Sharp estimates for the gradient of the generalized Poisson integral for a half-space

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Dedicated to Vakhtang Kokilashvili on the occasion of his 80th birthday

Abstract. A representation of the sharp coefficient in a pointwise estimate for the gradient of the generalized Poisson integral of a function $f$ on $\mathbb{R}^n$ is obtained under the assumption that $f$ belongs to $L^p$. The explicit value of the coefficient is found for the cases $p = 1$ and $p = 2$.

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1 Introduction

In the paper [3] (see also [6]) a representation for the sharp coefficient $\mathcal{K}_p(x)$ in the inequality

$$|\nabla u(x)| \leq \mathcal{K}_p(x) \|u\|_p$$

was found, where $u$ is harmonic function in the half-space $\mathbb{R}^{n+1}_+ = \{x = (x', x_{n+1}) : x' \in \mathbb{R}^n, x_{n+1} > 0\}$, represented by the Poisson integral with boundary values in $L^p(\mathbb{R}^n)$, $\|\cdot\|_p$ is the norm in $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, $x \in \mathbb{R}^{n+1}_+$. It was shown that

$$\mathcal{K}_p(x) = \frac{K_p}{x_{n+1}^{(n+p)/p}}$$

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and explicit formulas for $K_1$ and $K_2$ were given. Namely,

$$K_1 = \frac{2n}{\omega_{n+1}}, \quad K_2 = \sqrt{\frac{n(n+1)}{2^{n+1}\omega_{n+1}}},$$

where $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere in $\mathbb{R}^n$.

In [3] it was shown that the sharp coefficients in pointwise estimates for the absolute value of the normal derivative and the modulus of the gradient of a harmonic function in the half-space coincide for the case $p = 1$ as well as for the case $p = 2$.

Similar results for the gradient and the radial derivative of a harmonic function in the multidimensional ball with boundary values from $L^p$ for $p = 1, 2$ in [4] were obtained.

Thus, the $L^1, L^2$-analogues of Khavinson’s problem [1] were solved in [3, 4] for harmonic functions in the multidimensional half-space and the ball.

We note that explicit sharp coefficients in the inequality for the first derivative of analytic function in the half-plane and the disk with boundary values from $L^p$ in [2, 5, 7] were found.

In this paper we treat a generalization of the problem considered in our work [3]. Here we consider the generalized Poisson integral

$$u_f(x) = k_{n,\alpha} \int_{\mathbb{R}^n} \frac{x_{n+1}^{\alpha}}{|y - x|^{n+\alpha}} f(y') dy'$$

with $f \in L^p(\mathbb{R}^n)$, $\alpha > -(n/p)$, $1 \leq p \leq \infty$, where $x \in \mathbb{R}^{n+1}_+, y = (y', 0), y' \in \mathbb{R}^n$, and $k_{n,\alpha}$ is a normalization constant. In the case $\alpha = 1$ the last integral coincides with the Poisson integral for a half-space.

In Section 2 we obtain a representation for the sharp coefficient $C_p(x)$ in the inequality

$$|\nabla u_f(x)| \leq C_p(x) \|f\|_p,$$

where

$$C_p(x) = \frac{C_p}{x^{(n+p)/p}}$$

and the constant $C_p$ is characterized in terms of an extremal problem on the unit sphere $S^n$ in $\mathbb{R}^{n+1}$.

In Section 3 we reduce this extremal problem to that of finding of the supremum of a certain double integral, depending on a scalar parameter and show that

$$C_1 = k_{n,\alpha} n$$

if $-n < \alpha \leq n$, and

$$C_2 = \sqrt{\omega_{n-1} k_{n,\alpha}} \left\{ \frac{\sqrt{\pi}(n + \alpha)n(n+2)\Gamma\left(\frac{n}{2} - 1\right)\Gamma\left(\frac{n}{2} + \alpha\right)}{8(n+1+\alpha)\Gamma(n+\alpha)} \right\}^{1/2}$$

if $-(n/2) < \alpha \leq n(n+1)/2$.

It is shown that the sharp coefficients in pointwise estimates for the absolute value of the normal derivative and the modulus of the gradient of the generalized Poisson integral for a half-space coincide in the case $p = 1$ as well as in the case $p = 2$. 

2
2 Representation for the sharp constant in inequality for the gradient in terms of an extremal problem on the unit sphere

We introduce some notation used henceforth. Let \( \mathbb{R}^{n+1}_+ = \{ x = (x', x_{n+1}) : x' = (x_1, \ldots, x_n) \in \mathbb{R}^n, x_{n+1} > 0 \} \), \( S^n = \{ x \in \mathbb{R}^{n+1} : |x| = 1 \} \), \( S^+_n = \{ x \in \mathbb{R}^{n+1} : |x| = 1, x_{n+1} > 0 \} \) and \( S^-_n = \{ x \in \mathbb{R}^{n+1} : |x| = 1, x_{n+1} < 0 \} \). Let \( e_\sigma \) stand for the \( n+1 \)-dimensional unit vector joining the origin to a point \( \sigma \) on the sphere \( S^n \).

By \( \| \cdot \|_p \) we denote the norm in the space \( L^p(\mathbb{R}^n) \), that is

\[
\| f \|_p = \left\{ \int_{\mathbb{R}^n} |f(x')|^p \, dx' \right\}^{1/p},
\]

if \( 1 \leq p < \infty \), and \( \| f \|_\infty = \operatorname{ess sup}\{ |f(x')| : x' \in \mathbb{R}^n \} \).

Let the function in \( \mathbb{R}^{n+1}_+ \) be represented as the generalized Poisson integral

\[
u_f(x) = k_{n,\alpha} \int_{\mathbb{R}^n} \frac{x_{n+1}^\alpha}{|y - x|^{n+\alpha}} f(y') \, dy',
\]

with \( f \in L^p(\mathbb{R}^n) \), \( 1 \leq p \leq \infty \), where \( y = (y', 0), y' \in \mathbb{R}^n \),

\[k_{n,\alpha} = \left\{ \int_{\mathbb{R}^n} \frac{x_{n+1}^\alpha}{|y - x|^{n+\alpha}} \, dy' \right\}^{-1} = \frac{\Gamma\left(\frac{n+\alpha}{2}\right)}{\pi^{n/2} \Gamma\left(\frac{\alpha}{2}\right)},\]

and

\[\alpha > -\frac{n}{p}.\]

Now, we find a representation for the best coefficient \( C_p(x; z) \) in the inequality for the absolute value of derivative of \( u_f(x) \) in an arbitrary direction \( z \in S^n \), \( x \in \mathbb{R}^{n+1}_+ \). In particular, we obtain a formula for the constant in a similar inequality for the modulus of the gradient.

**Proposition 1.** Let \( x \) be an arbitrary point in \( \mathbb{R}^{n+1}_+ \) and let \( z \in S^n \). The sharp coefficient \( C_p(x; z) \) in the inequality

\[| (\nabla u_f(x), z) | \leq C_p(x; z) \| f \|_p\]

is given by

\[C_p(x; z) = \frac{C_p(z)}{x_{n+1}^{(n+p)/p}},\]

where

\[C_1(z) = k_{n,\alpha} \sup_{\sigma \in S^n_+} \left| (\alpha e_{n+1} - (n + \alpha)(e_\sigma, e_{n+1}) e_\sigma, z) (e_\sigma, e_{n+1}) \right|^{n+\alpha},\]

\[C_p(z) = k_{n,\alpha} \left\{ \int_{S^n_+} \left| (\alpha e_{n+1} - (n + \alpha)(e_\sigma, e_{n+1}) e_\sigma, z) (e_\sigma, e_{n+1}) \right|^{p-1} \frac{(n-1)p+1}{p-1} d\sigma \right\}^{p-1} \]

for \( 1 < p < \infty \), and

\[C_\infty(z) = k_{n,\alpha} \int_{S^n_+} \left| (\alpha e_{n+1} - (n + \alpha)(e_\sigma, e_{n+1}) e_\sigma, z) (e_\sigma, e_{n+1}) \right|^{\alpha-1} d\sigma.\]
In particular, the sharp coefficient \( C_p(x) \) in the inequality
\[
|\nabla u_f(x)| \leq C_p(x)\|f\|_p
\]
is given by
\[
C_p(x) = \frac{C_p}{x^{(n+p)/p}}, \tag{2.8}
\]
where
\[
C_p = \sup_{|z|=1} C_p(z). \tag{2.9}
\]

Proof. Let \( x = (x', x_{n+1}) \) be a fixed point in \( \mathbb{R}^{n+1}_+ \). The representation (2.1) implies
\[
\frac{\partial u_f}{\partial x_i} = k_{n,\alpha} \int_{\mathbb{R}^n} \left[ \frac{\partial}{\partial x_i} \cdot \left( \frac{\alpha e_{n+1} - (n + \alpha)(e_{xy}, e_{n+1})e_{xy}}{|y - x|^{n+\alpha}} \right) \right] f(y')dy',
\]
that is
\[
\nabla u_f(x) = k_{n,\alpha} x_{n+1}^{-\alpha-1} \int_{\mathbb{R}^n} \left[ \frac{\alpha e_{n+1} - (n + \alpha)(e_{xy}, e_{n+1})e_{xy}}{|y - x|^{n+\alpha}} \right] f(y')dy',
\]
where \( e_{xy} = (y - x)|y - x|^{-1} \). For any \( z \in S^n \),
\[
(\nabla u_f(x), z) = k_{n,\alpha} x_{n+1}^{-\alpha-1} \int_{\mathbb{R}^n} \left[ \frac{(\alpha e_{n+1} - (n + \alpha)(e_{xy}, e_{n+1})e_{xy}) e_{xy}}{|y - x|^{n+\alpha}} \right] f(y')dy'. \tag{2.10}
\]
Hence,
\[
C_1(x ; z) = k_{n,\alpha} x_{n+1}^{-\alpha-1} \sup_{y \in \partial \mathbb{R}^n} \left| \frac{(\alpha e_{n+1} - (n + \alpha)(e_{xy}, e_{n+1})e_{xy}) e_{xy}}{|y - x|^{n+\alpha}} \right|,
\]
and
\[
C_p(x ; z) = k_{n,\alpha} x_{n+1}^{-\alpha-1} \left\{ \int_{\mathbb{R}^n} \left[ \frac{(\alpha e_{n+1} - (n + \alpha)(e_{xy}, e_{n+1})e_{xy}) e_{xy}}{|y - x|^{n+\alpha}} \right]^q dy' \right\}^{1/q}, \tag{2.12}
\]
for \( 1 < p \leq \infty \), where \( p^{-1} + q^{-1} = 1 \).

Taking into account the equality
\[
\frac{x_{n+1}}{|y - x|} = (e_{xy}, -e_{n+1}), \tag{2.13}
\]
by (2.11) we obtain
\[
C_1(x ; z) = k_{n,\alpha} x_{n+1}^{-\alpha-1} \sup_{y \in \partial \mathbb{R}^n} \left| \frac{(\alpha e_{n+1} - (n + \alpha)(e_{xy}, e_{n+1})e_{xy}) e_{xy}}{x_{n+1}^{n+\alpha}} \right| \left( \frac{x_{n+1}}{|y - x|} \right)^{n+\alpha}
\]
\[
= k_{n,\alpha} \sup_{x_{n+1} \sigma \in S^n} \left| (\alpha e_{n+1} - (n + \alpha)(e_{\sigma}, e_{n+1})e_{\sigma}) e_{\sigma}, z \right| |(e_{\sigma}, -e_{n+1})|^{n+\alpha}.
\]

Replacing here \( e_{\sigma} \) by \(-e_{\sigma}\), we arrive at (2.4) for \( p = 1 \) with the sharp constant (2.5).

Let \( 1 < p \leq \infty \). Using (2.13) and the equality

\[
\frac{1}{|y - x|^{(n+\alpha)q}} = \frac{1}{x^{(n+\alpha)q-n}} \left( \frac{x_{n+1}}{|y - x|} \right)^{(n+\alpha)q-n-1} \frac{x_{n+1}}{|y - x|^{n+1}},
\]

and replacing \( q \) by \( p/(p-1) \) in (2.12), we conclude that (2.4) holds with the sharp constant

\[
C_p(z) = k_{n,\alpha} \left\{ \int_{S^n} \left| (\alpha e_{n+1} - (n+\alpha)(e_{\sigma}, e_{n+1})e_{\sigma}, z) \right|^{\frac{p}{p-1}} (e_{\sigma}, -e_{n+1})^{\frac{(n+1)p-n+1}{p-1}} d\sigma \right\}^{\frac{p-1}{p}},
\]

where \( S^n = \{ \sigma \in S^n : (e_{\sigma}, e_{n+1}) < 0 \} \). Replacing here \( e_{\sigma} \) by \(-e_{\sigma}\), we arrive at (2.6) for \( 1 < p < \infty \) and at (2.7) for \( p = \infty \).

By (2.10) we have

\[
\nabla u_j(x) = k_{n,\alpha} x^{\alpha-1} \sup_{|z| = 1} \left( \int_{\mathbb{R}^n} \frac{|(\alpha e_{n+1} - (n+\alpha)(e_{xy}, e_{n+1})e_{xy}, z)|}{|y - x|^{n+\alpha}} f(y')dy' \right)^{1/q}
\]

Hence, by the permutation of suprema, (2.12), (2.11) and (2.4),

\[
C_p(x) = k_{n,\alpha} x^{\alpha-1} \sup_{|z| = 1} \left( \int_{\mathbb{R}^n} \frac{|(\alpha e_{n+1} - (n+\alpha)(e_{xy}, e_{n+1})e_{xy}, z)|^q}{|y - x|^{(n+\alpha)q}} dy' \right)^{1/q} = \sup_{|z| = 1} C_p(x; z) = \sup_{|z| = 1} C_p(z) x^{-(n+p)/p} \tag{2.14}
\]

for \( 1 < p \leq \infty \), and

\[
C_1(x) = k_{n,\alpha} x^{\alpha-1} \sup_{|z| = 1} \sup_{y \in \partial \mathbb{R}^n} \frac{|(\alpha e_{n+1} - (n+\alpha)(e_{xy}, e_{n+1})e_{xy}, z)|}{|y - x|^{n+\alpha}} = \sup_{|z| = 1} C_1(x; z) = \sup_{|z| = 1} C_1(z) x^{-(n+1)} \tag{2.15}
\]

Using the notation (2.9) in (2.14) and (2.15), we arrive at (2.8).

**Remark.** Formula (2.6) for the coefficient \( C_p(z) \), \( 1 < p < \infty \), can be written with the integral over the whole sphere \( S^n \) in \( \mathbb{R}^{n+1} \),

\[
C_p(z) = \frac{k_{n,\alpha}}{2^{(p-1)/p}} \left\{ \int_{S^n} \left| (\alpha e_{n+1} - (n+\alpha)(e_{\sigma}, e_{n+1})e_{\sigma}, z) \right|^{\frac{p}{p-1}} (e_{\sigma}, e_{n+1})^{\frac{(n-1)p-n+1}{p-1}} d\sigma \right\}^{\frac{p-1}{p}}
\]

A similar remark relates to (2.7):

\[
C_\infty(z) = k_{n,\alpha} \left( \int_{S^n} \left| (\alpha e_{n+1} - (n+\alpha)(e_{\sigma}, e_{n+1})e_{\sigma}, z) \right| \left| (e_{\sigma}, e_{n+1}) \right|^{\alpha-1} d\sigma, \tag{2.16}
\]

as well as formula (2.5):

\[
C_1(z) = k_{n,\alpha} \sup_{\sigma \in S^n} \left| (\alpha e_{n+1} - (n+\alpha)(e_{\sigma}, e_{n+1})e_{\sigma}, z) \right| \left| (e_{\sigma}, e_{n+1}) \right|^{n+\alpha}.
\]

5
3 Reduction of the extremal problem to finding of the supremum by parameter of a double integral. The cases $p = 1$ and $p = 2$

The next assertion is based on the representation for $C_p$, obtained in Proposition 1.

**Proposition 2.** Let $f \in L^p(\mathbb{R}^n)$, and let $x$ be an arbitrary point in $\mathbb{R}^{n+1}$. The sharp coefficient $C_p(x)$ in the inequality

$$|\nabla u_f(x)| \leq C_p(x)\|f\|_p$$

is given by

$$C_p(x) = \frac{C_p}{x^{(n+p)/p}},$$

where

$$C_p = (\omega_{n-1})^{(p-1)/p}k_{n,\alpha} \sup_{\gamma \geq 0} \left\{ \int_0^\pi d\varphi \int_0^{\pi/2} \mathcal{F}_{n,p}(\varphi, \vartheta; \gamma) \, d\vartheta \right\}^{\frac{p-1}{p}},$$

if $1 < p < \infty$. Here

$$\mathcal{F}_{n,p}(\varphi, \vartheta; \gamma) = \left| \mathcal{G}_n(\varphi, \vartheta; \gamma) \right|^{p/(p-1)} \cos((n-1)p+1)/(p-1) \vartheta \sin^{n-1} \vartheta \sin^{n-2} \varphi$$

with

$$\mathcal{G}_n(\varphi, \vartheta; \gamma) = ((n + \alpha) \cos^2 \vartheta - \alpha) + \gamma(n + \alpha) \cos \vartheta \sin \vartheta \cos \varphi.$$ 

In addition,

$$C_1 = k_{n,\alpha} n$$

if $-n < \alpha \leq n$.

In particular,

$$C_2 = \sqrt{\omega_{n-1}}k_{n,\alpha} \left\{ \frac{\sqrt{\pi}(n + \alpha)n(n + 2)\Gamma \left( \frac{n}{2} - 1 \right) \Gamma \left( \frac{n}{2} + \alpha \right)}{8(n + 1 + \alpha)\Gamma(n + \alpha)} \right\}^{1/2}$$

for $-(n/2) < \alpha \leq n(n + 1)/2$.

For $p = 1$ and $p = 2$ the coefficient $C_p(x)$ is sharp in conditions of the Proposition also in the weaker inequality obtained from (3.1) by replacing $\nabla u_f$ by $\partial u_f/\partial x_{n+1}$.

**Proof.** The equality (3.2) was proved in Proposition 1.

(i) Let $p = 1$. Using (2.5), (2.9) and the permutability of two suprema, we find

$$C_1 = k_{n,\alpha} \sup_{|z|=1} \sup_{\sigma \in S^n_+} |(\alpha e_{n+1} - (n + \alpha)(e_\sigma, e_{n+1})e_\sigma, z)| (e_\sigma, e_{n+1})^{n+\alpha}$$

$$= k_{n,\alpha} \sup_{\sigma \in S^n_+} |(\alpha e_{n+1} - (n + \alpha)(e_\sigma, e_{n+1})e_\sigma| (e_\sigma, e_{n+1})^{n+\alpha}. $$ (3.7)
Taking into account the equality

$$|\alpha e_{n+1} - (n + \alpha)(e_\sigma, e_{n+1})e_\sigma|$$

$$= \left(\alpha e_{n+1} - (n + \alpha)(e_\sigma, e_{n+1})e_\sigma, \alpha e_{n+1} - (n + \alpha)(e_\sigma, e_{n+1})e_\sigma\right)^{1/2}$$

$$= \left(\alpha^2 + ((n + \alpha)^2 - 2\alpha(n + \alpha))(e_\sigma, e_{n+1})^2\right)^{1/2},$$

and using (2.3), (3.7), we arrive at the sharp constant (3.6) for $-n < \alpha \leq n$.

Furthermore, by (2.5),

$$C_1(e_{n+1}) = k_{n,\alpha} \sup_{\sigma \in \mathbb{S}_{++}^n} |\alpha - (n + \alpha)(e_\sigma, e_{n+1})^2|\left(e_\sigma, e_{n+1}\right)^{n+\alpha} \geq k_{n,\alpha} n.$$

Hence, by $C_1 \geq C_1(e_{n+1})$ and by (3.6) we obtain $C_1 = C_1(e_{n+1})$, which completes the proof in the case $p = 1$.

(ii) Let $1 < p < \infty$. Since the integrand in (2.6) does not change when $z \in \mathbb{S}^n$ is replaced by $-z$, we may assume that $z_{n+1} = (e_{n+1}, z) > 0$ in (2.9).

Let $z' = z - z_{n+1}e_{n+1}$. Then $(z', e_{n+1}) = 0$ and hence $z_{n+1}^2 + |z'|^2 = 1$. Analogously, with $\sigma = (\sigma_1, \ldots, \sigma_n, \sigma_{n+1}) \in \mathbb{S}_{++}^n$, we associate the vector $\sigma' = e_\sigma - \sigma_{n+1}e_{n+1}$.

Using the equalities $(\sigma', e_{n+1}) = 0$, $\sigma_{n+1} = \sqrt{1 - |\sigma'|^2}$ and $(z', e_{n+1}) = 0$, we find an expression for $(\alpha e_{n+1} - (n + \alpha)(e_\sigma, e_{n+1})e_\sigma, z)$ as a function of $\sigma'$:

$$(\alpha e_{n+1} - (n + \alpha)(e_\sigma, e_{n+1})e_\sigma, z') = \alpha z_{n+1} - (n + \alpha)\sigma_{n+1}(e_\sigma, z')$$

$$= \alpha z_{n+1} - (n + \alpha)\sigma_{n+1}\left((\sigma', z') + z_{n+1}\sigma_{n+1}\right)$$

$$= -\left[(n + \alpha)(1 - |\sigma'|^2) - \alpha\right]z_{n+1} - (n + \alpha)\sqrt{1 - |\sigma'|^2}(\sigma', z').$$

Let $\mathbb{B}^n = \{x' = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x'| < 1\}$. By (2.6) and (3.8), taking into account that $d\sigma = d\sigma' / \sqrt{1 - |\sigma'|^2}$, we may write (2.9) as

$$C_p = k_{n,\alpha} \sup_{z \in \mathbb{S}_{++}^n} \left\{ \int_{\mathbb{B}^n} \mathcal{H}_{n,p}(|\sigma'|, (\sigma', z')) \left(1 - |\sigma'|^2\right)^{(\alpha p + n + 1)/2(p - 1)} d\sigma' \right\}^{p - 1 \over p}$$

$$= k_{n,\alpha} \sup_{z \in \mathbb{S}_{++}^n} \left\{ \int_{\mathbb{B}^n} \mathcal{H}_{n,p}(|\sigma'|, (\sigma', z')) \left(1 - |\sigma'|^2\right)^{(\alpha - 2)p + n + 2)/2(p - 1)} d\sigma' \right\}^{p - 1 \over p},$$

where

$$\mathcal{H}_{n,p}(|\sigma'|, (\sigma', z')) = \left|((n + \alpha)(1 - |\sigma'|^2) - \alpha)z_{n+1} + ((n + \alpha)\sqrt{1 - |\sigma'|^2}(\sigma', z'))^{\alpha p - 1 \over 2(p - 1)}.

Using the well known formula (see e.g. [8], 3.3.2(3)),

$$\int_{\mathbb{B}^n} g(|x|, (\alpha, x)) dx = \omega_{n-1} \int_0^1 r^{n-1} dr \int_0^\pi g(r, |\alpha| r \cos \varphi) \sin^{n-2} \varphi \, d\varphi,$$

$$7$$
The last equality and (3.13) imply
\[
\int_{B_n} \mathcal{H}_{n,p}(|\sigma'|, (\sigma', z')) (1 - |\sigma'|) ((\alpha - 2)p + n + 2)/(p - 1) \, d\sigma'
\]
\[
= \omega_{n-1} \int_0^1 r^{n-1} (1 - r^2) ((\alpha - 2)p + n + 2)/(p - 1) \, dr \int_0^{\pi} \mathcal{H}_{n,p}(r, r|z'| \cos \varphi) \sin^{n-2} \varphi d\varphi.
\]
Making the change of variable \( r = \sin \vartheta \) in the right-hand side of the last equality, we find
\[
\int_{B_n} \mathcal{H}_{n,p}(|\sigma'|, (\sigma', z')) (1 - |\sigma'|) ((\alpha - 2)p + n + 2)/(2(p - 1)) \, d\sigma'
\]
\[
= \omega_{n-1} \int_0^{\pi/2} \sin^{n-2} \varphi d\varphi \int_0^{\pi/2} \mathcal{H}_{n,p}(\sin \vartheta, |z'| \sin \vartheta \cos \varphi) \sin^{n-1} \vartheta \cos^{(\alpha - 1)p + n + 1}/p! \, \vartheta d\vartheta,
\]
where, by (3.10),
\[
\mathcal{H}_{n,p}(\sin \vartheta, |z'| \sin \vartheta \cos \varphi) = |((n + \alpha) \cos^2 \vartheta - \alpha) z_{n+1} + (n + \alpha)|z'| \cos \vartheta \sin \vartheta \cos \varphi|^{p/(p - 1)}.
\]
Introducing here the parameter \( \gamma = |z'|/z_{n+1} \) and using the equality \(|z'|^2 + z_{n+1}^2 = 1\), we obtain
\[
\mathcal{H}_{n,p}(\sin \vartheta, |z'| \sin \vartheta \cos \varphi) = (1 + \gamma^2)^{-p/(p - 1)} |G_n(\varphi, \vartheta; \gamma)|^{p/(p - 1)},
\]
where \( G_n(\varphi, \vartheta; \gamma) \) is given by (3.5).
By (3.9), taking into account (3.11) and (3.12), we arrive at (3.3).
(iii) Let \( p = 2 \). By (3.3), (3.4) and (3.5),
\[
C_2 = \sqrt{\omega_{n-1}} k_{n,\alpha} \sup_{\gamma \geq 0} \left\{ \int_0^{\pi/2} d\varphi \int_0^{\pi/2} \mathcal{F}_{n,2}(\varphi, \vartheta; \gamma) \, d\vartheta \right\}^{1/2},
\]
where
\[
\mathcal{F}_{n,2}(\varphi, \vartheta; \gamma) = \left[ ((n + \alpha) \cos^2 \vartheta - \alpha) + \gamma(n + \alpha) \cos \vartheta \sin \vartheta \cos \varphi \right]^2 \cos^{n-1+2\alpha} \vartheta \sin^{n-1} \vartheta \sin^{\gamma - 2} \varphi.
\]
The last equality and (3.13) imply
\[
C_2 = \sqrt{\omega_{n-1}} k_{n,\alpha} \sup_{\gamma \geq 0} \frac{1}{\sqrt{1 + \gamma^2}} \left\{ \mathcal{I}_1 + \gamma^2 \mathcal{I}_2 \right\}^{1/2},
\]
where
\[
\mathcal{I}_1 = \int_0^{\pi} \sin^{n-2} \varphi \, d\varphi \int_0^{\pi/2} ((n + \alpha) \cos^2 \vartheta - \alpha)^2 \sin^{n-1} \vartheta \cos^{n-1+2\alpha} \vartheta \, d\vartheta
\]
\[
= \frac{\sqrt{\pi} n(n + 2)(n + \alpha) \Gamma \left( \frac{n-1}{2} \right) \Gamma \left( \frac{n+2+\alpha}{2} \right)}{4(n + 2\alpha)(n + 1 + \alpha)\Gamma(n + \alpha)}
\]
(3.15)
\begin{align*}
\mathcal{I}_2 &= (n + \alpha)^2 \int_0^\pi \sin^{n-2} \varphi \cos^2 \varphi \, d\varphi \int_0^{\pi/2} \sin^{n+1} \theta \cos^{n+2\alpha} \theta \, d\theta \\
&= \frac{\sqrt{\pi} (n + \alpha) \Gamma \left( \frac{n+1}{2} \right) \Gamma \left( \frac{n+2+2\alpha}{2} \right)}{4(n + 1 + \alpha)\Gamma(n + \alpha)}.
\end{align*}

(3.16)

By (3.14) we have

\[ C_2 = \sqrt{\omega_{n-1}} k_{n,\alpha} \max \{ \mathcal{I}_1^{1/2}, \mathcal{I}_2^{1/2} \}. \]

(3.17)

Further, by (3.15) and (3.16),

\[ \frac{\mathcal{I}_1}{\mathcal{I}_2} = \frac{n(n+2)}{n+2\alpha}. \]

Therefore,

\[ \frac{\mathcal{I}_1}{\mathcal{I}_2} - 1 = \frac{n^2 + n - 2\alpha}{n + 2\alpha}. \]

Taking into account (3.17) and that \( n + 2\alpha > 0 \) for \( p = 2 \) by (2.3), we see that inequality

\[ \frac{\mathcal{I}_1}{\mathcal{I}_2} \geq 1 \]

holds for \( \alpha \leq n(n + 1)/2 \). So, we arrive at the representation for \( C_2 \) with \( -(n/2) < \alpha \leq n(n + 1)/2 \) given in formulation of the Proposition.

Since \( z \in S^n \) and the supremum in \( \gamma = |z'|/z_{n+1} \) in (3.13) is attained for \( \gamma = 0 \), we have \( C_2 = C_2(e_{n+1}) \) under requirements of the Proposition. \qed

References


