Sharp estimates for the gradient of the generalized Poisson integral for a half-space

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Dedicated to Vakhtang Kokilashvili on the occasion of his 80th birthday

Abstract. A representation of the sharp coefficient in a pointwise estimate for the gradient of the generalized Poisson integral of a function f on $\mathbb{R}^n$ is obtained under the assumption that f belongs to L^p. The explicit value of the coefficient is found for the cases $p = 1$ and $p = 2$.

Keywords: generalized Poisson integral, sharp estimate for the gradient, half-space

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1 Introduction

In the paper [3] (see also [6]) a representation for the sharp coefficient $\mathcal{K}_p(x)$ in the inequality

$$|\nabla u(x)| \leq \mathcal{K}_p(x)\big\|u\big\|_p$$

was found, where u is harmonic function in the half-space $\mathbb{R}^{n+1}_+ = \big\{x = (x', x_{n+1}) : x' \in \mathbb{R}^n, x_{n+1} > 0\big\}$, represented by the Poisson integral with boundary values in $L^p(\mathbb{R}^n)$, $|| \cdot ||_p$ is the norm in $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, $x \in \mathbb{R}^{n+1}_+$. It was shown that

$$\mathcal{K}_p(x) = \frac{K_p}{x_{n+1}^{(n+p)/p}}$$

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and explicit formulas for K_1 and K_2 were given. Namely,

$$K_1 = \frac{2n}{\omega_{n+1}}\ , \qquad\qquad K_2 = \sqrt{\frac{n(n+1)}{2^{n+1}\omega_{n+1}}}\ ,$$

where $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere in $\mathbb{R}^n$.

In [3] it was shown that the sharp coefficients in pointwise estimates for the absolute value of the normal derivative and the modulus of the gradient of a harmonic function in the half-space coincide for the case $p = 1$ as well as for the case $p = 2$.

Similar results for the gradient and the radial derivative of a harmonic function in the multidimensional ball with boundary values from L^p for $p = 1, 2$ in [4] were obtained.

Thus, the L^1, L^2-analogues of Khavinson's problem [1] were solved in [3, 4] for harmonic functions in the multidimensional half-space and the ball.

We note that explicit sharp coefficients in the inequality for the first derivative of analytic function in the half-plane and the disk with boundary values of the real-part from L^p in [2, 5, 7] were found.

In this paper we treat a generalization of the problem considered in our work [3]. Here we consider the generalized Poisson integral

$$u_{_f}(x) = k_{n,\alpha} \int_{\mathbb{R}^n} \frac{x^{\alpha}_{n+1}}{|y-x|^{n+\alpha}}\ f(y')dy'$$

with $f \in L^p(\mathbb{R}^n)$, $\alpha > -(n/p)$, $1 \le p \le \infty$, where $x \in \mathbb{R}^{n+1}_+$, $y = (y', 0)$, $y' \in \mathbb{R}^n$, and $k_{n,\alpha}$ is a normalization constant. In the case $\alpha = 1$ the last integral coincides with the Poisson integral for a half-space.

In Section 2 we obtain a representation for the sharp coefficient $\mathcal{C}_p(x)$ in the inequality

$$\big|\nabla u_{_f}(x)\big| \le \mathcal{C}_p(x)\big\|f\big\|_p\ ,$$

where

$$\mathcal{C}_p(x) = \frac{C_p}{x^{(n+p)/p}_{n+1}}$$

and the constant C_p is characterized in terms of an extremal problem on the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$.

In Section 3 we reduce this extremal problem to that of finding of the supremum of a certain double integral, depending on a scalar parameter and show that

$$C_1 = k_{n,\alpha} n$$

if $-n < \alpha \le n$, and

$$C_2 = \sqrt{\omega_{n-1}} k_{n,\alpha} \left\{ \frac{\sqrt{\pi}(n+\alpha)n(n+2)\Gamma\left(\frac{n}{2}-1\right)\Gamma\left(\frac{n}{2}+\alpha\right)}{8(n+1+\alpha)\Gamma(n+\alpha)} \right\}^{1/2}$$

if $-(n/2) < \alpha \le n(n+1)/2$.

It is shown that the sharp coefficients in pointwise estimates for the absolute value of the normal derivative and the modulus of the gradient of the generalized Poisson integral for a half-space coincide in the case $p = 1$ as well as in the case $p = 2$.

2 Representation for the sharp constant in inequality for the gradient in terms of an extremal problem on the unit sphere

We introduce some notation used henceforth. Let $\mathbb{R}^{n+1}_+ = \big\{x = (x', x_{n+1}) : x' = (x_1, \dots, x_n) \in \mathbb{R}^n, x_{n+1} > 0\big\}$, $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, $\mathbb{S}^n_+ = \{x \in \mathbb{R}^{n+1} : |x| = 1,\ x_{n+1} > 0\}$ and $\mathbb{S}^n_- = \{x \in \mathbb{R}^{n+1} : |x| = 1,\ x_{n+1} < 0\}$. Let $\boldsymbol{e}_\sigma$ stand for the $n+1$-dimensional unit vector joining the origin to a point σ on the sphere $\mathbb{S}^n$.

By $||\cdot||_p$ we denote the norm in the space $L^p(\mathbb{R}^n)$, that is

$$||f||_p = \left\{ \int_{\mathbb{R}^n} |f(x')|^p \, dx' \right\}^{1/p},$$

if $1 \le p < \infty$, and $||f||_\infty = \text{ess sup}\{|f(x')| : x' \in \mathbb{R}^n\}$.

Let the function in $\mathbb{R}^{n+1}_+$ be represented as the generalized Poisson integral

$$u_f(x) = k_{n,\alpha} \int_{\mathbb{R}^n} \frac{x_{n+1}^\alpha}{|y - x|^{n+\alpha}} \, f(y') dy' \tag{2.1}$$

with $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, where $y = (y', 0)$, $y' \in \mathbb{R}^n$,

$$k_{n,\alpha} = \left\{ \int_{\mathbb{R}^n} \frac{x_{n+1}^\alpha}{|y - x|^{n+\alpha}} dy' \right\}^{-1} = \frac{\Gamma\left(\frac{n+\alpha}{2}\right)}{\pi^{n/2}\Gamma\left(\frac{\alpha}{2}\right)}\ , \tag{2.2}$$

and

$$\alpha > -\frac{n}{p}\ . \tag{2.3}$$

Now, we find a representation for the best coefficient $\mathcal{C}_p(x; \boldsymbol{z})$ in the inequality for the absolute value of derivative of $u_f(x)$ in an arbitrary direction $\boldsymbol{z} \in \mathbb{S}^n$, $x \in \mathbb{R}^{n+1}_+$. In particular, we obtain a formula for the constant in a similar inequality for the modulus of the gradient.

Proposition 1. *Let x be an arbitrary point in $\mathbb{R}^{n+1}_+$ and let $\boldsymbol{z} \in \mathbb{S}^n$. The sharp coefficient $\mathcal{C}_p(x; \boldsymbol{z})$ in the inequality*

$$\left| \big(\nabla u_f(x), \boldsymbol{z}\big) \right| \le \mathcal{C}_p(x; \boldsymbol{z}) \big\| f \big\|_p$$

is given by

$$\mathcal{C}_p(x; \boldsymbol{z}) = \frac{C_p(\boldsymbol{z})}{x_{n+1}^{(n+p)/p}}, \tag{2.4}$$

where

$$C_1(\boldsymbol{z}) = k_{n,\alpha} \sup_{\sigma \in \mathbb{S}^n_+} \left| \big(\alpha \boldsymbol{e}_{n+1} - (n+\alpha)(\boldsymbol{e}_\sigma, \boldsymbol{e}_{n+1})\boldsymbol{e}_\sigma,\ \boldsymbol{z}\big) \right| \big(\boldsymbol{e}_\sigma, \boldsymbol{e}_{n+1}\big)^{n+\alpha}, \tag{2.5}$$

$$C_p(\boldsymbol{z}) = k_{n,\alpha} \left\{ \int_{\mathbb{S}^n_+} \left| \big(\alpha \boldsymbol{e}_{n+1} - (n+\alpha)(\boldsymbol{e}_\sigma, \boldsymbol{e}_{n+1})\boldsymbol{e}_\sigma, \boldsymbol{z}\big) \right|^{\frac{p}{p-1}} \big(\boldsymbol{e}_\sigma, \boldsymbol{e}_{n+1}\big)^{\frac{(\alpha-1)p+n+1}{p-1}} d\sigma \right\}^{\frac{p-1}{p}} \tag{2.6}$$

for $1 < p < \infty$, and

$$C_\infty(\boldsymbol{z}) = k_{n,\alpha} \int_{\mathbb{S}^n_+} \left| \big(\alpha \boldsymbol{e}_{n+1} - (n+\alpha)(\boldsymbol{e}_\sigma, \boldsymbol{e}_{n+1})\boldsymbol{e}_\sigma,\ \boldsymbol{z}\big) \right| \big(\boldsymbol{e}_\sigma, \boldsymbol{e}_{n+1}\big)^{\alpha-1} d\sigma. \tag{2.7}$$

In particular, the sharp coefficient $\mathcal{C}_p(x)$ *in the inequality*

$$\left|\nabla u_{_f}(x)\right| \leq \mathcal{C}_p(x)\left\|f\right\|_p$$

is given by

$$\mathcal{C}_p(x) = \frac{C_p}{x_{n+1}^{(n+p)/p}}, \tag{2.8}$$

where

$$C_p = \sup_{|\boldsymbol{z}|=1} C_p(\boldsymbol{z}). \tag{2.9}$$

Proof. Let $x = (x', x_{n+1})$ be a fixed point in $\mathbb{R}^{n+1}_+$. The representation (2.1) implies

$$\frac{\partial u_{_f}}{\partial x_i} = k_{n,\alpha}\int_{\mathbb{R}^n}\left[\frac{\delta_{ni}\alpha x_{n+1}^{\alpha-1}}{|y-x|^{n+\alpha}} + \frac{(n+\alpha)x_{n+1}^{\alpha}(y_i - x_i)}{|y-x|^{n+2+\alpha}}\right] f(y')dy',$$

that is

$$\begin{aligned}\nabla u_{_f}(x) &= k_{n,\alpha}x_{n+1}^{\alpha-1}\int_{\mathbb{R}^n}\left[\frac{\alpha\boldsymbol{e}_{n+1}}{|y-x|^{n+\alpha}} + \frac{(n+\alpha)x_{n+1}(y-x)}{|y-x|^{n+2+\alpha}}\right] f(y')dy' \\ &= k_{n,\alpha}x_{n+1}^{\alpha-1}\int_{\mathbb{R}^n}\frac{\alpha\boldsymbol{e}_{n+1} - (n+\alpha)(\boldsymbol{e}_{xy},\boldsymbol{e}_{n+1})\boldsymbol{e}_{xy}}{|y-x|^{n+\alpha}} f(y')dy',\end{aligned}$$

where $\boldsymbol{e}_{xy} = (y-x)|y-x|^{-1}$. For any $\boldsymbol{z} \in \mathbb{S}^n$,

$$(\nabla u_{_f}(x), \boldsymbol{z}) = k_{n,\alpha}x_{n+1}^{\alpha-1}\int_{\mathbb{R}^n}\frac{(\alpha\boldsymbol{e}_{n+1} - (n+\alpha)\boldsymbol{e}_{xy},\boldsymbol{e}_{n+1})\boldsymbol{e}_{xy},\ \boldsymbol{z})}{|y-x|^{n+\alpha}} f(y')dy'. \tag{2.10}$$

Hence,

$$\mathcal{C}_1(x;\boldsymbol{z}) = k_{n,\alpha}x_{n+1}^{\alpha-1}\sup_{y\in\partial\mathbb{R}^n}\frac{|(\alpha\boldsymbol{e}_{n+1} - (n+\alpha)(\boldsymbol{e}_{xy},\boldsymbol{e}_{n+1})\boldsymbol{e}_{xy},\ \boldsymbol{z})|}{|y-x|^{n+\alpha}}\ , \tag{2.11}$$

and

$$\mathcal{C}_p(x;\boldsymbol{z}) = k_{n,\alpha}x_{n+1}^{\alpha-1}\left\{\int_{\mathbb{R}^n}\frac{\left|\left(\alpha\boldsymbol{e}_{n+1} - (n+\alpha)(\boldsymbol{e}_{xy},\boldsymbol{e}_{n+1})\boldsymbol{e}_{xy},\boldsymbol{z}\right)\right|^q}{|y-x|^{(n+\alpha)q}}\ dy'\right\}^{1/q} \tag{2.12}$$

for $1 < p \leq \infty$, where $p^{-1} + q^{-1} = 1$.

Taking into account the equality

$$\frac{x_{n+1}}{|y-x|} = (\boldsymbol{e}_{xy}, -\boldsymbol{e}_{n+1}), \tag{2.13}$$

by (2.11) we obtain

$$\begin{aligned}\mathcal{C}_1(x;\boldsymbol{z}) &= k_{n,\alpha}x_{n+1}^{\alpha-1}\sup_{y\in\partial\mathbb{R}^n}\frac{|(\alpha\boldsymbol{e}_{n+1} - (n+\alpha)(\boldsymbol{e}_{xy},\boldsymbol{e}_{n+1})\boldsymbol{e}_{xy},\ \boldsymbol{z})|}{x_{n+1}^{n+\alpha}}\left(\frac{x_{n+1}}{|y-x|}\right)^{n+\alpha} \\ &= \frac{k_{n,\alpha}}{x_{n+1}^{n+1}}\sup_{\sigma\in\mathbb{S}^n_-}\left|\left(\alpha\boldsymbol{e}_{n+1} - (n+\alpha)(\boldsymbol{e}_\sigma,\boldsymbol{e}_{n+1})\boldsymbol{e}_\sigma,\ \boldsymbol{z}\right)\right|\left(\boldsymbol{e}_\sigma, -\boldsymbol{e}_{n+1}\right)^{n+\alpha}.\end{aligned}$$

Replacing here $\boldsymbol{e}_\sigma$ by $-\boldsymbol{e}_\sigma$, we arrive at (2.4) for $p = 1$ with the sharp constant (2.5).

Let $1 < p \le \infty$. Using (2.13) and the equality

$$\frac{1}{|y-x|^{(n+\alpha)q}} = \frac{1}{x_{n+1}^{(n+\alpha)q-n}} \left(\frac{x_{n+1}}{|y-x|}\right)^{(n+\alpha)q-n-1} \frac{x_{n+1}}{|y-x|^{n+1}} \ ,$$

and replacing q by $p/(p-1)$ in (2.12), we conclude that (2.4) holds with the sharp constant

$$C_p(\boldsymbol{z}) = k_{n,\alpha} \left\{ \int_{\mathbb{S}^n_-} \left|\big(\alpha \boldsymbol{e}_{n+1} - (n+\alpha)(\boldsymbol{e}_\sigma, \boldsymbol{e}_{n+1})\boldsymbol{e}_\sigma,\ \boldsymbol{z}\big)\right|^{\frac{p}{p-1}} \big(\boldsymbol{e}_\sigma, -\boldsymbol{e}_{n+1}\big)^{\frac{(\alpha-1)p+n+1}{p-1}} d\sigma \right\}^{\frac{p-1}{p}} ,$$

where $\mathbb{S}^n_- = \{\sigma \in \mathbb{S}^n : (\boldsymbol{e}_\sigma, \boldsymbol{e}_{n+1}) < 0\}$. Replacing here $\boldsymbol{e}_\sigma$ by $-\boldsymbol{e}_\sigma$, we arrive at (2.6) for $1 < p < \infty$ and at (2.7) for $p = \infty$.

By (2.10) we have

$$\left|\nabla u_{_f}(x)\right| = k_{n,\alpha} x_{n+1}^{\alpha-1} \sup_{|\boldsymbol{z}|=1} \int_{\mathbb{R}^n} \frac{(\alpha \boldsymbol{e}_{n+1} - (n+\alpha)(\boldsymbol{e}_{xy}, \boldsymbol{e}_{n+1})\boldsymbol{e}_{xy},\ \boldsymbol{z})}{|y-x|^{n+\alpha}} f(y')dy'.$$

Hence, by the permutation of suprema, (2.12), (2.11) and (2.4),

$$\begin{aligned} \mathcal{C}_p(x) &= k_{n,\alpha} x_{n+1}^{\alpha-1} \sup_{|\boldsymbol{z}|=1} \left\{ \int_{\mathbb{R}^{n+1}} \frac{\left|\big(\alpha \boldsymbol{e}_{n+1} - (n+\alpha)(\boldsymbol{e}_{xy}, \boldsymbol{e}_{n+1})\boldsymbol{e}_{xy}, \boldsymbol{z}\big)\right|^q}{|y-x|^{(n+\alpha)q}} dy' \right\}^{1/q} \\ &= \sup_{|\boldsymbol{z}|=1} \mathcal{C}_p(x; \boldsymbol{z}) = \sup_{|\boldsymbol{z}|=1} C_p(\boldsymbol{z}) x_{n+1}^{-(n+p)/p} \end{aligned} \tag{2.14}$$

for $1 < p \le \infty$, and

$$\begin{aligned} \mathcal{C}_1(x) &= k_{n,\alpha} x_{n+1}^{\alpha-1} \sup_{|\boldsymbol{z}|=1} \sup_{y \in \partial\mathbb{R}^n} \frac{|(\alpha \boldsymbol{e}_{n+1} - (n+\alpha)(\boldsymbol{e}_{xy}, \boldsymbol{e}_{n+1})\boldsymbol{e}_{xy},\ \boldsymbol{z})|}{|y-x|^{n+\alpha}} \\ &= \sup_{|\boldsymbol{z}|=1} \mathcal{C}_1(x; \boldsymbol{z}) = \sup_{|\boldsymbol{z}|=1} C_1(\boldsymbol{z}) x_{n+1}^{-(n+1)}. \end{aligned} \tag{2.15}$$

Using the notation (2.9) in (2.14) and (2.15), we arrive at (2.8). □

Remark. Formula (2.6) for the coefficient $C_p(\boldsymbol{z}), 1 < p < \infty$, can be written with the integral over the whole sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$,

$$C_p(\boldsymbol{z}) = \frac{k_{n,\alpha}}{2^{(p-1)/p}} \left\{ \int_{\mathbb{S}^n} \left|\big(\alpha \boldsymbol{e}_{n+1} - (n+\alpha)(\boldsymbol{e}_\sigma, \boldsymbol{e}_{n+1})\boldsymbol{e}_\sigma, \boldsymbol{z}\big)\right|^{\frac{p}{p-1}} \big(\boldsymbol{e}_\sigma, \boldsymbol{e}_{n+1}\big)^{\frac{(\alpha-1)p+n+1}{p-1}} d\sigma \right\}^{\frac{p-1}{p}}$$

A similar remark relates (2.7):

$$C_\infty(\boldsymbol{z}) = \frac{k_{n,\alpha}}{2} \int_{\mathbb{S}^n} \left|\big(\alpha \boldsymbol{e}_{n+1} - (n+\alpha)(\boldsymbol{e}_\sigma, \boldsymbol{e}_{n+1})\boldsymbol{e}_\sigma,\ \boldsymbol{z}\big)\right| \left|\big(\boldsymbol{e}_\sigma, \boldsymbol{e}_{n+1}\big)\right|^{\alpha-1} d\sigma, \tag{2.16}$$

as well as formula (2.5):

$$C_1(\boldsymbol{z}) = k_{n,\alpha} \sup_{\sigma \in \mathbb{S}^n} \left|\big(\alpha \boldsymbol{e}_{n+1} - (n+\alpha)(\boldsymbol{e}_\sigma, \boldsymbol{e}_{n+1})\boldsymbol{e}_\sigma,\ \boldsymbol{z}\big)\right| \left|\big(\boldsymbol{e}_\sigma, \boldsymbol{e}_{n+1}\big)\right|^{n+\alpha} .$$

3 Reduction of the extremal problem to finding of the supremum by parameter of a double integral. The cases $p = 1$ and $p = 2$

The next assertion is based on the representation for C_p, obtained in Proposition 1.

Proposition 2. *Let $f \in L^p(\mathbb{R}^n)$, and let x be an arbitrary point in $\mathbb{R}^{n+1}_+$. The sharp coefficient $\mathcal{C}_p(x)$ in the inequality*

$$|\nabla u_{_f}(x)| \leq \mathcal{C}_p(x)\big\| f\big\|_p \tag{3.1}$$

is given by

$$\mathcal{C}_p(x) = \frac{C_p}{x_{n+1}^{(n+p)/p}} \ , \tag{3.2}$$

where

$$C_p = (\omega_{n-1})^{(p-1)/p} k_{n,\alpha} \sup_{\gamma\geq 0} \frac{1}{\sqrt{1+\gamma^2}} \left\{ \int_0^{\pi} d\varphi \int_0^{\pi/2} \mathcal{F}_{n,p}(\varphi,\vartheta;\gamma)\, d\vartheta \right\}^{\frac{p-1}{p}} , \tag{3.3}$$

if $1 < p < \infty$. Here

$$\mathcal{F}_{n,p}(\varphi,\vartheta;\gamma) = \big|\mathcal{G}_n(\varphi,\vartheta;\gamma)\big|^{p/(p-1)} \cos^{((\alpha-1)p+n+1)/(p-1)}\vartheta \sin^{n-1}\vartheta \sin^{n-2}\varphi \tag{3.4}$$

with

$$\mathcal{G}_n(\varphi,\vartheta;\gamma) = \big((n+\alpha)\cos^2\vartheta - \alpha\big) + \gamma(n+\alpha)\cos\vartheta\sin\vartheta\cos\varphi \ . \tag{3.5}$$

In addition,

$$C_1 = k_{n,\alpha} n \tag{3.6}$$

if $-n < \alpha \leq n$.

In particular,

$$C_2 = \sqrt{\omega_{n-1}} k_{n,\alpha} \left\{ \frac{\sqrt{\pi}(n+\alpha)n(n+2)\Gamma\left(\frac{n}{2}-1\right)\Gamma\left(\frac{n}{2}+\alpha\right)}{8(n+1+\alpha)\Gamma(n+\alpha)} \right\}^{1/2}$$

for $-(n/2) < \alpha \leq n(n+1)/2$.

For $p = 1$ and $p = 2$ the coefficient $\mathcal{C}_p(x)$ is sharp in conditions of the Proposition also in the weaker inequality obtained from (3.1) by replacing $\nabla u_{_f}$ by $\partial u_{_f}/\partial x_{n+1}$.

Proof. The equality (3.2) was proved in Proposition 1.

(i) Let $p = 1$. Using (2.5), (2.9) and the permutability of two suprema, we find

$$\begin{aligned} C_1 &= k_{n,\alpha} \sup_{|\boldsymbol{z}|=1} \sup_{\sigma\in\mathbb{S}^n_+} \big|\big(\alpha \boldsymbol{e}_{n+1} - (n+\alpha)(\boldsymbol{e}_\sigma, \boldsymbol{e}_{n+1})\boldsymbol{e}_\sigma,\ \boldsymbol{z}\big)\big| \big(\boldsymbol{e}_\sigma, \boldsymbol{e}_{n+1}\big)^{n+\alpha} \\ &= k_{n,\alpha} \sup_{\sigma\in\mathbb{S}^n_+} \big|\alpha \boldsymbol{e}_{n+1} - (n+\alpha)(\boldsymbol{e}_\sigma, \boldsymbol{e}_{n+1})\boldsymbol{e}_\sigma\big| \big(\boldsymbol{e}_\sigma, \boldsymbol{e}_{n+1}\big)^{n+\alpha} \ . \end{aligned} \tag{3.7}$$

Taking into account the equality

$$\begin{aligned}&\big|\alpha \boldsymbol{e}_{n+1}-(n+\alpha)(\boldsymbol{e}_\sigma,\boldsymbol{e}_{n+1})\boldsymbol{e}_\sigma\big|\\ &=\Big(\alpha \boldsymbol{e}_{n+1}-(n+\alpha)(\boldsymbol{e}_\sigma,\boldsymbol{e}_{n+1})\boldsymbol{e}_\sigma,\ \alpha \boldsymbol{e}_{n+1}-(n+\alpha)(\boldsymbol{e}_\sigma,\boldsymbol{e}_{n+1})\boldsymbol{e}_\sigma\Big)^{1/2}\\ &=\Big(\alpha^2+((n+\alpha)^2-2\alpha(n+\alpha))(\boldsymbol{e}_\sigma,\ \boldsymbol{e}_{n+1})^2\Big)^{1/2},\end{aligned}$$

and using (2.3), (3.7), we arrive at the sharp constant (3.6) for $-n<\alpha\le n$.

Furthermore, by (2.5),

$$C_1(\boldsymbol{e}_{n+1})=k_{n,\alpha}\sup_{\sigma\in\mathbb{S}^n_+}\big|\alpha-(n+\alpha)(\boldsymbol{e}_\sigma,\boldsymbol{e}_{n+1})^2\big|\big(\boldsymbol{e}_\sigma,\boldsymbol{e}_{n+1}\big)^{n+\alpha}\ge k_{n,\alpha}n.$$

Hence, by $C_1\ge C_1(\boldsymbol{e}_{n+1})$ and by (3.6) we obtain $C_1=C_1(\boldsymbol{e}_{n+1})$, which completes the proof in the case $p=1$.

(ii) Let $1<p<\infty$. Since the integrand in (2.6) does not change when $\boldsymbol{z}\in\mathbb{S}^n$ is replaced by $-\boldsymbol{z}$, we may assume that $z_{n+1}=(\boldsymbol{e}_{n+1},\boldsymbol{z})>0$ in (2.9).

Let $\boldsymbol{z}'=\boldsymbol{z}-z_{n+1}\boldsymbol{e}_{n+1}$. Then $(\boldsymbol{z}',\boldsymbol{e}_{n+1})=0$ and hence $z^2_{n+1}+|\boldsymbol{z}'|^2=1$. Analogously, with $\sigma=(\sigma_1,\dots,\sigma_n,\sigma_{n+1})\in\mathbb{S}^n_+$, we associate the vector $\boldsymbol{\sigma}'=\boldsymbol{e}_\sigma-\sigma_{n+1}\boldsymbol{e}_{n+1}$.

Using the equalities $(\boldsymbol{\sigma}',\boldsymbol{e}_{n+1})=0$, $\sigma_{n+1}=\sqrt{1-|\boldsymbol{\sigma}'|^2}$ and $(\boldsymbol{z}',\boldsymbol{e}_{n+1})=0$, we find an expression for $(\alpha\boldsymbol{e}_{n+1}-(n+\alpha)(\boldsymbol{e}_\sigma,\boldsymbol{e}_{n+1})\boldsymbol{e}_\sigma,\ \boldsymbol{z}\big)$ as a function of $\boldsymbol{\sigma}'$:

$$\begin{aligned}&(\alpha\boldsymbol{e}_{n+1}-(n+\alpha)(\boldsymbol{e}_\sigma,\boldsymbol{e}_{n+1})\boldsymbol{e}_\sigma,\ \boldsymbol{z}\big)=\alpha z_{n+1}-(n+\alpha)\sigma_{n+1}\big(\boldsymbol{e}_\sigma,\boldsymbol{z}\big)\\ &=\alpha z_{n+1}-(n+\alpha)\sigma_{n+1}\big(\boldsymbol{\sigma}'+\sigma_{n+1}\boldsymbol{e}_{n+1},\ \boldsymbol{z}'+z_{n+1}\boldsymbol{e}_{n+1}\big)\\ &=\alpha z_{n+1}-(n+\alpha)\sigma_{n+1}\big[\big(\boldsymbol{\sigma}',\boldsymbol{z}'\big)+z_{n+1}\sigma_{n+1}\big]\\ &=-\big[(n+\alpha)(1-|\boldsymbol{\sigma}'|^2)-\alpha\big]z_{n+1}-(n+\alpha)\sqrt{1-|\boldsymbol{\sigma}'|^2}\ \big(\boldsymbol{\sigma}',\boldsymbol{z}'\big).\end{aligned}\tag{3.8}$$

Let $\mathbb{B}^n=\{x'=(x_1,\dots,x_n)\in\mathbb{R}^n:|x'|<1\}$. By (2.6) and (3.8), taking into account that $d\sigma=d\sigma'/\sqrt{1-|\boldsymbol{\sigma}'|^2}$, we may write (2.9) as

$$\begin{aligned}C_p&=k_{n,\alpha}\sup_{\boldsymbol{z}\in\mathbb{S}^n_+}\left\{\int_{\mathbb{B}^n}\frac{\mathcal{H}_{n,p}\big(|\boldsymbol{\sigma}'|,(\boldsymbol{\sigma}',\boldsymbol{z}')\big)\big(1-|\boldsymbol{\sigma}'|^2\big)^{(\alpha p+n+1)/2(p-1)}}{\sqrt{1-|\boldsymbol{\sigma}'|^2}}\,d\sigma'\right\}^{\frac{p-1}{p}}\\ &=k_{n,\alpha}\sup_{\boldsymbol{z}\in\mathbb{S}^n_+}\left\{\int_{\mathbb{B}^n}\mathcal{H}_{n,p}\big(|\boldsymbol{\sigma}'|,(\boldsymbol{\sigma}',\boldsymbol{z}')\big)\big(1-|\boldsymbol{\sigma}'|^2\big)^{((\alpha-2)p+n+2)/2(p-1)}d\sigma'\right\}^{\frac{p-1}{p}},\end{aligned}\tag{3.9}$$

where

$$\mathcal{H}_{n,p}\big(|\boldsymbol{\sigma}'|,(\boldsymbol{\sigma}',\boldsymbol{z}')\big)=\Big|\big[(n+\alpha)(1-|\boldsymbol{\sigma}'|^2)-\alpha\big]z_{n+1}+(n+\alpha)\sqrt{1-|\boldsymbol{\sigma}'|^2}\ \big(\boldsymbol{\sigma}',\boldsymbol{z}'\big)\Big|^{p/(p-1)}.\tag{3.10}$$

Using the well known formula (see e.g. [8], **3.3.2(3)**),

$$\int_{B^n}g\big(|\boldsymbol{x}|,(\boldsymbol{a},\boldsymbol{x})\big)dx=\omega_{n-1}\int_0^1 r^{n-1}dr\int_0^\pi g\big(r,|\boldsymbol{a}|r\cos\varphi\big)\sin^{n-2}\varphi\ d\varphi\ ,$$

we obtain

$$\int_{\mathbb{B}^n} \mathcal{H}_{n,p}\big(|\boldsymbol{\sigma}'|, (\boldsymbol{\sigma}', \boldsymbol{z}')\big)\big(1-|\boldsymbol{\sigma}'|^2\big)^{((\alpha-2)p+n+2)/2(p-1)}\, d\sigma'$$

$$= \omega_{n-1}\int_0^1 r^{n-1}\big(1-r^2\big)^{((\alpha-2)p+n+2)/2(p-1)}dr\int_0^\pi \mathcal{H}_{n,p}\big(r, r|\boldsymbol{z}'|\cos\varphi\big)\sin^{n-2}\varphi d\varphi\ .$$

Making the change of variable $r=\sin\vartheta$ in the right-hand side of the last equality, we find

$$\int_{\mathbb{B}^n} \mathcal{H}_{n,p}\big(|\boldsymbol{\sigma}'|, (\boldsymbol{\sigma}', \boldsymbol{z}')\big)\big(1-|\boldsymbol{\sigma}'|^2\big)^{\frac{(\alpha-2)p+n+2}{2(p-1)}}\, d\sigma' \tag{3.11}$$

$$= \omega_{n-1}\int_0^\pi \sin^{n-2}\varphi d\varphi\int_0^{\pi/2} \mathcal{H}_{n,p}\big(\sin\vartheta,\ |\boldsymbol{z}'|\sin\vartheta\cos\varphi\big)\sin^{n-1}\vartheta\cos^{\frac{(\alpha-1)p+n+1}{p-1}}\vartheta d\vartheta\ ,$$

where, by (3.10),

$$\mathcal{H}_{n,p}\big(\sin\vartheta,\ |\boldsymbol{z}'|\sin\vartheta\cos\varphi\big) = \Big|\big((n+\alpha)\cos^2\vartheta-\alpha\big)z_{n+1}+(n+\alpha)|\boldsymbol{z}'|\cos\vartheta\sin\vartheta\cos\varphi\Big|^{p/(p-1)}.$$

Introducing here the parameter $\gamma=|\boldsymbol{z}'|/z_{n+1}$ and using the equality $|\boldsymbol{z}'|^2+z^2_{n+1}=1$, we obtain

$$\mathcal{H}_{n,p}\big(\sin\vartheta,\ |\boldsymbol{z}'|\sin\vartheta\cos\varphi\big) = (1+\gamma^2)^{-p/2(p-1)}\big|\mathcal{G}_n(\varphi,\vartheta;\gamma)\big|^{p/(p-1)}, \tag{3.12}$$

where $\mathcal{G}_n(\varphi,\vartheta;\gamma)$ is given by (3.5).

By (3.9), taking into account (3.11) and (3.12), we arrive at (3.3).

(iii) Let $p=2$. By (3.3), (3.4) and (3.5),

$$C_2 = \sqrt{\omega_{n-1}}\, k_{n,\alpha}\ \sup_{\gamma\geq 0}\ \frac{1}{\sqrt{1+\gamma^2}}\left\{\int_0^\pi d\varphi\int_0^{\pi/2}\mathcal{F}_{n,2}(\varphi,\vartheta;\gamma)\, d\vartheta\right\}^{1/2}, \tag{3.13}$$

where

$$\mathcal{F}_{n,2}(\varphi,\vartheta;\gamma)=\big[\big((n+\alpha)\cos^2\vartheta-\alpha\big)+\gamma(n+\alpha)\cos\vartheta\sin\vartheta\cos\varphi\big]^2\cos^{n-1+2\alpha}\vartheta\sin^{n-1}\vartheta\sin^{n-2}\varphi.$$

The last equality and (3.13) imply

$$C_2 = \sqrt{\omega_{n-1}}\, k_{n,\alpha}\ \sup_{\gamma\geq 0}\ \frac{1}{\sqrt{1+\gamma^2}}\left\{\mathcal{I}_1+\gamma^2\mathcal{I}_2\right\}^{1/2}, \tag{3.14}$$

where

$$\mathcal{I}_1 = \int_0^\pi \sin^{n-2}\varphi\, d\varphi\int_0^{\pi/2}\big((n+\alpha)\cos^2\vartheta-\alpha\big)^2\sin^{n-1}\vartheta\cos^{n-1+2\alpha}\vartheta\, d\vartheta$$

$$= \frac{\sqrt{\pi}n(n+2)(n+\alpha)\,\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n+2+\alpha}{2}\right)}{4(n+2\alpha)(n+1+\alpha)\Gamma(n+\alpha)} \tag{3.15}$$

and

$$\begin{aligned}\mathcal{I}_2 &= (n+\alpha)^2 \int_0^{\pi} \sin^{n-2}\varphi \cos^2\varphi \, d\varphi \int_0^{\pi/2} \sin^{n+1}\vartheta \cos^{n+1+2\alpha}\vartheta \, d\vartheta \\ &= \frac{\sqrt{\pi}\,(n+\alpha)\,\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n+2+2\alpha}{2}\right)}{4(n+1+\alpha)\Gamma(n+\alpha)}. \end{aligned} \tag{3.16}$$

By (3.14) we have

$$C_2 = \sqrt{\omega_{n-1}}\, k_{n,\alpha}\ \max\left\{\mathcal{I}_1^{1/2}, \mathcal{I}_2^{1/2}\right\}. \tag{3.17}$$

Further, by (3.15) and (3.16),

$$\frac{\mathcal{I}_1}{\mathcal{I}_2} = \frac{n(n+2)}{n+2\alpha}.$$

Therefore,

$$\frac{\mathcal{I}_1}{\mathcal{I}_2} - 1 = \frac{n^2+n-2\alpha}{n+2\alpha}.$$

Taking into account (3.17) and that $n+2\alpha > 0$ for $p=2$ by (2.3), we see that inequality

$$\frac{\mathcal{I}_1}{\mathcal{I}_2} \geq 1$$

holds for $\alpha \leq n(n+1)/2$. So, we arrive at the representation for C_2 with $-(n/2) < \alpha \leq n(n+1)/2$ given in formulation of the Proposition.

Since $\boldsymbol{z} \in \mathbb{S}^n$ and the supremum in $\gamma = |\boldsymbol{z}'|/z_{n+1}$ in (3.13) is attained for $\gamma = 0$, we have $C_2 = C_2(\boldsymbol{e}_{n+1})$ under requirements of the Proposition. □

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