Solvability criteria for the Neumann p-Laplacian with irregular data

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Abstract

Necessary and sufficient conditions for the unique solvability of the Neumann problem for the p-Laplace operator are found. They characterize both the domain and measures on the right-hand sides. The present paper complements sufficient conditions of solvability obtained in [ACMM] for the case when the right-hand side of the equation belongs to $L^q(\Omega)$.

1 Criterion in terms of level surfaces

Let Ω be an open connected set in $\mathbb{R}^n$ with compact closure $\overline{\Omega}$ and boundary $\partial\Omega$. By $L^{1,p}(\Omega)$ we denote the space of distributions in Ω with $\nabla u \in L^p(\Omega)$ supplied with the seminorm $\|\nabla u\|_{L^p(\Omega)}$. It is well known that the intersection $L^\infty(\Omega) \cap L^{1,p}(\Omega)$ is dense in $L^{1,p}(\Omega)$ for any Ω (see [M], Sect. 1.1.6, 1.1.8).

Let f be a finite mesure defined on Ω with $f(\Omega) = 0$. First we shall discuss the Neumann problem for

$$-\Delta_p u = f \qquad \text{in } \Omega$$

with zero boundary data. Here $p \in (1, \infty)$ and

$$\Delta_p u = -\text{div}\big(|\nabla u|^{p-2}\nabla u\big)$$

is the p-Laplacian. By a weak solution of this problem we mean an element of the quotient space $L^{1,p}(\Omega)/\mathbb{R}^1$ satisfying

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx = \int_\Omega v \, f(dx) \tag{1.1}$$

for an arbitrary $v \in L^\infty(\Omega) \cap C^\infty(\Omega) \cap L^{1,p}(\Omega)$. In what follows, we use the term "the Neumann problem" for this identity. The uniqueness of the solution follows from the monotonicity of Δ_p and is standard.

In case f is absolutely continuous with respect to the n-dimensional Lebesgue measure and its density f' belongs to $L^q(\Omega)$ for some $q > 1$, problem (1.1) is solvable if the Poincaré inequality

$$\inf_{c\in\mathbb{R}^1} \|w-c\|_{L^{q/(q-1)}(\Omega)} \leq const\,\|\nabla w\|_{L^p(\Omega)} \tag{1.2}$$

holds for all $w \in L^{1,p}(\Omega)$.

Inequality (1.2) is also necessary provided (1.1) is solvable for all absolutely continuous f with $f' \in L^{q/(q-1)}(\Omega)$. Necessary and sufficient conditions for the validity of (1.2) can be found in Sect. 3.4.3 [M]. These conditions are

$$\Omega \in \mathcal{J}_{p,1/q'} \quad \text{if } \ p \leq q' \qquad \text{and} \qquad \Omega \in \mathcal{H}_{p,1/q'} \quad \text{if } \ p > q',$$

where $\mathcal{J}_{p,\alpha}$ and $\mathcal{H}_{p,\alpha}$ are classes of domains introduced in Sect. 6.4.3 [M].

In the following theorem, formulated in [M1] for $p = 2$, we obtain an individual solvability criterion for a measure f. For the proofs of other results in [M1] see [M2].

Theorem 1 *Problem (1.1) is solvable if and only if*

$$\sup_{\{v\}} \int_0^\infty \frac{|f(M_\tau)|^{p/(p-1)} d\,\tau}{\left(\int_{E_\tau} |\nabla v|^{p-1}\, H_{n-1}(dx)\right)^{1/(p-1)}} < \infty. \tag{1.3}$$

Here $M_\tau = \{x \in \Omega : \ v(x) \geq \tau\}$, $E_\tau = \{x \in \Omega : \ v(x) = \tau\}$, H_{n-1} *is the* $(n-1)$*-dimensional Hausdorf measure, and the supremum is taken over all functions* $v \in L^\infty(\Omega) \cap L^{1,p}(\Omega) \cap C^\infty(\Omega)$.

Proof. Without loss of generality we assume that both $m_n\{x : \ v(x) > 0\}$ and $m_n\{x : \ v(x) < 0\}$ do not exceed $\frac{1}{2}\, m_n(\Omega)$, where m_n is the n-dimensional Lebesgue measure.

Sufficiency. We show that (1.3) implies the continuity of the functional

$$v \to \Phi(v) = \int_\Omega v\, f(dx)$$

in the space $L^{1,p}(\Omega)/\mathbb{R}^1$.

Clearly,

$$|f(v)| \leq \Big| \int_\Omega v_+\, f(dx) \Big| + \Big| \int_\Omega v_-\, f(dx) \Big|$$

with

$$v_+ = v \quad \text{in } M_0, \qquad v_+ = 0 \quad \text{in } \Omega \backslash M_0$$

and

$$v_- = 0 \quad \text{in } M_0, \qquad v_- = -v \quad \text{in } \Omega \backslash M_0.$$

We shall use the notation

$$\psi(t) = \int_0^t \frac{d\,\tau}{\left(\int_{E_\tau} |\nabla v|^{p-1}\, H_{n-1}(dx)\right)^{1/(p-1)}}.$$

Let us show that $\psi(t) < \infty$ for every $t < \sup v$. Repeating the proof of Lemma 2.2.3 in [M], we arrive at the inequality

$$\psi(t) \le \int_0^t \frac{d}{d\tau} m_n(M_\tau) \frac{d\tau}{\big(H_{n-1}(E_\tau)\big)^{p/(p-1)}}.$$

Since $M_t \subset M_\tau$ for $t > \tau$, it follows that

$$\inf_{\{\tau:\tau\le t\}} H_{n-1}(E_\tau) \ge \lambda_{\frac{1}{2}m_n(\Omega)}\big(m_n(M_t)\big),$$

where $\lambda_{\frac{1}{2}m_n(\Omega)}$ is the area minimizing function introduced in Sect. 5.2.4 [M]. Therefore, by Lemma 5.2.4 [M],

$$\inf H_{n-1}(E_\tau) \ge \lambda_{\frac{1}{2}m_n(\Omega)}(s) > 0 \text{ for } s > 0.$$

Hence

$$\psi(t) \le \frac{m_n(\Omega)}{2\big(\lambda_{\frac{1}{2}m_n(\Omega)}\big(m_n(M_\tau)\big)\big)^{p/(p-1)}} < \infty.$$

Following the argument used in the proof of Lemma 2.2.2/1 in [M], we show that the function $t(\psi)$, inverse of $\psi(t)$, is absolutely continuous and that

$$\int_\Omega |\nabla v_+|^p dx = \int_0^{\sup \psi} \big(t'(\psi)\big)^p d\psi. \tag{1.4}$$

It follows from the definition of the Lebesgue integral that

$$\int_\Omega v_+ f(dx) = \int_0^\infty f(M_\tau)\, d\tau.$$

Using the absolute continuity of $t(\psi)$, we arrive at

$$\Big|\int_\Omega v_+ f(dx)\Big| = \Big|\int_0^{\sup \psi} t'(\psi) f\big(M_{t(\psi)}\big) d\psi\Big|$$

$$\le \Big(\int_0^{\sup \psi} \big(t'(\psi)\big)^p d\psi\Big)^{1/p} \Big(\int_0^{\sup \psi} \big|f\big(M_{t(\psi)}\big)\big|^{p/(p-1)} d\psi\Big)^{(p-1)/p}.$$

Now, by (1.4),

$$\Big|\int_\Omega v_+ f(dx)\Big| \le \|\nabla v_+\|_{L^p(\Omega)} \int_0^\infty \frac{|f(M_t)|\, dt}{\big(\int_{E_t} |\nabla v|^{p-1} H_{n-1}(dx)\big)^{1/(p-1)}}.$$

The functional

$$\Big|\int_\Omega v_- f(dx)\Big|$$

is majorized in the same way. Hence the functional

$$\int_\Omega v\, f(dx)$$

is bounded in $L^{1,p}(\Omega)/\mathbb{R}^1$ which implies the solvability of problem (1.1).

Necessity. Let $C^{0,1}$ be the space of locally Lipschitz functions in Ω. If problem (1.1) is solvable in $L^{1,p}(\Omega)/\mathbb{R}^1$ then there exists a constant $C \in \mathbb{R}^1$ such that for all functions $w \in C^{0,1}(\Omega) \cap L^{1,p}(\Omega)$

$$\Big|\int_\Omega w\, f(dx)\Big| \le C\,\|\nabla w\|_{L^p(\Omega)}.$$

Let ξ be a function, measurable, positive and bounded on $(0,\infty)$ and let

$$\gamma(t) = \int_0^t \xi(\tau)\, d\tau.$$

We use the notation w for the function given by $w(x) = \gamma(v_+(x))$, where v_+ is the positive part of the function $w \in C^\infty(\Omega) \cap L^{1,p}(\Omega)$. Since $w \in C^{0,1}(\Omega) \cap L^{1,p}(\Omega)$, we have

$$\Big|\int_\Omega \gamma(v_+) f(dx)\Big|^p \le C^p \int_\Omega |\xi(v_+)|^p |\nabla v_+|^p dx. \tag{1.5}$$

Using the identity

$$\int_\Omega \gamma(v_+) f(dx) = \int_0^\infty \xi(t)\, f(M_t)\, dt,$$

we derive from (1.5) that

$$\Big|\int_0^\infty \xi(t)\, f(M_t)\, dt\Big|^p \le C^p \int_0^\infty |\xi(t)|^p \int_{E_t} |\nabla v_+|^{p-1} H_{n-1}(dx).$$

Putting

$$g(t) = \int_{E_t} |\nabla v_+|^{p-1} H_{n-1}(dx), \qquad h(t) = f(M_t),$$

we obtain

$$\Big|\int_0^\infty \xi(t)\, h(t) dt\Big| \le C\Big(\int_0^\infty |\xi(t)|^p g(t)\, dt\Big)^{1/p}$$

for all positive bounded functions ξ and therefore, for all bounded ξ. This means that the functional

$$\xi \to \int_0^\infty \xi(t)\, h(t)\, dt$$

is bounded in $L^p(\mathbb{R}^1_+, g(t)dt)$, and we find that

$$\int_0^\infty \frac{\big(h(t)\big)^{p/(p-1)} dt}{\big(g(t)\big)^{1/(p-1)}} \le C^{p/(p-1)}.$$

This inequality can be written as

$$\int_0^\infty \frac{|f(M_t)|^{p/(p-1)} dt}{\Big(\int_{E_t} |\nabla v|^{p-1} H_{n-1}(dx)\Big)^{1/(p-1)}} \le C^{p/(p-1)}.$$

The proof is complete.

2 Sufficient condition in terms of an isocapacitary function

Let G be an open subset of Ω subject to the condition

$$m_n(G) \leq \frac{1}{2} m_n(\Omega)$$

and let F be a subset of G closed in Ω. The set $K = G \backslash F$ will be called a condenser. We introduce the p-capacity of this condenser

$$c_p(K) = \inf\{\|\nabla u\|^p_{L^p(G)} : \ u \in C^{0,1}(\Omega), \ u \geq 1 \text{ on } F \ \text{ and } u \leq 0 \text{ on } \Omega \backslash G\}.$$

Theorem 2 *Let the function*

$$\mathbb{R}^1_+ \ni s \to \varkappa_p(s) = \sup_{\{K: c_p(K) \leq s\}} |f(F)|$$

satisfy

$$\int_0^\infty \Big(\frac{\varkappa_p(s)}{s}\Big)^{p/(p-1)} ds < \infty. \tag{2.1}$$

Then the Neumann problem (1.1) is uniquely solvable in $L^{1,p}(\Omega)/\mathbb{R}^1$.

Proof. We shall use the notation introduced in the proof of Theorem 1. In that proof we obtained the inequality

$$\Big|\int_\Omega v_+ f(dx)\Big| \leq \Big(\int_{M_0} |\nabla v|^p\Big)^{1/p} \int_0^\infty \frac{|f(M_t)|^{p/(p-1)} dt}{\Big(\int_{E_t} |\nabla v_+|^{p-1} H_{n-1}(dx)\Big)^{1/(p-1)}}.$$

Since $m_n(M_t) \leq \frac{1}{2} m_n(\Omega)$, we have

$$|f(M_t)| \leq \varkappa_p\big(c_p(K_{t,\varepsilon})\big),$$

where $K_{t,\varepsilon}$ is the condenser with

$$G = \{x : v_+ > \varepsilon\} \qquad \text{and } F = M_t,$$

with $t > \varepsilon > 0$.

By Lemma 6.1.3/1 [M],

$$c_p(K_{t,\varepsilon}) \leq \big(\psi(t) - \psi(\varepsilon)\big)^{1-p}$$

and therefore,

$$|f(M_t)| \leq \varkappa_p\Big(\frac{1}{\psi(t)^{p-1}}\Big).$$

Hence

$$\int_0^\infty \frac{|f(M_t)|^{p/(p-1)} dt}{\Big(\int_{E_t} |\nabla v_+|^{p-1} H_{n-1}(dx)\Big)^{1/(p-1)}} \leq \int_0^\infty \Big(\varkappa_p\Big(\frac{1}{\psi(t)^{p-1}}\Big)\Big)^{1/(p-1)} d\psi(t)$$

$$\leq \int_0^\infty \Big(\frac{\varkappa_p(s)}{s}\Big)^{p/(p-1)} ds.$$

The value

$$\Big| \int_\Omega v_- f(dx) \Big|$$

is estimated in the same way. The result follows from Theorem 1.

Definition. Let $\nu_p(s)$ be the infimum of $c_p(K)$ taken over the collection of all condensers $K = G \backslash F$ with

$$m_n(F) \geq s \qquad \text{and} \qquad m_n(G) \leq \frac{1}{2} m_n(\Omega).$$

As an obvious consequence of Theorem 2, using this condenser capacitary minimizing function, we can state the following sufficient solvability condition which involves a relation between the measure f and the Lebesgue measure m_n.

Corollary 1 *Let for all sets F closed in Ω subject to $m_n(F) \leq \frac{1}{2} m_n(\Omega)$ there hold the inequality*

$$|f(F)| \leq k\Big[\nu_p\big(m_n(F)\big)\Big], \tag{2.2}$$

where k is a nondecreasing function satisfying

$$\int_0^\infty \Big(\frac{k(s)}{s}\Big)^{p/(p-1)} ds < \infty. \tag{2.3}$$

Then problem (1.1) is uniquely solvable in $L^{1,p}(\Omega)/\mathbb{R}^1$.

Remark Assume that for small positive s

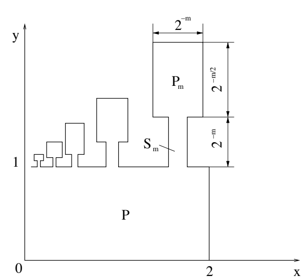
$$\nu_p(s) \geq const\, s^{p/q'}, \tag{2.4}$$

where $p \leq q'$. Using terminology of Sect. 4.3 [M], the domain Ω belongs to the class $\mathcal{J}_{p,1/q'}$. As mentioned above, this condition is equivalent to the Poincaré type inequality (1.3), which in its turn is necessary and sufficient for the solvability of (1.1) with arbitrary absolutely continuous f with $f' \in L^{q'}(\Omega)$. An individual sufficient condition for solvability of (1.1) in $\Omega \in \mathcal{J}_{p,1/q'}$ which stems from (2.4) and (2.2) is the inequality

$$|f(F)| \leq k\Big(\big((m_n(F)\big)^{p/q'}\Big) \tag{2.5}$$

with increasing k subject to (2.3).

Example 1. Consider the planar domain depicted in Fig.1. Let us assume that the width of the passage S_m is equal to $2^{-\alpha m}$, where $\alpha = p + 1/2$. Arguing as in Sect. 6.4.3 [M], one can show that $\Omega \in \mathcal{J}_{p,1/p}$. Hence problem (1.1) is solvable if the measure f satisfies



$$|f(F)| \leq k\big(m_n(F)\big)$$

with increasing k satisfying (2.3).

Example 2. Consider the n-dimensional "whirlpool" depicted in Fig.2

$$\Omega = \{x = (x', x_n) : |x'| < x_n^\beta,\ 0 < x_n < 1\},$$

where $\beta \geq 1$.

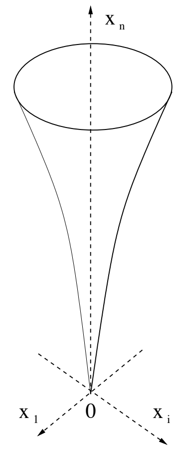
Let $p < \beta(n-1)+1$. Then by Example 6.3.6/1 [M], $\Omega \in \mathcal{J}_{p,1/q'}$, i.e. for small s

$$\nu_p(s) \geq const\, s^{p/q'}$$

with

$$\frac{1}{q'} = \frac{1}{p} - \frac{1}{\beta(n-1)+1}. \tag{2.6}$$

Hence the solvability condition (2.5) for the β-whirlpool involves q' given by (2.6).



3 Solvability criterion in terms of the modulus of a family of surfaces

Let $\{F_t\}_{t\geq 0}$ denote a decreasing family of subsets of Ω closed in Ω, with smooth surface $\Omega \cap \partial F_t$ subject to

$$m_n(F_0) \leq \frac{1}{2} m_n(\Omega).$$

We call $\{F_t\}_{t\geq 0}$ admissible if there exists a function $v \in C^\infty(\Omega)$ such that

$$F_t = \{x \in \Omega : v(x) \geq t\}.$$

Now, we are able to formulate a necessary and sufficient condition of solvability of (1.1) in the spirit of the theory of modulus of families of surfaces (see [F]).

Theorem 3 *Problem (1.1) is uniquely solvable if and only if there exists a Borel nonnegative function $g \in L^{p/(p-1)}(\Omega)$ such that for any admissible family $\{F_t\}_{t\geq 0}$*

$$|f(F_t)| \leq \int_{\Omega\cap\partial F_t} g(x) H_{n-1}(dx). \tag{3.1}$$

Proof. *Necessity.* Let v be an arbitrary smooth function in (1.1) and let $\lambda(t)$ be an increasing function on $\mathbb{R}^1$ equal to zero for $t < 0$. Then, by (1.1) and the coarea formula, we have

$$\int_0^\infty \lambda'(t) \int_{E_t} |\nabla u|^{p-2} \nabla u \cdot \mathbf{N}\, H_{n-1}(dx)\, dt = \int_\Omega \lambda(v) f(dx),$$

where $\mathbf{N}$ is the normal vector to $E_t = \{x \in \Omega : v(x) = t\}$. The right-hand side can be written as

$$\int_0^\infty f(F_t)\, \lambda'(t)\, dt,$$

where $F_t = \{x \in \Omega : v(x) \geq t\}$. Hence, for almost every $t > 0$ we obtain

$$f(F_t) = \int_{\Omega\cap\partial F_t} |\nabla u|^{p-2} \nabla u \cdot \mathbf{N}\, H_{n-1}(dx)$$

which implies (3.1) with $g(x) = |\nabla u(x)|^{p-1}$.

Sufficiency. Since $\{M_t\}_{t>0}$ is an admissible family, we have $E_t = \Omega \cap \partial F_t$ for almost all t. Take $g \in L^{p'}(\Omega)$ satisfying (3.1).

Let us check condition (1.3). We have

$$\int_0^\infty \frac{|f(M_\tau)|^{p/(p-1)}\, d\tau}{\left(\int_{E_\tau} |\nabla v|^{p-1} H_{n-1}(dx)\right)^{1/(p-1)}} \le \int_0^\infty \frac{\left(\int_{E_\tau} g(x) H_{n-1}(dx)\right)^{p/(p-1)} d\tau}{\left(\int_{E_\tau} |\nabla v|^{p-1} H_{n-1}(dx)\right)^{1/(p-1)}}.$$

By Hölder's inequality,

$$\left(\int_{E_\tau} g(x) H_{n-1}(dx)\right)^{p/(p-1)} \le \int_{E_\tau} \frac{g^{p/(p-1)}}{|\nabla v|} H_{n-1}(dx) \left(\int_{E_\tau} |\nabla v|^{p-1} H_{n-1}(dx)\right)^{1/(p-1)}.$$

Hence

$$\int_0^\infty \frac{|f(M_\tau)|^{p/(p-1)}\, d\tau}{\left(\int_{E_\tau} |\nabla v|^{p-1} H_{n-1}(dx)\right)^{1/(p-1)}} \le \int_0^\infty \int_{E_\tau} \frac{g^{p/(p-1)}}{|\nabla v|} H_{n-1}(dx) d\tau.$$

By the coarea formula, the last integral is equal to

$$\int_{M_0} g^{p/(p-1)} dx.$$

The result follows.

4 Neumann problem with nonhomogeneous boundary condition

It is straightforward to reformulate the previous statement to the Neumann problem

$$\begin{aligned} \Delta_p u &= 0, && \text{in } \Omega \\ |\nabla u|^{p-2} \nabla u \cdot \mathbf{N} &= \varphi, && \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

where φ is a finite measure supported by $\partial\Omega$, $\varphi(\partial\Omega) = 0$ and $\mathbf{N}$ is the outer normal with respect to Ω.

The role of the space $L^{1,p}(\Omega)$ will be played by the space $\tilde{L}^{1,p}(\Omega)$ obtained by completion of the intersection $L^{1,p}(\Omega) \cap C^1(\Omega) \cap C(\overline{\Omega})$ in the metrics of $L^{1,p}(\Omega)$. We define a weak solution to problem (4.1) as a function $u \in \tilde{L}^{1,p}(\Omega)$ satisfying

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v dx = \int_{\partial\Omega} v\, \varphi(dx) \tag{4.2}$$

for all $v \in L^{1,p}(\Omega) \cap C^1(\Omega) \cap C(\overline{\Omega})$.

The following criterion is a direct analogue of Theorem 1 and is proved in the same way.

Theorem 4 *Problem (4.2) is solvable if and only if*

$$\sup_{\{v\}} \int_0^\infty \frac{|\varphi(M_\tau)|^{p/(p-1)} d\,\tau}{\left(\int_{E_\tau} |\nabla v|^{p-1} H_{n-1}(dx)\right)^{1/(p-1)}} < \infty.$$

Here $M_\tau = \{x \in \partial\Omega : v(x) \geq \tau\}$ *and, as before,* $E_\tau = \{x \in \Omega : v(x) = \tau\}$.

Theorem 3 holds along with its proof if the criterion (3.1) is replaced by

$$|\varphi(\partial\Omega \cap \partial G_t)| \leq \int_{\Omega\cap\partial G_t} g(x)\, H_{n-1}(dx),$$

where g is a non-negative Borel function in $L^{p/(p-1)}(\Omega)$.

Remark All formulations and proofs in the present paper hold if Ω is an open subset of an n-dimensional smooth compact manifold.

In conclusion I mention the article [MP], where the Neumann problem for the p-Laplacian is studied for interior and exterior cuspidal domains. In [MP] a complete characterisation of distributional data on the right-hand side of the equation and the Neumann condition is obtained.

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