

ON THE FUNDAMENTAL SOLUTION OF AN ELLIPTIC EQUATION IN NONDIVERGENCE FORM

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ABSTRACT. For $n \geq 3$, we consider the existence and asymptotics for the fundamental solution of an elliptic operator in nondivergence form, $\mathcal{L}(x, \partial_x) = a_{ij}(x)\partial_i\partial_j$. We assume that the coefficients are continuous with modulus of continuity $\omega(r)$ satisfying the square Dini condition $\int_0^1 \omega^2(t)t^{-1} dt < \infty$. For fixed y , we construct a solution of $\mathcal{L}Z_y(x) = 0$ for $0 < |x-y| < \gamma$ with explicit leading order term which is $O(|x-y|^{2-n}e^{I(x,y)})$ as $x \rightarrow y$, where the behavior of $I(x, y)$ as $x \rightarrow y$ is given by an improper integral and plays an important role for the fundamental solution: if $I(x, y)$ approaches a finite limit as $x \rightarrow y$, then we can solve $\mathcal{L}(x, \partial_x)F(x, y) = \delta(x - y)$, and $F(x, y)$ is asymptotic as $x \rightarrow y$ to the fundamental solution for the constant coefficient operator $\mathcal{L}(y, \partial_x)$. On the other hand, if $I(x, y) \rightarrow -\infty$ as $x \rightarrow y$ then the solution violates the “extended maximum principle” of Gilbarg & Serrin.

0. INTRODUCTION

Background. Consider an elliptic operator in nondivergence form

$$(1) \quad \mathcal{L}(x, \partial_x) u(x) = a_{ij}(x) \partial_i \partial_j u(x),$$

where $\partial_i = \partial/\partial x_i$ and we have used the summation convention for repeated indices. The coefficients $a_{ij} = a_{ji}$ are real-valued functions defined on \mathbb{R}^n for $n \geq 3$, and we denote the symmetric and positive definite matrix $(a_{ij}(x))$ by \mathbf{A}_x . A *fundamental solution* for \mathcal{L} in an open set U is a function $F(x, y)$ satisfying $F(x, \cdot) \in L^1_{loc}(U)$ and

$$(2) \quad -\mathcal{L}(x, \partial_x)F(x, y) = \delta(x - y) \quad \text{for } x, y \in U$$

in a distributional sense that needs to be made clear; for this some regularity of the coefficients will be required. If $F(x, y)$ satisfying (2) exists, then the operator Q defined by

$$(3) \quad Q\phi(x) = - \int_U F(x, y) \phi(y) dy$$

provides a right-inverse for \mathcal{L} on $C_0(U)$, the space of continuous functions with compact support in U .

In the “classical” case that the coefficient functions are λ -Hölder continuous in a bounded domain U for some $\lambda \in (0, 1)$, it is well-known (cf.[22]) that a fundamental solution exists in U and is asymptotic (as $x \rightarrow y$) to the fundamental solution for the constant coefficient operator obtained by freezing the coefficients a_{ij} at y : for $n \geq 3$ this means

$$(4) \quad F(x, y) = \tilde{F}_y(x - y)(1 + H(x, y)),$$

where, letting \langle, \rangle denote the inner product in \mathbb{R}^n ,

$$(5) \quad \tilde{F}_y(x) = \frac{\langle \mathbf{A}_y^{-1} x, x \rangle^{\frac{2-n}{2}}}{(n-2) |S^{n-1}| \sqrt{\det \mathbf{A}_y}}$$

gives the fundamental solution for the constant coefficient operator $\mathcal{L}(y, \partial_x) = a_{ij}(y) \partial_i \partial_j$, and the remainder term $H(x, y)$ in (4) satisfies

$$(6) \quad |H(x, y)| + r|\nabla_x H(x, y)| + r^2|\nabla_x^2 H(x, y)| \leq cr^\lambda \quad \text{as } r = |x - y| \rightarrow 0,$$

for all y in a compact subset of U . This fundamental solution may be used (cf. [22]) to show the classical regularity result: if $u \in C^2(U)$ and $\mathcal{L}u$ is λ -Hölder continuous in U , then $\partial_i \partial_j u$ is λ -Hölder continuous in U .

The Hölder continuity may be generalized by assuming the coefficients have a weaker modulus of continuity, i.e. $a_{ij} \in C^\omega(U)$ where $\omega(r)$ is a continuous, increasing function for $0 \leq r < 1$ satisfying $\omega(0) = 0$, and

$$(7) \quad C^\omega(U) = \{f \in C(U) : |f(x) - f(y)| \leq c\omega(|x - y|) \text{ for } x, y \in U\}.$$

If ω satisfies the Dini condition at zero, i.e. $\int_0^1 \omega(t)t^{-1} dt < \infty$, then we say that the coefficients are *Dini continuous*. In this case, there are regularity results analogous to the case of Hölder continuity (cf. Proposition 1.14 in Chapter 3 of [25]); however, we could not find in the literature an asymptotic description of the fundamental solution such as (4) with estimates on $H(x, y)$.

Dini continuity is also essential for the “extended maximum principle” of Gilbarg & Serrin [8] to hold: a C^2 -solution of

$$\mathcal{L}u \geq 0 \text{ for } 0 < |x| \leq r_0$$

with

$$u(x) = o(|x|^{2-n}) \text{ as } |x| \rightarrow 0$$

must satisfy

$$u(x) < M := \max\{u(y) : |y| = r_0\} \text{ for } 0 < |x| \leq r_0,$$

and $\limsup_{|x| \rightarrow 0} u(x) < M$. In fact, they give an example (which we will discuss in Section 2) in which the coefficients are not Dini continuous and the extended maximum principle fails.

The above regularity assumptions (Hölder or Dini continuity) on the coefficients are required to study the behavior of the fundamental solution as a function of x (for fixed y). If we instead fix x and consider the behavior in y , then regularity of the coefficients a_{ij} is not required; however, we cannot expect to achieve as precise an asymptotic description as (4). This is most conveniently described in terms of the *Green's function* for (1) on a smooth, bounded domain U , which may be defined (as in [4]) to be $G(x, \cdot) \in L^1_{\text{loc}}(U)$ satisfying

$$(8) \quad \phi(x) = - \int_U G(x, y) \mathcal{L}(y, \partial_y) \phi(y) dy \quad \text{for any } \phi \in C^2(\bar{U}) \text{ with } \phi = 0 \text{ on } \partial U.$$

Notice that (8) can be expressed formally as $-\mathcal{L}^*(y, \partial_y)G(x, y) = \delta(x - y)$ and implies that $Q\phi(x) = - \int_U G(x, y) \phi(y) dy$ defines a left-inverse for \mathcal{L} on $C^2_0(U)$. When the a_{ij} are measurable, bounded, and uniformly elliptic in U , then Fabes & Strook [6] showed that $G(x, \cdot) \in L^q(U)$ for some $q > n/(n - 1)$, while Bauman [1], [2], [3], and Escauriaza [4] obtained pointwise estimates on $G(x, y)$ as $y \rightarrow x$ in terms of a nonnegative “adjoint solution” $W(y)$ which satisfies $\mathcal{L}^*(y, \partial_y)W(y) = 0$ in U . However, our paper is not concerned with such general coefficients, and for us a Green's function will also be a fundamental solution in the sense of (2).

Our results. In this paper, we allow the coefficients a_{ij} to be less regular than Dini continuous, and we want to study the solutions of $\mathcal{L}u(x) = 0$ with an isolated singularity at $x = y$, as well as the existence and asymptotics of a fundamental solution $F(x, y)$ satisfying (2) in an appropriate distributional sense. We assume that the coefficients have modulus of continuity ω satisfying the “square-Dini condition”

$$(9) \quad \int_0^1 \frac{\omega^2(t)}{t} dt < \infty.$$

Condition (9) has been encountered by other authors in different contexts: cf. [5], [7], [12], [24].

To construct our solution of (2), we first fix y and seek a solution of

$$(10) \quad \mathcal{L}(x, \partial_x)Z_y(x) = 0 \quad \text{for } x \in B_\varepsilon(y) \setminus \{y\},$$

where $B_\varepsilon(y) = \{x : |x - y| < \varepsilon\}$ for ε sufficiently small, and $Z_y(x)$ has the appropriate singularity as $x \rightarrow y$. Assuming that the modulus of continuity at y satisfies (9), we shall construct (cf. Theorem 1 and Corollary 1) a solution of (10) with the asymptotic description

$$(11) \quad Z_y(x) \sim \langle \mathbf{A}_y^{-1}(x - y), (x - y) \rangle^{\frac{2-n}{2}} e^{I(x,y)} \quad \text{as } x \rightarrow y,$$

where the factor $e^{I(x,y)}$ adjusts for lack of regularity in the coefficients: if the a_{ij} are Hölder continuous, then we can take $I(x,y) \equiv 0$ and $c_y Z_y(x)$ is asymptotic to $\tilde{F}_y(x - y)$ as $x \rightarrow y$. In general, however, we find that

$$(12) \quad I(x,y) = I_y \left(\sqrt{\langle \mathbf{A}_y^{-1}(x - y), (x - y) \rangle} \right),$$

where

$$(13) \quad I_y(r) = \frac{1}{|S^{n-1}|} \int_{r < |z-y| < \varepsilon} \left(\text{tr}(\mathbf{A}_z \mathbf{A}_y^{-1}) - n \frac{\langle \mathbf{A}_z \mathbf{A}_y^{-1/2}(z - y), \mathbf{A}_y^{-1/2}(z - y) \rangle}{|z - y|^2} \right) \frac{dz}{|z - y|^n}$$

with tr denoting matrix trace. As $r \rightarrow 0$, $I_y(r)$ need not even be bounded, so the singularity of $Z_y(x)$ need not be $O(|x - y|^{2-n})$ as it was in the Hölder case. These formulas simplify significantly if we use an affine change of variables in which y corresponds to $x = 0$ and $a_{ij}(0) = \delta_{ij}$:

$$(14) \quad Z(x) \sim |x|^{2-n} e^{I(|x|)} \quad \text{as } |x| \rightarrow 0,$$

where

$$(15) \quad I(r) = \frac{1}{|S^{n-1}|} \int_{r < |z| < \varepsilon} \left(\text{tr}(\mathbf{A}_z) - n \frac{\langle \mathbf{A}_z z, z \rangle}{|z|^2} \right) \frac{dz}{|z|^n}.$$

We can verify that the absolute value of the integrand in (15) is bounded by $\omega(|z|)$, so the coefficients being Dini continuous implies that the improper integral defining $I(0)$ converges absolutely. Even if $I(0)$ does not converge, we shall see that for any $\lambda > 0$ there exist $C_\lambda > 0$ such that

$$(16) \quad |I(r)| \leq \lambda |\log r| + C_\lambda \quad \text{for } 0 < r < \varepsilon,$$

so the singularity of Z at $x = 0$ is never very far from $|x|^{2-n}$. Nevertheless, the behavior of $I(r)$ as $r \rightarrow 0$ plays an important role in the extended maximum principle and the existence of a fundamental solution. There are three principal cases to consider.

1. $I(0) = \lim_{r \rightarrow 0} I(r)$ exists and is finite.

In this case, we can scale $Z(x)$ by a constant multiple to obtain a solution that is asymptotic to the fundamental solution for the Laplacian. In fact, we can solve the distributional equation

$$(17) \quad -\mathcal{L}(x, \partial_x)Z(x) = C_0 \delta(x),$$

and find

$$(18) \quad C_0 = (n - 2) |S^{n-1}| e^{I(0)}.$$

Note that the improper integral defining $I(0)$ may converge even if the modulus of continuity does *not* satisfy the Dini condition.

2. $I(r) \rightarrow -\infty$ as $r \rightarrow 0$.

We see that $Z(x) = o(|x|^{2-n})$ as $|x| \rightarrow 0$; we still have $Z(x) \rightarrow +\infty$ as $x \rightarrow 0$, so this violates the extended maximum principle of [8]. Nevertheless, we can solve (17) to find $C_0 = 0$.

3. $I(r) \rightarrow \infty$ as $r \rightarrow 0$.

Now we find $Z(x)|x|^{n-2} \rightarrow \infty$ as $|x| \rightarrow 0$, so this solution grows more rapidly than the fundamental solution for the Laplacian. However, we cannot solve (17) for C_0 in this case.

We next allow y to vary over U . Provided that we are in Case 1 at each $y \in U$, we can use $Z_y(x)$ to construct the fundamental solution $F(x, y)$ in U . Our main result (Theorem 3) states that, provided $a_{ij} \in C^\omega(U)$ where ω satisfies (9) and $I_y(0) = \lim_{r \rightarrow 0} I_y(r)$ exists and is finite for each $y \in U$, a fundamental solution $F(x, y)$ exists in the form (4) where the remainder term $H(x, y)$ may be estimated in L^p for any $p \in (1, \infty)$ in terms of ω and the rate of convergence $I_y(r) \rightarrow I_y(0)$. More specifically, let us assume $\omega(r)r^{-1+\kappa}$ is nonincreasing for $0 < r < 1$ where $\kappa \in (0, 1)$, and introduce

$$(19) \quad \sigma(r) = \int_0^r \frac{\omega^2(t)}{t} dt.$$

Further, let us assume that

$$(20) \quad |I_y(r) - I_y(0)| \leq \theta(r) \quad \text{for all } y \in U,$$

where $\theta(r)$ is a nondecreasing function of r satisfying $\theta(0) = 0$. Then for y in a compact subset of U , we have

$$(21) \quad r^2 M_p(\nabla^2 H(\cdot, y), r; y) \leq c \max(\omega(r), \sigma(r), \theta(r)) \quad \text{as } r \rightarrow 0,$$

where $M_p(f(\cdot, y), r; y)$ denotes the L^p -mean of $f(x, y)$ as a function of x (for fixed y) over the annulus $A_r(y) = \{x : r < |x - y| < 2r\}$. (In fact, when ω satisfies the Dini condition, the right hand side of (21) reduces to just $c\omega(r)$.) Taking $p > n$, we obtain pointwise bounds on $|H(x, y) + r|\nabla_x H(x, y)|$, but we no longer have pointwise bounds on $\nabla_x^2 H(x, y)$ as we did in (6) when the coefficients a_{ij} were Hölder continuous.

Note. In (21) and throughout this paper, c is used to denote a constant whose value may change line-by-line. It may depend upon n and the a_{ij} , but not on r .

Organization of this paper and comparison with other works. The organization of this paper is as follows. In Section 1 we discuss some preliminary estimates for solutions of the Poisson equation $\Delta u = f$. In Section 2 we construct the solution $Z_y(x)$ of (10) by first considering the case $a_{ij}(0) = \delta_{ij}$ and then performing a change of coordinates. In Section 3 we analyze the equation $-\mathcal{L}(x, \partial_x)Z_y(x) = C_y\delta(x - y)$ and calculate C_y when $I_y(0)$ is finite or $-\infty$. Finally, in Section 4 we apply the previous results to construct our fundamental solution $F(x, y)$ in the form (4) and obtain estimates on the remainder term $H(x, y)$. The analysis in each section makes use of L_p -means on annuli $A_r = \{x : r < |x| < 2r\}$ to measure the growth and decay of functions as $r \rightarrow 0$; these are defined in Section 1.

The results obtained and the techniques used in this paper are closely related to those in our previous paper [18]. In [18] we studied the asymptotics for solutions of the adjoint equation for \mathcal{L} , and discussed their relationship to the estimates obtained in [4] as $y \rightarrow x$. In fact, the condition that $I_y(r)$ is bounded (above and below) as $r \rightarrow 0$ not only allows one to conclude that the solution Z_y of (10) satisfies $c_y|x - y|^{2-n} \leq Z_y(x) \leq c'_y|x - y|^{2-n}$ as $x \rightarrow y$, but that $\mathcal{L}^*(x, \partial_x)u(x) = 0$ admits solutions in $B_\varepsilon(y)$ whose L_p -mean is bounded between positive constants as $r = |x - y| \rightarrow 0$; if $I_y(0)$ exists and is finite for every y , then this bounded solution of $\mathcal{L}^*(x, \partial_x)u(x) = 0$ is continuous.

Both [18] and the present work are independent of, but related to, the asymptotic theory developed in [15]. In particular, L_p -means were extensively used in [13] and [14]. The asymptotic formulas that we obtain are analogous to those of [16], where an asymptotic representation near the boundary was obtained for solutions to the Dirichlet problem for elliptic equations in divergence form with discontinuous coefficients. In particular, note that we do not make use of the maximum principle in this paper; in fact, most results of this paper hold for complex-valued coefficients, although this introduces some technical difficulties which we have chosen to avoid.

Keywords. Fundamental solution, maximum principle, nondivergence form, elliptic operator, modulus of continuity, asymptotics, Dini condition, square-Dini condition.

1. PRELIMINARIES

Throughout this paper, to measure the growth of functions as $x \rightarrow y$, it will be convenient to use the L^p -mean for some $p \in (1, \infty)$:

$$(22) \quad M_p(w, r; y) = \left(\int_{A_r(y)} |w(x)|^p dx \right)^{1/p},$$

where $A_r(y) = \{x : r < |x - y| < 2r\}$; here (and elsewhere in this paper) the slashed integral denotes mean value. It is natural to also introduce

$$(23) \quad M_\infty(w, r; y) = \sup_{A_r(y)} |w(x)|.$$

We may apply (22) and (23) to vector or matrix valued functions w with $|w|$ denoting the norm. We also need to estimate derivatives, so we define

$$(24) \quad M_{1,\infty}(w, r; y) = rM_\infty(Dw, r; y) + M_\infty(w, r; y),$$

where Dw represents the gradient of w , and for $p \in (1, \infty)$

$$(25) \quad M_{2,p}(w, r; y) = r^2 M_p(D^2w, r; y) + rM_p(Dw, r; y) + M_p(w, r; y),$$

where D^2w represents the Hessian matrix of w . Sobolev estimates show that

$$(26) \quad p > n \quad \Rightarrow \quad M_{1,\infty}(w, r; y) \leq c M_{2,p}(w, r; y).$$

When $y = 0$, we shall abbreviate $M_p(w, r; 0)$ as $M_p(w, r)$ (and similarly for $M_{1,\infty}$ and $M_{2,p}$).

For $x \in \mathbb{R}^n \setminus \{0\}$, let $\theta = x/|x| \in S^{n-1}$ and let $d\theta$ denote the standard surface measure on S^{n-1} . We will use the spherical mean of a function w :

$$(27) \quad \bar{w}(r) = \int_{S^{n-1}} w(r\theta) d\theta.$$

In particular, in this section we consider the equation

$$(28) \quad \Delta v = f \quad \text{in } \mathbb{R}^n \setminus \{0\}$$

when $\bar{f} = 0$, and investigate the behavior of the L^p -mean of the solution as $x \rightarrow 0$; our results are quite analogous to those of [13] and [15]. We shall let $\Gamma(|x|) = c_n |x|^{2-n}$ denote the fundamental solution for the Laplacian in \mathbb{R}^n .

Proposition 1. *Suppose $n \geq 2$ and $f \in L^p_{loc}(\mathbb{R}^n \setminus \{0\})$ satisfies $\bar{f} = 0$,*

$$(29) \quad \int_{|x|<1} |x| |f(x)| dx < \infty, \quad \text{and} \quad \int_{|x|>1} \frac{|f(x)|}{|x|^{n-1}} dx < \infty.$$

Then $v = Kf = \Gamma \star f$ defines a distribution solution of (28) that satisfies

$$(30) \quad M_{2,p}(Kf, r) \leq c \left(r^2 \tilde{M}_p(f, r) + r^{1-n} \int_{|x|<r} |x| |f(x)| dx + r \int_{|x|>r} \frac{|f(x)|}{|x|^{n-1}} dx \right),$$

where we have introduced

$$\tilde{M}_p(w, r) := \left(\int_{r/2 < |x| < 4r} |w(x)|^p dx \right)^{1/p}.$$

As indicated in Section 1 of [18], the integrals in (30) can be estimated by integrals of M_p .

Proposition 2. *Under the conditions of Proposition 1,*

$$(31) \quad M_{2,p}(Kf, r) \leq c \left(r^{1-n} \int_0^r M_p(f, \rho) \rho^n d\rho + r \int_r^\infty M_p(f, \rho) d\rho \right)$$

Remark 1. *If we no longer assume $\bar{f} = 0$, then we can replace (29) by $\int_{|x|<1} |f(x)| dx < \infty$ and $\int_{|x|>1} |f(x)| \cdot |x|^{2-n} dx < \infty$, and conclude (for $n \geq 3$) that*

$$M_{2,p}(Kf, r) \leq c \left(r^2 \tilde{M}_p(f, r) + r^{2-n} \int_{|x|<r} |f(x)| dx + \int_{|x|>r} \frac{|f(x)|}{|x|^{n-2}} dx \right).$$

However, this estimate is not strong enough for the purposes of this paper.

Proof of Proposition 1. Let $v = Kf$ and let χ denote the characteristic function for the annulus $\bar{A}_r = B_{4r} \setminus B_{r/2}$. Using $\bar{f} = 0$, we see that

$$\begin{aligned} v(x) &= \int_{\mathbb{R}^n} \Gamma(|x-y|) \chi(y) f(y) dy + \int_{|y|<r/2} (\Gamma(|x-y|) - \Gamma(|x|)) f(y) dy \\ &\quad + \int_{|y|>4r} (\Gamma(|x-y|) - \Gamma(|y|)) f(y) dy. \end{aligned}$$

We want to estimate $M_p(v, r)$, $rM_p(Dv, r)$, and $r^2M_p(D^2v, r)$, and show that each is bounded by the right hand side of (30).

Using Stein's inequality [23], for $0 < a < n/p$ and $0 < b < n/p'$ with $a + b = 2$ we have

$$\begin{aligned} \left\| \int_{\mathbb{R}^n} \Gamma(|x-y|) \chi(y) f(y) dy \right\|_{L^p(A_r)} &\leq cr^a \|\chi(y) |y|^b f(y)\|_{L^p(\mathbb{R}^n)} \\ &= cr^a \||y|^b f(y)\|_{L^p(\bar{A}_r)} \leq cr^2 \|f\|_{L^p(\bar{A}_r)}. \end{aligned}$$

It is elementary to show that for $|y| < r/2$ and $r < |x| < 2r$ we have $|\Gamma(|x-y|) - \Gamma(|x|)| \leq c|x|^{1-n}|y|$, where c is independent of r , so for $x \in A_r$ we have

$$\left| \int_{|y|<r/2} (\Gamma(|x-y|) - \Gamma(|x|)) f(y) dy \right| \leq c|x|^{1-n} \int_{|y|<r/2} |f(y)| |y| dy.$$

Similarly, we can show that for $|y| > 4r$ and $r < |x| < 2r$ we have $|\Gamma(|x-y|) - \Gamma(|y|)| \leq c|x||y|^{1-n}$, so $x \in A_r$ implies

$$\left| \int_{B_{4r}^c} (\Gamma(|x-y|) - \Gamma(|y|)) f(y) dy \right| \leq c|x| \int_{|y|>4r} \frac{|f(y)|}{|y|^{n-1}} dy.$$

Using these estimates, we easily conclude that $M_p(v, r)$ is bounded by the right hand side of (30).

Next we consider

$$\frac{\partial v}{\partial x_i} = \int_{\mathbb{R}^n} \Gamma_i(x-y) \chi(y) f(y) dy + \int_{B_{r/2}} \Gamma_i(x-y) f(y) dy + \int_{B_{4r}^c} \Gamma_i(x-y) f(y) dy,$$

where

$$\Gamma_i(x) = \Gamma'(|x|) \frac{x_i}{|x|}.$$

Applying Stein's inequality as above but with $a + b = 1$, we conclude

$$\left\| \int_{\mathbb{R}^n} \Gamma_i(x-y) \chi(y) f(y) dy \right\|_{L^p(A_r)} \leq cr \|f\|_{L^p(\bar{A}_r)}.$$

Elementary estimates for $r < |x| < 2r$ show that

$$\left| \int_{B_{r/2}} \Gamma_i(x-y)f(y) dy \right| = \left| \int_{B_{r/2}} (\Gamma_i(x-y) - \Gamma_i(x))f(y) dy \right| \leq c|x|^{-n} \int_{|y|<r/2} |f(y)||y| dy$$

and

$$\left| \int_{B_{4r}^c} (\Gamma_i(x-y)f(y) dy \right| = \left| \int_{B_{4r}^c} (\Gamma_i(x-y) - \Gamma_i(y))f(y) dy \right| \leq c \int_{|y|>4r} \frac{|f(y)|}{|y|^{n-1}} dy.$$

From these estimates we easily conclude that $rM_p(Dv, r)$ is bounded by the right hand side of (30).

Finally, we consider

$$\frac{\partial^2 v}{\partial x_i \partial x_j} = \int_{\mathbb{R}^n} \Gamma_{ij}(x-y)\chi(y)f(y) dy + \int_{B_{r/2}} \Gamma_{ij}(x-y)f(y) dy + \int_{B_{4r}^c} \Gamma_{ij}(x-y)f(y) dy,$$

where Γ_{ij} is the singular kernel given by

$$\Gamma_{ij}(x) = \Gamma''(|x|) \frac{x_i x_j}{|x|^2} + \Gamma'(|x|) \frac{\delta_{ij}|x|^2 - x_i x_j}{|x|^3}.$$

Using the L^p -boundedness of singular integral operators, we conclude

$$\left\| \int_{\mathbb{R}^n} \Gamma_{ij}(x-y)\chi(y)f(y) dy \right\|_{L^p(A_r)} \leq c \|f\|_{L^p(\bar{A}_r)}.$$

Elementary estimates for $r < |x| < 2r$ show that

$$\left| \int_{B_{r/2}} \Gamma_{ij}(x-y)f(y) dy \right| = \left| \int_{B_{r/2}} (\Gamma_{ij}(x-y) - \Gamma_{ij}(x))f(y) dy \right| \leq c|x|^{-n-1} \int_{|y|<r/2} |f(y)||y| dy$$

and

$$\left| \int_{B_{4r}^c} (\Gamma_{ij}(x-y)f(y) dy \right| = \left| \int_{B_{4r}^c} (\Gamma_{ij}(x-y) - \Gamma_{ij}(|y|))f(y) dy \right| \leq c|x|^{-1} \int_{|y|>4r} \frac{|f(y)|}{|y|^{n-1}} dy.$$

These estimates show that $r^2 M_p(D^2 v, r)$ is bounded by the right hand side of (30). This completes the proof. \square

2. CONSTRUCTING THE SINGULAR SOLUTION $Z_y(x)$ IN $B_\varepsilon(y) \setminus \{y\}$

In this section, we fix y and construct a solution of (10) for ε sufficiently small. Since the result is localized near y , the ellipticity and continuity of the coefficients of \mathcal{L} need only be assumed at y . In fact, we first consider the case when $y = 0$ and $a_{ij}(0) = \delta_{ij}$:

$$(32) \quad \sup_{|x|=r} \|\mathbf{A}_x - \mathbf{I}\| \leq \omega(r) \quad \text{for } 0 < r < 1,$$

where \mathbf{I} is the identity matrix and ω is a positive, nondecreasing function for $0 < r < 1$ satisfying the square-Dini condition (9). For convenience, we shall also assume the monotonicity condition that for some $\kappa \in (0, 1)$ we have

$$(33) \quad \omega(r) r^{-1+\kappa} \quad \text{is nonincreasing for } 0 < r < 1.$$

The significance of (33) is that it requires $\omega(r)$ to vanish more slowly than r as $r \rightarrow 0$, which is a natural constraint. (As long as $\omega(r) r^{-1+\kappa}$ is nonincreasing for $0 < r < \eta$ with some $\eta > 0$, then ω may be modified for $\eta < r < 1$ to satisfy (33).) We seek a solution of

$$(34) \quad \mathcal{L}(x, \partial_x) Z(x) = 0 \quad \text{for } 0 < |x| < \varepsilon,$$

where ε is sufficiently small; although we generally assume that \mathcal{L} has real-valued coefficients, the theorem below holds when the a_{ij} are complex-valued.

Theorem 1. *For $n \geq 3$ and $p \in (1, \infty)$, suppose that the coefficients $a_{ij}(x)$ are bounded, measurable functions satisfying (32). For $\varepsilon > 0$ sufficiently small, there exists a solution of (34) in the form*

$$(35) \quad Z(x) = h(|x|) + v(x),$$

where h is defined by

$$(36) \quad h(r) = \int_r^\varepsilon s^{1-n} e^{I(s)} ds (1 + \zeta(r)),$$

with $I(r)$ given by (15) and

$$(37) \quad M_{2,p}(\zeta, r) \leq c \max(\omega(r), \sigma(r)),$$

where σ is given in (19), and v in (35) satisfies

$$(38) \quad M_{2,p}(v, r) \leq c r^{2-n} e^{I(r)} \omega(r).$$

Moreover, for any $u \in W_{loc}^{2,p}(\overline{B_\varepsilon} \setminus \{0\})$ that is a strong solution of $\mathcal{L}(x, \partial_x)u = 0$ in $\overline{B_\varepsilon} \setminus \{0\}$ subject to the growth condition

$$(39) \quad M_{2,p}(u, r) \leq c r^{1-n+\varepsilon_0} \quad \text{where } \varepsilon_0 > 0,$$

there exist constants C, C_0, C_1, \dots, C_n (depending on u) such that

$$(40) \quad u(x) = CZ(x) + C_0 + \sum_{j=1}^n C_j x_j + w(x),$$

where w satisfies

$$(41) \quad M_{2,p}(w, r) \leq c r^{2-\varepsilon_1} \quad \text{for any } \varepsilon_1 > 0.$$

We shall prove this theorem below, but first let us make some observations. In general, we do not know whether $I(r)$ is bounded as $r \rightarrow 0$, but we can verify that $|I'(r)| \leq c r^{-1} \omega(r)$, so integration by parts in (36) shows that

$$(42) \quad h(r) = \frac{r^{2-n}}{n-2} e^{I(r)} + h_1(r),$$

where $h_1(r)$ satisfies $M_{1,\infty}(h_1, r) \leq c r^{2-n} e^{I(r)} \max(\omega(r), \sigma(r))$. If we take $p > n$ and apply (26) to v , we conclude that

$$(43) \quad Z(x) = \frac{|x|^{2-n} e^{I(|x|)}}{n-2} (1 + \xi(x)) \quad \text{as } |x| \rightarrow 0.$$

where $M_{1,\infty}(\xi, r) \leq c \max(\omega(r), \sigma(r))$. Obviously, we can multiply the Z of (43) by $n-2$ to obtain the Z of (14).

Even when $I(r)$ is not bounded as $r \rightarrow 0$, we can derive useful bounds on $Z(x)$ as $|x| \rightarrow 0$. It is not difficult to verify that a symmetric matrix \mathbf{A} satisfies

$$(44) \quad -2(n-1)\|\mathbf{A} - \mathbf{I}\| \leq \text{tr}(\mathbf{A}) - n\langle \mathbf{A}y, y \rangle |y|^{-2} \leq 2(n-1)\|\mathbf{A} - \mathbf{I}\| \quad \text{for } |y| = 1,$$

so there exist constants $c, C > 0$ so that Z satisfies (assuming $n \geq 3$)

$$(45) \quad c|x|^{2-n} \exp\left(-c_n \int_{|x|}^\varepsilon \omega(t) \frac{dt}{t}\right) \leq |Z(x)| \leq C|x|^{2-n} \exp\left(c_n \int_{|x|}^\varepsilon \omega(t) \frac{dt}{t}\right)$$

as $x \rightarrow 0$, where $c_n = 2(n-1)/|S^{n-1}|$. Using (45) and the fact that $\omega(r) \rightarrow 0$ as $r \rightarrow 0$, we obtain (16), which shows that the singularity of Z at $x = 0$ is very close to the classical case.

An interesting class of examples is obtained by letting

$$(46) \quad a_{ij}(x) = \delta_{ij} + g(|x|)x_i x_j |x|^{-2},$$

where $g(0) = 0$ but vanishes slowly as $r \rightarrow 0$. Gilbarg & Serrin [8] used (46) with certain specific functions g to show that Dini continuity is essential for their extended maximum principle to hold. In our formulation,

$$(47) \quad \operatorname{tr}(\mathbf{A}_z) - n \frac{\langle \mathbf{A}_z z, z \rangle}{|z|^2} = (1-n)g(|z|),$$

so

$$(48) \quad I(r) = \int_r^\varepsilon (1-n) \frac{g(\rho)}{\rho} d\rho.$$

Thus any $g(r) > 0$ which does *not* satisfy the Dini condition at $r = 0$ (but does satisfy the square-Dini condition) will yield $I(r) \rightarrow -\infty$ as $r \rightarrow 0$, so the $Z(x)$ of Theorem 1 gives an example of a solution of (34) with singularity at $x = 0$ even though $Z(x) = o(|x|^{2-n})$ as $|x| \rightarrow 0$, i.e. the extended maximum principle fails; the specific function in [8] is $g(r) = -(1 + (n-1) \log r)^{-1}$.

It is natural to suppose that Theorem 1 also holds for $n = 2$; indeed much of the proof of Theorem 1 holds without modification. However, in general when $n = 2$, we do not know whether $\log r$ or $e^{I(r)}$ is dominant as $r \rightarrow 0$, so the asymptotic analysis behind (42) (which is also used in deriving (36)) is more delicate.

Proof of Theorem 1. Instead of showing the existence of Z in a very small ball B_ε , we shall replace the condition that ω satisfies the square-Dini condition by

$$(49) \quad \sigma(1) = \int_0^1 \frac{\omega^2(t)}{t} dt < \delta,$$

where δ is sufficiently small, and show existence in B_1 . In fact, using (49) and (33), we see that

$$(50) \quad \omega(r) < c_{\kappa,n} \sqrt{\delta} \quad \text{for } 0 < r < 1,$$

where $c_{\kappa,n}$ depends only on κ and n :

$$\delta > \int_{r/2}^r \frac{\omega^2(t)}{t} dt \geq \omega^2(r) r^{-2+2\kappa} \int_{r/2}^r t^{1-2\kappa} dt = \omega^2(r) c'_{\kappa,n}.$$

Moreover, it will be useful to consider \mathcal{L} as defined on all of \mathbb{R}^n with $\mathcal{L} = \Delta$ outside of B_1 . Therefore, we shall assume that

$$(51) \quad a_{ij}(x) = \delta_{ij} \quad \text{for } |x| > 1,$$

and investigate a solution of $\mathcal{L}Z = 0$ in $\mathbb{R}^n \setminus \{0\}$.

To construct $Z(x)$, we let $h(r) = \overline{Z}(r)$ denote the spherical mean as in (27), and let $v(x) = Z(x) - h(|x|)$, so $\overline{v}(r) = 0$. We shall reduce the problem to solving an operator equation of the form $(I + S + T)v = f$ where S and T have small operator norm on a Banach space X defined as follows: for fixed $p \in (1, \infty)$, let us consider the functions v in $W_{loc}^{2,p}(\mathbb{R}^n \setminus \{0\})$ for which the norm

$$(52) \quad \|v\|_X = \sup_{0 < r < 1} \frac{M_{2,p}(v, r) r^{n-2}}{\omega(r) e^{I(r)}} + \sup_{r > 1} \frac{M_{2,p}(v, r) r^{n-1}}{\sqrt{\delta}}$$

is finite, and take the closure to form a Banach space X . (Note: if $n = 2$, the definition of X is still given by (52).) The reduction to the operator equation will take several steps.

If we take the spherical mean of $\mathcal{L}(h + v) = 0$, we obtain an ordinary differential equation for h :

$$\alpha(r)h'' + \frac{\alpha_n(r) - \alpha(r)}{r}h' + \overline{a_{ij}\partial_i\partial_j v}(r) = 0,$$

where

$$(53) \quad \alpha_n(r) = \int_{S^{n-1}} a_{ii}(r\theta) d\theta \quad \text{and} \quad \alpha(r) = \int_{S^{n-1}} a_{ij}(r\theta)\theta_i\theta_j d\theta.$$

Using the fact that $\overline{\Delta v} = \Delta \bar{v} = 0$, the equation for h becomes

$$(54) \quad \alpha(r)h'' + \frac{\alpha_n(r) - \alpha(r)}{r}h' + \overline{\beta_{ij}\partial_i\partial_j v}(r) = 0,$$

where $\beta_{ij}(x) = a_{ij}(x) - \delta_{ij}$ satisfies $|\beta_{ij}(x)| \leq \omega(|x|)$ by (32). Notice that v satisfies the partial differential equation

$$(55) \quad -\Delta v = \beta_{ij}\partial_i\partial_j h - \overline{\beta_{ij}\partial_i\partial_j h} + \beta_{ij}\partial_i\partial_j v - \overline{\beta_{ij}\partial_i\partial_j v}.$$

We shall first solve (54) and use that to eliminate h from (55); then we will be able to apply the results of Section 1.

Let us introduce $g = h'$ and rewrite (54) as

$$(56) \quad g' + \frac{n-1+R(r)}{r}g = B[D^2v](r),$$

where

$$(57) \quad R(r) = \frac{\alpha_n(r)}{\alpha(r)} - n$$

and

$$B[D^2v](r) = -\alpha^{-1}(r)\overline{\beta_{ij}\partial_i\partial_j v}(r).$$

It follows from (32) that $|\alpha_n(r) - n| \leq \omega(r)$ and $|\alpha(r) - 1| \leq \omega(r)$ for $0 < r < 1/2$, and from (51) that $R(r) = 0$ for $r > 1$; consequently,

$$(58) \quad |R(r)| \leq c\omega(r) \quad \text{for } 0 < r < \infty$$

where we have defined $\omega(r) = 0$ for $r > 1$. Similarly,

$$|B[D^2v](r)| \leq c\omega(r)|\overline{D^2v}(r)| \quad \text{for } 0 < r < \infty;$$

and since the monotonicity of $\omega(r)$ together with (33) imply

$$\max_{r \leq \rho \leq 2r} \omega(\rho) \leq c\omega(r),$$

we consequently obtain

$$(59) \quad M_p(B[D^2v], r) \leq c\omega(r)M_p(D^2v, r) \quad \text{for } 0 < r < \infty.$$

To solve (56), let us introduce

$$(60) \quad E_{\pm}(r) = \exp \left[\pm \int_r^\infty \frac{R(t)}{t} dt \right] = \exp \left[\pm \int_r^1 \frac{R(t)}{t} dt \right] = \frac{1}{E_{\mp}(r)}$$

and observe that $E_{\pm}(r) \equiv 1$ for $r > 1$. It will be more convenient for us to use $E_{\pm}(r)$ than $e^{\pm I(r)}$, but these functions are equivalent: if we note that (15) can be written as

$$(61) \quad I(r) = \int_r^1 [\alpha_n(\rho) - n\alpha(\rho)] \frac{d\rho}{\rho},$$

then we see that

$$(62) \quad E_+(r) = A e^{I(r)}(1 + \tau(r)),$$

where $A = \exp[\int_0^1 R(\rho)[1 - \alpha(\rho)]\rho^{-1}d\rho]$ is finite and positive, and

$$\tau(r) = \exp \left[- \int_0^r R(\rho)(1 - \alpha(\rho)) \frac{d\rho}{\rho} \right] - 1$$

satisfies $|\tau(r)| \leq c\sigma(r)$. Thus for some constants c_1, c_2 we have

$$(63) \quad c_1 E_+(r) \leq e^{I(r)} \leq c_2 E_+(r) \quad \text{for } 0 < r < 1.$$

Now if we introduce $\phi(r) = r^{n-1} E_-(r)g(r)$, then we can rewrite (56) as

$$(64) \quad \phi'(r) = r^{n-1} E_-(r)B[D^2v](r).$$

But (64) may be integrated to obtain

$$(65) \quad \phi(r) = \phi(0) + \int_0^r \rho^{n-1} E_-(\rho)B[D^2v](\rho) d\rho,$$

where $\phi(0)$ is an arbitrary constant. Of course, to conclude (65), we must verify that ϕ' is integrable on $(0, 1)$. But $v \in X$ implies $M_p(D^2v, r) \leq c\omega(r)r^{-n}E_+(r)$, so we can use (59) to conclude $M_p(B[D^2v], r) \leq c\omega(r)r^{-n}E_+(r)$, and then invoke Hölder's inequality to obtain

$$\int_r^{2r} \rho^{n-1} E_-(\rho)B[D^2v](\rho) d\rho \leq c\omega^2(r).$$

Now if we write

$$\int_0^s \rho^{n-1} E_-(\rho)B[D^2v](\rho) d\rho = \sum_{j=0}^{\infty} \int_{s/2^{j+1}}^{s/2^j} \rho^{n-1} E_-(\rho)B[D^2v](\rho) d\rho,$$

then we obtain the estimate

$$(66) \quad \left| \int_0^s \rho^{n-1} E_-(\rho)B[D^2v](\rho) d\rho \right| \leq c \sum_{j=0}^{\infty} \omega^2 \left(\frac{s}{2^{j+1}} \right) \leq c \int_0^s \frac{\omega^2(\rho)}{\rho} d\rho = c\sigma(s) < c\delta.$$

In particular, we see that ϕ' is integrable and so (65) is valid. This enables us to write

$$(67) \quad h'(r) = g(r) = r^{1-n} E_+(r) \left[\phi(0) + \int_0^r \rho^{n-1} E_-(\rho)B[D^2v](\rho) d\rho \right]$$

and

$$(68) \quad h''(r) = (1 - n - R(r)) r^{-n} E_+(r) \left[\phi(0) + \int_0^r \rho^{n-1} E_-(\rho)B[D^2v](\rho) d\rho \right] + B[D^2v](r).$$

We can now use (67) and (68) to express

$$\beta_{ij} \partial_i \partial_j h = r^{-n} E_+(r) \left[\phi(0) + \int_0^r \rho^{n-1} E_-(\rho)B[D^2v](\rho) d\rho \right] \sigma(r\theta) + B[D^2v](r) \beta_{ij} \theta_i \theta_j,$$

where $\psi(r\theta) = \beta_{ii}(r\theta) - (n + R(r))\beta_{ij}(r\theta)\theta_i\theta_j$ also satisfies $|\psi(r\theta)| \leq c\omega(r)$. Thus we can apply $K = \Gamma\star$ to (55) to obtain

$$(69) \quad v + Sv + Tv = \phi(0)w,$$

where

$$w(x) = K_{y \rightarrow x} [|y|^{-n} E_+(|y|)(\psi(y) - \overline{\psi}(|y|))],$$

$$Sv(x) = -K_{y \rightarrow x} \left[|y|^{-n} E_+(|y|) \int_0^{|y|} \rho^{n-1} E_-(\rho)B[D^2v](\rho) d\rho (\psi(y) - \overline{\psi}(|y|)) \right],$$

and

$$Tv = -K [B[D^2v] (\beta_{ij}\theta_i\theta_j - \overline{\beta_{ij}\theta_i\theta_j}) + \beta_{ij}\partial_i\partial_jv - \overline{\beta_{ij}\partial_i\partial_jv}].$$

For a given value of $\phi(0)$, we can solve (69) provided we show that $w \in X$ and both $S : X \rightarrow X$ and $T : X \rightarrow X$ have small operator norms.

To show $w \in X$, we must estimate $M_{2,p}(w, r)$ for $0 < r < 1$ and for $r > 1$. For this it is useful to observe that

$$E_-(r)E_+(\rho) = \exp\left(\int_\rho^r \frac{R(t)}{t} dt\right),$$

so as a consequence of (58) and (50), we obtain

$$(70) \quad \left(\frac{\rho}{r}\right)^{c\sqrt{\delta}} \leq \exp\left(\pm \int_\rho^r \frac{R(t)}{t} dt\right) \leq \left(\frac{r}{\rho}\right)^{c\sqrt{\delta}} \quad \text{for } 0 < \rho \leq r \leq 1.$$

In particular, $c_1 E_\pm(r) \leq E_\pm(2r) \leq c_2 E_\pm(r)$ as $r \rightarrow 0$, and for any $g \in L_{loc}^p(\mathbb{R}^n \setminus \{0\})$ we can readily verify that

$$(71) \quad M_p(|x|^\nu E_\pm(|x|)g(x), r) \leq c r^\nu E_\pm(r) M_p(g, r),$$

for any fixed $\nu \in \mathbb{R}$. For $0 < r < 1$ we use Proposition 2 with $f(x) = |x|^{-n} E_+(|x|)(\psi(x) - \bar{\psi}(|x|))$ (which vanishes for $|x| > 1$) to conclude

$$M_{2,p}(w, r) \leq c \left(r^{1-n} \int_0^r E_+(\rho)\omega(\rho) d\rho + r \int_r^1 E_+(\rho)\omega(\rho)\rho^{-n} d\rho \right).$$

Using (70) and the fact that $r^{1-\kappa}\omega(r)$ is nondecreasing (since both $r^{1-\kappa}$ and ω are), we find

$$\int_0^r E_+(\rho)\omega(\rho) d\rho \leq E_+(r) r^{c\sqrt{\delta}} \int_0^r \rho^{-c\sqrt{\delta}}\omega(\rho) d\rho \leq E_+(r) \omega(r) r^{c\sqrt{\delta}+1-\kappa} \int_0^r \rho^{-c\sqrt{\delta}-1+\kappa} d\rho.$$

Taking δ small enough that $\varepsilon - \sqrt{c\delta} > 0$, we obtain

$$(72) \quad \int_0^r E_+(\rho)\omega(\rho) d\rho \leq c r E_+(r) \omega(r).$$

Using (70) and the fact that $r^{-1+\kappa}\omega(r)$ is nonincreasing (by (33)), we find

$$\int_r^1 E_+(\rho)\omega(\rho)\rho^{-n} d\rho \leq E_+(r) r^{-c\sqrt{\delta}} \int_r^1 \rho^{c\sqrt{\delta}-n}\omega(\rho) d\rho \leq E_+(r) \omega(r) r^{-c\sqrt{\delta}-1+\kappa} \int_r^1 \rho^{c\sqrt{\delta}+1-\kappa-n} d\rho.$$

For δ small enough that $n - 2 + \kappa - c\sqrt{\delta} > 0$, we have

$$(73) \quad \int_r^1 E_+(\rho)\omega(\rho)\rho^{-n} d\rho \leq c r^{1-n} E_+(r) \omega(r).$$

Using (72) and (73), we obtain

$$\frac{M_{2,p}(w, r)r^{n-2}}{\omega(r)E_+(r)} \leq c \quad \text{for all } 0 < r < 1.$$

We can then use (63) to replace $E_+(r)$ by $e^{I(r)}$ as required in the norm for X in (52). Meanwhile, for $r > 1$ we use $\omega(r)$ nondecreasing, $\omega(1) \leq c\sqrt{\delta}$, and $E_+(\rho) \leq \rho^{-c\sqrt{\delta}}$ to conclude

$$M_{2,p}(w, r) \leq c r^{1-n} \int_0^1 E_+(\rho)\omega(\rho) d\rho \leq c\sqrt{\delta} r^{1-n} \int_0^1 \rho^{-c\sqrt{\delta}} d\rho \leq c\sqrt{\delta} r^{1-n},$$

provided δ is sufficiently small. Consequently,

$$\frac{M_{2,p}(w, r)r^{n-1}}{\sqrt{\delta}} \leq c \quad \text{for all } r > 1,$$

and this confirms that $w \in X$.

Next let us show that S maps X to itself with small operator norm. We suppose that $\|v\|_X \leq 1$ and estimate $M_{2,p}(Sv, r)$ separately for $0 < r < 1$ and for $r > 1$. For $0 < r < 1$, we examine

the proof of (66) and observe that the condition $\|v\|_X \leq 1$ enables us to choose the constant c independent of v . Thus

$$f_1(y) = |y|^{-n} E_+(|y|) \int_0^{|y|} \rho^{n-1} E_-(\rho) B[D^2 v](\rho) d\rho (\sigma(y) - \bar{\sigma}(|y|))$$

satisfies

$$M_p(f_1, r) \leq c \delta E_+(r) \omega(r) r^{-n} \text{ for } 0 < r < 1$$

and $M_p(f_1, r) = 0$ for $r > 1$. For $0 < r < 1$, we apply Proposition 2 to $Sv = -Kf_1$ to obtain

$$M_{2,p}(Sv, r) \leq c \delta \left(r^{1-n} \int_0^r E_+(\rho) \omega(\rho) d\rho + r \int_r^1 E_+(\rho) \omega(\rho) \rho^{-n} d\rho \right),$$

and then use (72) and (73) to conclude (for δ sufficiently small)

$$\frac{M_{2,p}(Sv, r) r^{n-2}}{\omega(r) E_+(r)} \leq c \delta \text{ for all } 0 < r < 1.$$

On the other hand, for $r > 1$, Proposition 2 implies (for δ sufficiently small)

$$M_{2,p}(Sv, r) \leq c \delta r^{1-n} \int_0^1 E_+(\rho) \omega(\rho) d\rho \leq c \delta^{3/2} r^{1-n} \int_0^1 \rho^{-c\sqrt{\delta}} d\rho \leq c \delta^{3/2} r^{1-n}.$$

Thus we have

$$\frac{M_{2,p}(Sv, r) r^{n-1}}{\sqrt{\delta}} \leq c \delta \text{ for all } r > 1.$$

Combining these inequalities, we see that $S : X \rightarrow X$ has small operator norm.

Finally, we show that T maps X to itself with small operator norm. We suppose that $\|v\|_X \leq 1$ and estimate $M_{2,p}(Tv, r)$ separately for $0 < r < 1$ and for $r > 1$. Notice that

$$f_2 = B[D^2 v] (\beta_{ij} \theta_i \theta_j - \overline{\beta_{ij} \theta_i \theta_j})$$

satisfies

$$M_p(f_2, r) \leq \omega(r) M_p(B[D^2 v], r) \leq c \omega^3(r) E_+(r) r^{-n} \text{ for } 0 < r < 1,$$

where c is independent of v , and $M_p(f_2, r) = 0$ for $r > 1$. Similarly,

$$f_3 = \beta_{ij} \partial_i \partial_j v - \overline{\beta_{ij} \partial_i \partial_j v}$$

satisfies

$$M_p(f_3, r) \leq \omega(r) M_p(D^2 v, r) \leq \omega^2(r) E_+(r) r^{-n} \text{ for } 0 < r < 1,$$

and $M_p(f_3, r) = 0$ for $r > 1$. For $0 < r < 1$, we apply Proposition 2 to $Tv = -K(f_2 + f_3)$ to obtain

$$M_{2,p}(Tv, r) \leq c \left(r^{1-n} \int_0^r \omega^2(\rho) E_+(\rho) d\rho + r \int_r^1 \omega^2(\rho) E_+(\rho) \rho^{-n} d\rho \right).$$

Using (50), (72), and (73),

$$\frac{M_{2,p}(Tv, r) r^{n-2}}{\omega(r) E_+(r)} \leq c \sqrt{\delta} \text{ for all } 0 < r < 1.$$

On the other hand, for $r > 1$, we use (50) and (70) to estimate

$$M_{2,p}(Tv, r) \leq c r^{1-n} \int_0^1 \omega^2(\rho) E_+(\rho) d\rho \leq c \delta r^{1-n} \int_0^1 \rho^{-c\sqrt{\delta}} d\rho \leq c \delta r^{1-n}.$$

Consequently,

$$\frac{M_{2,p}(Tv, r) r^{n-1}}{\sqrt{\delta}} \leq c \sqrt{\delta} \text{ for all } r > 1.$$

Combining these estimates, we see that $T : X \rightarrow X$ has small operator norm.

Since both S and T have small operator norms on X , we conclude that (69) has a unique solution v , depending on the choice of the constant $c_1 = \phi(0)$. But once c_1 and v are known, we obtain $g(r)$ from (67), and $h(r)$ by integration of $g(r)$:

$$(74) \quad h(r) = \int_r^\infty s^{1-n} E_+(s) \left[c_1 + \int_0^s \rho^{n-1} E_-(\rho) B[D^2 v](\rho) d\rho \right] ds + c_2.$$

where c_2 is an arbitrary constant. To obtain the desired solution of Theorem I, we choose c_1 to enable us to replace $E_+(r)$ by $e^{I(r)}$ for $0 < r < 1$. Using (62) we see that we should choose $c_1 = A^{-1}$ and write $h(r) = h_0(r) + h_1(r) + c$ where

$$(75) \quad h_0(r) = \int_r^1 s^{1-n} e^{I(s)} ds$$

and

$$(76) \quad h_1(r) = \int_r^1 s^{1-n} e^{I(s)} \tau(s) ds + \int_r^1 s^{1-n} E_+(s) \int_0^s \rho^{n-1} E_-(\rho) B[D^2 v](\rho) d\rho ds.$$

Now integrate by parts to obtain

$$h_0(r) = \frac{r^{2-n}}{n-2} e^{I(r)} + c + \frac{1}{n-2} \int_r^1 s^{2-n} e^{I(s)} I'(s) ds.$$

But (61) implies $|I'(s)| \leq c\omega(s)/s$ and similar to (70) we can show that

$$e^{I(s)} e^{-I(r)} \leq \left(\frac{s}{r} \right)^{c\sqrt{\delta}} \quad \text{for } s > r,$$

so we may use (33) to obtain

$$\begin{aligned} \left| \int_r^1 s^{2-n} e^{I(s)} I'(s) ds \right| &\leq c \int_r^1 s^{1-n} e^{I(s)} \omega(s) ds \leq c r^{-c\sqrt{\delta}} e^{I(r)} \int_r^1 s^{1-n+c\sqrt{\delta}} \omega(s) ds \\ &\leq c\omega(r) e^{I(r)} r^{-c\sqrt{\delta}-1+\kappa} [r^{3-n+c\sqrt{\delta}-\kappa} + 1] \leq c r^{2-n} \omega(r) e^{I(r)} \end{aligned}$$

provided δ is sufficiently small. Thus we find

$$(77) \quad \left| h_0(r) - \frac{r^{2-n}}{n-2} e^{I(r)} \right| \leq c r^{2-n} e^{I(r)} \omega(r) \quad \text{for } 0 < r < 1.$$

To estimate h_1 we use $|\tau(s)| \leq c\sigma(s)$ together with (66) and a similar analysis to the above to obtain

$$(78) \quad |h_1(r)| \leq c r^{2-n} e^{I(r)} \max(\omega(r), \sigma(r)) \quad \text{for } 0 < r < 1.$$

Define $\zeta(r)$ by

$$(79) \quad \zeta(r) = \frac{h_1(r)}{h_0(r)} \quad \text{for } 0 < r < 1.$$

Using (75) – (78) we can estimate $|\zeta(r)|, |r\zeta'(r)| \leq c \max(\omega(r), \sigma(r))$. To estimate ζ'' , we write $h_0\zeta'' = h_1'' - h_0''\zeta - 2h_0'\zeta'$. The terms $h_0''\zeta$ and $2h_0'\zeta'$ may be estimated pointwise as before, but $h_1''(r)$ involves the term $B[D^2 v](r)$, which cannot be estimated pointwise. However, we know $M_p(B[D^2 v], r) \leq c\omega(r)r^{-n}e^{I(r)}$, so we conclude $M_p(r^2\zeta'', r) \leq c \max(\omega(r), \sigma(r))$. Putting this together with the lower order derivatives, we obtain the desired estimate, $M_{2,p}(\zeta, r) \leq c \max(\omega(r), \sigma(r))$. Summarizing so far, we have found a solution Z of (34) in the desired form (35).

Next we need to verify that any strong solution $u \in W_{loc}^{2,p}(\overline{B_1} \setminus \{0\})$ of $\mathcal{L}u = 0$ that satisfies the growth estimate (39) must be of the form (40). To do this, we shall invoke well-known results for weighted Sobolev spaces. To begin with, let us introduce the weighted L^p -norm on $B_\circ = B_1 \setminus \{0\}$:

$$(80) \quad \|u\|_{L_\beta^p(B_\circ)}^p = \int_{0 < |x| < 1} |x|^{\beta p} |u(x)|^p dx.$$

To relate this to the M_p -norm, notice that

$$\int_{0 < |x| < 1} |x|^{\beta p} |u(x)|^p dx = \sum_{k=1}^{\infty} \int_{2^{-k} < |x| < 2^{-k+1}} |x|^{\beta p} |u(x)|^p dx.$$

Moreover, there exist constants c_1, c_2 (independent of k) such that

$$c_1 2^{-k(\beta p + n)} M_p(u, 2^{-k})^p \leq \int_{2^{-k} < |x| < 2^{-k+1}} |x|^{\beta p} |u(x)|^p dx \leq c_2 2^{-k(\beta p + n)} M_p(u, 2^{-k})^p.$$

Consequently, $M_p(u, r) \leq c r^\alpha$ for $0 < r < 1$ implies $u \in L_\beta^p(B_\circ)$ if $\alpha + \beta > -n/p$, and conversely, $u \in L_\beta^p(B_\circ)$ implies $M_p(u, 2^{-k}) \leq c 2^{k\beta + kn/p}$ which implies $M_p(u, r) \leq c_\alpha r^\alpha$ for $0 < r < 1$ if we choose $\alpha = -\beta - n/p$. We obtain analogous relationships between the M_p -norm and the L_β^p -norm of the terms $|x|^{|\alpha|} |\partial^\alpha u(x)|$ for $|\alpha| \leq 2$.

Now let us introduce a weighted L^p -norm for functions on $\mathbb{R}_\circ^n = \mathbb{R}^n \setminus \{0\}$ with separate weights at the origin and infinity:

$$(81) \quad \|u\|_{L_{\beta,\gamma}^p(\mathbb{R}_\circ^n)}^p = \|u\|_{L_\beta^p(B_\circ)}^p + \|u\|_{L_\gamma^p(B^c)}^p = \int_{0 < |x| < 1} |u(x)|^p |x|^{\beta p} dx + \int_{|x| > 1} |u(x)|^p |x|^{\gamma p} dx,$$

where $B^c = \mathbb{R}^n \setminus \overline{B_1}$. We then define the weighted Sobolev space $W_{\beta,\gamma}^{2,p}(\mathbb{R}_\circ^n)$ to be those functions in $W_{loc}^{2,p}(\mathbb{R}_\circ^n)$ for which the norm

$$(82) \quad \|u\|_{W_{\beta,\gamma}^{2,p}} = \sum_{|\alpha| \leq 2} \| |x|^{|\alpha|} \partial^\alpha u \|_{L_{\beta,\gamma}^p(\mathbb{R}_\circ^n)}$$

is finite. Many authors have used similar weighted Sobolev spaces to study operators like the Laplacian on \mathbb{R}^n , \mathbb{R}_\circ^n , and other noncompact manifolds with conical or cylindrical ends. Using the analysis in [19], [20] or [17], for example, it is easily verified that the bounded operator

$$(83) \quad \Delta : W_{\beta,\gamma}^{2,p}(\mathbb{R}_\circ^n) \rightarrow L_{\beta+2,\gamma+2}^p(\mathbb{R}_\circ^n)$$

is Fredholm (finite nullity and finite deficiency) for all values of β and γ *except* for the values $-2 + \frac{n}{q} + k$ and $-\frac{n}{p} - k$ where $q = p/(p-1)$ and k is any nonnegative integer. In fact, (83) is an isomorphism for $-n/p < \beta, \gamma < -2 + n/q$ (recall that we are assuming $n \geq 3$, so such β, γ exist). Since we are principally interested in the behavior of functions at the origin, we will fix $\gamma_0 \in (-n/p, -2 + n/q)$. Then

$$\beta_- < \beta_+ \quad \Rightarrow \quad W_{\beta_-, \gamma_0}^{2,p}(\mathbb{R}_\circ^n) \subset W_{\beta_+, \gamma_0}^{2,p}(\mathbb{R}_\circ^n).$$

Moreover, for $\beta_+ \in (-2 + n/q, -1 + n/q)$, we find that (83) is surjective with a one-dimensional nullspace spanned by $|x|^{2-n} \in W_{\beta,\gamma_0}^{2,p}(\mathbb{R}_\circ^n)$; for $\beta_- \in (-n/p - 2, -n/p - 1)$, we find that (83) is injective with codimension equal to $n+1$.

Next we use perturbation theory (cf. [11], Ch.IV, Sec.5) to conclude not only that the operator

$$(84) \quad \mathcal{L} : W_{\beta,\gamma}^{2,p}(\mathbb{R}_\circ^n) \rightarrow L_{\beta+2,\gamma+2}^p(\mathbb{R}_\circ^n)$$

is Fredholm for exactly the same values of β and γ as for (83), but the nullity and deficiency of (83) and (84) agree, provided δ is sufficiently small. So, in addition to the fixed $\gamma_0 \in (-n/p, -2 + n/q)$,

let us now choose $\beta_0 \in (-n/p, -2 + n/q)$, as well as $\beta_- \in (-1 - n/p, -n/p)$ and $\beta_+ \in (-2 + n/q, -1 + n/q)$. Then we assume that δ is so small that

$$(85) \quad \mathcal{L} : W_{\beta_0, \gamma_0}^{2,p}(\mathbb{R}_o^n) \rightarrow L_{\beta_0+2, \gamma_0+2}^p(\mathbb{R}_o^n) \quad \text{is an isomorphism,}$$

$$(86) \quad \mathcal{L} : W_{\beta_-, \gamma_0}^{2,p}(\mathbb{R}_o^n) \rightarrow L_{\beta_-+2, \gamma_0+2}^p(\mathbb{R}_o^n) \quad \text{is injective with } \text{codim} = n + 1,$$

and

$$(87) \quad \mathcal{L} : W_{\beta_+, \gamma_0}^{2,p}(\mathbb{R}_o^n) \rightarrow L_{\beta_++2, \gamma_0+2}^p(\mathbb{R}_o^n) \quad \text{is surjective with nullity} = 1.$$

We claim that $Z \in W_{\beta_+, \gamma_0}^{2,p}(\mathbb{R}_o^n)$. In fact, this is quite simple to check given the explicit formulas (75) and (76), and the fact that $v \in X$, where X has the norm (52). Thus Z is a basis vector for the one-dimensional nullspace of (87).

Now suppose $u \in W_{loc}^{2,p}(\overline{B_1} \setminus \{0\})$ satisfies $\mathcal{L}u = 0$ and the growth estimate (39) with $\varepsilon_0 \in (0, 1)$. Introduce a cut-off function $\chi \in C_0^\infty(B_1)$ equal to 1 on $B_{1/2}$. If we now specify that $\beta_+ \in (-1 - \varepsilon_0 + n/q, -1 + n/q)$, then $\chi u \in W_{\beta_+, \gamma_0}^{2,p}(\mathbb{R}_o^n)$. Let $f = \mathcal{L}(\chi u)$. Since $f = 0$ for $|x| < 1/2$ and for $|x| > 1$, $f \in L_{\beta_0+2, \gamma_0+2}^p(\mathbb{R}_o^n)$. But (85) is an isomorphism, so we can find $v = \mathcal{L}^{-1}f \in W_{\beta_0, \gamma_0}^{2,p}(\mathbb{R}_o^n)$. Now $\chi u - v \in W_{\beta_+, \gamma_0}^{2,p}(\mathbb{R}_o^n)$ satisfies $\mathcal{L}(\chi u - v) = 0$. Since the nullspace of (87) is spanned by Z , there exists a constant C such that $\chi u - v = CZ$. But this means in particular that $u = CZ + v$ for $0 < |x| < 1/2$.

Now let us describe v asymptotically. Let $\zeta_0, \zeta_1, \dots, \zeta_n$ denote a basis for the cokernel of (86), i.e. the ζ_i are linear functionals on $L_{\beta_-+2, \gamma_0+2}^p(\mathbb{R}_o^n)$ that are linearly independent and vanish on the image of (86). Now we want to choose C_0, C_1, \dots, C_n so that $\mathcal{L}(\chi(v - C_0 - \sum_{j=1}^n C_j x_j))$ is in the range of (86), i.e.

$$(88) \quad \zeta_i[\mathcal{L}(\chi(C_0 + \sum_{j=1}^n C_j x_j))] = \zeta_i[\mathcal{L}(\chi v)] \quad \text{for } i = 0, \dots, n.$$

To be able to solve the linear system (88) for C_0, \dots, C_n , we need to verify that the finite-dimensional linear map

$$(89) \quad (C_0, \dots, C_n) \rightarrow \left(\zeta_0[\mathcal{L}(\chi(C_0 + \sum_{j=1}^n C_j x_j))], \dots, \zeta_n[\mathcal{L}(\chi(C_0 + \sum_{j=1}^n C_j x_j))] \right)$$

is nonsingular. Suppose (C_0, \dots, C_n) satisfies $\zeta_j[\mathcal{L}(\chi(C_0 + \sum_{j=1}^n C_j x_j))] = 0$ for $j = 0, \dots, n$. Then $[\mathcal{L}(\chi(C_0 + \sum_{j=1}^n C_j x_j))]$ is in the range of (86) and there exists $\psi \in W_{\beta_-, \gamma_0}^{2,p}(\mathbb{R}_o^n)$ such that $\mathcal{L}(\psi) = \mathcal{L}(\chi(C_0 + \sum_{j=1}^n C_j x_j))$, i.e. $\mathcal{L}(\psi - \chi(C_0 + \sum_{j=1}^n C_j x_j)) = 0$. But $\psi - \chi(C_0 + \sum_{j=1}^n C_j x_j) \in W_{\beta_0, \gamma_0}^{2,p}(\mathbb{R}_o^n)$ and (85) is an isomorphism, so $\psi = \chi(C_0 + \sum_{j=1}^n C_j x_j)$. But χ and χx_j are *not* in $W_{\beta_-, \gamma_0}^{2,p}(\mathbb{R}_o^n)$, so the only way that we can have $\psi = \chi(C_0 + \sum_{j=1}^n C_j x_j)$ is to have $C_0 = C_1 = \dots = C_n = 0$. Consequently, the linear map (89) is nonsingular.

Thus we can find C_0, \dots, C_n and $w \in W_{\beta_-, \gamma_0}^{2,p}(\mathbb{R}_o^n)$ so that $\mathcal{L}(w) = \mathcal{L}(\chi(v - C_0 - C_1 x_1 - \dots - C_n x_n))$. But (85) is an isomorphism, so we obtain $v = C_0 + C_1 x_1 + \dots + C_n x_n + w$ for $0 < |x| < 1/2$. This yields (40) and we only need to verify $M_{2,p}(w, r) \leq cr^{2-\varepsilon_1}$. But recall that $w \in W_{\beta_-, \gamma_0}^{2,p}(\mathbb{R}_o^n)$ implies $M_{2,p}(w, r) \leq cr^{-\beta_- - n/p}$ for $0 < r < 1/2$, and we can then let $\beta_- = -2 - n/p + \varepsilon_1$ for any $\varepsilon_1 \in (0, 1)$ to obtain the desired estimate. \square

Now let us formulate the result for a general point $y \in \mathbb{R}^n$ where we do not assume $a_{ij}(y) = \delta_{ij}$. With y fixed and the same conditions (9) and (33) on ω , we now assume

$$(90) \quad \sup_{|x-y|=r} \|\mathbf{A}_x - \mathbf{A}_y\| \leq \omega(r) \quad \text{for } 0 < r < 1,$$

and we want to construct a solution of

$$(91) \quad \mathcal{L}(x, \partial_x)Z_y(x) = 0 \quad \text{for } x \in B_\varepsilon(y) \setminus \{y\},$$

for ε sufficiently small. (Unlike Theorem 1, in the Corollary below we need to assume that the coefficients are real-valued so that we can choose coordinates in which $a_{ij}(0) = \delta_{ij}$.)

Corollary 1. *For $n \geq 3$ and $p \in (1, \infty)$, fix $y \in \mathbb{R}^n$ and suppose that the constant coefficient operator $\mathcal{L}(y, \partial_x)$ is elliptic and the coefficients $a_{ij}(x)$ are bounded, measurable functions satisfying (90). For $\varepsilon > 0$ sufficiently small, there exists a solution of (91) in the form*

$$(92) \quad Z_y(x) = h_y(|\mathbf{A}_y^{-1/2}(x-y)|) + v(x)$$

where h_y is defined by

$$(93) \quad h_y(r) = \int_r^\varepsilon s^{1-n} e^{I_y(s)} ds (1 + \zeta_y(r)),$$

with $I_y(r)$ given by (13) and

$$(94) \quad M_{2,p}(\zeta_y, r; y) \leq c \max(\omega(r), \sigma(r)),$$

and v in (92) satisfies

$$(95) \quad M_{2,p}(v, r; y) \leq c r^{2-n} e^{I_y(r)} \omega(r).$$

Moreover, for any $u \in W_{loc}^{2,p}(\overline{B_\varepsilon(y)} \setminus \{y\})$ that is a strong solution of $\mathcal{L}(x, \partial_x)u = 0$ in $\overline{B_\varepsilon(y)} \setminus \{y\}$ subject to the growth condition

$$M_{2,p}(u, r; y) \leq c r^{1-n+\varepsilon_0}$$

where $\varepsilon_0 > 0$, there exist constants C, C_0, C_1, \dots, C_n (depending on u) such that

$$(96) \quad u(x) = CZ_y(x) + C_0 + \sum_{j=1}^n C_j x_j + w(x) \quad \text{for } 0 < |x-y| < \varepsilon,$$

where w satisfies

$$(97) \quad M_{2,p}(w, r) \leq c r^{2-\varepsilon_1} \quad \text{for any } \varepsilon_1 > 0.$$

If we use integration by parts, we can write the solution in Corollary 1 as

$$(98) \quad Z_y(x) = \frac{\langle \mathbf{A}_y^{-1}(x-y), (x-y) \rangle^{\frac{2-n}{2}}}{(n-2)} e^{I_y(\sqrt{\langle \mathbf{A}_y^{-1}(x-y), (x-y) \rangle})} (1 + \xi_y(x)),$$

where $M_{1,\infty}(\xi_y, r; y) \leq c \max(\omega(r), \sigma(r))$ for $0 < r < \varepsilon$. Notice that, if $I_y(r)$ has a finite limit as $r \rightarrow 0$, then the leading term in (98) is just a constant times $F_y(x-y)$, the fundamental solution for $\mathcal{L}(y, \partial_x)$ (cf. (5)).

Proof of Corollary 1. For now we continue to assume $y = 0$, i.e.

$$(99) \quad \sup_{|x|=r} \|\mathbf{A}_x - \mathbf{A}_0\| \leq \omega(r),$$

and let us assume \mathbf{A}_0 is positive definite. Let $\mathbf{B} = (b_{ij}) = \mathbf{A}_0^{-1/2}$ so that $\mathbf{B}\mathbf{A}_0\mathbf{B} = \mathbf{I}$. Introduce new independent variables $\tilde{x} = \mathbf{B}x$ and the matrix $\tilde{\mathbf{A}}_{\tilde{x}} = (\tilde{a}_{ij}(\tilde{x})) = \mathbf{B}\mathbf{A}_x\mathbf{B}$, which satisfies $\tilde{a}_{ij}(0) = \delta_{ij}$. Using $\partial/\partial x_i = b_{ik}\partial/\partial \tilde{x}_k = b_{ki}\partial/\partial \tilde{x}_k$, we can then write

$$(100) \quad \mathcal{L}(x, \partial_x) = a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} = \tilde{a}_{kl}(\tilde{x}) \frac{\partial^2}{\partial \tilde{x}_k \partial \tilde{x}_l} = \tilde{\mathcal{L}}(\tilde{x}, \partial_{\tilde{x}})$$

and apply Theorem 1 to $\tilde{\mathcal{L}}$ in the coordinates \tilde{x} . We conclude that for $\varepsilon > 0$ sufficiently small, there is a solution \tilde{Z} of $\tilde{\mathcal{L}}(\tilde{x}, \partial_{\tilde{x}})\tilde{Z}(\tilde{x}) = 0$ for $0 < |\tilde{x}| < \varepsilon$, of the form $\tilde{Z}(\tilde{x}) = \tilde{h}(|\tilde{x}|) + \tilde{v}(\tilde{x})$ where $M_{2,p}(\tilde{v}, r) \leq c r^{2-n} e^{I(r)} \omega(r)$ and $\tilde{h}(r)$ is of the form (36) with

$$(101) \quad I(r) = \frac{1}{|S^{n-1}|} \int_{r < |\tilde{z}| < \varepsilon} \left(\operatorname{tr} \tilde{\mathbf{A}}_{\tilde{z}} - n \langle \tilde{\mathbf{A}}_{\tilde{z}} \tilde{z}, \tilde{z} \rangle |\tilde{z}|^{-2} \right) \frac{d\tilde{z}}{|\tilde{z}|^n}.$$

Expressed in terms of the original variables x , we obtain $Z(x) = \tilde{Z}(\tilde{x}) = \tilde{h}(|\mathbf{B}x|) + \tilde{v}(\mathbf{B}x)$ that satisfies $\mathcal{L}(x, \partial_x)Z(x) = 0$ for $0 < |\mathbf{B}x| < \varepsilon$; but choosing ε_0 sufficiently small, we conclude that $\mathcal{L}(x, \partial_x)Z(x) = 0$ for $0 < |x| < \varepsilon_0$.

Finally, if y is a general point in \mathbb{R}^n , then let $\tilde{x} = \mathbf{B}(x - y)$ with $\mathbf{B} = \mathbf{A}_y^{-1/2}$ and let $\tilde{\mathbf{A}}_{\tilde{x}} = (\tilde{a}_{ij}(\tilde{x})) = \mathbf{B} \mathbf{A}_x \mathbf{B}$; since $\tilde{x} = 0$ corresponds to $x = y$, we have $\tilde{a}_{ij}(0) = \delta_{ij}$ and we can apply Theorem I to $\tilde{\mathcal{L}}(\tilde{x}, \partial_{\tilde{x}}) = \mathcal{L}(x, \partial_x)$. We obtain the solution $Z(x) = \tilde{h}(|\mathbf{B}(x - y)|) + \tilde{v}(\mathbf{B}(x - y))$ where $\tilde{h}(r)$ involves $I(r)$ as in (101). To transform (101) to the original variables, replace $\tilde{\mathbf{A}}_{\tilde{z}}$ by \mathbf{A}_z and every other occurrence of \tilde{z} by $z - y$; we find that (101) is of the desired form (13), so we may let $h_y(r) = \tilde{h}(r)$. Moreover, since \tilde{v} satisfies $M_{2,p}(\tilde{v}, r) \leq c r^{2-n} e^{I(r)} \omega(r)$, it is clear that $v(x) = \tilde{v}(\mathbf{B}(x - y))$ satisfies $M_{2,p}(v, r; y) \leq c r^{2-n} e^{I(r)} \omega(r)$, completing the proof. \square

3. FINDING THE CONSTANT C_y SO THAT $\mathcal{L}Z_y(x) = C_y \delta(x - y)$ IN $B_\varepsilon(y)$

Let us now begin to discuss the role that $Z_y(x)$ plays in finding the fundamental solution. As before, we first consider $y = 0$ with $a_{ij}(0) = \delta_{ij}$ and then use a change of variables to consider a general fixed $y \in \mathbb{R}^n$; as in the proof of Theorem 1, we shall assume $\varepsilon = 1$. We first want to see whether the function $Z(x)$ found in Theorem 1 satisfies

$$(102) \quad -\mathcal{L}(x, \partial_x)Z(x) = C_0 \delta(x) \quad \text{for } x \in B_1(0)$$

for some constant C_0 .

It is not immediately clear how the left-hand side of (102) is defined. Recall from the proof of Theorem 1 the decomposition $Z(x) = h(|x|) + v(x)$, where h is given by (74) and $M_{2,p}(v, r) \leq c r^{2-n} e^{I(r)} \omega(r)$. We can easily calculate $\partial_i \partial_j Z$ and show that for any $\mu > 0$ there is a constant C_μ so that

$$(103) \quad M_p(\partial_i \partial_j Z, r) \leq C_\mu r^{-n-\mu}.$$

In fact, since the a_{ij} are bounded functions, we conclude that for each i, j the function

$$(104) \quad F_{ij}(x) = a_{ij}(x) \partial_i \partial_j Z(x) \quad \text{for } x \neq 0$$

satisfies

$$(105) \quad M_p(F_{ij}, r) \leq C_\mu r^{-n-\mu} \quad \text{for } 0 < r < 1.$$

But this estimate implies that F_{ij} can be regularized at $x = 0$ to give a distribution \mathcal{F}_{ij} , in particular as a continuous linear functional on the space of λ -Hölder continuous functions of compact support in $U = B_1(0)$:

$$(106) \quad \langle \mathcal{F}_{ij}, \phi \rangle = \int_{|x| < 1} F_{ij}(x) [\phi(x) - \phi(0) \chi(|x|)] dx$$

where $\chi(r)$ is a cut-off function which is identically 1 near $r = 0$ but vanishes for $r > 1/2$. (Since $|\phi(x) - \phi(0) \chi(|x|)| \leq C|x|^\lambda$, by choosing $0 < \mu < \lambda$ we see that the integral in (106) converges.) Now let us take the sum over all i, j :

$$(107) \quad \mathcal{F}_0 = \sum_{i,j} \mathcal{F}_{ij}.$$

Then \mathcal{F}_0 is a regularization of $\mathcal{L}(x, \partial_x)Z(x) = 0$, and so it vanishes:

$$\langle \mathcal{F}_0, \phi \rangle = \sum_{i,j} \langle \mathcal{F}_{ij}, \phi \rangle = \int \sum_{i,j} F_{ij}(x) [\phi(x) - \phi(0)\chi(|x|)] dx = 0.$$

Of course, regularization effects the distribution only at $x = 0$, so if we can interpret the expression (108)

$$\mathcal{F} = \mathcal{L}(x, \partial_x)Z(x),$$

as a distribution, then \mathcal{F} is supported only at $x = 0$. As such, it is a linear combination of the delta distribution and its derivatives. But since \mathcal{F} is a continuous linear functional on λ -Hölder functions with $\lambda \in (0, 1)$, it must only involve the delta distribution itself, i.e. (102) must hold for some constant C_0 . But we still have two problems: (i) how is \mathcal{F} itself defined as a distribution? and (ii) how do we calculate the constant C_0 ?

The difficulty in defining \mathcal{F} as a distribution in U is a consequence of the lack of regularity of the a_{ij} . In particular, there is no difficulty in defining the distributional derivatives of Z :

$$(109) \quad \langle \partial_i \partial_j Z, \phi \rangle = - \int_U \partial_j Z(x) \partial_i \phi(x) dx$$

for $\phi \in C_0^1(U)$, since the integral on the right converges. To handle the a_{ij} , we can write

$$(110) \quad \langle \mathcal{L}Z, \phi \rangle = \int_U [(a_{ij}(x) - \delta_{ij}) \partial_i \partial_j Z(x) \phi(x) - \partial_i Z(x) \partial_i \phi(x)] dx.$$

Of course, the integral in (110) is actually an improper integral due to the singularity of $\partial_i \partial_j Z$ at $x = 0$; but provided this integral converges, we conclude that $-\mathcal{L}Z = C_0 \delta$ in U , and we can calculate C_0 by

$$(111) \quad C_0 = \lim_{\varepsilon \rightarrow 0} \int_{|x| < 1} [(-a_{ij}(x) + \delta_{ij}) \partial_i \partial_j Z(x) \phi_\varepsilon(|x|) + \partial_i Z(x) \partial_i \phi_\varepsilon(|x|)] dx,$$

where $\phi_\varepsilon(|x|) = \chi(|x|/\varepsilon)$ with the cut-off function χ introduced above; for these purposes, we are able to assume $\phi(x) = \phi(|x|)$ since we can write $\phi(x) = \phi_0(|x|) + \phi_1(x)$ with $|\phi_1(x)| + |x| |\nabla \phi_1(x)| \leq c|x|$ for $|x| < 1$, which shows that $\langle \mathcal{L}Z, \phi_1 \rangle$ is well-defined and ϕ_1 contributes nothing to C_0 . We shall now prove the following.

Theorem 2. *Suppose the conditions of Theorem 1 hold and Z is the function found there. If $I(0) = \lim_{r \rightarrow 0} I(r)$ exists and is finite then the integral in (110) converges and we can calculate $C_0 = |S^{n-1}| e^{I(0)}$. If $I(r) \rightarrow -\infty$ as $r \rightarrow 0$, then the integral in (110) converges and we can calculate $C_0 = 0$.*

Proof of Theorem 2. Recall from the proof of Theorem I the decomposition $Z(x) = h(|x|) + v(x)$, where h is given by (74) and v satisfies (38). Since we always assume that $I(r)$ is bounded above, we obtain from (38)

$$(112) \quad M_{2,p}(v, r) \leq c r^{2-n} \omega(r) \quad \text{for } 0 < r < 1.$$

We shall separately consider the roles of v and h . In the estimates below, $\int_{|x| < \varepsilon}$ should actually be considered as an improper integral $\lim_{\eta \rightarrow 0} \int_{\eta < |x| < \varepsilon}$, but we avoid such cumbersome notation.

For v , we use (112) to conclude that $M_p(\partial_i v, r) \leq c r^{1-n} \omega(r)$ and so

$$\left| \int_{|x| < \varepsilon} \partial_i v \partial_i \phi_\varepsilon dx \right| \leq c \varepsilon^{-1} \int_0^\varepsilon \omega(r) dr \leq c \omega(\varepsilon) \rightarrow 0.$$

We also have $M_p(\partial_i \partial_j v, r) \leq c r^{-n} \omega(r)$ and so

$$\left| \int_{|x| < \varepsilon} (a_{ij} - \delta_{ij}) \partial_i \partial_j v \phi_\varepsilon dx \right| \leq c \int_0^\varepsilon \frac{\omega^2(r)}{r} dr = c \sigma(\varepsilon) \rightarrow 0.$$

We conclude that v makes no contribution to the value of C_0 .

To determine the effect of h , let us write $h(r) = h_0(r) + h_1(r) + c$ where

$$(113) \quad h_0(r) = c_1 \int_r^1 s^{1-n} E_+(s) ds$$

with c_1 chosen as in the proof of Theorem I so that $c_1 E_+(r) = e^{I(r)} + o(1)$ as $r \rightarrow 0$, and

$$(114) \quad h_1(r) = \int_r^1 s^{1-n} E_+(s) \int_0^s \rho^{n-1} E_-(\rho) B[D^2 v](\rho) d\rho.$$

(Notice that this choice of h_0 and h_1 differs slightly from (75) and (76).) Let us consider h_1 first:

$$\partial_i h_1 = -x_i r^{-n} E_+(r) \int_0^r \rho^{n-1} E_-(\rho) B[D^2 v](\rho) d\rho$$

and

$$\partial_i \partial_j h_1 = -r^{-n} E_+(r) \left(\delta_{ij} - n \frac{x_i x_j}{r^2} + \frac{x_i x_j}{r^2} R(r) \right) \int_0^r \rho^{n-1} E_-(\rho) B[D^2 v](\rho) d\rho - \frac{x_i x_j}{r^2} B[D^2 v](r).$$

The calculation of $\partial_i h_1$ combined with (66) and the assumption that $I(r)$ is bounded shows

$$(115) \quad \left| \int_{|x| < \varepsilon} \partial_i h_1 \partial_i \phi_\varepsilon dx \right| \leq c \varepsilon^{-1} \int_0^\varepsilon \sigma(r) dr \leq c \sigma(\varepsilon) \rightarrow 0.$$

On the other hand, the ‘‘worst’’ term in $(a_{ij} - \delta_{ij}) \partial_i \partial_j h_1$ is

$$W(x) = -r^{-n} E_+(r) \left(a_{ii} - n \frac{a_{ij} x_i x_j}{r^2} - \left(\frac{a_{ij} x_i x_j}{r^2} - 1 \right) R \right) \int_0^r \rho^{n-1} E_-(\rho) B[D^2 v](\rho) d\rho.$$

Using the fact that ϕ_ε only depends on r and observing that

$$\int_{S^{n-1}} \left(a_{ii} - n \frac{a_{ij} x_i x_j}{r^2} - \left(\frac{a_{ij} x_i x_j}{r^2} - 1 \right) R \right) ds = R(r),$$

we can calculate

$$\begin{aligned} \langle W, \phi_\varepsilon \rangle &= -|S^{n-1}| \int_0^\varepsilon E_+(r) \frac{R(r)}{r} \phi_\varepsilon(r) \int_0^r \rho^{n-1} E_-(\rho) B[D^2 v](\rho) d\rho dr, \\ &= |S^{n-1}| \int_0^\varepsilon (E_+)'(r) \phi_\varepsilon(r) \int_0^r \rho^{n-1} E_-(\rho) B[D^2 v](\rho) d\rho dr. \end{aligned}$$

Now we can integrate by parts to obtain

$$\langle W, \phi_\varepsilon \rangle = |S^{n-1}| \left(\int_0^\varepsilon E_+(r) \phi_\varepsilon'(r) \int_0^r \rho^{n-1} E_-(\rho) B[D^2 v](\rho) d\rho dr + \int_0^\varepsilon \phi_\varepsilon(r) r^{n-1} B[D^2 v](r) dr \right).$$

But, using $|\phi_\varepsilon'| \leq c \varepsilon^{-1}$ and $|E_+| \leq c$, we find

$$\left| \int_0^\varepsilon E_+(r) \phi_\varepsilon'(r) \int_0^r \rho^{n-1} E_-(\rho) B[D^2 v](\rho) d\rho dr \right| \leq c \varepsilon^{-1} \int_0^\varepsilon \sigma(r) dr \leq c \sigma(\varepsilon) \rightarrow 0,$$

and

$$\left| \int_0^\varepsilon \phi_\varepsilon(r) r^{n-1} B[D^2 v](r) dr \right| \leq c \sigma(\varepsilon) \rightarrow 0.$$

We conclude that h_1 makes no contribution to C_0 .

Now consider h_0 . We compute $\partial_i h_0 = -c_1 x_i r^{-n} E_+(r)$ and

$$\partial_i \partial_j h_0 = -c_1 r^{-n} E_+(r) \left(\delta_{ij} - n \frac{x_i x_j}{r^2} - \frac{x_i x_j}{r^2} R(r) \right),$$

so

$$(a_{ij} - \delta_{ij}) \partial_i \partial_j h_0 = -c_1 r^{-n} E_+(r) \left(a_{ii} - n \frac{a_{ij} x_i x_j}{r^2} - \left(\frac{a_{ij} x_i x_j}{r^2} - 1 \right) R \right).$$

Notice that

$$\int_{|x|<\varepsilon} r^{-n} E_+(r) \left(a_{ii} - n \frac{a_{ij}x_i x_j}{r^2} - \left(\frac{a_{ij}x_i x_j}{r^2} - 1 \right) R \right) \chi\left(\frac{r}{\varepsilon}\right) dx = |S^{n-1}| \int_0^\varepsilon E_+(r) \frac{R(r)}{r} \chi\left(\frac{r}{\varepsilon}\right) dr,$$

so we now may compute

$$\begin{aligned} -\langle \mathcal{L}h_0, \phi_\varepsilon \rangle &= c_1 |S^{n-1}| \int_0^\varepsilon \left(E_+(r) \frac{R(r)}{r} \chi\left(\frac{r}{\varepsilon}\right) - E_+(r) \left[\chi\left(\frac{r}{\varepsilon}\right) \right]' \right) dr \\ &= -c_1 |S^{n-1}| \int_0^\varepsilon \left[E_+(r) \chi\left(\frac{r}{\varepsilon}\right) \right]' dr. \end{aligned}$$

Now, if $I(0)$ exists (and is finite), then $c_1 E_+(0) = e^{I(0)}$, and we conclude $-\langle \mathcal{L}h_0, \phi_\varepsilon \rangle = |S^{n-1}| e^{I(0)}$. On the other hand, if $I(r) \rightarrow -\infty$ as $r \rightarrow 0$, then $E_+(r) \rightarrow 0$, so $C_0 = 0$. This completes the proof of Theorem 2. \square

Now let us consider a general fixed $y \in \mathbb{R}^n$ and try to find the constant C_y so that

$$(116) \quad -\mathcal{L}(x, \partial_x) Z_y(x) = C_y \delta(x) \quad \text{for } x \in B_1(y).$$

We replace (110) by

$$(117) \quad \langle \mathcal{L}Z_y, \phi \rangle = \int_U [(a_{ij}(x) - a_{ij}(y)) \partial_i \partial_j Z_y(x) \phi(x) - a_{ij}(y) \partial_i Z_y(x) \partial_j \phi(x)] dx.$$

Corollary 2. *Suppose the conditions of Corollary 1 hold and Z_y is the function found there. If $I_y(0) = \lim_{r \rightarrow 0} I_y(r)$ exists and is finite then the integral in (117) converges and Z_y satisfies (116) with C_y given by*

$$(118) \quad C_y = (n-2) |S^{n-1}| \sqrt{\det \mathbf{A}_y} e^{I_y(0)}.$$

If $I_y(r) \rightarrow -\infty$ as $r \rightarrow 0$ then the integral in (117) converges and Z_y satisfies (116) with $C_y = 0$.

Proof of Corollary 2. We need only show that $\langle -\mathcal{L}Z_y, \phi \rangle = |S^{n-1}| (\det \mathbf{A}_y)^{1/2} e^{I_y(0)} \phi(y)$ for some $\phi \in C_0^\infty(B_{\varepsilon_y}(y))$. Let us recall the change of coordinates used in the proof of Theorem I, namely $\tilde{x} = \mathbf{B}(x - y)$ where $\mathbf{B} = \mathbf{A}_y^{-1/2}$, and let $\tilde{\phi}(\tilde{x}) = \phi(x)$; then

$$-\int \mathcal{L}(x, \partial_x) Z_y(x) \phi(x) dx = -(\det \mathbf{A}_y)^{1/2} \int \tilde{\mathcal{L}}(\tilde{x}, \partial_{\tilde{x}}) \tilde{Z}_0(\tilde{x}) \tilde{\phi}(\tilde{x}) d\tilde{x}.$$

But we can apply Theorem 2 to conclude $-\langle \tilde{\mathcal{L}}\tilde{Z}_0, \tilde{\phi} \rangle = |S^{n-1}| e^{I_y(0)} \tilde{\phi}(0)$. Since $\tilde{\phi}(0) = \phi(y)$, we obtain the desired result. \square

4. CONSTRUCTING THE FUNDAMENTAL SOLUTION

Now we are in a position to construct the fundamental solution in a bounded open set $U \subset \mathbb{R}^n$ when $a_{ij} \in C^\omega(U)$, assuming that ω satisfies the square-Dini condition (9) and for every $y \in U$ we have that $I_y(0)$ exists and is finite. Given the results in the preceding section, it is natural to seek the fundamental solution in the form $F(x, y) = Z_y(x)/C_y + v(x, y)$ where $\mathcal{L}(x, \partial_x)v(x, y) = 0$. In fact, it will be convenient to replace $F(x, y)$ with the Green's function $G(x, y)$, in the sense of (8), for a smooth, bounded domain V that contains U .

But first let us observe that the additional assumption that $I_y(0)$ exists and is finite allows us to improve the asymptotic description of $Z_y(x)$. In fact, let us fix $y = 0$ with $a_{ij}(0) = \delta_{ij}$, and assume

$$(119) \quad |I(r) - I(0)| \leq \theta(r)$$

where $\theta(r)$ is a positive, nondecreasing function for $0 < r < 1$ such that $\theta(0) = 0$; as with ω we additionally assume that for some $\nu \in (0, 1)$ we have

$$(120) \quad \theta(r) r^{-1+\nu} \quad \text{is nonincreasing for } 0 < r < 1.$$

(If $\omega(r)$ satisfies the Dini condition, then we can take $\theta(r) = \omega(r)$.)

Lemma 1. *Under the conditions of Theorem 1, let us additionally assume that (20) and (120) hold. Then the solution Z found in Theorem 1 satisfies*

$$(121) \quad Z(x) = \frac{|x|^{2-n} e^{I(0)}}{n-2} (1 + \xi(x)),$$

where for any $p \in (1, \infty)$ we can estimate ξ by

$$(122) \quad M_{2,p}(\xi, r) \leq c \max(\omega(r), \sigma(r), \theta(r)) \quad \text{as } r \rightarrow 0,$$

where c depends only on ω , θ , n , and $I(0)$.

Proof of Lemma 1. In the decomposition $Z(x) = h(|x|) + v(x)$ as in (35), it is elementary to use (38) with (20) to show that (122) applies to $\xi = c|x|^{n-2}v$, so we focus on h . Recall the decomposition $h = h_0 + h_1$ using (75) and (76). If we differentiate h_0 twice we obtain

$$h_0''(r) = (n-1)r^{-n}e^{I(r)} - r^{1-n}e^{I(r)}I'(r).$$

Now use (20) to conclude that $e^{I(r)} \leq (1 + 2\theta(r))e^{I(0)}$ for r sufficiently small, and use (44) to conclude $|I'(r)| \leq 2(n-1)\omega(r)/r$. Consequently, we obtain

$$|h_0''(r) - (n-1)r^{-n}e^{I(0)}| \leq 2(n-1)r^{-n}e^{I(0)}(\theta(r) + \omega(r) + \theta(r)\omega(r)) \leq cr^{-n} \max(\theta(r), \omega(r));$$

here (and henceforth) c depends only on ω , θ , n , and $I(0)$. Using (33) and (120), we can integrate this twice to obtain

$$\left| h_0(r) - \frac{r^{2-n}e^{I(0)}}{n-2} \right| \leq cr^{2-n} \max(\theta(r), \omega(r)).$$

In fact, this argument has shown that $\xi_0(r) = (n-2)r^{n-2}e^{-I(0)}h_0(r) - 1$ satisfies the pointwise estimate

$$M_{2,\infty}(\xi_0, r) \leq c \max(\omega(r), \theta(r)).$$

Similarly, we may differentiate h_1 twice and can estimate most of the terms pointwise; however the term $B[D^2v](r)$ can only be estimated in M_p , so $\xi_1 = (n-2)r^{n-2}e^{-I(0)}h_1(r)$ satisfies

$$M_{2,p}(\xi_1, r) \leq c \max(\omega(r), \sigma(r), \theta(r)).$$

These may be combined to yield (122), so the Lemma is proved. \square

Now we are ready to construct our fundamental solution in a bounded open set.

Theorem 3. *Suppose (1) is uniformly elliptic in a bounded open set $U \subset \mathbb{R}^n$ for $n \geq 3$, where the coefficients a_{ij} are continuous functions with uniform modulus of continuity ω satisfying (9). If $I_y(0)$ exists at every $y \in U$ and*

$$(123) \quad |I_y(r) - I_y(0)| \leq \theta(r) \quad \text{for all } y \in U,$$

where θ is a positive, nondecreasing function for $0 < r < 1$ with $\theta(0) = 0$ and (120), then there is a function $F(x, y)$ satisfying (2) for $x, y \in U$; moreover, $F(x, y)$ admits the asymptotic description

$$(124) \quad F(x, y) = \frac{\langle \mathbf{A}_y^{-1}(x-y), (x-y) \rangle^{\frac{2-n}{2}}}{(n-2)|S^{n-1}| \sqrt{\det \mathbf{A}_y}} (1 + H(x, y)),$$

where for any $p \in (1, \infty)$ and any compact set $K \subset U$ we have

$$(125) \quad M_{2,p}(H(\cdot, y), r; y) \leq c \max(\omega(r), \sigma(r), \theta(r)) \quad \text{as } r \rightarrow 0,$$

with constant c independent of $y \in K$.

Proof of Theorem 3. For each $y \in U$, denote the ε in Corollary 1 by ε_y . Note that the size of ε_y depends on the behavior of the coefficients a_{ij} near y through their ellipticity (i.e. the norm of $\mathbf{A}_y^{-1/2}$) and their continuity (i.e. ω). Since the ellipticity and modulus of continuity are uniform on U , we can find $\varepsilon > 0$ that is independent of $y \in U$. In fact, if we choose a smooth, bounded domain V with $V \supset \overline{U}$ and $\text{dist}(U, \partial V) > \varepsilon$, then we can extend the coefficients a_{ij} to V in such a way as to maintain the uniform ellipticity as well as the modulus of continuity $\omega(r)$ (cf. [21]). Thus we can construct $Z_y(x)$ in $B_\varepsilon^\circ(y)$ for all $y \in \overline{U}$. In fact, repeating this argument with a neighborhood of V , we may assume that $Z_y(x)$ is defined for all $y \in \overline{V}$.

For each $y \in \overline{V}$ let us use Corollary 2 to calculate $C_y > 0$, and conclude that $-\mathcal{L}(x, \partial_x)Z_y(x)/C_y = \delta(x - y)$ for all $x, y \in \overline{V}$ with $|x - y| < \varepsilon$. We shall construct $G(x, y)$ as the Green's function for \mathcal{L} in V . For fixed $y \in V$, let us introduce a smooth cut-off function $\eta_y(r)$ satisfying $\eta_y(r) = 1$ for sufficiently small $r > 0$ but $\eta_y(|\cdot - y|)$ has compact support in V . Then let us define

$$(126) \quad G(x, y) = \eta_y(|x - y|)Z_y(x)/C_y + v(x, y),$$

where $v(x, y)$ is to be determined. But if we apply $-\mathcal{L}(x, \partial_x)$ to $G(x, y)$ we obtain

$$-\mathcal{L}(x, \partial_x)G(x, y) = \delta(x - y) + \psi(x, y) - \mathcal{L}(x, \partial_x)v(x, y),$$

where $\psi(\cdot, y) \in L^p(V)$. So, for fixed $y \in V$, consider the Dirichlet problem for v :

$$(127) \quad \begin{aligned} \mathcal{L}(x, \partial_x)v(x, y) &= \psi(x, y) \quad \text{for } x \in V, \\ v(x, y) &= 0 \quad \text{for } x \in \partial V. \end{aligned}$$

It is well-known (cf. Theorem 9.15 in [9]) that (127) has a unique solution $v(\cdot, y) \in W^{2,p}(V) \cap W_0^{1,p}(V)$, so using this $v(x, y)$ in (126) not only ensures that (8) holds for $x, y \in V$ (and hence for $x, y \in U$), but the following: for any $f \in C(\overline{V})$, $u(x) = -\int_V G(x, y)f(y) dy$ satisfies $\mathcal{L}u = f$ in V and $u = 0$ on ∂V . Now if we pick $\phi \in C^2(\overline{V})$ with $\phi = 0$ on ∂V and let $f = \mathcal{L}\phi$, then uniqueness of the solution shows $u = \phi$ in V , i.e. (8) holds. In other words, the $G(x, y)$ that we have constructed is just the Green's function for \mathcal{L} in V ; in particular, (2) holds.

Now from (126) and Corollaries 1 and 2 we see that

$$(128) \quad G(x, y) = \frac{\langle \mathbf{A}_y^{-1}(x - y), (x - y) \rangle^{\frac{2-n}{2}}}{(n-2)|S^{n-1}|\sqrt{\det \mathbf{A}_y}} e^{J_y(\sqrt{\langle \mathbf{A}_y^{-1}(x-y), (x-y) \rangle})} (1 + \xi_y(x)),$$

where $M_{1,\infty}(\xi_y, r; y) \leq c \max(\omega(r), \sigma(r))$ for $0 < r < \varepsilon$ and $J_y(r) = I_y(r) - I_y(0)$, i.e.

$$(129) \quad J_y(r) = \frac{-1}{|S^{n-1}|} \int_{0 < |z-y| < r} \left(\text{tr}(\mathbf{A}_z \mathbf{A}_y^{-1}) - n \frac{\langle \mathbf{A}_z \mathbf{A}_y^{-1/2}(z-y), \mathbf{A}_y^{-1/2}(z-y) \rangle}{|z-y|^2} \right) \frac{dz}{|z-y|^n}.$$

However, if we choose coordinates in which $y = 0$ and $a_{ij}(0) = \delta_{ij}$, then we may apply Lemma 1 to absorb the exponential term in (128) into $1 + \xi_y$ and obtain

$$(130) \quad G(x, y) = \frac{\langle \mathbf{A}_y^{-1}(x - y), (x - y) \rangle^{\frac{2-n}{2}}}{(n-2)|S^{n-1}|\sqrt{\det \mathbf{A}_y}} (1 + \xi_y(x)) \quad \text{as } x \rightarrow y,$$

where $M_{2,p}(\xi_y, r; y) \leq c_y \max(\omega(r), \sigma(r), \theta(r))$ for $0 < r < \varepsilon$ with c_y depending on ω , θ , n , and $I_y(0)$. But if we select a compact subset $K \subset U$, then c_y may be taken independent of $y \in K$, so we can replace $\xi_y(x)$ by $H(x, y)$ and obtain (124), (125) as $|x - y| \rightarrow 0$.

Letting $F(x, y) = G(x, y)$ for $x, y \in U \subset V$, we have our fundamental solution in U . \square

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