral operators involved have small norm on an appropriate function space. Consider the functions w in $L^p_{loc}(\mathbb{R}\backslash\{0\})$ for which the norm

$$\|w\|_{p,\Omega} := \sup_{0<r<1} \frac{M_p(w,r)}{\Omega(r)} + \sup_{r>1} \frac{M_p(w,r)}{\sqrt{\delta}\, r^{-n}} \tag{53}$$

is finite, and take the closure to form a Banach space X. We want to show that the right hand side of (51) is in X and that the integral operators S_1 and T_2 map X to itself with small norm. It will be useful to observe that the continuity and positivity of $E(r)$ implies that, for any $w \in L^p_{\mathrm{loc}}(\mathbb{R}\backslash\{0\})$, we have

$$M_p(Ew,r) = E(\tilde r)M_p(w,r) \quad \text{for some } \tilde r = \tilde r_w \in (r,2r), \tag{54}$$

with an analogous statement for E^{-1}.

To show $T_1(1) \in X$, we must estimate $M_p(T_1(1),r)$ separately for $0<r<1$ and $r>1$. For $0<r<1$, we can use (54) to find $\tilde r \in (r,2r)$ so that

$$M_p(T_1(1),r) = E^{-1}(\tilde r)M_p(T_{\rho\to r}[E(\rho)],r),$$

and then use (40) to estimate

$$\begin{aligned} M_p(T_1(1),r) &\le c\,E^{-1}(\tilde r)\left(r\int_r^\infty \Omega(\rho)M_p(E,\rho)\rho^{-2}\,d\rho + r^{-n}\int_0^r \Omega(\rho)M_p(E,\rho)\rho^{n-1}\,d\rho\right) \\ &= c\,E^{-1}(\tilde r)\left(r\int_r^\infty \Omega(\rho)E(\tilde\rho)\rho^{-2}\,d\rho + r^{-n}\int_0^r \Omega(\rho)E(\tilde\rho)\rho^{n-1}\,d\rho\right), \end{aligned}$$

where $\tilde\rho \in (\rho,2\rho)$ by (54). But now we can use (42) and (44) to conclude

$$M_p(T_1(1),r) \le c\left(r\int_r^1 \Omega(\rho)(\rho/r)^{c\sqrt\delta}\rho^{-2}\,d\rho + r^{-n}\int_0^r \Omega(\rho)(r/\rho)^{c\sqrt\delta}\rho^{n-1}\,d\rho\right)$$

Using the monotonicty properties (5) and (6), we obtain

$$M_p(T_1(1),r) \le c\left(\Omega(r)r^{\varepsilon-c\sqrt\delta}\int_r^1 \rho^{-\varepsilon-1+c\sqrt\delta}d\rho + \Omega(r)r^{c\sqrt\delta-\varepsilon}\int_0^r \rho^{\varepsilon-c\sqrt\delta-1}d\rho\right).$$

Provided δ is sufficiently small that $\varepsilon - c\sqrt\delta > 0$, we conclude that

$$M_p(T_1(1),r) \le c\,\Omega(r) \quad \text{for } 0<r<1. \tag{55}$$

On the other hand, for $r>1$ we use $E^{-1}(r)\equiv 1$ and $\Omega(r)\equiv 0$ to estimate

$$\begin{aligned} M_p(T_1(1),r) = M_p(T_{\rho\to r}[E(\rho)],r) &\le c\,r^{-n}\int_0^1 \Omega(\rho)M_p(E,\rho)\rho^{n-1}\,d\rho \\ = c\,r^{-n}\int_0^1 \Omega(\rho)E(\tilde\rho)\rho^{n-1}\,d\rho &\le c\,r^{-n}\int_0^1 \Omega(\rho)\rho^{n-c\sqrt\delta-1}\,d\rho. \end{aligned}$$

Provided $n - c\sqrt\delta > 0$, we obtain

$$M_p(T_1(1),r) \le c\sqrt\delta\, r^{-n} \quad \text{for } r>1. \tag{56}$$

The two estimates (55) and (56) together confirm that $T_1(1)\in X$.

Now let us show that S_1 maps X to itself with small operator norm. We suppose that $\|V\|_{p,\Omega} \le 1$ and estimate $M_p(S_1V,r)$ separately for $0<r<1$ and $r>1$. For $0<r<1$ we have $M_p(V,r)\le\Omega(r)$ and we can argue as in the previous paragraph to obtain

$$M_p(S_1V,r) \le c\left(r\int_r^1 \Omega^2(\rho)(\rho/r)^{c\sqrt\delta}\rho^{-2}\,d\rho + r^{-n}\int_0^r \Omega^2(\rho)(r/\rho)^{c\sqrt\delta}\rho^{n-1}\,d\rho\right).$$

Using (43) and the monotonicity analysis as above, we conclude that

$$\frac{M_p(S_1V,r)}{\Omega(r)} \le c\sqrt\delta \quad \text{for } 0<r<1. \tag{57}$$

For $r>1$ we use (44) with $r=1$ and (43) to obtain

$$M_p(S_1V,r) \le c\,r^{-n}\int_0^1 \Omega^2(\rho)\rho^{n-c\sqrt\delta-1}\,d\rho \le c\,r^{-n}\delta\int_0^1 \rho^{n-c\sqrt\delta-1}\,d\rho \le c\,r^{-n}\delta$$

provided δ is sufficiently small, and we conclude that

$$\frac{M_p(S_1 V, r)}{\sqrt{\delta}\, r^{-n}} \le c\sqrt{\delta} \quad \text{for } r > 1. \tag{58}$$

The estimates (57) and (58) together show that S_1 maps X to itself with small operator norm.

Finally, we estimate T_2. For $0 < r < 1$ and $M_p(V, r) \le \Omega(r)$, we argue as before to write

$$M_p(T_2 V, r) \le c\left(r\int_r^\infty \Omega(\rho)\,(\rho/r)^{c\sqrt{\delta}}\, M_p\left[\int_0^\rho KV(t)\frac{dt}{t}, \rho\right]\rho^{-2}\, d\rho \right.$$
$$\left. + r^{-n}\int_0^r \Omega(\rho)\,(r/\rho)^{c\sqrt{\delta}}\, M_p\left[\int_0^\rho KV(t)\frac{dt}{t}, \rho\right]\rho^{n-1}\, d\rho\right).$$

But we can use (31) to obtain $M_p(KV, \rho) \le c\,\Omega(\rho) M_p(V, \rho) \le c\,\Omega^2(\rho)$ so that

$$M_p(T_2 V, r) \le c\left(r\int_r^1 \Omega(\rho)\ (\rho/r)^{c\sqrt{\delta}}\left[\int_0^\rho \frac{\Omega^2(t)}{t}\, dt\right]\rho^{-2}\, d\rho + \right.$$
$$\left. r^{-n}\int_0^r \Omega(\rho)\ (r/\rho)^{c\sqrt{\delta}}\left[\int_0^\rho \frac{\Omega^2(t)}{t}\, dt\right]\rho^{n-1}\, d\rho\right) \le c\,\delta\,\Omega(r).$$

Using (43) and the monotonicity argument, we have

$$\frac{M_p(T_2 V, r)}{\Omega(r)} \le c\,\delta \quad \text{for } 0 < r < 1. \tag{59}$$

For $r > 1$, we use (44) with $r = 1$ and (30) to obtain

$$M_p(T_2 V, r) \le c\, r^{-n}\int_0^1 \Omega^2(\rho)\rho^{-c\sqrt{\delta}+n-2}\, d\rho \le c\, r^{-n}\delta\int_0^1 \rho^{-c\sqrt{\delta}+n-2}\, d\rho \le c\, r^{-n}\delta,$$

provided δ is sufficiently small. These estimates show that T_2 maps X to itself with small operator norm.

Since S_1 and T_2 have small operator norms on X, we conclude that (51) admits a unique solution $V \in X$. Let us now investigate the implications for the weak solution $Z(x) = \overline{Z}(|x|) + v(x)$ that we are trying to construct. Tracing back through the definitions, we see that our solution of (1) is given by

$$Z(x) = \frac{E(r)}{\alpha(r)}\left(Y(0) + Y_1(r) - \fint_{\partial B_1} V(r\theta) a_{ij}(r\theta)\theta_i\theta_j\, ds + \alpha(r) V(x)\right), \tag{60}$$

where

$$Y_1(r) = \int_0^r \frac{KV(\rho)}{\rho}\, d\rho.$$

Recall that $|\alpha(r) - 1| \le c\,\Omega(r)$ and V satisfies $M_p(V, r) \le c\,\Omega(r)$ as $r \to 0$. Moreover,

$$M_p\left[\int_0^r \frac{KV(\rho)}{\rho}\, d\rho, r\right] \le c\int_0^r \frac{\Omega^2(\rho)}{\rho}\, d\rho,$$

so all the terms after the $Y(0)$ inside the parentheses of (60) are bounded in M_p either by $\Omega(r)$ or by $\int_0^r \Omega^2(\rho)\rho^{-1}\, d\rho$ for $0 < r < 1$. Let us explore $E(r)$. Notice that

$$\int_r^1 \frac{Q(\rho)}{\rho} d\rho = \int_r^1 [-\alpha_0(\rho) + n\alpha(\rho)]\frac{d\rho}{\rho} + \int_r^1 Q(\rho)[1 - \alpha(\rho)]\frac{d\rho}{\rho}$$
$$= c_0\int_{B_1\setminus B_r} (n a_{ij}(y) y_i y_j |y|^{-2} - a_{ii}(y))\frac{dy}{|y|^n} + \int_r^1 Q(\rho)[1 - \alpha(\rho)]\frac{d\rho}{\rho}.$$

But $|Q(\rho)[1-\alpha(\rho)]| \le c\,\Omega^2(\rho)$, so

$$\int_r^1 Q(\rho)[1-\alpha(\rho)]\,\frac{d\rho}{\rho} = C - \int_0^r Q(\rho)[1-\alpha(\rho)]|\,\frac{d\rho}{\rho}.$$

Exponentiating, we obtain

$$E(r) = \exp\left[\int_r^1 \frac{Q(\rho)}{\rho}\,d\rho\right] = C\exp\left[c_0\int_{B_1\setminus B_r}(na_{ij}(y)y_iy_j|y|^{-2} - a_{ii}(y))\frac{dy}{|y|^n}\right](1+\zeta_0(r)),$$

where $|\zeta_0(r)| \le c\int_0^r \Omega^2(\rho)\rho^{-1}\,d\rho$. Since we can rescale Z to set $CY(0)=1$, we have the formula (10). This completes the proof of Theorem 1.

Remark 4. *Choosing $C = -|\partial B_1|^{-1}$ in the proof of Theorem 1 and modifying the rest of the argument, we could obtain the asymptotic representation of the fundamental solution of equation (1):*

$$\Gamma_{\mathcal{A}}(x) = \Gamma(x) + \Delta^{-1}\partial_i\partial_j((a_{ij}-\delta_{ij})\Gamma) + v(x),$$

where Γ is the fundamental solution of Δ and

$$M_p(v,r) \le c\,\Omega^2(r)\,r^{2-n} \quad \textit{for } r\in(0,1).$$

Here the condition (4) is not necessary; we only need the smallness of $\Omega(r)$. However, this formula, as well as the additional terms in the asymptotic expansion of $\Gamma_{\mathcal{A}}$ can be obtained more easily by iteration from the equation $\mathcal{A}u(x) = \delta(x)$.

3. Proof of Theorem 2

Let $q = p/(p-1)$ and $\beta,\gamma\in\mathbb{R}$. Let us introduce the weighted L^p norm for functions on $\mathbb{R}^n$ with separate weights at the origin and infinity

$$\|u\|^q_{L^q_{\beta,\gamma}(\mathbb{R}^n)} = \|u\|^q_{L^q_\beta(B_1)} + \|u\|^q_{L^q_\gamma(B_1^c)} = \int_{|x|<1}|u(x)|^q\,|x|^{\beta q}\,dx + \int_{|x|>1}|u(x)|^q\,|x|^{\gamma q}\,dx, \tag{61}$$

and the weighted Sobolev space $W^{2,q}_{\beta,\gamma}(\mathbb{R}^n\backslash\{0\})$ with norm

$$\sum_{|\alpha|\le 2}\|r^{|\alpha|}\partial^\alpha u\|_{L^q_{\beta,\gamma}(\mathbb{R}^n)}. \tag{62}$$

Notice that $L^p_{-\beta,-\gamma}(\mathbb{R}^n)$ is the dual space for $L^q_{\beta,\gamma}(\mathbb{R}^n)$ and the notation $W^{-2,p}_{-\beta,-\gamma}(\mathbb{R}^n\backslash\{0\})$ will be used for the dual of $W^{2,q}_{\beta,\gamma}(\mathbb{R}^n\backslash\{0\})$. Many authors have used similar weighted Sobolev spaces to study operators like the Laplacian on $\mathbb{R}^n$, $\mathbb{R}^n\backslash\{0\}$, and other noncompact manifolds with conical or cylindrical ends.

Using the analysis in [22], [21] or [20], for example, it is easily verified that the bounded operator

$$\Delta : W^{2,q}_{\beta,\gamma}(\mathbb{R}^n\backslash\{0\}) \to L^q_{\beta+2,\gamma+2}(\mathbb{R}^n) \tag{63}$$

is Fredholm (finite nullity and finite deficiency) for all values of β and γ *except* for the values $-2+\frac{n}{p}+k$ and $-\frac{n}{q}-k$ where k is any nonnegative integer. In fact, (63) is an isomorphism for $-n/q<\beta,\gamma<-2+n/p$ (recall that we are assuming $n\ge 3$, so such β,γ exist). Since we are principally interested in the behavior of functions at the origin, we will fix $\gamma_0\in(-n/q,-2+n/p)$. Then, for $\beta\in(-2+n/p,-1+n/p)$, we find that (63) is surjective with a one-dimensional nullspace spanned by $|x|^{2-n}$.

Now let us consider the formal adjoint of $\mathcal{A}$, which also defines a bounded operator on these spaces

$$\mathcal{L}^* = \overline{a}_{ij}(x)\partial_i\partial_j : W^{2,q}_{\beta,\gamma}(\mathbb{R}^n\backslash\{0\}) \to L_{\beta+2,\gamma+2}(\mathbb{R}^n), \tag{64}$$

where $\overline{a}_{ij}$, of course, denotes the complex conjugate of a_{ij}. Because $a_{ij}(x) = \delta_{ij}$ for $|x| > 1$ and $a_{ij}(x) - \delta_{ij}$ vanishes as $x \to 0$, the analysis in the above references shows that the operator (64) is Fredholm for exactly the same values of β and γ as for (63). In fact, for fixed nonexceptional values of β and γ, we may take δ sufficiently small and use perturbation theory (cf. [14], Ch.IV, Sec.5) to conclude that the nullity and deficiency of (63) and (64) agree.

So, in addition to the fixed $\gamma_0 \in (-n/q, -2+n/p)$, let us now fix $\beta_1 \in (-n/q, -2+n/p)$ and $\beta_2 \in (-2+n/p, -1+n/p)$, and denote the adjoints of the corresponding operators (64) by $\mathcal{A}_1$ and $\mathcal{A}_2$:

$$\mathcal{A}_1 : L^p_{-\beta_1-2,-\gamma_0-2}(\mathbb{R}^n) \to W^{-2,p}_{-\beta_1,-\gamma_0}(\mathbb{R}^n\backslash\{0\}) \tag{65}$$

is an isomorphism, and

$$\mathcal{A}_2 : L^p_{-\beta_2-2,-\gamma_0-2}(\mathbb{R}^n) \to W^{-2,p}_{-\beta_2,-\gamma_0}(\mathbb{R}^n\backslash\{0\}) \tag{66}$$

is injective with a one-dimensional cokernel. An arbitrary non-zero functional in $\operatorname{Coker}\mathcal{A}_2$ will be denoted by ζ.

We introduce a cut-off function $\eta \in C_0^\infty(B_1)$ equal to 1 on $B_{1/2}$. It follows from (60) that $\eta Z \in L^p_{-\beta_1-2}(B_1)$ but it is not in $L^p_{-\beta_2-2}(B_1)$. Let $F = \mathcal{A}((1-\eta)Z) = -\mathcal{A}(\eta Z)$. Since $F = 0$ on $B_{1/2}$ and on B_1^c, it follows that $F \in W^{-2,p}_{-\beta_2,-\gamma_0}(\mathbb{R}^n\backslash\{0\}) \subset W^{-2,p}_{-\beta_1,-\gamma_0}(\mathbb{R}^n\backslash\{0\})$. But (65) is an isomorphism, so ηZ is the only solution of $\mathcal{A}_1 v = -F$. Since $L^p_{-\beta_2-2}(B_1) \subset L^p_{-\beta_1-2}(B_1)$, $\mathcal{A}_2 v = -F$ has no solutions, which means that $\zeta(F) \neq 0$.

Now let $u \in L^p_{\rm loc}(B_1)$ be a weak solution of $\mathcal{A}u = 0$ satisfying $M_p(u,r) \leq c\, r^{2-n+\varepsilon_0}$. This estimate on u implies that $u \in L^p_{-\beta-2}(B_1)$ for $\beta < -n/q + \varepsilon_0$. So let us restrict our choice of β_1 to the interval $(-n/q, -n/q+\varepsilon_0)$. Denote by C an arbitrary constant. Since $\mathcal{A}(\eta(u-CZ)) = 0$ on $B_{1/2}$ and on B_1^c, we have $\mathcal{A}(\eta(u-CZ)) \in W^{-2,p}_{-\beta_2-2,-\gamma_0-2}(\mathbb{R}^n\backslash\{0\})$. Choosing C to satisfy

$$C\zeta(F) = \zeta(\mathcal{A}(\eta u)),$$

we obtain $\zeta(\mathcal{A}(\eta(u-CZ))) = 0$, which implies $\eta(u-CZ) \in \operatorname{Dom}(\mathcal{A}_2)$, and in particular

$$\eta(u-CZ) \in L^p_{-\beta_2-2}(B_1).$$

But this implies that $w = u - CZ$ satisfies $M_p(w,r) \leq c\, r^{\beta_2+2-n/p}$. Assuming that we had fixed $\beta_2 = \frac{n}{p} - 1 - \varepsilon_1$ where $\varepsilon_1 \in (0,1)$, we conclude that

$$M_p(w,r) \leq c\, r^{1-\varepsilon_1}. \tag{67}$$

This completes the proof.

References

[1] A.Aleksandrov, *Majorants of solutions of linear equations of order two*, Vestnik Leningrad University **21** 1966, 15-25.

[2] I.Y.Bakelman, *Theory of quasilinear elliptic equations*, Siberian Math. J. **2** 1961, 179-186.

[3] P.Bauman, *Positive solutions of elliptic equations in nondivergent form and their adjoints*, Ark. Mat. **22** 1984, 153-173.

[4] P.Bauman, *Equivalence of the Green's function for diffusion operators in $\mathbb{R}^n$: a counterexample*, Proc. Amer. Math. Soc. **91** 1984, 64-68.

[5] P.Bauman, *A Wiener test for nondivergence structure second-order elliptic equations*, Indiana U. Math. J. **34** 1985, 825-844.

[6] L.Caffarelli, X.Cabré, *Fully Nonlinear Elliptic Equations*, AMS Colloquium Publications, Vol. 43, AMS, Providence, RI, 1995.

[7] M.Cerutti, L.Escauriaza, E.Fabes, *Uniqueness in the Dirichlet problem for some elliptic operators with discontinuous coefficient*, Ann. Mat. Pura Appl. **CLXIII** 1993, 161-180.

[8] L.Escauriaza, *Weak type-(1,1) inequalities and regularity properties of adjoint and normalized adjoint solutions to linear nondivergence form operators with VMO coefficients*, Duke Math. J. **74** 1994, 177-201.

[9] L.Escauriaza, *Bounds for the fundamental solution of elliptic and parabolic equations in nondivergence form*, Comm. Part. Diff. Equat. **25** 2000, 821-845.

[10] L.Escauriaza, C.Kenig, *Area integral estimates for solutions and normalized adjoint solutions to nondivergence form elliptic equations*, Ark. Mat. **31** 1993, 275-296.

[11] E.Fabes, N.Garofalo, S.Marín-Malavé, S.Salsa, *Fatou theorems for some nonlinear elliptic equations*, Rev. Mat. Ib. **4** 1988, 227-242.

[12] E.Fabes, D.Stroock, *The L^p-integrability of Green's functions and fundamental solutions for elliptic and parabolic equations*, Duke Math. J. **51** 1984, 977-1016.

[13] D.Gilbarg, N.Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 2001.

[14] T. Kato, *Perturbation Theory for Linear Operators,* Springer-Verlag, New York, 1966.

[15] V.Kozlov, V.Maz'ya, *The estimates for L_p-averages and asymptotics of solutions of elliptic boundary value problems in a cone I* (Russian), Seminar Analysis Operator Equat. and Numer. Anal. 1985/86 Karl-Weierstrass-Institut für Mathematik, 55-92,

[16] V.Kozlov, V.Maz'ya, *The estimates for L_p-averages and asymptotics of solutions of elliptic boundary value problems in a cone II* (Russian), Math. Nachr. **137** 1980, 113-139.

[17] V.Kozlov, V.Maz'ya, *Differential Equations with Operator Coefficients,* Springer Monographs in Mathematics, Springer-Verlag, 1999.

[18] V.Kozlov, V.Maz'ya, *Asymptotic formula for solutions to the Dirichlet problem for elliptic equations with discontinuous coefficients near the boundary* (English summary), Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **2** 2003, 551-600.

[19] N.Krylov, M.Safonov, *A certain property of solutions of parabolic equations with measurable coefficients*, Math. USSR Izv. **16** 1981, 151-164.

[20] R.Lockhart, R.McOwen, *Elliptic differential operators on noncompact manifolds*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) **12** 1985, 409-447.

[21] R.McOwen, *The behavior of the Laplacian on weighted Sobolev spaces*, Comm. Pure Appl. Math. **32** 1979, 783-795.

[22] V.Maz'ya, B.Plamenevski, *Estimates in L_p and Hölder classes and the Miranda-Agmon maximum principle solutions of elliptic boundary problems in domains with singular points on the boundary* (in Russian), Math. Nachr. **81** 1978, 25-82.

[23] C.Pucci, *Limitazioni per soluzioni di equaziones ellittiche*, Ann. Mat. Pura Appl. **74** 1966, 15-30.

[24] P.Sjögren, *On the adjoint of an elliptic linear differential operator and its potential theory*, Ark. Mat. **11** 1973, 153-165.

[25] D.Strook, S.Varadhan, *Diffusion processes with continuous coefficients, I*, Comm. Pure Appl. Math. **22** 1969, 345-400.

[26] D.Strook, S.Varadhan, *Diffusion processes with continuous coefficients, II*, Comm. Pure Appl. Math. **22** 1969, 479-530.

[27] E.Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

Linköping University, Ohio State University, University of Liverpool

Northeastern University