# BOUNDARY CHARACTERISTIC POINT REGULARITY FOR NAVIER–STOKES EQUATIONS: BLOW-UP SCALING AND PETROVSKII-TYPE CRITERION (A FORMAL APPROACH)

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Abstract. The *three-dimensional* (3D) *Navier–Stokes equations*

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \Delta \mathbf{u}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad Q_0, \tag{0.1}$$

where $\mathbf{u} = [u, v, w]^T$ is the vector field and $p$ is the pressure, are considered. Here, $Q_0 \subset \mathbb{R}^3 \times [-1, 0)$ is a smooth domain of a typical *backward paraboloid* shape, with the vertex $(0, 0)$ being its only *characteristic point*: the plane $\{t = 0\}$ is tangent to $\partial Q_0$ at the origin, and other characteristics for $t \in [0, -1)$ intersect $\partial Q_0$ transversely. Dirichlet boundary conditions on the lateral boundary $\partial Q_0$ and smooth initial data are prescribed:

$$\mathbf{u} = 0 \quad \text{on} \quad \partial Q_0, \quad \text{and} \quad \mathbf{u}(x, -1) = \mathbf{u}_0(x) \quad \text{in} \quad Q_0 \cap \{t = -1\} \quad (\operatorname{div} \mathbf{u}_0 = 0). \tag{0.2}$$

Existence, uniqueness, and regularity studies of (0.1) in *non-cylindrical domains* were initiated in the 1960s in pioneering works by J.L. Lions, Sather, Ladyzhenskaya, and Fujita–Sauer. However, the problem of a *characteristic vertex* regularity remained open.

In this paper, the classic problem of regularity (in Wiener's sense) of the vertex $(0, 0)$ for (0.1), (0.2) is considered. Petrovskii's famous "$2\sqrt{\log \log}$-criterion" of boundary regularity for the heat equation (1934) is shown to apply. Namely, after a blow-up scaling and a special matching with a boundary layer near $\partial Q_0$, the regularity problem reduces to a 3D perturbed nonlinear dynamical system for the first Fourier-type coefficients of the solutions expanded using solenoidal Hermite polynomials. Finally, this confirms that the nonlinear convection term gets an exponentially decaying factor and is then negligible. Therefore, the regularity of the vertex is entirely dependent on the linear terms and hence remains the same for Stokes' and purely parabolic problems.

Well-posed Burnett equations with the minus bi-Laplacian in (0.1) are also discussed.

## 1. Introduction: vertex regularity for the Navier–Stokes equations

**1.1. Navier–Stokes equations inside a non-cylindrical backward paraboloid: first history since 1960s.** We consider 3D Navier–Stokes equations (the 3D NSEs)

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \Delta \mathbf{u}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad Q_0, \tag{1.1}$$

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where $\mathbf{u} = [u, v, w]^T(x, t)$ is the vector field and $p = p(x, t)$ is the corresponding pressure.

The NSEs (1.1) are posed in a smooth *non-cylindrical* domain

$$Q_0 \subset \mathbb{R}^3 \times [-1, 0)$$

of a typical *backward paraboloid* shape, with the vertex $(0, 0)$ being its only *characteristic point*: the plane $\{t = 0\}$ is tangent to $\partial Q_0$ at the origin. No characteristic points of $\partial Q_0$ are assumed to exist for $t \in [-1, 0)$, i.e., other characteristics $\{t = \tau\}$, for any $\tau \in [-1, 0)$, intersect $\partial Q_0$ transversely, in a natural sense. Next, the zero Dirichlet boundary conditions on the lateral boundary $\partial Q_0$ and smooth initial data at $t = -1$ are prescribed:

$$\begin{aligned} &\mathbf{u} = 0 \quad \text{on} \quad \partial Q_0, \quad \text{and} \\ &\mathbf{u}(x, -1) = \mathbf{u}_0(x) \quad \text{in} \quad Q_0 \cap \{t = -1\}, \quad \text{where} \quad \operatorname{div} \mathbf{u}_0 = 0. \end{aligned} \tag{1.2}$$

The questions of solvability, uniqueness, and regularity for the Navier–Stokes equations in *non-cylindrical* (and non-characteristic) domains, i.e., in our case, up to the vertex, for $t \le -\delta_0 < 0$, were actively studied since the 1960s. J.L. Lions began this study in 1963; see references in his classic monograph [44, Ch. 3] concerning elliptic regularization–penalization methods; as well as Fujita–Sauer [15, 16] (1969) as one of the first such study of weak solutions via a penalization. Another alternative, as was pointed out in [44, Ch. 3, § 8.1], is a "rather careful using" Galerkin methods with time dependent basis functions; see Sather [64] (1963). In 1968, Ladyzhenskaya [39] proved local existence (global for $N = 2$ and for small initial data if $N = 3$) and uniqueness of strong solutions for time-dependent domains using a different method. See Neustupa [59] for more recent results, references, and other related problems.

However, the problem of regularity of a characteristic boundary point for the NSEs in any dimension $N \ge 2$ was not addressed elsewhere and remained open. Naturally, in order to proceed with regularity issues concerning the paraboloid vertex $(0, 0)$, we have to assume that a unique smooth bounded solution of (1.1), (1.2) exists in $Q_0$, i.e., with no $L^\infty$-blow-up for $t < 0$[1]. In particular, as is well-known (see [39, 45, 66]), global smooth solutions always exist for sufficiently small initial data, so we can directly proceed with the vertex regularity, at least, for this class of solutions.

1.2. **Regularity of the characteristic paraboloid vertex.** Thus, the classic problem of regularity (in *Wiener's sense*, see [47]) of a boundary characteristic point for the NSEs problem (1.1), (1.2) is under consideration.

**Definition (vertex regularity/irregularity).** *According to Wiener* [70]*, the vertex* $(0, 0)$ *of the given backward paraboloid* $Q_0$ *for the NSEs problem* (1.1)*,* (1.2) *is* **regular** *if, for any bounded data* $\mathbf{u}_0(x)$*,*

$$\mathbf{u}(0, 0^-) = 0, \tag{1.3}$$

*and* **irregular** *otherwise, i.e., at least for some initial data,* (1.3) *fails.*

[1] But the solution is formally allowed to blow-up at the vertex $(0, 0^-)$.

The boundary and other regularity issues for the Navier–Stokes equations in $\mathbb{R}^2$ and $\mathbb{R}^3$ have been and remain key and very popular in modern mathematical literature, since J. Leray's seminal papers in 1933-34 [42, 43]. Among various regularity and partial regularity results for the NSEs, the boundary regularity properties in piecewise smooth or Lipschitz domains and those with thin channels, or other non-regular domains (as we will show, such settings are key for our study) always played a special role. Mentioning Kondratiev's first study of 1967 [33], we refer to advanced results, further references, and reviews in recent papers [7, 31, 38, 49, 50, 52, 56] and [71]. See also [37, 57] for the related linear Stokes problem

$$\mathbf{u}_t = -\nabla p + \Delta \mathbf{u}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad Q_0, \quad \mathbf{u}(0,x) = \mathbf{u}_0(x) \;\; (\operatorname{div} \mathbf{u}_0 = 0). \tag{1.4}$$

Concerning compressible flows and other related problems, see a good survey in [38], where 2D NSEs in a polygon domain with a convex vertex were studied.

Note that, and this is key for us in what follows, J. Leray in [42, 43] actually posed a deep problem on both *backward* and *forward continuation* phenomena, which sound modern and advanced nowadays for general nonlinear PDE theory:

$$\begin{aligned} &\textbf{Leray's blow-up scenario:} \quad \text{self-similar blow-up as } t \to T^- \;\; (t < T) \\ &\qquad \text{and similarity collapse of this singularity as } t \to T^+ \;\; (t > T); \end{aligned} \tag{1.5}$$

see his precise statements and a discussion on these principal issues in [18, § 2.2]. In this connection, such "backward blow-up scaling approaches" will be key later on.

According to our approach, we deal with a typical asymptotic problem of clarifying a generic behaviour of solutions near a "blow-up singularity" $(0,0)$ (a "micro-scale structure" of nonlinear PDEs involved). Of course, the vertex regularity problem setting essentially and crucially depends on the *a priori* given shape of the prescribed backward paraboloids, which affects our methods of matched blow-up expansions. Anyway, we hope that our blow-up analysis on shrinking as $t \to 0^-$ subsets will eventually help to better understand the possible nature of other plausible blow-up singularities of the NSEs. As a common feature of our blow-up analysis, we will see that a complete/closed set of vector solenoidal Hermite polynomials[2] in $\mathbb{R}^3$ can play an important part.

Our main goal here is as follows: using techniques of blow-up scaling and matched asymptotic expansions from reaction-diffusion theory, *to show that a Petrovskii's-like criterion of boundary regularity for the heat equation* (1934) *does occur for the NSEs* (1.1).

1.3. **Petrovskii's criterion of 1934 for the heat equation: a first discussion.** For the *heat equation*,

$$u_t = \Delta u \quad \text{in} \quad Q_0, \tag{1.6}$$

the regularity problem of the characteristic vertex $(0,0)$ was *optimally* solved for dimensions $N = 1$ and 2 by *Ivan Georgievich Petrovskii* in 1934-35 [60, 61][3], who introduced

[2]These are the same vector polynomials that occur in the study of multiple zero formation for the Stokes equations and NSEs, [18, § 3].

[3]Compare with the dates of Leray's pioneering study, "1933-34"; almost a perfect coincidence!

his famous *Petrovskii's regularity criterion*; see [17] and [20] for a full history and further developments in general parabolic theory. This is the so-called "$2\sqrt{\log\log}$–criterion" (see (2.8) and (2.9) below), which we are going to achieve, at least formally, for the 3D NSEs.

The following issues naturally occur:

**(i)** On one hand, Petrovskii's-like criterion can be expected, since (1.1), similar to (1.6), is indeed a (vector) parabolic second-order equation;

**(ii)** On the other hand, (1.1) is a *nonlocal* parabolic PDE for solenoidal vector fields, and it is not straightforward from the beginning that this cannot affect the regularity standing; and

**(iii)** Finally and most essentially, (1.1) contains both linear (the second-order Laplacian) and nonlinear (convective) operators, so that the regularity of the vertex $(0,0)$ inevitably will depend on both, which makes the analysis more difficult.

Note that both the issues **(i)** and **(ii)** apply to the linear Stokes problem without the quadratic convection (1.4), for which our regularity results turn out to be new as well.

1.4. **Layout of the paper.** In Section 2, we perform first a *blow-up* scaling near the characteristic vertex $(0,0)$. In Sections 3 and 4, the main goal is to show how the convection term in the NSEs (1.1) can affect the regularity conditions by deriving sharp formal asymptotics of solutions near the characteristic point. A necessary and already well-existing spectral theory involving a complete set of vector *solenoidal Hermite polynomials* as eigenfunctions of the linear Hermite operator is described in Appendix A.

For the sake of our regularity study, we apply a method of a matched asymptotic (blow-up) expansion, where a *Boundary Layer* behaviour close to the lateral boundary $\partial Q_0$ (Section 3) is matched, as $t \to 0^-$, with a *"centre subspace behaviour"* in an *Inner Region*, developed in Section 4. This leads to a perturbed 3D nonlinear dynamical system for the first Fourier-like coefficients in the eigenfunction expansions of the vector field $\mathbf{u}(x,t)$ via standard solenoidal Hermite polynomials. This approach falls into the scope of typical ideas of asymptotic PDE theory, which got a full mathematical justification for many problems of interest. We refer to a most general asymptotic analysis performed in [34], and also to a number of complicated blow-up asymptotics in reaction-diffusion theory [23]. According to the classification in [34], our matched blow-up approach corresponds to perturbed three-dimensional dynamical systems, i.e., to a rather not-that-advanced case being however a constructive one that has given a number of new asymptotic/regularity results. We propose a final, more general discussion of various (boundary and interior) blow-up singularities for the NSEs in Section 5.

In addition, for more clear expressing our regularity techniques and their applicability in general PDE theory, we develop in Appendix B (the C one contains the corresponding Hermite spectral analysis) at the paper end, as a natural extension, a similar regularity analysis of the *well-posed Burnett equations* in $Q_0$ with zero Dirichlet conditions

$$
\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p - \Delta^2 \mathbf{u}, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad Q_0, \\ \mathbf{u} = \nabla\mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial Q_0, \quad \mathbf{u}(x,-1) = \mathbf{u}_0(x), \end{cases} \tag{1.7}
$$

where $\mathbf{n}$ denotes the unit inward normal vector to $\partial Q_0 \cap \{t\}$. Here we have the *bi-harmonic* diffusion operator $-\Delta^2 \mathbf{u}$ on the right-hand side of the $\mathbf{u}$-equation. It turns out that our general scheme of the boundary regularity analysis can be applied; however, with harder asymptotics and more formal nature of the final difficult estimates.

Concerning other problems and techniques of modern regularity theory, we refer to monographs [27, 36, 37, 51, 54] and [35], [46]–[53] as an update guide to elliptic regularity theory including higher-order equations, as well as to references/results in [28, 38, 41, 40, 12, 69] and [17, 20] for linear and semilinear parabolic PDEs.

## 2. First blow-up scaling: Sturm's backward scaling variable, paraboloid geometry, and the Cauchy problem

### 2.1. **First blow-up scaling: exponentially small perturbations of a rescaled parabolic flow.**

We perform *blow-up scaling* in (1.1) for the regularity analysis

$$\mathbf{u}(x,t) = \mathbf{v}(y,\tau), \quad y = \tfrac{x}{\sqrt{-t}}, \quad \tau = -\ln(-t) : (-1,0) \to \mathbb{R}_+, \tag{2.1}$$

where $y$ is, indeed, *Sturm's backward rescaled variable* introduced him in 1836 [65] in the study of zero sets of solutions of linear parabolic equations such as (1.6) for $N=1$.

Thus, scaling (2.1) yields the following **exponentially perturbed** rescaled equation[4]:

$$\begin{gathered} \mathbf{v}_\tau + \mathrm{e}^{-\frac{\tau}{2}}\,(\mathbf{v}\cdot\nabla)\mathbf{v} = -\mathrm{e}^{-\frac{\tau}{2}}\,\nabla p + \mathbf{B}^*\mathbf{v}, \quad \operatorname{div}\mathbf{v} = 0 \quad \text{in} \quad \hat{Q}_0, \\ \text{where} \quad \mathbf{v} = [v^1, v^2, v^3]^T \quad \text{and} \quad \mathbf{B}^* = \Delta - \tfrac{1}{2}\, y\cdot\nabla \end{gathered} \tag{2.2}$$

is Hermite's classic symmetric (self-adjoint) operator, [4, p. 48].

### 2.2. **Backward paraboloid geometry and a slow growing factor $\varphi(\tau)$.**

According to Petrovskii [60, 61], the backward paraboloid $\partial Q_0$ will be defined as follows: it is a perturbation of the standard *fundamental* backward one,

$$S(t) = \partial Q_0 \cap \{t\} : \quad q_0(x) \equiv \sqrt{\textstyle\sum_{i=1}^3 a_i x_i^2} = (-t)^{\frac{1}{2}}\,\varphi(\tau), \tag{2.3}$$

where $a_i \in (0,1]$ are normalized constants. We can treat more general *convex* paraboloids, but, for simplicity, will restrict to the basic ones as in (2.3), which are also of a challenge. For difficult estimates to follow, we will use *radially symmetric* paraboloids with $a_i = 1$:

$$S(t) = \partial Q_0 \cap \{t\} : \quad q_0(x) \equiv |x| = (-t)^{\frac{1}{2}}\,\varphi(\tau). \tag{2.4}$$

In (2.3) and (2.4), $\varphi(\tau) > 0$ is a slow growing function satisfying

$$\varphi(\tau) \to +\infty, \quad \varphi'(\tau) > 0, \quad \varphi'(\tau) \to 0, \quad \text{and} \quad \tfrac{\varphi'(\tau)}{\varphi(\tau)} \to 0 \quad \text{as} \quad \tau \to +\infty. \tag{2.5}$$

Moreover, as a sharper characterization of the above class of *slow growing functions*, we use the following criterion:

$$\left(\tfrac{\varphi(\tau)}{\varphi'(\tau)}\right)' \to \infty \quad \text{as} \quad \tau \to +\infty. \tag{2.6}$$

[4]Here, (2.2) *is not* the rescaled one (5.3) written in Leray's variable (5.2), a difference to be discussed.

This is a typical condition in blow-up analysis distinguishing classes of exponential (the limit is 0), power-like (a constant $\neq 0$), and slow-growing functions. See [63, pp. 390-400], where in Lemma 1 on p. 400, extra properties of slow-growing functions (2.6) are proved. For instance, one can derive the following comparison of such $\varphi(\tau)$ with any power:

$$\text{for any } \alpha > 0, \quad \varphi(\tau) \ll \tau^{\alpha} \quad \text{and} \quad \varphi'(\tau) \ll \tau^{\alpha-1} \quad \text{for} \quad \tau \gg 1. \tag{2.7}$$

Such estimates are useful in evaluating perturbation terms in the rescaled equations.

In Petrovskii's criterion for the heat equation (1.6), for any $N \geq 1$, the "almost optimal" function, satisfying (2.5), (2.6) and delivering a regular vertex $(0,0)$, is

$$\boxed{\varphi_*(\tau) = 2\sqrt{\ln \tau} \quad \text{as} \quad \tau \to +\infty.} \tag{2.8}$$

Replacing this *fundamental constant* "2" by $2+\varepsilon$, for an arbitrarily small constant $\varepsilon > 0$, makes $(0,0)$ irregular for the heat equation (1.6). In the general case of arbitrary $\varphi(\tau)$, Petrovskii's criterion, for our $N = 3$, for the radially symmetric paraboloid (2.4) reads[5]

$$\boxed{\int^{\infty} \varphi^3(\tau)\, \mathrm{e}^{-\varphi^2(\tau)/4}\, \mathrm{d}\tau = \infty.} \tag{2.9}$$

These dependencies will be compared with those obtained for the NSEs (1.1).

Thus, the monotone positive function $\varphi(\tau)$ in (2.3) is assumed to determine a sharp behaviour of the boundary of $Q_0$ near the vertex $(0,0^-)$ to guarantee its regularity. It follows that the rescaled equation (2.2) is set in an expanding rescaled domain

$$\hat{S}(\tau) \equiv \partial \hat{Q}_0 \cap \{\tau\}: \quad q_0(y) \equiv \sqrt{\textstyle\sum_{i=1}^{3} a_i y_i^2} = \varphi(\tau) \to +\infty \quad \text{as} \quad \tau \to +\infty. \tag{2.10}$$

By $\mathbf{n}$ we denote the inward unit normal to $\hat{S}(\tau)$. In the limit $\tau = +\infty$, we arrive at the equation (2.2) in the whole space $\mathbb{R}^3$, which requires some spectral theory (Appendix A).

2.3. **Towards the Cauchy problem.** In *Inner Region* (see Section 3 for details), described by compact subsets in the variable $y$ in (2.1), we deal with the original rescaled problem (2.2) in the unboundently expanding domains (2.10). As usual and customary in potential and general PDE theory (see e.g., Vladimirov [68]), it is convenient to consider the NSEs in whole space $\mathbb{R}^3 \times [-1,0)$. Note that, in the study of the NSEs in non-cylindrical domains, Fujita–Sauer [15, 16] also extended the problem to $\mathbb{R}^3$ by introducing a *strong absorption term* $-n\,\mathbf{u}$ in the complementary domain on the right-hand side in (1.1) and passing to the limit $n \to +\infty$. Then, in view of the control of the total kinetic $L^2$-energy, this "regularized" solution $\mathbf{u} = \mathbf{u}^{(n)} \to 0$ as $n \to +\infty$ outside the given non-cylindrical domain, and hence, in the limit, the zero Dirichlet conditions on the boundary are restored.

Thus, we extend $\mathbf{v}(y,\tau)$ by 0 beyond the boundary, i.e., set

$$\hat{\mathbf{v}}(y,\tau) = \mathbf{v}(y,\tau) H(\varphi(\tau) - q_0(y)) = \begin{cases} \mathbf{v}(y,\tau) & \text{for} \quad 0 \leq q_0(y) < \varphi(\tau), \\ 0 & \text{for} \quad q_0(y) \geq \varphi(\tau), \end{cases} \tag{2.11}$$

[5] For $Q_0 \subset \mathbb{R}^N \times [-1,0)$, the multiplier $\varphi^3$ in this integral criterion is replaced by $\varphi^N$.

where $H(\cdot)$ is the Heaviside function. Then, since $\mathbf{v}=0$ on the lateral boundary $\hat{S}(\tau)=\{q_0(y)=\varphi(\tau)\}$, one can check that, in the sense of distributions (see e.g., [68, § 6.5]),

$$\hat{\mathbf{v}}_\tau=\mathbf{v}_\tau H, \quad \nabla\hat{\mathbf{v}}=\nabla\mathbf{v}H, \quad \Delta\hat{\mathbf{v}}=\Delta\mathbf{v}H+\tfrac{\partial\mathbf{v}}{\partial\mathbf{n}}\,\delta_{\hat{S}(\tau)}, \tag{2.12}$$

where a single-layer potential with the density $\mu=\frac{\partial\mathbf{v}}{\partial\mathbf{n}}$ acts as follows: for any $\phi\in C_0^\infty$,

$$\langle\mu\delta_{\hat{S}(\tau)},\phi\rangle=\textstyle\int_{\hat{S}(\tau)}\mu\phi\,\mathrm{d}s. \tag{2.13}$$

In order to avoid a pressure trace on $\hat{S}(\tau)$, we perform a *continuous* extension of $p(y,\tau)$ by solving the Laplace equation in the outer domain:

$$\Delta\hat{p}=0 \quad \text{in} \quad \mathbb{R}^3\setminus(\overline{\hat{Q}_0(\tau)\cap\{\tau\}}), \quad \hat{p}=p \ \text{ on } \ \hat{S}(\tau). \tag{2.14}$$

This outer Dirichlet problem is known to admit a unique solution $\hat{p}(y,\tau)$ vanishing at infinity, [68, § 28], so that we are given the continuous pressure $\hat{p}(y,\tau)$ defined in the whole $\mathbb{R}^3\times\mathbb{R}_+$. It then follows that

$$\begin{gathered}\nabla\hat{p}=\nabla p \ \text{ in } \ \overline{\hat{Q}_0(\tau)\cap\{\tau\}} \quad (\text{since the jump is zero: } [\hat{p}]_{\hat{S}(\tau)}=0), \ \text{ and}\\ \nabla\hat{p} \ \text{ is div-free in } \mathbb{R}^3 \text{ as a distribution}: \ \langle\nabla\hat{p},\phi\rangle=0, \ \forall\phi\in C_0^\infty, \ \operatorname{div}\phi=0.\end{gathered} \tag{2.15}$$

Thus, $\{\hat{\mathbf{v}},\hat{p}\}$ satisfies the following Cauchy problem in $\mathbb{R}^3\times\mathbb{R}_+$:

$$\hat{\mathbf{v}}_\tau+\mathrm{e}^{-\frac{\tau}{2}}\,(\hat{\mathbf{v}}\cdot\nabla)\hat{\mathbf{v}}=-\mathrm{e}^{-\frac{\tau}{2}}\,\nabla\hat{p}+\mathbf{B}^*\hat{\mathbf{v}}-\tfrac{\partial\mathbf{v}}{\partial\mathbf{n}}\,\delta_{\hat{S}(\tau)}, \quad \operatorname{div}\hat{\mathbf{v}}=0. \tag{2.16}$$

Hence, we obtain a single perturbation term on the right-hand side expressed in terms of a simple layer potential with the prescribed density on the surface (2.10), changing with the time $\tau$. Clearly, various linear and nonlinear "interactions" of all these and other operators in (2.16) will define regularity of the vertex.

2.4. **The Cauchy problem in Leray's nonlocal setting.** Using Leray's nonlocal formulation of NSEs [45, p. 32], we next apply to the Cauchy problem (2.16) the operator

$$\mathbb{P}=I-\nabla\Delta^{-1}(\nabla\cdot I) \quad (\|\mathbb{P}\|=1)$$

being the Leray–Hopf projector of $(L^2(\mathbb{R}^3))^3$ onto the subspace $\{\mathbf{w}\in(L^2)^3: \ \operatorname{div}\mathbf{w}=0\}$ of solenoidal vector fields. Let us note another representation of the projector $\mathbb{P}$ therein:

$$\mathbb{P}\mathbf{w}=[v_1-R_1\sigma, v_2-R_2\sigma, v_3-R_3\sigma]^T, \quad \text{where} \quad \sigma=R_1w_1+R_2w_2+R_3w_3,$$

and $R_j$ are the Riesz transforms, with symbols $\frac{\xi_j}{|\xi|}$. Using the fundamental solution of $\Delta$ in $\mathbb{R}^N$, $N\ge3$, and denoting by $\sigma_N$ the surface area of the unit ball $B_1\subset\mathbb{R}^N$,

$$b_N(y)=-\tfrac{1}{(N-2)\sigma_N}\,\tfrac{1}{|y|^{N-2}}, \quad \text{where} \quad \sigma_N=\tfrac{2\pi^{N/2}}{\Gamma(N/2)} \quad (\sigma_3=4\pi), \tag{2.17}$$

the projection of the convective term reads:

$$\begin{gathered}-\mathbb{P}(\mathbf{v}\cdot\nabla)\mathbf{v}=-(\mathbf{v}\cdot\nabla)\mathbf{v}+C_3\int\limits_{\mathbb{R}^3}\tfrac{y-z}{|y-z|^3}\ \mathrm{tr}(\nabla\mathbf{v}(z,\tau))^2\,\mathrm{d}z,\\ \text{where} \quad \mathrm{tr}(\nabla\mathbf{v}(z,\tau))^2=\textstyle\sum_{(i,j)}v^i_{z_j}v^j_{z_i} \quad \text{and} \quad C_N=\tfrac{1}{\sigma_N}>0 \ \left(C_3=\tfrac{1}{4\pi}\right).\end{gathered} \tag{2.18}$$

This is a more convenient form for some estimates.

Using the projection $\mathbb{P}$ eliminates the pressure term $\nabla \hat{p}$ in (2.16), and we obtain the Cauchy problem for the following perturbed nonlocal parabolic equation for $\hat{\mathbf{v}}$:

$$\boxed{\hat{\mathbf{v}}_\tau = \mathbf{B}^* \hat{\mathbf{v}} - \mathrm{e}^{-\frac{\tau}{2}}\, \mathbb{P}\,(\hat{\mathbf{v}} \cdot \nabla)\hat{\mathbf{v}} - \tfrac{\partial \mathbf{v}}{\partial \mathbf{n}}\, \delta_{\hat{S}(\tau)} \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}_+.} \tag{2.19}$$

Since, by construction, the last term *is solenoidal*, we have omitted the projection $\mathbb{P}$ therein. We recall again that local existence and uniqueness of a classic solution $\hat{\mathbf{v}}(y,\tau)$ of (2.19) are guaranteed by known local regularity properties of the NSEs. Moreover, for any sufficiently small data $\mathbf{v}_0$, solutions of (2.19) are well defined for all $\tau \in \mathbb{R}_+$, i.e., up to the boundary blow-up moment $t = 0^-$ ($\tau = +\infty$). For other solutions, in general, we assume that $\hat{\mathbf{v}}(y,\tau)$ are well defined and do not blow-up at a finite $\tau > 0$, so we need to study their behaviour as $\tau \to +\infty$.

It then follows that Wiener's regularity of the vertex $(0,0)$ is equivalent to the following:

$$\boxed{\begin{array}{l} 0 \text{ is } \textit{globally asymptotically stable} \text{ for (2.19), i.e.,} \\ \text{any such solution of (2.19) satisfies } \hat{\mathbf{v}}(y,\tau) \to 0 \text{ as } \tau \to +\infty \text{ uniformly in } y. \end{array}} \tag{2.20}$$

2.5. **A full pressure representation in $\mathbb{R}^3 \times \mathbb{R}_+$.** As usual [45, p. 30], once the vector field $\hat{\mathbf{v}}$ has been obtained from (2.19), the pressure is then given by the corresponding Poisson equation. Since it is slightly more technical to get it from the rescaled equation (2.19) containing extra operators, we consider first the original (non-rescaled) problem for $\hat{\mathbf{u}} = \mathbf{u} H(\cdot)$, where the Heaviside function $H$ is concentrated on $Q_0(t) \cap \{t\}$, and construct a harmonic extension $\hat{p}$ as in (2.14) for $S(t)$. This equation, which will be also in use, is

$$\hat{\mathbf{u}}_t + (\hat{\mathbf{u}} \cdot \nabla)\hat{\mathbf{u}} = -\nabla \hat{p} + \Delta \hat{\mathbf{u}} - \tfrac{\partial \mathbf{u}}{\partial \mathbf{n}}\, \delta_{S(t)}, \tag{2.21}$$

where we keep the same notation and arguments as in (2.19). Then, taking div [45, p. 30], we obtain the following equation with two extra densities of special potentials:

$$-\Delta_x \hat{p} = \mathrm{tr}\,(\nabla_x \hat{\mathbf{u}})^2 + \mathrm{div}_x \big(\tfrac{\partial \mathbf{u}}{\partial \mathbf{n}_x}\, \delta_{S(t)}\big) - [\nabla_x p]_{S(t)}\, \delta_{S(t)}. \tag{2.22}$$

Observe that the jump of the pressure gradient $[\nabla_x p]_{S(t)}$ enters the second density, that makes this elliptic problem more nonlocal and hence more difficult.

After scaling in (2.1), i.e., setting $x = y\sqrt{-t}$ when approaching the vertex $(0,0)$, we have from (2.22), taking into account that $\delta_{S(t)} = (-t)\, \delta_{\hat{S}(\tau)}$,

$$-\Delta_y \hat{p} = \mathrm{tr}\,(\nabla_y \hat{\mathbf{v}})^2 + \mathrm{e}^{-\tau}\, \mathrm{div}_y \big(\tfrac{\partial \mathbf{v}}{\partial \mathbf{n}_y}\, \delta_{\hat{S}(\tau)}\big) - \mathrm{e}^{-\frac{3\tau}{2}}\, [\nabla_y p]_{\hat{S}(\tau)}\, \delta_{\hat{S}(\tau)} \quad \text{in} \quad \mathbb{R}^3. \tag{2.23}$$

Therefore, this *nonlocal* problem for $\hat{p}$ is reduced to a Fredholm linear integral equation (with a positive kernel) of the second kind,

$$\hat{p} = (-\Delta_y)^{-1} \big[\mathrm{tr}\,(\nabla_y \hat{\mathbf{v}})^2 + \mathrm{e}^{-\tau}\, \mathrm{div}_y \big(\tfrac{\partial \mathbf{v}}{\partial \mathbf{n}_y}\, \delta_{\hat{S}(\tau)}\big) - \mathrm{e}^{-\frac{3\tau}{2}}\, [\nabla_y p]_{\hat{S}(\tau)}\, \delta_{\hat{S}(\tau)}\big], \tag{2.24}$$

where $(-\Delta_y)^{-1}$ is defined by the convolution with the fundamental solution (2.17), $N = 3$. It follows that the behaviour of $p$ as $\tau \to +\infty$ is dependent on the trace of its gradient on the expanded boundary $\hat{S}(\tau)$. Fortunately, as $\tau \to +\infty$, the jump of the gradient $[\nabla_y p]_{\hat{S}(\tau)}$ on the right-hand side of (2.24) has an exponentially small influence on $\hat{p}$, so that (2.24)/(2.23) are "almost" standard integral/elliptic equations with good positive kernels/Laplacian. However, we will need to justify that, nevertheless, the corresponding densities of these single and "double-layer-type" potentials do not get exponentially large, thus undermining their exponentially small factors in front of them. This can be done *a posteriori*, when the independent rescaled $\hat{\mathbf{v}}$-problem (2.19) has been solved.

On the other hand, introducing in (2.23) the following integral operator[6]:

$$\mathbb{M}(\tau)p \equiv -\Delta_y \hat{p} + \mathrm{e}^{-\frac{3\tau}{2}} \left[\nabla_y p\right]_{\hat{S}(\tau)} \delta_{\hat{S}(\tau)}, \tag{2.25}$$

the pressure is given by

$$\hat{p} = \mathbb{M}^{-1}(\tau) \big(\mathrm{tr}\,(\nabla_y \hat{\mathbf{v}})^2 + \mathrm{e}^{-\tau}\, \mathrm{div}_y \left(\tfrac{\partial \mathbf{v}}{\partial \mathbf{n}_y}\, \delta_{\hat{S}(\tau)}\right)\big). \tag{2.26}$$

Indeed, this is the equivalent pressure representation, since $\mathbb{M}^{-1}(\tau)$ is well defined by the local well-posedness of the NSEs in bounded domains with smooth non-characteristic boundaries.

## 3. Boundary layer expansion close to $\partial Q_0$

3.1. **Two region expansion.** As we have mentioned, by the divergence of $\varphi(\tau)$ in (2.5), sharp asymptotics of solutions close to the vertex $(0,0)$ will essentially depend on the spectral properties of the linear operator $\mathbf{B}^*$ in the whole space $\mathbb{R}^3$ (see Appendix A), as well as on the nonlinear projected convective term. This "interaction" between linear and nonlinear operators in the NSEs, together with the paraboloid shape, are key for us.

Studying asymptotics of solutions of the rescaled problem (2.2), as rather often occurs in difficult blow-up expansions in nonlinear PDE theory, this singularity problem is solved by matching of expansions of solutions in two regions:

(i) In an *Inner Region*, which is situated around the origin $y = 0$, and

(ii) In a *Boundary Region* close to the boundary (2.10), where a *Boundary Layer* occurs.

In other words, we show that generic behaviour of solutions of the NSEs in shrinking neighbourhoods of the paraboloid vertex $(0,0)$ *is not of any self-similar form*, and hence gets more complicated and demands novel non-group-similarity techniques to detect.

Actually, such a two-region structure (i)–(ii) above, with the asymptotics specified below, defines the class of generic solutions under consideration. We begin with a simpler analysis in the Boundary Region (ii).

3.2. **Boundary layer (BL) variables and perturbed equation.** Sufficiently close to the lateral boundary of $Q_0$, it is natural to introduce the next rescaled variables

$$z = \tfrac{y}{\varphi(\tau)} \quad \text{and} \quad \hat{\mathbf{v}}(y,\tau) = \mathbf{w}(z,\tau). \tag{3.1}$$

This makes the corresponding rescaled paraboloid (2.10) fixed:

$$\tilde{S}: \quad \sqrt{\textstyle\sum_{i=1}^3 a_i z_i^2} = 1. \tag{3.2}$$

[6]For $\tau \gg 1$, this is just an asymptotically small perturbation of the Laplacian; though proving that $p$ on $\hat{S}(\tau)$ does not grow exponentially fast is a part of the problem.

The rescaled vector field $\mathbf{w}$ now solves a perturbed equation:

$$\mathbf{w}_\tau = \tfrac{1}{\varphi^2}\,\Delta_z \mathbf{w} - \tfrac{1}{2}\,z\cdot\nabla_z \mathbf{w} + \tfrac{\varphi'}{\varphi}\,z\cdot\nabla_z\mathbf{w} - \tfrac{1}{\varphi}\,\mathrm{e}^{-\frac{\tau}{2}}\,\mathbb{P}(\mathbf{w}\cdot\nabla_z)\mathbf{w} - \varphi\,\tfrac{\partial \mathbf{w}}{\partial \mathbf{n}}\,\delta_{\tilde S}. \tag{3.3}$$

Let us introduce the BL-variables: fixing a point $z_0 \in \tilde S$ on the boundary (3.2), we set

$$\xi = \varphi^2(\tau)(z_0 - z) \equiv \varphi(\varphi z_0 - y), \quad \varphi^2(\tau)\mathrm{d}\tau = \mathrm{d}s, \quad \mathbf{w}(z,\tau) = \rho(s)\mathbf{g}(\xi,s), \tag{3.4}$$

where $\rho(s) = [\rho^1(s), \rho^2(s), \rho^3(s)]^T \in \mathbb{R}^3$ for $s \gg 1$ is an unknown scaling slow varying/decaying (in the same natural sense, associated with (2.6)) time-factor depending on the function $\varphi(\tau)$. Cf. e.g., $\sim \frac{1}{\varphi(\tau)}$ as a clue. As usual, this $\rho$-scaling is chosen to get uniformly bounded rescaled solutions, so, we naturally assume that, for each component,

$$\sup_\xi\, g^j(\xi,s) = 1 \quad \text{for all} \quad s \gg 1, \quad j = 1,\,2,\,3. \tag{3.5}$$

On substitution into the PDE in (3.3), we obtain the following perturbation of a linear uniformly parabolic equation:

$$\begin{aligned} &\mathbf{g}_s = \mathbf{A}\mathbf{g} - \tfrac{1}{2}\,\tfrac{1}{\varphi^2}\,\xi\cdot\nabla_\xi\mathbf{g} - \tfrac{\varphi'_\tau}{\varphi}\left(z_0 - \tfrac{\xi}{\varphi^2}\right)\cdot\nabla_\xi\mathbf{g} - \tfrac{2\varphi'_\tau}{\varphi^3}\,\xi\cdot\nabla_\xi\mathbf{g} - \tfrac{\rho'_s}{\rho}\,\mathbf{g} \\ &-\tfrac{\rho}{\varphi}\,\mathrm{e}^{-\frac{\tau}{2}}\,\mathbb{P}(\mathbf{g}\cdot\nabla_\xi)\mathbf{g} - \tfrac{1}{\varphi^3}\,\tfrac{\partial\mathbf{g}}{\partial\mathbf{n}_\xi}\,\delta_{S_\xi(\tau)}, \;\; \text{where} \;\; \mathbf{A}\mathbf{g} = \Delta_\xi\mathbf{g} + \tfrac{1}{2}\,z_0\cdot\nabla_\xi\mathbf{g}, \end{aligned} \tag{3.6}$$

and $S_\xi(\tau)$ is the boundary (3.2) expressed in terms of the BL-variable $\xi$ in (3.4), so that

$$\delta_{S_\xi(\tau)} = \tfrac{1}{\varphi^3(\tau)}\,\delta_{\tilde S}.$$

As usual in boundary layer theory, the BL-scaling (3.4) means that we are looking for a generic pattern of the behaviour described by the perturbed equation (3.6) on compact subsets, shrinking (focusing) to a fixed $z_0$ on the lateral boundary,

$$|\xi| = o\big(\tfrac{1}{\varphi^2(\tau)}\big) \to 0 \quad \Longrightarrow \quad |z - z_0| = o\big(\tfrac{1}{\varphi^4(\tau)}\big) \to 0 \quad \text{as} \quad \tau \to +\infty. \tag{3.7}$$

Thus, in (3.6), we arrive at a linear uniformly parabolic equation perturbed by a number of linear and nonlinear terms, which, under given and other special hypothesis to be specified, are *asymptotically small*. Indeed, on the space-time compact subsets (3.7), the second term on the right-hand side of (3.6) becomes asymptotically small, while all the other linear ones even smaller in view of the slow growth/decay assumptions such as (2.6) for $\varphi(\tau)$ and $\rho(s)$. In particular, the rescaled nonlinear convective term in (3.6) is asymptotically small on bounded rescaled vector fields $\mathbf{g}$ in view of an exponentially decaying factor and by the hypotheses (2.5).

However, the last term in (3.6), given by a density of a simple layer potential, requires a special treatment. Indeed, this density depends upon a still unknown gradient of $\mathbf{w} = \rho\,\mathbf{g}$ on the boundary, which is under scrutiny in the present BL-analysis. However, the nature of our BL-scaling assumes that we deal with uniformly bounded rescaled function $\mathbf{g}$, on which the last term is asymptotically small and is of order $\sim \frac{1}{\varphi(\tau)} \to 0$ as $\tau \to +\infty$.

The BL-representation (3.4), by using the rescaling and (3.5), naturally leads to the following asymptotic behaviour at infinity:

$$\lim_{s\to+\infty} g^j(\xi,s) \to 1 \quad \text{as} \quad \xi \to \infty, \tag{3.8}$$

where all the derivatives also vanish by the standard interior parabolic regularity. Actually, the nature of the BL-scaling (3.4) near the point $z_0 \in \tilde S$ implies that, asymptotically, the limit problem becomes one-dimensional, depending on the space variable

$$\eta = \xi \cdot \mathbf{n}, \tag{3.9}$$

where $\mathbf{n}$ is the unit inward normal to the smooth boundary $\tilde S$ in (3.2). Therefore, the limit $\xi \to \infty$ in (3.8) should be also understood in the sense of the single variable (3.9). This essentially simplifies the BL-structure to appear.

Moreover, since according to the BL-scaling (3.7), as $s \to +\infty$, the rescaled solution becomes constant (see (3.8)) a.e. and hence solenoidal, *we do not need to require the limit BL-profile to be solenoidal as well.* Moreover, we will see that, in the actual boundary layer, the BL-asymptotic is "almost" solenoidal, up to an exponentially small perturbation.

3.3. **Passing to the limit: generic solutions.** Thus, we arrive at the problem of passing to the limit as $s \to +\infty$ in the problem (3.6), (3.8). Since, by the definition in (3.4), the rescaled orbit $\{\mathbf{g}(s),\ s > 0\}$ is uniformly bounded, by classic parabolic interior theory [9, 10, 11], one can pass to the limit in (3.6) along a subsequence $\{s_k\} \to +\infty$. Namely, we have that, uniformly on compact subsets defined in (3.7), as $k \to \infty$,

$$\mathbf{g}(s_k + s) \to \mathbf{h}(s), \quad \text{where} \quad \mathbf{h}_s = \mathbf{A}\mathbf{h}, \quad \mathbf{h} = 0 \ \text{at} \ \eta = 0, \quad h^j|_{\eta=+\infty} = 1. \tag{3.10}$$

Consider this one-dimensional *limit* (at $s = +\infty$) *equation* obtained from (3.6):

$$\mathbf{h}_s = \mathbf{A}\mathbf{h} \equiv \mathbf{h}_{\eta\eta} + \tfrac{1}{2}\mathbf{h}_\eta \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}_+, \quad \mathbf{h}(0, s) = 0, \quad h^j(+\infty, s) = 1. \tag{3.11}$$

It is a linear parabolic PDE in the unbounded domain $\mathbb{R}_+$, governed by the operator $\mathbf{A}$ admitting a standard symmetric representation in a weighted space. Namely, we have:

**Proposition 3.1.** (i) (3.11) *is a gradient system in a weighted* $(L^2)^3$*-space, and*

(ii) *for bounded orbits, the* $\omega$*-limit set* $\Omega_0$ *of* (3.11) *consists of a unique stationary profile*

$$g_0^j(\xi) = 1 - \mathrm{e}^{-\eta/2}, \quad j = 1, 2, 3, \tag{3.12}$$

*and* $\Omega_0$ *is uniformly stable in the Lyapunov sense in a weighted* $(L^2)^3$*-space.*

*Proof.* (i) As a second-order equation, (3.11) can be written in the symmetric form

$$\mathrm{e}^{\eta/2}\mathbf{h}_s = (\mathrm{e}^{\eta/2}\mathbf{h}_\eta)_\eta, \tag{3.13}$$

and hence admits multiplication by $\mathbf{h}_s$ in $(L^2)^3$ that yields a monotone Lyapunov function:

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}s} \int \mathrm{e}^{\eta/2}(h^j_\eta)^2 = -\int \mathrm{e}^{\eta/2}(h^j_s)^2 \le 0. \tag{3.14}$$

Note that, regardless that the weight $\mathrm{e}^{\eta/2}$ is exponentially growing as $\eta \to +\infty$, on the limit profile (3.12), all the functionals in (3.14) are well defined. In other words, the problem (3.6) is a perturbed *gradient system*, that makes much easier to pass to the limit $s \to +\infty$ by using power tools of gradient system theory; see e.g., Hale [29].

(ii) For a given bounded orbit $\{h(s)\}$, denote $h^j(s) = g_0^j + w^j(s)$, so that $\mathbf{w}(s)$ solves the same equation (3.13). Multiplying by $\mathbf{w}(s)$ in $(L^2)^3$ yields

$$\tfrac{1}{2}\,\tfrac{\mathrm{d}}{\mathrm{d}s}\int \mathrm{e}^{\eta/2}(w^j)^2\,\mathrm{d}\eta = -\int \mathrm{e}^{\eta/2}(w^j_\eta)^2\,\mathrm{d}\eta < 0 \tag{3.15}$$

for any nontrivial solution, whence the uniform stability (contractivity) property. $\square$

Finally, we state the main stabilization result in the boundary layer, which also establishes the actual class of *generic* solutions we are dealing with.

**Proposition 3.2.** *Under specified above assumptions and hypotheses, there exists a class of solutions of the perturbed equation* (3.6)*, for which, in a weighted* $(L^2)^3$*-space and uniformly on compact subsets,*

$$g^j(\xi,s) \to g_0^j(\xi) \quad as \quad s \to +\infty \quad (j=1,2,3). \tag{3.16}$$

*Proof.* (i) Under given hypotheses, the uniform stability result in (ii) of Proposition 3.1 implies [23, Ch. 1] that the $\omega$-limit set of the asymptotically perturbed equation (3.6) is contained in that for the limit one (3.11), which, under the given hypotheses, consists of the unique profile (3.12). $\square$

3.4. **BL-behaviour is "almost" divergence free.** For the future convenience, we state again the asymptotic BL-behaviour: in the rescaled sense, by (3.16), (3.12), and (4.8),

$$\hat{\mathbf{v}}(y,\tau) = \mathbf{c}_0(\tau)\,\mathbf{g}_0(y,\tau) + \ldots, \quad \text{where} \quad g_0^j(y,\tau) = 1 - \mathrm{e}^{-\frac{\varphi(\tau)}{2}\,\mathrm{dist}\{y,\partial\hat{Q}_0(\tau)\}}. \tag{3.17}$$

It is important that, by (3.17), the first term is "a.e." an exponentially small perturbation of a divergence-free flow. Indeed, differentiating (3.17) at any point staying away from the boundary by an arbitrarily small fixed $\mathrm{dist}\{\cdot\} = \delta_0 > 0$, we have

$$\mathrm{div}_y\,\hat{\mathbf{v}}(y,\tau) = O\big(c_0(\tau)\varphi(\tau)\mathrm{e}^{-\frac{\varphi(\tau)}{2}\,\delta_0}\big) \to 0 \quad \text{as} \quad \tau \to +\infty, \tag{3.18}$$

provided that $c_0(\tau)$ is not exponentially large (this does not happen, as we will show). In other words, not that surprisingly, the BL-expansion well keeps solenoidal features of originally divergence-free solutions $\hat{\mathbf{v}}$ and, as customary, just makes an asymptotically (i.e., exponentially as in (3.18)) small perturbation of the div. Thus, a somehow essential violation of the solenoidal property can happen only in an asymptotically small $O\big(\frac{1}{\varphi(\tau)}\big)$-neighbourhood of the boundary, which is negligible and plays no role for $\tau \gg 1$.

## 4. Inner Region expansion: towards ODEs regularity criterion

4.1. **A standard semigroup approach leads to a more complicated problem.** Let us first perform necessary manipulations using a standard semigroup approach. Applying to (2.21) the projector $\mathbb{P}$ yields

$$\hat{\mathbf{u}}_t = \Delta\hat{\mathbf{u}} - \tfrac{\partial \mathbf{u}}{\partial \mathbf{n}}\,\delta_{S(t)} - \mathbb{P}\,(\hat{\mathbf{u}}\cdot\nabla)\hat{\mathbf{u}}. \tag{4.1}$$

Therefore, using the fundamental solution $b(x,t)$ of the heat operator $D_t - \Delta$ with the rescaled kernel (the Gaussian) as in (A.5) gives the following convolution representation of the solution of (4.1):

$$\hat{\mathbf{u}}(t) = b(t) * \mathbf{u}_0 - \int_0^t b(t-s) * \tfrac{\partial \mathbf{u}}{\partial \mathbf{n}}(s)\, \delta_{S(s)}\, \mathrm{d}s - \int_0^t b(t-s) * \mathbb{P}\,(\hat{\mathbf{u}}(s) \cdot \nabla)\hat{\mathbf{u}}(s)\, \mathrm{d}s. \tag{4.2}$$

In particular, taking div, we see that the second term on the right-hand side, containing a surface integral, is div-free, since $\mathbf{u}$ is. Finally, sharply estimating the normal derivative therein and in the third term from the core of BL-theory, the asymptotics (3.17), one can study the asymptotic behaviour of solutions, after using the necessary scaling (2.1).

However, it turns out that this standard *integral* semigroup approach leads to a more complicated analysis, than a *differential* one we will perform by using known spectral properties of the rescaled operator $\mathbf{B}^*$ involved in (2.19). Nevertheless, it is worth mentioning that such an approach can be translated to the integral equation (4.2), with the clear advantage of a more reliable rigorous justification by using obviously smoother properties of solutions and, as a result, their better uniform estimates in stronger metrics. It should be noted that some principle difficulties cannot be avoided in such a way, and, overall, technical technical questions become more dominant for (4.2).

4.2. **Eigenfunction expansion: derivation of a 3D dynamical system.** Thus, in the Inner Region, we deal with the original rescaled Cauchy problem (2.19). Since, by construction, the extended solution orbit (2.11) is uniformly bounded in $(L^2_{\rho^*}(\mathbb{R}^3))^3$, we can use the converging in the mean (and uniformly on compact subsets in $y$) eigenfunction expansion via the solenoidal Hermite polynomials as in (A.17):

$$\hat{\mathbf{v}}(y,\tau) = \textstyle\sum_{(\beta)} \mathbf{c}_\beta(\tau)\mathbf{v}^*_\beta(y), \tag{4.3}$$

where we use the convention (A.18). Substituting (4.3) into (2.16) and using the orthonormality of these polynomials yield the following dynamical system for the expansion coefficients: for all $|\beta| \ge 0$,

$$\dot{\mathbf{c}}_\beta = \lambda_\beta \mathbf{c}_\beta - \langle \tfrac{\partial \mathbf{v}}{\partial \mathbf{n}}\, \delta_{\hat{S}(\tau)}, \mathbf{v}_\beta \rangle - \mathrm{e}^{-\frac{\tau}{2}} \langle \mathbb{P}(\hat{\mathbf{v}} \cdot \nabla_y)\hat{\mathbf{v}}H, \mathbf{v}_\beta \rangle, \tag{4.4}$$

where $\lambda_\beta = -\frac{|\beta|}{2}$ are the real eigenvalues as in (A.2).

Recalling that eigenvalues in (4.4) satisfy $\lambda_\beta \le -\frac{1}{2}$ for all $|\beta| \ge 1$, it follows that we need to concentrate on the "maximal" first Fourier generic pattern associated with the first constant Hermite polynomial $\mathbf{v}^*_0$ in (A.14),

$$k = |\beta| = 0: \quad \lambda_0 = 0 \quad \text{and} \quad \mathbf{v}^*_0(y) \equiv \mathbf{e} = [1,1,1]^T. \tag{4.5}$$

The normalized eigenfunction of the $L^2$-adjoint operator $\mathbf{B}$ is then

$$\mathbf{v}_0(y) = F(y)\, \mathbf{e}, \tag{4.6}$$

where $F(y)$ is the Gaussian in (A.6). Actually, as follows from (3.16), this corresponds to a naturally understood "centre subspace behaviour" for the operator $\mathbf{B}^*$ in (2.16):

$$\hat{\mathbf{v}}(y,\tau) = \mathbf{c}_0(\tau)\mathbf{e} + \mathbf{w}^\perp(y,\tau), \quad \text{where} \quad \mathbf{w}^\perp \in \mathrm{Span}\{\mathbf{v}^*_\beta,\ |\beta| \ge 1\}, \tag{4.7}$$

and $\mathbf{w}^\perp(y,\tau)$ is then negligible relative to $\mathbf{c}_0(\tau)$ for $\tau \gg 1$.

Fortunately, we actually do not need such a literal using of those "centre subspace issues" for a difficult non-autonomous equation like (2.19), since the asymptotics of solutions (4.7) is directly dictated by Proposition 3.2. In its turn, this is an equivalent characterization of our class of generic patterns, and, in particular, the following holds:

**Proposition 4.1.** *Under the given above assumptions and hypotheses, for the prescribed class of generic solutions defined in Proposition 3.2,* (4.7) *holds with* $w^{\perp j}(\tau) = o(|c_0^j(\tau)|)$ *as* $\tau \to +\infty$*, and then the matching with the boundary layer behaviour in* (3.4) *requires*

$$\tag{4.8} \tfrac{a_0^j(\tau)}{\rho^j(s)} \to 1 \quad as \quad \tau \to +\infty \quad \Longrightarrow \quad \rho^j(s) = a_0^j(\tau)(1+o(1)), \quad j=1,2,3.$$

*Proof.* This follows from the construction of the boundary layer by comparing the solution representations in (3.4), (3.16), and (4.7). $\square$

Thus, the equation for $\mathbf{c}_0(\tau)$, with $\lambda_0 = 0$, takes the form:

$$\tag{4.9} \dot{\mathbf{c}}_0 = -\langle \tfrac{\partial \mathbf{v}}{\partial \mathbf{n}}\, \delta_{\hat S(\tau)}, \mathbf{v}_0 \rangle - \mathrm{e}^{-\frac{\tau}{2}} \langle \mathbb{P}(\hat{\mathbf{v}} \cdot \nabla_y)\hat{\mathbf{v}} H, \mathbf{v}_0 \rangle,$$

where the first adjoint eigenfunction $\mathbf{v}_0$ is as in (4.6).

We now need to return to BL-theory in Section 3 establishing the boundary behaviour (3.4) for $\tau \gg 1$, which has the form (3.17). Then the convergence (3.17), which by a standard parabolic regularity is also true for the spatial derivatives, yields, in the natural rescaled sense,

$$\tag{4.10} \tfrac{\partial \mathbf{v}}{\partial \mathbf{n}} = -\tfrac{1}{2}\, \mathbf{c}_0(\tau)\varphi(\tau) \tfrac{y \cdot \mathbf{n}(y)}{|y|} + \dots, \quad \text{where} \quad y \cdot \mathbf{n}(y) < 0 \ \text{(by convexity)}.$$

Therefore, the first (linear) term in (4.9) asymptotically reads

$$\tag{4.11} -\langle \tfrac{\partial \mathbf{v}}{\partial \mathbf{n}}\, \delta_{\hat S(\tau)}, \mathbf{v}_0 \rangle = \tfrac{\mathbf{c}_0 \varphi(\tau)}{2(4\pi)^{3/2}} \int\limits_{\hat S(\tau)} \tfrac{s \cdot \mathbf{n}(s)}{|s|}\, \mathrm{e}^{-|s|^2/4}\, \mathrm{d}s + \dots .$$

Due to the normalization in (2.3), we have that $a_i \le 1$, so that the last term can be estimates above as: for any component $j=1,2,3$,

$$\tag{4.12} \tfrac{\varphi(\tau)}{2(4\pi)^{3/2}} \int\limits_{\hat S(\tau)} \tfrac{s \cdot \mathbf{n}(s)}{|s|}\, \mathrm{e}^{-|s|^2/4}\, \mathrm{d}s \le -\gamma_1 \varphi^3(\tau)\, \mathrm{e}^{-\varphi^2(\tau)/4} + \dots, \quad \gamma_1 = \text{const.} > 0.$$

Recall that the extra factor $\varphi^2$ is obtained via integration over a closed *surface* in $\mathbb{R}^3$.

For the simple radial paraboloid shape in (2.10) and (3.2), with $a_i = 1$, i.e., by (2.4),

$$\tag{4.13} \hat S(\tau) = \partial \hat Q_0 \cap \{\tau\}: \quad q_0(y) \equiv |y| = \varphi(\tau), \quad \text{and} \quad \tilde S: \quad |z| = 1,$$

(4.12) presents a sharp asymptotics behaviour rather than an estimate: for $j=1,2,3$,

$$\tag{4.14} \tfrac{\mathbf{c}_0 \varphi(\tau)}{2(4\pi)^{3/2}} \int\limits_{\hat S(\tau)} \tfrac{s \cdot \mathbf{n}(s)}{|s|}\, \mathrm{e}^{-|s|^2/4}\, \mathrm{d}s = -\mathbf{c}_0 \gamma_1 \varphi^3(\tau)\, \mathrm{e}^{-\varphi^2(\tau)/4} + \dots .$$

Naturally, it is possible to derive most sharp asymptotics for this radial case.

Finally, we need to estimate the last nonlinear quadratic term in (4.9): by (3.17),

$$\mathrm{e}^{-\frac{\tau}{2}}\,\langle \mathbb{P}(\hat{\mathbf{v}}\cdot\nabla_y)\hat{\mathbf{v}}H,\mathbf{v}_0\rangle = \mathrm{e}^{-\frac{\tau}{2}}\,(\mathbf{c}_0\cdot\mathbf{e})\mathbf{c}_0\,\tfrac{1}{(4\pi)^{3/2}}\int\limits_{\mathbb{R}^3}\mathbb{P}[(\mathbf{g}_0\cdot\nabla_y)\mathbf{g}_0H]\,\mathrm{e}^{-|y|^2/4}\,\mathrm{d}y+...\,, \tag{4.15}$$

where the quadratic term $(\mathbf{c}_0\cdot\mathbf{e})\mathbf{c}_0$ denotes the vector $(c_0^1+c_0^2+c_0^3)\mathbf{c}_0\in\mathbb{R}^3$. Fortunately, since this nonlinear term enjoys having a fast decaying exponential factor $\mathrm{e}^{-\frac{\tau}{2}}$, we do not need its better sharp estimates. We just need to show that, on the generic solutions obeying the BL-behaviour (3.17), this term always remains exponentially small, so does not play any role for the regularity conclusions.

We restrict to the radial case, though the asymptotic smallness of the convection can be similarly shown for more general convex paraboloids. Thus, using the representation of $\mathbb{P}(\hat{\mathbf{v}}\cdot\nabla)\hat{\mathbf{v}}$ given in (2.18) and using the change as in (3.1), i.e., setting $y=\varphi(\tau)z$, we have from (3.17), for $\tau\gg 1$,

$$\begin{aligned}\mathrm{e}^{-\frac{\tau}{2}}\,(\hat{\mathbf{v}}\cdot\nabla)\hat{\mathbf{v}} &= \mathrm{e}^{-\frac{\tau}{2}}\,(\mathbf{c}_0\cdot\mathbf{e})\mathbf{c}_0\,\varphi\big[\big(1-\mathrm{e}^{-\frac{\varphi^2}{2}\,d_z}\big)\mathrm{e}^{-\frac{\varphi^2}{2}\,d_z}(\mathbf{n}\cdot\mathbf{e})\\ &-C_3\varphi^2\int\tfrac{z-\zeta}{|z-\zeta|^3}\,\mathrm{e}^{-\varphi^2\,d_\zeta}\textstyle\sum_{(i,j)}(\mathbf{n}\cdot e_i)(\mathbf{n}\cdot e_j)\,\mathrm{d}\zeta\big]+...\equiv J_1+J_2,\end{aligned} \tag{4.16}$$

where $d_z$ (and $d_\zeta$ in the integral) denotes the distance: $d_z=\mathrm{dist}\,\{z,\tilde{S}\}$. For such a rough estimate from above, one can omit the projector $\mathbb{P}$, using the fact that $\|\mathbb{P}\|=1$.

It is not difficult to see that, the $J_1$-term in (4.16) is leading to the following integral:

$$J_1\sim \mathrm{e}^{-\frac{\tau}{2}}\,(\mathbf{c}_0\cdot\mathbf{e})\mathbf{c}_0\varphi^4\int\limits_{\{|z|\le 1\}}\mathrm{e}^{-\frac{\varphi^2}{4}\,|z|^2}\big(1-\mathrm{e}^{-\frac{\varphi^2}{2}\,d_z}\big)\mathrm{e}^{-\frac{\varphi^2}{2}\,d_z}(\mathbf{n}\cdot\mathbf{e})\,\mathrm{d}z, \tag{4.17}$$

where, in the radial geometry, we may put $d_z=1-|z|$. Reducing the integral in (4.17) to a standard 1D one, it can be estimated above as follows: for some constant $\gamma_2>0$,

$$|J_1(\tau)|\le\gamma_2\,\mathrm{e}^{-\frac{\tau}{2}}\,|(\mathbf{c}_0\cdot\mathbf{e})\mathbf{c}_0|\,\varphi^4(\tau)\,\mathrm{e}^{-\frac{\varphi^2(\tau)}{4}}\quad\text{as}\quad\tau\to+\infty. \tag{4.18}$$

Consider the last term $J_2$ in (4.16). The corresponding upper estimate is: for $\gamma_{3,4}>0$,

$$\begin{aligned}|J_2(\tau)| &\le\gamma_3\mathrm{e}^{-\frac{\tau}{2}}\,|(\mathbf{c}_0\cdot\mathbf{e})\mathbf{c}_0|\,\varphi^6\,\Big|\int\limits_{\{|z|\le 1\}}\mathrm{e}^{-\frac{\varphi^2}{4}\,|z|^2}\int\tfrac{z-\zeta}{|z-\zeta|^3}\,\mathrm{e}^{-\varphi^2 d_\zeta}\textstyle\sum_{i,j}(\cdot)\,\mathrm{d}\zeta\,\mathrm{d}z\Big|\\ &\le\gamma_4\mathrm{e}^{-\frac{\tau}{2}}\,|(\mathbf{c}_0\cdot\mathbf{e})\mathbf{c}_0|\,\varphi^6(\tau)\,\mathrm{e}^{-\frac{\varphi^2(\tau)}{4}}.\end{aligned} \tag{4.19}$$

We do not guarantee that the $\varphi^6$ multiplier in the final estimate in (4.19) is any sharp (as well as $\varphi^4$ in (4.18)), but it is sufficient for showing the convection neglect near the vertex. Indeed, any such very rough estimates (or omitting the projector $\mathbb{P}$ as we did above) cannot undermine the principal fact: extra multipliers containing **any** power of $\varphi(\tau)$, being a *slow* growing function, do not practically affect the exponentially decaying factor $\mathrm{e}^{-\tau/2}$ as $\tau\to+\infty$ in (4.15) and (4.16). Comparing with (4.17), (4.18) yields that (4.19) supplies us with the leading term as $\tau\to+\infty$.

Thus, bearing in mind all above assumptions and estimates for generic patterns, we obtain the following asymptotically approximate dynamical system for the first expansion coefficients $\mathbf{c}_0(\tau)$: for $\tau \gg 1$,

$$\boxed{\dot{\mathbf{c}}_0 \sim -\gamma_1\, \mathbf{c}_0 \varphi^3(\tau)\, \mathrm{e}^{-\varphi^2(\tau)/4} + \gamma_4\, \mathrm{e}^{-\frac{\tau}{2}}\, (\mathbf{c}_0 \cdot \mathbf{e})\mathbf{c}_0\, \varphi^6(\tau)\, \mathrm{e}^{-\varphi^2(\tau)/4} + \dots,} \tag{4.20}$$

where we now omit all higher-order terms appeared via the above hypotheses. The sign "$\sim$" in (4.20) means that, in the presentation of the influence of the nonlinear convection term, we used the estimate (4.19), rather than a sharp asymptotics. However, this estimate suffices to declare that the convection term is negligible in the regularity analysis.

4.3. **3D regularity criterion.** Thus, according to (2.20), the following conclusion holds:

**Theorem 4.1.** *Under the assumed above hypotheses and conditions, the vertex* $(0,0)$ *is regular for the NSEs problem* (1.1), (1.2) *in the class of generic solutions, iff*

$$\boxed{\textit{the origin is globally asymptotically stable for the 3D dynamical system (4.20)},} \tag{4.21}$$

*i.e., any its solution satisfies:*

$$\mathbf{c}_0(\tau) \to 0 \quad \textit{as} \quad \tau \to +\infty. \tag{4.22}$$

4.4. **Two regularity conclusions.** We begin with a simpler linear one.

1. Linear Stokes problem. As a first consequence, we confirm that Petrovskii's criterion (2.9) remains valid in the linear case. Recall that here, our analysis do not include more difficult "nonlinear" estimates used in (4.15).

**Theorem 4.2.** *For the linear Stokes problem* (1.4), *under our hypotheses on generic solutions, the regularity criterion of the vertex* $(0,0)$ *is Petrovskii's one* (2.9).

*Proof.* Introducing the new time,

$$\varphi^3(\tau)\, \mathrm{e}^{-\varphi^2(\tau)/4}\, \mathrm{d}\tau = \mathrm{d}s \quad \Longrightarrow \quad s = \int_0^\tau \varphi^3(\zeta)\, \mathrm{e}^{-\varphi^2(\zeta)/4}\, \mathrm{d}\zeta, \tag{4.23}$$

from (4.20) (with no quadratic term), we obtain a linear diagonal autonomous system

$$\tfrac{\mathrm{d}}{\mathrm{d}s}\, \mathbf{c}_0 = -\gamma_1\, \mathbf{c}_0 \quad \Longrightarrow \quad \mathbf{c}_0(s) = \mathbf{c}_0(0)\, \mathrm{e}^{-\gamma_1 s} \to 0 \tag{4.24}$$

if and only if $s \to +\infty$ as $\tau \to +\infty$, and the divergence in (2.9) follows. $\square$

2. Navier–Stokes equations. Performing the change (4.23) for the full dynamical system (4.20) yields

$$\tfrac{\mathrm{d}}{\mathrm{d}s}\, \mathbf{c}_0 = -\gamma_1\, \mathbf{c}_0 + \gamma_4\, \mathrm{e}^{-\frac{\tau}{2}}\, (\mathbf{c}_0 \cdot \mathbf{e})\mathbf{c}_0\, \varphi^3(\tau). \tag{4.25}$$

An elementary balancing of the linear and nonlinear term on the right-hand side shows that the nonlinear one can be efficiently involved into the *regular asymptotics* if it has at least an exponential growth

$$|\mathbf{c}_0(\tau)| \gg \mathrm{e}^{\frac{\tau}{2}}\, \tfrac{1}{\varphi^3(\tau)} \gg 1 \quad \text{for} \quad \tau \gg 1, \tag{4.26}$$

since, by assumptions, $\varphi(\tau)$ is a slow growing function. Of course, (4.26) is impossible, since by the regularity assumption $\mathbf{c}_0(\tau) \to 0$.

Note that, if $\mathbf{v}(\tau)$ gets vanishing as $\tau \to +\infty$, then the same is true for the pressure via (2.26) (then (2.25) is a perturbation of the Laplacian), i.e., the pressure influence in the full equation (with no convection) is truly exponentially negligible.

Thus, this is our final conclusion for the NSEs: under given hypothesis and for generic solutions[7], *the nonlinear convection term cannot affect the regularity of the vertex* $(0,0)$*, so that Petrovskii's criterion* (2.9) *remains true for the Navier–Stokes equations* (1.1).

## 5. Final discussion: two key blow-up problems for the 3D NSEs

It is now worth and natural to look how the characteristic boundary regularity analysis stands and fits in the wide area around the *Millennium Prize Problem for the Clay Institute* (the MPPCI) on *global existence or nonexistence of bounded smooth solutions for the NSEs* (1.1); see Fefferman [14]. Both deal with settings presented in a similar fashion:

**($\mathbb{BR}$)** **$\mathbb{B}$oundary point $\mathbb{R}$egularity**: sharp "blow-up" asymptotic expansions of possible solutions of (1.1) *near a characteristic boundary point* of $Q_0$. Then, mathematically speaking, after blow-up scaling (2.1), we arrive at an exponentially small perturbation of a solenoidal heat equation in (2.2), which is convenient to write down in its full presentation

$$\boxed{\mathbf{v}_\tau = \Delta \mathbf{v} - \tfrac{1}{2}\, y \cdot \nabla \mathbf{v} - \boxed{\mathrm{e}^{-\frac{\tau}{2}}}\, \mathbb{P}(\mathbf{v} \cdot \nabla)\mathbf{v} \quad \text{in} \quad \hat{Q}_0.} \tag{5.1}$$

We observe here a crucial moment: in the vertex regularity study, the key nonlinear convection term gets an extra fast exponentially decaying factor $\mathrm{e}^{-\frac{\tau}{2}}$ (boxed in (5.1)), that, indeed, essentially simplifies the matched asymptotic analysis. As we have seen, our goal was then simply to show that this nonlinear exponentially small perturbation does not affect the regularity of the vertex $(0,0)$ at all, so it becomes pure parabolic (Petrovskii's) one, i.e., governed as $\tau \to +\infty$ by the rescaled solenoidal heat equation

$$\mathbf{v}_\tau = \Delta \mathbf{v} - \tfrac{1}{2}\, y \cdot \nabla \mathbf{v} \quad \text{in} \quad \hat{Q}_0.$$

**($\mathbb{IR}$)** **$\mathbb{I}$nterior point $\mathbb{R}$egularity (the MPPCI)**: asymptotic expansion of possible solutions in *any interior point to check whether finite time blow-up in* $L^\infty$ *is possible or not.* Then, a full Leray's self-similar blow-up scaling must be performed[8] [43]:

$$\mathbf{u}(x,t) = \tfrac{1}{\sqrt{T-t}}\, \mathbf{v}(y,\tau), \quad y = \tfrac{x}{\sqrt{T-t}}, \quad \tau = -\ln(T-t), \tag{5.2}$$

---

[7]For the full Navier–Stokes models, in view of an essential difficulties to justify nonlinear convection estimates for calculating (4.15), we rather hesitate to state this even as a formal asymptotics; further mathematical research is necessary and is desirable.

[8]This Leray's blow-up self-similarity led him to a conjecture on *existence of a blow-up similarity solution* as $t \to T^-$, and on *existence of a self-similar extension* for $t > T$ (with $(T-t) \mapsto (t-T)$ in (5.2)); see some extra details and a discussion in [18, § 2].

where $T > 0$ is the assumed finite blow-up time of the vector field $\mathbf{u}(x,t)$ (used to be always $T = 0$ beforehand). This leads to a different *autonomous* rescaled equation:

$$\boxed{\mathbf{v}_\tau = \Delta \mathbf{v} - \tfrac{1}{2}\, y \cdot \nabla \mathbf{v} - \tfrac{1}{2}\, \mathbf{v} - \boxed{\phantom{a}}\, \mathbb{P}(\mathbf{v} \cdot \nabla)\mathbf{v} \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}_+,} \tag{5.3}$$

where, unlike (5.1), *the box is empty*: no exponentially decaying factor therein!

This is the main principal difference between both key regularity problems: in the rescaled equation (5.3), the convection term **is not accompanied** by an exponentially small time-factor as in (5.1). Therefore, it can play a vital role in creating a possible $L^\infty$-blow-up singularity. It turned out that a self-similar blow-up of Leray's type (5.2), with a nontrivial profile $\mathbf{v} = \mathbf{v}(y)$ solving the stationary equation (5.3), does not exist:

$$\Delta \mathbf{v} - \tfrac{1}{2}\, y \cdot \nabla \mathbf{v} - \tfrac{1}{2}\, \mathbf{v} - \mathbb{P}(\mathbf{v} \cdot \nabla)\mathbf{v} = 0 \quad \text{in} \quad \mathbb{R}^3, \;\; \mathbf{v} \in L^2(\mathbb{R}^3) \quad \Longrightarrow \quad \mathbf{v} = 0; \tag{5.4}$$

see [58, 67, 55, 30]. Therefore, it seems that a leading idea how to create a blow-up singularity for the NSEs is to deal with the so-called Type II blow-up solutions, i.e., those which violate a uniform similarity estimate for Type I solutions:

$$\textbf{Type I:} \quad \text{for some constant } C > 0, \quad |\mathbf{u}(x,t)| \le \tfrac{C}{\sqrt{T-t}} \text{ in } \mathbb{R}^3 \times (0,T), \tag{5.5}$$

$$\text{i.e., } \textbf{Type II blow-up}: \quad \limsup_{t \to T^-} \sqrt{T-t}\, \sup_{x \in \mathbb{R}^3} |\mathbf{u}(x,t)| = +\infty. \tag{5.6}$$

In the rescaled variables (5.2), Type II blow-up means looking for a global solution of (5.3) that "blows up" as $\tau \to +\infty$. This problem is open, though some formal scenarios of such a Type II blow-up in the NSEs have been discussed for a while; see, e.g., references and discussions in [18].

Overall, we expect that the present study of simpler *boundary point regularity* governed by an exponentially perturbed rescaled equation (5.1) is an inevitable and important step towards solving the main open *interior point regularity* blow-up problem for (5.3). Indeed, for (5.1), it turned out that the asymptotic behaviour near the vertex was spatially governed by the simplest *first solenoidal Hermite polynomial* (actually, a constant). While, in order to understand blow-up in the "MPPCI equation" (5.3), it seems that an essential involvement of *all other* solenoidal Hermite polynomials $\{\mathbf{v}^*_\beta(y)\}$ as eigenfunctions of the linear rescaled operator $\mathbf{B}^*$ (see Appendix A) should be detected first, *before* attacking a possible essentially nonlinear structure of a Type-II blow-up singularity for the NSEs.

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## References

[1] S. Angenent, *Large-time asymptotics of the porous medium equation*, In: Nonl. Differ. Equat. and Their Equilib. States I (Berkeley, CA, 1986), W.-M. Ni, L.A. Peletier and J. Serrin Eds, MSRI Publ. **12**, Springer Verlag, New York/Tokyo, 1988, pp. 21–34.

[2] G. Barbatis, *Explicit estimates on the fundamental solution of higher-order parabolic equations with measurable coefficients*, J. Differ. Equat., **174** (2001), 442–463.

[3] G. Barbatis, *Sharp heat-kernel estimates for higher-order operators with singular coefficients*, Proc. Edinb. Math. Soc. (2), **47** (2004), 53–67.

[4] M.S. Birman and M.Z. Solomjak, Spectral Theory of Self-Adjoint Operators in Hilbert Space, D. Reidel, Dordrecht/Tokyo, 1987.

[5] L. Brandolese, *Fine properties of self-similar solutions of the Navier–Stokes equations*, Arch. Ration. Mech. Anal., **192** (2009), 375–401.

[6] L. Brandolese and F. Vigneron, *New asymptotic profiles of nonstationary solutions of the Navier–Stokes system*, J. Math. Pures Appl., **88** (2007), 64–86.

[7] P. Deuring and W. von Wahl, *Strong solutions of the Navier-Stokes system in Lipschitz bounded domains*, Math. Nachr., **171** (1995), 111–148.

[8] Yu.V. Egorov, V.A. Galaktionov, V.A. Kondratiev, and S.I. Pohozaev, *Global solutions of higher-order semilinear parabolic equations in the supercritical range*, Adv. Differ. Equat., **9** (2004), 1009–1038.

[9] A. Friedman, Partial Differential Equations, Robert E. Krieger Publ. Comp., Malabar, 1983.

[10] S.D. Eidelman, Parabolic Systems, North-Holland Publ. Comp., Amsterdam/London, 1969.

[11] S.D. Eidelman, Parabolic equations, In: Modern Problems of Math. Fundam. Achiev., Vol. **63** (Part. Differ. Equat.), VINITI AN SSSR: 1990, Moscow, p. 244. English transl.: J. Soviet Math., 1991.

[12] L.C Evans and R.F. Gariepy, *Wieners criterion for the heat equation*, Arch. Ration. Mech. Anal., **78** (1982), 293-314.

[13] M.V. Fedoryuk, *Singularities of the kernels of Fourier integral operators and the asymptotic behaviour of the solution of the mixed problem*, Russian Math. Surveys, **32** (1977), 67–120.

[14] C. Fefferman, *Existence & smoothness of the Navier–Stokes equation*, The Clay Math. Inst., `http://www.esi2.us.es/∼mbilbao/claymath.htm`.

[15] H. Fujita and N. Sauer, *Construction of weak solutions of the Navier–Stokes equation in a non cylindrical domain*, Bull. Amer. Math. Soc., **75** (1969), 465–468.

[16] H. Fujita and N. Sauer, *On existence of weak solutions of the Navier–Stokes equations in regions with moving boundaries*, J. Fac. Sci., Univ. Tokyo, Sect. IA, **17** (1970), 403–420.

[17] V.A. Galaktionov, *On regularity of a boundary point in higher-order parabolic equations: towards Petrovskii-type criterion by blow-up approach*, NoDEA, **16** (2009), 597–655 (arXiv:0901.3986).

[18] V.A. Galaktionov, *On blow-up "twistors" for the Navier–Stokes equations in $\mathbb{R}^3$: a view from reaction-diffusion theory*, arXiv:0901.4286.

[19] V.A. Galaktionov, *Five types of blow-up in a semilinear fourth-order reaction-diffusion equation: an analytic-numerical approach*, Nonlinearity, **22** (2009), 1695–1741 (arXiv:0901.4307).

[20] V. Galaktionov and V. Maz'ya, *Boundary characteristic point regularity for semilinear reaction-diffusion equations: towards an ODE criterion*, J. Math. Sci. (New York, Springer), **175**, No. 3 (2011) (arXiv:1106.4696).

[21] V.A. Galaktionov, E. Mitidieri, and S.I. Pohozaev, *On global solutions and blow-up for Kuramoto–Sivashinsky–type models and well-posed Burnett equations*, Nonl. Anal., **70** (2009), 2930–2952 (arXiv:0902.0257).

[22] V.A. Galaktionov and S.R. Svirshchevskii, Exact Solutions and Invariant Subspaces of Nonlinear Partial Differential Equations in Mechanics and Physics, Chapman & Hall/CRC, Boca Raton, Florida, 2007.

[23] V.A. Galaktionov and J.L. Vazquez, A Stability Technique for Evolution Partial Differential Equations. A Dynamical Systems Approach, Progr. in Nonl. Differ. Equat. and Their Appl., Vol. **56**, Birkhäuser, Boston/Berlin, 2004.

[24] T. Gallay and C.E. Wayne, *Long-time asymptotics of the Navier-Stokes and vorticity equations on* $\mathbb{R}^3$, Phil. Trans. Roy. Soc. London, **360** (2002), 2155–2188.

[25] T. Gallay and C.E. Wayne, *Invariant manifolds and long-time asymptotics of the Navier-Stokes and vorticity equations on* $\mathbb{R}^2$, Arch. Ration. Mech. Anal., **163** (2002), 209–258.

[26] T. Gallay and C.E. Wayne, *Long-time asymptotics of the Navier-Stokes equations in* $\mathbb{R}^2$ *and* $\mathbb{R}^3$, Z. Angew. Math. Mech., **86** (2006), 256–257.

[27] P. Grisvard, Elliptic Problems in Non-Smooth Domains, Pitman, Boston, 1985.

[28] P. Grisvard, Singularities in Boundary Value Problems, PMA, Vol. **22**, Springer-Verlag, New York, 1992.

[29] J.K. Hale, Asymptotic Behavior of Dissipative Systems, AMS, Providence, RI, 1988.

[30] T.Y. Hou and R. Li, *Nonexistence of locally self-similar blow-up for the 3D incompressible Navier-Stokes equations*, Disc. Cont. Dynam. Syst., **18** (2007), 637–642 (full text: arXiv:math/0603126v1).

[31] M. Kohr and C. Pintea, *Stokes-Brinkman transmission problems on Lipschitz and* $C^1$ *domains in Riemannian manifolds*, Comm. Pure Appl. Anal., **9** (2010), 493–537.

[32] A.N. Kolmogorov and S.V. Fomin, Elements of the Theory of Functions and Functional Analysis, Nauka, Moscow, 1976.

[33] V.A. Kondrat'ev, *Asymptotics of solutions of the Navier–Stokes equation near an angular point on the boundary*, J. Appl. Math. Mech., **31** (1967), 125–129.

[34] V. Kozlov and V. Maz'ya, *An asymptotic theory of higher-order operator differential equations with nonsmooth nonlinearities*, J. Funct. Anal., **217** (2004), 448–488.

[35] V. Kozlov and V. Maz'ya, *Asymptotic formula for solutions to elliptic equations near the Lipschitz boundary*, Ann. Mat. Pura Appl. (4), **184** (2005), 185–213.

[36] V.A. Kozlov, V.G. Maz'ya, and J. Rossmann, Elliptic Boundary Value Problems in Domains with Point Singularities, Math. Surveys Monogr., Vol. **52**, Amer. Math. Soc., Providence, RI, 1997.

[37] V.A. Kozlov, V.G. Maz'ya, and J. Rossmann, Spectral Problems with Corner Singularities of Solutions to Elliptic Equations, Math. Surveys Monogr., Vol. **85**, Amer. Math. Soc., Providence, RI, 2001.

[38] J.R. Kweon, *Regularity of solutions for the Navier–Stokes system of incompressible flows on a polygon*, J. Differ. Equat., **235** (2007), 166–198.

[39] O.A. Ladyženskaja, *An initial-boundary value problem for the Navier-Stokes equations in domains with boundary changing in time*, Zap. Naučn. Semin. Leningr. Otdel. Steklov Matem. Inst. (LOMI), **11** (1968), 97–128 (in Russian).

[40] E. Lanconelli, *Sul problema di Dirichlet per l'equazione del calore*, Ann. Mat. pura Appl., **97** (1973), 83–114.

[41] E.M. Landis, *Necessary and sufficient conditions for regularity of a boundary point in the Dirichlet problem for the heat-conduction equation*, Dokl. Akad. Nauk SSSR, **185** (1969), 517-520; Soviet Math., **10** (1969), 380–384.

[42] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, C. R. Acad. Sci. Paris, **196** (1933), 527.

[43] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math., **63** (1934), 193–248.

[44] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Gauthier–Villars, Paris, 1969.

[45] A.J. Majda and A.L. Bertozzi, Vosticity and Incompressible Flow, Cambridge Univ. Press., Cambridge, 2002.

[46] S. Mayboroda and V. Maz'ya, *Boundedness of the gradient of a solution and Wiener test of order one for the biharmonic equation*, Invent. Math., **175** (2009), 287–334.

[47] V. Maz'ya, *On the Wiener type regularity of a boundary point for the polyharmonic operator*, Appl. Anal., **71** (1999), 149–165.

[48] V. Maz'ya, *The Wiener test for higher order elliptic equations*, Duke Math. J., **115** (2002), 479–512.

[49] V. Maz'ya and J. Rossmann, *Mixed boundary value problems for the stationary Navier–Stokes system in polyhedral domains*, Arch. Ration. Mech. Anal., **194** (2009), 669–712.

[50] V. Maz'ya and J. Rossmann, *A maximum modulus estimate for solutions of the Navier–Stokes system in domains of polyhedral type*, Math. Nachr., **282** (2009), 459–469.

[51] V. Maz'ya and J. Rossmann, Elliptic Equations in Polyhedral Domains, Mathematical Surveys and Monographs, Vol. 162. American Mathematical Society, Providence, RI, 2010.

[52] V.G. Maz'ya and A.S. Slutskii, *Asymptotic analysis of the Navier-Stokes system in a plane domain with thin channels*, Asympt. Anal., **23** (2000), 59–89.

[53] V. Maz'ya and A.S. Slutskii, *Asymptotic solution to the Dirichlet problem for a two-dimensional Riccatti's type equation near a corner point*, Asympt. Anal., **39** (2004), 169–185.

[54] V.G. Maz'ya and A.A. Soloviev, Boundary Integral Equations on Contours with Peaks, Operator Theory: Advances and Applications, Vol. 196, Birkhuser Verlag, Basel, 2010.

[55] J.R. Miller, M. O'Leary, and M. Schonbek, *Nonexistence of singular pseudo-self-similar solutions of the Navier–Stokes system*, Math. Ann., **319** (2001), 809–815.

[56] M. Mitrea and S. Monniaux, *The regularity of the Stokes operator and the Fujita–Kato approach to the Navier–Stokes initial value problem in Lipschitz domains*, J. Funct. Anal., **254** (2008), 1522–1574.

[57] M. Mitrea and S. Monniaux, *On the analyticity of the semigroup generated by the Stokes operator with Neumann-type boundary conditions on Lipschitz subdomains of Riemannian manifolds*, Trans. Amer. Math. Soc., **361** (2009), 3125–3157.

[58] J. Nečas, M. Ružička, and V. Šverák, *On Leray's self-similar solutions of the Navier-Stokes equations*, Acta Math., **176** (1996), 283–294.

[59] J. Neustupa, *Existence of a weak solution of the Navier–Stokes equation in a general time-varying domain by the Rothe method*, Math. Meth. Appl. Sci., **32** (2009), 653–683.

[60] J. Petrowsky, *Über die Lösungen der ersten Randwertaufgabe der Wärmeleitungsgleichung*, Učenye Zapiski Moscovsk. Gosud. Univ., No. **2** (1934), 55–59, Moscow, USSR (in German, with Russian summary).

[61] I.G. Petrovsky, *Zur ersten Randwertaufgabe der Wärmeleitungsleichung*, Compositio Math., **1** (1935), 383–419.

[62] P. Rosenau, *Extending hydrodynamics via the regularization of the Chapman-Enskog expansions*, Phys. Lett. A, **40** (1989), 7193–7196.

[63] A.A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov, and A.P. Mikhailov, Blow-up in Quasilinear Parabolic Equations, Walter de Gruyter, Berlin/New York, 1995.

[64] J.O. Sather, The Initial-Bpundary Value Problem for the Navier-Stokes Equations in Regions with Moving Boundaries, Thesis (Ph.D), University of Minnesota, 1963, 76 pp.

[65] C. Sturm, *Mémoire sur une classe d'équations à différences partielles,* J. Math. Pures Appl., **1** (1836), 373–444.

[66] R. Temam, Navier–Stokes Equations, Theory and Numerical Analysis, North-Holland, Amsterdam, 1985.

[67] T.-P. Tsai, *On Leray's self-similar solutions of the Navier–Stokes equations satisfying local energy estimates*, Arch. Ration. Mech. Anal., **143** (1998), 29–51.

[68] V.S. Vladimirov, Equations of Mathematical Physics, Marcel Dekker, Inc., New York, 1971.

[69] H. Watanabe, *The initial-boundary value problems for the heat operator in non-cylindrical domains*, J. Math. Soc. Japan, **49** (1997), 399–430.

[70] N. Wiener, *The Dirichlet problem*, J. Math. and Phys. Mass. Inst. Tech., **3** (1924), 127–146; reprinted in: N. Wiener, Collected Works with Commentaries, Vol. I, ed. P. Masani, Mathematicians of Our Time **10**, MIT Press, Cambridge, Mass., 1976, pp. 394–413.

[71] J. Wolf, *On the boundary regularity of suitable weak solutions to the Navier-Stokes equations*, Ann. Univ. Ferrara, **56** (2010), 97–139.

## Appendix A: Hermitian spectral theory of the linear rescaled operator $\mathbf{B}^*$: point spectrum and solenoidal Hermite polynomials

Thus, approaching the characteristic vertex $(0,0)$ in the blow-up manner (2.1), one observes Hermite's operator $\mathbf{B}^*$ as the principal linear part of the rescaled equation (2.2). Writing it the corresponding divergent form,

$$\mathbf{B}^*\mathbf{v} \equiv \tfrac{1}{\rho^*}\, \nabla \cdot (\rho^* \nabla \mathbf{v}), \tag{A.1}$$

where the weight is $\rho^*(y) = \mathrm{e}^{-\frac{|y|^2}{4}} > 0$, we observe that the actual rescaled evolution is now restricted to the weighted $L^2$-space $L^2_{\rho^*}(\mathbb{R}^3)$, with the exponentially decaying weight $\rho^*(y)$. Here, $\mathbf{B}^*$ is the ("adjoint") Hermite operator with the point spectrum [4, p. 48]

$$\sigma(\mathbf{B}^*) = \big\{\lambda_k = -\tfrac{k}{2}, \quad k = |\beta| = 0,1,2,...\big\} \quad (\beta \text{ is a multiindex in } \mathbb{R}^3), \tag{A.2}$$

where each $\lambda_k$ has the multiplicity $\frac{(k+1)(k+2)}{2}$ for $N = 3$, or the binomial number $C^k_{N+k-1}$. The corresponding complete and closed set of eigenfunctions $\Phi^* = \{\psi^*_\beta(y)\}$ is composed from separable Hermite polynomials. Note another important property of Hermite polynomials:

$$\forall\, \psi^*_\beta, \quad \text{any derivative } D^\gamma \psi^*_\beta \quad \text{is also an eigenfunction with } k = |\beta| - |\gamma| \ge 0. \tag{A.3}$$

Recall that [4]

$$\text{polynomial set } \ \Phi^* \ \text{ is complete and closed in } \ L^2_{\rho^*}(\mathbb{R}^3). \tag{A.4}$$

Further spectral properties are convenient to demonstrate using the linear operator $\mathbf{B}$,

$$\mathbf{B} = \Delta + \tfrac{1}{2}\, y \cdot \nabla + \tfrac{3}{2}\, I \quad \text{in} \quad L^2_\rho(\mathbb{R}^3), \quad \text{where} \quad \rho = \tfrac{1}{\rho^*}, \tag{A.5}$$

which is adjoint to $\mathbf{B}^*$ in the dual $L^2$-metric. It has the same point spectrum and the corresponding eigenfunctions are multiple of the same Hermite polynomials according to the well-known *generating formula*:

$$\psi_\beta(y) = \tfrac{(-1)^{|\beta|}}{\sqrt{\beta!}}\, D^\beta F(y) \equiv \psi^*_\beta(y) F(y), \quad \text{where} \quad F(y) = \tfrac{1}{(4\pi)^{3/2}}\, \mathrm{e}^{-|y|^2/4} \tag{A.6}$$

is the rescaled kernel of the fundamental solutions of $D_t - \Delta$ in $\mathbb{R}^3 \times \mathbb{R}_+$. Then, the bi-orthonormality holds:

$$\langle \psi_\beta^*, \psi_\gamma \rangle = \delta_{\beta\gamma} \quad \text{for any} \quad \beta,\, \gamma. \tag{A.7}$$

Indeed, this dual metric can be also treated as that in $L^2_{\rho^*}(\mathbb{R}^3)$ for the self-adjoint case, but we prefer to keep " $L^2$-dual" notations (for using also in non-symmetric Burnett cases; cf. (1.7)).

Obviously, one needs to consider eigenfunction expansions in the solenoidal restriction

$$\hat{L}^2_{\rho^*}(\mathbb{R}^3) = L^2_{\rho^*}(\mathbb{R}^3)^3 \cap \{\operatorname{div} \mathbf{v} = 0\}. \tag{A.8}$$

Indeed, among the polynomials $\Phi^* = \{\psi_\beta^*\}$ there are many that well-suit the solenoidal fields. Namely, introducing the eigenspaces

$$\Phi_k^* = \operatorname{Span}\{\psi_\beta^*,\ |\beta| = k\}, \quad k \ge 1,$$

in view of (A.3), div plays a role of a "shift operator" in the sense that

$$\operatorname{div} : \Phi_k^{*3} \to \Phi_{k-1}^*. \tag{A.9}$$

We next define the corresponding solenoidal eigenspaces as follows:

$$\mathcal{S}_k^* = \{\mathbf{v}^* = [v_1^*, v_2^*, v_3^*]^T : \quad \operatorname{div} \mathbf{v}^* = 0,\ v_i^* \in \Phi_k^*\}, \quad \text{where} \quad \dim \mathcal{S}_k^* = k(k+2); \tag{A.10}$$

see [24, 25, 26] and further references therein.

Actually, the paper [24] deals with *global* asymptotics of NSEs solutions as $t \to +\infty$, where the adjoint operator $\mathbf{B}$ in (A.5) occurs. Since $\mathbf{B}$ is self-adjoint in $L^2_\rho(\mathbb{R}^3)$, many results from [25, Append. A] can be applied to $\mathbf{B}^*$. For a full collection, see [5, 6] for further asymptotic expansions and self-similar solutions. In particular, this made it possible to construct therein fast decaying solutions of the NSEs on each 1D stable manifolds with the asymptotic behaviour[9]

$$\mathbf{u}_\beta(x,t) \sim t^{\lambda_k - \frac{1}{2}}\, \mathbf{v}_\beta\big(\tfrac{x}{\sqrt{t}}\big) + ... \quad \text{as} \quad t \to \infty, \ \text{ where } \mathbf{v}_\beta = \mathbf{v}_\beta^* F \in \mathcal{S}_k \tag{A.11}$$

are solenoidal eigenfunctions of $\mathbf{B}$. Namely, taking

$$\mathbf{v} = [v_1, v_2, v_3]^T \in \mathcal{S}_k,\ \ v_i \in \Phi_k = \operatorname{Span}\big\{\psi_\beta = \tfrac{(-1)^{|\beta|}}{\sqrt{\beta!}}\, D^\beta F(y),\ |\beta| = k\big\}, \tag{A.12}$$

where $F$ stands for the rescaled Gaussian in (A.6), we have that

$$\operatorname{div} \mathbf{v} = (v_1)_{y_1} + (v_2)_{y_2} + (v_3)_{y_3} = \operatorname{div}(\mathbf{v}^* F) \equiv (\operatorname{div} \mathbf{v}^*)F - \tfrac{1}{2}\, y \cdot \mathbf{v}^*\, F. \tag{A.13}$$

This establishes a one-to-one correspondence between solenoidal eigenfunction classes $\mathcal{S}_k^*$ in (A.10) for $\mathbf{B}^*$ and $\mathcal{S}_k$ in (A.11) for $\mathbf{B}$; see (A.14)–(A.16) below for the first eigenfunctions $\mathbf{v}_\beta = \mathbf{v}_\beta^* F$. Therefore, $\dim \mathcal{S}_k = k(k+2)$, etc.; see details and rather involved proofs of the asymptotics (A.11) for $k = 1$ and 2 in [24].

In particular, those solenoidal Hermite polynomial eigenfunctions of $\mathbf{B}^*$ can be chosen as follows [25, p. 2166-69] (the choice is obviously not unique, normalization constants are omitted):

$$\underline{\lambda_0 = 0:} \quad \mathbf{v}_0^* = [1,1,1]^T = \mathbf{e} \quad \text{(the first solenoidal Hermite polynomial)}, \tag{A.14}$$

[9] We present here only the first term of expansion; as usual in dynamical system theory, other terms in the case of "resonance" can contain $\ln t$-factors (*q.v.* [1] for a typical PDE application); this phenomenon was shown to exist for the NSEs in $\mathbb{R}^2$ [25, p. 236].

$$\text{(A.15)}\qquad \underline{\lambda_1 = -\tfrac{1}{2}:}\quad \mathbf{v}_{11}^* = \begin{bmatrix} 0 \\ -y_3 \\ y_2 \end{bmatrix}, \quad \mathbf{v}_{12}^* = \begin{bmatrix} y_3 \\ 0 \\ -y_1 \end{bmatrix}, \quad \mathbf{v}_{13}^* = \begin{bmatrix} -y_2 \\ y_1 \\ 0 \end{bmatrix} \ (\dim \mathcal{S}_1^* = 3);$$

$$\underline{\lambda_2 = -1:}\ \mathbf{v}_{21}^* = \begin{bmatrix} 4 - y_2^2 - y_3^2 \\ y_1 y_2 \\ -y_1 y_3 \end{bmatrix}, \ \mathbf{v}_{22}^* = \begin{bmatrix} y_1 y_2 \\ 4 - y_1^2 - y_3^2 \\ -y_2 y_3 \end{bmatrix}, \ \mathbf{v}_{23}^* = \begin{bmatrix} y_1 y_3 \\ -y_2 y_3 \\ 4 - y_1^2 - y_2^2 \end{bmatrix},$$

$$\text{(A.16)}\qquad \mathbf{v}_{24}^* = -\begin{bmatrix} 0 \\ -y_1 y_3 \\ y_1 y_2 \end{bmatrix}, \quad \mathbf{v}_{25}^* = -\begin{bmatrix} y_2 y_3 \\ 0 \\ -y_2 y_1 \end{bmatrix},$$

$$\mathbf{v}_{26}^* = \begin{bmatrix} -y_2 y_3 \\ y_2 y_3 \\ y_1^2 - y_2^2 \end{bmatrix}, \ \mathbf{v}_{27}^* = \begin{bmatrix} y_1 y_2 \\ y_3^2 - y_1^2 \\ -y_2 y_3 \end{bmatrix}, \ \mathbf{v}_{28}^* = \begin{bmatrix} y_2^2 - y_3^2 \\ -y_1 y_2 \\ y_1 y_3 \end{bmatrix} \ \ (\dim \mathcal{S}_2^* = 8), \quad \text{etc.}$$

We need the following final conclusion. By (A.4), the set of vectors $\Phi^{*3}$ is complete and closed in[10] $L^2_{\rho^*}(\mathbb{R}^3)^3$, so that

$$\text{(A.17)}\qquad \forall\, \mathbf{v} \in L^2_{\rho^*}(\mathbb{R}^3)^3 \quad \Longrightarrow \quad \mathbf{v} = \textstyle\sum_{(\beta)} \mathbf{c}_\beta \mathbf{v}_\beta^*, \quad \mathbf{v}_\beta^* \in \Phi_k^{*3}, \ \ k = |\beta| \ge 0,$$

where we use the vector notation

$$\text{(A.18)}\qquad \mathbf{c}_\beta(\tau) = [c_\beta^1(\tau), c_\beta^2(\tau), c_\beta^3(\tau)]^T \quad \Longrightarrow \quad \mathbf{c}_\beta \mathbf{v}_\beta^* \equiv [c_\beta^1(\tau) v_{\beta 1}^*, c_\beta^2(\tau) v_{\beta 2}^*, c_\beta^3(\tau) v_{\beta 3}^*]^T.$$

It then follows from (A.7)–(A.9) that

$$\text{(A.19)}\qquad \text{polynomial set } \ \hat{\Phi}^* = \Phi^{*3} \cap \{\operatorname{div} \mathbf{v} = 0\} \ \text{ is complete and closed in } \ \hat{L}^2_{\rho^*}(\mathbb{R}^3).$$

In what follows, we always assume that we deal with "solenoidal" asymptotics involving eigenfunctions as in (A.10).

## Appendix B: Vertex regularity for Burnett equations

B.1. **Burnett equations in a hierarchy of hydrodynamic models.** The Burnett equations (1.7) appear as the *second approximation* (the NSEs (1.1) being the *first one*) of the corresponding kinetic equations on the basis of Grad's method in Chapman–Enskog expansions for hydrodynamics. Namely, Grad's method applied to kinetic equations, by expanding the kernel of the integral operators involved in terms of those with pointwise supports, yields, in addition to the classic operators of the Euler equations, other viscosity parts as follows:

$$D_t \mathbf{u} \equiv \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \sum_{n=0}^{\infty} \varepsilon^{2n+1} \Delta^n (\mu_n \Delta \mathbf{u}) + \ldots = \varepsilon\big(\mu_0 \Delta \mathbf{u} + \varepsilon^2 \mu_1 \Delta^2 \mathbf{u} + \ldots\big) + \ldots,$$

where $\varepsilon > 0$ is essentially the *Knudsen number* Kn; see details in Rosenau's regularization approach, [62]. In a full model, truncating such series at $n = 0$ leads to the Navier–Stokes equations (1.1) (with $\mu_0 > 0$), while $n = 1$ is associated with the Burnett equations (1.7). Note that Burnett-type equations, with a small parameter appeared as higher-order viscosity

[10]Note a standard result of functional analysis: polynomials are complete in any weighted $L^p$-space with an exponentially decaying weight; see the analyticity argument in Kolmogorov–Fomin [32, p. 431].

approximations of the Navier–Stokes equations, is an effective tool for proving existence of their weak ("turbulent" in Leray's sense) solutions; see Lions' monograph [44, § 6, Ch. 1]. Note that the "Problem on blow-up/non-blow-up for Burnett equations (1.7) at an interior point" starts from dimensions $N = 7$: for $N \le 6$, there exists a unique global smooth $L^2$-solution, [21, § 6].

The final finite-dimensional dynamical system for first Fourier coefficients of solutions to (1.7) is derived similarly, but the regularity conclusions are shown to be more difficult and even rather obscure. The necessary spectral properties and vector solenoidal generalized Hermite polynomials as eigenfunctions of the rescaled non-self adjoint operator $\mathbf{B}^*$ are introduced below in Appendix C.

More briefly, we now list below main steps of the regularity analysis for (1.7).

B.2. **First blow-up scaling: an exponentially perturbed parabolic equation.** The first *blow-up* scaling in (1.7) is now

$$\mathbf{u}(x,t) = \mathbf{v}(y,\tau), \quad y = \tfrac{x}{(-t)^{1/4}}, \quad \tau = -\ln(-t) : (-1,0) \to \mathbb{R}_+, \tag{B.1}$$

which yields the following rescaled equation:

$$\mathbf{v}_\tau = -\mathrm{e}^{-\frac{3\tau}{4}} \nabla p + \mathbf{B}^*\mathbf{v} - \mathrm{e}^{-\frac{3\tau}{4}} (\mathbf{v}\cdot\nabla)\mathbf{v} \quad \text{in} \quad \hat{Q}_0, \quad \text{where} \quad \mathbf{B}^* = -\Delta^2 - \tfrac{1}{4}\, y \cdot \nabla \tag{B.2}$$

is the adjoint operator (C.1) for $m = 2$ with good spectral properties given in Appendix C.

B.3. **Backward paraboloid.** Here, $\partial Q_0$ is defined as follows:

$$S(t) = \partial Q_0 \cap \{t\} : \quad q_0(x) \equiv \big(\textstyle\sum_{i=1}^N a_i x_i^4\big)^{\frac{1}{4}} = (-t)^{\frac{1}{4}}\, \varphi(\tau) \quad (\textstyle\sum a_i^4 = 1), \tag{B.3}$$

with the same slow growing functions $\varphi(\tau)$ as in (2.3). The rescaled equation (B.2) is then set in an expanding domain,

$$\hat{S}(t) = \partial \hat{Q}_0 \cap \{\tau\} : \quad q_0(y) \equiv \big(\textstyle\sum_{i=1}^N a_i y_i^4\big)^{\frac{1}{4}} = \varphi(\tau) \to +\infty \quad \text{as} \quad \tau \to +\infty. \tag{B.4}$$

B.4. **The Cauchy problem setting.** Extending to the Cauchy problem by using the variables (2.12), we use Green's second formula: for any $\chi \in C_0^\infty$,

$$\textstyle\int_{\{q_0(y) \le \varphi(\tau)\}} (\mathbf{v}\Delta^2\chi - \chi\Delta^2 v)\mathrm{d}y = \int_{\hat{S}(t)} \big(\Delta\chi \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - \mathbf{v}\frac{\partial \Delta\chi}{\partial \mathbf{n}} - \Delta\mathbf{v}\frac{\partial \chi}{\partial \mathbf{n}} + \chi\frac{\partial \Delta\mathbf{v}}{\partial \mathbf{n}}\big)\,\mathrm{d}s. \tag{B.5}$$

In view of the Dirichlet boundary conditions in (1.7), in the sense of distributions, (B.5) reads:

$$\Delta^2\hat{\mathbf{v}} = \Delta^2\mathbf{v}H + \big(\tfrac{\partial}{\partial \mathbf{n}}\Delta\mathbf{v}\big)\,\delta_{\hat{S}(\tau)} + \tfrac{\partial}{\partial \mathbf{n}}(\Delta\mathbf{v}\,\delta_{\hat{S}(\tau)}), \tag{B.6}$$

where densities of a single- and a double-layer potential now depend on the Laplacian $\Delta\mathbf{v}$ instead of $\mathbf{v}$ in (2.12). Using the same harmonic pressure extension, we obtain

$$\hat{\mathbf{v}}_\tau = \mathbf{B}^*\hat{\mathbf{v}} - \mathrm{e}^{-\frac{3\tau}{4}}\,\mathbb{P}\,(\hat{\mathbf{v}}\cdot\nabla)\hat{\mathbf{v}} - \mathbb{P}\,(\tfrac{\partial}{\partial \mathbf{n}}\Delta\mathbf{v})\,\delta_{\hat{S}(\tau)} - \mathbb{P}\,\tfrac{\partial}{\partial \mathbf{n}}(\Delta\mathbf{v}\,\delta_{\hat{S}(\tau)}). \tag{B.7}$$

As for the NSEs with $m = 1$, this problem is always locally well-posed, and is guaranteed to be globally well posed either for $N \le 6$ or for any sufficiently smooth initial data for $N \ge 7$.

B.5. **Boundary layer and a perturbed rescaled equation.** Close to the lateral boundary of $Q_0$, the next rescaled variables are

$$z = \frac{y}{\varphi(\tau)} \quad \text{and} \quad \hat{\mathbf{v}}(y,\tau) = \mathbf{w}(z,\tau). \tag{B.8}$$

This makes the corresponding rescaled paraboloid (B.4) fixed:

$$\tilde{S}: \quad \left(\sum_{i=1}^N a_i z_i^4\right)^{\frac{1}{4}} = 1. \tag{B.9}$$

The rescaled vector field $\mathbf{w}$ satisfies a perturbed equation:

$$\begin{gathered} \mathbf{w}_\tau = -\tfrac{1}{\varphi^4}\,\Delta_z^2\mathbf{w} - \tfrac{1}{4}\,z\cdot\nabla_z\mathbf{w} + \tfrac{\varphi'}{\varphi}\,z\cdot\nabla_z\mathbf{w} - \tfrac{1}{\varphi}\,\mathrm{e}^{-\frac{3\tau}{4}}\,\mathbb{P}(\mathbf{w}\cdot\nabla_z)\mathbf{w} \\ -\varphi^{N-4}\,\mathbb{P}\left(\tfrac{\partial}{\partial\mathbf{n}}\Delta\mathbf{v}\right)\delta_{\tilde{S}} - \varphi^{N-4}\,\mathbb{P}\,\tfrac{\partial}{\partial\mathbf{n}}(\Delta\mathbf{v}\,\delta_{\tilde{S}}). \end{gathered} \tag{B.10}$$

The BL-variables for a fixed point $z_0 \in \tilde{S}$ on the boundary (3.2) are

$$\xi = \varphi^{\frac{4}{3}}(\tau)(z_0 - z), \quad \varphi^{\frac{4}{3}}(\tau)\mathrm{d}\tau = \mathrm{d}s, \quad \text{and} \quad \mathbf{w}(z,\tau) = \rho(s)\mathbf{g}(\xi,s). \tag{B.11}$$

In this boundary layer, we are looking for a generic pattern of the behaviour described by (B.10) on compact subsets near the lateral boundary, satisfying

$$|\xi| = o\big(\varphi^{-\frac{4}{3}}(\tau)\big) \to 0 \quad \Longrightarrow \quad |z - z_0| = o\big(\varphi^{-\frac{8}{3}}(\tau)\big) \to 0 \quad \text{as} \quad \tau \to +\infty. \tag{B.12}$$

Substituting (B.11) into the PDE (B.10) yields

$$\begin{gathered} \mathbf{g}_s = \mathbf{A}\mathbf{g} - \tfrac{1}{4}\,\tfrac{1}{\varphi^{4/3}}\,\xi\cdot\nabla_\xi\mathbf{g} - \tfrac{\varphi'_\tau}{\varphi}\left(z_0 - \tfrac{\xi}{\varphi^{4/3}}\right)\cdot\nabla_\xi\mathbf{g} \\ -\tfrac{4}{3}\,\tfrac{\varphi'_\tau}{\varphi^{1/3}}\,\xi\cdot\nabla_\xi\mathbf{g} - \tfrac{\rho'_s}{\rho}\,\mathbf{g} + \tfrac{\rho}{\varphi}\,\mathrm{e}^{-\frac{3\tau}{4}}\,\mathbb{P}(\mathbf{g}\cdot\nabla_\xi)\mathbf{g} - \tfrac{1}{\rho}\,\varphi^{N-4-\frac{4}{3}}\,\mathbb{P}\left(\tfrac{\partial}{\partial\mathbf{n}}\Delta\mathbf{v}\right)\delta_{\tilde{S}} \\ -\tfrac{1}{\rho}\,\varphi^{N-4-\frac{4}{3}}\,\mathbb{P}\,\tfrac{\partial}{\partial\mathbf{n}}(\Delta\mathbf{v}\,\delta_{\tilde{S}}), \quad \text{where} \quad \mathbf{A}\mathbf{g} = -\Delta^2\mathbf{g} + \tfrac{1}{4}\,z_0\cdot\nabla_\xi\mathbf{g}. \end{gathered} \tag{B.13}$$

Again, in (B.13), we observe a perturbed linear uniformly parabolic equation. As usual, here one needs to check that all the perturbation terms are asymptotically small as $s$ (or $\tau$) $\to +\infty$, relative to the stationary autonomous operator $\mathbf{A}$. This is done similarly to $m = 1$ above. Overall, the BL representation (B.11) and (3.5) imply that (3.8) holds. Moreover, asymptotically, the limit problem becomes one-dimensional, depending on the space variable (3.9).

We again pose the same asymptotic behaviour (3.8) at infinity. According to the scaling (B.11), let us fix a uniformly bounded rescaled orbit $\{\mathbf{g}(s),\ s > 0\}$[11]. Then, by parabolic theory [9, 10], we can again pass to the limit in (B.13) along a subsequence $\{s_k\} \to +\infty$, removing small perturbations. Therefore, uniformly on compact subsets defined in (B.12), as $k \to \infty$,

$$\mathbf{g}(s_k + s) \to \mathbf{h}(s), \quad \text{where} \quad \mathbf{h}_s = \mathbf{A}\mathbf{h}, \quad \mathbf{h} = \tfrac{\partial\mathbf{h}}{\partial\mathbf{n}} = 0 \text{ at } \xi = 0, \quad h^j|_{\xi=+\infty} = 1. \tag{B.14}$$

The limit equation obtained from (B.13),

$$\mathbf{h}_s = \mathbf{A}\mathbf{h} \equiv -\mathbf{h}_{\eta\eta\eta\eta} + \tfrac{1}{4}\,h_\eta \tag{B.15}$$

is again a standard linear parabolic PDE in $\mathbb{R}^N \times \mathbb{R}_+$, with a non self-adjoint operator $\mathbf{A}$, so (B.15) is not a gradient system in $L^2$. We then need to show that, in an appropriate weighted

[11]As usual, the scaled function $\rho(s)$ remains unknown and to be determined by matching with the inner region behaviour.

$L^2$-space if necessary and under the hypothesis (3.8), the stabilization holds, i.e., the $\omega$-limit set of the orbit $\{\mathbf{h}(s)\}_{s>0}$ consists of a single equilibrium: as $s \to +\infty$,

$$\text{(B.16)} \qquad \begin{cases} \mathbf{h}(\xi, s) \to \mathbf{g}_0(\xi), \quad \text{where} \quad \mathbf{A}\mathbf{g}_0 = 0 \ \text{ for } \ \eta \in \mathbb{R}, \\ \mathbf{g}_0 = \mathbf{g}_0' = 0 \quad \text{for} \quad \eta = 0, \quad g_0^j(+\infty) = 1. \end{cases}$$

This gives the unique solution of (B.16) (see [17, § 7] and [20, § 5]): for, e.g., $z_0 = [1, 1, 1]^T$,

$$\text{(B.17)} \qquad g_0^j(\xi) = 1 - \mathrm{e}^{-\frac{\eta}{2^{5/3}}} \left[ \cos\left(\tfrac{\sqrt{3}\,\eta}{2^{5/3}}\right) + \tfrac{1}{\sqrt{3}} \sin\left(\tfrac{\sqrt{3}\,\eta}{2^{5/3}}\right) \right], \quad j = 1, 2, \ldots, N.$$

It turns out that the limit problem (B.15) possesses a number of strong gradient and contractivity properties. Namely setting by linearization

$$\text{(B.18)} \qquad \mathbf{h}(s) = \mathbf{g}_0 + \mathbf{w}(s) \quad \Longrightarrow \quad \mathbf{w}_s = \mathbf{A}\mathbf{w} \equiv -\mathbf{w}_{\eta\eta\eta\eta} + \tfrac{1}{4}\,\mathbf{w}_\eta, \quad \mathbf{w} = \mathbf{w}_\eta = 0 \ \text{ at } \ \eta = 0,$$

we arrive at the following (cf. Proposition 3.1 for $m = 1$):

**Proposition B.1.** (i) (B.18) *is a gradient system in $L^2$, and*

(ii) *In the given class of solutions, the $\omega$-limit set $\Omega_0$ of* (B.18) *consists of the origin only and is uniformly stable.*

*Proof.* (i) One can see that (B.18) admits a monotone Lyapunov function obtained by multiplying by $\mathbf{w}_{\eta\eta}$ in $L^2$:

$$\text{(B.19)} \qquad \tfrac{1}{2}\,\tfrac{\mathrm{d}}{\mathrm{d}s} \int ((w^j)_\eta)^2 = -\int ((w^j)_{\eta\eta\eta})^2 \le 0.$$

Hence, (ii) also follows. $\square$

Thus, quite similar to the second-order case, under given assumptions, we can pass to the limit $s \to +\infty$ along any sequence in the perturbed gradient system (B.13). Then, again similarly to $m = 1$, the uniform stability of the stationary point $g_0$ in the limit autonomous system (B.15) in a suitable metric guarantees that the asymptotically small perturbations do not affect the omega-limit set; see [23, Ch. 1]. However, at this moment, we cannot avoid the following convention, which for $m = 2$ is much more key than for $m = 1$. Actually, the convergence (B.14) and (B.16) for the perturbed dynamical system (B.13) should be considered as the main Hypothesis, which characterizes the class of generic patterns under consideration, and then the normalization (3.8) is its partial consequence. For bi-harmonic flows, a more clear characterization of this class of generic patterns is very difficult. It seems that a correct language of doing this (in fact, for both cases $m = 1$ and $m \ge 2$) is to reinforce the corresponding centre subspace behaviour as in (4.7).

Finally, we summarize these conclusions as follows:

**Proposition B.2.** *Under the given hypothesis and conditions, the problem* (B.13) *admits a family of solutions* (*called generic*) *satisfying* (B.16).

Such a definition of generic patterns looks rather non-constructive, which is unavoidable for such higher-order nonlocal PDEs. However, (B.16) is expected to occur for “almost all” solutions.

Thus, we stop further discussions concerning the passage to the limit $s \to +\infty$ in (B.13), which, as we have shown, under the given hypotheses on the asymptotic smallness of perturbations available, reduces to a linear stability analysis of the nontrivial equilibrium $\mathbf{g}_0$ of the linear rescaled operator $\mathbf{A}$ in (B.13). This has been resolved for a class of generic solutions. More generally, we have to deal with solutions of (B.13) from the *stable subset* $\mathcal{W}_0$ of $\mathbf{g}_0$ within the prescribed perturbed equations (B.13). A clear, constructive, and full identification of $\mathcal{W}_0$ is not possible for such higher-order nonlocal perturbed parabolic equations.

We summarize the conclusions as follows: *in what follows, under the given hypothesis and conditions, we will deal with a family of solutions $\mathcal{W}_0$ of* (B.13) (*called generic*), *for which* (B.16) *holds.*

B.6. **Inner region analysis.** In the Inner Region, the original rescaled problem (B.7) occurs. For any extended solution orbit (2.11) uniformly bounded in $L^2_{\rho^*}(\mathbb{R})$, we use the eigenfunction expansion (4.3) via the generalized solenoidal Hermite polynomials (C.21). Substituting (4.3) into (B.7) and using the orthonormality property (C.20) yield a dynamical system: for any multiindex $\beta$, with $|\beta| \ge 0$,

$$\Big\{ \dot{\mathbf{c}}_\beta = \lambda_\beta \mathbf{c}_\beta - \langle \mathbb{P}\left(\tfrac{\partial}{\partial \mathbf{n}} \Delta \mathbf{v}\right) \delta_{\hat S(\tau)}, \mathbf{v}_\beta \rangle - \langle \mathbb{P}\, \tfrac{\partial}{\partial \mathbf{n}} (\Delta \mathbf{v}\, \delta_{\hat S(\tau)}), \mathbf{v}_\beta \rangle - \mathrm{e}^{-\frac{3\tau}{4}} \langle \mathbb{P}(\hat{\mathbf{v}} \cdot \nabla) \hat{\mathbf{v}}, \mathbf{v}_\beta \rangle, \tag{B.20}$$

where $\lambda_\beta = -\frac{|\beta|}{4}$ by (C.14), so that $\lambda_\beta < 0$ for any $|\beta| \ge 1$. As for $m = 1$, one then needs to concentrate on the first Fourier generic patterns associated with the centre subspace for $\mathbf{B}^*$ (cf. (C.23) for $N = 3$)

$$k = 0: \quad \lambda_0 = 0 \quad \text{and} \quad \mathbf{v}_0^*(y) = \mathbf{e} = [1, 1, ..., 1]^T, \quad \mathbf{v}_0(y) = F(y)\mathbf{e}. \tag{B.21}$$

This reflects another characterization of our class of generic patterns. The equation for $\mathbf{c}_0(\tau)$ then takes the form:

$$\dot{\mathbf{c}}_0 = - \int\limits_{\hat S(\tau)} \left(\tfrac{\partial}{\partial \mathbf{n}} \Delta \mathbf{v}\right) \mathbf{v}_0 \,\mathrm{d}s - \int\limits_{\hat S(\tau)} \Delta \mathbf{v}\, \tfrac{\partial}{\partial \mathbf{n}} \mathbf{v}_0 \,\mathrm{d}s - \mathrm{e}^{-\frac{3\tau}{4}} \int\limits_{\mathbb{R}^N} \mathbb{P}(\hat{\mathbf{v}} \cdot \nabla) \hat{\mathbf{v}}\, \mathbf{v}_0 \,\mathrm{d}y. \tag{B.22}$$

Note that first two terms on the right-hand side in (B.22) have a pure solenoidal parabolic (bi-harmonic) nature, while the only Navier–Stokes influence is presented by the last nonlinear term with an exponentially decaying factor.

Using next the boundary behaviour (B.16) with the 1D profile (B.17) for $\tau \gg 1$: in the rescaled sense, on the given compact subsets, (3.17) holds, with $\mathbf{g}_0$ given by (B.17), where $\eta$ stands for the rescaled distance:

$$\eta = \varphi^{\frac{1}{3}}(\tau)\, \mathrm{dist}\{y, \hat S(\tau)\}. \tag{B.23}$$

Similar to (3.18), such a BL-asymptotics is "almost" solenoidal for $\tau \gg 1$, i.e., perturbed by an exponentially small factor at any distance $\delta_0 > 0$ from the boundary. By the matching of both Regions for such generic patterns, (4.8) has to remain valid.

Performing, similar to (4.10)–(4.19), proper estimating of all the three terms on the right-hand side of (B.22) (the last, the Navier–Stokes one can be again estimated rather roughly) yields the following dynamical system for the first Fourier coefficients:

$$\Big\{ \dot{\mathbf{c}}_0 = -\gamma_{11} \mathbf{c}_0 \varphi^N \mathbf{v}_0(\varphi) - \gamma_{12} \mathbf{c}_0 \varphi^{N - \frac{2}{3}} \mathbf{v}_0'(\varphi) - \gamma_2\, \mathrm{e}^{-\frac{3\tau}{4}} (\mathbf{c}_0 \cdot \mathbf{e}) \mathbf{c}_0\, \varphi^{N+3}\, \mathbf{v}_0(\varphi) + ... , \tag{B.24}$$

where, again, the first two terms are purely "parabolic". Similar to (4.20), we replace surface integrals by some "average" values, actually assuming the radial dependence on $|y|$ with the rescaled surface $\hat{S}(\tau)$ : $\{|y| = \varphi(\tau)\}$, as in (4.13). As above, we do not guarantee that the multiplier $\varphi^{N+3}$ in the last term in (B.24) is any optimal one (since the whole term will be shown to be negligible anyway).

Finally, using the expansion of the rescaled kernel given in (C.11) and keeping the leading term only, the asymptotic dynamical system reads

$$
\text{(B.25)} \quad \begin{cases} \dot{\mathbf{c}}_0 = \frac{4d_0}{3}\gamma_1 \mathbf{c}_0 \varphi^{N-\delta_0}(\tau)\, \mathrm{e}^{-d_0\varphi^{4/3}(\tau)}[C_1 \sin(b_0\varphi^{\frac{4}{3}}(\tau)) + C_2 \cos(b_0 \varphi^{\frac{4}{3}}(\tau))] \\ \qquad -\gamma_2\, \mathrm{e}^{-\frac{3\tau}{4}}\, (\mathbf{c}_0 \cdot \mathbf{e})\mathbf{c}_0\, \varphi^{N+3-\delta_0}(\tau)\, \mathrm{e}^{-d_0\varphi^{4/3}(\tau)}[C_1 \sin(b_0\varphi^{\frac{4}{3}}(\tau)) + C_2 \cos(b_0\varphi^{\frac{4}{3}}(\tau))], \end{cases}
$$

where, as usual, $\gamma_{1,2} \in \mathbb{R}^N$ are some constant vectors.

We then arrive at a typical and simple **ODE regularity criterion**: *under the given hypotheses and conventions of our asymptotic analysis, the vertex* $(0, 0)$ *is regular in the class of generic solutions iff any solution of the non-autonomous* $3D$ *dynamical system* (B.25) *vanishes as* $\tau \to +\infty$, i.e., 0 is globally asymptotically stable for (B.25).

B.7. **Two regularity conclusions. 1.** The first regularity conclusion is straightforward: in the absence of the convection term, i.e., for the *linear fourth-order Stokes problem*,

$$
\text{(B.26)} \qquad \mathbf{u}_t = -\nabla p - \Delta^2 \mathbf{u}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad Q_0,
$$

with the Dirichlet boundary conditions as in (1.7), the vertex $(0, 0)$ is regular provided that the *following integral diverges to* $-\infty$ (cf. (2.9)):

$$
\text{(B.27)} \qquad \boxed{\tfrac{4d_0}{3}\gamma_1 \int^{+\infty} \varphi^{N+\frac{1}{3}-\delta_0}(s)\, \mathrm{e}^{-d_0\varphi^{4/3}(s)}[C_1 \sin(b_0\varphi^{\frac{4}{3}}(s)) + C_2 \cos(b_0\varphi^{\frac{4}{3}}(s))]\, \mathrm{d}s = -\infty.}
$$

Since the function under the integrals is strongly oscillatory and, in general, is of changing sign for $s \gg 1$, (B.27) may require a special procedure of "oscillatory cut-off" of the given $\varphi(\tau)$ to delete a possible positive part of the divergent integral; see [17, § 7]. In other words, to get the regularity conclusion (B.27), the behaviour of $\varphi(\tau)$ as $\tau \to +\infty$ must be very carefully adjusted with the nonmonotone and oscillatory behaviour of the rescaled kernel $F(\varphi(\tau))$ of the fundamental solution (C.4) of the bi-harmonic operator.

If (B.27) fails, then the vertex is not regular. One can see that the transition from a possible regularity (after an oscillatory cut-off) to the guaranteed irregularity occurs at the following "critical" paraboloid with

$$
\text{(B.28)} \qquad d_0\varphi_*^{\frac{4}{3}}(\tau) = \ln \tau \quad \Longrightarrow \quad \varphi_*(\tau) = d_0^{-\frac{3}{4}}(\ln \tau)^{\frac{3}{4}} \equiv 2^{\frac{11}{4}}\, 3^{-\frac{3}{4}}\, (\ln \tau)^{\frac{3}{4}} \quad \text{as} \quad \tau \to +\infty.
$$

This is the fourth-order analogy of Petrovskii's function (2.8), so that the constant therein,

$$
\text{(B.29)} \qquad C_* = 2^{\frac{11}{4}}\, 3^{-\frac{3}{4}}
$$

is optimal (similar to the "2" in (2.8)): replacing it by any larger one $C_* + \varepsilon$, with an $\varepsilon > 0$, guarantees convergence in (B.27) and hence the irregularity of the vertex $(0, 0)$. Indeed, for such $\varphi(\tau)$, the integral in (B.27) simply *converges*, i.e., on almost all such centre subspace orbits,

$$
\text{(B.30)} \qquad \mathbf{c}_0(\tau) \not\to 0 \quad \text{as} \quad \tau \to +\infty.
$$

**2.** Concerning the full nonlinear Burnett problem (1.7), we arrive at the same conclusion as for the NSEs. Namely, it follows by balancing two terms on the right-hand side of (B.25), that the last nonlinear one may be leading provided that

$$|\mathbf{c}_0(\tau)| \gg \mathrm{e}^{\frac{3\tau}{4}} \, \tfrac{1}{\varphi^3(\tau)} \to +\infty \quad \text{as} \quad \tau \to +\infty, \tag{B.31}$$

which never happens in the case of the "linear" regularity described by (B.27). Hence, the *convective term cannot change the vertex regularity*, thus leading to the same regularity criterion.

B.8. **Nonexistence of similarity blow-up for Burnett equations: Type II singularities are also needed.** For (1.7), Leray's-type blow-up scaling (5.2) takes the form

$$\mathbf{u}(x,t) = (T-t)^{-\frac{3}{4}} \, \mathbf{v}(y,\tau), \quad y = \tfrac{x}{(T-t)^{1/4}}, \quad \tau = -\ln(T-t), \tag{B.32}$$

where $\mathbf{v}(y,\tau)$ solves the rescaled equation

$$\mathbf{v}_\tau = \mathbf{B}^* \mathbf{v} - \tfrac{3}{4}\, \mathbf{v} - \mathbb{P}\, (\mathbf{v} \cdot \nabla) \mathbf{v} \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+. \tag{B.33}$$

Here $\mathbf{B}^*$ is the linear rescaled operator (C.1) with the known point spectrum and eigenfunctions being generalized Hermite polynomials.

As for the NSEs (1.1) (see Section 5), the first question is whether a nontrivial Type I self-similar blow-up exists, i.e., whether a nontrivial stationary solution $\mathbf{v} = \mathbf{v}(y)$ of (B.33) exists:

$$-\Delta^2 \mathbf{v} - \tfrac{1}{4}\, y \cdot \nabla \mathbf{v} - \tfrac{3}{4}\, \mathbf{v} - \mathbb{P}\, (\mathbf{v} \cdot \nabla) \mathbf{v} = 0 \quad \text{in} \quad \mathbb{R}^N, \quad \mathbf{v} \in L^2(\mathbb{R}^N). \tag{B.34}$$

It is curious that a negative answer (i.e., similar for the NSEs in Section 5) can be obtained rather convincingly just by a local asymptotic analysis of the elliptic equation (B.34). As happens in practically all blow-up problems for reaction-diffusion and other nonlinear PDEs (see examples in, e.g., [19, 23, 63]), a "generic" behaviour of its solutions as $z = |y| \to +\infty$ is governed by the leading lower-order linear terms, i.e., in the radial representation, this means that, for $z \gg 1$,

$$-\tfrac{1}{4}\, z \mathbf{v}_z' - \tfrac{3}{4}\, \mathbf{v} + ... = 0 \quad \Longrightarrow \quad \mathbf{v}(z) \sim \tfrac{\mathbf{C}}{z^3} \quad \text{as} \quad z \to +\infty, \tag{B.35}$$

where $\mathbf{C} \in \mathbb{R}^3$ is a constant vector. Of course, (B.35) is just a rough radial estimate, so an extra "angular separation" is necessary to produce all asymptotics like that at infinity. However, (B.35) is sufficient for a key negative conclusion: via the local behaviour (B.35), for any $\mathbf{C} \ne 0$,

$$\tfrac{\mathbf{C}}{|y|^3} \in L^2(\{|y| > 1\}) \quad \text{iff} \quad N < 6. \tag{B.36}$$

In other words, in the "blow-up case"[12] $N \ge 7$, blow-up cannot be of a self-similar (Type I) form (B.32) with a nontrivial asymptotics (B.35).

Surely, this is not a proof of such a nonexistence, since the special single case $\mathbf{C} = 0$ in (B.35) has not been ruled out. Indeed, formally, it can happen that, for $\mathbf{C} = 0$, the similarity profile $\mathbf{v}(y)$ solving (B.34) may reach an exponential decay at infinity (on derivation, see Appendix C)

$$\mathbf{v}(y) \sim \mathbf{C_1}\, \tfrac{1}{|y|}\, \mathrm{e}^{-a_0 |y|^{4/3}}, \quad \text{where} \quad a_0 = 3 \cdot 2^{-\frac{8}{3}} \quad \text{and} \quad \mathbf{C_1} \in \mathbb{R}^3. \tag{B.37}$$

[12]For $N \le 6$, solutions of (1.7) do not blow-up in $L^\infty(\mathbb{R}^N)$, [21].

**Example: a diversion to blow-up in a related semilinear bi-harmonic flow.** As is known from similarity blow-up in semilinear bi-harmonic equations such as

$$u_t = -\Delta^2 u + |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \quad \text{where} \quad p > 1, \tag{B.38}$$

such a behaviour is highly unlikely. To explain this, consider its self-similar blowing-up solutions

$$u(x,t) = (T-t)^{-\frac{1}{p-1}}\, v(y),\ y = \tfrac{x}{(T-t)^{1/4}} \implies -\Delta^2 v - \tfrac{1}{4}\, y \cdot \nabla v - \tfrac{1}{p-1}\, v + |v|^{p-1}v = 0. \tag{B.39}$$

Checking the asymptotic behaviour as $y \to +\infty$, we again obtain the generic algebraic decay similar to (B.35), which is governed by two linear terms: as $z = |y| \to +\infty$,

$$-\tfrac{1}{4}\, z v' - \tfrac{1}{p-1}\, v + ... = 0 \quad \Longrightarrow \quad v(y) \sim C\,|y|^{-\frac{4}{p-1}}. \tag{B.40}$$

Similarly, for $C = 0$, the behaviour gets exponentially decaying (cf. (B.37)):

$$v(y) \sim C_1 |y|^{\delta} \mathrm{e}^{-a_0 |y|^{4/3}}, \quad \text{where} \quad \delta = -\tfrac{2}{3}\left(N - \tfrac{2}{p-1}\right), \tag{B.41}$$

with the same $a_0$ as in (B.37). See [19, § 2.3] and Appendix C for a derivation of such two-scale WKBJ-asymptotics (B.41) and (B.37). Therefore, the asymptotic bundle of exponentially decaying solutions (B.41) of the ODE in (B.39) contains a unique parameter $C_1 \in \mathbb{R}$, i.e., it is **one**-dimensional. Thus, this is not enough to "shoot" *two* symmetry conditions at the origin:

$$v'(0) = v'''(0) = 0, \tag{B.42}$$

so an extra parameter should be at hand, and this is $p$. As shown in [19, § 2] by a careful numerical analysis of the ODE in (B.39) for $N = 1$, there exists a *unique value* of the exponent

$$p = p_\delta = 1.40... , \tag{B.43}$$

for which (B.39) admits a solution with the exponential decay (B.41), with some $C_1 \neq 0$. Then, in the sense of bounded measures in $\mathbb{R}$, for $p = p_\delta$,

$$|u(x,t)|^{\frac{p-1}{4}} \to D\delta(x) \quad \text{as} \quad t \to T^-, \quad \text{with the constant } D = \textstyle\int |v(y)|^{\frac{p-1}{4}}\, \mathrm{d}y > 0. \tag{B.44}$$

**Back to blow-up in Burnett equations.** We expect that a similar phenomenon **does not** exist for similarity blow-up in the Burnett equations, i.e., exponentially decaying similarity profiles (B.37) do not exist. Recall that, unlike (B.38), the equations (1.7) and (B.34) **do not** contain any free parameter (like $p$ in (B.38)), which could allow to get such a solution at least for some its values. Of course, (B.34) is a system of three solenoidal fourth-order semilinear elliptic equations, and a definite negative nonexistence conclusion is very difficult to justify rigorously[13].

Overall, we arrive at the following plausible situation: *similar to the NSEs* (1.1) *in dimensions* $N \geq 3$ (*see discussion in Section 5*), *blow-up in the Burnett equations* (1.7) *in dimensions* $N \geq 7$

[13]An extra parameter may be "hidden" in a kind of "symmetry group" in the $\mathbb{R}^N$-geometry admitted by these PDEs (anyway, this looks not that convincing). Overall, existence of a pure self-similar blow-up for Burnett equations for $N = 7$ is a too simple way to settle this new "fourth-order Millennium Problem", and (at least one of the) authors would like to rule out such a trivial solution of it.

*cannot be self-similar and requires constructing (or proving their nonexistence) non-self-similar Type II blow-up singularities*[14].

## Appendix C: Solenoidal Hermitian spectral theory for operator pair $\{\mathbf{B}, \mathbf{B}^*\}$

We describe the necessary spectral properties of the linear $2m$th-order differential operator in $\mathbb{R}^N$ ($m = 2$ for the Burnett equations (1.7))

$$\mathbf{B}^* = (-1)^{m+1} \Delta_y^m - \tfrac{1}{2m}\, y \cdot \nabla_y, \tag{C.1}$$

and of its $L^2$-adjoint $\mathbf{B}$ given by

$$\mathbf{B} = (-1)^{m+1} \Delta_y^m + \tfrac{1}{2m}\, y \cdot \nabla_y + \tfrac{N}{2m}\, I. \tag{C.2}$$

As we have seen, for $m = 1$, (C.1) and (C.2) are classic Hermite self-adjoint operators with completely known spectral properties, [4, p. 48]. For any $m \ge 2$, both operators (C.1) and (C.2), though looking very similar to those for $m = 1$, *are not symmetric* and do not admit a self-adjoint extension, so we follow [8] in presenting spectral theory.

### C.1. Fundamental solution, rescaled kernel, and first estimates.

The fundamental solution $b(x,t)$ of the linear poly-harmonic parabolic equation

$$u_t = -(-\Delta)^m u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+ \tag{C.3}$$

takes the standard similarity form

$$b(x,t) = t^{-\frac{N}{2m}} F(y), \quad y = \tfrac{x}{t^{1/2m}}. \tag{C.4}$$

The rescaled kernel $F$ is the unique radial solution of the elliptic equation

$$\mathbf{B}F \equiv -(-\Delta)^m F + \tfrac{1}{2m}\, y \cdot \nabla F + \tfrac{N}{2m}\, F = 0 \ \text{ in } \mathbb{R}^N, \quad \text{with } \textstyle\int F = 1. \tag{C.5}$$

For $m \ge 2$, the rescaled kernel function $F(|y|)$ is oscillatory as $|y| \to \infty$ and satisfies [10, 13]

$$|F(y)| < D\, \mathrm{e}^{-d_0 |y|^\alpha} \ \text{ in } \mathbb{R}^N, \quad \text{where } \alpha = \tfrac{2m}{2m-1} \in (1,2), \tag{C.6}$$

for some positive constants $D$ and $d_0$ depending on $m$ and $N$.

[14]Here, there occurs a *4th-order Blow-up Problem for the Burnett equations* (1.7) that may be much more difficult mathematically than the *Millennium Prize Problem for the Navier–Stokes ones* (1.1). Of course, unlike the classic one in the actual $\mathbb{R}^3$, a "non-realistic" dimension $N = 7$ makes the 4th-order Problem less attractive for applications and for a general public, but, mathematically, it can be even more fundamental for PDE theory, since represents less understood features and principles of interaction of a higher-order viscosity-diffusion operator with a nonlinear convection one gathered in a nonlocal fashion.

C.2. **Some constants.** As we have seen, the rescaled kernel $F(y)$ satisfies (C.6), where $d_0$ admits an explicit expression; see below. Such optimal exponential estimates of the fundamental solutions of higher-order parabolic equations are well-known and were first obtained by Evgrafov–Postnikov (1970) and Tintarev (1982); see Barbatis [2, 3] for key references.

As a crucial issue for the boundary point regularity study, we will need a sharper, than given by (C.6), asymptotic behaviour of the rescaled kernel $F(y)$ as $y \to +\infty$. To get that, we keep four leading terms in (C.5) and obtain, in terms of the radial variable $y \mapsto |y| > 0$:

$$\text{(C.7)} \qquad (-1)^{m+1}\big[F^{(2m)} + m\tfrac{N-1}{y}\,F^{(2m-1)} + ...\big] + \tfrac{1}{2m}\,yF' + \tfrac{N}{2m}\,F = 0 \quad \text{for} \quad y \gg 1.$$

Using standard classic WKBJ asymptotics, we substitute into (C.7) the function

$$\text{(C.8)} \qquad F(y) = y^{-\delta_0}\,\mathrm{e}^{ay^{\alpha}} + ... \quad \text{as} \quad y \to +\infty,$$

exhibiting two scales. Balancing two leading terms gives the algebraic equation for $a$ and $\delta_0$:

$$\text{(C.9)} \qquad (-1)^m(\alpha a)^{2m-1} = \tfrac{1}{2m} \quad \text{and} \quad \delta_0 = \tfrac{m(2N-1)-N}{2m-1} > 0\,.$$

By construction, one needs to get the root $a$ of (C.9) with the maximal $\mathrm{Re}\,a < 0$. This yields (see e.g., [2, 3] and [22, p. 141])

$$\text{(C.10)} \qquad a = \tfrac{2m-1}{(2m)^{\alpha}}\big[-\sin\big(\tfrac{\pi}{2(2m-1)}\big) + \mathrm{i}\,\cos\big(\tfrac{\pi}{2(2m-1)}\big)\big] \equiv -d_0 + \mathrm{i}\,b_0 \quad (d_0 > 0).$$

Finally, this gives the following double-scale asymptotic of the kernel:

$$\text{(C.11)} \qquad F(y) = y^{-\delta_0}\,\mathrm{e}^{-d_0 y^{\alpha}}\big[C_1\sin(b_0 y^{\alpha}) + C_2\cos(b_0 y^{\alpha})\big] + ... \quad \text{as} \quad y = |y| \to +\infty,$$

where $C_{1,2}$ are real constants, $|C_1| + |C_2| \ne 0$. In (C.11), we present the first two leading terms from the $m$-dimensional bundle of exponentially decaying asymptotics.

In particular, for the Burnett equations (1.7) in $\mathbb{R}^3$, we have

$$\text{(C.12)} \qquad m = 2,\ N = 3: \quad \alpha = \tfrac{4}{3}, \quad d_0 = 3\cdot 2^{-\frac{11}{3}}, \quad b_0 = 3^{\frac{3}{2}}\cdot 2^{-\frac{11}{3}}, \quad \text{and} \quad \delta_0 = \tfrac{7}{3}.$$

C.3. **The discrete real spectrum and eigenfunctions of B.** For $m \ge 2$, $\mathbf{B}$ is considered in the weighted space $L^2_\rho(\mathbb{R}^N)$ with the exponentially growing weight function

$$\text{(C.13)} \qquad \rho(y) = \mathrm{e}^{a|y|^{\alpha}} > 0 \quad \text{in } \mathbb{R}^N,$$

where $a \in (0, 2d_0)$ is a fixed constant. We next introduce a standard Hilbert (a weighted Sobolev) space of functions $H^{2m}_\rho(\mathbb{R}^N)$ with the inner product and the induced norm

$$\langle v, w\rangle_\rho = \int\limits_{\mathbb{R}^N} \rho(y) \sum_{k=0}^{2m} D^k_y v(y)\,\overline{D^k_y w(y)}\,\mathrm{d}y \quad \text{and} \quad \|v\|^2_\rho = \int\limits_{\mathbb{R}^N} \rho(y) \sum_{k=0}^{2m} |D^k_y v(y)|^2\,\mathrm{d}y.$$

Then $H^{2m}_\rho(\mathbb{R}^N) \subset L^2_\rho(\mathbb{R}^N) \subset L^2(\mathbb{R}^N)$, and $\mathbf{B}$ is a bounded linear operator from $H^{2m}_\rho(\mathbb{R}^N)$ to $L^2_\rho(\mathbb{R}^N)$. Key spectral properties of the operator $\mathbf{B}$ are as follows [8]:

**Lemma C.1.** (i) *The spectrum of* $\mathbf{B}$ *comprises real simple eigenvalues only,*

$$\text{(C.14)} \qquad \sigma(\mathbf{B}) = \big\{\lambda_\beta = -\tfrac{k}{2m},\ k = |\beta| = 0, 1, 2, ...\big\}.$$

(ii) *The eigenfunctions* $\psi_\beta(y)$ *are given by*

$$\text{(C.15)} \qquad \psi_\beta(y) = \tfrac{(-1)^{|\beta|}}{\sqrt{\beta!}}\,D^\beta F(y), \quad \textit{for any } |\beta| = k \ge 0.$$

(iii) *Eigenfunction subset* (C.14) *is complete in* $L^2(\mathbb{R})$ *and in* $L^2_\rho(\mathbb{R})$.
(iv) *The resolvent* $(\mathbf{B} - \lambda I)^{-1}$ *for* $\lambda \notin \sigma(\mathbf{B})$ *is a compact integral operator in* $L^2_\rho(\mathbb{R}^N)$.

By Lemma C.1, the centre and stable subspaces of $\mathbf{B}$ are given by

$$E^c = \mathrm{Span}\{\psi_0 = F\} \quad \text{and} \quad E^s = \mathrm{Span}\{\psi_\beta,\, |\beta| > 0\}. \tag{C.16}$$

C.4. **Polynomial eigenfunctions of the operator $\mathbf{B}^*$.** Consider the operator (C.1) in the weighted space $L^2_{\rho^*}(\mathbb{R}^N)$, where $\langle \cdot, \cdot \rangle_{\rho^*}$ and $\| \cdot \|_{\rho^*}$ being the inner product and the norm, with the "adjoint" exponentially decaying weight function

$$\rho^*(y) \equiv \tfrac{1}{\rho(y)} = \mathrm{e}^{-a|y|^\alpha} > 0. \tag{C.17}$$

We ascribe to $\mathbf{B}^*$ the domain $H^{2m}_{\rho^*}(\mathbb{R}^N)$, which is dense in $L^2_{\rho^*}(\mathbb{R}^N)$, and then

$$\mathbf{B}^* : \; H^{2m}_{\rho^*}(\mathbb{R}^N) \to L^2_{\rho^*}(\mathbb{R}^N)$$

is a bounded linear operator. $\mathbf{B}$ is adjoint to $\mathbf{B}^*$ in the usual sense: denoting by $\langle \cdot, \cdot \rangle$ the inner product in the dual space $L^2(\mathbb{R}^N)$, we have

$$\langle \mathbf{B} v, w \rangle = \langle v, \mathbf{B}^* w \rangle \quad \text{for any } \; v \in H^{2m}_\rho(\mathbb{R}^N) \quad \text{and} \quad w \in H^{2m}_{\rho^*}(\mathbb{R}^N). \tag{C.18}$$

The eigenfunctions of $\mathbf{B}^*$ take a particularly simple finite polynomial form and are as follows:

**Lemma C.2.** (i) $\sigma(\mathbf{B}^*) = \sigma(\mathbf{B})$.
(ii) *The eigenfunctions* $\psi^*_\beta(y)$ *of* $\mathbf{B}^*$ *are generalized Hermite polynomials of degree* $|\beta|$ *given by*

$$\psi^*_\beta(y) = \tfrac{1}{\sqrt{\beta!}} \Big[ y^\beta + \textstyle\sum_{j=1}^{[|\beta|/2m]} \tfrac{1}{j!} (-\Delta)^{mj} y^\beta \Big] \quad \textit{for any} \quad \beta. \tag{C.19}$$

(iii) *Eigenfunction subset* (C.19) *is complete in* $L^2_{\rho^*}(\mathbb{R}^N)$.
(iv) $\mathbf{B}^*$ *has a compact resolvent* $(\mathbf{B}^* - \lambda I)^{-1}$ *in* $L^2_{\rho^*}(\mathbb{R}^N)$ *for* $\lambda \notin \sigma(\mathbf{B}^*)$.
(v) *The bi-orthonormality of the bases* $\{\psi_\beta\}$ *and* $\{\psi^*_\gamma\}$ *holds in the dual* $L^2$*-metric:*

$$\langle \psi_\beta, \psi^*_\gamma \rangle = \delta_{\beta\gamma} \quad \textit{for any} \quad \beta, \, \gamma. \tag{C.20}$$

**Remark on closure.** This is an important issue for using eigenfunction expansions of solutions. Firstly, in the self-adjoint case $m = 1$, the sets of eigenfunctions are closed in the corresponding spaces, [4] (and we have used this in our previous NSEs study).

Secondly, for $m \geq 2$, one needs some extra details. Namely, using (C.20), we can introduce the subspaces of eigenfunction expansions and begin with the operator $\mathbf{B}$. We denote by $\tilde{L}^2_\rho$ the subspace of eigenfunction expansions $v = \sum c_\beta \psi_\beta$ with coefficients $c_\beta = \langle v, \psi^* \rangle$ defined as the closure of the finite sums $\{\sum_{|\beta| \leq M} c_\beta \psi_\beta\}$ in the norm of $L^2_\rho$. Similarly, for the adjoint operator $\mathbf{B}^*$, we define the subspace $\tilde{L}^2_{\rho^*} \subseteq L^2_{\rho^*}$. Note that since the operators are not self-adjoint and the eigenfunction subsets are not orthonormal, in general, these subspaces can be different from $L^2_\rho$ and $L^2_{\rho^*}$, and particularly the equality is guaranteed in the self-adjoint case $m = 1$, $a = \frac{1}{4}$.

Thus, for $m \geq 2$, in the above subspaces obtained via a suitable closure, we can apply standard eigenfunction expansion techniques as in the classic self-adjoint case $m = 1$.

C.5. **Solenoidal Hermite polynomials.** The vector solenoidal Hermite polynomials are constructed from (C.19) in a manner similar to that for $m = 1$; cf (A.14)–(A.16). Namely, given a vector polynomial

$$\mathbf{v}^*_\beta = [\psi^*_{\beta_1}, \psi^*_{\beta_2}, ..., \psi^*_{\beta_N}]^T, \quad \text{where} \quad |\beta_1| = |\beta_2| = ... = |\beta_N| = |\beta|, \tag{C.21}$$

it gets solenoidal provided that

$$\operatorname{div} \mathbf{v}^*_\beta \equiv \sum_{i=1}^{N} (\psi_{\beta_i})_{y_i} = 0. \tag{C.22}$$

For instance, for the Burnett case $m = 2$ and $N = 3$, some pairs are (not all linearly independent eigenfunctions are presented, normalization constants are omitted):

$$\lambda_0 = 0: \quad \mathbf{v}^*_0 = [1, 1, 1]^T, \tag{C.23}$$

$$\lambda_1 = -\tfrac{1}{4}: \mathbf{v}^*_{11} = [y_2, -y_3, y_2]^T, \quad \mathbf{v}^*_{12} = [y_3, y_3, -y_1]^T, \quad \mathbf{v}^*_{13} = [-y_2, y_1, y_1]^T, \tag{C.24}$$

$$\lambda_2 = -\tfrac{1}{2}: \quad \mathbf{v}^*_{21} = [-y_1^2 - y_3^2, y_1 y_2, y_1 y_3]^T, \ \mathbf{v}^*_{22} = [y_1 y_2, -y_2^2 - y_3^2, y_2 y_3]^T, \ \text{etc.} \tag{C.25}$$

$$\lambda_3 = -\tfrac{3}{4}: \quad \mathbf{v}^*_{31} = [y_2^3, y_3^3, y_1^3], \quad \mathbf{v}^*_{32} = [y_1 y_2^2, y_2 y_1^2, -y_3(y_1^2 + y_2^2)], \ \text{etc.} \tag{C.26}$$

$$\lambda_4 = -1: \quad \mathbf{v}^*_{41} = [y_2^4 + 4!, y_3^4 + 4!, y_1^4 + 4!]^T, \ \mathbf{v}^*_{42} = [y_1 y_2^3, y_2 y_1^3, -y_3(y_1^3 + y_2^3)]^T, \quad \text{etc.} \tag{C.27}$$

As in the self-adjoint case $m = 1$, some technical efforts are necessary toward completeness/closure of generalized solenoidal Hermite polynomials in suitable spaces. We omit details.

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