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Angle singularities of solutions to the Neumann problem for the two-dimensional Riccati's equation

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Abstract. A two-dimensional Riccati's equation with Neumann boundary data is considered in a domain with an angular point. Asymptotic formulas for an arbitrary solution near the vertex are obtained.

1. Introduction

Let K_δ be the sector

$$\{x = (x_1, x_2) \in \mathbb{R}^2\colon\ 0 < r < \delta,\ \theta \in (0, \varphi)\},$$

where (r, θ) are the polar coordinates of x and $\varphi \in (0, 2\pi]$. Consider the nonlinear boundary value problem

$$\Delta u + \alpha(x)(\partial_{x_1} u)^2 + 2\beta(x)\partial_{x_1} u \partial_{x_2} u + \gamma(x)(\partial_{x_2} u)^2 = 0 \quad \text{on } K_\delta, \tag{1}$$

$$\partial_\theta u|_{\theta=0} = \partial_\theta u|_{\theta=\varphi} = 0 \quad \text{for } r < \delta. \tag{2}$$

Here α, β and γ are measurable functions. We suppose that for almost all $x \in K_\delta$ and for all $(\xi_1, \xi_2) \in \mathbb{R}^2$,

$$\lambda\,(\xi_1^2 + \xi_2^2) \leqslant \alpha(x)\xi_1^2 + 2\beta(x)\xi_1\xi_2 + \gamma(x)\xi_2^2 \leqslant \Lambda\,(\xi_1^2 + \xi_2^2) \tag{3}$$

with positive constants λ and Λ. We assume everywhere that u belongs to the Sobolev space $H^2(G)$ for any open set G such that $\overline{G} \subset \overline{K}_\delta \setminus \{O\}$.

Our aim is to describe the asymptotic behaviour of u near the vertex O without a priori restrictions on its growth. We show that there exist two possibilities: either u is unbounded and then

$$u(x) = Q(r) + c_* + \mathrm{o}(1), \tag{4}$$

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where

$$Q(r) = \varphi \int_r^{\delta/\varepsilon} \frac{\mathrm{d}s}{s} \left(\iint_{x \in K_\delta \setminus K_s} \frac{\alpha(x)x_1^2 + 2\beta(x)x_1x_2 + \gamma(x)x_2^2}{|x|^4} \,\mathrm{d}x_1 \,\mathrm{d}x_2 \right)^{-1}, \tag{5}$$

or u is bounded and has the same asymptotics

$$u(x) = c_0 + c_1 r^{\pi/\varphi} \cos(\pi\theta/\varphi) + \mathrm{o}\big(r^{\pi/\varphi}\big) \tag{6}$$

as in the case of the Neumann problem for $\Delta u = 0$. Here c_*, c_0 and c_1 are real constants.

If the coefficients α, β and γ are constant we write the asymptotic expansion for unbounded solutions:

$$u(x) \sim d \log\log r^{-1} + c_* + \sum_{k=1}^{\infty} \frac{P_k(\log\log r^{-1}, \theta)}{(\log r^{-1})^k}, \tag{7}$$

where

$$d = \left(\frac{\alpha + \gamma}{2} + \beta \,\frac{\sin^2\varphi}{\varphi} + \frac{\alpha - \gamma}{4} \,\frac{\sin 2\varphi}{\varphi} \right)^{-1}, \tag{8}$$

and $P_k(\tau, \theta)$ are polynomials of degree $\leqslant k$ in τ whose coefficients are smooth functions of $\theta \in [0, \varphi]$. If u is bounded it admits the asymptotic representation

$$u(x) \sim c_0 + \sum_{k=1}^{\infty} r^{k\pi/\varphi} p_{k-1}(\log r, \theta), \tag{9}$$

where $c_k = \text{const}$ and p_k are polynomials in the first argument with smooth coefficients on $[0, \varphi]$.

We note that all our results and their proofs extend to the case when O is the center of the disk $K_\delta = \{x\colon r < \delta\}$. One should only put $\varphi = \pi$ in (6) and (9). In other words, we also describe the asymptotic behaviour of solutions to Eq. (1) which are either bounded at O or have an isolated singularity there.

We finish this paper by showing that problem (1), (2) has solutions with asymptotics (4).

It is worth noting that Eq. (1) and the Neumann conditions as well as assumption (3) about the coefficients α, β and γ are preserved under conformal mappings. Therefore, (4) and (6) along with asymptotics of conformal mappings (see [2]) imply asymptotic representations of solutions at infinity and near boundary singularities other than corners, for example, cusps.

2. Auxiliary ordinary differential equation

Lemma 2.1. *Let g be a locally integrable non-negative function on the interval $[t_0, \infty)$. Suppose that an absolutely continuous function $z = z(t)$, which is not identically zero for large t, satisfies the inequality*

$$\dot{z}(t) \leqslant -qz^2(t) - g(t) \quad \textit{for } t \geqslant t_0, \tag{10}$$

where q is a positive constant. Then z is a positive function and

$$z(t) \leqslant \frac{1}{q(t-c)} \quad \textit{for } t \geqslant t_0 \tag{11}$$

with

$$c = t_0 - \frac{1}{qz(t_0)}. \tag{12}$$

Moreover,

$$\int_t^\infty g(\tau)\,\mathrm{d}\tau \leqslant z(t) \quad \textit{for } t \geqslant t_0. \tag{13}$$

Proof. We show that z is non-negative. Let $z(t_1) < 0$ for a certain $t_1 \geqslant t_0$. From (10) it follows that z does not increase and, hence, $z(t) \leqslant z(t_1)$ for $t \geqslant t_1$. By (10), $\dot{z}(t)/z^2(t) \leqslant -q$. Integrating the last inequality over the interval (t, t_2), $t_1 \leqslant t \leqslant t_2$, we obtain

$$\frac{1}{z(t)} + q(t_2 - t) \leqslant \frac{1}{z(t_2)}.$$

The left-hand side tends to $+\infty$ when $t = t_2/2$ and $t_2 \to +\infty$, but the right-hand side is bounded. This contradiction shows that z is non-negative.

We prove that z is positive. Indeed, if $z(t_1) = 0$ for some $t_1 \geqslant t_0$ then $z(t) = 0$ for all $t \geqslant t_1$, since z does not increase and is non-negative. Thus $z > 0$.

We turn to inequality (11). The function $y(t) = q^{-1}(t-c)^{-1}$ satisfies

$$\dot{y}(t) = -qy^2(t), \qquad y(t_0) = z(t_0).$$

Therefore,

$$\dot{z} - \dot{y} \leqslant -q(z+y)(z-y). \tag{14}$$

If $y(t) \leqslant z(t)$ on an interval (t_0, t_3) then, by (14), $\dot{z} \leqslant \dot{y}$ on the same interval. Since $z(t_0) = y(t_0)$ it follows that $z(t) \leqslant y(t)$, $t \in (t_0, t_3)$. This proves (11).

In order to obtain (13) it suffices to integrate (10) over $(t, +\infty)$. □

Let us consider the equation

$$\dot{z}(t) + \mathcal{R}(t)z^2(t) + f(t) = 0 \quad \text{for } t > t_0, \tag{15}$$

where $\mathcal{R}$ and f are real-valued, measurable, bounded functions on $[t_0, \infty)$. We shall suppose that

$$\mathcal{R}(t) \geqslant q > 0 \quad \text{for } t \geqslant t_0 \quad \text{and} \quad f(t) = \mathrm{O}\big(t^{-3}\big) \quad \text{as } t \to +\infty. \tag{16}$$

We need the following standard comparison principle.

Lemma 2.2. *Let z and y be absolutely continuous non-negative functions on $[t_0,\infty)$ such that*

$$\dot z \leqslant -\mathcal{R}z^2 - f, \quad \dot y \geqslant -\mathcal{R}y^2 - f \quad \text{on } (t_0,\infty),$$

and $y(t_0) \geqslant z(t_0)$. Then $y(t) \geqslant z(t)$ for $t \geqslant t_0$.

Proof. Suppose that $y(t_1) = z(t_1)$ for some $t_1 \geqslant t_0$ and $y(t) < z(t)$ for $t \in (t_1,t_2)$. Then

$$\dot z - \dot y \leqslant -\mathcal{R}(z+y)(z-y) \leqslant 0 \quad \text{for } t \in (t_1,t_2).$$

Consequently, $y > z$ on (t_1,t_2). The result follows by contradiction. □

Lemma 2.3. *Let z be a non-negative solution of* (15)*. Then either*

$$\lim_{t\to+\infty} z(t)\int_{t_0}^{t} \mathcal{R}(\tau)\,\mathrm{d}\tau = 1 \tag{17}$$

or

$$z(t) = \mathrm{O}\big(t^{-2}\big) \quad \text{as } t \to +\infty. \tag{18}$$

Proof. First, we prove that

$$z(t) \leqslant \left(\int_{t_0}^{t} \mathcal{R}(\tau)\,\mathrm{d}\tau\right)^{-1} + C\left(\int_{t_0}^{t} \mathcal{R}(\tau)\,\mathrm{d}\tau\right)^{-3/2} \quad \text{for } t \geqslant t_1 \tag{19}$$

with some $C > 0$ and with t_1 being sufficiently large. Denote by $y(t)$ the right-hand side of (19). Then

$$\dot y + \mathcal{R}y^2 + f = \frac{1}{2}C\mathcal{R}\left(\int_{t_0}^{t} \mathcal{R}(\tau)\,\mathrm{d}\tau\right)^{-5/2} + C^2\mathcal{R}\left(\int_{t_0}^{t} \mathcal{R}(\tau)\,\mathrm{d}\tau\right)^{-3} + \mathrm{O}\big(t^{-3}\big) \geqslant 0,$$

provided C is sufficiently large and $t \geqslant t_1$. Moreover, we can suppose that (19) is valid for $t = t_1$. Reference to Lemma 2.2 proves (19) for all $t \geqslant t_1$.

Inequality (19) implies

$$\limsup_{t\to\infty} z(t)\int_{t_0}^{t} \mathcal{R}(\tau)\,\mathrm{d}\tau \leqslant 1.$$

If, additionally,

$$\liminf_{t\to\infty} z(t)\int_{t_0}^{t} \mathcal{R}(\tau)\,\mathrm{d}\tau \geqslant 1, \tag{20}$$

then we arrive at (17).

We suppose that (20) fails and prove (18). Let there exist a sequence $\{t_j\}_{j\geqslant 1}$ such that $t_j \to \infty$ as $j \to \infty$ and

$$z(t_j) \leqslant \varepsilon\biggl(\int_{t_0}^{t_j} \mathcal{R}(\tau)\,\mathrm{d}\tau\biggr)^{-1} \tag{21}$$

with $\varepsilon \in (0, 1)$. We put $y(t) = \varepsilon(\int_{t_0}^{t} \mathcal{R}(\tau)\,\mathrm{d}\tau)^{-1}$. Then

$$\dot{y} + \mathcal{R}y^2 + f = (\varepsilon^2 - \varepsilon)\mathcal{R}\biggl(\int_{t_0}^{t} \mathcal{R}(\tau)\,\mathrm{d}\tau\biggr)^{-2} + f \leqslant 0,$$

provided t is sufficiently large. By Lemma 2.2 this and (21) give the inequality

$$z(t) \leqslant \varepsilon\biggl(\int_{t_0}^{t} \mathcal{R}(\tau)\,\mathrm{d}\tau\biggr)^{-1} \quad \text{for } t \geqslant t_1,$$

where t_1 is sufficiently large. Using the last estimate we derive from (15)

$$\dot{z}(t) + \varepsilon\mathcal{R}\biggl(\int_{t_0}^{t} \mathcal{R}(\tau)\,\mathrm{d}\tau\biggr)^{-1} z(t) \geqslant -f(t) \quad \text{for } t \geqslant t_1$$

or, equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\biggl(z(t)\biggl(\int_{t_0}^{t} \mathcal{R}(\tau)\,\mathrm{d}\tau\biggr)^{\varepsilon}\biggr) \geqslant -\biggl(\int_{t_0}^{t} \mathcal{R}(\tau)\,\mathrm{d}\tau\biggr)^{\varepsilon} f(t). \tag{22}$$

By (19),

$$z(t)\biggl(\int_{t_0}^{t} \mathcal{R}(\tau)\,\mathrm{d}\tau\biggr)^{\varepsilon} \to 0 \quad \text{as } t \to \infty.$$

Integrating (22) over (t, ∞) we get

$$z(t) \leqslant \biggl(\int_{t_0}^{t} \mathcal{R}(\tau)\,\mathrm{d}\tau\biggr)^{-\varepsilon} \int_{t}^{\infty} \biggl(\int_{t_0}^{\tau} \mathcal{R}(s)\,\mathrm{d}s\biggr)^{\varepsilon} |f(\tau)|\,\mathrm{d}\tau.$$

Since $\varepsilon \in (0, 1)$, the left-hand side is $\mathrm{O}(t^{-2})$ and we arrive at (18). □

Lemma 2.4. *Let z be a non-negative solution of* (15) *and let* (17) *hold. Then*

$$z(t) = \biggl(\int_{t_0}^{t} \mathcal{R}(\tau)\,\mathrm{d}\tau\biggr)^{-1} + \mathrm{O}\bigl(t^{-2}\log t\bigr). \tag{23}$$

Proof. We represent z as

$$z(t) = \Big(\int_{t_0}^{t} \mathcal{R}(\tau)\,\mathrm{d}\tau \Big)^{-1} \big(1 + \zeta(t)\big).$$

By (17), the function $\zeta(t)$ tends to 0 as $t \to \infty$ and satisfies

$$\dot{\zeta}(t) = -\mathcal{R}(t)\Big(\int_{t_0}^{t} \mathcal{R}(\tau)\,\mathrm{d}\tau \Big)^{-1} \big(\zeta(t) + \zeta^2(t)\big) - \Big(\int_{t_0}^{t} \mathcal{R}(\tau)\,\mathrm{d}\tau \Big) f(t). \tag{24}$$

We denote

$$G(t) = \int_{t_1}^{t} \mathcal{R}(\tau)\Big(\int_{t_0}^{\tau} \mathcal{R}(s)\,\mathrm{d}s \Big)^{-1} \big(1 + \zeta(\tau)\big)\,\mathrm{d}\tau,$$

where $t_1 > t_0$. Then

$$\lim_{t\to+\infty} G(t)\Big(\log\Big(\int_{t_0}^{t} \mathcal{R}(\tau)\,\mathrm{d}\tau \Big)\Big)^{-1} = 1, \tag{25}$$

and Eq. (24) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{e}^{G(t)}\zeta(t) \Big) = -\mathrm{e}^{G(t)}\Big(\int_{t_0}^{t} \mathcal{R}(\tau)\,\mathrm{d}\tau \Big) f(t). \tag{26}$$

Hence, and by (25), $\zeta(t) = \mathrm{O}(t^{-1+\varepsilon})$ as $t \to \infty$, where ε is an arbitrary positive number. Using this relation we can improve (25) as follows:

$$G(t) = \log\Big(\int_{t_0}^{t} \mathcal{R}(\tau)\,\mathrm{d}\tau \Big) + \mathrm{O}(1). \tag{27}$$

Hence, and by (26), we have $\zeta(t) = \mathrm{O}(t^{-1}\log t)$. The proof is complete. □

Lemma 2.5. *Let z be a non-negative solution of* (15) *and let* (18) *hold. Then*

$$|z(t)| \leqslant c \int_{t}^{\infty} |f(\tau)|\,\mathrm{d}\tau, \tag{28}$$

where c is independent of t and f.

Proof. From (15) we derive that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{e}^{\int_{t_0}^{t} \mathcal{R}(\tau)z(\tau)\,\mathrm{d}\tau} z(t) \Big) = -\mathrm{e}^{\int_{t_0}^{t} \mathcal{R}(\tau)z(\tau)\,\mathrm{d}\tau} f(t).$$

Due to (18), the function $t \to \int_{t_0}^{t} \mathcal{R}(\tau)z(\tau)\,\mathrm{d}\tau$ is bounded. By integrating the last equality we arrive at (28). □

3. Splitting of Eq. (1)

Using the coordinates $t = -\log r$ and θ we rewrite (1) and (2) as

$$(\partial_t^2 + \partial_\theta^2)u + A(\partial_t u)^2 + 2B\partial_t u \partial_\theta u + C(\partial_\theta u)^2 = 0 \quad \text{for } t > t_0,\ 0 < \theta < \varphi, \tag{29}$$

$$\partial_\theta u|_{\theta=0} = \partial_\theta u|_{\theta=\varphi} = 0 \quad \text{for } t > t_0, \tag{30}$$

where $t_0 = -\log\delta$ and

$$A(t,\theta) = \alpha\cos^2\theta + 2\beta\sin\theta\cos\theta + \gamma\sin^2\theta,$$
$$B(t,\theta) = (\alpha-\gamma)\sin\theta\cos\theta + \beta(\sin^2\theta - \cos^2\theta),$$
$$C(t,\theta) = \alpha\sin^2\theta - 2\beta\sin\theta\cos\theta + \gamma\cos^2\theta.$$

The functions $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ have to be calculated at $x = (\mathrm{e}^{-t}\cos\theta, \mathrm{e}^{-t}\sin\theta)$. From (3) it follows that

$$\lambda(\xi_1^2+\xi_2^2) \leqslant A\xi_1^2 + 2B\xi_1\xi_2 + C\xi_2^2 \leqslant \Lambda(\xi_1^2+\xi_2^2). \tag{31}$$

We represent the function u as

$$u(t,\theta) = h(t) + v(t,\theta), \tag{32}$$

where $h(t) = \frac{1}{\varphi}\int_0^\varphi u(t,\theta)\,\mathrm{d}\theta$. Then

$$\int_0^\varphi v(t,\theta)\,\mathrm{d}\theta = 0 \tag{33}$$

and

$$\partial_\theta v(t,0) = \partial_\theta v(t,\varphi) = 0 \quad \text{for } t > t_0. \tag{34}$$

Inserting (32) in (29) and then integrating with respect to θ over the interval $(0,\varphi)$ we arrive at the equation

$$\ddot{h} + \overline{A}(t)\dot{h}^2 + f(t) = 0 \quad \text{for } t > t_0, \tag{35}$$

where $\overline{A}(t) = (1/\varphi)\int_0^\varphi A(t,\theta)\,\mathrm{d}\theta$ and

$$f(t) = \frac{1}{\varphi}\int_0^\varphi \bigl\{A\bigl(2\dot{h}\partial_t v + (\partial_t v)^2\bigr) + 2B\bigl(\dot{h} + \partial_t v\bigr)\partial_\theta v + C(\partial_\theta v)^2\bigr\}\,\mathrm{d}\theta. \tag{36}$$

Subtracting (35) from (29) we obtain

$$\partial_t^2 v + \partial_\theta^2 v = (\overline{A} - A)\dot{h}^2 - \mathfrak{f}, \tag{37}$$

where

$$\mathfrak{f} = A\big(2\dot{h}\partial_t v + (\partial_t v)^2\big) + 2B\big(\dot{h} + \partial_t v\big)\partial_\theta v + C(\partial_\theta v)^2 - f(t). \tag{38}$$

Clearly,

$$\int_0^\varphi \mathfrak{f}(t,\theta)\,\mathrm{d}\theta = 0. \tag{39}$$

Thus, the boundary value problem (29), (30) is split into system (35), (37) completed by conditions (33) and (34).

4. Auxiliary estimates for h and v

Lemma 4.1. *The following estimates hold*:

$$\int_t^\infty \int_0^\varphi \big((\partial_\tau v)^2 + (\partial_\theta v)^2\big)\,\mathrm{d}\theta\,\mathrm{d}\tau \leqslant \frac{\varphi}{\lambda^2(t-c)} \tag{40}$$

and

$$0 \leqslant \dot{h}(t) \leqslant \frac{1}{\lambda(t-c)} \tag{41}$$

for $t > t_0$, *where* c *is defined by* (12).

Proof. From (29) and (31) we derive $(\partial_t^2 + \partial_\theta^2)u + \lambda((\partial_t u)^2 + (\partial_\theta u)^2) \leqslant 0$. Integrating this inequality over the interval $(0,\varphi)$ and using (30) we arrive at

$$\ddot{h}(t) + \lambda\dot{h}^2(t) + g(t) \leqslant 0 \quad \text{for } t > t_0, \tag{42}$$

where

$$g(t) = \frac{\lambda}{\varphi}\int_0^\varphi \big((\partial_t v)^2(t,\theta) + (\partial_\theta v)^2(t,\theta)\big)\,\mathrm{d}\theta.$$

Applying Lemma 2.1 to inequality (42) we obtain (40) and (41). □

5. Pointwise estimate for the gradient

We shall use the notations

$$\mathcal{C} = \big\{(t,\theta)\colon\ t \in \mathbb{R},\ \theta \in (0,\varphi)\big\} \quad \text{and} \quad \mathcal{C}_t = \big\{(\tau,\theta)\colon\ t < \tau < t+1,\ \theta \in (0,\varphi)\big\}.$$

Theorem 5.1. *Let* $u \in H^2(\mathcal{C}_t)$ *for any* $t > t_0$ *be a solution of* (29), (30). *Then*

$$\|\nabla_2 u\|_{L_2(\mathcal{C}_t)} = \mathrm{O}\bigl(t^{-1/2}\bigr) \tag{43}$$

and

$$\max_{0<\theta<\varphi} \bigl|\nabla u(t,\theta)\bigr| = \mathrm{O}\bigl(t^{-1/2}\bigr). \tag{44}$$

Proof. Let

$$\Phi(\xi,\zeta) = A\xi_1\zeta_1 + B(\xi_1\zeta_2 + \xi_2\zeta_1) + C\xi_2\zeta_2,$$

where ξ and ζ are points in $\mathbb{R}^2$. We introduce a cut-off function $\eta \in C_0^\infty(-1,2)$, $\eta = 1$ on $(0,1)$, and set $\eta_t(\tau) = \eta(\tau - t)$. It will be convenient to make u orthogonal to 1 on the set

$$\mathcal{C}'_t = \bigl\{(\tau,\theta)\colon\ t-1 < \tau < t+2,\ \theta \in (0,\varphi)\bigr\}.$$

We rewrite (29) as

$$-\Delta(\eta_t u) = \Phi\bigl(\nabla u, \nabla(\eta_t u) - u\nabla\eta_t\bigr) - 2\nabla\eta_t\nabla u - u\Delta\eta_t. \tag{45}$$

Hence, for $p \in (1,2)$,

$$\|\eta_t u\|_{W_p^2(\mathcal{C}'_t)} \leqslant c\bigl(\bigl\| |\nabla u|\, |\nabla(\eta_t u)| \bigr\|_{L_p(\mathcal{C}'_t)} + \|u\nabla u\|_{L_p(\mathcal{C}'_t)} + \|u\|_{W_p^1(\mathcal{C}'_t)}\bigr).$$

The first norm on the right does not exceed

$$c\|\nabla u\|_{L_2(\mathcal{C}'_t)}\bigl\|\nabla(\eta_t u)\bigr\|_{L_{2p/(2-p)}(\mathcal{C}'_t)},$$

which is majorized by $ct^{-1/2}\|\eta_t u\|_{W_p^2(\mathcal{C}'_t)}$ due to (40), (41) and Sobolev's embedding $W_p^1(\mathcal{C}'_t)$ into $L_{2p/(2-p)}(\mathcal{C}'_t)$. Therefore, for sufficiently large t,

$$\|\eta_t u\|_{W_p^2(\mathcal{C}'_t)} \leqslant c\bigl(\|u\nabla u\|_{L_p(\mathcal{C}'_t)} + \|u\|_{W_p^1(\mathcal{C}'_t)}\bigr).$$

By Sobolev's embedding theorem and by (40), (41),

$$\|u\nabla u\|_{L_p(\mathcal{C}'_t)} \leqslant c\|\nabla u\|_{L_2(\mathcal{C}'_t)}\|u\|_{L_{2p/(2-p)}(\mathcal{C}'_t)} \leqslant c\|\nabla u\|^2_{L_2(\mathcal{C}'_t)} \leqslant ct^{-1}.$$

Furthermore,

$$\|u\|_{W_p^1(\mathcal{C}'_t)} \leqslant c\|\nabla u\|_{L_2(\mathcal{C}'_t)} \leqslant ct^{-1/2}.$$

Hence,

$$\|u\|_{W_p^2(\mathcal{C}'_t)} + \|u\|_{W^1_{2p/(2-p)}(\mathcal{C}'_t)} \leqslant ct^{-1/2}.$$

Therefore, the right-hand side of (45) belongs to $L_{p/(2-p)}(\mathcal{C}'_t)$ and its norm in this space does not exceed $ct^{-1/2}$. Thus,

$$\|\eta_t u\|_{W^2_{p/(2-p)}(\mathcal{C}'_t)} \leqslant ct^{-1/2}$$

for all $p \in (1,2)$. Finally, (44) follows by Sobolev's embedding theorem. □

6. On the Neumann problem for the Laplace operator in the strip

Here we consider the boundary value problem

$$\begin{cases} -(\partial_t^2 + \partial_\theta^2)v = F & \text{on } \mathcal{C}, \\ \partial_\theta v(t,0) = \partial_\theta v(t,\varphi) = 0 & \text{for } t \in \mathbb{R}, \end{cases} \tag{46}$$

where $\mathcal{C}$ is the same as in the previous section. We shall study this problem assuming that

$$\int_0^\varphi v(t,\theta)\,\mathrm{d}\theta = \int_0^\varphi F(t,\theta)\,\mathrm{d}\theta = 0. \tag{47}$$

We need the following assertion on the solvability of problem (46), (47).

Theorem 6.1. (i) Existence. *Let $F \in L_{2,\mathrm{loc}}(\mathcal{C})$ satisfy* (47) *and let*

$$\int_{-\infty}^{\infty} \mathrm{e}^{-(\pi/\varphi)|\tau|} \|F\|_{L_2(\mathcal{C}_\tau)}\,\mathrm{d}\tau < \infty. \tag{48}$$

Then problem (46) *has a solution $v \in H^2_{\mathrm{loc}}(\overline{\mathcal{C}})$ such that*

$$\|v\|_{H^2(\mathcal{C}_t)} \leqslant C \int_{-\infty}^{\infty} \mathrm{e}^{-(\pi/\varphi)|t-\tau|} \|F\|_{L_2(\mathcal{C}_\tau)}\,\mathrm{d}\tau, \tag{49}$$

where C is a positive constant which depends only on φ.

(ii) Uniqueness. *Solution $v \in H^2_{\mathrm{loc}}(\overline{\mathcal{C}})$ satisfying* (46), (47) *and subject to*

$$\|v\|_{L_2(\mathcal{C}_t)} = \mathrm{o}\big(\mathrm{e}^{(\pi/\varphi)|t|}\big) \quad \text{as } t \to \pm\infty \tag{50}$$

is unique.

Proof. (i) The system $\{w_k(\theta)\}_{k=0}^{\infty}$, where

$$w_0(\theta) = \frac{1}{\sqrt{\varphi}}, \qquad w_k(\theta) = \sqrt{\frac{2}{\varphi}} \cos\frac{k\pi\theta}{\varphi}, \quad k > 0,$$

forms an orthonormal basis in $L_2(0,\varphi)$. Due to (47) we can represent the function F as the Fourier series

$$F(t,\theta) = \sum_{k=1}^{\infty} F_k(t) w_k(\theta) \tag{51}$$

with $F_k(t) = \int_0^\varphi F(t,\theta) w_k(\theta)\,\mathrm{d}\theta$. It is clear that $F_k \in L_{2,\mathrm{loc}}(\mathbb{R})$ and that

$$\|F\|_{L_2(\mathcal{C}_t)} = \left(\sum_{k=1}^{\infty} \|F_k\|^2_{L_2(t,t+1)}\right)^{1/2}.$$

We are looking for a solution v in the form

$$v(t,\theta) = \sum_{k=1}^{\infty} v_k(t) w_k(\theta). \tag{52}$$

It satisfies (47) and there exist positive constants C_1 and C_2 such that

$$C_1\|v\|^2_{H^2(\mathcal{C}_t)} \leqslant \sum_{k=1}^{\infty}\left\{\left(1+\left(\frac{k\pi}{\varphi}\right)^4\right)\|v_k\|^2_{L_2(t,t+1)} + \|\ddot{v}_k\|^2_{L_2(t,t+1)}\right\} \leqslant C_2\|v\|^2_{H^2(\mathcal{C}_t)}. \tag{53}$$

Inserting (52) and (51) in (46) we obtain the equation for v_k:

$$-\ddot{v}_k(t) + \left(\frac{k\pi}{\varphi}\right)^2 v_k(t) = F_k(t).$$

Hence,

$$v_k(t) = \frac{\varphi}{2k\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-(k\pi/\varphi)|t-\tau|} F_k(\tau)\,\mathrm{d}\tau.$$

By Minkowski's inequality for the norm

$$\left\|\{g_k\}_{k=1}^{\infty}\right\| = \left(\sum_{k=1}^{\infty}\int_t^{t+1} \left|g_k(\tau)\right|^2 \mathrm{d}\tau\right)^{1/2},$$

we obtain

$$\left(\sum_{k=1}^{\infty}\left(\frac{k\pi}{\varphi}\right)^4 \int_t^{t+1} \left|v_k(x)\right|^2 \mathrm{d}x\right)^{1/2}$$
$$\leqslant \frac{1}{2}\int_{-\infty}^{\infty}\left(\sum_{k=1}^{\infty}\left(\frac{k\pi}{\varphi}\right)^2\left(\int_t^{t+1} \mathrm{e}^{-(k\pi/\varphi)|x-\tau|}\left|F_k(\tau)\right|\mathrm{d}x\right)^2\right)^{1/2} \mathrm{d}\tau.$$

By direct calculation one can verify that

$$\int_t^{t+1} \mathrm{e}^{-(k\pi/\varphi)|x-\tau|}\,\mathrm{d}x \leqslant \frac{C(\varphi)}{k}\mathrm{e}^{-(\pi/\varphi)|t-\tau|}.$$

Hence,

$$\|\partial_\theta^2 v\|_{L_2(\mathcal{C}_t)} \leqslant C\int_{-\infty}^{\infty} \mathrm{e}^{-(\pi/\varphi)|t-\tau|}\|F(\tau,\cdot)\|_{L_2(0,\varphi)}\,\mathrm{d}\tau \leqslant C\int_{-\infty}^{\infty} \mathrm{e}^{-(\pi/\varphi)|t-\tau|}\|F\|_{L_2(\mathcal{C}_\tau)}\,\mathrm{d}\tau.$$

The norms $\|v\|_{L_2(\mathcal{C}_t)}$, $\|\partial_t^2 v\|_{L_2(\mathcal{C}_t)}$ can be estimated analogously. Thus series (52) belongs to $H^2_{\mathrm{loc}}(\overline{\mathcal{C}})$ and satisfies (46), (47) and (49).

(ii) By (46) with $F = 0$, we have

$$\ddot{v}_k - \left(\frac{k\pi}{\varphi}\right)^2 v_k = 0.$$

Moreover, by (50),

$$v_k(t) = \mathrm{o}\big(\mathrm{e}^{-(\pi/\varphi)|t|}\big) \quad \text{as } t \to \pm\infty.$$

This gives $v_k = 0$. Hence $v = 0$. □

Here is a version of Theorem 6.1, which deals with the Neumann problem for the semistrip $S = (t_0,\infty)\times(0,\varphi)$.

Theorem 6.2. *Let $F \in L_2(\overline{S})$ and $v \in H^2_{\mathrm{loc}}(\overline{S})$ satisfy* (47) *for $t > t_0$ and*

$$\int_{t_0}^{\infty} \mathrm{e}^{-(\pi/\varphi)|\tau|}\|F\|_{L_2(\mathcal{C}_\tau)}\,\mathrm{d}\tau < \infty. \tag{54}$$

Let v be a solution of

$$\begin{cases} -(\partial_\theta^2 + \partial_t^2)v = F & \text{on } S, \\ \partial_\theta v|_{\theta=0} = \partial_\theta v|_{\theta=\varphi} = 0 & \text{for } t > t_0, \end{cases} \tag{55}$$

subject to

$$\|v\|_{L_2(\mathcal{C}_t)} = \mathrm{o}\big(\mathrm{e}^{(\pi/\varphi)t}\big) \quad \text{as } t \to +\infty. \tag{56}$$

Then

$$\|v\|_{H^2(\mathcal{C}_t)} \leqslant C\left(\int_{t_0}^{\infty} \mathrm{e}^{-(\pi/\varphi)|t-\tau|}\|F\|_{L_2(\mathcal{C}_t)}\,\mathrm{d}\tau + \mathrm{e}^{-(\pi/\varphi)t}\|v\|_{H^2(\mathcal{C}_{t_0})}\right) \tag{57}$$

for $t > t_0$.

Proof. Let $\eta = \eta(t)$ be a smooth function equal to 1 for $t > t_0 + 1$ and 0 for $t < t_0$. Then

$$-\big(\partial_\theta^2 + \partial_t^2\big)(\eta v) = \eta F - 2\dot{\eta}\partial_t v - \ddot{\eta} v \quad \text{on } \mathcal{C} \quad \text{and} \quad \partial_\theta(\eta v) = 0 \quad \text{on } \partial\mathcal{C}.$$

Moreover, ηv satisfies (50). Applying Theorem 6.1 we arrive at (57). □

7. Estimate for v

Here, as in Sections 3 and 4, the pair (h, v) is a solution of the boundary value problem (33)–(35), (37).

Lemma 7.1. *Let ε be an arbitrary positive number. Then there exists a number $t_1 > t_0$ depending on ε such that the estimate*

$$\|v\|_{H^2(\mathcal{C}_t)} \leqslant C_\varepsilon \biggl(\int_{t_1}^{\infty} \mathrm{e}^{-((\pi/\varphi)-\varepsilon)|t-\tau|} \|\dot{h}^2\|_{L_2(\tau,\tau+1)} \,\mathrm{d}\tau + \mathrm{e}^{-((\pi/\varphi)-\varepsilon)t} \biggr) \tag{58}$$

holds for $t > t_1$, where c_ε depends on ε.

Proof. Using (41) one can estimate function (38) as

$$\|\mathfrak{f}\|_{L_2(\mathcal{C}_t)} \leqslant Ct^{-1}\|v\|_{H^1(\mathcal{C}_t)} + C\bigl\|(\partial_\theta v)^2 + (\partial_\tau v)^2\bigr\|_{L_2(\mathcal{C}_t)}. \tag{59}$$

Applying the Gagliardo–Nirenberg inequality

$$\|w^2\|_{L_2(\mathcal{C}_t)} \leqslant C\|w\|_{L_2(\mathcal{C}_t)}\|w\|_{H^1(\mathcal{C}_t)},$$

and using (40) in order to estimate the second term in the right-hand side of (59), we get

$$\bigl\|(\partial_\theta v)^2 + (\partial_\tau v)^2\bigr\|_{L_2(\mathcal{C}_t)} \leqslant C\bigl(\|\partial_\theta v\|_{L_2(\mathcal{C}_t)} + \|\partial_\tau v\|_{L_2(\mathcal{C}_t)}\bigr)\|v\|_{H^2(\mathcal{C}_t)} \leqslant Ct^{-1/2}\|v\|_{H^2(\mathcal{C}_t)}. \tag{60}$$

Hence,

$$\|\mathfrak{f}\|_{L_2(\mathcal{C}_t)} \leqslant Ct^{-1/2}\|v\|_{H^2(\mathcal{C}_t)} \quad \text{for large } t. \tag{61}$$

Let $t_1 > t_0$. Due to Theorem 5.1 and (41) the right-hand side of (37) satisfies (54) and v is subject to (56). By (61) and by Theorem 6.2 with t_0 replaced by t_1 we obtain

$$\|v\|_{H^2(\mathcal{C}_t)} \leqslant C\biggl(\int_{t_1}^{\infty} \mathrm{e}^{-(\pi/\varphi)|t-\tau|}\bigl(\tau^{-1/2}\|v\|_{H^2(\mathcal{C}_\tau)} + \|\dot{h}^2\|_{L_2(\tau,\tau+1)}\bigr)\,\mathrm{d}\tau + \mathrm{e}^{-(\pi/\varphi)(t-t_1)}\|v\|_{H^2(\mathcal{C}_{t_1})}\biggr)$$

for $t > t_1$. We choose t_1 such that $t_1^{-1/2} = \kappa\varepsilon$, where κ is a constant depending only on φ. Then

$$\|v\|_{H^2(\mathcal{C}_t)} \leqslant C\kappa\varepsilon \int_{t_1}^{\infty} \mathrm{e}^{-(\pi/\varphi)|t-\tau|}\|v\|_{H^2(\mathcal{C}_\tau)}\,\mathrm{d}\tau + \Psi(t), \tag{62}$$

where

$$\Psi(t) = C\biggl(\int_{t_1}^{\infty} \mathrm{e}^{-(\pi/\varphi)|t-\tau|}\|\dot{h}^2\|_{L_2(\tau,\tau+1)}\,\mathrm{d}\tau + \mathrm{e}^{-(\pi/\varphi)(t-t_1)}\|v\|_{H^2(\mathcal{C}_{t_1})}\biggr). \tag{63}$$

Hence,

$$\|v\|_{H^2(\mathcal{C}_t)} \leqslant \Psi(t) + \sum_{j=1}^{N}(C\kappa\varepsilon)^j \int_{t_1}^{\infty}\cdots\int_{t_1}^{\infty} \mathrm{e}^{-(\pi/\varphi)(|t-\tau_1|+\cdots+|\tau_{j-1}-\tau_j|)}\Psi(\tau_j)\,\mathrm{d}\tau_1\ldots\mathrm{d}\tau_j$$
$$+ (C\kappa\varepsilon)^{N+1}\int_{t_1}^{\infty}\cdots\int_{t_1}^{\infty} \mathrm{e}^{-(\pi/\varphi)(|t-\tau_1|+\cdots+|\tau_N-\tau|)}\|v\|_{H^2(\mathcal{C}_\tau)}\,\mathrm{d}\tau_1\ldots\mathrm{d}\tau_N\,\mathrm{d}\tau. \tag{64}$$

The last multiple integral can be majorized by

$$\left(2\frac{\pi}{\varphi}\right)^{N+1}\int_{t_1}^{\infty} g_N(t-\tau)\|v\|_{H^2(\mathcal{C}_\tau)}\,\mathrm{d}\tau,$$

where g_N is Green's function of the operator $\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\frac{\pi^2}{\varphi^2}\right)^{N+1}$. This Green's function is given by

$$g_N(t) = \left(\frac{\varphi}{\pi}\right)^{2N+1}\mathrm{e}^{-(\pi/\varphi)|t|}p_N\left(\frac{\pi}{\varphi}|t|\right),$$

where

$$p_N(\tau) = \sum_{q=0}^{N}\frac{t^q}{q!}\binom{2N-q}{N}2^{-2N-1+q} \leqslant \sum_{q=0}^{N}\frac{\tau^q}{q!}, \quad \tau \geqslant 0.$$

Hence, and by the uniform boundedness of $\|v\|_{H^2(\mathcal{C}_\tau)}$, the last term in (64) tends to zero as $N\to\infty$ for sufficiently small ε. By taking the limit in (64) as $N\to\infty$ we arrive at

$$\|v\|_{H^2(\mathcal{C}_t)} \leqslant C\left(\int_{t_1}^{\infty} g_\varepsilon(t-\tau)\|\dot{h}^2\|_{L_2(\tau,\tau+1)}\,\mathrm{d}\tau + g_\varepsilon(t-t_1)\|v\|_{H^2(\mathcal{C}_{t_1})}\right), \tag{65}$$

where

$$g_\varepsilon(t-\tau) = g_0(t-\tau) + \sum_{k=1}^{\infty}\left(\frac{2\pi C\kappa\varepsilon}{\varphi}\right)^k\int_{\mathbb{R}^k} g_0(t-\tau_1)g_0(\tau_1-\tau_2)\cdots g_0(\tau_k-\tau)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\ldots\mathrm{d}\tau_k \tag{66}$$

with $g_0(t) = (\varphi/2\pi)\mathrm{e}^{-(\pi/\varphi)|t|}$. One can see that g_ε is Green's function of the operator $-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\frac{\pi^2}{\varphi^2}-\frac{2\pi C\kappa\varepsilon}{\varphi}$. Hence, we get

$$g_\varepsilon(t) = \frac{1}{2}\left(\frac{\pi^2}{\varphi^2}-\frac{2\pi C\kappa\varepsilon}{\varphi}\right)^{-1/2}\exp\left(-\left(\frac{\pi^2}{\varphi^2}-\frac{2\pi C\kappa\varepsilon}{\varphi}\right)^{1/2}|t|\right). \tag{67}$$

By choosing κ sufficiently small we have

$$g_\varepsilon(t) \leqslant C\exp\left(-\left(\frac{\pi}{\varphi}-\varepsilon\right)|t|\right).$$

The result follows from (65). □

Estimates (41) and (58) immediately give

Corollary 7.2. *The estimate*

$$\|v\|_{H^2(\mathcal{C}_t)} \leqslant Ct^{-2} \tag{68}$$

holds for $t > t_0$.

Since $|f(t)| \leqslant C\|v\|_{H^2(\mathcal{C}_t)}(\|v\|_{H^2(\mathcal{C}_t)} + \|\dot{h}\|_{L_2(\mathcal{C}_t)})$, estimates (41) and (68) imply

$$|f(t)| \leqslant Ct^{-3} \quad \text{for } t > t_0. \tag{69}$$

Proposition 7.3. *Let there exist a constant* $c > 0$ *such that*

$$|\dot{h}(t)| \leqslant ct^{-2} \quad \textit{for } t > t_0. \tag{70}$$

Then the estimate

$$\|v\|_{H^2(\mathcal{C}_t)} \leqslant C\left(\int_{t_1}^{\infty} \mathrm{e}^{-(\pi/\varphi)|t-\tau|}\|\dot{h}^2\|_{L_2(\tau,\tau+1)}\,\mathrm{d}\tau + \mathrm{e}^{-(\pi/\varphi)t}\right) \tag{71}$$

holds, where t_1 *is sufficiently large and* C *is a constant independent of* h.

Proof. Using (70) and (68) and reasoning as in the beginning of the proof of Proposition 7.1 we obtain

$$\|\mathfrak{f}\|_{L_2(\mathcal{C}_t)} \leqslant Ct^{-2}\|v\|_{H^2(\mathcal{C}_t)}.$$

Hence, the right-hand side F in (37) admits the estimate

$$\|F\|_{L_2(\mathcal{C}_t)} \leqslant Ct^{-2}\|v\|_{H^2(\mathcal{C}_t)} + C\|\dot{h}^2\|_{L_2(t,t+1)}.$$

Applying Theorem 6.2 to Eq. (37) on the semiaxis $t > t_1$ we arrive at the estimate

$$\|v\|_{H^2(\mathcal{C}_t)} \leqslant \Psi(t) + C\int_{t_1}^{\infty} \mathrm{e}^{-(\pi/\varphi)|t-\tau|}\tau^{-2}\|v\|_{H^2(\mathcal{C}_\tau)}\,\mathrm{d}\tau, \tag{72}$$

where Ψ is defined by (63). Iterating this inequality and arguing as in the proof of Lemma 7.1 we obtain

$$\|v\|_{H^2(\mathcal{C}_t)} \leqslant C\int_{t_1}^{\infty} g_\omega(t,\tau)\|\dot{h}^2\|_{L_2(\tau,\tau+1)}\,\mathrm{d}\tau + Cg_\omega(t,t_1)\|v\|_{H^2(\mathcal{C}_{t_1})}. \tag{73}$$

Here

$$g_\omega(t,\tau) = g_0(t-\tau) + \sum_{k=1}^{\infty}\int_{\mathbb{R}^k} g_0(t-\tau_1)\omega(\tau_1)g_0(\tau_1-\tau_2)\cdots\omega(\tau_k)g_0(\tau_k-\tau)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\ldots\mathrm{d}\tau_k \tag{74}$$

and

$$\omega(\tau) = \begin{cases} 2\pi C\varphi^{-1}\tau^{-2} & \text{for } \tau > t_1, \\ 0 & \text{otherwise.} \end{cases}$$

One can see that g_ω is Green's function of the operator $-\frac{d^2}{dt^2} + \frac{\pi^2}{\varphi^2} - \omega(t)$. Since $\omega(t) = O(t^{-2})$ as $t \to \infty$ and $\omega(t) < \pi^2/\varphi^2$, it follows from [1, Theorem 6.4.1] that

$$g_\omega(t,\tau) \leqslant C e^{-(\pi/\varphi)|t-\tau|} \quad \text{for } t, \tau > t_1.$$

This together with (73) leads to (71). □

Proposition 7.4. *Any solution h of* (35) *satisfies one of two alternatives*:

(i) *the relation*

$$\dot{h}(t) = \left(\int_{t_0}^{t} \overline{A}(\tau)\, d\tau \right)^{-1} + O\big(t^{-2}\log t\big) \tag{75}$$

holds;

(ii) *the estimate*

$$\big|\dot{h}(t)\big| \leqslant C e^{-(\pi/\varphi)t} \tag{76}$$

holds for large t.

Proof. Equation (35) coincides with (15) where $\mathcal{R} = \overline{A}$, $z = \dot{h}$, and f satisfies (16) by (69). By Lemma 2.3 there are two alternatives:

(i) Let (17) with $\mathcal{R} = \overline{A}$ be valid. Then, by Lemma 2.4, relation (75) holds.

(ii) Let (18) be valid. Then, by (68), (70) and by (71),

$$\big|f(t)\big| \leqslant Ct^{-2}\left(\int_{t_1}^{\infty} e^{-(\pi/\varphi)|t-\tau|} \big\|\dot{h}^2\big\|_{L_2(\tau,\tau+1)}\, d\tau + e^{-(\pi/\varphi)t} \right),$$

where t_1 is a large positive number. This together with (28) gives

$$\big|\dot{h}(t)\big| \leqslant Ct^{-1}\left(\int_{t_1}^{\infty} e^{-(\pi/\varphi)|t-\tau|} \big\|\dot{h}^2\big\|_{L_2(\tau,\tau+1)}\, d\tau + e^{-(\pi/\varphi)t} \right). \tag{77}$$

We introduce the function $w(t) = \max_{t\leqslant\tau\leqslant t+1} |\dot{h}|$. By (77) and (18) we have

$$w(t) \leqslant C\left(\int_{t_0}^{\infty} e^{-(\pi/\varphi)|t-\tau|} \tau^{-2} w(\tau)\, d\tau + e^{-(\pi/\varphi)t} \right).$$

Iterating this inequality (compare with the proof of Proposition 7.3) we obtain

$$w(t) \leqslant C\biggl(\int_{t_1}^{\infty} g_\omega(t,\tau)\mathrm{e}^{-(\pi/\varphi)\tau}\,\mathrm{d}\tau + \mathrm{e}^{-(\pi/\varphi)t}\biggr),$$

where g_ω is given by (74). This leads to

$$w(t) \leqslant C\mathrm{e}^{-(\pi/\varphi)t} \quad \text{for large } t,$$

which completes the proof. □

8. Principal terms of the asymptotics

The main result is the following

Theorem 8.1. *Let u be a solution of the boundary value problem* (1), (2). *Then*

$$u(x) = Q(r) + c_1 + w_1(x) \tag{78}$$

or

$$u(x) = c_2 + c_3 r^{\pi/\varphi}\cos(\pi\theta/\varphi) + w_2(x), \tag{79}$$

where Q is given by (5) *and c_1, c_2 and c_3 are constants. The remainder terms w_1 and w_2 admit the estimates*

$$\sum_{j+k\leqslant 2} r^{-j-k-1}\bigl\|\partial_{x_1}^j\partial_{x_2}^k w_1\bigr\|_{L_2(K_r\setminus K_{r/\mathrm{e}})} \leqslant C\frac{\log\log r^{-1}}{\log r^{-1}}$$

and

$$\sum_{j+k\leqslant 2} r^{-j-k-1}\bigl\|\partial_{x_1}^j\partial_{x_2}^k w_2\bigr\|_{L_2(K_r\setminus K_{r/\mathrm{e}})} \leqslant C_\varepsilon r^{(2\pi/\varphi)-\varepsilon}.$$

Here ε is an arbitrary positive number.

We reformulate this theorem in the coordinates (t,θ).

Theorem 8.2. *Let $u \in H^2_{\mathrm{loc}}([t_0,\infty)\times[0,\varphi])$ be a solution of* (29), (30). *Then one of the alternatives holds*:

(i) (unbounded solution)

$$u(t) = \int_{t_0}^{t}\biggl(\int_{t_0}^{\tau}\overline{A}(s)\,\mathrm{d}s\biggr)^{-1}\mathrm{d}\tau + c_1 + \rho(t,\theta), \tag{80}$$

where c_1 *is a constant and*

$$\|\rho\|_{H^2(\mathcal{C}_t)} \leqslant Ct^{-1}\log t \quad \textit{for } t > t_0; \tag{81}$$

(ii) (bounded solution)

$$u(t,\theta) = c_2 + c_3 \mathrm{e}^{-(\pi/\varphi)t}\cos(\pi\theta/\varphi) + \rho(t,\theta), \tag{82}$$

where

$$\|\rho\|_{H^2(\mathcal{C}_t)} \leqslant C_\varepsilon \mathrm{e}^{-((2\pi/\varphi)-\varepsilon)t} \quad \textit{for } t > t_0.$$

Proof. (i) Suppose that the alternative (i) in Proposition 7.4 is valid. By (75),

$$h(t) = \int_{t_0}^{t}\left(\int_{t_0}^{\tau}\overline{A}(s)\,\mathrm{d}s\right)^{-1}\mathrm{d}\tau + c_1 + h_1(t),$$

where $\|h_1\|_{H^2(t,t+1)} \leqslant Ct^{-1}\log t$. This, together with (68), implies (80) and (81) with $\rho = h_1 + v$.

(ii) Let estimate (76) hold. Then we can rewrite problem (29), (30) as

$$\begin{cases}(\partial_t^2+\partial_\theta^2)u = f(t,\theta) & \text{for } t>t_0,\ \theta\in(0,\varphi),\\ \partial_\theta u = 0 & \text{for } \theta = 0,\varphi \text{ and for } t>t_0,\end{cases} \tag{83}$$

where f satisfies $\|f\|_{L_2(t,t+1)} \leqslant C\mathrm{e}^{-2(\pi/\varphi)t}$. By (76) and (71),

$$\|v\|_{H^2(\mathcal{C}_t)} = \mathrm{o}\big(\mathrm{e}^{-(\pi/\varphi)t}\big) \quad \text{for large } t.$$

Representation (82) follows from well-known results on asymptotic behaviour of solutions to elliptic problems in a strip (see, for example, [3, Theorem 5.4.1]). □

9. Complete asymptotic expansion

Let us consider Eq. (1) with constant coefficients α, β and γ. In this case one can write the whole asymptotic representation for u.

Theorem 9.1. *If* α, β *and* γ *are constant then an arbitrary unbounded solution* u *of* (1), (2) *admits the asymptotic expansion*

$$u(x) = d\log\log r^{-1} + c + \sum_{k=1}^{N}\frac{P_k(\log\log r^{-1},\theta)}{(\log r^{-1})^k} + R_N(x_1,x_2),$$

where d *is given by* (8) *and* $P_k(\xi,\theta)$ *are polynomials of degree* $\leqslant k$ *in* ξ *whose coefficients are smooth functions of* $\theta\in[0,\varphi]$. *The remainder term* R_N *satisfies*

$$\sum_{k+j\leqslant 2} r^{k+j-1}\|\partial_{x_1}^k\partial_{x_2}^j R_N\|_{L_2(K_r\setminus K_{r/\mathrm{e}})} \leqslant C_N\frac{(\log\log r^{-1})^{N+1}}{(\log r^{-1})^{N+1}} \quad \textit{for small } r.$$

In order to prove Theorem 9.1 we need two lemmas.

Lemma 9.2. *Let* χ *be a solution of the equation*

$$\dot{\chi}(t) + d^{-1}\chi^2(t) + g(t) = 0 \quad \textit{for } t > t_0 > 1, \tag{84}$$

subject to

$$\chi(t) = \frac{d}{t} + \mathrm{O}\left(\frac{\log t}{t^2}\right). \tag{85}$$

Assume that

$$g(t) = \frac{\kappa}{t^3} + \sum_{k=4}^{N} \frac{q_{k-2}(\log t)}{t^k} + \mathrm{O}\left(\frac{(\log t)^{N-1}}{t^{N+1}}\right), \tag{86}$$

where $\kappa = \mathrm{const}$ *and* q_s *are polynomials of degree* $\leqslant s$*. Then*

$$\chi(t) = \frac{d}{t} + \sum_{k=2}^{N-1} \frac{p_{k-1}(\log t)}{t^k} + \chi_N(t), \tag{87}$$

where p_s *are polynomials of degree* $\leqslant s$ *and*

$$\chi_N(t) = \mathrm{O}\left(\frac{(\log t)^{N-1}}{t^N}\right). \tag{88}$$

Proof. By putting $\chi(t) = dt^{-1} + t^{-2}\psi(t)$ we arrive at the equation

$$\dot{\psi} = -d^{-1}t^{-2}\psi^2 - t^2 g \quad \text{for } t > t_0, \tag{89}$$

where $\psi(t) = \mathrm{O}(\log t)$. By integrating (89) over (t_0, t) we obtain

$$\psi(t) = -\kappa \log t + \mathrm{const} + \psi_1(t),$$

where $\psi_1(t) = \mathrm{O}(t^{-1}(\log t)^2)$. This gives (87) with $N = 3$, $\chi_3(t) = t^{-2}\psi_1(t)$ and $p_1(\xi) = -\kappa\xi + \mathrm{const}$. Clearly,

$$\psi_1(t) = d^{-1}\int_t^\infty \tau^{-2}\left(-\kappa\log\tau + \mathrm{const} + \psi_1(\tau)\right)^2 \mathrm{d}\tau + \int_t^\infty \left(\tau^2 g(\tau) + \kappa\tau^{-1}\right)\mathrm{d}\tau. \tag{90}$$

The result follows from (90) by induction in N. □

Lemma 9.3. *Let* v *be a solution of* (55) *with*

$$F(t,\theta) = \sum_{k=2}^{N} \frac{f_{k-2}(\log t, \theta)}{t^k} + F_N(t,\theta), \tag{91}$$

where f_s are polynomials in ξ of degree $\leqslant s$ with smooth coefficients on $[0,\varphi]$ and

$$\|F_N\|_{L_2(\mathcal{C}_t)} = \mathrm{O}\biggl(\frac{(\log t)^{N-1}}{t^{N+1}}\biggr). \tag{92}$$

Also let

$$\int_0^{\varphi} v(t,\theta)\,\mathrm{d}\theta = 0 \tag{93}$$

and $\|v\|_{H^2(\mathcal{C}_t)} = \mathrm{O}(1)$ for large t. Then

$$v(t,\theta) = \sum_{k=2}^{N} \frac{v_{k-2}(\log t,\theta)}{t^k} + V_N(t,\theta), \tag{94}$$

where $v_s(\xi,\theta)$ are polynomials in ξ of degree $\leqslant s$ with smooth coefficients on $[0,\varphi]$ and

$$\|V_N\|_{H^2(\mathcal{C}_t)} = \mathrm{O}\biggl(\frac{(\log t)^{N-1}}{t^{N+1}}\biggr) \tag{95}$$

for large t.

Proof. From (93) and (55) one derives

$$\int_0^{\varphi} f_s(\xi,\theta)\,\mathrm{d}\theta = \int_0^{\varphi} F_N(t,\theta)\,\mathrm{d}\theta = 0$$

for $s = 0,1,\ldots,N-2$ and for all ξ and $t > t_0$. Inserting (94) and (91) into (55) and equating terms with the same power of t we get

$$\begin{cases} -\partial_\theta^2 v_{k-2} = (k-2)(k-1)v_{k-4} - (2k-3)\dot{v}_{k-4} + \ddot{v}_{k-4} + f_{k-2}, \\ \partial_\theta v_{k-2}|_{\theta=0,\varphi} = 0 \end{cases} \tag{96}$$

for $k = 2,3,\ldots,N$ (here $v_{-2} = v_{-1} = 0$) and

$$\begin{cases} -(\partial_t^2 + \partial_\theta^2)V_N = F_N + \sum_{k=N-1}^{N}(k(k+1)v_{k-2} - (2k+1)\dot{v}_{k-2} + \ddot{v}_{k-2})t^{-k-2}, \\ \partial_\theta V_N|_{\theta=0,\varphi} = 0. \end{cases} \tag{97}$$

From (96) one can find all v_s, $s = 0,1,\ldots,N-2$, subject to $\int_0^{\varphi} v_s(\xi,\theta)\,\mathrm{d}\theta = 0$. Applying Theorem 6.2 to (97) we obtain estimate (95) for V_N. $\square$

Proof of Theorem 9.1. By Theorem 8.2 it suffices to obtain the required asymptotic expansion for the solution u of (29), (30) given by (80). This means that *a priori* u is given by

$$u(t,\theta) = d\log t + c + \rho(t,\theta),$$

where ρ satisfies (81). By (32) it is sufficient to establish the following asymptotic representation for $\dot{h}$ and v:

$$\dot{h}(t) = \frac{d}{t} + \sum_{k=2}^{N+1} \frac{p_{k-1}(\log t)}{t^k} + \mathrm{O}\biggl(\frac{(\log t)^{N+1}}{t^{N+2}}\biggr), \tag{98}$$

$$v(t) = \sum_{k=2}^{N} \frac{v_{k-2}(\log t, \theta)}{t^k} + V_N(t, \theta), \tag{99}$$

where $p_s(\xi)$ and $v_s(\xi, \theta)$ are polynomials in ξ of degree $\leqslant s$. The coefficients of v_s are smooth functions of $\theta \in [0, \varphi]$. The remainder term V_N should satisfy estimate (95). Since u is unbounded, $\dot{h}$ satisfies (75) which becomes in our case

$$\dot{h} = dt^{-1} + \mathrm{O}\bigl(t^{-2} \log t\bigr).$$

By this and (68) we obtain the asymptotic representation for the right-hand side in (37):

$$\bigl(d^{-1} - A\bigr) \frac{d^2}{t^2} + \mathrm{O}\bigl(t^{-3} \log t\bigr).$$

Applying Theorem 6.2 we get

$$v(t, \theta) = v_0(\theta) t^{-2} + \mathrm{O}\bigl(t^{-3} \log t\bigr), \tag{100}$$

where v_0 is the solution of

$$\begin{cases} \partial_\theta^2 v_0(\theta) = d^2\bigl(d^{-1} - A(\theta)\bigr) & \text{on } (0, \varphi), \\ \partial_\theta v_0|_{\theta=0,\varphi} = 0, \end{cases}$$

subject to the orthogonality condition $\int_0^\varphi v_0(\theta)\, \mathrm{d}\theta = 0$. By (36) and (100) the function f in (35) has the asymptotics

$$f(t) = \frac{2\dot{h}(t)}{\varphi t^2} \int_0^\varphi B(\theta) \partial_\theta v_0(\theta)\, \mathrm{d}\theta + \mathrm{O}\bigl(t^{-4} \log t\bigr).$$

Using Lemma 9.2 we obtain

$$\dot{h}(t) = dt^{-1} + \frac{p_1(\log t)}{t^2} + \mathrm{O}\biggl(\frac{(\log t)^2}{t^3}\biggr),$$

where p_1 is a linear polynomial in $\log t$. Now we can improve the asymptotic representation for the right-hand side in (37) and then the asymptotics for v. Continuing this iterative procedure we arrive at (98) and (99). □

Theorem 9.4. *If α, β and γ are constant functions then an arbitrary bounded solution u of* (1), (2) *admits the asymptotic expansion*

$$u(x) = c_0 + \sum_{k=1}^{N} r^{k\pi/\varphi} p_{k-1}(\log r, \theta) + w_N(r, \theta),$$

where c_0 is a constant, $p_s(\xi, \theta)$ is a polynomial of degree $\leqslant s$ in ξ whose coefficients are smooth of $\theta \in [0, \varphi]$ and

$$\sum_{k+j\leqslant 2} r^{k+j-1} \|\partial_{x_1}^k \partial_{x_2}^j w_N\|_{L_2(K_r \setminus K_{r/e})} \leqslant C_N r^{(N+1)\pi/\varphi} |\log r|^N$$

for small r.

Proof. By Theorem 8.1, representation (79) is valid. The result follows from Lemma 5.1.6 and Theorem 5.4.1 in [3]. □

10. Existence of a solution with prescribed asymptotics

We begin with an existence result from [2] (see Theorem 2.2) adjusted to problem (33), (34), (37). Let F denote the right-hand side in (37). The only information on F we need is the trivial inequality

$$\|F\|_{L_2(\mathcal{C}_t)} \leqslant C\big(\|\dot{h}\|_{L_2(t,t+1)} + \|v\|_{H^2(\mathcal{C}_t)}\big)^2.$$

Here the constant C depends only on Λ.

Theorem 10.1. *Let s be a continuous function on $[t_0, \infty)$ satisfying*

$$s(t) \geqslant \mathcal{C} \int_{t_0-1}^{\infty} \mathrm{e}^{(\pi/\varphi)|t-\tau|} \big(\|\dot{h}\|_{L_2(\tau,\tau+1)} + s(\tau)\big)\, \mathrm{d}\tau, \tag{101}$$

where $\mathcal{C}$ is a constant depending on Λ and φ. Let also

$$s(t) = \mathrm{o}\big(\mathrm{e}^{-(\pi/\varphi)t}\big) \quad \text{as } t \to \infty.$$

Then problem (33), (34), (37) *has a solution $v \in H^2_{\mathrm{loc}}([t_0, \infty) \times [0, \varphi])$ such that*

$$\|v\|_{H^2(\mathcal{C}_t)} \leqslant s(t).$$

Corollary 10.2. *Let t_0 be sufficiently large and let $h \in C^1(t_0, \infty)$ be subject to*

$$|\dot{h}(t)| \leqslant 2(\lambda t)^{-1} \tag{102}$$

for $t > t_0$. Then there exists a solution of problem (33), (34), (37) *satisfying*

$$\|v\|_{H^2(\mathcal{C}_t)} \leqslant C t^{-2} \quad \text{for } t > t_0.$$

Proof. Due to (102) inequality (101) has the solution $s(t) = ct^{-2}$ with c depending only on λ, Λ and φ. The result follows from Theorem 10.1. □

We prove the main existence result of the present section.

Theorem 10.3. *For sufficiently small* δ *there exists a solution of* (1), (2) *with the asymptotics*

$$u(x) = Q(r) + \mathrm{o}(1).$$

Proof. We are looking for a solution h of (34) in the form

$$\dot{h} = \left(\int_{t_0}^{t} \overline{A}(\tau)\,\mathrm{d}\tau \right)^{-1} \bigl(1 + z(t)\bigr),$$

where z is subject to $z(t) \leqslant Ct^{-1/2}$ for $t \geqslant t_0$ with a fixed constant C. Then for a sufficiently large t_0 the function $\dot{h}$ is subject to (102) and z satisfies the equation

$$\dot{z}(t) + \overline{A}(t)\left(\int_{t_0}^{t} \overline{A}(\tau)\,\mathrm{d}\tau \right)^{-1} \bigl(z(t) + z^2(t)\bigr) + \mathcal{F}(z)(t) = 0,$$

where $\mathcal{F}$ is a nonlinear operator subject to the estimate $|\mathcal{F}(z)(t)| \leqslant Ct^{-2}$ by Corollary 10.2 and by (35). Now the result follows by a standard fixed point argument. □

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