Computation of volume potentials over bounded domains via approximate approximations

F. Lanzara[1] , V. Maz'ya[2] , G. Schmidt[3]

[1] *Department of Mathematics, Sapienza University of Rome,*
Piazzale Aldo Moro 2, 00185 Rome, Italy
lanzara@mat.uniroma1.it

[2] *Department of Mathematics, University of Linköping,*
581 83 Linköping, Sweden;
Department of Mathematical Sciences, M&O Building,
University of Liverpool, Liverpool L69 3BX, UK;
vlmaz@mai.liu.se

[3] *Weierstrass Institute for Applied Analysis and Stochastics,*
Mohrenstr. 39, 10117 Berlin, Germany
schmidt@wias-berlin.de

Abstract. We obtain cubature formulas of volume potentials over bounded domains combining the basis functions introduced in the theory of approximate approximations with their integration over the tangential-halfspace. Then the computation is reduced to the quadrature of one dimensional integrals over the halfline. We conclude the paper providing numerical tests which show that these formulas give very accurate approximations and confirm the predicted order of convergence.

1 Introduction

We consider the volume potential of modified Helmholtz operators in $\mathbb{R}^n$

$$\mathcal{A}_n = -\Delta + \lambda^2$$

with $\lambda \in \mathbb{C}$. If $\lambda^2 \neq 0$, then the fundamental solution $\kappa_\lambda(\mathbf{x})$ is given by

$$\kappa_\lambda(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} \Big(\frac{|\mathbf{x}|}{\lambda} \Big)^{1-n/2} K_{n/2-1}(\lambda |\mathbf{x}|) \,,$$

where $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ and K_ν is the modified Bessel function of the second kind. The fundamental solutions of the Laplacian ($\lambda = 0$) are well-known

$$\kappa_0(\mathbf{x}) = \begin{cases} \dfrac{1}{2\pi} \log \dfrac{1}{|\mathbf{x}|} \,, & n = 2 \,, \\ \dfrac{\Gamma(\frac{n}{2} - 1)}{4\pi^{n/2}} \dfrac{1}{|\mathbf{x}|^{n-2}} \,, & n \geq 3 \,. \end{cases}$$

For $f \in C^1(\Omega)$, $\Omega \subset \mathbb{R}^n$, the volume potential

$$u(\mathbf{x}) = \mathcal{K}_\lambda f(\mathbf{x}) = \int_\Omega \kappa_\lambda(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d\mathbf{y} \tag{1.1}$$

provides a solution of

$$\mathcal{A}_n u = \begin{cases} f(\mathbf{x}) & \mathbf{x} \in \Omega \\ \\ 0 & otherwise \end{cases}$$

We study cubature formulas for the volume potential (1.1) using the concept *approximate approximations* (see [4]). The cubature of volume potentials over the full space and over high-dimensional halfspaces has already been studied in [1] and [2], respectively. In [5] cubature formulas based on approximate approximations for the single layer harmonic potential were considered.

Assume $f \in C^N(\Omega)$. We extend f outside Ω with preserved smoothness and we denote by $\widetilde{f} \in C_0^N(\mathbb{R}^n)$ the continuation of f. Assume that there exists $C > 0$ such that

$$||\widetilde{f}||_{W^N_\infty(\mathbb{R}^n)} \leq C\, ||f||_{W^N_\infty(\Omega)}.$$

We introduce a uniform grid $\{h\mathbf{m}\}$ with step h. A cubature formula for (1.1) can be obtained if we replace f by the approximate quasi-interpolant

$$\mathcal{M}_{h,\mathcal{D}}\widetilde{f}(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{\mathbf{m}\in\mathbb{Z}^n} \widetilde{f}(h\mathbf{m}) \eta\left(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}\right) \tag{1.2}$$

where $\eta \in \mathcal{S}(\mathbb{R}^n)$ and satisfies the moment conditions of order N

$$\int_{\mathbb{R}^n} \eta(\mathbf{x})\, \mathbf{x}^\alpha d\mathbf{x} = \delta_{0,\alpha}, \quad 0 \leq |\alpha| < N. \tag{1.3}$$

The quasi-interpolant (1.2) approximates f in Ω. It is known ([4]) that

$$|f(\mathbf{x}) - \mathcal{M}_{h,\mathcal{D}}\widetilde{f}(\mathbf{x})| \leq c(\sqrt{\mathcal{D}}h)^N \|\nabla_N f\|_{L^\infty} + \sum_{k=0}^{N-1} \varepsilon_k (\sqrt{\mathcal{D}}h)^k \big|\nabla_k f(\mathbf{x})\big|$$

with

$$\varepsilon_k \leq \sum_{\mathbf{m}\in\mathbb{Z}^n\setminus\{0\}} \big|\nabla_k \mathcal{F}\eta(\sqrt{\mathcal{D}}\mathbf{m})\big|\,;\, \lim_{\mathcal{D}\to\infty} \sum_{\mathbf{m}\in\mathbb{Z}^n\setminus\{0\}} \big|\nabla_k \mathcal{F}\eta(\sqrt{\mathcal{D}}\mathbf{m}) = 0.$$

Since η is a smooth and rapidly decaying function, for any error $\epsilon > 0$ one can fix $r > 0$ and the parameter $\mathcal{D} > 0$ such that the quasi-interpolant with nodes in a neighborhood of Ω

$$\mathcal{M}^r_{h,\mathcal{D}}\widetilde{f}(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{d(h\mathbf{m},\Omega)\leq r\, h\sqrt{\mathcal{D}}} \widetilde{f}(h\mathbf{m}) \eta\left(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}\right) \tag{1.4}$$

approximates f with

$$|f(\mathbf{x}) - \mathcal{M}^r_{h,\mathcal{D}}\widetilde{f}(\mathbf{x})| = \mathcal{O}((\sqrt{\mathcal{D}}h)^N + \epsilon)\|f\|_{W^N_\infty} \tag{1.5}$$

for all $x \in \Omega$.

We use the quasi-interpolant (1.4) of the density to obtain a cubature formula of the volume potential (1.1)

$$\mathcal{K}_{\lambda,h}\widetilde{f}(\mathbf{x}) = \mathcal{K}_\lambda(\mathcal{M}^r_{h,\mathcal{D}}\widetilde{f})(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{\substack{\mathbf{m}\in\mathbb{Z}^n \\ d(h\mathbf{m},\Omega)\le r\, h\sqrt{\mathcal{D}}}} \widetilde{f}(h\mathbf{m}) \int_\Omega \kappa_\lambda(\mathbf{x}-\mathbf{y})\,\eta\left(\frac{\mathbf{y}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}\right) d\mathbf{y}. \quad (1.6)$$

Since $\mathcal{K}_\lambda$ is a bounded mapping between suitable function spaces, the differences $\mathcal{K}_{\lambda,h}\widetilde{f}(\mathbf{x}) - \mathcal{K}_\lambda f(\mathbf{x})$ behave like estimate (1.5). We use the radial generating functions

$$\eta_{2M}(\mathbf{x}) = \pi^{-n/2} L^{(n/2)}_{M-1}(|\mathbf{x}|^2)\,\mathrm{e}^{-|\mathbf{x}|^2}, \quad (1.7)$$

where $L^{(\gamma)}_k$ are the generalized Laguerre polynomials

$$L^{(\gamma)}_k(y) = \frac{\mathrm{e}^y\, y^{-\gamma}}{k!}\left(\frac{d}{dy}\right)^k\left(\mathrm{e}^{-y}\, y^{k+\gamma}\right), \quad \gamma > -1, \quad (1.8)$$

which satisfy the moment condition (1.3) with $N = 2M$. Formula (1.6) would give a cubature of (1.1), if the integrals

$$\int_\Omega \kappa_\lambda(\mathbf{x}-\mathbf{y})\,\eta\left(\frac{\mathbf{y}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}\right) d\mathbf{y}$$

could be computed efficiently for nodes with $d(h\mathbf{m},\Omega) \le r\, h\sqrt{\mathcal{D}}$.

For smooth domains, we propose to replace the integrals in (1.6) by integrals over the *tangential-halfspace* at a point of $\partial\Omega$ with minimal distance to $h\mathbf{m}$. It is proved that these formulas approximate (1.1) with the order $\mathcal{O}((h\sqrt{\mathcal{D}})^2)$. We conclude the paper providing numerical tests which show that these formulas give very accurate approximations and the order of convergence cannot be improved.

2 Cubature based on (1.2)

In this section we study the approximation of the integral (1.1) over a bounded region $\Omega \subset \mathbb{R}^n$ with smooth boundary by the sum (1.6) for appropriately chosen $r > 0$.

Denote by $P_{h\mathbf{m}}$ a point of $\partial\Omega$ with minimal distance to $h\mathbf{m}$ and by $n_{P_{h\mathbf{m}}}$ the normal at $P_{h\mathbf{m}}$ directed towards the interior of Ω. Let $T_{h\mathbf{m}}$ be the halfspace bounded by the tangential plane at the point $P_{h\mathbf{m}}$ such that the inner normal at $P_{h\mathbf{m}}$ coincides with $n_{P_{h\mathbf{m}}}$.

We define the following cubature formula for the volume potential (1.1)

$$\widetilde{\mathcal{K}}_{\lambda,h}\widetilde{f}(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{d(h\mathbf{m},\Omega)\le r\, h\sqrt{\mathcal{D}}} \widetilde{f}(h\mathbf{m}) \int_{T_{h\mathbf{m}}} \kappa_\lambda(\mathbf{x}-\mathbf{y})\,\eta\left(\frac{\mathbf{y}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}\right) d\mathbf{y}. \quad (2.1)$$

To derive explicit expressions for the integrals in (2.1), as basis functions we use (1.7), satisfying the moment condition (1.3) with $N = 2M$ (cf. [4]). We have

Lemma 2.1. *Let* $\operatorname{Re}\lambda^2 \geq 0$, $n \geq 1$ *or* $\operatorname{Re}\lambda^2 = 0$, $n \geq 3$. *The solution of the equation*

$$(-\Delta + \lambda^2)u = \begin{cases} \eta_{2M}(\mathbf{x}), & x_n \geq a, \\ 0, & x_n < a \end{cases}$$

is given by the one-dimensional integral

$$\frac{1}{8\pi^{n/2}} \int_0^\infty \mathrm{e}^{-\lambda^2 t/4}\, \mathrm{e}^{-|\mathbf{x}|^2/(1+t)} \Big(\operatorname{erfc}\big(F(t,x_n,a)\big)\mathcal{P}_M(\mathbf{x},t) + \frac{\mathrm{e}^{-F^2(t,x_n,a)}}{\sqrt{\pi}} \mathcal{Q}_M(\mathbf{x},t,a) \Big)\, dt \tag{2.2}$$

where

$$\mathcal{P}_M(\mathbf{x},t) = \sum_{k=0}^{M-1} \frac{1}{(1+t)^{k+n/2}} L_k^{(n/2-1)}\Big(\frac{|\mathbf{x}|^2}{1+t}\Big)\,,$$

$$\mathcal{Q}_M(\mathbf{x},t,a) = \frac{2}{(1+t)^{(n-1)/2}} \sum_{k=0}^{M-1}\sum_{l=0}^{k} \frac{(-1)^{k-l}}{(k-l)!4^{k-l}} L_l^{((n-3)/2)}\Big(\frac{|\mathbf{x}'|^2}{1+t}\Big) \sum_{j=1}^{2(k-l)} \frac{(-1)^j}{t^{j/2}}$$
$$\times \Big(\binom{2(k-l)}{j} H_{2(k-l)-j}\Big(\frac{x_n}{\sqrt{1+t}}\Big) \frac{H_{j-1}(F(t,x_n,a))}{(1+t)^{k+1/2}} - H_{j-1}\Big(\frac{a-x_n}{\sqrt{t}}\Big) \frac{H_{2(k-l)-j}(a)}{(1+t)^l} \Big)$$

with the function

$$F(t,x,a) = \sqrt{\frac{1+t}{t}}\Big(a - \frac{x}{1+t}\Big). \tag{2.3}$$

H_k *denote the Hermite polynomials*

$$H_k(x) = (-1)^k\, \mathrm{e}^{x^2} \frac{d^k}{dx^k}\, \mathrm{e}^{-x^2}\ . \tag{2.4}$$

Proof. We consider the heat equation in $\mathbb{R}^n$

$$\partial_t z - \Delta z = 0\,, \quad z(\mathbf{x},0) = \begin{cases} \eta_{2M}(\mathbf{x}), & x_n \geq a, \\ 0, & x_n < a, \end{cases}$$

whose solution is given by the Poisson integral

$$z(\mathbf{x},t) = \frac{1}{(4\pi t)^{n/2}} \int\limits_{y_n > a} \mathrm{e}^{-|\mathbf{x}-\mathbf{y}|^2/(4t)}\, \eta_{2M}(\mathbf{y})\, d\mathbf{y}.$$

The representation ([4, p.55])

$$\eta_{2M}(\mathbf{x}) = \pi^{-n/2} \sum_{j=0}^{M-1} \frac{(-1)^j}{j!4^j} \Delta^j\, \mathrm{e}^{-|\mathbf{x}|^2}, \tag{2.5}$$

shows that

$$\eta_{2M}(\mathbf{x}',x_n) = \pi^{-n/2} \sum_{j=0}^{M-1} \frac{(-1)^j}{j!4^j} \sum_{l=0}^{j} \binom{j}{l} \Delta_{\mathbf{x}'}^l\, \mathrm{e}^{-|\mathbf{x}'|^2} \frac{d^{2(j-l)}}{dx_n^{2(j-l)}}\, \mathrm{e}^{-x_n^2}\,.$$

Hence

$$z(\mathbf{x},t) = \frac{1}{(4t)^{n/2}\pi^n} \sum_{j=0}^{M-1} \frac{(-1)^j}{j!4^j} \sum_{l=0}^{j} \binom{j}{l} \varphi_{j-l}(x_n, 4t, a) \int_{\mathbb{R}^{n-1}} \mathrm{e}^{-|\mathbf{x}'-\mathbf{y}'|^2/(4t)}\, \Delta^l_{\mathbf{y}'}\, \mathrm{e}^{-|\mathbf{y}'|^2}\, d\mathbf{y}'$$

with

$$\varphi_k(x,t,p) = \int_p^\infty \mathrm{e}^{-(x-y)^2/t}\, \frac{d^{2k}}{dy^{2k}}\, \mathrm{e}^{-y^2}\; dy. \tag{2.6}$$

From

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} \mathrm{e}^{-|\mathbf{x}'-\mathbf{y}'|^2/(4t)}\, \Delta^l_{\mathbf{y}'}\, \mathrm{e}^{-|\mathbf{y}'|^2}\, d\mathbf{y}' &= \Delta^l_{\mathbf{x}'} \int_{\mathbb{R}^{n-1}} \mathrm{e}^{-|\mathbf{x}'-\mathbf{y}'|^2/(4t)}\, \mathrm{e}^{-|\mathbf{y}'|^2}\, d\mathbf{y}' \\ &= \Big(\frac{4\pi t}{1+4t}\Big)^{(n-1)/2} \Delta^l_{\mathbf{x}'}\, \mathrm{e}^{-|\mathbf{x}'|^2/(1+4t)}\end{aligned}$$

and the relation

$$\Delta^j\, \mathrm{e}^{-|\mathbf{x}'|^2/(1+t)} = \frac{(-1)^j\, j!4^j}{(1+t)^j}\, \mathrm{e}^{-|\mathbf{x}'|^2/(1+t)}\, L_j^{((n-3)/2)}\Big(\frac{|\mathbf{x}'|^2}{1+t}\Big)$$

([4, p.121]), we obtain

$$\begin{aligned}z(\mathbf{x},t) = \frac{1}{(4t)^{n/2}\pi^n} &\frac{\mathrm{e}^{-|\mathbf{x}'|^2/(1+4t)}}{(1+4t)^{(n-1)/2}} \sum_{j=0}^{M-1} \frac{(-1)^j}{j!4^j} \\ &\times \sum_{l=0}^{j} \binom{j}{l} \frac{(-1)^l\, l!4^l}{(1+4t)^l} \varphi_{j-l}(x_n,4t,a) L_l^{((n-3)/2)}\Big(\frac{|\mathbf{x}'|^2}{1+4t}\Big).\end{aligned}$$

From

$$\varphi_0(x,t,p) = \int_p^\infty \mathrm{e}^{-(x-y)^2/t}\, \mathrm{e}^{-y^2}\; dy = \frac{\sqrt{\pi}}{2}\sqrt{\frac{t}{1+t}}\, \mathrm{e}^{-x^2/(1+t)}\, \mathrm{erfc}\,\big(F(t,x,p)\big)\,,$$

for $k \geq 1$, integration by parts leads to

$$\int_p^\infty \mathrm{e}^{-(x-y)^2/t}\, \frac{d^{2k}}{dy^{2k}}\, \mathrm{e}^{-y^2}\; dy_j = \frac{\partial^{2k}}{\partial x^{2k}} \varphi_0(x,t,p) - \sum_{\ell=0}^{2k-1} (-1)^\ell \frac{\partial^\ell}{\partial y^\ell}\, \mathrm{e}^{-(x-y)^2/t}\, \frac{d^{2k-\ell-1}}{dy^{2k-\ell-1}}\, \mathrm{e}^{-y^2} \bigg|_{y=p}\,,$$

and the definition (2.4) gives

$$\frac{d^{2k-\ell-1}}{dy^{2k-\ell-1}}\, \mathrm{e}^{-y^2} = (-1)^{2k-\ell-1}\, \mathrm{e}^{-y^2}\, H_{2k-\ell-1}(y)\,,\quad \frac{\partial^\ell}{\partial y^\ell}\, \mathrm{e}^{-(x-y)^2/t} = \frac{(-1)^\ell\, \mathrm{e}^{-(x-y)^2/t}}{t^{\ell/2}}\, H_\ell\Big(\frac{y-x}{\sqrt{t}}\Big).$$

In view of

$$\frac{d^\ell}{dx^\ell}\, \mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}(-1)^\ell\, \mathrm{e}^{-x^2}\, H_{\ell-1}(x)), \quad \ell \geq 1\,,$$

one gets for $\ell < 2k$

$$\frac{\partial^{2k-\ell}}{\partial x^{2k-\ell}}\,\mathrm{erfc}\big(F(t,x,p)\big) = \frac{(-1)^{2k-\ell}}{(t(1+t))^{k-\ell/2}}\left[\frac{d^{2k-\ell}}{dz^{2k-\ell}}\,\mathrm{erfc}(z)\right]_{z=F(t,x,p)}$$
$$= \frac{2\,\mathrm{e}^{-F^2(t,x,p)}}{\sqrt{\pi}(t(1+t))^{k-\ell/2}}\,H_{2k-\ell-1}(F(t,x,p))\,.$$

Therefore, since

$$\frac{d^\ell}{dx^\ell}\,\mathrm{e}^{-x^2/(1+t)} = \frac{(-1)^\ell\,\mathrm{e}^{-x^2/(1+t)}}{(1+t)^{\ell/2}}H_\ell\Big(\frac{x}{\sqrt{1+t}}\Big)\,,$$

we obtain

$$\frac{\partial^{2k}}{\partial x^{2k}}\,\varphi_0(x,t,p) = \frac{\sqrt{\pi t}}{2}\,\frac{\mathrm{e}^{-x^2/(1+t)}}{(1+t)^{k+1/2}}H_{2k}\Big(\frac{x}{\sqrt{1+t}}\Big)\,\mathrm{erfc}(F(t,x,p))$$
$$-\,\frac{\sqrt{t}\,\mathrm{e}^{-x^2/(1+t)}\,\mathrm{e}^{-F^2(t,x,p)}}{(1+t)^{k+1/2}}\sum_{\ell=0}^{2k-1}\binom{2k}{\ell}\frac{(-1)^\ell}{t^{k-\ell/2}}H_\ell\Big(\frac{x}{\sqrt{1+t}}\Big)H_{2k-\ell-1}(F(t,x,p))\,.$$

Thus simple transformations give

$$\varphi_k(x,t,p) = \mathrm{e}^{-x^2/(1+t)}\Bigg(\mathrm{erfc}\big(F(t,x,p)\big)H_{2k}\Big(\frac{x}{\sqrt{1+t}}\Big)\frac{\sqrt{\pi\,t}}{2(1+t)^{k+1/2}}$$
$$+\,\mathrm{e}^{-F^2(t,x,p)}\sum_{\ell=1}^{2k}\frac{(-1)^\ell}{t^{(\ell-1)/2}}\bigg(\binom{2k}{\ell}H_{2k-\ell}\Big(\frac{x}{\sqrt{1+t}}\Big)\frac{H_{\ell-1}(F(t,x,p))}{(1+t)^{k+1/2}} - H_{\ell-1}\Big(\frac{p-x}{\sqrt{t}}\Big)H_{2k-\ell}(p)\bigg)\Bigg). \tag{2.7}$$

Using (2.7), the relations

$$H_{2j}(x) = (-1)^j\,4^j\,j!\,L_j^{(-1/2)}(x^2) \quad \text{and} \quad L_j^{(a+b+1)}(x+y) = \sum_{l=0}^{j} L_l^{(a)}(x)L_{j-l}^{(b)}(y)$$

we find

$$\sum_{j=0}^{M-1}\frac{(-1)^j}{j!4^j}\frac{1}{(1+t)^{j+n/2}}\sum_{l=0}^{j}\binom{j}{l}(-1)^l\,l!4^l L_l^{((n-3)/2)}\Big(\frac{|\mathbf{x}'|^2}{1+t}\Big)H_{2(j-l)}\Big(\frac{x_n}{\sqrt{1+t}}\Big)$$
$$= \sum_{j=0}^{M-1}\frac{1}{(1+t)^{j+n/2}}\sum_{l=0}^{j}L_l^{((n-3)/2)}\Big(\frac{|\mathbf{x}'|^2}{1+t}\Big)L_{j-l}^{(-1/2)}\Big(\frac{x_n^2}{1+t}\Big) = \mathcal{P}_M(\mathbf{x},t)$$

leading to

$$z(\mathbf{x},t) = \frac{\mathrm{e}^{-|\mathbf{x}|^2/(1+4t)}}{2\pi^{n/2}}\Big(\mathrm{erfc}\big(F(4t,x_n,a)\big)\mathcal{P}_M(\mathbf{x},4t) + \frac{\mathrm{e}^{-F^2(4t,x_n,a)}}{\sqrt{\pi}}\,\mathcal{Q}_M(\mathbf{x},4t,a)\Big).$$

$\square$

In the particular case $n=2$

$$\mathcal{P}_1(\mathbf{x},t) = \frac{1}{1+t}\,; \quad \mathcal{P}_2(\mathbf{x},t) = \frac{1}{1+t}\Big(1+\frac{1}{1+t}-\frac{|\mathbf{x}|^2}{(1+t)^2}\Big)\,;$$
$$\mathcal{P}_3(\mathbf{x},t) = \mathcal{P}_2(\mathbf{x},t) + \frac{1}{1+t}\Big(\frac{1}{(1+t)^2} - 2\frac{|\mathbf{x}|^2}{(1+t)^3} + \frac{|\mathbf{x}|^4}{2\,(1+t)^4}\Big)$$
$$= \frac{1}{1+t}\Big(1+\frac{1}{1+t}-\frac{|\mathbf{x}|^2}{(1+t)^2} + \frac{1}{(1+t)^2} - 2\frac{|\mathbf{x}|^2}{(1+t)^3} + \frac{|\mathbf{x}|^4}{2\,(1+t)^4}\Big)\,;$$
$$\mathcal{Q}_1(\mathbf{x},t,a) = 0; \quad \mathcal{Q}_2(\mathbf{x},t,a) = -\frac{\sqrt{t}}{(t+1)^{3/2}}\big(a+\frac{x_2}{1+t}\big);$$
$$\mathcal{Q}_3(\mathbf{x},t,a) = \frac{1}{4}\frac{\sqrt{t}}{(1+t)^{3/2}}\,\left(\frac{-2\,a\,t}{(1+t)} + \Big(a+\frac{x_2}{1+t}\Big)\Big(\frac{4\,|\mathbf{x}|^2-2\,x_2^2}{(1+t)^2} - \frac{7}{1+t} + 2\,a^2 - 5\Big)\right).$$

Remark 2.1. *For sufficiently large $|a|>r$ the integrands in (2.2) are approximated by*

$$\begin{cases} 0\,, & a\ge r\,,\\ 2e^{-\lambda^2 t/4}\,\mathrm{e}^{-|\mathbf{x}|^2/(1+t)}\,\mathcal{P}_M(\mathbf{x},t), & a\le -r\,,\end{cases}$$

with the error $\mathcal{O}(\mathrm{e}^{-a^2})$, which is in accordance with

$$\int_{\mathbb{R}^n} \kappa_\lambda(\mathbf{x}-\mathbf{y})\eta_{2M}(\mathbf{y})\,d\mathbf{y} = \frac{1}{4\pi^{n/2}}\int_0^\infty \mathrm{e}^{-\lambda^2 t/4}\,\mathrm{e}^{-|\mathbf{x}|^2/(1+t)}\mathcal{P}_M(\mathbf{x},t)\,dt.$$

Theorem 2.1. *Suppose that the generating function $\eta\in\mathcal{S}(\mathbb{R}^n)$ satisfies the moment condition* (1.3) *with $N\ge 2$. Then the integral* (1.1) *is approximated by the sum*

$$\widetilde{\mathcal{K}}_{\lambda,h}\widetilde{f}(\mathbf{x}) = \mathcal{D}^{-n/2}\sum_{d(h\mathbf{m},\Omega)\le r\,h\sqrt{\mathcal{D}}} \widetilde{f}(h\mathbf{m})\int_{T_{h\mathbf{m}}} \kappa_\lambda(\mathbf{x}-\mathbf{y})\,\eta\left(\frac{\mathbf{y}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}\right)d\mathbf{y} \tag{2.8}$$

with the error estimate

$$|\widetilde{\mathcal{K}}_{h,\lambda}\widetilde{f}(\mathbf{x}) - \mathcal{K}_\lambda f(\mathbf{x})| = \mathcal{O}(\epsilon + c_1\,(h\sqrt{\mathcal{D}})^2) \tag{2.9}$$

provided $\partial\Omega$ has C^2-smoothness. The saturation term ϵ can be negligible small if $\mathcal{D}$ is large enough.

Proof. We study the difference

$$\widetilde{\mathcal{K}}_{\lambda,h}\widetilde{f}(\mathbf{x}) - \mathcal{K}_{\lambda,h}\widetilde{f}(\mathbf{x}) = \mathcal{D}^{-n/2}\sum_{d(h\mathbf{m},\Omega)\le r\,h\sqrt{\mathcal{D}}} \widetilde{f}(h\mathbf{m})\int_{T_{h\mathbf{m}}} \kappa_\lambda(\mathbf{x}-\mathbf{y})\,\eta\left(\frac{\mathbf{y}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}\right)d\mathbf{y}$$
$$-\mathcal{D}^{-n/2}\sum_{d(h\mathbf{m},\Omega)\le r\,h\sqrt{\mathcal{D}}} \widetilde{f}(h\mathbf{m})\int_{\Omega} \kappa_\lambda(\mathbf{x}-\mathbf{y})\,\eta\left(\frac{\mathbf{y}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}\right)d\mathbf{y}$$

as $h\to 0$.

Since $\eta(\mathbf{y})$ is supported in the small neighborhood of the origin $|\mathbf{y}| \le r$, if $h\mathbf{m} \in \Omega$ such that $d(h\mathbf{m}, \partial\Omega) > rh\sqrt{\mathcal{D}}$ we replace each integral

$$\int_\Omega \kappa_\lambda(\mathbf{x}-\mathbf{y})\,\eta\left(\frac{\mathbf{y}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}\right) d\mathbf{y}$$

by integrals over $T_{h\mathbf{m}}$ without loss accuracy.

Assume $\mathbf{m} \in \mathbb{Z}^n : d(h\mathbf{m}, \partial\Omega) \le rh\sqrt{\mathcal{D}}$. We choose a local coordinate system with the origin $\mathbf{0}$ at the point $h\mathbf{m}$ and the normal at the nearest point of $\partial\Omega$ from $\mathbf{0}$ coincides with the x_n- axis $(\mathbf{0}', 1)$, $\mathbf{0}' \in \mathbb{R}^{n-1}$. In these coordinates system the halfspace is defined by $T = T_{h\mathbf{m}} = \{\mathbf{y} = (\mathbf{y}', y_n) : \mathbf{y}' \in \mathbb{R}^{n-1}, y_n > \rho_{h\mathbf{m}}\}$ where

$$\rho = \rho_{h\mathbf{m}} = \begin{cases} dist(\mathbf{0}', \partial\Omega) & \text{if } \mathbf{0} \notin \Omega \\ -dist(\mathbf{0}', \partial\Omega) & \text{if } \mathbf{0} \in \Omega. \end{cases}$$

We have $|\rho| < rh\sqrt{\mathcal{D}}$. We assume that in a neighborhood U of the point $(\mathbf{0}', \rho) \in \partial\Omega$ the domain Ω is given by

$$y_n \ge \varphi(\mathbf{y}'), \quad \varphi(\mathbf{0}) = \rho,\ \nabla\varphi(\mathbf{0}) = 0. \tag{2.10}$$

We introduce the change of variable $\mathbf{z} = \mathbf{z}(\mathbf{y})$ defined as

$$\mathbf{z}' = \mathbf{y}',\ z_n = y_n - \varphi(\mathbf{y}') + \rho$$

and its inverse $\mathbf{y} = \mathbf{y}(\mathbf{z})$

$$\mathbf{y}' = \mathbf{z}',\ y_n = z_n + \varphi(\mathbf{z}') - \rho.$$

Choose h such that $B_{hr\sqrt{\mathcal{D}}} \cap \Omega \subset U$ and assume $r_0 \ge r$ such that $\mathbf{z}(\Omega \cap B_{hr\sqrt{\mathcal{D}}}) = T \cap B_{hr_0\sqrt{\mathcal{D}}}$. Then the difference

$$\int_T \kappa_\lambda((\mathbf{x}-h\mathbf{m})-\mathbf{y})\,\eta\left(\frac{\mathbf{y}}{h\sqrt{\mathcal{D}}}\right) d\mathbf{y} - \int_\Omega \kappa_\lambda((\mathbf{x}-h\mathbf{m})-\mathbf{y})\,\eta\left(\frac{\mathbf{y}}{h\sqrt{\mathcal{D}}}\right) d\mathbf{y}$$

takes the form

$$\int_{T\cap B_{r_0 h\sqrt{\mathcal{D}}}} \left[\kappa_\lambda((\mathbf{x}-h\mathbf{m})-\mathbf{z})\,\eta\left(\frac{\mathbf{z}}{h\sqrt{\mathcal{D}}}\right) - \kappa_\lambda((\mathbf{x}-h\mathbf{m})-\mathbf{y}(\mathbf{z}))\,\eta\left(\frac{\mathbf{y}(\mathbf{z})}{h\sqrt{\mathcal{D}}}\right)\right] d\mathbf{z}$$

$$= \int_{T\cap B_{r_0 h\sqrt{\mathcal{D}}}(\mathbf{0})} \kappa_\lambda((\mathbf{x}-h\mathbf{m})-\mathbf{z}))\,\left[\eta\left(\frac{\mathbf{z}}{h\sqrt{\mathcal{D}}}\right) - \eta\left(\frac{\mathbf{y}(\mathbf{z})}{h\sqrt{\mathcal{D}}}\right)\right] d\mathbf{z} \tag{2.11}$$

$$+ \int_{T\cap B_{r_0 h\sqrt{\mathcal{D}}}(\mathbf{0})} [\kappa_\lambda((\mathbf{x}-h\mathbf{m})-\mathbf{z}) - \kappa_\lambda((\mathbf{x}-h\mathbf{m})-\mathbf{y}(\mathbf{z})]\,\eta\left(\frac{\mathbf{y}(\mathbf{z})}{h\sqrt{\mathcal{D}}}\right) d\mathbf{z}. \tag{2.12}$$

Consider the integral in (2.11). In view of (2.10) we can consider locally

$$\varphi(\mathbf{z}') = \rho + \frac{1}{2} K\mathbf{z}'\cdot\mathbf{z}', \quad K = \{\partial_{ij}\varphi(\mathbf{0})\}_{i,j=1}^{n-1}.$$

Therefore we have

$$|\mathbf{z}-\mathbf{y}(\mathbf{z})| = |\varphi(\mathbf{z}') - \rho| \le c_1\,(h\sqrt{\mathcal{D}})^2,\ \mathbf{z} \in B_{r_0 h\sqrt{\mathcal{D}}}(\mathbf{0}) \tag{2.13}$$

which leads to

$$\left|\eta\left(\frac{\mathbf{z}}{h\sqrt{\mathcal{D}}}\right)-\eta\left(\frac{\mathbf{y}(\mathbf{z})}{h\sqrt{\mathcal{D}}}\right)\right|\le c_2\, h\sqrt{\mathcal{D}}. \tag{2.14}$$

Suppose $|\mathbf{x}-h\mathbf{m}|\ge 2r_0h\sqrt{\mathcal{D}}$. Since $|(\mathbf{x}-h\mathbf{m})-\mathbf{z}|\ge|\mathbf{x}-h\mathbf{m}|/2$, we obtain

$$\int_{T\cap B_{r_0h\sqrt{\mathcal{D}}}(\mathbf{0})}\kappa_\lambda((\mathbf{x}-h\mathbf{m})-\mathbf{z})\,d\mathbf{z}\le c\frac{(h\sqrt{\mathcal{D}})^n}{|\mathbf{x}-h\mathbf{m}|^{n-2}}.$$

If $|\mathbf{x}-h\mathbf{m}|<2r_0h\sqrt{\mathcal{D}}$ then, for $a>r_0$,

$$\int_{T\cap B_{r_0h\sqrt{\mathcal{D}}}(\mathbf{0})}\kappa_\lambda((\mathbf{x}-h\mathbf{m})-\mathbf{z})\,d\mathbf{z}$$

$$\le c\int_{B_{ah\sqrt{\mathcal{D}}}(\mathbf{x}-h\mathbf{m})}\frac{d\mathbf{z}}{|(\mathbf{x}-h\mathbf{m})-\mathbf{z}|^{n-2}}+c\int_{B_{r_0h\sqrt{\mathcal{D}}}(\mathbf{0})\setminus B_{ah\sqrt{\mathcal{D}}}(\mathbf{x}-h\mathbf{m})}\frac{d\mathbf{z}}{|(\mathbf{x}-h\mathbf{m})-\mathbf{z}|^{n-2}}.$$

Obviously the first integral in the right-hand side is $\mathcal{O}((h\sqrt{\mathcal{D}})^2)$. To estimate the second integral we use the relation

$$|\mathbf{x}-\mathbf{z}|\ge c(|\mathbf{x}'-\mathbf{z}'|+h\sqrt{\mathcal{D}}),\ \forall\mathbf{z}\in\mathbb{R}^n\setminus B_{ah\sqrt{\mathcal{D}}}(\mathbf{x}), \tag{2.15}$$

which implies ([3, (4.2.13)])

$$\int_{B_{r_0h\sqrt{\mathcal{D}}}(\mathbf{0})\setminus B_{ah\sqrt{\mathcal{D}}}(\mathbf{x})}\frac{d\mathbf{z}}{|\mathbf{x}-\mathbf{z}|^{n-2}}$$

$$\le c\int_{B_{r_0h\sqrt{\mathcal{D}}}(\mathbf{0})\setminus B_{ah\sqrt{\mathcal{D}}}(\mathbf{x})}\frac{d\mathbf{z}}{(|\mathbf{x}'-\mathbf{z}'|+h\sqrt{\mathcal{D}})^{n-2}}\le c(h\sqrt{\mathcal{D}})^2.$$

Thus, for all $\mathbf{x}\in\mathbb{R}^n$,

$$\int_{T\cap B_{r_0h\sqrt{\mathcal{D}}}(\mathbf{0})}\kappa_\lambda((\mathbf{x}-h\mathbf{m})-\mathbf{z})\,d\mathbf{z}\le c\frac{(h\sqrt{\mathcal{D}})^n}{(|\mathbf{x}-h\mathbf{m}|+h\sqrt{\mathcal{D}})^{n-2}}.$$

Therefore, if S_h denotes the strip $S_h=\{\mathbf{x}\in\mathbb{R}^n:d(\mathbf{x},\partial\Omega)\le rh\sqrt{\mathcal{D}}\}$, keeping in mind (2.14) we have

$$\sum_{h\mathbf{m}\in S_h}|\widetilde{f}(h\mathbf{m})|\int_{T\cap B_{r_0h\sqrt{\mathcal{D}}}}\kappa_\lambda((\mathbf{x}-h\mathbf{m})-\mathbf{z})\left|\eta\left(\frac{\mathbf{z}}{h\sqrt{\mathcal{D}}}\right)-\eta\left(\frac{\mathbf{y}(\mathbf{z})}{h\sqrt{\mathcal{D}}}\right)\right|d\mathbf{z}$$

$$\le C\,||f||_\infty(h\sqrt{\mathcal{D}})^{n+1}\sum_{h\mathbf{m}\in S_h}\frac{1}{(|\mathbf{x}-h\mathbf{m}|+h\sqrt{\mathcal{D}})^{n-2}}$$

$$\le C\,||f||_\infty(h\sqrt{\mathcal{D}})\int_{S_h}\frac{d\mathbf{y}}{(|\mathbf{x}-\mathbf{y}|+h\sqrt{\mathcal{D}})^{n-2}}\le C\,||f||_\infty(h\sqrt{\mathcal{D}})^2. \tag{2.16}$$

It remains to consider the integral in (2.12). We use the inequality (see [3, p.80])

$$|\kappa_\lambda((\mathbf{x}-h\mathbf{m})-\mathbf{z})-\kappa_\lambda((\mathbf{x}-h\mathbf{m})-\mathbf{y}(\mathbf{z})| \tag{2.17}$$

$$\leq \; c\,(h\sqrt{\mathcal{D}})^2 \left[\frac{1}{|(\mathbf{x}-h\mathbf{m})-\mathbf{z}|^{n-1}} + \frac{1}{|(\mathbf{x}-h\mathbf{m})-\mathbf{y}(\mathbf{z})|^{n-1}} \right]$$

and obtain

$$\int_{T\cap B_{rh\sqrt{\mathcal{D}}}(\mathbf{0})} |\kappa_\lambda((\mathbf{x}-h\mathbf{m})-\mathbf{z}) - \kappa_\lambda((\mathbf{x}-h\mathbf{m})-\mathbf{y}(\mathbf{z})|\,|\eta\left(\frac{\mathbf{y}(\mathbf{z})}{h\sqrt{\mathcal{D}}}\right)|\,d\mathbf{z}$$

$$\leq c_1\,(h\sqrt{\mathcal{D}})^2 \int_{B_{r_0 h\sqrt{\mathcal{D}}}(\mathbf{0})} \frac{1}{|(\mathbf{x}-h\mathbf{m})-\mathbf{z}|^{n-1}} d\mathbf{z}.$$

If $|\mathbf{x}-h\mathbf{m}| \geq 2r_0 h\sqrt{\mathcal{D}}$ then

$$\int_{B_{r_0 h\sqrt{\mathcal{D}}}(\mathbf{0})} \frac{d\mathbf{z}}{|(\mathbf{x}-h\mathbf{m})-\mathbf{z}|^{n-1}} \leq c\frac{(h\sqrt{\mathcal{D}})^n}{|\mathbf{x}-h\mathbf{m}|^{n-1}}.$$

If $|\mathbf{x}-h\mathbf{m}| < 2r_0 h\sqrt{\mathcal{D}}$, by using (2.15) we obtain that

$$\int_{B_{r_0 h\sqrt{\mathcal{D}}}(\mathbf{0})} \frac{d\mathbf{z}}{|(\mathbf{x}-h\mathbf{m})-\mathbf{z}|^{n-1}} \leq c(h\sqrt{\mathcal{D}}).$$

We deduce that, for all $\mathbf{x} \in \mathbb{R}^n$,

$$\int_{B_{r_0 h\sqrt{\mathcal{D}}}(\mathbf{0})} \frac{1}{|(\mathbf{x}-h\mathbf{m})-\mathbf{z}|^{n-1}} d\mathbf{z} \leq c\frac{(h\sqrt{\mathcal{D}})^n}{(|\mathbf{x}-h\mathbf{m}|+h\sqrt{\mathcal{D}})^{n-1}}.$$

Therefore

$$\sum_{h\mathbf{m}\in S_h} |\widetilde{f}(h\mathbf{m})| \int_{T\cap B_{r_0 h\sqrt{\mathcal{D}}}} |\kappa_\lambda((\mathbf{x}-h\mathbf{m})-\mathbf{z}) - \kappa_\lambda((\mathbf{x}-h\mathbf{m})-\mathbf{y}(\mathbf{z})|\,|\eta\left(\frac{\mathbf{y}(\mathbf{z})}{h\sqrt{\mathcal{D}}}\right)|d\mathbf{z}$$

$$\leq c_1\,(h\sqrt{\mathcal{D}})^2 \sum_{h\mathbf{m}\in S_h} |\widetilde{f}(h\mathbf{m})| \int_{B_{r_0 h\sqrt{\mathcal{D}}}(\mathbf{0})} \frac{1}{|(\mathbf{x}-h\mathbf{m})-\mathbf{z}|^{n-1}}\, d\mathbf{z}$$

$$\leq C\,||f||_\infty (h\sqrt{\mathcal{D}})^{n+2} \sum_{h\mathbf{m}\in S_h} \frac{1}{(|\mathbf{x}-h\mathbf{m}|+h\sqrt{\mathcal{D}})^{n-1}}$$

$$\leq C\,||f||_\infty (h\sqrt{\mathcal{D}})^2 \int_{S_h} \frac{d\mathbf{y}}{(|\mathbf{x}-\mathbf{y}|+h\sqrt{\mathcal{D}})^{n-1}}$$

$$\leq C\,||f||_\infty (h\sqrt{\mathcal{D}})^3 |\log(\max(h\sqrt{\mathcal{D}}, dist(\mathbf{x},\partial\Omega))|.$$

The last inequality and (2.16) complete the proof.

□

3 Implementation and numerical examples

From Lemma 2.1 and Remark 2.1 we obtain the following one dimensional integral representation for the integrals in (2.8) if $h\mathbf{m} \in \Omega$ and $d(h\mathbf{m}, \partial\Omega) \geq rh\sqrt{\mathcal{D}}$

$$
\begin{aligned}
&\mathcal{D}^{-n/2} \int_{\mathbb{R}^n} \kappa_\lambda(\mathbf{x}-\mathbf{y})\, \eta_{2M}\left(\frac{\mathbf{y}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}\right) d\mathbf{y} \\
&= \frac{\lambda^{n/2-1}}{(2\pi\mathcal{D})^{n/2}} \int_{\mathbb{R}^n} \frac{K_{n/2-1}(\lambda|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|^{n/2-1}}\, \eta_{2M}\left(\frac{\mathbf{y}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}\right) d\mathbf{y} \\
&= \frac{h^2}{\mathcal{D}^{n/2-1}} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{(\lambda\, h\sqrt{\mathcal{D}})^{n/2-1}}{|r_{\mathbf{m}}-\mathbf{y}|^{n/2-1}} K_{n/2-1}(\ \lambda h\sqrt{\mathcal{D}}|r_{\mathbf{m}}-\mathbf{y}|)\, \eta_{2M}\,(\mathbf{y})\, d\mathbf{y} \\
&= \frac{h^2\mathcal{D}}{4(\pi\mathcal{D})^{n/2}} \int_0^\infty \mathrm{e}^{-\lambda^2 h^2 \mathcal{D}t/4}\, \mathrm{e}^{-|r_{\mathbf{m}}|^2/(1+t)} \mathcal{P}_M(r_{\mathbf{m}}, t)\, dt
\end{aligned}
$$

with

$$
r_{\mathbf{m}} = \frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}.
$$

To apply Lemma 2.1 for nodes $h\mathbf{m}$ in the $rh\sqrt{\mathcal{D}}$ neighborhood of $\partial\Omega$ we perform a coordinate transformation such that the center $h\mathbf{m}$ is the origin in the new coordinates $\boldsymbol{\xi} = (\xi_1, ..., \xi_n)$ and $T_{h\mathbf{m}} = \{\boldsymbol{\xi} : \xi_n > \rho_{h\mathbf{m}}\}$, where $\rho_{h\mathbf{m}}$ is the distance of the center $h\mathbf{m}$ to $\partial\Omega$. Since η_{2M} is radial we obtain after the change of variable $\mathbf{y} - h\,\mathbf{m} = \omega\boldsymbol{\xi}h\sqrt{\mathcal{D}}$, where ω is the rotation matrix,

$$
\begin{aligned}
&\frac{\lambda^{n/2-1}}{(2\pi\,\mathcal{D})^{n/2}} \int_{T_{h\mathbf{m}}} \frac{K_{n/2-1}(\lambda|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|^{n/2-1}}\, \eta_{2M}\left(\frac{\mathbf{y}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}\right) d\mathbf{y} \\
&= \frac{\lambda^{n/2-1}}{(2\pi\,\mathcal{D})^{n/2}} (h\sqrt{\mathcal{D}})^n \int_{\zeta_n > \frac{\rho_{h\mathbf{m}}}{h\sqrt{\mathcal{D}}}} \frac{K_{n/2-1}(\lambda h\sqrt{\mathcal{D}}|\omega^{-1}r_{\mathbf{m}}-\boldsymbol{\xi}|)}{|\omega^{-1}r_{\mathbf{m}}-\boldsymbol{\xi}|^{n/2-1}} \eta_{2M}(\boldsymbol{\xi}) d\boldsymbol{\xi} \\
&= \frac{h^n}{8\pi^{n/2}} \int_0^\infty \mathrm{e}^{-\lambda^2 h^2 \mathcal{D}\,t/4}\, \mathrm{e}^{-|r_{\mathbf{m}}|^2/(1+t)} \\
&\qquad \times \Big(\operatorname{erfc}\big(F_{\mathbf{m}}(t)\big) \mathcal{P}_M(\omega^{-1}r_{\mathbf{m}}, t) + \frac{\mathrm{e}^{-F^2_{\mathbf{m}}(t)}}{\sqrt{\pi}} \mathcal{Q}_M\big(\omega^{-1}r_{\mathbf{m}}, t, \frac{\rho_{h\mathbf{m}}}{h\sqrt{\mathcal{D}}})\Big)\, dt,
\end{aligned}
$$

where we set

$$
F_{\mathbf{m}}(t) = F(t, (\omega^{-1}r_{\mathbf{m}})_n, \frac{\rho_{h\mathbf{m}}}{h\sqrt{\mathcal{D}}})\,.
$$

Then the cubature formula (2.8) reduces to the computation of integrals over the half-line

$$
\begin{aligned}
\widetilde{\mathcal{K}}_{h,\lambda}\widetilde{f}(\mathbf{x}) &= \frac{h^2\mathcal{D}}{4(\pi\mathcal{D})^{n/2}} \sum_{\substack{d(h\mathbf{m},\partial\Omega)\geq r\,h\sqrt{\mathcal{D}} \\ h\mathbf{m}\in\Omega}} f(h\mathbf{m}) \int_0^\infty \mathrm{e}^{-\lambda^2 h^2 \mathcal{D}t/4}\, \mathrm{e}^{-|r_{\mathbf{m}}|^2/(1+t)} \mathcal{P}_M(r_{\mathbf{m}}, t)\, dt \\
&+ \frac{h^n}{8\pi^{n/2}} \sum_{d(h\mathbf{m},\partial\Omega)< r\,h\sqrt{\mathcal{D}}} \widetilde{f}(h\mathbf{m}) \int_0^\infty \mathrm{e}^{-\lambda^2 h^2 \mathcal{D}\,t/4}\, \mathrm{e}^{-|r_{\mathbf{m}}|^2/(1+t)} \\
&\qquad \times \Big(\operatorname{erfc}\big(F_{\mathbf{m}}(t)\big) \mathcal{P}_M(\omega^{-1}r_{\mathbf{m}}, t) + \frac{\mathrm{e}^{-F^2_{\mathbf{m}}(t)}}{\sqrt{\pi}} \mathcal{Q}_M(\omega^{-1}r_{\mathbf{m}}, t, \frac{\rho_{h\mathbf{m}}}{h\sqrt{\mathcal{D}}})\Big)\, dt.
\end{aligned}
$$

For computing at the grid $\{h\mathbf{k}\}$ we introduce $\eta_{\mathbf{k}-\mathbf{m}} = (\omega^{-1}(\mathbf{k}-\mathbf{m}))_n$ as the projection of $\mathbf{k}-\mathbf{m}$ onto the normal to the tangential plane, i.e. the ξ_n-axis, and set

$$\begin{aligned}
a_{\mathbf{k}}^{(M)} &= \frac{1}{4}\int_0^\infty \mathrm{e}^{-\lambda^2 h^2\,\mathcal{D}t/4}\,\mathrm{e}^{-|\mathbf{k}|^2/(\mathcal{D}(1+t))}\,\mathcal{P}_M(\frac{\mathbf{k}}{\sqrt{\mathcal{D}}},t)\,dt\\
b_{\mathbf{k},\mathbf{m}}^{(M)} &= \frac{1}{8}\int_0^\infty \mathrm{e}^{-\lambda^2 h^2\,\mathcal{D}t/4}\,\mathrm{e}^{-|\mathbf{k}-\mathbf{m}|^2/(\mathcal{D}(1+t))}\\
&\quad\times\Big(\operatorname{erfc}\big(F_{\mathbf{k},\mathbf{m}}(t)\big)\mathcal{P}_M(\frac{\mathbf{k}-\mathbf{m}}{\sqrt{\mathcal{D}}},t)+\frac{\mathrm{e}^{-F^2_{\mathbf{k},\mathbf{m}}(t)}}{\sqrt{\pi}}\mathcal{Q}_M(\omega^{-1}\frac{\mathbf{k}-\mathbf{m}}{\sqrt{\mathcal{D}}},t,\frac{\rho_{h\mathbf{m}}}{h\sqrt{\mathcal{D}}})\Big)\,dt,
\end{aligned}$$

where

$$F_{\mathbf{k},\mathbf{m}}(t) = \sqrt{\frac{1+t}{t}}\Big(\frac{\rho_{h\mathbf{m}}}{h\sqrt{\mathcal{D}}}-\frac{\eta_{\mathbf{k}-\mathbf{m}}}{\sqrt{\mathcal{D}}(1+t)}\Big).$$

Hence a cubature for the volume potential at the uniform grid $\{h\mathbf{k}\}$ is

$$\mathcal{K}_{h,\lambda}^{(M)}f(h\mathbf{k}) = \frac{h^2}{\pi^{n/2}\,\mathcal{D}^{n/2-1}}\sum_{\substack{d(h\mathbf{m},\partial\Omega)\ge r\,h\sqrt{\mathcal{D}}\\ h\mathbf{m}\in\Omega}} f(h\mathbf{m})\,a_{\mathbf{k}-\mathbf{m}}^{(M)} + \frac{h^n}{\pi^{n/2}}\sum_{d(h\mathbf{m},\partial\Omega)<r\,h\sqrt{\mathcal{D}}}\widetilde{f}(h\mathbf{m})b_{\mathbf{k},\mathbf{m}}^{(M)}. \tag{3.1}$$

We transform the one-dimensional integral representation of $a_{\mathbf{k}-\mathbf{m}}^{(M)}$ and $b_{\mathbf{k},\mathbf{m}}^{(M)}$ to integrals over $\mathbb{R}$ with doubly exponentially decaying integrands by making the substitutions

$$t = \mathrm{e}^{\xi},\quad \xi = \alpha(\sigma+\mathrm{e}^{\sigma}),\quad \sigma = \beta(u-\mathrm{e}^{-u}), \tag{3.2}$$

with certain positive constants a and b and the computation is based on the classical trapezoidal rule. We get after the substitutions

$$\begin{aligned}
a_{\mathbf{k}}^{(M)} &= \frac{1}{4}\int_{-\infty}^\infty \mathrm{e}^{-\lambda^2 h^2\,\mathcal{D}\Phi(u)/4}\,\mathrm{e}^{-|\mathbf{k}|^2/(\mathcal{D}(1+\Phi(u)))}\,\mathcal{P}_M(\frac{\mathbf{k}}{\sqrt{\mathcal{D}}},\Phi(u))\,\Phi'(u)du\\
b_{\mathbf{k},\mathbf{m}}^{(M)} &= \frac{1}{8}\int_{-\infty}^\infty \mathrm{e}^{-\lambda^2 h^2\,\mathcal{D}\Phi(u)/4}\,\mathrm{e}^{-|\mathbf{k}-\mathbf{m}|^2/(\mathcal{D}(1+\Phi(u)))}\\
&\quad\times\Big(\operatorname{erfc}\big(F_{\mathbf{k},\mathbf{m}}(\Phi(u))\big)\mathcal{P}_M(\frac{\mathbf{k}-\mathbf{m}}{\sqrt{\mathcal{D}}},\Phi(u))+\frac{\mathrm{e}^{-F^2_{\mathbf{k},\mathbf{m}}(\Phi(u))}}{\sqrt{\pi}}\mathcal{Q}_M(\omega^{-1}\frac{\mathbf{k}-\mathbf{m}}{\sqrt{\mathcal{D}}},\Phi(u),\frac{\rho_{h\mathbf{m}}}{h\sqrt{\mathcal{D}}})\Big)\,\Phi'(u)du,
\end{aligned}$$

where we set

$$\begin{aligned}
\Phi(u) &= \exp(\alpha\beta(u-\exp(-u))+\alpha\exp(\beta(u-\exp(-u)))),\\
\Phi'(u) &= \Phi(u)\alpha\beta(1+\mathrm{e}^{-u})(1+\exp(\beta(u-\exp(-u)))).
\end{aligned}$$

The quadrature with the trapezoidal rule with step size τ gives

$$\begin{aligned}
a_{\mathbf{k}}^{(M)} &\approx \frac{\tau}{4}\sum_{s=-N_0}^{N_1}\mathrm{e}^{-\lambda^2\,h^2\,\mathcal{D}\,\Phi(s\,\tau)/4}\,\mathrm{e}^{-|\mathbf{k}|^2/(\mathcal{D}(1+\Phi(s\,\tau)))}\,\mathcal{P}_M(\frac{\mathbf{k}}{\sqrt{\mathcal{D}}},\Phi(s\,\tau))\,\Phi'(s\,\tau)\,,\\
b_{\mathbf{k},\mathbf{m}}^{(M)} &\approx \frac{\tau}{8}\sum_{s=-N_0}^{N_1}\mathrm{e}^{-\lambda^2\,h^2\mathcal{D}\,\Phi(s\,\tau)/4}\,\mathrm{e}^{-|\mathbf{k}-\mathbf{m}|^2/(\mathcal{D}(1+\Phi(s\,\tau)))}\,\Phi'(s\,\tau)\times\\
&\Big(\operatorname{erfc}\big(F_{\mathbf{k},\mathbf{m}}(\Phi(s\,\tau))\big)\mathcal{P}_M\big(\frac{\mathbf{k}-\mathbf{m}}{\sqrt{\mathcal{D}}},\Phi(s\,\tau)\big)+\frac{\mathrm{e}^{-F^2_{\mathbf{k},\mathbf{m}}(\Phi(s\,\tau))}}{\sqrt{\pi}}\mathcal{Q}_M\big(\omega^{-1}\frac{\mathbf{k}-\mathbf{m}}{\sqrt{\mathcal{D}}},\Phi(s\,\tau),\frac{\rho_{h\mathbf{m}}}{h\sqrt{\mathcal{D}}}\big)\Big).
\end{aligned}$$

We have tested formula (3.1) in the ellipse

$$\Omega = \{\mathbf{x} \in \mathbb{R}^2 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \leq 1\}, \quad 0 < b \leq a.$$

If $u(\mathbf{x}) = 0$ and $\nabla u(\mathbf{x}) = 0$ on $\partial\Omega$ and 0 outside the ellipse, then the density

$$f(\mathbf{x}) = (-\Delta + \lambda^2) u(\mathbf{x}), \quad \mathbf{x} \in \Omega$$

satisfies

$$\mathcal{K}_\lambda f(\mathbf{x}) = \begin{cases} u(\mathbf{x}) & \mathbf{x} \in \Omega \\ 0 & \mathbf{x} \notin \Omega \end{cases}$$

We have tested the approximation of the potential (1.1) with the density

$$f(\mathbf{x}) = (-\Delta + \lambda^2) \sin\left(1 - \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2}\right)^2 \tag{3.3}$$

which provides the exact value

$$\mathcal{K}_\lambda f(\mathbf{x}) = \begin{cases} \sin\left(1 - \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2}\right)^2 & \mathbf{x} \in \Omega \\ 0 & \mathbf{x} \notin \Omega \end{cases}$$

and we assumed f itself as the extension outside the ellipse.

The results in Table 1 show the accuracy of the method. They were obtained with the parameters $\alpha = 4$, $\beta = 2$ in (3.2) and the quadrature parameters $\tau = 0.01$, $N_0 = -80$, $N_1 = 100$. We assumed $h = 2^{-7}$ and different a, b and λ. We chose $r = 6$ and $\mathcal{D} = 3$, which gives the saturation error less than 10^{-10}.

In Table 2 and 3 we report on the relative errors and approximation rates for $\mathcal{K}_\lambda f(0.5, 0)$ and $\mathcal{K}_\lambda f(0.25, 0.25)$, respectively, for different a and b, and $\lambda = 1$. The approximated values are computed by the cubature formulas $\mathcal{K}_{\lambda,h}^{(M)}$ in (3.1) for $M = 1, 2, 3$, with $\mathcal{D} = 4$. We have chosen $\alpha = 4$, $\beta = 2$ in (3.2) and the quadrature parameters $\tau = 0.006$, $N_0 = -160$, $N_1 = 200$. We obtain rate of convergence $\mathcal{O}(h^{2M})$ although Theorem 2.1. This result comes from the structure of the error $\mathcal{E} = |\widetilde{\mathcal{K}}_{\lambda,h} \widetilde{f}(\mathbf{x}) - \mathcal{K}_\lambda f(\mathbf{x})|$ which is given by $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$ where $\mathcal{E}_1 = \mathcal{O}(\epsilon + h^{2M})$ and $\mathcal{E}_2 = \mathcal{O}(h^2)$. Numerical experiments show that $\mathcal{E}_2 << \mathcal{E}_1$.

Analogous results are obtained in Table 4 for the approximation of $\mathcal{K}_\lambda g(0, 0) = 1$, with

$$g(x_1, x_2) = (-\Delta + \lambda^2)(1 - \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2})^2/(1 + |\mathbf{x}|^2)^{-1}. \tag{3.4}$$

To check the rate of convergence we considered the function

$$f(x_1, x_2) = (-\Delta + \lambda^2) \cos(30\,\pi\, x_1) \cos(30\,\pi\, x_2). \tag{3.5}$$

The results in Table 5 show that $\mathcal{K}_{\lambda,h}^{(M)} \widetilde{f}(\mathbf{x})$ approximates $\mathcal{K}_\lambda f(\mathbf{x})$ with the predicted approximation rate 2, for $M = 1, 2, 3$, in agreement with Theorem 2.1.

$a = b = 1.5$

			$\lambda^2 = 0.2$		$\lambda^2 = 2$	
x_1	x_2	exact	approximation	error	approximation	error
0.00	0.00	0.8414709848	0.8414709850	0.258E-09	0.8414709848	0.470E-10
0.25	0.00	0.8106234643	0.8106234645	0.267E-09	0.8106234643	0.481E-10
0.50	0.00	0.7104401615	0.7104401617	0.300E-09	0.7104401615	0.519E-10
0.75	0.00	0.5333026735	0.5333026737	0.396E-09	0.5333026736	0.687E-10
1.00	0.00	0.3037650630	0.3037650633	0.718E-09	0.3037650631	0.156E-09
1.25	0.00	0.0932286160	0.0932286162	0.248E-08	0.0932286161	0.711E-09
0.25	0.25	0.7783135137	0.7783135139	0.277E-09	0.7783135138	0.492E-10
0.50	0.50	0.5687113034	0.5687113037	0.371E-09	0.5687113035	0.635E-10
0.75	0.75	0.2474039593	0.2474039595	0.895E-09	0.2474039593	0.210E-09
1.00	1.00	0.0123453654	0.0123453656	0.187E-07	0.0123453655	0.581E-08

$a = 1.5, b = 1$

			$\lambda^2 = 0.2$		$\lambda^2 = 2$	
x_1	x_2	exact	approximation	error	approximation	error
0.00	0.00	0.8414709848	0.8414709867	0.219E-08	0.8414709852	0.519E-09
0.25	0.00	0.8106234643	0.8106234661	0.225E-08	0.8106234647	0.528E-09
0.50	0.00	0.7104401615	0.7104401632	0.248E-08	0.7104401619	0.566E-09
0.75	0.00	0.5333026735	0.5333026752	0.312E-08	0.5333026739	0.679E-09
1.00	0.00	0.3037650630	0.3037650646	0.511E-08	0.3037650634	0.108E-08
1.25	0.00	0.0932286160	0.0932286174	0.154E-07	0.0932286163	0.321E-08
0.25	0.25	0.7363058387	0.7363058405	0.247E-08	0.7363058391	0.580E-09
0.50	0.50	0.3969386023	0.3969386041	0.457E-08	0.3969386028	0.117E-08
0.75	0.75	0.0351490085	0.0351490104	0.541E-07	0.0351490091	0.182E-07

$a = 1.5, b = 0.5$

			$\lambda^2 = 0.2$		$\lambda^2 = 2$	
x_1	x_2	exact	approximation	error	approximation	error
0.00	0.00	0.8414709848	0.8414712254	0.286E-06	0.8414710692	0.100E-06
0.25	0.00	0.8106234643	0.8106237000	0.291E-06	0.8106235459	0.101E-06
0.50	0.00	0.7104401615	0.7104403833	0.312E-06	0.7104402355	0.104E-06
0.75	0.00	0.5333026735	0.5333028743	0.376E-06	0.5333027358	0.117E-06
1.00	0.00	0.3037650630	0.3037652384	0.577E-06	0.3037651117	0.160E-06
1.25	0.00	0.0932286160	0.0932287650	0.160E-05	0.0932286511	0.377E-06
0.25	0.25	0.4982722935	0.4982725309	0.476E-06	0.4982723784	0.170E-06

Table 1: Exact and approximated values of $\mathcal{K}_\lambda f(x_1, x_2)$ and relative error using $\mathcal{K}^{(3)}_{\lambda,h} f(x_1, x_2)$, when $h = 2^{-7}$ and f is given in (3.3)

	$a = 1.5$	$b = 1.5$	$a = 1.5$	$b = 1$	$a = 1.5$	$b = 0.5$
			$M = 1$			
h^{-1}	error	rate	error	rate	error	rate
2^4	0.439E-01		0.968E-01		0.572E+00	
2^5	0.110E-01	1.997	0.243E-01	1.993	0.167E+00	1.781
2^6	0.275E-02	2.000	0.608E-02	2.000	0.419E-01	1.990
2^7	0.688E-03	2.000	0.152E-02	2.000	0.105E-01	2.000
2^8	0.172E-03	2.000	0.380E-03	2.000	0.262E-02	2.000
2^9	0.430E-04	2.000	0.950E-04	2.000	0.655E-03	2.000

$M = 2$

	$a = 1.5$	$b = 1.5$	$a = 1.5$	$b = 1$	$a = 1.5$	$b = 0.5$
h^{-1}	error	rate	error	rate	error	rate
2^4	0.174E-03		0.114E-02		0.186E+00	
2^5	0.443E-05	5.294	0.500E-05	7.831	0.288E-02	6.013
2^6	0.183E-06	4.601	0.626E-06	2.996	0.218E-04	7.044
2^7	0.996E-08	4.196	0.534E-07	3.552	0.919E-06	4.570
2^8	0.600E-09	4.053	0.356E-08	3.908	0.922E-07	3.317
2^9	0.371E-10	4.013	0.226E-09	3.979	0.630E-08	3.871

$M = 3$

	$a = 1.5$	$b = 1.5$	$a = 1.5$	$b = 1$	$a = 1.5$	$b = 0.5$
h^{-1}	error	rate	error	rate	error	rate
2^4	0.719E-04		0.747E-03		0.469E-01	
2^5	0.102E-05	6.138	0.102E-04	6.189	0.177E-02	4.732
2^6	0.155E-07	6.039	0.153E-06	6.066	0.248E-04	6.152
2^7	0.241E-09	6.010	0.236E-08	6.018	0.373E-06	6.057
2^8	0.376E-11	6.001	0.368E-10	6.003	0.577E-08	6.015
2^9	0.936E-13	5.328	0.507E-12	6.180	0.899E-10	6.003

Table 2: Relative errors and approximation rates for $\mathcal{K}_\lambda f(0.5,0) = 0.7104401614873481$ using $\mathcal{K}^{(M)}_{\lambda,h} f(0.5,0)$ in $\Omega = \{\mathbf{x} \in \mathbb{R}^2 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \le 1\}$, $\lambda = 1$ and f in (3.3)

$M = 1$

	$a = 1.5$	$b = 1.5$	$a = 1.5$	$b = 1$	$a = 1.5$	$b = 0.5$
h^{-1}	error	rate	error	rate	error	rate
2^4	0.387E-01		0.955E-01		0.822E+00	
2^5	0.967E-02	2.000	0.240E-01	1.991	0.246E+00	1.738
2^6	0.242E-02	2.000	0.601E-02	1.999	0.623E-01	1.983
2^7	0.604E-03	2.000	0.150E-02	2.000	0.156E-01	1.999
2^8	0.151E-03	2.000	0.376E-03	2.000	0.390E-02	2.000
2^9	0.378E-04	2.000	0.939E-04	2.000	0.974E-03	2.000

$M = 2$

	$a = 1.5$	$b = 1.5$	$a = 1.5$	$b = 1$	$a = 1.5$	$b = 0.5$
h^{-1}	error	rate	error	rate	error	rate
2^4	0.593E-04		0.139E-02		0.312E+00	
2^5	0.225E-05	4.721	0.192E-04	6.174	0.586E-02	5.735
2^6	0.228E-06	3.301	0.244E-06	6.297	0.114E-03	5.682
2^7	0.156E-07	3.870	0.765E-09	8.318	0.350E-05	5.026
2^8	0.997E-09	3.969	0.177E-09	2.113	0.164E-06	4.420
2^9	0.627E-10	3.992	0.145E-10	3.612	0.937E-08	4.126

$M = 3$

	$a = 1.5$	$b = 1.5$	$a = 1.5$	$b = 1$	$a = 1.5$	$b = 0.5$
h^{-1}	error	rate	error	rate	error	rate
2^4	0.663E-04		0.762E-03		0.677E-01	
2^5	0.947E-06	6.130	0.104E-04	6.191	0.281E-02	4.589
2^6	0.144E-07	6.037	0.156E-06	6.067	0.395E-04	6.154
2^7	0.224E-09	6.009	0.240E-08	6.018	0.593E-06	6.058
2^8	0.348E-11	6.008	0.374E-10	6.004	0.917E-08	6.016
2^9	0.117E-12	4.896	0.558E-12	6.067	0.143E-09	6.003

Table 3: Relative errors and approximation rates for $\mathcal{K}_\lambda f(0.25,0.25)$ using $\mathcal{K}^{(M)}_{\lambda,h} f(0.25,0.25)$ in $\Omega = \{\mathbf{x} \in \mathbb{R}^2 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \le 1\}$, $\lambda = 1$ and f in (3.3).

$M = 1$

	$a = 1.5$	$b = 1.5$	$a = 1.5$	$b = 1$	$a = 1.5$	$b = 0.5$
h^{-1}	error	rate	error	rate	error	rate
2^2	0.415E+00		0.648E+00		0.368E+01	
2^3	0.139E+00	1.574	0.216E+00	1.585	0.131E+01	1.489
2^4	0.386E-01	1.854	0.594E-01	1.862	0.366E+00	1.841
2^5	0.992E-02	1.959	0.153E-01	1.961	0.942E-01	1.959
2^6	0.250E-02	1.989	0.384E-02	1.990	0.237E-01	1.989
2^7	0.626E-03	1.997	0.962E-03	1.997	0.594E-02	1.997
2^8	0.157E-03	1.999	0.241E-03	1.999	0.149E-02	1.999
2^9	0.391E-04	2.000	0.602E-04	2.000	0.372E-03	2.000

$M = 2$

	$a = 1.5$	$b = 1.5$	$a = 1.5$	$b = 1$	$a = 1.5$	$b = 0.5$
h^{-1}	error	rate	error	rate	error	rate
2^2	0.134E+00		0.210E+00		0.141E+01	
2^3	0.173E-01	2.952	0.254E-01	3.050	0.187E+00	2.920
2^4	0.143E-02	3.599	0.207E-02	3.616	0.138E-01	3.756
2^5	0.968E-04	3.882	0.140E-03	3.885	0.914E-03	3.918
2^6	0.618E-05	3.969	0.895E-05	3.970	0.580E-04	3.978
2^7	0.388E-06	3.992	0.562E-06	3.992	0.364E-05	3.995
2^8	0.243E-07	3.998	0.352E-07	3.998	0.228E-06	3.999
2^9	0.152E-08	3.999	0.220E-08	3.999	0.142E-07	4.000

$M = 3$

	$a = 1.5$	$b = 1.5$	$a = 1.5$	$b = 1$	$a = 1.5$	$b = 0.5$
h^{-1}	error	rate	error	rate	error	rate
2^2	0.495E-01		0.703E-01		0.577E+00	
2^3	0.284E-02	4.120	0.394E-02	4.157	0.229E-01	4.659
2^4	0.751E-04	5.244	0.106E-03	5.221	0.499E-03	5.517
2^5	0.138E-05	5.768	0.195E-05	5.762	0.903E-05	5.789
2^6	0.225E-07	5.938	0.318E-07	5.937	0.145E-06	5.960
2^7	0.355E-09	5.984	0.502E-09	5.984	0.228E-08	5.990
2^8	0.549E-11	6.014	0.790E-11	5.992	0.357E-10	5.998
2^9	0.150E-12	5.190	0.858E-13	6.524	0.576E-12	5.954

Table 4: Relative errors and approximation rates for $\mathcal{K}_\lambda g(0,0) = 1$ using $\mathcal{K}_{\lambda,h}^{(M)} g(0,0)$ in $\Omega = \{\mathbf{x} \in \mathbb{R}^2 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \leq 1\}$, $\lambda = 1$ and g in (3.4).

$M = 1$

		$a = 1.5$	$b = 1.5$	$a = 1.5$	$b = 1$	$a = 1.5$	$b = 0.5$
h^{-1}	x_1	error	rate	error	rate	error	rate
2^8	0.250	0.171E-05		0.171E-05		0.171E-05	
”	0.750	0.395E-05		0.395E-05		0.395E-05	
”	1.250	0.268E-05		0.268E-05		0.268E-05	
”	1.500	0.569E-08		0.592E-08		0.688E-08	
2^9	0.250	0.762E-06	1.163	0.762E-06	1.163	0.762E-06	1.163
”	0.750	0.176E-05	1.163	0.176E-05	1.163	0.176E-05	1.162
”	1.250	0.120E-05	1.162	0.120E-05	1.162	0.120E-05	1.162
”	1.500	0.140E-07	-1.302	0.144E-07	-1.284	0.163E-07	-1.241
2^{10}	0.250	0.217E-06	1.811	0.217E-06	1.811	0.217E-06	1.811
”	0.750	0.503E-06	1.811	0.503E-06	1.811	0.503E-06	1.811
”	1.250	0.341E-06	1.811	0.341E-06	1.811	0.341E-06	1.811
”	1.500	0.460E-08	1.609	0.467E-08	1.627	0.504E-08	1.690
2^{11}	0.250	0.561E-07	1.954	0.561E-07	1.954	0.561E-07	1.954
”	0.750	0.130E-06	1.954	0.130E-06	1.954	0.130E-06	1.954
”	1.250	0.881E-07	1.954	0.881E-07	1.954	0.881E-07	1.954
”	1.500	0.122E-08	1.915	0.123E-08	1.926	0.128E-08	1.975

$M = 2$

		$a = 1.5$	$b = 1.5$	$a = 1.5$	$b = 1$	$a = 1.5$	$b = 0.5$
h^{-1}	x_1	error	rate	error	rate	error	rate
2^8	0.250	0.316E-05		0.316E-05		0.316E-05	
”	0.750	0.731E-05		0.731E-05		0.731E-05	
”	1.250	0.497E-05		0.497E-05		0.496E-05	
”	1.500	0.534E-07		0.555E-07		0.644E-07	
2^9	0.250	0.897E-06	1.816	0.897E-06	1.816	0.897E-06	1.816
”	0.750	0.208E-05	1.816	0.208E-05	1.816	0.208E-05	1.816
”	1.250	0.141E-05	1.816	0.141E-05	1.816	0.141E-05	1.815
”	1.500	0.194E-07	1.464	0.196E-07	1.503	0.210E-07	1.619
2^{10}	0.250	0.227E-06	1.986	0.227E-06	1.986	0.227E-06	1.986
”	0.750	0.524E-06	1.986	0.524E-06	1.986	0.524E-06	1.986
”	1.250	0.356E-06	1.986	0.356E-06	1.986	0.356E-06	1.986
”	1.500	0.494E-08	1.970	0.495E-08	1.984	0.504E-08	2.056
2^{11}	0.250	0.567E-07	1.999	0.567E-07	1.999	0.567E-07	1.999
”	0.750	0.131E-06	1.999	0.131E-06	1.999	0.131E-06	1.999
”	1.250	0.890E-07	1.999	0.890E-07	1.999	0.890E-07	1.999
”	1.500	0.124E-08	1.999	0.124E-08	2.001	0.124E-08	2.022

$M = 3$

		$a = 1.5$	$b = 1.5$	$a = 1.5$	$b = 1$	$a = 1.5$	$b = 0.5$
h^{-1}	x_1	error	rate	error	rate	error	rate
2^8	0.250	0.356E-05		0.356E-05		0.355E-05	
”	0.750	0.823E-05		0.823E-05		0.823E-05	
”	1.250	0.559E-05		0.559E-05		0.559E-05	
”	1.500	0.751E-07		0.770E-07		0.862E-07	
2^9	0.250	0.906E-06	1.972	0.906E-06	1.972	0.906E-06	1.972
”	0.750	0.210E-05	1.972	0.210E-05	1.972	0.210E-05	1.972
”	1.250	0.142E-05	1.972	0.142E-05	1.972	0.142E-05	1.972
”	1.500	0.198E-07	1.925	0.198E-07	1.959	0.203E-07	2.087
2^{10}	0.250	0.227E-06	1.999	0.227E-06	1.999	0.227E-06	1.999
”	0.750	0.525E-06	1.999	0.525E-06	1.999	0.525E-06	1.999
”	1.250	0.356E-06	1.999	0.356E-06	1.999	0.356E-06	1.999
”	1.500	0.495E-08	1.999	0.495E-08	2.002	0.495E-08	2.035
2^{11}	0.250	0.567E-07	2.000	0.567E-07	2.000	0.567E-07	2.000
”	0.750	0.131E-06	2.000	0.131E-06	2.000	0.131E-06	2.000
”	1.250	0.891E-07	2.000	0.891E-07	2.000	0.891E-07	2.000
”	1.500	0.124E-08	2.000	0.124E-08	2.000	0.124E-08	2.001

Table 5: Absolute errors and approximation rates for $\mathcal{K}_1 f(x_1, 0)$ using $\mathcal{K}_{1,h}^{(M)} f(x_1, 0)$ in $\Omega = \{\mathbf{x} \in \mathbb{R}^2 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \leq 1\}$, with f given in (3.5).

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