The L^p-dissipativity of certain differential and integral operators

Alberto Cialdea and Vladimir Maz'ya

In Memory of Selim G. Krein

Abstract. After surveying some of the results previously obtained by the authors which concerns the L^p-dissipativity of scalar and metrics partial differential operators, we give necessary and, separately, sufficient conditions for the L^p-dissipativity of the "complex oblique derivative" operator. In the case of real coefficients we provide a necessary and sufficient condition. We prove also the L^p-positivity for a certain class of integral operators.

1. Introduction

In a series of papers [**2, 3, 4, 6**] we have studied the problem of characterizing the L^p-dissipativity of scalar and metrics partial differential operators. The main result we have obtained is that the algebraic condition

$$|p-2|\,|\langle \mathscr{I}m\, \mathscr{A}\, \xi, \xi\rangle| \leq 2\sqrt{p-1}\,\langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi\rangle \tag{1.1}$$

for any $\xi \in \mathbb{R}^n$, is necessary and sufficient for the L^p-dissipativity of the Dirichlet problem for the scalar differential operator $\nabla^t(\mathscr{A}\,\nabla)$, where $\mathscr{A}$ is a matrix whose entries are complex measures and whose imaginary parts is symmetric. Specifically we have proved that condition (1.1) is necessary and sufficient for the L^p-dissipativity of the related sesquilinear form

$$\mathscr{L}(u,v) = \int_\Omega \langle \mathscr{A}\,\nabla u, \nabla v\rangle\,.$$

Such condition characterizes the L^p-dissipativity individually, for each p, while in the literature previous results dealt with the L^p-dissipativity for any $p \in [1,+\infty)$. Later on we have considered more general operators. Our results are described in the monograph [**5**].

We remark that, if $\mathscr{I}m\,\mathscr{A}$ is symmetric, (1.1) is equivalent to the the condition

$$\frac{4}{p\,p'}\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle + \langle \mathscr{R}e\,\mathscr{A}\,\eta,\eta\rangle + 2\langle (p^{-1}\,\mathscr{I}m\,\mathscr{A} + p'^{-1}\,\mathscr{I}m\,\mathscr{A}^*)\xi,\eta\rangle \geq 0 \tag{1.2}$$

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for any $\xi, \eta \in \mathbb{R}^n$. If the matrix $\mathscr{I}m\,\mathscr{A}$ is not symmetric, condition (1.2) is only sufficient for the L^p-dissipativity of the corresponding form.

Let us consider the class of partial differential operators of the second order whose principal part is such that the form (1.2) is not merely non-negative, but strictly positive. This class of operators, which could be called *p-strongly elliptic*, was recently considered by Carbonaro and Dragičević [**1**], and Dindoš and Pipher [**7**].

In what follows, saying the L^p-dissipativity of an operator A, we mean the L^p-dissipativity of the corresponding form $\mathscr{L}$, just to simplify the terminology.

In the present paper, after surveying some of our more recent results for systems which are not contained in [**5**], we give some new theorems.

The first ones concern the "complex oblique derivative" operator

$$\lambda \cdot \nabla u = \frac{\partial u}{\partial x_n} + \sum_{j=1}^{n-1} a_j \frac{\partial u}{\partial x_j}$$

where $\lambda = (1, a_1, \dots, a_{n-1})$ and a_j are complex valued functions. We give necessary and, separately, sufficient conditions under which such boundary operator is L^p-dissipative on $\mathbb{R}^{n-1}$. If the coefficients a_j are real valued, we provide a necessary and sufficient condition.

The last result concerns a class of integral operators which can be written as

$$\int_{\mathbb{R}^n}^{*} [u(x) - u(y)]\, K(dx, dy) \tag{1.3}$$

where the integral has to be understood as a principal value in the sense of Cauchy and the kernel $K(dx, dy)$ is a Borel positive measure defined on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying certain conditions. The class of operators we consider includes the fractional powers of Laplacian $(-\Delta)^s$, with $0 < s < 1$. We establish the L^p-positivity of operator (1.3), extending in this way a result we obtained in [**5**, p.230–231].

The paper is organized as follows. Section 2 presents a review of our main results concerning scalar differential operators of the second order.

Section 3 is dedicated to systems. In particular we describe some necessary and sufficient conditions for the L^p-dissipativity of systems of partial differential operators of the first order, recently obtained in [**6**].

The topic of Section 4 is elasticity system. After recalling the necessary and sufficient conditions we previously obtained in the planar case, we describe sufficient conditions holding in any dimension and proved in [**4**]. Finding necessary and sufficient conditions for the L^p-dissipativity of elasticity system in the three-dimensional case is still an open problem.

In Sections 5 and 6 we prove the above mentioned results concerning the "complex oblique derivative" operator and the operator (1.3) respectively.

2. The scalar operators

Let Ω be an open set in $\mathbb{R}^n$. Consider the sesquilinear form

$$\mathscr{L}(u,v) = \int_\Omega (\langle \mathscr{A}\, \nabla u, \nabla v\rangle - \langle \mathbf{b}\nabla u, v\rangle + \langle u, \overline{\mathbf{c}}\nabla v\rangle - a\langle u, v\rangle)$$

defined on $C_0^1(\Omega) \times C_0^1(\Omega)$. Here $\mathscr{A}$ is a $n \times n$ matrix function with complex valued entries $a^{hk} \in (C_0(\Omega))^*$, $\mathbf{b} = (b_1, \dots, b_n)$ and $\mathbf{c} = (c_1, \dots, c_n)$ are complex valued vectors with $b_j, c_j \in (C_0(\Omega))^*$ and a is a complex valued scalar distribution in $(C_0^1(\Omega))^*$. The symbol $\langle\cdot,\cdot\rangle$ denotes the inner product either in $\mathbb{C}^n$ or in $\mathbb{C}$.

In what follows, if $p \in (1,\infty)$, p' denotes its conjugate exponent $p/(p-1)$.

The integrals appearing in this definition have to be understood in a proper way. The entries a^{hk} being measures, the meaning of the first term is

$$\int_\Omega \langle \mathscr{A}\,\nabla u, \nabla v\rangle = \int_\Omega \partial_k u\, \partial_h \overline{v}\, da^{hk}\,.$$

Similar meanings have the terms involving $\mathbf{b}$ and $\mathbf{c}$. Finally, the last term is the action of the distribution $a \in (C_0^1(\Omega))^*$ on the functions $u\,\overline{v}$ belonging to $C_0^1(\Omega)$.

We say that the form $\mathscr{L}$ is L^p*-dissipative* $(1<p<\infty)$ if for all $u \in C_0^1(\Omega)$

$$\mathscr{R}e\, \mathscr{L}(u, |u|^{p-2}u) \geq 0 \qquad \text{if } p \geq 2; \tag{2.1}$$

$$\mathscr{R}e\, \mathscr{L}(|u|^{p'-2}u, u) \geq 0 \qquad \text{if } 1<p<2 \tag{2.2}$$

(we use here that $|u|^{q-2}u \in C_0^1(\Omega)$ for $q \geq 2$ and $u \in C_0^1(\Omega)$).

The form $\mathscr{L}$ is related to the operator

$$Au = \operatorname{div}(\mathscr{A}\,\nabla u) + \mathbf{b}\nabla u + \operatorname{div}(\mathbf{c}u) + au. \tag{2.3}$$

where div denotes the divergence operator. The operator A acts from $C_0^1(\Omega)$ to $(C_0^1(\Omega))^*$ through the relation

$$\mathscr{L}(u,v) = -\int_\Omega \langle Au, v\rangle$$

for any $u, v \in C_0^1(\Omega)$. The integration is understood in the sense of distributions.

As we already remarked, saying the L^p-dissipativity of the operator A, we mean theL^p-dissipativity of the form $\mathscr{L}$.

The following Lemma provides a necessary and sufficient condition for the L^p-dissipativity of the form $\mathscr{L}$.

Lemma 2.1 ([**2**]). *The operator A is L^p-dissipative if and only if for all $v \in C_0^1(\Omega)$*

$$\mathscr{R}e \int_\Omega \Big[\langle \mathscr{A}\,\nabla v, \nabla v\rangle - (1-2/p)\langle(\mathscr{A}-\mathscr{A}^*)\nabla(|v|), |v|^{-1}\overline{v}\nabla v\rangle -$$
$$(1-2/p)^2\langle \mathscr{A}\,\nabla(|v|), \nabla(|v|)\rangle\Big] + \int_\Omega \langle \mathscr{I}m(\mathbf{b}+\mathbf{c}), \mathscr{I}m(\overline{v}\nabla v)\rangle +$$
$$\int_\Omega \mathscr{R}e(\operatorname{div}(\mathbf{b}/p - \mathbf{c}/p') - a)|v|^2 \geq 0.$$

Here and in what follows the integrand is extended by zero on the set where v vanishes.

This result has several consequences. The first one is a necessary condition for the L^p-dissipativity.

Corollary 2.2 ([**2**]). *If the operator A is L^p-dissipative, we have*

$$\langle \mathscr{R}e\,\mathscr{A}\,\xi, \xi\rangle \geq 0 \tag{2.4}$$

(in the sense of measures) for any $\xi \in \mathbb{R}^n$.

Obviously condition (2.4) is not sufficient for the L^p-dissipativity of the form $\mathscr{L}$.

COROLLARY 2.3 ([**2**]). *Let α,β two real constants. If*

$$
\begin{aligned}
&\frac{4}{p\,p'}\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle + \langle \mathscr{R}e\,\mathscr{A}\,\eta,\eta\rangle + 2\langle (p^{-1}\,\mathscr{I}m\,\mathscr{A} + p'^{-1}\,\mathscr{I}m\,\mathscr{A}^*)\xi,\eta\rangle + \\
&\qquad \langle \mathscr{I}m(\mathbf{b}+\mathbf{c}),\eta\rangle - 2\langle \mathscr{R}e(\alpha\mathbf{b}/p - \beta\mathbf{c}/p'),\xi\rangle + \\
&\qquad \mathscr{R}e\,[\operatorname{div}((1-\alpha)\mathbf{b}/p - (1-\beta)\mathbf{c}/p') - a] \geq 0
\end{aligned}
\tag{2.5}
$$

for any $\xi,\eta\in\mathbb{R}^n$, the operator A is L^p-dissipative.

Generally speaking, conditions (2.5) are not necessary, as the following example shows.

EXAMPLE 2.4. let $n=2$ and

$$
\mathscr{A} = \begin{pmatrix} 1 & i\gamma \\ -i\gamma & 1 \end{pmatrix}
$$

where γ is a real constant, $\mathbf{b}=\mathbf{c}=a=0$. In this case polynomial (2.5) is given by

$$
(\eta_1-\gamma\xi_2)^2 + (\eta_2-\gamma\xi_1)^2 - (\gamma^2 - 4/(pp'))|\xi|^2.
$$

Taking $\gamma^2 > 4/(pp')$, condition (2.5) is not satisfied, while we have the L^p-dissipativity, because the corresponding operator A is the Laplacian.

Note that in this example the matrix $\mathscr{I}m\,\mathscr{A}$ is not symmetric. Later we give another example showing that, even for symmetric matrices $\mathscr{I}m\,\mathscr{A}$, conditions (2.5) are not necessary for L^p-dissipativity (see Example 2.11). Nevertheless in the next section we show that the conditions are necessary for the L^p-dissipativity, provided the operator A has no lower order terms and the matrix $\mathscr{I}m\,\mathscr{A}$ is symmetric (see Theorem 2.5 and Remark 2.6).

In the case of an operator (2.3) without lower order terms:

$$
Au = \operatorname{div}(\mathscr{A}\,\nabla u)
\tag{2.6}
$$

with the coefficients $a^{hk}\in (C_0(\Omega))^*$, we can give an algebraic necessary and sufficient condition for the L^p-dissipativity.

THEOREM 2.5 ([**2**]). *Let the matrix $\mathscr{I}m\,\mathscr{A}$ be symmetric, i.e. $\mathscr{I}m\,\mathscr{A}^t = \mathscr{I}m\,\mathscr{A}$. The form*

$$
\mathscr{L}(u,v) = \int_\Omega \langle \mathscr{A}\,\nabla u,\nabla v\rangle
$$

is L^p-dissipative if and only if

$$
|p-2|\,|\langle \mathscr{I}m\,\mathscr{A}\,\xi,\xi\rangle| \leq 2\sqrt{p-1}\,\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle
\tag{2.7}
$$

for any $\xi\in\mathbb{R}^n$, where $|\cdot|$ denotes the total variation.

REMARK 2.6. One can prove that condition (2.7) holds if and only if

$$
\frac{4}{p\,p'}\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle + \langle \mathscr{R}e\,\mathscr{A}\,\eta,\eta\rangle - 2(1-2/p)\langle \mathscr{I}m\,\mathscr{A}\,\xi,\eta\rangle \geq 0
\tag{2.8}
$$

for any $\xi,\eta\in\mathbb{R}^n$. This means that conditions (2.5) are necessary and sufficient for the operators considered in Theorem 2.5.

REMARK 2.7. Let us assume that either A has lower order terms or they are absent and $\mathscr{I}m\,\mathscr{A}$ is not symmetric. Using the same arguments as in Theorem 2.5, one could prove that (2.7) (or, equivalenty, (2.8)) is still a necessary condition for A to be L^p-dissipative. However, in general, it is not sufficient. This is shown by the next example (see also Theorem 2.9 below for the particular case of constant coefficients).

EXAMPLE 2.8. Let $n=2$ and let Ω be a bounded domain. Denote by σ a not identically vanishing real function in $C_0^2(\Omega)$ and let $\lambda \in \mathbb{R}$. Consider operator (2.6) with

$$\mathscr{A} = \begin{pmatrix} 1 & i\lambda\partial_1(\sigma^2) \\ -i\lambda\partial_1(\sigma^2) & 1 \end{pmatrix}$$

i.e.

$$Au = \partial_1(\partial_1 u + i\lambda\partial_1(\sigma^2)\,\partial_2 u) + \partial_2(-i\lambda\partial_1(\sigma^2)\,\partial_1 u + \partial_2 u),$$

where $\partial_i = \partial/\partial x_i$ $(i=1,2)$.

By definition, we have L^2-dissipativity if and only if

$$\mathscr{R}e \int_\Omega ((\partial_1 u + i\lambda\partial_1(\sigma^2)\,\partial_2 u)\partial_1\overline{u} + (-i\lambda\partial_1(\sigma^2)\,\partial_1 u + \partial_2 u)\partial_2\overline{u})\,dx \geq 0$$

for any $u \in C_0^1(\Omega)$, i.e. if and only if

$$\int_\Omega |\nabla u|^2 dx - 2\lambda \int_\Omega \partial_1(\sigma^2)\,\mathscr{I}m(\partial_1\overline{u}\,\partial_2 u)\,dx \geq 0$$

for any $u \in C_0^1(\Omega)$. Taking $u = \sigma\,\exp(itx_2)$ $(t \in \mathbb{R})$, we obtain, in particular,

$$t^2 \int_\Omega \sigma^2 dx - t\lambda \int_\Omega (\partial_1(\sigma^2))^2 dx + \int_\Omega |\nabla\sigma|^2 dx \geq 0. \tag{2.9}$$

Since

$$\int_\Omega (\partial_1(\sigma^2))^2 dx > 0,$$

we can choose $\lambda \in \mathbb{R}$ so that (2.9) is impossible for all $t \in \mathbb{R}$. Thus A is not L^2-dissipative, although (2.7) is satisfied.

Since A can be written as

$$Au = \Delta u - i\lambda(\partial_{21}(\sigma^2)\,\partial_1 u - \partial_{11}(\sigma^2)\,\partial_2 u),$$

the same example shows that (2.7) is not sufficient for the L^2-dissipativity in the presence of lower order terms, even if $\mathscr{I}m\,\mathscr{A}$ is symmetric.

Generally speaking, it is impossible to obtain an algebraic characterization for an operator with lower order terms. Indeed, let us consider, for example, the operator

$$Au = \Delta u + a(x)u$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with zero Dirichlet boundary data. Denote by λ_1 the first eigenvalue of the Dirichlet problem for the Laplace equation in Ω. A sufficient condition for the L^2-dissipativity of A has the form $\mathrm{Re}\,a \leq \lambda_1$, and we cannot give an algebraic characterization of λ_1.

Consider, as another example, the operator

$$A = \Delta + \mu$$

where μ is a nonnegative Radon measure on Ω. The operator A is L^p-dissipative if and only if

$$\int_\Omega |w|^2 d\mu \leq \frac{4}{p\,p'} \int_\Omega |\nabla w|^2 dx \tag{2.10}$$

for any $w \in C_0{}^\infty(\Omega)$ (cf. Lemma 2.1). Maz'ya [**9, 10, 11**] proved that the following condition is sufficient for (2.10):

$$\frac{\mu(F)}{\mathrm{cap}_\Omega(F)} \leq \frac{1}{p\,p'} \tag{2.11}$$

for all compact set $F \subset \Omega$ and the following condition is necessary:

$$\frac{\mu(F)}{\mathrm{cap}_\Omega(F)} \leq \frac{4}{p\,p'} \tag{2.12}$$

for all compact set $F \subset \Omega$. Here, $\mathrm{cap}_\Omega(F)$ is the capacity of F with respect to Ω, i.e.,

$$\mathrm{cap}_\Omega(F) = \inf\Big\{ \int_\Omega |\nabla u|^2 dx \, : \, u \in C_0{}^\infty(\Omega), \ u \geq 1 \text{ on } F \Big\}. \tag{2.13}$$

The condition (2.11) is not necessary and the condition (2.12) is not sufficient.

However, it is possible to find necessary and sufficient conditions in the case of constant coefficients. Namely, let A be the differential operator

$$Au = \nabla^t(\mathscr{A}\,\nabla u) + \mathbf{b}\nabla u + au$$

with constant complex coefficients. Without loss of generality we assume that the matrix $\mathscr{A}$ is symmetric.

THEOREM 2.9 ([**2**]). *Let Ω be an open set in $\mathbb{R}^n$ which contains balls of arbitrarily large radius. The operator A is L^p-dissipative if and only if there exists a real constant vector V such that*

$$2\,\mathscr{R}e\,\mathscr{A}\,V + \mathscr{I}m\,\mathbf{b} = 0$$
$$\mathscr{R}e\,a + \langle \mathscr{R}e\,\mathscr{A}\,V, V\rangle \leq 0$$

and the inequality

$$|p-2|\,|\langle \mathscr{I}m\,\mathscr{A}\,\xi,\xi\rangle| \leq 2\sqrt{p-1}\,\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle \tag{2.14}$$

holds for any $\xi \in \mathbb{R}^n$.

COROLLARY 2.10 ([**2**]). *Let Ω be an open set in $\mathbb{R}^n$ which contains balls of arbitrarily large radius. Let us suppose that the matrix $\mathscr{R}e\,\mathscr{A}$ is not singular. The operator A is L^p-dissipative if and only if (2.14) holds and*

$$4\,\mathscr{R}e\,a \leq -\langle (\mathscr{R}e\,\mathscr{A})^{-1}\,\mathscr{I}m\,\mathbf{b}, \mathscr{I}m\,\mathbf{b}\rangle\,. \tag{2.15}$$

EXAMPLE 2.11. Let $n = 1$ and $\Omega = \mathbb{R}^1$. Consider the operator

$$\left(1 + 2\,\frac{\sqrt{p-1}}{p-2}\,i\right) u'' + 2iu' - u,$$

where $p \neq 2$ is fixed. Conditions (2.14) and (2.15) are satisfied and this operator is L^p-dissipative, in view of Corollary 2.10.

On the other hand, the polynomial considered in Corollary 2.3 is

$$Q(\xi,\eta) = \left(2\,\frac{\sqrt{p-1}}{p}\,\xi - \eta\right)^2 + 2\eta + 1$$

which is not nonnegative for any $\xi, \eta \in \mathbb{R}$. This shows that, in general, condition (2.5) is not necessary for the L^p-dissipativity, even if the matrix $\mathscr{I}m\,\mathscr{A}$ is symmetric.

3. L^p-dissipativity for systems

In this Section we describe criteria we have obtained for some systems of partial differential equations.

3.1. Systems of the first order. Let $\mathscr{B}^h$ and $\mathscr{C}^h$ $(h = 1, \dots, n)$ be $m \times m$ matrices whith complex-valued entries $b^h_{ij}, c^h_{ij} \in (C_0(\Omega))^*$ $(1 \leq i, j \leq m)$. Let $\mathscr{D}$ stand for a matrix whose elements d_{ij} are complex-valued distributions in $(C^1_0(\Omega))^*$.

Let $\mathscr{L}(u,v)$ be the sesquilinear form

$$\mathscr{L}(u,v) = \int_\Omega \langle \mathscr{B}^h \, \partial_h u, v\rangle - \langle \mathscr{C}^h \, u, \partial_h v\rangle + \langle \mathscr{D}\, u, v\rangle \tag{3.1}$$

defined in $(C^1_0(\Omega))^m \times (C^1_0(\Omega))^m$, where $\partial_h = \partial/\partial x_h$.

The form $\mathscr{L}$ is related to the system of partial differential operators of the first order:

$$Eu = \mathscr{B}^h \, \partial_h u + \partial_h(\mathscr{C}^h \, u) + \mathscr{D}\, u \tag{3.2}$$

As in the scalar case, we say that the form $\mathscr{L}$ is L^p-dissipative if (2.1)-(2.2) hold for all $u \in (C^1_0(\Omega))^m$.

We have found necessary and sufficient conditions for the L^p-dissipativity when

$$Eu = \mathscr{B}^h \, \partial_h u + \mathscr{D}\, u$$

and the entries of the matrices $\mathscr{B}^h, \mathscr{D}$ are locally integrable functions. Moreover we suppose that also $\partial_h \, \mathscr{B}^h$ (where the derivatives are in the sense of distributions) is a matrix with locally integrable entries.

Theorem 3.1 ([**6**]). *The form*

$$\mathscr{L}(u,v) = \int_\Omega \langle \mathscr{B}^h \, \partial_h u, v\rangle + \langle \mathscr{D}\, u, v\rangle$$

is $L^p - $dissipative if, and only if, the following conditions are satisfied:

(1)

$$\mathscr{B}^h(x) = b_h(x)\, I, \qquad \text{if } p \neq 2, \tag{3.3}$$

$$\mathscr{B}^h(x) = (\mathscr{B}^h)^*(x), \qquad \text{if } p = 2, \tag{3.4}$$

for almost any $x \in \Omega$ and $h = 1, \dots, n$. Here b_h are real locally integrable functions $(1 \leq h \leq n)$.

(2)

$$\mathscr{R}e\langle (p^{-1}\partial_h \, \mathscr{B}^h(x) - \mathscr{D}(x))\zeta, \zeta\rangle \geq 0 \tag{3.5}$$

for any $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$ and for almost any $x \in \Omega$.

As far as the more general operator (3.2) is concerned, we have the following result, under the assumption that $\mathscr{B}^h$, $\mathscr{C}^h$, $\mathscr{D}$, $\partial_h \, \mathscr{B}^h$ and $\partial_h \, \mathscr{C}^h$ are matrices with complex locally integrable entries.

Theorem 3.2 ([**6**]). *The form (3.1) is L^p-dissipative if, and only if, the following conditions are satisfied*

(1)

$$\mathscr{B}^h(x) + \mathscr{C}^h(x) = b_h(x)\, I, \qquad \text{if } p \neq 2,$$

$$\mathscr{B}^h(x) + \mathscr{C}^h(x) = (\mathscr{B}^h)^*(x) + (\mathscr{C}^h)^*(x), \qquad \text{if } p = 2,$$

for almost any $x \in \Omega$ and $h = 1, \dots, n$. Here b_h are real locally integrable functions ($1 \leq h \leq n$).

(2)

$$\mathscr{R}e\langle (p^{-1}\partial_h \mathscr{B}^h(x) - p'^{-1}\partial_h \mathscr{C}^h(x) - \mathscr{D}(x))\zeta, \zeta\rangle \geq 0$$

for any $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$ and for almost any $x \in \Omega$.

3.2. Systems of the second order. In this section we consider the class of systems of partial differential equations of the form

$$Eu = \partial_h(\mathscr{A}^h(x)\partial_h u) + \mathscr{B}^h(x)\partial_h u + \mathscr{D}(x)u, \tag{3.6}$$

where $\mathscr{A}^h$, $\mathscr{B}^h$ and $\mathscr{D}$ are $m \times m$ matrices with complex locally integrable entries.

If the operator (3.6) has no lower order terms, we have necessary and sufficient conditions:

THEOREM 3.3 ([**3**]). *The operator*

$$\partial_h(\mathscr{A}^h(x)\partial_h u)$$

is L^p-dissipative if and only if

$$\begin{aligned} &\mathscr{R}e\langle \mathscr{A}^h(x)\lambda, \lambda\rangle - (1-2/p)^2\, \mathscr{R}e\langle \mathscr{A}^h(x)\omega, \omega\rangle (\mathscr{R}e\langle \lambda, \omega\rangle)^2 \\ &-(1-2/p)\, \mathscr{R}e(\langle \mathscr{A}^h(x)\omega, \lambda\rangle - \langle \mathscr{A}^h(x)\lambda, \omega\rangle)\, \mathscr{R}e\langle \lambda, \omega\rangle \geq 0 \end{aligned} \tag{3.7}$$

for almost every $x \in \Omega$ and for every $\lambda, \omega \in \mathbb{C}^m$, $|\omega| = 1$, $h = 1, \dots, n$.

Combining this result with Theorem 3.1 we find

THEOREM 3.4 ([**6**]). *Let E be the operator* (3.6)*, where $\mathscr{A}^h$ are $m \times m$ matrices with complex locally integrable entries and the matrices $\mathscr{B}^h(x)$, $\mathscr{D}(x)$ satisfy the hypothesis of Theorem 3.1. If* (3.7) *holds for almost every $x \in \Omega$ and for every $\lambda, \omega \in \mathbb{C}^m$, $|\omega| = 1$, $h = 1, \dots, n$, and if conditions* (3.3)-(3.4) *and* (3.5) *are satisfied, the operator E is L^p-dissipative.*

Consider now the operator (3.6) in the scalar case (i.e. $m = 1$)

$$\partial_h(a^h(x)\partial_h u) + b^h(x)\partial_h u + d(x)u$$

(a^h, b^h and d being scalar functions). In this case such an operator can be written in the form

$$Eu = \operatorname{div}(\mathscr{A}(x)\nabla u) + \mathscr{B}(x)\nabla u + d(x)\, u \tag{3.8}$$

where $\mathscr{A} = \{c_{hk}\}$, $c_{hh} = a^h$, $c_{hk} = 0$ if $h \neq k$ and $\mathscr{B} = \{b^h\}$. For such an operator one can show that (3.7) is equivalent to

$$\frac{4}{pp'}\langle \mathscr{R}e\, \mathscr{A}(x)\xi, \xi\rangle + \langle \mathscr{R}e\, \mathscr{A}(x)\eta, \eta\rangle - 2(1-2/p)\langle \mathscr{I}m\, \mathscr{A}(x)\xi, \eta\rangle \geq 0 \tag{3.9}$$

for almost any $x \in \Omega$ and for any $\xi, \eta \in \mathbb{R}^n$ (see [**5**, Remark 4.21, p.115]). Condition (3.9) is in turn equivalent to the inequality:

$$|p-2|\, |\langle \mathscr{I}m\, \mathscr{A}(x)\xi, \xi\rangle| \leq 2\sqrt{p-1}\, \langle \mathscr{R}e\, \mathscr{A}(x)\xi, \xi\rangle \tag{3.10}$$

for almost any $x \in \Omega$ and for any $\xi \in \mathbb{R}^n$ (see [**5**, Remark 2.8, p.42]). We have then

THEOREM 3.5 ([**6**]). *Let E be the scalar operator* (3.8) *where $\mathscr{A}$ is a diagonal matrix. If inequality* (3.10) *and conditions* (3.3)-(3.4) *and* (3.5) *are satisfied, the operator E is L^p-dissipative.*

More generally, consider the scalar operator (3.8) with a matrix $\mathscr{A} = \{a_{hk}\}$ not necessarily diagonal. The following result holds true.

THEOREM 3.6 ([**6**]). *Let the matrix $\mathscr{I}m\,\mathscr{A}$ be symmetric. If inequality* (3.10) *and conditions* (3.3)-(3.4) *and* (3.5) *are satisfied, the operator* (3.8) *is L^p-dissipative.*

4. The L^p-dissipativity of the Lamé operator

Let us consider the classical operator of linear elasticity

$$Eu = \Delta u + (1-2\nu)^{-1}\nabla \operatorname{div} u \tag{4.1}$$

where ν is the Poisson ratio. We assume that either $\nu > 1$ or $\nu < 1/2$. It is well known that E is strongly elliptic if and only if this condition is satisfied.

We remark that the elasticity system is not of the form considered in the subsection 3.2.

Let $\mathscr{L}$ be the bilinear form associated with operator (4.1), i.e.

$$\mathscr{L}(u,v) = -\int_\Omega (\langle \nabla u, \nabla v\rangle + (1-2\nu)^{-1} \operatorname{div} u \operatorname{div} v)\, dx\,,$$

The following lemma holds in any dimensions:

LEMMA 4.1 ([**3**]). *Let Ω be a domain of $\mathbb{R}^n$. The operator* (4.1) *is L^p-dissipative if and only if*

$$\int_\Omega [C_p|\nabla|v||^2 - \sum_{j=1}^2 |\nabla v_j|^2 + \gamma\, C_p\, |v|^{-2}|v_h\partial_h|v||^2 - \gamma\, |\operatorname{div} v|^2]\, dx \leq 0$$

for any $v \in (C_0^1(\Omega))^2$, where

$$C_p = (1-2/p)^2, \qquad \gamma = (1-2\nu)^{-1}. \tag{4.2}$$

More precise results are known in the case of planar elasticity. At first we have an algebraic necessary condition:

LEMMA 4.2 ([**3**]). *Let Ω be a domain of $\mathbb{R}^2$. If the operator* (4.1) *is L^p-dissipative, we have*

$$C_p[|\xi|^2 + \gamma\,\langle\xi,\omega\rangle^2]\langle\lambda,\omega\rangle^2 - |\xi|^2|\lambda|^2 - \gamma\,\langle\xi,\lambda\rangle^2 \leq 0$$

for any ξ, λ, $\omega \in \mathbb{R}^2$, $|\omega| = 1$ (the constants C_p and γ being given by (4.2)*).*

Hinging on Lemmas 4.1 and 4.2, we proved

THEOREM 4.3 ([**3**]). *Let Ω be a domain of $\mathbb{R}^2$. The operator* (4.1) *is L^p-dissipative if and only if*

$$\left(\frac{1}{2}-\frac{1}{p}\right)^2 \leq \frac{2(\nu-1)(2\nu-1)}{(3-4\nu)^2}\,. \tag{4.3}$$

Concerning the elasticity system in any dimension, the next Theorem shows that condition (4.3) is necessary, even in the case of a non constant Poisson ratio. Here Ω is a bounded domain in $\mathbb{R}^n$ whose boundary is in the class C^2.

THEOREM 4.4 ([**4**]). *Suppose $\nu = \nu(x)$ is a continuos function defined in Ω such that*

$$\inf_{x\in\Omega} |2\nu(x) - 1| > 0.$$

If the operator (4.1) *is L^p-dissipative in Ω, then*

$$\left(\frac{1}{2}-\frac{1}{p}\right)^2 \leq \inf_{x\in\Omega} \frac{2(\nu(x)-1)(2\nu(x)-1)}{(3-4\nu(x))^2} .$$

We do not know if condition (4.3) is sufficient for the L^p-dissipativity of the n-dimensional elasticity. The next Theorem provides a more strict sufficient condition.

THEOREM 4.5 ([**4**]). *Let Ω be a domain in $\mathbb{R}^n$. If*

$$(1-2/p)^2 \leq \begin{cases} \dfrac{1-2\nu}{2(1-\nu)} & \text{if } \nu < 1/2 \\ \\ \dfrac{2(1-\nu)}{1-2\nu} & \text{if } \nu > 1. \end{cases}$$

the operator (4.1) *is L^p-dissipative.*

We have also a kind of weighted L^p-negativity of elasticity system defined on rotationally symmetric vector functions.

Let Φ be a point on the $(n-2)-$dimensional unit sphere S^{n-2} with spherical coordinates $\{\vartheta_j\}_{j=1,\dots,n-3}$ and φ, where $\vartheta_j \in (0,\pi)$ and $\varphi \in [0,2\pi)$. A point $x\in\mathbb{R}^n$ is represented as a triple (ϱ,ϑ,Φ), where $\varrho>0$ and $\vartheta\in[0,\pi]$. Correspondingly, a vector u can be written as $u=(u_\varrho,u_\vartheta,u_\Phi)$ with $u_\Phi=(u_{\vartheta_{n-3}},\dots,u_{\vartheta_1},u_\varphi)$. We call $u_\varrho, u_\vartheta, u_\Phi$ the spherical components of the vector u.

THEOREM 4.6 ([**4**]). *Let the spherical components u_ϑ and u_Φ of the vector u vanish, i.e. $u=(u_\varrho,0,0)$, and let u_ϱ depend only on the variable ϱ. Then, if $\alpha \geq n-2$, we have*

$$\int_{\mathbb{R}^n} \left(\Delta u + (1-2\nu)^{-1}\nabla \operatorname{div} u\right) |u|^{p-2} u \, \frac{dx}{|x|^\alpha} \leq 0$$

for any $u\in (C_0^\infty(\mathbb{R}^n\setminus\{0\}))^n$ satisfying the aforesaid symmetric conditions, if and only if

$$-(p-1)(n+p'-2) \leq \alpha \leq n+p-2.$$

If $\alpha < n-2$ the same result holds replacing $(C_0^\infty(\mathbb{R}^n\setminus\{0\}))^n$ by $(C_0^\infty(\mathbb{R}^n))^n$.

5. The L^p-dissipativity of the "complex oblique derivative" operator

In this section we consider the L^p-dissipativity of the "complex oblique derivative" operator, i.e. of the boundary operator

$$\lambda\cdot\nabla u = \frac{\partial u}{\partial x_n} + \sum_{j=1}^{n-1} a_j \frac{\partial u}{\partial x_j} , \tag{5.1}$$

the coefficients a_j being L^∞ complex valued functions defined on $\mathbb{R}^{n-1}$.

We start with a Lemma, in which we use the concept of multiplier (see [**12**]). In particular we consider the space - we denote by $\mathscr{M}$ - of the multipliers acting form $H^{1/2}(\mathbb{R}^{n-1})$ into itself. Necessary and sufficient conditions for a function to be a multiplier and equivalent expressions for the relevant norms are given in [**12**] (see, in particular, Theorem 4.1.1, p.134).

LEMMA 5.1. *Let $a = (a_1, \dots, a_{n-1})$ be a vector multiplier belonging to $\mathscr{M}$. We have*

$$\left| \int_{\mathbb{R}^{n-1}} a_j\, f\, \partial_j g\, dx' \right| \leq \|a\|_{\mathscr{M}} \|\nabla f\|_{L^2(\mathbb{R}^n_+)} \|\nabla g\|_{L^2(\mathbb{R}^n_+)}$$

for any $f, g \in H^1(\mathbb{R}^n_+)$, $j = 1, \dots, n-1$. Here the derivatives are understood in the sense of distributions.

PROOF. Let us denote by Λ the operator $\sqrt{-\Delta}$ and write

$$\int_{\mathbb{R}^{n-1}} a_j\, f\, \partial_j g\, dx' = \int_{\mathbb{R}^{n-1}} \Lambda^{1/2}(a_j\, f)\, \Lambda^{-1/2}(\partial_j g)\, dx' .$$

We have

$$\left| \int_{\mathbb{R}^{n-1}} a_j\, f\, \partial_j g\, dx' \right| \leq \|\Lambda^{1/2}(a_j\, f)\|_{L^2(\mathbb{R}^{n-1})} \|\Lambda^{-1/2}(\partial_j g)\|_{L^2(\mathbb{R}^{n-1})} \leq$$
$$\|a\|_{\mathscr{M}} \|\Lambda^{1/2} f\|_{L^2(\mathbb{R}^{n-1})} \|\Lambda^{1/2} g\|_{L^2(\mathbb{R}^{n-1})} \leq \|a\|_{\mathscr{M}} \|\nabla f\|_{L^2(\mathbb{R}^n_+)} \|\nabla g\|_{L^2(\mathbb{R}^n_+)} .$$

□

The next Theorem provides a sufficient condition for the L^p-dissipativity of operator (5.1) under the assumption that

$$\| \mathscr{I}m\, a\|_{\mathscr{M}} < \frac{4}{p\, p'} . \tag{5.2}$$

THEOREM 5.2. *Suppose condition* (5.2) *is satisfied. If there exists a real vector $\Gamma \in L^2_{loc}(\mathbb{R}^n)$ such that*

$$-\partial_j(\mathscr{R}e\, a_j)\, \delta(x_n) \leq \frac{p}{2} \left(\frac{4}{p\, p'} - \| \mathscr{I}m\, a\|_{\mathscr{M}} \right) (\operatorname{div} \Gamma - |\Gamma|^2) \tag{5.3}$$

in $\mathbb{R}^n$, in the sense of distributions, then the operator (5.1) *is L^p-dissipative.*

PROOF. It is well known that $-\partial/\partial_{x_n} u(x', 0) = \Lambda(u)$ where, as in the previous Lemma, $\Lambda = \sqrt{-\Delta}$. This permits us to introduce the sesquilinear form

$$\mathscr{L}(u, v) = -\int_{\mathbb{R}^{n-1}} \langle \Lambda^{1/2} u, \Lambda^{1/2} v\rangle\, dx' + \int_{\mathbb{R}^{n-1}} \langle \sum_{j=1}^{n-1} a_j \partial_j u, v\rangle\, dx' . \tag{5.4}$$

We say that the operator $\lambda \cdot \nabla$ is L^p-dissipative if conditions (2.1) and (2.2) are satisfied for any $u \in C^1_0(\mathbb{R}^{n-1})$, the form $\mathscr{L}$ being given by (5.4).

Suppose $p \geq 2$. Denote by U the harmonic extension of u, i.e.

$$U(x', x_n) = \frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} u(y')\, \frac{x_n}{(|x' - y'|^2 + x_n^2)^{n/2}}\, dy'$$

ω_n being the measure of the unit sphere in $\mathbb{R}^n$.

Integrating by parts we get

$$\mathscr{R}e\, \mathscr{L}(u, |u|^{p-2} u) =$$
$$\mathscr{R}e \int_{\mathbb{R}^{n-1}} \frac{\partial U}{\partial x_n}\, |u|^{p-2} \overline{u}\, dx' + \mathscr{R}e \int_{\mathbb{R}^{n-1}} \sum_{j=1}^{n-1} a_j \frac{\partial u}{\partial x_j}\, |u|^{p-2} \overline{u}\, dx' =$$
$$-\mathscr{R}e \int_{\mathbb{R}^n_+} \nabla U \cdot \nabla(|U|^{p-2} \overline{U})\, dx + \mathscr{R}e \int_{\mathbb{R}^{n-1}} \sum_{j=1}^{n-1} a_j \frac{\partial u}{\partial x_j}\, |u|^{p-2} \overline{u}\, dx' .$$

Therefore, for $p \geq 2$, the operator $\lambda \cdot \nabla$ is L^p-dissipative if and only if

$$\mathscr{R}e \int_{\mathbb{R}^{n-1}} \sum_{j=1}^{n-1} a_j \frac{\partial u}{\partial x_j} |u|^{p-2} \overline{u}\, dx' \leq \mathscr{R}e \int_{\mathbb{R}^n_+} \nabla U \cdot \nabla(|U|^{p-2}\overline{U})\, dx \tag{5.5}$$

for any $u \in C_0^1(\mathbb{R}^{n-1})$, U being the harmonic extension of u to $\mathbb{R}^n_+$.

Setting $V = |U|^{(p-2)/2} U$ and $v = |u|^{(p-2)/2} u$, we get

$$|u|^{p-2}\overline{u}\, \partial_j u = -(1-2/p)\, |v|\, \partial_j |v| + \overline{v}\, \partial_j v$$

and then

$$\mathscr{R}e(|u|^{p-2}\overline{u}\, \partial_j u) = -(1-2/p)\, |v|\, \partial_j |v| + \mathscr{R}e(\overline{v}\, \partial_j v) = \frac{1}{p} \partial_j(|v|^2).$$

With similar computations we find

$$\mathscr{R}e(\nabla U \cdot \nabla(|U|^{p-2}\overline{U})) = |\nabla V|^2 - (1-2/p)^2 |\, \nabla |V|\, |^2$$

Inequality (5.5) becomes

$$\begin{aligned} -\frac{1}{p} \int_{\mathbb{R}^{n-1}} \partial_j(\mathscr{R}e\, a_j)\, |v|^2\, dx' - \int_{\mathbb{R}^{n-1}} \mathscr{I}m\, a_j\ \mathscr{I}m(\overline{v}\, \partial_j v)\, dx' \leq \\ \int_{\mathbb{R}^n_+} (|\nabla V|^2 - (1-2/p)^2 |\, \nabla|V|\, |^2)\, dx\, . \end{aligned} \tag{5.6}$$

Lemma 5.1 implies that

$$\left| \int_{\mathbb{R}^{n-1}} \mathscr{I}m\, a_j\ \mathscr{I}m(\overline{v}\, \partial_j v)\, dx' \right| \leq \|\, \mathscr{I}m\, a\|_{\mathscr{M}} \int_{\mathbb{R}^n_+} |\nabla V|^2 dx \tag{5.7}$$

On the other hand, inequality (5.3) is the necessary and sufficient condition for the validity of the inequality

$$-\int_{\mathbb{R}^{n-1}} \partial_j(\mathscr{R}e\, a_j)\, |v|^2\, dx' \leq \frac{p}{2} \left(\frac{4}{p\, p'} - \|\, \mathscr{I}m\, a\|_{\mathscr{M}} \right) \int_{\mathbb{R}^n} |\nabla V|^2\, dx \tag{5.8}$$

for any $V \in C_0^\infty(\mathbb{R}^n)$, v being the restriction of V on $\mathbb{R}^{n-1}$ (see [**8**, Th. 5.1]). In particular, we find

$$-\int_{\mathbb{R}^{n-1}} \partial_j(\mathscr{R}e\, a_j)\, |v|^2\, dx' \leq p \left(\frac{4}{p\, p'} - \|\, \mathscr{I}m\, a\|_{\mathscr{M}} \right) \int_{\mathbb{R}^n_+} |\nabla V|^2\, dx \tag{5.9}$$

for any function $V \in C_0^\infty(\mathbb{R}^n)$ which is even with respect to x_n. Since $|\, \nabla |V|\, | \leq |\nabla V|$ and $1 - (1-2/p)^2 = 4/(p\, p')$, we have also

$$\frac{4}{p\, p'} \int_{\mathbb{R}^n_+} |\nabla V|^2\, dx \leq \int_{\mathbb{R}^n_+} (|\nabla V|^2 - (1-2/p)^2 |\, \nabla |V|\, |^2)\, dx\, .$$

This inequality, together with (5.7) and (5.9), show that (5.6) holds and the operator $\lambda \cdot \nabla$ is L^p-dissipative.

If $1 < p < 2$ we have to show that

$$\mathscr{R}e\, \mathscr{L}(|u|^{p'-2}u, u) \leq 0$$

for any $u \in C_0^1(\mathbb{R}^{n-1})$.

Arguing as for (5.5) we find that the operator $\lambda \cdot \nabla$ is L^p-dissipative if and only if

$$\mathscr{R}e \int_{\mathbb{R}^{n-1}} \sum_{j=1}^{n-1} a_j \frac{\partial (|u|^{p'-2}u)}{\partial x_j}\, \overline{u}\, dx' \leq \mathscr{R}e \int_{\mathbb{R}^n_+} \nabla(|U|^{p'-2}U) \cdot \nabla \overline{U}\, dx\, .$$

Setting $V=|V|^{p'-2}V$ and $v=|u|^{p'-2}v$, we have

$$\mathscr{R}e(|u|^{p'-2}u\,\partial_j\overline{u})=(1-2/p')\,|v|\,\partial_j|v|+\mathscr{R}e(v\,\partial_j\overline{v})=\frac{1}{p}\partial_j(|v|^2),$$
$$\mathscr{R}e(\nabla(|U|^{p'-2}U)\cdot\nabla\overline{U})=|\nabla V|^2-(1-2/p)^2|\,\nabla|V|\,|^2.$$

Therefore the operator $\lambda\cdot\nabla$ is L^p-dissipative if and only if (5.6) holds and the proof proceeds as in the case $p\geq 2$. □

The next Theorem provides a necessary condition, similar to the previous one, but which contains a different constant. We remark that in the next Theorem we do not require the smallness of $\|\,\mathscr{I}m\,a\|_{\mathscr{M}}$.

Theorem 5.3. *If the operator $\lambda\cdot\nabla$ is L^p-dissipative, then there exists a real vector $\Gamma\in L^2_{loc}(\mathbb{R}^n)$ such that we have the inequality*

$$-\partial_j(\mathscr{R}e\,a_j)\,\delta(x_n)\leq\frac{p}{2}\,(1+\|\,\mathscr{I}m\,a\|_{\mathscr{M}})\,(\operatorname{div}\Gamma-|\Gamma|^2)$$

in the sense of distributions.

Proof. If $\lambda\cdot\nabla$ is L^p-dissipative, we have the inequality (5.5). Keeping in mind (5.7) we find

$$-\frac{1}{p}\int_{\mathbb{R}^{n-1}}\partial_j(\mathscr{R}e\,a_j)\,|v|^2\,dx'\leq\int_{\mathbb{R}^n_+}(|\nabla V|^2-(1-2/p)^2|\,\nabla|V|\,|^2)\,dx+$$
$$\int_{\mathbb{R}^{n-1}}\mathscr{I}m\,a_j\;\mathscr{I}m(\overline{v}\,\partial_j v)\,dx'\leq(1+\|\,\mathscr{I}m\,a\|_{\mathscr{M}})\int_{\mathbb{R}^n_+}|\nabla V|^2\,dx\,.$$

Let us now consider $V\in C_0^\infty(\mathbb{R}^n)$ and write $V=V_o+V_e$, where V_o and V_e are odd and even respectively. The last inequality we have written leads to

$$-\frac{1}{p}\int_{\mathbb{R}^{n-1}}\partial_j(\mathscr{R}e\,a_j)\,|v|^2\,dx'\leq\frac{1+\|\,\mathscr{I}m\,a\|_{\mathscr{M}}}{2}\int_{\mathbb{R}^n}|\nabla V_o|^2\,dx\leq$$
$$\frac{1+\|\,\mathscr{I}m\,a\|_{\mathscr{M}}}{2}\int_{\mathbb{R}^n}|\nabla V|^2\,dx$$

and the thesis follows from the already quoted result [**8**, Th. 5.1].

□

A different sufficient condition can be obtained by using the concept of capacity (see (2.13)).

Theorem 5.4. *Suppose condition* (5.2) *is satisfied. If*

$$-\frac{1}{\operatorname{cap}_\Omega(F)}\int_{F\cap\mathbb{R}^{n-1}}\partial_j(\mathscr{R}e\,a_j)\,dx'\leq\frac{p}{8}\left(\frac{4}{p\,p'}-\|\,\mathscr{I}m\,a\|_{\mathscr{M}}\right)\tag{5.10}$$

for all compact sets $F\subset\mathbb{R}^n$, then the operator $\lambda\cdot\nabla$ is L^p-dissipative.

Proof. We know that inequality (2.10) holds for any test function w if condition (2.11) is satisfied for all compact sets $F\subset\Omega$. As remarked in [**11**, Remark 5.2], (2.11) implies (2.10) even without the requirement that $\mu\geq 0$, i.e. μ can be an arbitrary locally finite real valued charge.

Therefore, condition (5.10) implies inequality (5.8) and, as in Theorem 5.2, the result follows. □

In the case of the real oblique derivative problem we have a necessary and sufficient condition.

THEOREM 5.5. *Let us suppose that the coefficients a_j in* (5.1) *are real valued* ($j = 1, \dots, n-1$)*. The operator $\lambda \cdot \nabla$ is L^p-dissipative if and only if there exists a real vector $\Gamma \in L^2_{loc}(\mathbb{R}^n)$ such that*

$$-\partial_j(\mathscr{R}e\, a_j)\, \delta(x_n) \leq \frac{2}{p'}(\operatorname{div}\Gamma - |\Gamma|^2) \tag{5.11}$$

in the sense of distributions.

PROOF. The operator $\lambda \cdot \nabla$ being real, we consider the conditions (2.1) and (2.2) for real valued functions. We remark that, if V is a real valued function, we have $|\nabla V| = |\nabla |V||$ and then

$$\int_{\mathbb{R}^n_+} (|\nabla V|^2 - (1-2/p)^2 |\nabla |V||^2)\, dx = \frac{4}{p\,p'} \int_{\mathbb{R}^n_+} |\nabla V|^2 dx\,.$$

Therefore the L^p-dissipativity of $\lambda \cdot \nabla$ occurs if and only if

$$-\frac{1}{p} \int_{\mathbb{R}^{n-1}} \partial_j(\mathscr{R}e\, a_j)\, |v|^2\, dx' \leq \frac{4}{p\,p'} \int_{\mathbb{R}^n_+} |\nabla V|^2 dx\,. \tag{5.12}$$

Arguing as in the proof of Theorem 5.2, we find that (5.12) holds if and only if

$$-\int_{\mathbb{R}^{n-1}} \partial_j(\mathscr{R}e\, a_j)\, |v|^2\, dx' \leq \frac{2}{p'} \int_{\mathbb{R}^n} |\nabla V|^2 dx\,.$$

for any $V \in C^1_0(\mathbb{R}^n)$. Appealing again to [**8**, Th. 5.1], we see that this inequality holds if and only if condition (5.11) is satisfied. $\square$

6. L^p-positivity of certain integral operators

The aim of this section is to prove the L^p-positivity of the operator

$$Tu(E) = \int_E \int^*_{\mathbb{R}^n} [u(x) - u(y)]\, K(dx, dy)\,. \tag{6.1}$$

Here E is a Borel set in $\mathbb{R}^n$, $K(dx, dy)$ is a nonnegative Borel measure on $\mathbb{R}^n \times \mathbb{R}^n$, locally finite outside the diagonal $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid x = y\}$ and the function u belongs to $C^1_0(\mathbb{R}^n)$.

The integral in (6.1) has to be understood as a principal value in the sense of Cauchy

$$\lim_{\varepsilon \to 0^+} \int_E \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} [u(x) - u(y)]\, K(dx, dy)$$

and we assume that such singular integral does exist for any $u \in C^1_0(\mathbb{R}^n)$.

In what follows we shall make also the following assumptions on the kernel $K(dx, dy)$:

(i) $K(E, F) = K(F, E)$ for any Borel sets $E, F \subset \mathbb{R}^n$;
(ii) for any compact set $E \subset \mathbb{R}^n$ we have

$$\iint_{E \times E} |x - y|^2 K(dx, dy) < \infty\,;$$

(iii) for any $R > 0$ we have
$$\int_{|x|<R} \int_{|y-x|>2R} K(dx, dy) < \infty\, .$$

As an example, consider the measure $K(dx, dy) = |x-y|^{-n-2s} dxdy$ $(0 < s < 1)$; it satisfies all the previous conditions and in this case the operator T coincides, up to a constant factor, to the fractional power of Laplacian $(-\Delta)^s$ (see, e.g., [**5**, p.230]).

As in (2.1)-(2.2), we say that T is L^p-positive if

$$\begin{aligned}
&\int_{\mathbb{R}^n} \langle Tu, |u|^{p-2}u\rangle \geq 0, \quad \text{if } p \geq 2,\\
&\int_{\mathbb{R}^n} \langle T(|u|^{p'-2}u), u\rangle \geq 0, \quad \text{if } 1 < p < 2,
\end{aligned} \tag{6.2}$$

for any $u \in C^1_0(\mathbb{R}^n)$.

Theorem 6.1. *Let $K(dx, dy)$ be a kernel satisfying the previous conditions. Then the operator* (6.1) *is L^p-positive. More precisely we have the inequalities*
(6.3)
$$\int_{\mathbb{R}^n} \langle Tu, |u|^{p-2}u\rangle \geq \frac{4}{p\,p'} \iint_{\mathbb{R}^n\times\mathbb{R}^n} (|u(y)|^{p/2} - |u(x)|^{p/2})^2 K(dx, dy)\, , \textit{ if } p \geq 2;$$
$$\int_{\mathbb{R}^n} \langle T(|u|^{p'-2}u), u\rangle \geq \frac{4}{p\,p'} \iint_{\mathbb{R}^n\times\mathbb{R}^n} (|u(y)|^{p'/2} - |u(x)|^{p'/2})^2 K(dx, dy)\, , \textit{ if } 1 < p < 2.$$

Proof. Let us observe that, in view of (i), we may write
$$\int_{\mathbb{R}^n} \langle Tu, v\rangle = \frac{1}{2} \iint_{\mathbb{R}^n\times\mathbb{R}^n} [u(x) - u(y)][v(x) - v(y)]\, K(dx, dy) \tag{6.4}$$
for any $u, v \in C^1_0(\mathbb{R}^n)$. In fact, since u and v have compact support and thanks to conditions (ii) and (iii), the integral in the right hand side of (6.4) is absolutely convergent. Now we may appeal to the dominated convergence Theorem to obtain (6.4).

Let $p \geq 2$ and consider
$$\int_{\mathbb{R}^n} \langle Tu, |u|^{p-2}u\rangle =$$
$$\frac{1}{2} \iint_{\mathbb{R}^n\times\mathbb{R}^n} [u(x) - u(y)][|u(x)|^{p-2}u(x) - |u(y)|^{p-2}u(y)]\, K(dx, dy)$$
for any $u \in C^1_0(\mathbb{R}^n)$. Note that in this case $|u|^{p-2}u \in C^1_0(\mathbb{R}^n)$. Since
$$(x - y)(|x|^{p-2}x - |y|^{p-2}y) \geq \frac{4}{p\,p'}\, (|x|^{p/2} - |y|^{p/2})^2$$
for any $x, y \in \mathbb{R}$ (see [**5**, p.231]), we have that (6.2) holds for any $u \in C^1_0(\mathbb{R}^n)$.

If $1 < p < 2$, the second condition in (6.2) can be written as
$$\int_{\mathbb{R}^n} \langle Tv, |v|^{p'-2}v\rangle \geq 0$$
for any $v \in C^1_0(\mathbb{R}^n)$ and the result follows as in the case $p \geq 2$ already considered. □

As a Corollary, we have that under an additional condition, we have a lower estimate involving a Besov semi-norm.

Corollary 6.2. *Let the kernel $K(dx,dy)$ satisfy the conditions of Theorem 6.1. Moreover suppose that there exist $C>0$ and $s\in(0,1)$ such that*

$$K(dx,dy)\geq C\frac{dxdy}{|x-y|^{n+2s}}\quad \text{on } \mathbb{R}^n\times\mathbb{R}^n\,. \tag{6.5}$$

Then we have

$$\int_{\mathbb{R}^n}\langle Tu,u\rangle\,|u|^{p-2}\geq\frac{2C}{p\,p'}\,\|\,|u|^{p/2}\|^2_{\mathcal{L}^{s,2}(\mathbb{R}^n)}\,,\quad \text{if } p\geq 2;$$
$$\int_{\mathbb{R}^n}\langle T(|u|^{p'-2}u),u\rangle\geq\frac{2C}{p\,p'}\,\|\,|u|^{p'/2}\|^2_{\mathcal{L}^{s,2}(\mathbb{R}^n)}\,,\quad \text{if } 1<p<2,$$

where

$$\|v\|_{\mathcal{L}^{s,2}(\mathbb{R}^n)}=\left(\iint_{\mathbb{R}^n\times\mathbb{R}^n}|v(y)-v(x)|^2\frac{dxdy}{|y-x|^{n+2s}}\right)^{1/2}.$$

Proof. The result follows immediately from (6.3) and (6.5). □

We conclude with a remark. In Sections 5 and 6, the space $C_0^1(\mathbb{R}^n)$ was the class of admissible functions. Actually, in these cases, we could extend this class and consider more general functions like, for example, compactly supported Lipschitz functions or even bounded functions in proper Sobolev spaces.

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Dipartimento di Matematica, Economia ed Informatica, University of Basilicata, V.le dell'Ateneo Lucano 10, 85100, Potenza, Italy
E-mail address: cialdea@email.it

Department of Mathematical Sciences, M& O Building, University of Liverpool, Liverpool L69 7ZL, UK. Department of Mathematics, Linköping University, SE-581 83, Linköping, Sweden. RUDN University, 6 Miklukho-Maklay St, Moscow, 117198, Russia.
E-mail address: vladimir.mazya@liu.se