Asymptotic analysis of the Navier-Stokes system in a plane domain with thin channels

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Abstract. The flow of viscous incompressible fluid in a domain $\Omega_\varepsilon$ depending on a small parameter $\varepsilon$ is considered. The domain $\Omega_\varepsilon$ is the union of a domain $\Omega_0$ with piecewise smooth boundary and thin channels with width of order $\varepsilon$. Every channel contains one angle point of the domain $\Omega_0$ near the channels inlet. We prove the existence of a solution $(v_\varepsilon, p_\varepsilon)$ to the Navier-Stokes system such that in a neighbourhood of an angle point of the domain $\Omega_0$ the pair $(v_\varepsilon, p_\varepsilon)$ is equal, up to the term with finite kinetic energy, to the Jeffery-Hamel solution. In the channels the pair $(v_\varepsilon, p_\varepsilon)$ asymptotically coincides with the Poiseuille solution. Asymptotic expressions for the kinetic energy and the Dirichlet integral of $(v_\varepsilon, p_\varepsilon)$ is obtained.

Keywords. Navier-Stokes system, Jeffery-Hamel solution, Poiseuille solution, corner boundary points,
**Introduction**

We consider the flow of a viscous incompressible fluid in a domain $\Omega_\varepsilon$ depending on a small parameter $\varepsilon$. To describe $\Omega_\varepsilon$ we introduce a limit domain $\Omega_0$ and

![Diagram of domain $\Omega_\varepsilon$.](image)

Fig.1. Domain $\Omega_\varepsilon$.

thin channels. Let $\Omega_0$ be a domain in $\mathbb{R}^2$ with compact closure and boundary $\partial \Omega_0$. We assume that $\partial \Omega_0$ is a union of smooth closed arcs and by $\{A\}$ we denote the finite set of all end points of these arcs. With every point $A \in \{A\}$ we associate a thin channel $C^A_\varepsilon$ with $A$ inside $C^A_\varepsilon$ (see Fig.2, the formal description

![Diagram of channel $C^A_\varepsilon$.](image)

Fig.2. Channel $C^A_\varepsilon$.

of $C^A_\varepsilon$ will be given in Section 3).
Let \((r, \theta), |\theta| < \pi\), be the polar coordinates with origin at \(A\) and the polar axis directed inside \(\Omega_\varepsilon\). Suppose that the domain \(\Omega_0\) is given by \(-\omega^-_A < \theta < \omega^+_A\) in the disk with center \(A\) and diameter \(d_A\). We assume that \(0 < \omega^-_A < \omega^+_A < \pi/2\).

The domain \(\Omega_\varepsilon\) is introduced by
\[
\Omega_\varepsilon = \Omega_0 \cup \cup_{\{A\}} C^A_\varepsilon.
\]

We deal with the Navier-Stokes system
\[
\langle v_\varepsilon, \nabla \rangle v_\varepsilon = -\rho^{-1}\text{grad}p_\varepsilon + \nu\Delta v_\varepsilon \quad \text{on} \quad \Omega_\varepsilon, \quad (0.1)
\]
\[
\text{div} v_\varepsilon = 0 \quad \text{on} \quad \Omega_\varepsilon. \quad (0.2)
\]
Here \(\langle \cdot, \cdot \rangle\) is the scalar product in \(\mathbb{R}^2\), \(\nu\) is the viscosity, \(\rho\) is the density, \(v_\varepsilon\) is the velocity vector and \(p_\varepsilon\) is the pressure.

We assume that the vector-valued function \(v_\varepsilon\) satisfies the Dirichlet boundary condition at every interval \(B^A_\varepsilon\) (see Fig.2):
\[
v_\varepsilon = \varepsilon^{-1}\varphi^A(\varepsilon^{-1}(x - B)), \quad x \in B^A_\varepsilon, \quad (0.3)
\]
where
\[
\varphi^A \in (C^{1,\alpha}(-b^-_A, b^+_A))^2
\]
and \(\varphi^A\) is equal to zero at the end points of \(B^A_\varepsilon\). We suppose also that the velocity vector \(v_\varepsilon\) satisfies the homogeneous Dirichlet condition on the remaining part of the boundary \(\partial\Omega_\varepsilon\):
\[
v_\varepsilon(x) = 0, \quad x \in \partial\Omega_\varepsilon \setminus \cup_{\{A\}} B^A_\varepsilon. \quad (0.4)
\]
Let the pressure \(p_\varepsilon\) be subject to the condition
\[
p_\varepsilon = 0, \quad (0.5)
\]
where \(f\) is the mean value of the function \(f\) over the domain \(\Omega_\varepsilon\).

We introduce the notation
\[
\Upsilon_A = -\int_{-b^-_A}^{b^+_A} \varphi^A_n(t) \, dt, \quad (0.6)
\]
where (0.6) and henceforth $a_n$ stands for the normal component of the vector $a$. We assume that

$$\sum_{\{A\}} \gamma_A = 0.\quad (0.7)$$

We first construct an asymptotic solution $(V_\varepsilon, P_\varepsilon)$ of problem (0.1)−(0.5) such that in $\Omega_0$, outside the set $\{A\}$ there holds the asymptotic relation

$$(V_\varepsilon(x), P_\varepsilon(x)) \sim (v_0(x), p_0(x)), \quad \varepsilon \to 0,\quad (0.8)$$

where $(v_0, p_0)$ is a solution of system (0.1), (0.2) in the domain $\Omega_0$ with the flux

$$\int_{\{x \in \Omega_0 : |x - A| = \tau\}} \langle v_0, \frac{x - A}{|x - A|} \rangle ds_x = \gamma_A\quad (0.9)$$

![Diagram showing the domain $\Omega_0$ and vector $A$.](image)

Fig.3. "Model" domain $\Omega_0$.

given at every angle point $A$ ( $\tau$ being a sufficiently small positive number). Also let $v_0$ be subject to the boundary condition

$$v_0(x) = 0, \quad x \in \partial \Omega_0.$$  

In a neighbourhood of an angle point the pair $(v_0, p_0)$ is equal, up to the term with finite Dirichlet integral, to the well-known exact solution of the Navier-Stokes system obtained by Jeffery (1915) and Hamel (1916) (see [1,2]). This solution $(H^A, Q^A)$, which describes a plane viscous source (or sink) flow between straight walls has the following form in the polar coordinates $(r, \theta)$ with origin at $A$:

$$H^A_r(r, \theta) = r^{-1} V^A(\theta),$$

$$H^A_\theta(r, \theta) = 0,$$

$$Q^A(r, \theta) = r^{-2} J^A(\theta).\quad (0.10)$$
In a small neighbourhood of the point \( A \in \{ A \} \) we look for \((V_\varepsilon, P_\varepsilon)\) in the asymptotic form

\[
(V_\varepsilon(x), P_\varepsilon(x)) \sim (\varepsilon^{-1}v^A(\varepsilon^{-1}(x - A)), \varepsilon^{-2}p^A(\varepsilon^{-1}(x - A))), \quad \varepsilon \to 0.0(11)
\]

where \((v^A, p^A)\) is a solution of the Navier-Stokes system considered in the model

\[
\begin{align*}
\Omega_1(\Lambda_A) &= \{ y \in \Lambda_A : y_2 > 0, \ |y| = T_1 \}, \\
\Omega_2(\Lambda_A) &= \{ y \in \Lambda_A : y_1 \in (-b_A^-, b_A^+), \ y_2 = -T_2 \},
\end{align*}
\]

Fig.4. The "model" domain \( \Lambda_A \).

domain \( \Lambda_A \) depicted in Fig.4. The velocity \( v^A \) satisfies the boundary condition

\[
v^A(y) = 0, \quad y \in \partial \Lambda_A
\]

and the flux condition

\[
\Upsilon_A = \int_{y \in \Gamma_1(\Lambda_A)} \langle v^A, \frac{y}{|y|} \rangle ds_y,
\]

which is equivalent to

\[
\Upsilon_A = -\int_{y \in \Gamma_2(\Lambda_A)} v_2^A dy.
\]

Here \((y_1, y_2)\) are Cartesian coordinates with center \( A \) and with the axis \( Ay_2 \) directed along the axis of the channel (see Fig.4);
where $T_1$ and $T_2$ are sufficiently large positive numbers. By $a_j$, $j = 1, 2$, we denote the components of the vector $a$. In particular, $v_2$ in (0.14) is the second component of $v$.

The behavior of $(v^A, p^A)$ as $|y| \to \infty$, $y_2 > 0$, is described, up to terms with finite Dirichlet integral, by the Jeffery-Hamel solution (0.10).

In the channel $C_\varepsilon^A$ we have

$$
(V_\varepsilon(x), P_\varepsilon(x))
$$

$$
\sim (\varepsilon^{-1} v^C(\varepsilon^{-1}(x - C)), \varepsilon^{-2} p^C(\varepsilon^{-1}(x - C)) + \kappa^C \varepsilon^{-3}), \varepsilon \to 0,
$$

where $C$ is the middle point of the axis of the channel, $(v^C, p^C)$ is the Poiseuille solution to the Navier-Stokes system in an infinite strip, and $\kappa^C$ is a constant.

In order to construct the asymptotic solution $(V_\varepsilon, P_\varepsilon)$ near the end interval $B_\varepsilon^B$ of the channel $C_\varepsilon^A$ we introduce a solution $(v^B, p^B)$ of the Navier-Stokes system (0.1), (0.2) in the semi-strip $\Pi_B$ which does not depend on the parameter $\varepsilon$ (see Fig. 5. The "model" domain $\Pi_B$.

Fig. 5). In a small neighbourhood of the end interval $B_\varepsilon^B$ of the channel we have

$$
(V_\varepsilon(x), P_\varepsilon(x))
$$

$$
\sim (\varepsilon^{-1} v^B(\varepsilon^{-1}(x - B)), \varepsilon^{-2} p^B(\varepsilon^{-1}(x - B)) + \kappa^B \varepsilon^{-3}), \varepsilon \to 0,
$$
where $\kappa^B = \text{const.}$ On the basement of $\Pi_B$ the boundary condition

$$\mathbf{v}^B(t_1, 0) = \varphi^A(t_1), \quad t_1 \in (-b_A^-, b_A^+)$$

(0.17)

is satisfied, where $\varphi^A$ is the vector-valued function in the boundary condition (0.3) corresponding to the channel with the end interval $B^A_\varepsilon$. On the lateral sides of $\Pi_B$ the velocity vector $\mathbf{v}^B$ satisfies

$$\mathbf{v}^B(\pm b_A^\pm, t_2) = 0, \quad t_2 \in (0, +\infty)$$

(0.18)

and has the prescribed flux

$$\int_{t \in \Xi(\Pi_B)} \mathbf{v}_2^B \, dt = \Upsilon_A,$$

where

$$\Xi(\Pi_B) = \{t \in \Pi_B : t_1 \in (-b_A^-, b_A^+), t_2 = T\},$$

and $T > 0$.

We introduce a partition of unity $\{X_\varepsilon, \eta^A_\varepsilon, \mu^B_\varepsilon, \xi^B_\varepsilon\}$ in $\Omega_\varepsilon$, where $\eta^A_\varepsilon$ and $\xi^B_\varepsilon$ are cut-off functions supported by neighbourhoods of $A$ and $B$ respectively. By $X_\varepsilon$ we denote cut-off function which vanishes in a neighbourhood of $\{A\}$. The cut-off function $\mu^B_\varepsilon$ is equal to 1 outside a neighbourhood of $B_\varepsilon$.

We construct the asymptotic solution $(\mathbf{V}_\varepsilon, P_\varepsilon)$ of system (0.1)-(0.5) in the form

$$\mathbf{V}_\varepsilon(x) = \mathbf{v}_0(x) X_\varepsilon(x) + \varepsilon^{-1} \left\{ \eta^A_\varepsilon(x) \mu^B_\varepsilon(x) \mathbf{v}^A(\varepsilon^{-1}(x - A)) + \xi^B_\varepsilon(x) \mathbf{v}^B(\varepsilon^{-1}(x - B)) \right\},$$

(0.19)

$$P_\varepsilon(x) = p_0(x) X_\varepsilon(x) + \varepsilon^{-2} \left\{ \eta^A_\varepsilon(x) \mu^B_\varepsilon(x) p^A(\varepsilon^{-1}(x - A)) + \xi^B_\varepsilon(x) p^B(\varepsilon^{-1}(x - B)) \right\},$$

(0.20)

In (0.19), (0.20) and henceforth the summation is taken over all the channels i.e. over the set $\{A\}$.

We introduce the number

$$\mathcal{R} = \nu^{-1} \sum \|\varphi^A\|_{(C^{1,\alpha}((-b_A^-, b_A^+))^2)}^2$$
and suppose that \( \mathcal{R} \) is sufficiently small:

\[
\mathcal{R} \ll 1. \quad (0.21)
\]

Our basic result is the existence theorem for a solution \((v_\varepsilon, p_\varepsilon)\) of (0.1)–(0.5) such that

\[
v_\varepsilon(x) = V_\varepsilon(x) + w_\varepsilon(x),
\]

\[
p_\varepsilon(x) = P_\varepsilon(x) + q_\varepsilon(x),
\]

where

\[
\|w_\varepsilon\|_{(B^1(\Omega_\varepsilon))^2} + \|q_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq c_\varepsilon^\delta, \quad \delta > 0. \quad (0.23)
\]

We also obtain asymptotic expressions for two integral characteristics of the solution \((v_\varepsilon, p_\varepsilon)\). Let \( L_A = |AB| \) and \( b_A = \varepsilon^{-1}|B^A| \), i.e. \( L_A \) and \( \varepsilon b_A \) are the length and the width of \( C^A_\varepsilon \) respectively. We show that the kinetic energy

\[
\mathcal{E}(v_\varepsilon) = \frac{\rho}{2} \int_{\Omega_\varepsilon} |v_\varepsilon(x)|^2 \, dx \quad (0.24)
\]

admits the representation

\[
\mathcal{E}(v_\varepsilon) = \frac{3\rho}{5} \varepsilon \sum \frac{\gamma^2_A L_A b_A^{-1}}{\varepsilon} + \frac{\rho}{2} \log \frac{1}{\varepsilon} \sum \frac{\gamma^2_A}{\omega_A^+} \int_{\omega_A^-} (V^A(\theta))^2 \, d\theta + O(1). \quad (0.25)
\]

For the Dirichlet integral

\[
\mathcal{I}(v_\varepsilon) = \int_{\Omega_\varepsilon} (\nabla v_\varepsilon(x))^2 \, dx \quad (0.26)
\]

we obtain the asymptotic formula

\[
\mathcal{I}(v_\varepsilon) = 12\varepsilon^{-3} \sum \gamma^2_A L_A b_A^{-3} + O(\varepsilon^{-2}). \quad (0.27)
\]

In Section 1 we consider the Dirichlet problem with prescribed fluxes at the points \( A \) for the Navier-Stokes system in the domain \( \Omega_0 \). Auxiliary boundary value problems in model domains \( \Lambda_A \) and \( \Pi_B \) are considered in Section 2. The next Section 3 concerns the Stokes problem in \( \Omega_\varepsilon \). In Section
4 we derive the principal term \((V_\varepsilon, P_\varepsilon)\) in the representation (0.22). The auxiliary Section 5 is a preparation to the proof of our principal result, an existence theorem for the solution \((v_\varepsilon, p_\varepsilon)\) of problem (0.1) − (0.5) in the form (0.22). In the same section we study a boundary value problem for the remainder term \((w_\varepsilon, q_\varepsilon)\). In Section 6 we prove the existence of \((v_\varepsilon, p_\varepsilon)\). The last Section 7 contains a proof of asymptotic formulas (0.25), (0.27) for the kinetic energy and for the Dirichlet integral.

1 The flow in the limit domain \(\Omega_0\)

Consider the system

\[
\langle v, \nabla \rangle v = -\rho^{-1} \text{grad} p + \nu \Delta v \text{ on } \Omega_0, \tag{1.1}
\]

\[
\text{div } v = 0 \text{ on } \Omega_0 \tag{1.2}
\]

with \(p\) and \(\nabla v\) square summable outside any neighborhood of \(\{A\}\). Suppose that \(v\) satisfies

\[
v = 0 \text{ on } \partial \Omega_0 \setminus \{A\}. \tag{1.3}
\]

At every angle point \(A \in \{A\}\) we prescribe the flux \(M_A\),

\[
\int_{\{x \in \Omega_0 : |x - A| = \tau\}} \langle v, \frac{x - A}{|x - A|} \rangle dx = M_A \tag{1.4}
\]

and we suppose that

\[
\sum M_A = 0.
\]

Before proving the existence of the solution of (1.1)-(1.4) we note that the principal term of its asymptotics near the point \(A\) coincides with the Jeffery-Hamel solution \((H, Q)\) for the angle

\[
\{(r, \theta) : -\omega_- < \theta < \omega_+, \ 0 < r < +\infty\}
\]

which is defined as follows. The vector-function \(H\) satisfies the zero Dirichlet condition on the set

\[
\{(r, \theta) : \theta = \omega_\pm, \ 0 < r < +\infty\}
\]
and has the unit flux at $A$. The radial component $\mathcal{V}_r$ of the vector $\mathbf{V} = r\mathbf{H}$ satisfies

$$
(\partial^2 \mathcal{V}_r / \partial \theta^2)(\theta, R) + 4(\mathcal{V}_r(\theta, R) - K) + R(\mathcal{V}_r(\theta, R))^2 = 0,
$$

(1.5)

$$
\int_{\omega_+}^{\omega_-} \mathcal{V}_r(\theta, R) \, d\theta = \sigma,
$$

(1.6)

$$
\mathcal{V}_r(\pm \omega_{\pm}, R) = 0,
$$

(1.7)

where $K$ is an unknown constant depending on $R > 0$, $\sigma = 1$ in the case of the source and $\sigma = -1$ in the case of the sink. The angle component $\mathcal{V}_\theta$ of $\mathbf{V}$ is equal to zero and the function $J = r^2 Q$ is found from

$$
J = 2\rho \nu (\mathcal{V}_r - K).
$$

(1.8)

Properties of this solution, which is expressed in elliptic functions, have been investigated in detail in [3–6]. In particular, a complete information about its dependence on the Reynolds number has been obtained. A Jeffery-Hamel solution for the case of variable viscosity and density was considered in [7, 8].

By using the Jeffery-Hamel solution obtained in [4], L.E. Fraenkel [9, 10] and L.E. Fraenkel, P.M. Eagles [11] constructed an asymptotic series for the flow in channels with slightly curved walls. The stability of flow in an infinite channel of the same type was investigated in [12], [13]. In [14], P.M. Eagles showed that the Jeffery-Hamel solution appears as the first approximation of the boundary layer for the film flow over curved beds.

To study problem (1.1)–(1.4) we use weighted Hölder spaces $N^{j, \alpha}(\Omega_0)$ with $\alpha \in (0, 1), \tau \in \mathbb{R}^1$ and $j = 0$ or 1 of functions on $\Omega_0$ with finite norm

$$
\|u\|_{N^{j, \alpha}_r(\Omega_0)} = \sup_{x, y \in \Omega_0} |x - y|^{-\alpha} |\nabla^j (r^\tau(x)u(x)) - \nabla^j (r^\tau(y)u(y))| + \sup_{x \in \Omega_0} r^{\tau-j-\alpha}(x)|u(x)|,
$$

where $r(x) = \text{dist}\{x, \{A\}\}$, $\nabla^j u = \nabla u$ if $j = 1$ and $\nabla^j u = u$ if $j = 0$. By $N^{0, \alpha}_r(\Omega_0)$ we denote the subset of $N^{0, \alpha}_r(\Omega_0)$ containing functions equal zero on $\partial\Omega_0 \setminus \{A\}$. Also, let $N^{j, \alpha}_r(\Omega_0)$ be the space of distributions $\text{div} \mathbf{W} + r^{-1} \mathbf{W}_0$, where $\mathbf{W} \in (N^{0, \alpha}_r(\Omega_0))^2$, $\mathbf{W}_0 \in N^{0, \alpha}_r(\Omega_0)$. The following auxiliary result on the Stokes system in the plane domain with angle points is known (see [15], §5, where the three-dimensional case is considered).
**Lemma 1.1** The Stokes operator $S_0$ defined by

$$S_0(V, P) = (-\Delta V + (\nu \rho)^{-1} \text{grad} P, \text{div} V)$$

performs the isomorphism

$$D_\tau^\alpha = (N_\tau^{1,\alpha}(\Omega_0))^2 \times N_\tau^{0,\alpha}(\Omega_0) \rightarrow R_\tau^\alpha = (N_{\tau-1,\alpha}(\Omega_0))^2 \times N_{\tau,\perp}^{0,\alpha}(\Omega_0),$$

where $|\tau - 1 - \alpha| < 1$ and $N_\tau^{0,\alpha}(\Omega_0)$ is the space of functions $s \in N_\tau^{0,\alpha}(\Omega_0)$ satisfying the condition

$$\int_{\Omega_0} s(x) \, dx = 0.$$

Now we are in a position to construct a solution $(v, p)$ of problem (1.1)–(1.4) in $\Omega_0$. We formulate the principal result of this section. In its statement and in the sequel we put

$$\mathcal{M} = \sum |M_A|.$$

By $(\mathcal{V}_A, J_A)$ we denote the solution of problem (1.5)–(1.8), where $R = \nu^{-1} |M_A|$ and $\sigma = \text{sign} M_A$ for the angle corresponding to $A$.

Let $\zeta \in C_0^\infty(D_2(0))$ and let $\zeta(x) = 1$ for $x \in D_1(0)$ where $D_d(a)$ is the disk of diameter $d$ with center $a$.

We introduce the pair $(Y, \Theta)$ by

$$(Y, \Theta) = \sum |M_A| \zeta_A(H^A, Q^A),$$

where $\zeta_A(x) = \zeta(2d_A^{-1}(x - A)),$

$$(H^A, Q^A) = (r^{-1}V_A, r^{-2}J_A + c^A),$$

and $c^A$ is an arbitrary constant.

**Theorem 1.1** Let $\nu^{-1} \mathcal{M} < C_0$, where $C_0$ is a constant depending only on $\Omega_0$. Then there exists a solution $(v, p)$ of problem (1.1)–(1.4) represented in the form

$$(v, p) = (Y, \Theta) + (w, q),$$

where the pair $(w, q)$ belongs to $(N_\tau^{1,\alpha}(\Omega_0))^2 \times N_\tau^{0,\alpha}(\Omega_0)$ and satisfies the estimate

$$\|w\|_{(N_\tau^{1,\alpha}(\Omega_0))^2} + \|q\|_{N_\tau^{0,\alpha}(\Omega_0)} \leq c \mathcal{M}$$

with a constant $c$ independent of $\mathcal{M}.$
**Proof.** The pair \((w, q)\) satisfies the equation
\[
S_0(w, q) + \nu^{-1}T_0(w, q) = (\Phi, \psi),
\]  
(1.13)
where
\[
T_0(w, q) = (\langle w, \nabla \rangle w + \langle Y, \nabla \rangle w + \langle w, \nabla \rangle Y, 0),
\]

\[
\Phi = \sum |M_A| \left\{ H^A \Delta \zeta_A(x) + 2\langle \nabla \zeta_A, \nabla \rangle H^A - \nu^{-1}(\rho^{-1}Q^A \nabla \zeta_A + \eta_A(H^A, \nabla \zeta_A) + |M_A|(\zeta_A - 1)\langle H^A, \nabla \rangle H^A) \right\},
\]

\[
\psi = - \sum |M_A|\langle H^A, \nabla \zeta_A \rangle.
\]

For any \(S\) and \(T\) one has
\[
\langle S, \nabla \rangle T + \langle T, \nabla \rangle S = -S\text{div}T - T\text{div}S
\]
\[
+ (\text{div}(S_1T + T_1S), \text{div}(S_2T + T_2S)).
\]  
(1.14)
We put here \(S = w\), \(T = Y\) and \(S = w\), \(T = w\). Taking into account the resulting relations and equations
\[
\text{div}(Y + w) = 0, \quad \text{div}Y = \sum |M_A|\langle H^A, \nabla \zeta_A \rangle,
\]
we write (1.13) in the form
\[
S_0(w, q) + N_0(w, q) = (\Psi, \psi).
\]
Here
\[
\Psi = \Phi - \nu^{-1} \sum |M_A|\eta_A H^A \langle H^A, \nabla \zeta_A \rangle,
\]
and \(N_0 : D^\alpha_\tau \to R^\alpha_\tau\) is the operator defined by
\[
N_0(w, q) = \left( \text{div}(N^{(1)}(Y; (w, q))), \text{div}(N^{(2)}(Y; (w, q))) \right),
\]
where
\[
N^{(i)}((w, q)) = \nu^{-1}(Y^A_i w_j + Y^A_j w_i + w_i w_j).
\]
By using (1.14) and definition (1.9) of $H^4$ we represent $(\Psi, \psi)$ in the form

$$(\Psi, \psi) = \left( \text{div} \mathbf{X}^{(1)}(x), \text{div} \mathbf{X}^{(2)}(x), -\text{div} \Theta(x) \right) \Bigg|_{x \in \mathcal{Z}},$$

where $\mathcal{Z} = \bigcup_{\{A\}} \text{supp} \nabla \zeta_A$ and $\mathbf{X}^{(k)}$, $k = 1, 2$, are given by

$$\mathbf{X}^{(k)} = \sum |M_A| (\nabla \zeta_A H_k^A - \nu^{-1} (\rho^{-1} \zeta_A Q^A e^{(k)} - \zeta_A H_k^A H^A))$$

with

$$e^{(1)} = (1, 0), \quad e^{(2)} = (0, 1).$$

In accordance with the inequalities

$$\|\mathbf{X}^{(k)}\|_{N_r^0(\Omega_0)} \leq c M, \quad \|\text{div} \Theta\|_{N_r^{0,\alpha}(\Omega_0)} \leq c M$$

the estimates hold

$$\|\Psi\|_{N_r^{-1,\alpha}(\Omega_0)} + \|\psi\|_{N_r^{0,\alpha}(\Omega_0)} \leq c M.$$

Let $B_\delta$ be a ball in the space $D_r^\alpha$ of sufficiently small radius $\delta$ centered at $S_0^{-1}((\Psi, \psi))$. If $(w^{(j)}, q^{(j)}) \in B_\delta$, $j = 1, 2$, for sufficiently small $\nu^{-1}|M_A|$ and $\delta$, we obtain from the standard inequality

$$\|v^{-1}u\|_{\mathcal{S}_r^{0,\alpha}(\Omega_0)} \leq c \|u\|_{N_r^{1,\alpha}(\Omega_0)}$$

that

$$\|N_i^{(j)}((w^{(1)}, q^{(1)}) - N_i^{(j)}((w^{(2)}, q^{(2)}))\|_{(N_r^{0,\alpha}(\Omega_0))^2}$$

$$\leq m \|w^{(1)} - w^{(2)}\|_{(\mathcal{S}_r^{1,\alpha}(\Omega_0))^2}$$

for $m < 1$, and

$$\|N_i^{(j)}((w^{(j)}, q^{(j)}))\|_{(N_r^{0,\alpha}(\Omega_0))^2} \leq c \|w^{(j)}\|_{(\mathcal{S}_r^{1,\alpha}(\Omega_0))^2}.$$

Hence, the operator

$$S_0^{-1}(\mathcal{N}_0): D_r^\alpha \rightarrow D_r^\alpha$$

is a contraction mapping. Therefore, there exists one and only one solution $(w, q) \in B_\delta$ of equation (1.13) subject to (1.12).
**Remark 1.1** The Jeffery-Hamel solution \((\mathbf{H}^A, Q^A)\) is defined up to an arbitrary constant \(c^A\) (see (1.9)). Let \((\mathbf{H}_1^A, Q_1^A)\) and \((\mathbf{H}_2^A, Q_2^A)\) be the pairs defined by (1.9) with different constants \(c_1^A\) and \(c_2^A\). To the pairs \((\mathbf{H}_1^A, Q_1^A), (\mathbf{H}_2^A, Q_2^A)\) there correspond the solutions \((\mathbf{v}_1, p_1), (\mathbf{v}_2, p_2)\) given by (1.11) with the remainders \((\mathbf{w}_1, q_1)\) and \((\mathbf{w}_2, q_2)\) respectively. The pairs \((\mathbf{w}_1, q_1)\) and \((\mathbf{w}_2, q_2)\) can by found by (1.13) with the right-hand sides \((\Phi_1, 0)\) and \((\Phi_2, 0)\), subject to
\[
(\Phi_2, 0) = (\Phi_1, 0) + ((c_1^A - c_2^A)\nabla \zeta_A, 0).
\]
Hence and by (1.13)
\[
(\mathbf{w}_2, q_2) = (\mathbf{w}_1, q_1) + (0, (c_1^A - c_2^A)\zeta_A).
\]
Combining (1.9), (1.15) and (1.11) we have
\[
(\mathbf{v}_2, p_2) = (\mathbf{v}_1, p_1).
\]
Therefore the pressure does not depend on the choice of the constant \(c^A\) in (1.9) and we set \(c^A = 0\) in the sequel.

**Remark 1.2** Let the domain \(\Omega_0\) be prescribed by
\[
\lambda_-(r) - \omega/2 < \theta < \lambda_+(r) + \omega/2
\]
near the point \(A\), where \(\lambda_\pm\) are smooth functions, \(\lambda_\pm(0) = 0\). The difference between the present situation and Theorem 1.1 is that the function \(r^{-1}\mathbf{V}(\theta)\) does not satisfy the zero Dirichlet condition near \(A\) and therefore the principal term in the asymptotics of the solution \((\mathbf{v}, p)\) becomes more complicated.

One can show that the velocity vector and the pressure are represented in the form
\[
r^{-1}\mathbf{V}(\theta, R) + \mathbf{V}^*(\theta, R), \quad r^{-2}\mathbf{J}(\theta, R) + r^{-1}\mathbf{J}^*(\theta, R)
\]
modulo terms with finite energy. Here \(\mathbf{V}^*\) and \(\mathbf{J}^*\) are analytic in \(R\) at \(R = 0\) and
\[
\mathbf{V}^*_\theta(\theta, 0) = Z(\omega) \sum_{\pm} \pm \gamma_\pm(\omega(\theta \pm \omega/2) \sin(\theta \mp \omega/2) \sin(\theta \pm \omega/2)),
\]
\[
\mathbf{V}^*_r(\theta, 0) = -(d\mathbf{V}^*_\theta/d\theta)(\theta, 0),
\]
\[
\mathbf{J}^*(\theta, 0) = Z(\omega) \sum_{\pm} \gamma_\pm(\omega \sin(\theta \pm \omega/2) - \sin \omega \sin(\theta \mp \omega/2)),
\]
where
\[ Z(\omega) = \sin \omega / ((\sin \omega - \omega \cos \omega)(\sin^2 \omega - \omega^2)) \]
and \( \gamma_\pm \) is the curvature of the arc \( \theta = \pm \omega/2 + \lambda_\pm \) at the point \( A \), i.e. \( \gamma_\pm = 2(d\lambda_\pm/dr)(0) \).

In principle, our main result could be generalized to the case of curved angle considered here. However, we shall not dwell upon this extension for the sake of simplicity of presentation.

2 Navier-Stokes system in the model domains

2.1. Navier-Stokes system in an infinite channel. Let \( (z_1, z_2) \) be a Cartesian system and let \( \Sigma_A \) be the strip
\[ \Sigma_A = \{(z_1, z_2) : -b_A^- < z_1 < b_A^+, z_2 \in \mathbb{R}^1 \}. \]
By \( (U^A_M, P^A_M) \) we denote a solution of the Navier-Stokes system satisfying the zero Dirichlet condition on the boundary \( \partial \Sigma_A \) and such that
\[ M = \int_{z_1 \in (-b_A^-, b_A^+)} U(z) \, dz \]
This solution has the form
\[ (U^A_M, P^A_M) = M(U_A, P_A) + (0, \kappa), \] (2.1)
where \( \kappa \) is an arbitrary constant and \( (U_A, P_A) \) is explicitly given by
\[ U_A(z) = -6b_A^{-3}(0, (z_1 - b_A^+)(z_1 + b_A^-)), \] (2.2)
\[ P_A(z) = -12\rho \nu b_A^{-3} z_2 \]
(we remind that \( b_A = b_A^+ + b_A^- \)).

2.2. Navier-Stokes system in \( \Lambda_A \). We introduce a smooth partition of unity \{\( \zeta^+_A, \zeta^-_A, \zeta_0^A \)\} on the domain \( \Lambda_A \) (see Fig.4), where \( \zeta_0^A(y) = \zeta(b_A^{-1} y) \), \( \zeta^+_A(y) = 0 \) for positive \( y_2 \) and \( \zeta^+_A(y) = 0 \) for \( y_2 > b_A \).
Let \( w \) be a function on \( \Lambda_A \) and let
\[ |w|_\alpha = \sup_{y, z \in \Lambda_A} \frac{|w(y) - w(z)|}{|y - z|^\alpha}. \]
By $r(y)$ we denote the distance between $y$ and the nearest angle point on $\partial \Lambda_A$.

We say that a function $u$ on $\Lambda_A$ belongs to the space $K_{\delta,\tau,\beta}^l(\Lambda_A), l = 0, 1,$ and $\alpha \in (0, 1), \delta, \tau, \beta \in \mathbb{R}^1$, if it has the finite norm

$$
\|u\|_{K_{\delta,\tau,\beta}^l(\Lambda_A)} = \left| r^{l+\delta+\alpha+1} \nabla^l (\zeta_A^u) \right|_{\alpha} + \left| r^{l-\tau+\alpha} \nabla^l (\zeta_0^A u) \right|_{\alpha}
$$

$$
+ \left| e^{\beta r} \nabla^l (\zeta_A^u) \right|_{\alpha} + \left| r^{1+\delta} \xi_A^u \right|_{L^\infty(\Lambda_A)}
$$

$$
+ \left| r^{-\tau} \xi_0^A u \right|_{L^\infty(\Lambda_A)} + \left| e^{\beta r} \xi_A^u \right|_{L^\infty(\Lambda_A)}.
$$

The space of distributions $\text{div} \, h + r^{-1} h_0$, where

$$
h \in (K_{\delta,\tau,\beta}^0(\Lambda_A))^2, \quad h_0 \in K_{\delta,\tau,\beta}^0(\Lambda_A),
$$

will be denoted by $K_{\delta,\tau,\beta}^{-1,\alpha}(\Lambda_A)$. Let us consider the Dirichlet problem for the Stokes system

$$
\nu \Delta \mathbf{V} - \rho^{-1} \text{grad} \, P = \mathbf{F} \quad \text{on} \quad \Lambda_A,
$$

$$
\text{div} \, \mathbf{V} = f \quad \text{on} \quad \Lambda_A,
$$

$$
\mathbf{V} \big|_{\partial \Lambda_A} = 0.
$$

(2.3)

We suppose that the velocity $\mathbf{V}$ has the prescribed flux:

$$
M = \int_{y \in \Xi_1(\Lambda_A)} \langle \mathbf{V}, \frac{y}{|y|} \rangle \, ds_y,
$$

(2.4)

which is equivalent to

$$
M = - \int_{y \in \Xi_2(\Lambda_A)} V_2 \, dy.
$$

(2.5)

Let

$$
\mathcal{H}_0(\tau, \theta) = \frac{\mathbf{Q}}{\tau}, \quad \mathcal{Q}_0(\tau, \theta) = \frac{\mathbf{J}(\theta)}{\tau^2},
$$

where $\tau = |y|$ and $(\mathbf{Q}, \mathbf{J})$ is a solution of (1.5)–(1.8) for $R = 0$ and $\sigma = 1$. The following result is essentially known (see, for example,[15]).
Lemma 2.1 i) For $F = 0, f = 0$ and $M = 1$ there exists one and only one solution $(V_0, P_0)$ of problem (2.3)–(2.5) which can be represented in the form

$$(V_0, P_0) = \zeta_+^A(\mathcal{H}_0, Q_0) + \zeta_-^A(U_1^A, P_1^A) + \zeta_0^A(0, C_0) + (W_0, Q_0),$$

where $(W_0, Q_0) \in (K_{\delta, \tau, \beta}^{1, \alpha}(\Lambda_A))^2 \times K_{\delta + 1, \tau - 1, \beta}^{0, \alpha}(\Lambda_A)$ and

$$C_0 = 2 \int_{\Lambda_A} \{ V_0(Q_0 \nabla \zeta_+ + P_1^A \nabla \zeta_+ - \rho(\mathcal{H}_0 \Delta \zeta_+^A + U_1^A \Delta \zeta_-^A + 2(\langle \nabla \zeta_+^A, \nabla \rangle \mathcal{H}_0 + \langle \nabla \zeta_-^A, \nabla \rangle U_1^A)) \}
- (\zeta_+^A Q_0 + \zeta_-^A P_1^A + q_0)(\langle \mathcal{H}_0, \nabla \rangle \zeta_+ + (\langle U_1^A, \nabla \rangle \zeta_-^A) \} \ dx.$$

ii) Let

$$\int_{\Lambda_A} f(x) \ dx = 0.$$

For $(F, f) \in (K_{\delta + 1, \tau - 2, \beta}^{-1, \alpha}(\Lambda_A))^2 \times K_{\delta + 1, \tau - 1, \beta}^{0, \alpha}(\Lambda_A)$ there exists one and only one solution $(V, P)$ of problem (2.3)–(2.5) represented as

$$(V, P) = M \zeta_+^A(\mathcal{H}_0, Q_0) + M \zeta_-^A(U_1^A, P_1^A) + \zeta_0^A(0, C) + (W, Q).$$

Here

$$C = \int_{\Lambda_A} \{ \rho(F, V_0) + f P_0 \} \ dx + M C_0$$

and the pair $(W, Q) \in (K_{\delta + 1, \tau, \beta}^{1, \alpha}(\Lambda_A))^2 \times K_{\delta + 1, \tau - 1, \beta}^{0, \alpha}(\Lambda_A)$ satisfies

$$\|W\|_{(K_{\delta + 1, \tau, \beta}^{1, \alpha}(\Lambda_A))^2} + \|Q\|_{K_{\delta + 1, \tau - 1, \beta}^{0, \alpha}(\Lambda_A)} \leq c \nu^{-1}(\|F\|_{(K_{\delta + 2, \tau - 2, \beta}^{-1, \alpha}(\Lambda_A))^2} + \|f\|_{K_{\delta + 1, \tau - 1, \beta}^{0, \alpha}(\Lambda_A)}),$$

where the constant $c$ depends only on $\rho$ and $\Lambda_A$.

Consider the Dirichlet problem

$$\nu \Delta v - \rho^{-1} \text{grad } p = \langle v, \nabla \rangle v \quad \text{on } \Lambda_A,$$

$$\text{div } v = 0 \quad \text{on } \Lambda_A,$$

$$v \big|_{\partial \Lambda_A} = 0.$$
Suppose that the velocity \( v \) satisfies (2.4) with a given \( M \).

Let
\[
\mathcal{H}_M^A(y) = |y|^{-1} \mathcal{V}_M^A(\theta), \\
\mathcal{Q}_M^A(y) = |y|^{-2} \mathcal{F}_M^A(\theta),
\]
where \((\mathcal{V}_M^A, \mathcal{F}_M^A)\) is the solution of (1.5)–(1.8) with \( \omega_\pm = \omega^A_\pm \), \( \sigma = \text{sign} M \) and \( R = \nu^{-1}|M| \).

By Lemma 2.1 and the contraction mapping principle we arrive at the following assertion

**Lemma 2.2** For sufficiently small positive values \( \alpha, \tau, \delta, \beta, \nu^{-1}|M| \) there exists a unique solution \((v, p)\) of problem (2.6), (2.4), (2.5) represented in the form
\[
(v, p) = (\mathfrak{M}_M, \mathfrak{P}_M) + (w, q) + \zeta^A(0, \mathbb{C}),
\]
where
\[
\mathfrak{M}_M(y) = |M| \zeta^A_+(y) \mathcal{H}_M^A(y) + M \zeta^A_+(y) \mathcal{U}_M^A(y), \\
\mathfrak{P}_M(y) = |M| \zeta^A_+(y) \mathcal{Q}_M^A(y) + M \zeta^A_+(y) \mathcal{P}_M^A(y)
\]
and \((w, q, \mathbb{C}) \in (K^{1,\alpha}_{\delta,\tau,\sigma}(\Lambda_A))^2 \times K^{0,\alpha}_{\delta+1,\tau-1,\beta}(\Lambda_A) \times \mathbb{R}^1 \). Moreover,
\[
\|w\|_{(K^{1,\alpha}_{\delta,\tau,\sigma}(\Lambda_A))^2} + \|q\|_{K^{0,\alpha}_{\delta+1,\tau-1,\beta}(\Lambda_A)} + |\mathbb{C}| \leq c|M|,
\]
where \( c \) is a constant independent of \( M \).

### 2.3. The case of the semistrip

Let \( \Pi_B \) be the semistrip \( \{(t_1, t_2) : -b^-_A < t_1 < b^+_A, t_2 > 0\} \). We shall use the space \( C^l,\alpha(\Pi_B), \ l = 0, 1, \ \alpha \in (0, 1) \) of functions on \( \Pi_B \) with finite norm
\[
\|u\|_{C^l,\alpha(\Pi_B)} = \sup_{t,s \in \Pi_B} |t-s|^{-\alpha} |\nabla^lu(t) - \nabla^lu(s)| + \sup_{t \in \Pi_B} |u(t)|.
\]
By definition, \( u \in C^l,\alpha(\Pi_B) \) if \( \exp(\delta t_2)u \in C^l,\alpha(\Pi_B) \).

Consider the boundary value problem
\[
\nu \Delta V - \rho^{-1} \text{grad} P = 0 \quad \text{on} \quad \Pi_B,
\]
\[
\text{div} V = 0 \quad \text{on} \quad \Pi_B,
\]
\[
V(t_1, 0) = g(t_1), \quad t_1 \in [-b^-_A, b^+_A],
\]
\[
V(\pm b^+_A, t_2) = 0, \quad t_2 \geq 0
\]
where $g \in (C^{1,\alpha}(b_A^{-}, b_A^{+})))^2$ and $g(\pm b_A^{\pm}) = 0$. Suppose that

$$
\int_{t \in \Xi(\Pi_B)} V_2(t) \, dt = M \tag{2.10}
$$

with

$$
M = - \int_{b_A^{+}}^{b_A^{-}} g_2(t) \, dt.
$$

The following result is well-known (see [17], [18]).

**Lemma 2.3** There exists one and only one solution of problem (2.9), (2.10) represented in the form

$$(V, P) = (M(U_M^A, P_M^A) + (W, Q),
$$

where $(W, Q) \in (C^{1,\alpha}_{\delta}(\Pi_B))^2 \times C^0_{\delta}(\Pi_B)$ and the estimate

$$
\|W\|_{(C^{1,\alpha}_{\delta}(\Pi_B))^2} + \|Q\|_{C^0_{\delta}(\Pi_B)} \leq c\|g\|_{(C^{1,\alpha}(b_A^{-}, b_A^{+}))^2}
$$

holds with a constant $c$ depending only on $\rho$ and the domain $\Pi_B$.

By this Lemma and contraction mapping principle we obtain the following solvability result for the Navier-Stokes system

$$
\nu \Delta v - \rho^{-1} \text{grad} p = \langle v, \nabla \rangle v \quad \text{on} \quad \Pi_B,
$$

$$
\text{div} v = 0 \quad \text{on} \quad \Pi_B, \tag{2.11}
$$

$$
v(t_1, 0) = g(t_1), \quad t_1 \in [-b_A^{-}, b_A^{+}],
$$

$$
v(\pm b_A^{\pm}, t_2) = 0, \quad t_2 \geq 0.
$$

**Lemma 2.4** If $\nu^{-1}M$ is sufficiently small, there exists a single solution $(v, p)$ of problem (2.9), (2.10) represented in the form

$$
v(t) = U_M^A(t) + w(t),
$$

$$
p(t) = P_M^A(t) + q(t),
$$

where $(w, q) \in (C^{1,\alpha}_{\delta}(\Pi_B))^2 \times C^0_{\delta}(\Pi_B)$, and the estimate

$$
\|w\|_{(C^{1,\alpha}_{\delta}(\Pi_B))^2} + \|q\|_{C^0_{\delta}(\Pi_B)} \leq c\|g\|_{(C^{1,\alpha}(b_A^{-}, b_A^{+}))^2} \tag{2.12}
$$

is valid.
3 Stokes system in $\Omega_\varepsilon$

Let $\Omega_\varepsilon$ be the domain depicted in Fig.1. In order to determine $C_\varepsilon^A$ we introduce a local system of Cartesian coordinates $(y_1^A, y_2^A)$ with origin $A$ and with the axis $Ay_2^A$ directed into $\Omega_0$. The thin channel $C_\varepsilon^A$ will be defined as

$$C_\varepsilon^A = \{(y_1^A, y_2^A) : -\varepsilon b_A^- < y_1^A < \varepsilon b_A^+, -L_A^- < y_2^A < L_A^+\}.$$  

The values $b_A^\pm, L_A^\pm$ are subject to the inequalities

$$b_A^\pm > b_0^A > 0, \quad L_A^\pm > L_0 > 0,$$

where $b_0, L_0$ are constants independent of $\varepsilon$. The interval $B_\varepsilon^A = \{(y_1^A, y_2^A) : -\varepsilon b_A^- < y_1^A < \varepsilon b_A^+, y_2 = -L_A^-\}$ will be called the end of the channel $C_\varepsilon^A$. This interval $B_\varepsilon^A$ is orthogonal to the walls and placed at a finite distance $L_A = L_A^\pm$ from $A$. By $B \in B_\varepsilon^A$ we denote the point with coordinates $(y_1^A, y_2^A) = (0, -L_A)$.

We introduce the norm in the Sobolev space $H^1(\Omega_\varepsilon)$:

$$
\|u\|_{H^1(\Omega_\varepsilon)} = \left( \int_{\Omega_\varepsilon} |\nabla u|^2 \, dx + \int_{\Omega_\varepsilon} r_\varepsilon^{-2} |u|^2 \, dx \right)^{1/2},
$$

where

$$r_\varepsilon(x) = \begin{cases} 
  r & \text{when } x \in \Omega_0 \cap (\mathbb{D}_d(x - A) \setminus \mathbb{D}_{\varepsilon_0}(x - A)) \\
  \varepsilon & \text{when } x \in (\Omega_\varepsilon \cap \mathbb{D}_{\varepsilon_0}(x - A)) \cup C_\varepsilon^A \\
  1 & \text{when } x \in \Omega_0 \setminus \cup_{\{A\}} \mathbb{D}_d(x - A)
\end{cases}$$

and

$$d = \min_{\{A\}} d_A, \quad a = 2\max_{\{A\}} \{b_0^A / \cos \omega_0^A\}.$$

By $\hat{H}^1(\Omega_\varepsilon)$ we denote the completion of $C^\infty(\Omega_\varepsilon)$ with respect to this norm and we set

$$\|\varphi\|_{(\hat{H}^1(\Omega_\varepsilon))^*} = \sup \{\varphi(u) : \|u\|_{\hat{H}^1(\Omega_\varepsilon)} = 1\}.$$  

Before studying the structure of the solutions to the Navier-Stokes problem (0.1)–(0.5) consider an auxiliary linear Stokes system in $\Omega_\varepsilon$.

Lemma 3.1 Let

$$S : (\hat{H}^1(\Omega_\varepsilon))^2 \times L^2(\Omega_\varepsilon) \rightarrow ((\hat{H}^1(\Omega_\varepsilon))^2)^* \times L^2(\Omega_\varepsilon)$$

(3.1)
be the operator, which transforms \((U_\varepsilon, \pi_\varepsilon)\) to \((-\Delta U_\varepsilon + \rho^{-1} \nu^{-1} \nabla \pi_\varepsilon, \text{div} U_\varepsilon)\).

Suppose that \((F_\varepsilon, f_\varepsilon) \in ((\tilde{H}^1(\Omega_\varepsilon))^2)^* \times L^2(\Omega_\varepsilon)\) and that \(f_\varepsilon\) is subject to

\[\mathcal{F}_\varepsilon = 0.\]  

(3.2)

Then there exists a single solution \((U_\varepsilon, \pi_\varepsilon) \in (\tilde{H}^1(\Omega_\varepsilon))^2 \times L^2(\Omega_\varepsilon)\) of the problem

\[\mathcal{S}(U_\varepsilon, \pi_\varepsilon) = (F_\varepsilon, f_\varepsilon), \quad \pi_\varepsilon = 0,\]  

(3.3)

and the estimate holds

\[\|\pi_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|U_\varepsilon\|_{(\tilde{H}^1(\Omega_\varepsilon))^2} \leq c(\|F_\varepsilon\|_{((\tilde{H}^1(\Omega_\varepsilon))^2)^*} + \|f_\varepsilon\|_{L^2(\Omega_\varepsilon)}),\]  

(3.4)

where \(c\) does not depend on \(\varepsilon\).

**Proof.** The unique solvability of (3.3) is well-known [20]. We only need to check estimate (3.4). By using an argument from [19] we shall construct a vector function \(Z_\varepsilon \in \tilde{H}^1(\Omega_\varepsilon)\) satisfying the equation

\[\text{div} Z_\varepsilon = f_\varepsilon\]  

(3.5)

and the inequality

\[\|Z_\varepsilon\|_{(\tilde{H}^1(\Omega_\varepsilon))^2} \leq c\|f_\varepsilon\|_{L^2(\Omega_\varepsilon)},\]  

(3.6)

where \(c\) does not depend on \(\varepsilon\) and \(f_\varepsilon\). We consider \(\Omega_0\) as a sum of domains \(\Omega(l)\) star-shaped with respect to a ball, \(l = 1, \ldots, L\). The channels \(c_\varepsilon^{(j)}\) are represented as unions of the squares \(T_\varepsilon^{(k)}\), \(k = 1, 2, \ldots, K\), with the side length \(\varepsilon\). So we have

\[\Omega_\varepsilon = \bigcup_{l=1}^L \Omega(l) \cup \bigcup_{k=1}^K T_\varepsilon^{(k)}.\]

By (3.2) \(f_\varepsilon\) can be written as

\[f_\varepsilon(x) = \sum_{l=1}^L F(l)(x) + \sum_{k=1}^K f_\varepsilon^{(k)}(x),\]

where \(\text{supp} F(l) \subset \Omega(l)\), \(\text{supp} f_\varepsilon^{(k)} \subset T_\varepsilon^{(k)}\) and

\[\int_{\Omega(l)} F(l)(x)dx = 0, \quad \int_{T^{(k)}} f_\varepsilon^{(k)}(x)dx = 0\]  

(3.7)
(see [19]). According to (3.7) there exist vector-functions $\mathbf{Z}^{(l)} \in (\tilde{H}^1(\Omega^{(l)}))^2$, $\mathbf{z}_\varepsilon^{(k)} \in (\tilde{H}^1(\Omega_{\varepsilon}^{(k)}))^2$ satisfying the equations

$$\text{div} \mathbf{Z}^{(l)} = F^{(l)}, \quad \text{div} \mathbf{z}_\varepsilon^{(k)} = f_\varepsilon^{(k)},$$

and the inequalities

$$\|\mathbf{Z}^{(l)}\|_{(\tilde{H}^1(\Omega^{(l)}))^2} \leq c\|F^{(l)}\|_{L^2(\Omega^{(l)})}, \quad \|\nabla \mathbf{z}_\varepsilon^{(k)}\|_{L^2(\Omega_{\varepsilon}^{(k)})} \leq c\|f_\varepsilon^{(k)}\|_{L^2(\Omega_{\varepsilon}^{(k)})}.$$  

([19], Lemma 1). We extend $\mathbf{Z}^{(l)}, \mathbf{z}_\varepsilon^{(k)}$ by zero to $\Omega_{\varepsilon}$. Then, the vector function

$$\mathbf{Z}_{\varepsilon} = \sum_{l=1}^L \mathbf{Z}^{(l)} + \sum_{k=1}^K \mathbf{z}_\varepsilon^{(k)}$$

satisfies both (3.5) and (3.6).

Let $(\mathbf{U}_\varepsilon, \pi_\varepsilon) \in (\tilde{H}^1(\Omega_{\varepsilon}))^2 \times L^2(\Omega_{\varepsilon})$ be a solution of (3.3). Then $(\mathbf{U}_\varepsilon + \mathbf{Z}_{\varepsilon}, \pi_\varepsilon)$ is a solution of

$$S(\mathbf{U}_\varepsilon + \mathbf{Z}_{\varepsilon}, \pi_\varepsilon) = (F_{\varepsilon} + \Delta \mathbf{Z}_\varepsilon, 0).$$

By the standard energy estimate

$$\|\mathbf{U}_\varepsilon\|_{(\tilde{H}^1(\Omega_{\varepsilon}))^2} \leq c \|F_{\varepsilon} + \Delta \mathbf{Z}_\varepsilon\|_{((\tilde{H}^1(\Omega_{\varepsilon}))^2)^*},$$

and by (3.5), it follows

$$\|\mathbf{U}_\varepsilon\|_{(\tilde{H}^1(\Omega_{\varepsilon}))^2} \leq c\|F_{\varepsilon}\|_{((\tilde{H}^1(\Omega_{\varepsilon}))^2)^*} + \|f_\varepsilon\|_{L^2(\Omega_{\varepsilon})}. \quad (3.8)$$

In order to estimate the pressure $\pi_\varepsilon$, we introduce a function $\mathbf{I}_\varepsilon \in (\tilde{H}^1(\Omega_{\varepsilon}))^2$ satisfying

$$\text{div} \mathbf{I}_\varepsilon = \pi_\varepsilon, \quad (3.9)$$

$$\|\mathbf{I}_\varepsilon\|_{(\tilde{H}^1(\Omega_{\varepsilon}))^2} \leq c\|\pi_\varepsilon\|_{L^2(\Omega_{\varepsilon})}. \quad (3.10)$$

By (3.9), we have

$$\|\pi_\varepsilon\|_{L^2(\Omega_{\varepsilon})}^2 = -\int_{\Omega_{\varepsilon}} \langle \nabla \pi_\varepsilon, \mathbf{I}_\varepsilon \rangle \, dx \leq c\|\nabla \pi_\varepsilon\|_{((\tilde{H}^1(\Omega_{\varepsilon}))^2)^*} \|\mathbf{I}_\varepsilon\|_{(\tilde{H}^1(\Omega_{\varepsilon}))^2}. $$

Hence and from (3.10) we obtain

$$\|\pi_\varepsilon\|_{L^2(\Omega_{\varepsilon})} \leq c\|\nabla \pi_\varepsilon\|_{((\tilde{H}^1(\Omega_{\varepsilon}))^2)^*}. $$

Now (3.4) follows from (3.3) and (3.8).
4 The flow in $\Omega_\varepsilon$. Calculation of the principal term $(V_\varepsilon, P_\varepsilon)$

As already mentioned in Introduction, the principal term $(V_\varepsilon, P_\varepsilon)$ of representation (0.22) for the solution $(v_\varepsilon, p_\varepsilon)$ to problem (0.1)-(0.5) is defined by (0.19), (0.20). We give now more details for calculation of the term in (0.19), (0.20) and study their asymptotic behavior.

We define $X_\varepsilon$, $\eta_\varepsilon^A$, $\mu_\varepsilon^B$ and $\xi_\varepsilon^B$ by formulas

$$
\eta_\varepsilon^A(x) = \begin{cases} 
\zeta(\varepsilon^{-1/2}(x - A)) & \text{for } x \in \Omega_\varepsilon \setminus \mathcal{C}_\varepsilon^A \\
1 & \text{for } x \in \mathcal{C}_\varepsilon^A,
\end{cases}
$$

$$
\chi_\varepsilon^A(x) = 1 - \zeta(\varepsilon^{-1/2}(x - A)), \quad \xi_\varepsilon^B(x) = \zeta(\varepsilon^{-1/2}(x - B)),
$$

$$
\mu_\varepsilon^B(x) = 1 - \zeta(\varepsilon^{-1/2}(x - B)), \quad X_\varepsilon(x) = \prod \chi_\varepsilon^A(x),
$$

where $\mathcal{C}_\varepsilon^A$ is the channel, which starts at the point $A$ and the product is taken over all points of the set $\{A\}$. By definition of the cut-off functions we have

$$
\mu_\varepsilon^B(x) + \xi_\varepsilon^B(x) = 1, \quad \eta_\varepsilon^A(x) = 1, \quad \chi_\varepsilon^A(x) = 0 \quad \text{for } x \in \mathcal{C}_\varepsilon^A
$$

and

$$
X_\varepsilon(x) + \sum_{A \in \{A\}} \eta_\varepsilon^A(x) = 1, \quad \mu_\varepsilon^B(x) = 1, \quad \xi_\varepsilon^B(x) = 0, \quad \text{for } x \in \Omega_\varepsilon.
$$

Hence, the collection of cut-off functions $\{X_\varepsilon, \eta_\varepsilon^A, \mu_\varepsilon^B, \xi_\varepsilon^B\}$ forms a partition of unity on $\Omega_\varepsilon$.

The pair $(v_0, p_0)$ is determined from problem (1.1)-(1.4), with the prescribed fluxes

$$
M_\varepsilon = Y_A
$$

at the points $A \in \{A\}$. According to Theorem 1.1, one has

$$
(v_0, p_0) = (Y_0, \Theta_0) + (w_0, q_0 + K_\varepsilon), \quad (4.1)
$$

where $(w_0, q_0) \in (\hat{N}_{1,\kappa}^\alpha(\Omega_0))^2 \times N_{\kappa_\perp}^\alpha(\Omega_0)$, the pair $(Y_0, \Theta_0)$ is defined by (1.9) with $M_\varepsilon = Y_A$ and $K_\varepsilon$ is a constant.

The term $(v^A, p^A)$ is a solution of problem (2.9), (2.10), there $M = Y_A$ in the domain $\Lambda_\varepsilon$ (cf. Fig.4). By Lemma 2.2 $(v^A, p^A)$ can be represented as

$$
(v^A, p^A) = (w^A, \Phi^A) + (w^A, q^A + k_\varepsilon^A) + \zeta(0, C_\varepsilon^A), \quad (4.2)
$$
where \((w^A, q^A), C_0^A\) satisfy (2.8) with \(M = \Upsilon_A\), \(k_\varepsilon^A\) is a constant and \((\tilde{\mathcal{W}}^A, \tilde{\mathcal{P}}^A) = (\tilde{\mathcal{W}}_M^A, \tilde{\mathcal{P}}_M^A)\), where \(M = \Upsilon_A\).

The pair \((\mathbf{v}^B, p^B)\) is sought from problem (2.11) in the domain \(\Pi_B\) with \(\mathbf{g} = \varphi^A\). According to Lemma 2.4 the solution \((\mathbf{v}^B, p^B)\) has the form

\[
(\mathbf{v}^B, p^B) = \Upsilon_A(\mathbf{U}^A, \mathcal{P}^A) + (w^B, q^B + k_\varepsilon^B),
\]

(4.3)

where \((w^B, q^B)\) is subject to (2.12) with \(\mathbf{g} = \varphi^A\), \((\mathbf{U}^A, \mathcal{P}^A) = (\mathbf{U}_M^A, \mathcal{P}_M^A)\) with \(M = \Upsilon_A\) and \(k_\varepsilon^B\) is a constant.

In order to obtain representation (0.22) of the solution \(\mathbf{v}_\varepsilon, p_\varepsilon\) of problem (0.1)–(0.5) satisfying estimate (0.23) we find the constants \(K_\varepsilon, k_\varepsilon^A, k_\varepsilon^B\) from the condition

\[
\overline{P}_\varepsilon = O(\varepsilon^D),
\]

(4.4)

where \(D\) is a positive number. By (4.1)–(4.3) one has

\[
\int_{\Omega_\varepsilon} P_\varepsilon(x) \, dx = \sum \{I_1^A + I_2^A + I_3^A + I^B\} + I_0 + J,
\]

(4.5)

where

\[
I_1^A = \int_{\Omega_\varepsilon} \zeta_A(x) \zeta_+^A(\varepsilon^{-1}(x - A)) \mathcal{Q}^A(x) \, dx,
I_0 = \int_{\Omega_\varepsilon} q_0(x) X_\varepsilon(x) \, dx,
I_2^A = \frac{1}{\varepsilon^2} \int_{C_\varepsilon^A} \zeta_A^A(\varepsilon^{-1}(x - A))(\mathcal{P}^A(x) + C^A) \, dx,
I_3^A = \frac{1}{\varepsilon^2} \int_{C_\varepsilon^A} \eta_\varepsilon^A(x) \mu_\varepsilon^B(x) q^A(x) \, dx,
I^B = \frac{1}{\varepsilon^2} \int_{C_\varepsilon^A} \xi_\varepsilon^B(x) q^B(x) \, dx,
J = \int_{\Omega_\varepsilon} \{K_\varepsilon X_\varepsilon(x) + \sum (\eta_\varepsilon^A(x) \mu_\varepsilon^B(x) k_\varepsilon^A + \xi_\varepsilon^A(x) k_\varepsilon^B)\} \, dx
\]

and \(\mathcal{Q}^A = \mathcal{Q}_M^A\) with \(M = \Upsilon_A\). We shall calculate the integral \(I_1^A\) and \(I_2^A\).
We have

\[ I^A_1 = \int_{\epsilon_b A}^{a} \int_{-\omega_A^{-}}^{\omega_A^{+}} \xi_A(x)\xi_A^A(\epsilon^{-1}(x-A))J^A(\theta)r^{-1}d\theta dr \]

\[ = \int_{a/2-\omega_A^{-}}^{a} \int_{-\omega_A^{-}}^{\omega_A^{+}} \xi_A(x)J^A(\theta)r^{-1}d\theta dr + \int_{2b_A^{-}\omega_A^{-}}^{2b_A} \int_{-\omega_A^{-}}^{\omega_A^{+}} J^A(\theta)r^{-1}d\theta dr \tag{4.6} \]

\[ + \int_{b_A^{-}\omega_A^{-}}^{2b_A} \int_{\omega_A^{-}}^{\omega_A^{+}} \rho^{-1}\xi_A^A(y(\rho,\theta))d\theta d\rho = \log 1/\epsilon \int_{-\omega_A^{-}}^{\omega_A^{+}} J^A(\theta) d\theta + c^A_1, \]

where \( J^A = J^A_M \) with \( M = \Upsilon_A \) and

\[ c^A_1 = \int_{a/2-\omega_A^{-}}^{a} \int_{-\omega_A^{-}}^{\omega_A^{+}} \log r \frac{\partial J^A}{\partial r}(r,\theta)J^A(\theta) d\theta dr \]

\[ + \int_{b_A^{-}\omega_A^{-}}^{2b_A} \int_{\omega_A^{-}}^{\omega_A^{+}} \log \rho \frac{\partial J^A}{\partial \rho}(\rho,\theta)J^A(\theta) d\theta d\rho. \]

By (2.2) with \( b = b_A \), the integral \( I^A_2 \) is

\[ I^A_2 = \epsilon^{-2}6\rho\nu L_A b_A^{-2} + \epsilon^{-1}C_A L_A b_A - \epsilon^{-2}C_A^2 \int_{c^A_{\epsilon}}^{C_A} \xi^B(x) dx + c^A_2, \tag{4.7} \]

where \( L_A \) is the distance between \( A \) and \( B \) and

\[ c^A_2 = \int_{-2b_A}^{0} \int_{b_A}^{b_A} (1 - \xi_A^{-}(y))(P^A(y) + C_A^A) dy_1 dy_2. \]
We pass to the estimates of $I_3^A$, $I^B$ and $I_0$. We begin with the equality

$$I_3^A - \int_{\Lambda_A} q^A(y) \, dy = \frac{1}{\varepsilon^2} \int_{\varepsilon^{-1/2}}^{\varepsilon^{-1/2} \varepsilon_{b_{\Lambda}}^+} \int_{-\varepsilon_{b_{\Lambda}}^-}^{\varepsilon_{b_{\Lambda}}^+} (1 - \eta^A_{\varepsilon}(x)) q^A(\varepsilon^{-1}(x - A)) r \, dr \, d\theta,$$

$$+ \frac{1}{\varepsilon^2} \int_{-\infty}^{-\varepsilon b_{\Lambda}} \int_{\varepsilon^{-1/2}}^{\varepsilon^{-1/2} \varepsilon_{b_{\Lambda}}^+} (1 - \mu^B_{\varepsilon}(x)) q^A(\varepsilon^{-1}(x - A)) \, dx_1 \, dx_2. \tag{4.8}$$

Since $q^A \in K_{\delta+1, \tau-1, \beta}(\Lambda_A)$, we have

$$|q^A(\varepsilon^{-1}(x - A))| \leq c\varepsilon^{\delta+2} r^{-\delta-2} \text{ for } x \in \text{supp } (1 - \eta^A_{\varepsilon}),$$

$$|q^A(\varepsilon^{-1}(x - A))| \leq c\varepsilon^{-\beta/\varepsilon} \text{ for } x \in \text{supp } (1 - \mu^B_{\varepsilon}). \tag{4.9}$$

Hence by (4.8), (4.9) we obtain

$$I_3^A = \int_{\Lambda_A} q^A(y) \, dy + O(\varepsilon^{\delta/2}). \tag{4.10}$$

Similarly, using the equality

$$I^B - \int_{\Pi_B} q^B(t) \, dt = \frac{1}{\varepsilon^2} \int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2} \varepsilon_{b_{\Lambda}}^+} \int_{\varepsilon_{b_{\Lambda}}^-}^{\varepsilon_{b_{\Lambda}}^+} (1 - \xi^B_{\varepsilon}(x)) q^B(\varepsilon^{-1}(x - B)) \, dx_1 \, dx_2$$

and the inclusion $q^B \in C^{0, \alpha}_{\delta}(\Pi_B)$, we find

$$I^B = \int_{\Pi_B} q^B(t) \, dt + O(\varepsilon^{\delta/2}). \tag{4.11}$$

Since $q_0 \in N_{\tau, \lambda, 1, |\tau - 1 - \alpha| < 1}$, it follows that the equality

$$\int_{\Omega_{\varepsilon}} X_{\varepsilon}(x) q_0(x) \, dx = \int_{\Omega_0} q_0(x) \, dx + \sum_{\omega^+_{\Lambda}} \int_{-\omega^+_{\Lambda}}^{2\varepsilon^{-1/2}} (1 - \chi^A_{\varepsilon}(x)) q_0(x) r \, dr \, d\theta$$
implies

\[ I_0 = O(\varepsilon). \]  

(4.12)

Thus, by (4.6), (4.7), (4.10)–(4.12) we arrive at the formula

\[
\int_{\Omega_\varepsilon} P_\varepsilon(x) \, dx = J + \sum \left\{ \varepsilon^{-2} 6 \rho \nu L_A^2 b_A^{-2} - \varepsilon^{-2} C^A \int_{C^A_\varepsilon} \xi^B(x) \, dx \right. \\
+ \varepsilon^{-1} C^A L_A b_A \log 1/\varepsilon \int_{-\omega^-_A}^{\omega^+_A} J^A(\theta) \, d\theta \\
+ c_1^A + c_2^A + \int_{\Lambda_A} q^A(y) \, dy + \int_{\Pi_B} q^B(t) \, dt \left\} + O(\varepsilon). \right.
\]

(4.13)

In order to equate \( p_0 \) to \( \varepsilon^{-2} p^A \) as well as \( \varepsilon^{-2} p^A \) to \( \varepsilon^{-2} p^B \) in the domains \( \text{supp} \nabla \eta^A_\varepsilon \) and \( \text{supp} \nabla \mu^B_\varepsilon \) respectively, we put

\[ k^A_\varepsilon = \varepsilon^2 K_\varepsilon, \quad k^B_\varepsilon = k^A_\varepsilon + C^A. \]  

(4.14)

Let us calculate the integral \( J \). Taking into consideration (4.14) we have

\[ J = K_\varepsilon |\Omega_\varepsilon| + \varepsilon^{-2} \sum \left\{ \int_{C^A_\varepsilon} \xi^B(x) \, dx \right. \]

(4.15)

By direct calculation we obtain

\[ |\Omega_\varepsilon| = |\Omega_0| + \varepsilon \sum b_A L_A + \varepsilon^2 \frac{1}{2} \sum ((b^+_A)^2 \text{ctg} \omega^+_A + (b^-_A)^2 \text{ctg} \omega^-_A). \]  

(4.16)

Let us substitute (4.15), (4.16) into (4.13). Condition (4.4) implies

\[
K_\varepsilon \{ |\Omega_0| + \varepsilon \sum b_A L_A + \varepsilon^2 \frac{1}{2} \sum ((b^+_A)^2 \text{ctg} \omega^+_A + (b^-_A)^2 \text{ctg} \omega^-_A) \}
\]

\[
= \varepsilon^{-2} 6 \rho \nu \sum L_A^2 b_A^{-2} + \varepsilon^{-1} \sum C^A L_A b_A + \log 1/\varepsilon \sum_{-\omega^-_A}^{\omega^+_A} J^A(\theta) \, d\theta \\
+ \sum \{ c_1^A + c_2^A + \int_{\Lambda_A} q^A(y) \, dy + \int_{\Pi_B} q^B(t) \, dt \}.
\]

(4.17)
Hence, we look for $K_\varepsilon$ in the form
\[
K_\varepsilon = K^{(2)} \varepsilon^{-2} + K^{(1)} \varepsilon^{-1} + K^{(\log)} \log 1/\varepsilon + K^{(0)}.
\tag{4.18}
\]
After substituting (4.18) into (4.17) we have
\[
K^{(2)} = -|\Omega_0|^{-1} 6 \rho \nu \sum (L_A/b_A)^2,
\]
\[
K^{(1)} = -|\Omega_0|^{-1} \sum b_A L_A (C^A + K^{(2)}),
\]
\[
K^{(\log)} = -|\Omega_0|^{-1} \sum \int \mathcal{F}^A(\theta) d\theta,
\tag{4.19}
\]
\[
K^{(0)} = -|\Omega_0|^{-1} \sum \left\{ c^A_1 + c^A_2 + \int q^A(y) dy + \int q^B(t) dt \right. \\
+ K^{(1)} b_A L_A + K^{(2)}/2((b^+_A)^2 \cot \omega_A^+ + (b^-_A)^2 \cot \omega_A^-) \}
\]
Thus, the constants $K_\varepsilon, k^A_\varepsilon, k^B_\varepsilon$ are defined by (4.19), (4.14).

5 The boundary value problem for the remainder $(w_\varepsilon, p_\varepsilon)$

In the previous section we were concerned with the principal term $(V_\varepsilon, P_\varepsilon)$ in the asymptotic representation (0.22) for the solution $(v_\varepsilon, p_\varepsilon)$ of problem (0.1)-(0.5). To justify representation (0.22), consider the problem for the remainder $(w_\varepsilon, q_\varepsilon)$. Let
\[
T : (\bar{H}^1(\Omega_\varepsilon))^2 \times L^2(\Omega_\varepsilon) \to ((H^1(\Omega_\varepsilon))^2)^* \times L^2(\Omega_\varepsilon)
\]
be the operator defined by
\[
T(w, q) = (\langle w, \nabla \rangle w + \langle V_\varepsilon, \nabla \rangle w + \langle w, \nabla \rangle V_\varepsilon, 0).
\]
The pair $(w_\varepsilon, q_\varepsilon)$ satisfies the equation
\[
S(w_\varepsilon, q_\varepsilon) + \nu^{-1} T(w_\varepsilon, q_\varepsilon) = (F_\varepsilon, h_\varepsilon),
\tag{5.1}
\]
where $S$ is the operator of the Stokes system in $\Omega_\varepsilon$ (cf. Section 3) and
\[
F_\varepsilon = -S(V_\varepsilon, P_\varepsilon) - \nu^{-1} \langle V_\varepsilon, \nabla \rangle V_\varepsilon, \quad h_\varepsilon = -\text{div} V_\varepsilon.
Using (1.14) with \( S = w_\varepsilon, T = V_\varepsilon \) and \( S = T = w_\varepsilon \) as well as the equality \( \text{div}(V_\varepsilon + w_\varepsilon) = 0 \), we write (5.1) in the form
\[
S(w_\varepsilon, q_\varepsilon) + \mathcal{N}(w_\varepsilon, q_\varepsilon) = (G_\varepsilon, h_\varepsilon).
\]
(5.2)
Here
\[
\mathcal{N} = (\text{div}\mathcal{N}^{(1)}, \text{div}\mathcal{N}^{(2)}, 0), \quad G_\varepsilon = \nu^{-1}(\text{div}\mathcal{G}_\varepsilon^{(1)}, \text{div}\mathcal{G}_\varepsilon^{(2)}),
\]
with
\[
\mathcal{N}^{(k)} = \nu^{-1}(w_{\varepsilon k}w_\varepsilon + V_{\varepsilon k}w_\varepsilon + w_{\varepsilon k}V_\varepsilon),
\]
\[
\mathcal{G}_\varepsilon^{(k)} = \nabla V_{\varepsilon k} - p_\varepsilon e^{(k)} - V_{\varepsilon k}V_\varepsilon,
\]
where \( k = 1, 2 \).

To estimate the right-hand side \((G_\varepsilon, h_\varepsilon)\) of (5.2) we represent \( \Omega_\varepsilon \) in the form
\[
\Omega_\varepsilon = \cup_{\{A\}} (\Gamma_\varepsilon^A \cup \mathcal{G}_\varepsilon^A) \cup \cup_{\{B\}} (\Gamma_\varepsilon^B \cup \mathcal{G}_\varepsilon^B),
\]
where
\[
\Gamma_\varepsilon^A = \{ x \in \Omega_\varepsilon : x \in \Omega_0 \cap (D_{2\varepsilon^{-1/2}}(x - A) \setminus \mathcal{D}_{\varepsilon^{-1/2}}(x - A)) \},
\]
\[
\Gamma_\varepsilon^B = \{ x \in \Omega_\varepsilon : x \in \mathcal{C}_\varepsilon^A \cap \mathcal{D}_{2\varepsilon^{-1/2}}(x - B) \},
\]
\[
\mathcal{G}_\varepsilon^0 = \Omega_0 \setminus \cup_{\{A\}} \mathcal{D}_{2\varepsilon^{-1/2}}(x - A), \quad \mathcal{G}_\varepsilon^B = \Omega_\varepsilon \cap \cup_{\{B\}} \mathcal{D}_{\varepsilon^{-1/2}}(x - B),
\]
\[
\mathcal{G}_\varepsilon^A = \cup_{\{A\}} (\Omega_0 \cap \mathcal{D}_{\varepsilon^{-1/2}}(x - A)) \cup (\cup_{\{A\}} \mathcal{C}_\varepsilon^A \setminus \cup_{\{B\}} \mathcal{D}_{2\varepsilon^{-1/2}}(x - B))
\]
and \{\{B\}\} is the union the points \( B \) with coordinates \((y_1^A, y_2^A) = (0, -L_A)\) which is extended over all channels \( C^A \).

According to (0.19), (0.20) we have
\[
(V_\varepsilon, P_\varepsilon) \equiv \begin{cases} 
(v_0, p_0) & \text{on } x \in \mathcal{G}_\varepsilon^0 \\
(\varepsilon^{-1}v_1^A, \varepsilon^{-2}p_1^A) & \text{on } x \in \mathcal{G}_\varepsilon^A \\
(\varepsilon^{-1}v_2^B, \varepsilon^{-2}p_2^B) & \text{on } x \in \mathcal{G}_\varepsilon^B.
\end{cases}
\]

Hence, by definition of \((v_0, p_0), (v_1^A, p_1^A)\) and \((v_2^B, p_2^B)\) we obtain
\[
(G_\varepsilon, h_\varepsilon) = 0 \text{ on } \mathcal{G}_\varepsilon^0 \cup \mathcal{G}_\varepsilon^A \cup \mathcal{G}_\varepsilon^B.
\]
(5.4)

To simplify the notation, in Section 5 we omit the indices \( A, B \) for \( \lambda_\varepsilon^A, \eta_\varepsilon^A, \mu_\varepsilon^B, \xi_\varepsilon^B \).
Lemma 5.1 The inequality
\[ \| \mathcal{G}_{\varepsilon}^{(1)} \|_{L^2(\Gamma_\varepsilon)} + \| \mathcal{G}_{\varepsilon}^{(2)} \|_{L^2(\Gamma_\varepsilon)} + \| h_{\varepsilon} \|_{L^2(\Gamma_\varepsilon)} \leq c \varepsilon^D \] (5.5)
is valid with \( D > 0 \) and with a constant \( c \) independent of \( \varepsilon \).

Proof. By (5.4)
\[ \text{supp}\{ (\mathcal{G}_{\varepsilon}, h_{\varepsilon}) \} = \bigcup_{\{A\}} \Gamma_{\varepsilon}^A \cup \bigcup_{\{B\}} \Gamma_{\varepsilon}^B. \]

For \( x \in \Gamma_{\varepsilon}^A \) one has
\[ \chi_{\varepsilon}(x) + \eta_{\varepsilon}(x) = 1, \quad \text{div} \, \mathcal{H}^A = 0, \]
\[ \text{div} (\nabla \mathcal{H}_k^A - \varepsilon^{-1} \{ \rho^{-1} Q^A e^{(k)} + \mathcal{H}_k^A \mathcal{H}^A \}) = 0, \]
where \( \mathcal{H}^A = \mathcal{H}_M^A \) with \( M = \Upsilon_A \). Consequently,
\[ \mathcal{G}_{\varepsilon}^{(k)} = g_{k,1}^A + g_{k,2}^A + g_{k,3}^A, \quad h_{\varepsilon} = -\varepsilon^{-1} \text{div} (\eta_{\varepsilon} w^A + \chi_{\varepsilon} w^0), \]
where
\[ g_{k,1}^A = \varepsilon^{-1} \{ \nabla (\eta_{\varepsilon} w_k^A) - \varepsilon^{-1} \{ \rho^{-1} e^{(k)} \eta_{\varepsilon} q^A \} + \mathcal{H}_k^A \eta_{\varepsilon} w^A + \eta_{\varepsilon} w_k^A \mathcal{H}^A + \eta_{\varepsilon}^2 w_k^A w^A \}, \]
\[ g_{k,2}^A = \varepsilon^{-1} \{ \nabla (\chi_{\varepsilon} w_0^A) - \varepsilon^{-1} \{ \rho^{-1} e^{(k)} \chi_{\varepsilon} q_0 \} + \mathcal{H}_k^A \chi_{\varepsilon} w_0^A + \chi_{\varepsilon} w_0^A \mathcal{H}^A + \chi_{\varepsilon}^2 w_0^A w_0 \}, \]
\[ g_{k,3}^A = 2 \varepsilon^{-2} \chi_{\varepsilon} \eta_{\varepsilon} w_0 w^A. \]

Estimate (2.8) implies
\[ \varepsilon^{-1} |\nabla^j w^A (\varepsilon^{-1} (x - A))| \leq c \varepsilon^{\delta + j} r^{-\delta - j - 1}, \quad j = 0, 1, \]
\[ \varepsilon^{-2} |q^A (\varepsilon^{-1} (x - A))| \leq c \varepsilon^{\delta + 1} r^{-\delta - 2} \]
for \( x \in \Gamma_{\varepsilon}^A \). Hence
\[ \| g_{k,1}^A \|_{L^2(\Gamma_{\varepsilon}^A)} \leq c \varepsilon^{\delta / 2}, \quad k = 1, 2, \quad \varepsilon^{-1} \| \text{div} (\eta_{\varepsilon} w^A) \|_{L^2(\Gamma_{\varepsilon}^A)} \leq c \varepsilon^{\delta / 2}. \] (5.7)
Since \((w_0, q_0) \in (\tilde{N}^{1,\alpha}(\Omega_0))^2 \times N^{0,\alpha}(\Omega_0)\), we have
\[
|\nabla^j w_0(x) | \leq c r^{-j}, \ j = 0, 1, \ |q_0(x) | \leq c r^{-1}
\] (5.8)
for \(x \in \Gamma^A_{\varepsilon}\). By (5.8)
\[
\|g_{k,2}^A\|_{L^2(\Gamma^A_{\varepsilon})} \leq c \varepsilon^{\delta/2}, \ k = 1, 2, \ \varepsilon^{-1}\|\text{div } (\chi_{\varepsilon} w_0)\|_{L^2(\Gamma^A_{\varepsilon})} \leq c \varepsilon^{\delta/2}.
\] (5.9)
The estimate
\[
\|g_{k,3}^A\|_{L^2(\Gamma^A_{\varepsilon})} \leq c \varepsilon^{\delta/2}
\] (5.10)
for \(x \in \Gamma^A_{\varepsilon}\) follows from (5.6), (5.8). Unifying (5.7), (5.9), (5.10) we have
\[
\|G^k_{\varepsilon}\|_{L^2(\Gamma^A_{\varepsilon})} \leq c \varepsilon^{\delta/2}, \ k = 1, 2, \ \|h_{\varepsilon}\|_{L^2(\Gamma^A_{\varepsilon})} \leq c \varepsilon^{\delta/2}.
\] (5.11)

For \(x \in \Gamma^B_{\varepsilon}\) using the equalities
\[
\mu_{\varepsilon}(x) + \xi_{\varepsilon}(x) = 1, \ \text{div } U^A = 0,
\]
\[
\text{div} (\nabla U^A - \varepsilon^{-1}\{\rho^{-1}p^A e^{(k)} + U^A_k U^A\}) = 0
\]
we find
\[
G^k_{\varepsilon} = g^B_{k,1} + g^B_{k,2} + g^B_{k,3}, \ h_{\varepsilon} = -\varepsilon^{-1}\text{div } (\mu_{\varepsilon} w^A + \xi_{\varepsilon} w^B),
\]
where
\[
g^B_{k,1} = \varepsilon^{-1}\{\nabla(\xi_{\varepsilon} w^B_{k}) - \varepsilon^{-1}\{\rho^{-1}e^{(k)} \xi_{\varepsilon} q^B + U^A_k \xi_{\varepsilon} w^B + \xi_{\varepsilon} w^B U^A + \xi_{\varepsilon}^2 w^B w^B\}\},
\]
\[
g^B_{k,2} = \varepsilon^{-1}\{\nabla(\mu_{\varepsilon} w^A_{k}) - \varepsilon^{-1}\{\rho^{-1}e^{(k)} \mu_{\varepsilon} q^A + U^A_k \mu_{\varepsilon} w^A + \mu_{\varepsilon} w^A U^A + \mu_{\varepsilon}^2 w^A w^A\}\},
\]
\[
g^B_{k,3} = 2\varepsilon^{-2} \mu_{\varepsilon} \xi_{\varepsilon} w^A_k w^B_k.
\]
By (2.8) for \(x \in \Gamma^B_{\varepsilon}\) we obtain
\[
|\nabla^j w^A(\varepsilon^{-1}(x - A)) | \leq c e^{-d/\varepsilon}, \ j = 0, 1, \ |q^A(\varepsilon^{-1}(x - A)) | \leq c e^{-d/\varepsilon}(5.12)
\]
with \( d > 0 \). The similar estimate
\[
|\nabla^j w^B(\varepsilon^{-1}(x - B))| \leq ce^{-d/\varepsilon}, \quad j = 0, 1,
\]
\[
|q^B(\varepsilon^{-1}(x - B))| \leq ce^{-d/\varepsilon}, \quad d > 0
\]
for \( x \in \Gamma^B_\varepsilon \) follows from (2.12). Using (5.12) for \( g_{k,2}^B \), (5.13) for \( g_{k,1}^B \) and both estimates for \( h_\varepsilon, g_{k,3}^B \) we arrive to the inequalities
\[
\|g_{k,m}^B\|_{L^2(\Gamma^B_\varepsilon)} \leq ce^{-d/\varepsilon}, \quad m = 1, 2, 3, \quad \|h_\varepsilon\|_{L^2(\Gamma^B_\varepsilon)} \leq ce^{-d/\varepsilon}.
\]
(5.14)

Unifying (5.11) and (5.14) we complete the proof. 

Thus, by Lemma 5.1 and representation (3.3) for the function \( G_\varepsilon \) the right-hand side of (5.2) admits the estimate
\[
\|(G_\varepsilon, h_\varepsilon)\|_{((\tilde{H}^1(\Omega_\varepsilon))^2)^* \times L^2(\Omega_\varepsilon)} \leq c\varepsilon^D.
\]
(5.15)

6 The existence theorem

In Section 4 we obtained the constants \( K_\varepsilon, k_\varepsilon^A, k_\varepsilon^B \) and the pairs \((v_0, p_0), (v^A, p^A), (v^B, p^B)\) which enter formulas (0.19), (0.20) for the principal term \((V_\varepsilon, P_\varepsilon)\) of representation (0.22). In Section 5 we considered the problem for the remainder term \((w_\varepsilon, p_\varepsilon)\). Now we are in a position to prove the main result of the paper.

**Theorem 6.1** There exists a solution \((v_\varepsilon, p_\varepsilon)\) of problem (0.1)–(0.5) represented in the form (0.22), where \((w_\varepsilon, q_\varepsilon) \in (\tilde{H}^1(\Omega_\varepsilon))^2 \times L^2(\Omega_\varepsilon)\) is subject to (0.23).

**Proof.** Let
\[
l_\varepsilon = q_\varepsilon + P_\varepsilon.
\]
(6.1)
The pair \((w_\varepsilon, l_\varepsilon)\) satisfies equation (5.2) with
\[
\mathcal{N} : (\tilde{H}^1(\Omega_\varepsilon))^2 \times L^2(\Omega_\varepsilon) \rightarrow ((H^1(\Omega_\varepsilon))^2)^* \times L^2(\Omega_\varepsilon)
\]
being the operator acting by formula (5.3). Let \( \mathcal{B}_\kappa \) be the ball in \((\tilde{H}^1(\Omega_\varepsilon))^2 \times L^2(\Omega_\varepsilon)\) with center at \( S^{-1}(G_\varepsilon, h_\varepsilon) \) and with a small radius \( \kappa \) and let \((U^{(j)}, T^{(j)}) \in \mathcal{B}_\kappa, \quad j = 1, 2\). We shall show that if the right-hand side of the boundary condition (0.3) satisfies (0.21), then, for a sufficiently small \( \kappa \), the operator
\[
S^{-1}(\mathcal{N}) : (\tilde{H}^1(\Omega_\varepsilon))^2 \times L^2(\Omega_\varepsilon) \rightarrow (\tilde{H}^1(\Omega_\varepsilon))^2 \times L^2(\Omega_\varepsilon)
\]
is a contraction operator in $\mathcal{B}_\kappa$, i.e. the inequality
\[
\|\mathcal{N}(U^{(1)}, T^{(1)}) - \mathcal{N}(U^{(2)}, T^{(2)})\|_{((\tilde{H}^1(\Omega_\varepsilon))^*) \times L_2(\Omega_\varepsilon)} \\
\leq k\| (U^{(1)}, T^{(1)}) - (U^{(2)}, T^{(2)}) \|_{(\tilde{H}^1(\Omega_\varepsilon))^* \times L_2(\Omega_\varepsilon)}
\] (6.2)
holds with a constant $m < 1$ and
\[
\|\mathcal{N}(U^{(j)}, T^{(j)})\|_{((\tilde{H}^1(\Omega_\varepsilon))^*) \times L_2(\Omega_\varepsilon)} \leq \kappa.
\] (6.3)
By (5.3) in order to prove (6.2) it is sufficient to check inequalities
\[
\nu^{-1}\| V_{kU}^{(j)} \|_{L_2(\Omega_\varepsilon)} \leq C_R\| U^{(j)} \|_{H^1(\Omega_\varepsilon)},
\] (6.4)
\[
\nu^{-1}\| U_i^{(j)} U_{m}^{(k)} \|_{L_2(\Omega_\varepsilon)} \leq C_\kappa\| U^{(j)} \|_{H^1(\Omega_\varepsilon)}
\] (6.5)
with $i, j, k, m = 1, 2$ and constants $C_R, C_\kappa$ satisfying the conditions
\[
C_R \to 0 \text{ as } \mathcal{R} \to 0, \quad C_\kappa \to 0 \text{ as } \kappa \to 0.
\]
We begin with (6.4). By (0.19), (4.1)-(4.3)
\[
\| V_{kU} \|_{L_2(\Omega_\varepsilon)} \leq c(\| w_0 \|_{L_2(\Omega_\varepsilon)}
\]
\[
+ \sum \left\{ \| \zeta_A \eta_A^2 \mathcal{A}^4 U \|_{L_2(\Omega_\varepsilon)} + \varepsilon^{-1} \left\{ \| \eta_A^2 \mu_A \mathcal{B}^4 w \mathcal{A}^4 U \|_{L_2(\Omega_\varepsilon)}
\right.\right. \\
\left.\left. + \| \zeta_A \mu_A \mathcal{A}^4 U \|_{L_2(\Omega_\varepsilon)} + \| \xi_B \mathcal{B}^4 U \|_{L_2(\Omega_\varepsilon)} \right\}\right\}
\] (6.6)
(To simplify the notation, in (6.6) and henceforth we have omitted the indices $j, k$ for $U^{(j)}_k$ as well as the index $k$ for the components $V_{kU}, \U_k, \mathcal{H}_k$ of the vectors $\mathcal{V}_\varepsilon, \mathcal{U}, \mathcal{H}$.) Using the estimates
\[
\| u \|_{L_2(\mathcal{A}_k)} \leq C\| \nabla u \|_{L_2(\mathcal{A}_k)}, \quad \| r^{-1} u \|_{L_2(\Omega_\varepsilon)} \leq C\| \nabla u \|_{L_2(\Omega_\varepsilon)}
\] (6.7)
for $u \in \tilde{H}^1(\Omega_\varepsilon)$, we find
\[
\varepsilon^{-2}\| \zeta_A \mathcal{A}^4 U \|_{L_2(\Omega_\varepsilon)}^2 + \| \zeta_A \eta_A^2 \mathcal{A}^4 U \|_{L_2(\Omega_\varepsilon)}^2 \leq C|\mathcal{Y}_A|\| \nabla U \|_{L_2(\Omega_\varepsilon)}. \quad (6.8)
\]
Here and below we denote constants independent of $\varepsilon, \nu, \varphi$ by $C$.
According to (2.12) with $\mathcal{G} = \varphi^A$ and the Sobolev inequality
\[
\| u \|_{L_2(\mathcal{A}_k)} \leq \varepsilon^{1/2} C\| \nabla u \|_{L_2(\mathcal{A}_k)}
\]
the last term in (6.6) is estimated as follows
\[ \| \xi^B w^{B} U \|_{L^2(\Omega_\epsilon)} \leq C \| w^B \|_{L^4(\mathcal{C}_1^A)} \| U \|_{L^4(\mathcal{C}_3^A)} \]
\[ \leq C \epsilon \| w^B \|_{L^4(\Pi_B)} \| \nabla U \|_{L^4(\mathcal{C}_3^A)} \leq C \epsilon \| w^B \|_{H^1(\Pi_B)} \| U \|_{H^1(\Omega_\epsilon)} \]  \tag{6.9}
\[ \leq C \epsilon \| \varphi^A \|_{(C^{1,\alpha}(-b^-_A,b^+_A))^2} \| U \|_{H^1(\Omega_\epsilon)}. \]

We represent the function $\eta^A_u w^A U$ in the form
\[ \eta^A_u w^A U = (1 - \zeta^A) \mu^B w^A U + \zeta^A \eta^A_u w^A U. \]

Using (2.8) with $M = \Upsilon_A$ and a chain of inequalities similar to (6.9) we obtain
\[ \|(1 - \zeta^A) \mu^B w^A U \|_{L^2(\Omega_\epsilon)} \leq C \epsilon \| \varphi^A \|_{(C^{1,\alpha}(-b^-_A,b^+_A))^2} \| U \|_{H^1(\Omega_\epsilon)}. \]  \tag{6.10}

By (1.12) with
\[ M = \sum | \Upsilon_A | \]
and the Sobolev inequality
\[ \| u \|_{L^4(\Omega_\epsilon)} \leq C \| u \|_{H^1(\Omega_\epsilon)} \]  \tag{6.11}
we have
\[ \| X w^A \|_{L^4(\Omega_\epsilon)} \leq C \| w_0 \|_{L^4(\Omega_0)} \| U \|_{L^4(\Omega_\epsilon)} \leq C \| U \|_{H^1(\Omega_\epsilon)} \sum | \Upsilon_A |. \]  \tag{6.12}

Let us introduce the set
\[ S^A_\epsilon = \{ x \in \Omega_\epsilon : x \in \Omega_0 \cap (\mathbb{D}_{2\epsilon^{1/2}}(x - A) \setminus \mathbb{D}_{b_A}(x - A)) \}. \]

By (2.8) with $M = \Upsilon_A$ the estimate
\[ w^A(\epsilon^{-1}(x - A)) \leq C | \Upsilon_A | (x/\epsilon)^{-1-\delta}, \quad x \in S^A_\epsilon \]
holds. Hence
\[ \| w^A \|_{L^4(S^A_\epsilon)} \leq C | \Upsilon_A | (x/\epsilon)^{3/4+\delta/2}. \]  \tag{6.13}

Using (6.13) and the inequality
\[ \| u \|_{L^4(S^A_\epsilon)} \leq \epsilon^{1/2} C \| \nabla u \|_{L^2(S^A_\epsilon)} \]
we arrive at

\[ \| \zeta^A \eta^A w^A U \|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon |\Upsilon_A| \| U \|_{\mathcal{H}^1(\Omega_\varepsilon)}. \] (6.14)

Since $|\Upsilon_A| \leq C \nu R$ and $\sum |\Upsilon_A| \leq C \nu R$, by combining (6.6) with (6.8)–(6.10), (6.12)–(6.14) we obtain (6.4) with $C_R = R C$.

The estimate (6.3) with a sufficiently small $\kappa$ and the estimate (6.5) with $C_\kappa = \kappa C$ follow from (6.11).

Thus, $\mathcal{N}$ is a contraction operator in $\mathfrak{B}_\kappa$ and therefore, according to the Banach principle, there exists a unique solution $(w_\varepsilon, l_\varepsilon) \in \mathfrak{B}_\kappa$ of equation (5.2). Putting $\kappa = \varepsilon^D$ and taking into account (6.1), (4.4), (5.15) we complete the proof.

7 Asymptotic representations for the kinetic energy and Dirichlet integral

The asymptotic behavior of the kinetic energy $\mathcal{E}(v_\varepsilon)$ is described in the following assertion.

**Theorem 7.1** Kinetic energy $\mathcal{E}(v_\varepsilon)$ of the fluid in the domain $\Omega_\varepsilon$ has the asymptotic representation (0.25), where $\mathcal{V}^A = \mathcal{V}^A_\Lambda$ with $M = \Upsilon_A$.

**Proof.** We write the velocity $v_\varepsilon$ in the form

\[ v_\varepsilon = u_\varepsilon + W_\varepsilon + w_\varepsilon, \] (7.1)

where

\[ u_\varepsilon(x) = \varepsilon^{-1} \sum \left\{ \zeta_A(x - A) \zeta^A(\varepsilon^{-1}(x - A)) \mathcal{H}(\varepsilon^{-1}(x - A)) \right\}, \]

\[ + \zeta^A(\varepsilon^{-1}(x - A)) \mathcal{U}(\varepsilon^{-1}(x - A)) \}, \]

\[ W_\varepsilon(x) = X_\varepsilon(x) w_\varepsilon(x) \]

\[ + \varepsilon^{-1} \sum \left\{ \eta^A(\varepsilon^{-1}(x - A) + B) (\varepsilon^{-1}(x - A)) + \eta^B(\varepsilon^{-1}(x - B)) \right\}. \]

We remind that the summation is taken over all the channels. By (7.1) we have

\[ \mathcal{E}(v_\varepsilon) = \frac{\rho}{2} (\| u_\varepsilon \|^2_{L^2(\Omega_\varepsilon)} + \| W_\varepsilon \|^2_{L^2(\Omega_\varepsilon)} + \| w_\varepsilon \|^2_{L^2(\Omega_\varepsilon)} + J_1 + J_2), \] (7.2)
where

\[ J_1 = 2 \int_{\Omega_\varepsilon} u_\varepsilon (w_\varepsilon + W_\varepsilon) \, dx, \quad J_2 = 2 \int_{\Omega_\varepsilon} w_\varepsilon W_\varepsilon \, dx. \]

Straightforward calculation gives

\[ \int_{\Omega_\varepsilon} u_\varepsilon^2 \, dx = \frac{6}{5} \varepsilon \sum \gamma_A^2 L_A b_A^{-1} + \log \frac{1}{\varepsilon} \sum \gamma_A^2 \int_{\omega_A^+} (\psi_A^A(\theta))^2 \, d\theta + O(1). \quad (7.3) \]

Now we estimate other terms in the right-hand side of (7.2). Since

\[ \|W_\varepsilon\|_{\bar{H}^1(\Omega_\varepsilon)^2} \leq c (\|w_0\|_{\bar{H}^1(\Omega_0)^2})^2 \]

\[ + \sum \{ \|w^A\|_{\bar{H}^1(\Lambda_A)^2} + \|w^B\|_{\bar{H}^1(\Pi_B)^2} \}], \]

then (1.12) with \( \mathcal{M} = \sum |\gamma_A| \), (2.8) with \( M = \gamma_A \) and (2.12) with \( g = \varphi^A \) imply

\[ \|W_\varepsilon\|_{\bar{H}^1(\Omega_\varepsilon)} \leq C. \quad (7.4) \]

By (0.23) we have

\[ \|w_\varepsilon\|_{\bar{H}^1(\Omega_\varepsilon)} \leq c \varepsilon^{\delta}. \quad (7.5) \]

The estimate

\[ |J_1| \leq c \]

follows from (7.4) and (7.5). According to (7.4), (7.5) and (6.7)

\[ |J_2| \leq c. \quad (7.7) \]

Unifying (7.2)–(7.7) we arrive at (0.25).

Now we calculate the principal term of the asymptotic representation of the Dirichlet integral \( I(\nu_\varepsilon) \) of problem (0.1)–(0.5).

**Theorem 7.2** Dirichlet integral (0.26) of problem (0.1)–(0.5) admits representation (0.27).
**Proof.** We make use of expression (7.1) for the velocity vector \( v_\varepsilon \). A straightforward calculation gives

\[
\|\nabla v_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} = 12\varepsilon^{-3} \sum \Upsilon^2_{L_A}b_A^{-3} + O(\varepsilon^{-2}).
\]  
(7.8)

It follows by (7.4), (7.5) that

\[
\mathcal{I}(w_\varepsilon + W_\varepsilon) \leq c.
\]  
(7.9)

The inequality

\[
|\mathcal{I}(v_\varepsilon) - \mathcal{I}(u_\varepsilon)| \leq c(\mathcal{I}(w_\varepsilon + W_\varepsilon)^{1/2}(\mathcal{I}(w_\varepsilon + W_\varepsilon)^{1/2} + \mathcal{I}(u_\varepsilon)^{1/2})
\]  
combined with (7.8), (7.9) completes the proof. \( \square \)
References


