Global gradient estimates in elliptic problems under minimal data and domain regularity

Andrea Cianchi
Dipartimento di Matematica e Informatica "U.Dini", Università di Firenze
Piazza Ghiberti 27, 50122 Firenze, Italy

Vladimir Maz'ya
Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden
and
Department of Mathematical Sciences, M&O Building
University of Liverpool, Liverpool L69 3BX, UK

1 Introduction

The present paper is mainly devoted to report on some contributions by the authors on integrability properties of the gradient of solutions to boundary value problems for nonlinear elliptic equations in divergence form. This is a very classical issue in the theory of partial differential equations, whose modern developments can be traced back to the work of several authors in the late fifties and early sixties of the last century. We do not attempt to provide an even partial account of the vast bibliography on this topic. We just refer to the the monographs [BF, Gia, GT, Gi, LU, Mo] and to the recent survey paper [Mi].

The leitmotif of our investigations is the aim at assumptions on the regularity of the ground domain and of the prescribed data, which are minimal, in a sense, for a certain gradient bound to hold. A distinctive feature of our approach is in the derivation of estimates which are flexible enough to be applied in the proof of gradient bounds for a wide choice of norms. Most of the relevant estimates are formulated in terms of pointwise inequalities for the distribution function of the length of the gradient, or, equivalently, for its decreasing rearrangement. With this tool at disposal, global bounds for any rearrangement invariant norm of the gradient of solutions to either Dirichlet or Neumann boundary value problems are simply reduced to one-dimensional inequalities for Hardy type operators. The latter depend both on the class of elliptic differential operators under consideration, and on the regularity of the domain.

The first set of results to be presented is focused on boundary value problems involving nonlinear differential operators with a quite general structure, and need not enjoy additional smoothness properties. When dealing with this kind of problems, no (uniform) gradient integrability can be expected beyond the natural energy level dictated by the nonlinearity of the problem. Moreover, domain regularity only plays a role when Neumann boundary conditions are imposed, and can effectively be prescribed in terms of either of two functions - the isoperimetric

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function and the isocapacitary function - which are associated with the domain and reflect some of its geometric-functional properties. These functions were introduced in [Ma1, Ma3] to provide necessary and sufficient conditions for the validity of Sobolev type embeddings.
The use of isoperimetric inequalities in the study of Dirichlet and Neumann problems for (linear) elliptic equations was initiated in [Ma2, Ma3], where various a priori estimates for solutions and unique solvability results were obtained under weak assumptions on coefficients, data and domains. The standard isoperimetric inequality in $\mathbb{R}^n$ was exploited in [Ta1] and [Ta2] to establish symmetrization comparison principles for linear and nonlinear, respectively, elliptic Dirichlet problems. This line of research has subsequently been developed in a rich literature; results and references an this topic can be found e.g. in the books [Ka, Ke], and in the survey paper [Tr]. After recalling and rephrasing some fundamental results along this direction, we describe an alternative approach in the analysis of Neumann problems, which exploits the isocapacitary function. Such an approach can actually lead to stronger results than those obtained via the isoperimetric function when domains with complicated geometric configurations are taken into account.

The second part of our survey deals with boundary value problems for differential operators having a special structure and some smoothness, which allows for higher gradient integrability. Under sharp integrability assumptions on the curvature of the boundary of the domain, a rearrangement estimate for the gradient of solutions, both to Dirichlet and Neumann problems, is exhibited. Noticeably, this estimate provides a pointwise bound for a power of the rearrangement of the gradient which depends linearly on the rearrangement of the datum. As a consequence, it translates verbatim the linear theory of integrability of the gradient of solutions to boundary value problems for the Laplace equation to a parallel theory for nonlinear problems.
As an intermediate step, of independent interest, we show an L^∞ bound for the gradient under minimal integrability assumptions on the right-hand side of the equation. We emphasize that the bound in question turns out to hold also for systems.

2 Part I: equations with general nonlinearities

This section is devoted to equations subject to customary ellipticity conditions, without any extra smoothness assumption. Specifically, we deal with equations of the form

$$-\operatorname{div}(a(x,\nabla u)) = f(x) \quad \text{in } \Omega\,, \tag{2.1}$$

where Ω is a domain, namely a connected open set in $\mathbb{R}^n$, $n \geq 2$, having finite Lebesgue measure $|\Omega|$, f is an integrable function, and $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function such that, for a.e. $x \in \Omega$,

$$a(x,\xi)\cdot\xi \geq |\xi|^p \quad \text{for } \xi \in \mathbb{R}^n\,, \tag{2.2}$$

for some exponent $p > 1$. Although no additional hypothesis on the function $a(x,\xi)$ is really needed for our estimates, in order to ensure that suitable notions of solutions are meaningful we also assume that a function $h \in L^{p'}(\Omega)$ and a constant C exist such that, for a.e. $x \in \Omega$,

$$|a(x,\xi)| \leq C|\xi|^{p-1} + h(x) \quad \text{for } \xi \in \mathbb{R}^n\,. \tag{2.3}$$

Here, $p' = \frac{p}{p-1}$, the Hölder conjugate of p.
The conclusions that can be derived on gradient summability properties of solutions to boundary value problems for equation (2.1) depend on whether Dirichlet or Neumann conditions are imposed. In the former case, only the measure of Ω is relevant, whereas in the latter case its regularity plays a role as well.

2.1 Dirichlet problems

Here, we are concerned with Dirichlet problems obtained by coupling equation (2.1) with homogeneous boundary conditions, namely

$$
\begin{cases} -\mathrm{div}(a(x,\nabla u)) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \, . \end{cases} \tag{2.4}
$$

If f has a sufficiently strong degree of integrability to belong to $W_0^{1,p}(\Omega)^*$, the dual of the Sobolev space $W_0^{1,p}(\Omega)$, then a weak solution $u \in W_0^{1,p}(\Omega)$ to problem (2.4) is well defined by requiring that

$$
\int_\Omega a(x,\nabla u) \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx \tag{2.5}
$$

for every $\varphi \in W_0^{1,p}(\Omega)$. For instance, by the standard Sobolev embedding theorem for $W_0^{1,p}(\Omega)$, this is certainly the case when $f \in L^q(\Omega)$ with $q \geq \frac{np}{np-n+p}$ if $p < n$, $q > 1$ if $p = n$, and $q \geq 1$ if $p > n$. Under the additional strict monotonicity assumption that

$$
[a(x,\xi) - a(x,\eta)] \cdot (\xi - \eta) > 0 \quad \text{for } \xi, \eta \in \mathbb{R}^n \text{ with } \xi \neq \eta \, , \tag{2.6}
$$

the existence and uniqueness of a weak solution to problem (2.4) is well known, and follows via the Browder-Minthy theory of monotone operators.
On the other hand, if $f \notin W_0^{1,p}(\Omega)^*$, then solutions to (2.4) can still be considered, but have to be interpreted in a suitable generalized sense. As shown by classical examples [Se], allowing solutions just in the distributional sense may lead to pathological, non-uniqueness phenomena. The notion of entropy solution, introduced in [BBGGPV], turns out to be well suited for our framework. Its definition is briefly recalled below. Let us mention that, a posteriori, such a definition turns out to be equivalent to other definitions (such as renormalized solutions [M1, M2], and solutions obtained as limits of approximations [DaA]) available in the literature.
Given any $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2$, let $T_{t_1,t_2} : \mathbb{R} \to \mathbb{R}$ be the function defined as

$$
T_{t_1,t_2}(s) = \begin{cases} t_1 & \text{if } s < t_1 \\ s & \text{if } t_1 \leq s \leq t_2 \\ t_2 & \text{if } t_2 < s \, . \end{cases} \tag{2.7}
$$

For $p \geq 1$, set

$$
W_{0,T}^{1,p}(\Omega) = \{u \text{ is a measurable function in } \Omega\text{: } T_{-t,t}(u) \in W_0^{1,p}(\Omega) \text{ for } t > 0\} \, . \tag{2.8}
$$

Then [BBGGPV, Lemma 2.1] ensures that, for every $u \in W_{0,T}^{1,p}(\Omega)$, a unique measurable function $V_u : \Omega \to \mathbb{R}^n$ exists such that

$$
\nabla\big(T_{-t,t}(u)\big) = V_u \chi_{\{|u|<t\}} \qquad \text{a.e. in } \Omega \tag{2.9}
$$

for every $t > 0$, where χ_E denotes the characteristic function of the set E. Furthermore, a function $u \in W_{0,T}^{1,p}(\Omega)$ belongs to $W_0^{1,p}(\Omega)$ if and only if $u \in L^p(\Omega)$ and $|V_u| \in L^p(\Omega)$. In this case, $V_u = \nabla u$, the weak gradient of u. In what follows, with abuse of notation, for every $u \in W_{0,T}^{1,p}(\Omega)$ we denote V_u by ∇u.

A function $u \in W^{1,p}_{0,T}(\Omega)$ is called an entropy solution to the Dirichlet problem (2.4) if

$$\int_\Omega a(x,u,\nabla u)\cdot \nabla\big(T_{t_1,t_2}(u-\varphi)\big)\,dx \leq \int_\Omega f(x)T_{t_1,t_2}(u-\varphi)\,dx \tag{2.10}$$

for every $\varphi \in W^{1,p}_0(\Omega)\cap L^\infty(\Omega)$ and every $t_1 < t_2$.
For any $f \in L^1(\Omega)$ there exists a unique (entropy) solution u to the Dirichlet problem (2.4) [BBGGPV, Theorem 6.1]. Let us warn that such a solution need not be a weakly differentiable function in general, and, even if it is so, its weak gradient need to belong to the energy space $L^p(\Omega)$ if f has not a sufficiently high integrability degree.

Uniform estimates for norms of ∇u stronger than $\|\cdot\|_{L^p(\Omega)}$ cannot hold without further assumptions on the function $a(x,\xi)$ (in addition to (2.2)), and on Ω. This fact is well known even in the case of linear equations [Me]. Thus, when $f \in W^{1,p}_0(\Omega)^*$, the solution u to (2.4) is in fact a weak solution, and hence $|\nabla u| \in L^p(\Omega)$; thus, there is not much to add as far as the integrability of $|\nabla u|$ is concerned. On the other hand, if $f \notin W^{1,p}_0(\Omega)^*$, the question arises of how the integrability degree of f is reflected by that of $|\nabla u|$. This question can be posed in the framework of all rearrangement-invariant (quasi-)norms, in a sense the widest class of (quasi-)norms which only depend on integrability properties of functions.

Let us briefly recall a few basic facts on this function spaces setting. A quasi-normed function space $X(\Omega)$ on Ω is a linear space of measurable functions on Ω equipped with a functional $\|\cdot\|_{X(\Omega)}$, a quasi-norm, enjoying the following properties:

(i) $\|g\|_{X(\Omega)} > 0$ if $g \neq 0$;
$\|\lambda g\|_{X(\Omega)} = |\lambda|\|g\|_{X(\Omega)}$ for every $\lambda \in \mathbb{R}$ and $g \in X(\Omega)$;
$\|g_1+g_2\|_{X(\Omega)} \leq c(\|g_1\|_{X(\Omega)}+\|g_2\|_{X(\Omega)})$ for some constant $c \geq 1$ and for every $g_1, g_2 \in X(\Omega)$;

(ii) $0 \leq |g_1| \leq |g_2|$ a.e. in Ω implies $\|g_1\|_{X(\Omega)} \leq \|g_2\|_{X(\Omega)}$;

(iii) $0 \leq g_k \nearrow g$ a.e. implies $\|g_k\|_{X(\Omega)} \nearrow \|g\|_{X(\Omega)}$ as $k \to \infty$;

(iv) if G is a measurable subset of Ω and $|G| < \infty$, then $\|\chi_G\|_{X(\Omega)} < \infty$;

(v) for every measurable subset G of Ω with $|G| < \infty$, there exists a constant C such that $\int_G |g|dx \leq C\|g\|_{X(\Omega)}$ for every $g \in X(\Omega)$.

The space $X(\Omega)$ is called a Banach function space if (i) holds with $c = 1$. In this case, the functional $\|\cdot\|_{X(\Omega)}$ is actually a norm which renders $X(\Omega)$ a Banach space.
The decreasing rearrangement $g^* : [0,\infty) \to [0,\infty]$ of a measurable function g on Ω is defined as

$$g^*(s) = \sup\{t \geq 0 : |\{x \in \Omega : |g(x)| > t\}| > s\} \qquad \text{for } s \geq 0.$$

In other words, g^* is the (unique) non increasing, right-continuous function in $[0,\infty)$ which is equimeasurable with g. We also set $g^{**}(s) = \frac{1}{s}\int_0^s g^*(r)dr$ for $s > 0$, and observe that $g^* \leq g^{**}$, since g^* is non-increasing.
A quasi-normed function space (in particular, a Banach function space) $X(\Omega)$ is said to be rearrangement-invariant if there exists a quasi-normed function space $\overline{X}(0,|\Omega|)$ on the interval $(0,|\Omega|)$, called the representation space of $X(\Omega)$, having the property that

$$\|g\|_{X(\Omega)} = \|g^*\|_{\overline{X}(0,|\Omega|)} \tag{2.11}$$

for every $g \in X(\Omega)$. Obviously, if $X(\Omega)$ is a rearrangement-invariant quasi-normed space, then

$$\|g_1\|_{X(\Omega)} = \|g_2\|_{X(\Omega)} \qquad \text{if} \quad g_1^* = g_2^*. \tag{2.12}$$

Let us mention that, if $X(\Omega)$ is a Banach function space, property (2.12) is in fact equivalent to the existence of a representation space $\overline{X}(0,|\Omega|)$ fulfilling (2.11). For customary spaces $X(\Omega)$, an expression for the quasi-norm $\|\cdot\|_{\overline{X}(0,|\Omega|)}$ is immediately derived from that of $\|\cdot\|_{X(\Omega)}$, via elementary properties of rearrangements. Customary instances of rearrangement-invariant quasi-normed spaces are Lebesgue spaces, Lorentz spaces, Lorentz-Zygmund spaces, Orlicz spaces. Their definitions will be recalled below. We refer the reader to [BS] for a comprehensive treatment of rearrangement-invariant spaces.

A paradigmatic result which can give the flavor of the material collected in this paper is the following theorem from [Ta2].

Theorem 2.1 *Let u be the solution to the Dirichlet problem* (2.4)*. If $0 < r \leq p$, then*

$$\|\nabla u\|_{L^r(\Omega)} \leq (n\omega_n^{1/n})^{-\frac{1}{p-1}} \bigg(\int_0^{|\Omega|} s^{-\frac{r}{n'(p-1)}} \bigg(\int_0^s f^*(\rho)d\rho \bigg)^{\frac{r}{p-1}} ds \bigg)^{\frac{1}{r}}. \tag{2.13}$$

Here, $\omega_n = \pi^{n/2}/\Gamma(1+\frac{n}{2})$, the measure of the unit ball in $\mathbb{R}^n$.

Remark 2.2 Theorem 2.1 can be interpreted as a comparison principle between gradient norms of the solution u to problem (2.4), and the (radially symmetric) solution to a "symmetrized" problem. Indeed, the right-hand side of inequality (2.13) agrees with the L^r norm of the gradient of the (radially decreasing) solution to the p-Laplace equation in a ball, with the same measure as Ω, and with a right-hand side which is radially decreasing and whose decreasing rearrangement equals f^*.

Owing to Theorem 2.1, bounds for any norm $\|\nabla u\|_{L^r(\Omega)}$, with $r \leq p$, in terms of another rearrangement invariant quasi-norm of the datum f, are reduced to one-dimensional Hardy-type inequalities.

Corollary 2.3 *Let $0 < r \leq p$ and let $X(\Omega)$ be a rearrangement invariant quasi-normed space such that*

$$\bigg\| s^{-\frac{1}{n'(p-1)}} \bigg(\int_0^s \phi(\rho)d\rho \bigg)^{\frac{1}{p-1}} \bigg\|_{L^r(0,|\Omega|)} \leq C\|\phi\|_{\overline{X}(0,|\Omega|)}^{\frac{1}{p-1}}, \tag{2.14}$$

for some constant C and every nonnegative and non-increasing function $\phi \in \overline{X}(0,|\Omega|)$. If $f \in X(\Omega)$ and u is the solution to the Dirichlet problem (2.4)*, then*

$$\|\nabla u\|_{L^r(\Omega)} \leq C(n\omega_n^{1/n})^{-\frac{1}{p-1}} \|f\|_{X(\Omega)}^{\frac{1}{p-1}}. \tag{2.15}$$

Corollary 2.3 can be exploited, for instance, to derive estimates for Lebesgue norms of the gradient in terms of Lorentz norms of the datum. Recall that, if either $q \in (1,\infty]$ and $k \in (0,\infty]$, or $q = 1$ and $k \in (0,1]$, the Lorentz space $L^{q,k}(\Omega)$ is defined as the set of all measurable functions g on Ω for which the expression

$$\|g\|_{L^{q,k}(\Omega)} = \|s^{\frac{1}{q}-\frac{1}{k}} g^*(s)\|_{L^k(0,|\Omega|)} \tag{2.16}$$

is finite. On has that

$$L^{q,q}(\Omega) = L^q(\Omega)$$

for every $q \in [1, \infty]$. The space $L^{q,\infty}(\Omega)$ is called Marcinkiewicz space, or weak Lebesgue space. Moreover, $L^{q,k_1}(\Omega) \subsetneqq L^{q,k_2}(\Omega)$ if $k_1 < k_2$, and, $L^{q_1,k_1}_{\rm loc}(\Omega) \subsetneqq L^{q_2,k_2}_{\rm loc}(\Omega)$ if $q_1 > q_2$ and k_1, k_2 are admissible exponents in $(0, \infty]$.
If $q > 1$, then

$$\|s^{\frac{1}{q}-\frac{1}{k}} g^*(s)\|_{L^k(0,|\Omega|)} \approx \|s^{\frac{1}{q}-\frac{1}{k}} g^{**}(s)\|_{L^k(0,|\Omega|)}, \tag{2.17}$$

up to multiplicative constants depending on q and k. Furthermore, if either $q > 1$ and $k \in [1, \infty]$, or $q = k = 1$, then $L^{q,k}(\Omega)$ is in fact a Banach function space, up to equivalent norms.

The following result appears in [Ta1] in the case of linear equations, and in [AFT] in the nonlinear case.

Theorem 2.4 *Let $p \in (1, n)$, and let $1 < q \le \frac{np}{np-n+p}$. Let u be the solution to the Dirichlet problem* (2.4)*, with $f \in L^{q,\frac{nq}{n-q}}(\Omega)$. Then there exists a constant C such that*

$$\|\nabla u\|_{L^{\frac{nq(p-1)}{n-q}}(\Omega)} \le C \|f\|^{\frac{1}{p-1}}_{L^{q,\frac{nq}{n-q}}(\Omega)}.$$

In particular,

$$\|\nabla u\|_{L^{\frac{nq(p-1)}{n-q}}(\Omega)} \le C \|f\|^{\frac{1}{p-1}}_{L^q(\Omega)}.$$

Gradient regularity in Marcinkiewicz spaces, for $q = 1$, is established in [BBGGPV]. It also follows from a counterpart for Dirichlet problems of Proposition 2.13, Subsection 2.2, which deals with Neumann problems. The case when $p = n$ and $q = 1$ can be found in [DHM].

Bounds for more general rearrangement-invariant norms of $|\nabla u|$, still weaker than $\|\cdot\|_{L^p(\Omega)}$, can be established through the following pointwise estimate for $|\nabla u|^*$. The proof of such estimate exploits an argument similar to that used for [ACMM, Theorem 3.3], and somewhat enhances a result in the same direction from [AFT].

Theorem 2.5 *Let u be the solution to the Dirichlet problem* (2.4)*. Then there exists a constant $C = C(n,p)$ such that*

$$|\nabla u|^*(s) \le C(n,p) \left(\frac{1}{s} \int_{\frac{s}{2}}^{|\Omega|} \tau^{-\frac{p'}{n'}} \left(\int_0^\tau f^*(\rho)\, d\rho \right)^{p'} d\tau \right)^{\frac{1}{p}} \quad \text{for } s \in (0, |\Omega|). \tag{2.18}$$

Theorem 2.5 has an obvious corollary, by the very definition of rearrangement-invariant quasi-norm.

Corollary 2.6 *Let $X(\Omega)$ be a quasi-normed rearrangement-invariant space and let $f \in X(\Omega)$. Let u be the solution to the Dirichlet problem* (2.4)*. Assume that $Y(\Omega)$ is another quasi-normed rearrangement-invariant space such that*

$$\left\| \left(\frac{1}{s} \int_{\frac{s}{2}}^{|\Omega|} \tau^{-\frac{p'}{n'}} \left(\int_0^\tau \phi(\rho)\, d\rho \right)^{p'} d\tau \right)^{\frac{1}{p}} \right\|_{\overline{Y}(0,|\Omega|)} \le C \|\phi\|^{\frac{1}{p-1}}_{\overline{X}(0,|\Omega|)}, \tag{2.19}$$

for some constant C and every nonnegative and non-increasing function $\phi \in \overline{X}(0, |\Omega|)$. Then there exists a constant $C_1 = C_1(C, n, p)$ such that

$$\|\nabla u\|_{Y(\Omega)} \le C_1 \|f\|^{\frac{1}{p-1}}_{X(\Omega)}. \tag{2.20}$$

As an application of Corollary 2.6, let us mention a gradient bound for Lorentz norms – see [Ta1] (linear equations) and [AFT] (nonlinear equations).

Theorem 2.7 *Let* $1 < p < n$, $\max\{1, n/(np-n+k)\} < q < np/(np-n+p)$ *and* $0 < k < \infty$. *Let* u *be the solution to the Dirichlet problem* (2.4), *with* $f \in L^{q,k}(\Omega)$. *Then there exists a constant* C *such that*

$$\|\nabla u\|_{L^{\frac{nq(p-1)}{n-q},(p-1)k}(\Omega)} \le C\|f\|_{L^{q,k}(\Omega)}^{\frac{1}{p-1}}.$$

Let us mention that gradient estimates in Lorentz spaces for local solution to nonlinear elliptic problems are proved in [Mi].

2.2 Neumann problems

We focus here on the Neumann problem

$$\begin{cases} -\mathrm{div}(a(x,\nabla u)) = f(x) & \text{in } \Omega \\ a(x,\nabla u)\cdot \mathbf{n} = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.21}$$

where $\mathbf{n}$ denotes the outward unit normal on $\partial\Omega$. Given $p \in [1,\infty]$, we define the Sobolev type space

$$V^{1,p}(\Omega) = \left\{ u \in W^{1,1}_{loc}(\Omega) : |\nabla u| \in L^p(\Omega) \right\}\,.$$

If Ω is connected, and B is any ball such that $\overline{B} \subset \Omega$, then $V^{1,p}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{V^{1,p}(\Omega)} = \|u\|_{L^p(B)} + \|\nabla u\|_{L^p(\Omega)}.$$

Note that replacing B by another ball results in an equivalent norm. Moreover, if Ω is regular enough, for instance a Lipschitz domain, then, by a Poincaré type inequality, $V^{1,p}(\Omega) \subset L^p(\Omega)$, and hence $V^{1,p}(\Omega) = W^{1,p}(\Omega)$, the usual Sobolev space.
When a satisfies assumptions (2.2)–(2.3), and f belongs to the topological dual $V^{1,p}(\Omega)^*$ of $V^{1,p}(\Omega)$, a weak solution to problem (2.21) is well defined as a function $u \in V^{1,p}(\Omega)$ such that

$$\int_\Omega a(x,\nabla u)\cdot\nabla\varphi\, dx = \int_\Omega f\varphi\, dx \tag{2.22}$$

for every $\varphi \in V^{1,p}(\Omega)$. Under the additional monotonicity assumption (2.6), and the compatibility condition

$$\int_\Omega f(x)dx = 0\,, \tag{2.23}$$

the existence and uniqueness (up to additive constants) of a weak solution to problem (2.21) can be derived via a rather standard application of the Browder-Minthy theory of monotone operators.
If $f \notin V^{1,p}(\Omega)^*$, a generalized notion of solution to the Neumann problem (2.21) has to be adopted in the spirit, for instance, of the entropy solutions recalled in Section 2.1. Let us set

$$V^{1,p}_T(\Omega) = \{u \text{ is a measurable function in } \Omega : T_{-t,t}(u) \in V^{1,p}(\Omega) \text{ for } t > 0\}\,. \tag{2.24}$$

An extended notion of gradient V_u, again simply denoted by ∇u, for functions $u \in V^{1,p}_T(\Omega)$ can be introduced via (2.9). One has that a function $u \in V^{1,p}_T(\Omega)$ belongs to $V^{1,p}(\Omega)$ if and only if $|V_u| \in L^p(\Omega)$, and, in this case, $V_u = \nabla u$, the weak gradient of u.

In analogy with (2.10), a function $u \in V^{1,p}_T(\Omega)$ is called an entropy solution to the Neumann problem (2.21) if

$$\int_\Omega a(x,u,\nabla u)\cdot \nabla\big(T_{t_1,t_2}(u-\varphi)\big)\,dx \le \int_\Omega f(x)T_{t_1,t_2}(u-\varphi)\,dx \tag{2.25}$$

for every $\varphi \in V^{1,p}(\Omega)\cap L^\infty(\Omega)$, and every $t_1 < t_2$.
If Ω is a regular – Lipschitz, say – domain, a unique (up to additive constants) entropy solution to the Neumann problem (2.21) can be shown to exist for every $f \in L^1(\Omega)$. On the other hand, unlike the case of Dirichlet problems, in general a balance between the regularity of Ω and the degree of integrability of f is needed in order to guarantee the existence of a solution to the Neumann problem (2.21). We are not going to discuss this issue here. We refer to the paper [ACMM], where the existence of generalized solutions to Neumann problems in possibly irregular domains is discussed. We limit ourselves to assuming that a solution does exist, and to describing the a priori gradient estimates which can be derived depending on the (ir)regularity of Ω and of f.

In our approach, the regularity of Ω is prescribed in terms of either the isoperimetric function, or the isocapacitary function of Ω. These functions are the optimal ones in inequalities between the measure of subsets of Ω and either their relative perimeter, or their condenser capacity, respectively. The use of each one of these two functions has its own advantages. The isoperimetric function has a transparent geometric character, and it is usually easier to investigate. The isocapacitary function can be less simple to study, but it is in a sense more appropriate, since it leads to finer conclusions in general. Let us incidentally mention that suitable versions of these functions have also recently been employed in the analysis of eigenvalue problems on noncompact Riemannian manifolds [CM3, CM5].

The isoperimetric function $\lambda_\Omega : [0,|\Omega|/2] \to [0,\infty)$ of Ω is defined as

$$\lambda_\Omega(s) = \inf\{P(E,\Omega) : s \le |E| \le |\Omega|/2\} \qquad \text{for } s \in [0,|\Omega|/2]\,. \tag{2.26}$$

Here, $P(E;\Omega)$ is the perimeter of E relative to Ω, which agrees with $\mathcal{H}^{n-1}(\partial^M E \cap \Omega)$, where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure, and $\partial^M E$ stands for the essential boundary of E in the sense of geometric measure theory. The relative isoperimetric inequality tells us that

$$\lambda_\Omega(|E|) \le P(E;\Omega) \qquad \text{for every measurable set } E \subset \Omega \text{ with } |E| \le |\Omega|/2, \tag{2.27}$$

and is a straightforward consequence of definition (2.26). The isoperimetric function λ_Ω is explicitly known only for very special domains, such as balls [Ma5, BuZa] and convex cones [LP]. However, various qualitative and quantitative properties of λ_Ω have been investigated (in the even more general Riemannian framework), in connection, for instance, with Sobolev inequalities [HK, Ma1, Ma5, MP] and eigenvalue estimates [Ch, Ci2, Ga].
The function λ_Ω is known to be strictly positive in $(0,|\Omega|/2]$ when Ω is connected [Ma5, Lemma 3.2.4]. The only piece of information on λ_Ω which is relevant in view of our applications is its asymptotic behavior as $s \to 0^+$, which depends on the regularity of Ω. Heuristically speaking, a faster decay of λ_Ω to 0 results in a set Ω with a more irregular geometry. The decay of λ_Ω for specific open sets, and customary classes of sets, is exhibited in the examples at the end of this Subsection.

A counterpart of Theorem 2.1 for Neumann problems, which makes use of the isoperimetric function, is the content of Theorem 2.8 below, and can be proved by the techniques of [Ma3] in the linear case. The nonlinear case for regular domains is contained in [Ma5]; general domains and nonlinearities are treated in [Ci2].

Theorem 2.8 *Let u be a solution to the Neumann problem* (2.21)*. If $0 < r \le p$, then there exists a constant $C = C(p, r)$ such that*

$$\|\nabla u\|_{L^r(\Omega)} \le C\bigg(\int_0^{|\Omega|} \lambda_\Omega(s)^{-\frac{r}{p-1}}\bigg(\int_0^s f^*(\rho)d\rho\bigg)^{\frac{r}{p-1}} ds\bigg)^{\frac{1}{r}}. \tag{2.28}$$

As a consequence of Theorem 2.8, bounds for L^r norms of the gradient of solutions to the Neumann problem (2.21) are reduced to one-dimensional inequalities for a Hardy type operator depending on Ω just through λ_Ω.

Corollary 2.9 *Let $0 < r \le p$ and let $X(\Omega)$ be a rearrangement invariant quasi-normed space such that*

$$\bigg\|\lambda_\Omega(s)^{-\frac{1}{p-1}}\bigg(\int_0^s \phi(\rho)d\rho\bigg)^{\frac{1}{p-1}}\bigg\|_{L^r(0,|\Omega|)} \le C\|\phi\|_{\overline{X}(0,|\Omega|)}^{\frac{1}{p-1}}, \tag{2.29}$$

for some constant C and every nonnegative and non-increasing function $\phi \in \overline{X}(0, |\Omega|)$. If $f \in X(\Omega)$ and u is the solution to the Neumann problem (2.21)*, then*

$$\|\nabla u\|_{L^r(\Omega)} \le C'\|f\|_{X(\Omega)}^{\frac{1}{p-1}}, \tag{2.30}$$

where $C' = C'(C, p, r)$.

Corollary 2.9 can be exploited, in combination with Hardy type inequalities for non-increasing functions (see e.g. [CPSS]), to establish bounds for Lebesgue norms of $|\nabla u|$ in terms of Lebesgue, or Lorentz norms of f, depending on λ_Ω. Some special cases are dealt with in the examples below.

An analogue, for Neumann problems, of the rearrangement pointwise estimate of Theorem 2.5 is the content of the next result. In what follows, we set

$$\mathrm{med}(u) = \sup\{t \in \mathbb{R} : |\{u > t\}| \ge |\Omega|/2\}\,, \tag{2.31}$$

a median of u. Furthermore, we define $u_+ = \frac{|u|+u}{2}$ and $u_- = \frac{|u|-u}{2}$, the positive and the negative part of u, respectively.

Theorem 2.10 *Let u be the solution to the Neumann problem* (2.21) *satisfying* $\mathrm{med}(u) = 0$*. Then*

$$|\nabla u_\pm|^*(s) \le \bigg(\frac{2}{s}\int_{\frac{s}{2}}^{|\Omega|/2} \lambda_\Omega(\tau)^{-p'}\bigg(\int_0^\tau f_\pm^*(\rho)\,d\rho\bigg)^{p'} d\tau\bigg)^{\frac{1}{p}} \qquad \text{for } s \in (0, |\Omega|). \tag{2.32}$$

Theorem 2.10 follows via a direct argument, which parallels the proof of Theorem 2.5, but with the standard isoperimetric inequality in $\mathbb{R}^n$ replaced with the relative isoperimetric inequality in Ω. Alternatively, it can be derived from Theorem 2.14 and inequality (2.52) below.

Corollary 2.11 *Let $X(\Omega)$ be a rearrangement-invariant space and let $f \in X(\Omega)$. Let u be a solution to the Neumann problem* (2.21)*. Assume that $Y(\Omega)$ is a rearrangement-invariant space such that*

$$\bigg\|\bigg(\frac{1}{s}\int_{\frac{s}{2}}^{|\Omega|} \lambda_\Omega(\tau)^{-p'}\bigg(\int_0^\tau \phi(\rho)\,d\rho\bigg)^{p'} d\tau\bigg)^{\frac{1}{p}}\bigg\|_{\overline{Y}(0,|\Omega|)} \le C\|\phi\|_{\overline{X}(0,|\Omega|)}^{\frac{1}{p-1}}, \tag{2.33}$$

for some constant C and every nonnegative and non-increasing function $\phi \in \overline{X}(0,|\Omega|)$. Then there exists a constant $C_1 = C_1(C)$ such that

$$\|\nabla u\|_{Y(\Omega)} \le C_1 \|f\|_{X(\Omega)}^{\frac{1}{p-1}}. \tag{2.34}$$

Let us now turn to gradient estimates obtained in terms of the isocapacitary function, which are established in [CM1] and [ACMM]. The definition of isocapacitary function of Ω is analogous to that of isoperimetric function, provided that the relative perimeter of subsets of Ω is replaced with their condenser capacity. Recall that the standard p-capacity of a set $E \subset \Omega$ can be defined, for $p \ge 1$, as

$$C_p(E) = \inf\left\{\int_\Omega |\nabla u|^p\,dx : u \in W_0^{1,p}(\Omega), u \ge 1 \ \textit{a.e.} \text{ in some neighbourhood of } E\right\}. \tag{2.35}$$

A property is said to hold C_p-quasi everywhere in Ω, C_p-q.e. for short, if it is fulfilled outside a set of p-capacity zero. Each function $u \in W^{1,p}(\Omega)$ has a representative $\widetilde{u}$, called the precise representative, which is C_p-quasi continuous, in the sense that for every $\varepsilon > 0$, there exists a set $A \subset \Omega$, with $C_p(A) < \varepsilon$, such that $f_{|\Omega\setminus A}$ is continuous in $\Omega \setminus A$. The function $\widetilde{u}$ is unique, up to subsets of p-capacity zero.
A standard result in the theory of capacity tells us that, for every set $E \subset \Omega$,

$$C_p(E) = \inf\left\{\int_\Omega |\nabla u|^p\,dx : u \in W_0^{1,p}(\Omega), u \ge 1 \ C_p\text{-q.e. in } E\right\}, \tag{2.36}$$

see e.g. [MZ, Corollary 2.25]. Consistently, the p-capacity $C_p(E,G)$ of the condenser (E,G), where $E \subset G \subset \Omega$, can be defined as

$$C_p(E,G) = \inf\left\{\int_\Omega |\nabla u|^p\,dx : u \in W^{1,p}(\Omega), u \ge 1 \ C_p\text{-q.e. in } E \text{ and } u \le 0 \ C_p\text{-q.e. in } \Omega \setminus G\right\}. \tag{2.37}$$

The p-isocapacitary function $\nu_{\Omega,p} : [0,|\Omega|/2) \to [0,\infty)$ of Ω is given by

$$\begin{aligned}\nu_{\Omega,p}(s) = \inf\{&C_p(E,G) : E \text{ and } G \text{ are measurable subsets of } \Omega \text{ such that} \\ &E \subset G \subset \Omega,\ s \le |E| \text{ and } |G| \le |\Omega|/2\} \qquad \text{for } s \in [0,|\Omega|/2).\end{aligned} \tag{2.38}$$

Clearly, the function $\nu_{\Omega,p}$ is non-decreasing. Definition 2.38 immediately yields the isocapacitary inequality, which tells us that

$$\nu_{\Omega,p}(|E|) \le C_p(E,G) \quad \text{for every measurable sets } E \subset G \subset \Omega \text{ with } |G| \le |\Omega|/2. \tag{2.39}$$

The decay to 0 of $\nu_{\Omega,p}(s)$ as $s \to 0^+$ is related to the regularity of Ω, and plays a role in our results. Examples of such decay for some families of domains are presented below.

Conditions, in terms of $\nu_{\Omega,p}$, for gradient bounds, in Lebesgue spaces, for the solution to the Neumann problem 2.21 read as follows.

Theorem 2.12 *Assume that $f \in L^q(\Omega)$ for some $q \in [1,\infty]$. Let u be a solution to the Neumann problem (2.21). Let $0 < r \le p$. Then there exists a constant C such that*

$$\|\nabla u\|_{L^r(\Omega)} \le C \|f\|_{L^q(\Omega)}^{\frac{1}{p-1}}, \tag{2.40}$$

if either
(i) $q > 1$, $q(p-1) \le r$ *and*

$$\sup_{0<s<\frac{|\Omega|}{2}} \frac{s^{1+\frac{p(p-1)}{r}-\frac{p}{q}}}{\nu_{\Omega,p}(s)} < \infty\,, \tag{2.41}$$

or
(ii) $1 < q < \infty$, $0 < r < q(p-1)$ *and*

$$\int_0^{|\Omega|/2} \left(\frac{s}{\nu_{\Omega,p}(s)}\right)^{\frac{rq}{p[q(p-1)-r]}} ds < \infty\,, \tag{2.42}$$

or
(iii) $q = \infty$ *and*

$$\int_0^{|\Omega|/2} \left(\frac{s}{\nu_{\Omega,p}(s)}\right)^{\frac{r}{p(p-1)}} ds < \infty\,, \tag{2.43}$$

or
(iv) $q = 1$ *and*

$$\int_0^{|\Omega|/2} \left(\frac{s}{\nu_{\Omega,p}(s)}\right)^{\frac{r}{p(p-1)}} \frac{ds}{s^{\frac{r}{p-1}}} < \infty\,. \tag{2.44}$$

Moreover the constant C in (2.40) *depends only on p, q, r and on the left-hand side either of* (2.41), *or* (2.42), *or* (2.43) *or* (2.44), *respectively.*

In the borderline situation when $q = 1$, case (iv) of Theorem 2.12 can be somewhat improved on calling into play Marcinkiewicz norms, which extend $\|\cdot\|_{L^{q,\infty}(\Omega)}$. Recall that, given a bounded non-decreasing function $\omega : (0, |\Omega|) \to (0, \infty)$, the Marcinkiewicz space $M_\omega(\Omega)$ associated with ω is the set of all measurable functions u in Ω such that the quantity

$$\|u\|_{M_\omega(\Omega)} = \sup_{s\in(0,|\Omega|)} \omega(s) u^*(s) \tag{2.45}$$

is finite. The expression (2.45) is equivalent to a norm, which makes $M_\omega(\Omega)$ a rearrangement-invariant space, if and only if $\sup_{s\in(0,|\Omega|)} \frac{\omega(s)}{s} \int_0^s \frac{d\rho}{\omega(\rho)} < \infty$. Clearly, $M_\omega = L^{r,\infty}(\Omega)$ if $\omega(s) = s^{\frac{1}{r}}$ for some $r \ge 1$.

Proposition 2.13 *Assume that $f \in L^1(\Omega)$. Let u be a solution to the Neumann problem* (2.21). *Let $\omega_{\Omega,p} : (0, |\Omega|) \to [0, \infty)$ be the function defined by*

$$\omega_{\Omega,p}(s) = (s{\nu_{\Omega,p}}^{\frac{1}{p-1}}(s/2))^{\frac{1}{p}} \qquad \text{for } s \in (0, |\Omega|). \tag{2.46}$$

Then there exists a constant $C = C(p, n)$ such that

$$\|\nabla u\|_{M_{\omega_{\Omega,p}}(\Omega)} \le C \|f\|_{L^1(\Omega)}^{\frac{1}{p-1}}\,. \tag{2.47}$$

Versions of Theorem 2.10 and the ensuing Corollary 2.11, which involve the isocapacitary function instead of the isoperimetric function of Ω, have the following form.

Theorem 2.14 *Let u be the weak solution to the Neumann problem* (2.21) *satisfying* $\mathrm{med}(u) = 0$. *Then*

$$(2.48)\qquad |\nabla u_\pm|^*(s) \le \left(\frac{2}{s}\int_{\frac{s}{2}}^{|\Omega|/2}\left(\int_0^\tau f_\pm^*(\rho)\,d\rho\right)^{p'} d(-D\nu_{\Omega,p}^{\frac{1}{1-p}})(\tau)\right)^{\frac{1}{p}} \qquad \text{for } s\in(0,|\Omega|).$$

Corollary 2.15 *Let $X(\Omega)$ be a rearrangement-invariant space and let $f \in X(\Omega)$. Let u be a solution to the Neumann problem* (2.21). *Assume that $Y(\Omega)$ is a rearrangement-invariant space such that*

$$(2.49)\qquad \left\|\left(\frac{1}{s}\int_s^{|\Omega|}\left(\int_0^r \phi(\rho)\,d\rho\right)^{p'} d(-D\nu_p^{\frac{1}{1-p}})(\tau)\right)^{\frac{1}{p}}\right\|_{\overline{Y}(0,|\Omega|)} \le C\|\phi\|_{\overline{X}(0,|\Omega|)}^{\frac{1}{p-1}},$$

for some constant C and every nonnegative and non-increasing function $\phi \in \overline{X}(0,|\Omega|)$. Then there exists a constant $C_1 = C_1(C)$ such that

$$(2.50)\qquad \|\nabla u\|_{Y(\Omega)} \le C_1\|f\|_{X(\Omega)}^{\frac{1}{p-1}}.$$

For any connected open set Ω with finite measure, we have that

$$(2.51)\qquad \nu_1(s) \approx \lambda_\Omega(s) \quad \text{as } s\to 0^+,$$

as shown by an easy variant of [Ma5, Lemma 2.2.5], and, if $p > 1$,

$$(2.52)\qquad \nu_{\Omega,p}(s) \ge \left(\int_s^{|\Omega|/2}\frac{d\rho}{\lambda_\Omega(\rho)^{p'}}\right)^{1-p} \qquad \text{for } s\in(0,|\Omega|/2)$$

[Ma5, Proposition 4.3.4/1]. Owing to inequality (2.52), one can show that

$$\int_{\frac{s}{2}}^{|\Omega|/2}\left(\int_0^\tau f_\pm^*(\rho)\,d\rho\right)^{p'} d(-D\nu_{\Omega,p}^{\frac{1}{1-p}})(\tau) \le \int_{\frac{s}{2}}^{|\Omega|/2}\left(\int_0^\tau f_\pm^*(\rho)\,d\rho\right)^{p'}\frac{d\tau}{\lambda_\Omega(\tau)^{p'}} \quad \text{for } s\in(0,|\Omega|/2),$$

for every $f \in L^1(\Omega)$. Thus, Theorem 2.14 and Corollary 2.15 are always at least as sharp as their counterparts, Theorem 2.10 and Corollary 2.11, involving the isoperimetric function.
A reverse inequality in (2.52) does not hold in general, even up to a multiplicative constant. This accounts for the fact that the results on the Neumann problem (2.21) which can be derived in terms of $\nu_{\Omega,p}$ can be stronger than those resting upon λ_Ω – see Examples 4 and 5, below. However, the two sides of (2.52) are equivalent when Ω is sufficiently regular, as in Examples 1-3. In this case, the gradient bounds which follow by exploiting the isoperimetric and the isocapacitary function coincide.

Example 1. (John domains).
A bounded open set Ω in $\mathbb{R}^n$ is called a John domain if there exist a constant $c \in (0,1)$ and a point $x_0 \in \Omega$ having the property that, for every $x \in \Omega$, there exists a rectifiable curve $\varpi : [0,l] \to \Omega$, parametrized by arclenght, such that $\varpi(0) = x$, $\varpi(l) = x_0$, and

$$\mathrm{dist}\,(\varpi(s),\partial\Omega) \ge cs \qquad \text{for } s\in[0,l].$$

Note that, in particular, any Lipschitz domain is a John domain. The notion of John domain arises in connection with the analysis of holomorphic dynamical systems and quasiconformal mappings, and has been used in recent years in the study of Sobolev inequalities. In particular, a result from [KM] (complementing [HK]) implies that, if Ω is a John domain in $\mathbb{R}^n$, then

$$\lambda_\Omega(s) \approx s^{\frac{n-1}{n}} \text{for } s \in (0, |\Omega|/2).$$

Moreover, if $1 \leq p < n$, then

$$\nu_{\Omega,p}(s) \approx s^{\frac{n-p}{n}} \qquad \text{for } s \in (0, |\Omega|/2),$$

and, if $p \geq n$, then, for every $\gamma > 0$,

$$\nu_{\Omega,p}(s) \succeq s^\gamma \qquad \text{for } s \in (0, |\Omega|/2),$$

where the notation " $\succeq$ " means that the left-hand side is bounded from below by a positive constant times the right-hand side. In fact, in the borderline case when $p = n$, one can show that

$$\nu_{\Omega,p}(s) \approx \log^{1-n}(\tfrac{1}{s}) \qquad \text{for } s \in (0, |\Omega|/2).$$

When $1 < p < n$, Theorem 2.12, cases (i) and (iv), yields the estimate

$$\|\nabla u\|_{L^{\frac{q(p-1)n}{n-q}}(\Omega)} \leq C\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}}, \tag{2.53}$$

if $1 < q \leq \frac{np}{np+p-n}$, and

$$\|\nabla u\|_{L^r(\Omega)} \leq C\|f\|_{L^1(\Omega)}^{\frac{1}{p-1}}, \tag{2.54}$$

if $0 < r < (p-1)n'$, for a solution u to the Neumann problem 2.21. Moreover, from Proposition 2.13, one obtains that

$$\|\nabla u\|_{L^{(p-1)n',\infty}(\Omega)} \leq C\|f\|_{L^1(\Omega)}^{\frac{1}{p-1}}.$$

Note that inequalities (2.53) and (2.54) can also be proved via Corollary 2.9.

Example 2. (Hölder domains).
Let Ω be a bounded domain in $\mathbb{R}^n$ with a Hölder boundary with exponent $\alpha \in (0,1)$, and let $1 < p < \frac{1}{\alpha}(n-1)+1$. By a Sobolev embedding theorem of [La], and [Ma5, Theorem 6.3.3/1], we have that

$$\nu_{\Omega,p}(s) \succeq s^{1-\frac{\alpha p}{n-1+\alpha}} \quad \text{for } s \in (0, |\Omega|/2). \tag{2.55}$$

Thus, owing to Theorem 2.12, if u is a solution to the Neumann problem 2.21, then

$$\|\nabla u\|_{L^r(\Omega)} \leq C\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}}, \tag{2.56}$$

if $q > 1$, where $r = \min\big\{\frac{q(p-1)(n-1+\alpha)}{n-1-\alpha(q-1)}, p\big\}$, and

$$\|\nabla u\|_{L^r(\Omega)} \leq C\|f\|_{L^1(\Omega)}^{\frac{1}{p-1}}, \tag{2.57}$$

if $0 < r < (p-1)\big(1 + \frac{\alpha}{(n-1)}\big)$. In fact, by Proposition 2.13, one also has that

$$\|\nabla u\|_{L^{(p-1)(1+\frac{\alpha}{(n-1)}),\infty}(\Omega)} \leq C\|f\|_{L^1(\Omega)}^{\frac{1}{p-1}}.$$

On the other hand, by [La] again and [Ma5, Corollary 5.2.3] (see also [Ci1, Theorem 1] for the case $n = 2$),

$$\lambda_\Omega(s) \succeq s^{\frac{n-1}{n-1+\alpha}} \quad \text{for } s \in (0, |\Omega|/2).$$

Thus, making use of Corollary 2.9 leads to the same conclusions (2.56) and (2.57).

Example 3 (An unbounded funnel).
Let $\zeta : [0, \infty) \to (0, \infty)$ be a differentiable convex function such that $\lim_{\rho\to\infty} \zeta(\rho) = 0$. Consider the unbounded set

$$\Omega = \{x \in \mathbb{R}^n : x_n > 0, |x'| < \zeta(x_n)\}$$

(see Figure 1), where $x = (x', x_n)$ and $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. Assume that

$$\int_0^\infty \zeta(\rho)^{n-1} d\rho < \infty\,, \tag{2.58}$$

whence $|\Omega| < \infty$. Let $\Upsilon : [0, \infty) \to [0, \infty)$ be the function given by

$$\Upsilon(\rho) = n\omega_n \int_\rho^\infty \zeta(\tau)^{n-1} d\tau \qquad \text{for } \rho > 0.$$

By [Ma5, Example 5.3.3/2],

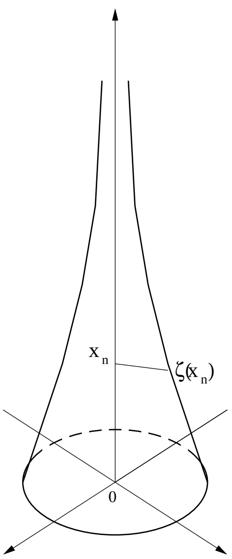

Figure 1: an unbounded funnel

$$\lambda_\Omega(s) \approx \big(\zeta(\Upsilon^{-1}(s))\big)^{n-1} \qquad \text{as } s \to 0^+,$$

and, by [Ma5, Example 6.3.6/2], if $p > 1$,

$$\nu_{\Omega,p}(s) \approx \left(\int_{\Upsilon^{-1}(|\Omega|/2)}^{\Upsilon^{-1}(s)} \zeta(\tau)^{\frac{1-n}{p-1}} d\tau\right)^{1-p} \quad \text{as } s \to 0^+.$$

Assume, for instance, that $\zeta(\rho) = \frac{1}{(1+\rho)^\beta}$ for some $\beta > \frac{1}{n-1}$. Then

$$\lambda_\Omega(s) \approx s^{\frac{\beta(n-1)}{\beta(n-1)-1}} \qquad \text{as } s \to 0^+,$$

and

$$\nu_{\Omega,p}(s) \approx s^{\frac{\beta(n-1)+p-1}{\beta(n-1)-1}} \quad \text{as } s \to 0^+.$$

An application of Theorem 2.12 tells us that, if u is a solution to the Neumann problem (**??**), then

$$\|\nabla u\|_{L^r(\Omega)} \leq C\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}},$$

provided that $r \leq p$ and $r < \frac{q(p-1)[\beta(n-1)-1]}{q+\beta(n-1)-1}$. The same conclusion can be derived via Proposition 2.13.

Example 4 (Courant-Hilbert)
Let us consider the Neumann problem (2.21) in the domain $\Omega \subset \mathbb{R}^2$ displayed in Figure 2. This set is exhibited in [CH] as an example of a domain in which the Poincaré inequality fails. In

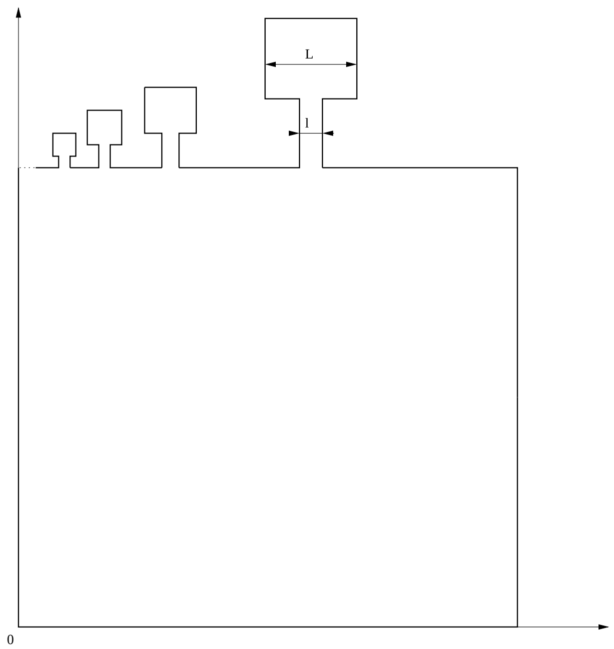

Figure 2: Courant-Hilbert domain

the figure, $L = 2^{-k}$ and $l = \delta(2^{-k})$, where $k \in \mathbb{N}$ and $\delta : [0,\infty) \to [0,\infty)$ is any function such that: $\delta(2s) \leq c\delta(s)$ for some $c > 0$ and for every $s > 0$; the function $\frac{s^{p+1}}{\delta(s)}$ is non-decreasing; the function $\frac{s^{1+\varepsilon}}{\delta(s)}$ is non-increasing for some $\varepsilon > 0$. One can show that, if $1 \leq p \leq 2$, then

$$\lambda_\Omega(s) \approx \delta\big(s^{1/2}\big) \qquad \text{as } s \to 0^+, \tag{2.59}$$

and

$$\nu_{\Omega,p}(s) \approx \delta\big(s^{1/2}\big) s^{\frac{1-p}{2}} \qquad \text{as } s \to 0^+ \tag{2.60}$$

[CM4]. Assume, for instance, that

$$\delta(s) \approx s^\gamma \quad \text{as } s \to 0^+,$$

for some $\gamma \in (1, p+1)$. Then, from Theorem 2.12 on can infer that

$$\|\nabla u\|_{L^r(\Omega)} \leq C\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}}, \tag{2.61}$$

if $q>1$, where $r=\min\left\{\frac{2pq(p-1)}{2p-q(p+1-\gamma)},p\right\}$, and, by Proposition 2.13,

$$\|\nabla u\|_{L^{\frac{2p(p-1)}{p-1+\gamma},\infty}(\Omega)} \le C\|f\|_{L^1(\Omega)}^{\frac{1}{p-1}}.$$

The use of Corollary 2.9 only yields

$$\|\nabla u\|_{L^r(\Omega)} \le C\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}},$$

if $q>1$, where $r=\min\left\{\frac{2q(p-1)}{\gamma q-2q+2},p\right\}$, a weaker conclusion then (2.61), since $\frac{2q(p-1)}{\gamma q-2q+2} < \frac{2pq(p-1)}{2p-q(p+1-\gamma)}$.

Example 5 (Nikodým)
We conclude with the highly irregular domain $\Omega\subset\mathbb{R}^2$ illustrated in Figure 3, which was introduced by Nikodým in his study of Sobolev embeddings.

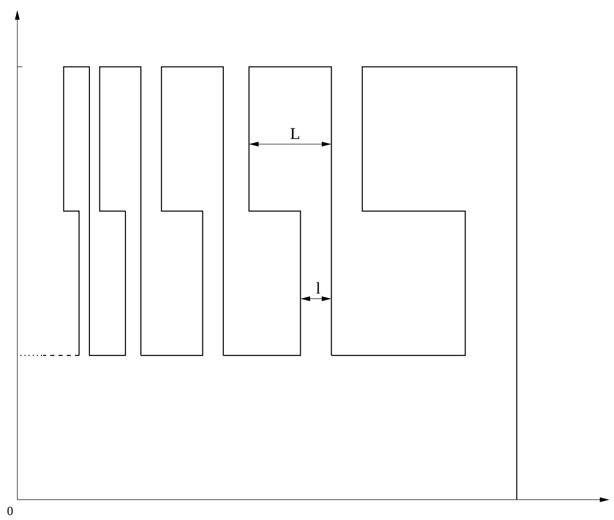

Figure 3: Nikodým domain

Here, $L=2^{-k}$ and $l=2^{-\beta k}$, where $\beta>1$ and $k\in\mathbb{N}$. One has that

$$\lambda_\Omega(s)\approx s^\beta \qquad \text{as } s\to 0^+, \tag{2.62}$$

and, if $1<p<2$,

$$\nu_{\Omega,p}(s)\approx s^\beta \qquad \text{as } s\to 0^+ \tag{2.63}$$

[Ma5, Section 6.5]. Thus, since

$$\left(\int_s^{|\Omega|/2}\frac{d\rho}{\lambda_\Omega(\rho)^{p'}}\right)^{1-p}\approx s^{p(\beta-1)+1} \qquad \text{as } s\to 0^+, \tag{2.64}$$

the isocapacitary function $\nu_{\Omega,p}(s)$ is not equivalent to $\left(\int_s^{|\Omega|/2}\frac{dr}{\lambda_\Omega(r)^{p'}}\right)^{1-p}$ for such a domain Ω. In fact, the estimates for the gradient of solutions to the Neumann problem (2.21), which can

be derived via the isocapacitary function, are stronger than those obtained by the isoperimetric function.
To see this, note that Theorem 2.12 implies that, if $q \geq 1$ and $r \leq p$, then

$$\|\nabla u\|_{L^r(\Omega)} \leq C\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}}, \tag{2.65}$$

provided that

$$r < \frac{pq(p-1)}{q(\beta-1)+p}. \tag{2.66}$$

Such a conclusion is stronger than what follows via Corollary 2.9, which yields (2.65) only for

$$r < \frac{q(p-1)}{q(\beta-1)+1}.$$

3 Part II: equations with special nonlinearities

We focus here on either Dirichlet problems of the form

$$\begin{cases} -\mathrm{div}(a(|\nabla u|)\nabla u) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

or Neumann problems of the form

$$\begin{cases} -\mathrm{div}(a(|\nabla u|)\nabla u) = f(x) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

We assume that $a : [0,\infty) \to [0,\infty)$ is of class $C^1(0,\infty)$,

$$-1 < \inf_{t>0} \frac{ta'(t)}{a(t)} \leq \sup_{t>0} \frac{ta'(t)}{a(t)} < \infty, \tag{3.3}$$

and there exist $p \in (1,\infty)$ and $c, C > 0$ such that

$$ct^{p-1} \leq ta(t) \leq C(t^{p-1}+1) \quad \text{for } t > 0. \tag{3.4}$$

In particular, the standard p-Laplace operator, corresponding to the choice $a(t) = t^{p-2}$, with $p > 1$, is included in this framework, since $i_a = s_a = p-2$ for this choice of a.

We shall present a rearrangement estimate for the gradient of solutions to problems (3.1) and (3.2), in the spirit of those exhibited in Section 2 for problems with general operators. Of course, the relevant estimate will be essentially stronger, and apt to establish any integrability property of the gradient, depending on the integrability of the right-hand side f. Our gradient rearrangement estimate is stated in Subsection 3.2. We preliminarily discuss the problem of the global boundednees of the gradient in Subsection 3.1. With this regard, let us point out that assumption (3.4) is irrelevant for the results of Subsection 3.1 to hold. We keep it in force just for simplicity of exposition, since dropping (3.4) entails the use of Orlicz-Sobolev spaces, instead of the standard Sobolev spaces, as a functional framework - see [CM2, CM6].

We pursue minimal regularity assumptions on $\partial\Omega$ ensuring gradient regularity. Bounded domains Ω whose boundary $\partial\Omega \in W^2L^{n-1,1}$ will be allowed. This means that Ω is locally the

subgraph of a function of $n-1$ variables whose second-order weak derivatives belong to the Lorentz space $L^{n-1,1}$. This is the weakest possible integrability assumption on second-order derivatives for the first-order derivatives to be continuous, and hence for $\partial\Omega \in C^{1,0}$ [CP]. Note that, by contrast, more standard results on global boundedeness of the gradient of solutions require that the functions whose subgraph locally agrees with Ω have a modulus of continuity satisfying a Dini condition – see [Li1, Section 3], [Li2, Theorem 5.1], and also [An, Remarks on Lemma A3.1].
The case of arbitrary convex domains is also covered by our results. This case is of special interest for the Neumann problem (3.2), for which, unlike the case of Dirichlet problems, even for bounded f an approach via standard barrier arguments does not apply. Partial contributions in this connection can be found in [Li4, Example, page 58, and Remark, page 62].

3.1 Global boundedness of the gradient

In this Subsection we exhibit a minimal integrability assumption on f ensuring the global boundedness of the gradient of solutions to (3.1) and (3.2), and hence the Lipschitz continuity of solutions [CM2, CM6]. The relevant assumption amounts to requiring that f belongs to the Lorentz space $L^{n,1}(\Omega)$, which is borderline for the family of Lebesgue spaces $L^q(\Omega)$ with $q > n$, inasmuch as $L^q(\Omega) \subsetneqq L^{n,1}(\Omega) \subsetneqq L^n(\Omega)$ for any such q. Let us notice that, if $f \in L^{n,1}(\Omega)$ and $\partial\Omega \in W^2L^{n-1,1}$, the existence and uniqueness (up to additive constants in the Neumann case) of a weak solution to problems (3.1) and (3.2) follows via classical minimization arguments for strictly convex functionals of which these problems are the Euler equations. We emphasize that our result sharpens more standard results available in the literature even for the Lapalce operator.

Theorem 3.1 *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, such that either $\partial\Omega \in W^2L^{n-1,1}$, or Ω is convex. Assume that the function $a \in C^1(0,\infty)$ and fulfills* (3.3)-(3.4). *Let $f \in L^{n,1}(\Omega)$, and let u be either the solution to the Dirichlet problem* (3.1) *or to the Neumann problem* (3.2). *Then*

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C \|f\|_{L^{n,1}(\Omega)}^{\frac{1}{p-1}} \tag{3.5}$$

for some constant $C = C(p,\Omega)$.

Some remarks about Theorem 3.1 are in order.

Remark 3.2 Theorem 3.1 holds, at least if a is monotone (either increasing or decreasing), even in the vector-valued case, namely if the equations in problem (3.1) or (3.2) are replaced by systems. This is possible, owing to the Uhlenbeck structure of the differential operator, which only depends on the length of the gradient. Recall that, in contrast with the scalar case, solutions to nonlinear elliptic systems with a more general structure can be irregular. Examples are produced in [SY], where the existence of nonlinear elliptic systems, with regular differential operators depending only on the gradient, but endowed with solutions which are not even bounded, is proved. Earlier examples of irregular solutions to elliptic systems are rooted in the paper [DeG], and include [GM] and [Ne].

Remark 3.3 A version of Theorem 3.1 also holds in the case when $n = 2$, under the slightly stronger assumption that $f \in L^q(\Omega)$ for some $q > n$.

Remark 3.4 The sharpness of assumption $f \in L^{n,1}(\Omega)$ for the boundedness of the gradient can be demonstrated by considering the Dirichlet problem for the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

when Ω is a ball in $\mathbb{R}^n$ [Ci3].

Remark 3.5 The assumption $\partial\Omega \in W^2L^{n-1,1}$ is optimal for the boundedness of the gradient, as long as the regularity of Ω is prescribed in terms of integrability properties of the second-order derivatives of the functions which locally represent its boundary. This can be shown by examples of Dirichlet and Neumann problems for the p-Laplace equation in domains whose boundaries have conical singularities [CM7]. Examples of the same nature also show that the conclusion of Theorem 3.1 may fail under slight local non-smooth perturbations of convex domains. Both examples involve a domain Ω whose boundary contains 0, is smooth outside a neighborhood of 0, and in such neighborhood Ω agrees with

$$\{x = (x', x_n) : x_n < L|x'|\}$$

for some number $L > 0$. In other words, in a neighborhood of 0 the domain Ω is bounded by an inward cone, whose aperture is $\arctan(1/L)$. The shape of Ω far from 0 is immaterial.
Assume first that $2 \leq p \leq n-1$ and that the cone in the definition of Ω is very sharp, namely that L is very large. Consider the Dirichlet problem

$$\text{(3.6)} \qquad \begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

One has that $\partial\Omega \in W^2L^q$ for every $q < n-1$, and, in fact, $\partial\Omega \in W^2L^{q,1}$ for every $q < n-1$, but $\partial\Omega \notin W^2L^{n-1,1}$. The function f can be chosen in such a way that it is smooth, vanishes in a neighborhood of 0, and the solution u to (3.6) satisfies

$$\text{(3.7)} \qquad u(x) \approx |x|^{\alpha(L)} F\big(x_n/|x|\big) \quad \text{as } x \to 0,$$

for some smooth function $F : \mathbb{R} \to \mathbb{R}$, and some exponent $\alpha(L) > 0$ such that

$$\lim_{L\to\infty} \alpha(L) \to 0,$$

see [KM]. Thus, given any $q > n$, we have that $|\nabla u| \notin L^q(\Omega)$ for sufficiently large L, even if f is very smooth. If, instead, $\partial\Omega \in W^2L^{n-1,1}$, then Theorem 3.1 ensures that $|\nabla u| \in L^\infty(\Omega)$ provided that $f \in L^{n,1}(\Omega)$, and hence, in particular, if $f \in L^\infty(\Omega)$.
Suppose next that the cone in the definition of Ω is almost flat, namely that L is very small. Clearly, Ω can be constructed in such a way that it is convex when $L = 0$. Consider the Neumann problem for the Laplace equation

$$\text{(3.8)} \qquad \begin{cases} -\Delta u = f(x) & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega\,. \end{cases}$$

One can show that there exist functions f which are smooth, vanish in a neighborhood of 0, and such that solution u to (3.8) satisfies

$$u(x) \approx |x|^{\beta(L)} F\big(x_n/|x|\big) \quad \text{as } x \to 0,$$

up to multiplicative constants independent of x, for some smooth function $F : \mathbb{R} \to \mathbb{R}$. Here, $\beta(L)$ is a positive exponent such that

$$\beta(L) < 1 \quad \text{if } L \text{ is sufficiently close to } 0,$$

see e.g. [KMR, Section 2.3.2]. Thus, if L is sufficiently small, there exists $q < \infty$ such that $|\nabla u| \notin L^q(\Omega)$. An analogous conclusion holds if the Neumann condition in (3.8) is replaced with the Dirichlet condition $u = 0$ on $\partial\Omega$.
This is another example showing that the regularity assumption on $\partial\Omega$ in Theorem 3.1 cannot be essentially relaxed. Indeed, boundedness, and high integrability, of $|\nabla u|$ need not be guaranteed, yet for the Laplace equation with a smooth right-hand side, even if $\partial\Omega$ is smooth everywhere, except at a single point, in a neighborhood of which $\partial\Omega$ is almost flat, and the regularity assumption $\partial\Omega \in W^2L^{n-1,1}$ is just slightly relaxed.
The same example also demonstrates that even a mild local perturbation of convexity may affect the conclusion of Theorem 3.1.

Remark 3.6 In [DM1], the assumption $f \in L^{n,1}_{loc}(\Omega)$ has independently, and by different techniques, been shown to ensure the local boundedness of the gradient of local solutions to nonlinear equations, and systems with special structure. The same assumption also yields the continuity of the gradient in Ω [DM2, KM2]. The question could thus be raised of whether the stronger conclusion $u \in C^1(\overline{\Omega})$ holds in Theorem 3.1. The answer is however negative in general, at least for convex domains. A counterexample can be produced as follows. Consider a disk $D \subset \mathbb{R}^2$, fix a point $\overline{x} \in \partial D$, and consider a sequence of points $x_k \in \partial D$ such that $x_k \to \overline{x}$. Let $\Omega \subset \mathbb{R}^2$ be the convex domain obtained as the union of D and of the sequence of sets bounded by ∂D and by the tangent straight-lines to ∂D at the points x_k. Let f be a nonnegative, smooth, radially symmetric function about the center of D, which has compact support in D. Consider the solution u to the (scalar) equation $-\Delta u = f(x)$ in Ω, subject to the Dirichlet condition $u = 0$ on $\partial\Omega$. If ∇u were continuous on $\partial\Omega$, then it should vanish at the intersection of the tangent straight-lines to ∂D at the points x_k, and hence one would have $|\nabla u(\overline{x})| = \left|\frac{\partial u}{\partial\nu}(\overline{x})\right| = 0$ as well. This contradicts the fact that $\left|\frac{\partial u}{\partial\nu}(\overline{x})\right| \geq \left|\frac{\partial v}{\partial\nu}(\overline{x})\right| > 0$, where v is the solution to the problem $-\Delta v = f(x)$ in D, with $v = 0$ on ∂D.
Variants of this example could be exhibited for Neumann problems. Also, the domain Ω can be modified into another domain whose boundary is of class C^1. Analogues in dimension $n \geq 3$ can be produced on replacing the disk with a ball, and the tangent straight-lines with the tangent hyperplanes.

Remark 3.7 A local estimate at the boundary for solutions to the Neumann problem (2.21), in convex domains, in the special case when $a(t) = t^{p-2}$ and $f = 0$, has recently been established in [BL].

3.2 A pointwise rearrangement estimate for the gradient

In order to grasp the spirit of the result of this section, based on [CM7], consider a prototypal problem given by the Poisson equation in the whole of $\mathbb{R}^n$. If $n \geq 3$, then the unique solution decaying to 0 at infinity to the equation

$$-\Delta u = f \qquad \text{in } \mathbb{R}^n \tag{3.9}$$

admits a representation formula in terms of a Riesz potential operator, namely

$$u(x) = \frac{1}{n(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}}\, dy \qquad \text{for } x \in \mathbb{R}^n. \tag{3.10}$$

Hence,

$$|\nabla u(x)| \leq \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-1}}\, dy \qquad \text{for a.e. } x \in \mathbb{R}^n. \tag{3.11}$$

A rearrangement inequality for convolutions [On] then implies that

$$|\nabla u|^*(s) \leq C \int_s^\infty f^{**}(\rho)\rho^{-\frac{1}{n'}}\, d\rho \quad \text{for } s > 0, \tag{3.12}$$

where C is a constant depending only on n.

Our pointwise gradient rearrangement estimate is a global nonlinear analogue of (3.12) for the solutions to (3.1) and (3.2), with $p \in [2, n)$.

Theorem 3.8 *Let Ω be a domain in $\mathbb{R}^n$, $n \geq 3$, such that either $\partial\Omega \in W^2L^{n-1,1}$, or Ω is convex. Assume that the function $a \in C^1(0,\infty)$ and fulfills* (3.3)-(3.4) *for some $p \in [2, n)$. Let $f \in L^1(\Omega)$, and let u be either the solution to the Dirichlet problem* (3.1) *or to the Neumann problem* (3.2)*. Then there exists a constant $C = C(\Omega, p)$ such that*

$$|\nabla u|^*(s)^{p-1} \leq C \int_s^{|\Omega|} f^{**}(\rho)\rho^{-\frac{1}{n'}}\, d\rho \quad \textit{for } s \in (0, |\Omega|). \tag{3.13}$$

Remark 3.9 A local estimate for the gradient of local solutions to nonlinear elliptic equations in terms of the Riesz potential of the right-hand side, which extends (3.11) in the same direction as (3.13) extends (3.12) for global solutions to nonlinear boundary value problems, is established in [KM1].

We present hereafter some gradient norm estimates which can be deduced thanks to Theorem 3.8. We begin with a general criterion which holds for arbitrary rearrangement-invariant quasi-norms, which is a straightforward consequence of Theorem 3.8.

Corollary 3.10 *Let Ω and u be as in Theorem 3.8. Let $X(\Omega)$ and $Y(\Omega)$ be rearrangement invariant quasi-normed spaces on Ω, and let $\overline{X}(0, |\Omega|)$ and $\overline{Y}(0, |\Omega|)$, respectively, be their representation spaces. Assume that there exists a constant C such that*

$$\left\| \int_s^{|\Omega|} \tau^{-1-\frac{1}{n'}} \int_0^\tau \phi(\rho)\, d\rho\, d\tau \right\|_{\overline{Y}(0,|\Omega|)} \leq C\|\phi\|_{\overline{X}(0,|\Omega|)} \tag{3.14}$$

for every non-decreasing function $\phi \in \overline{X}(0, |\Omega|)$. If $f \in X(\Omega)$, then there exists a constant $C' = C'(C, \Omega, p)$ such that

$$\|\, |\nabla u|^{p-1} \,\|_{Y(\Omega)} \leq C'\|f\|_{X(\Omega)}. \tag{3.15}$$

Let us notice that inequality (3.14) is equivalent to the pair of inequalities

$$\left\| \int_s^{|\Omega|} \phi(\rho)\rho^{-\frac{1}{n'}}\, d\rho \right\|_{\overline{Y}(0,|\Omega|)} \leq C\|\phi\|_{\overline{X}(0,|\Omega|)} \tag{3.16}$$

and

$$\left\| s^{-\frac{1}{n'}} \int_0^s \phi(\rho) d\rho \right\|_{\overline{Y}(0,|\Omega|)} \leq C\|\phi\|_{\overline{X}(0,|\Omega|)} \tag{3.17}$$

for every non-decreasing function $\phi \in \overline{X}(0,|\Omega|)$. This follows from Fubini's theorem applied to the double integral appearing in (3.14).
Inequality (3.14) is stronger, in general, than just (3.16), since, if $\phi : (0,|\Omega|) \to [0,\infty)$ is non-increasing, then

$$\phi(s) \leq \frac{1}{s}\int_0^s \phi(\rho)d\rho \qquad \text{for } s > 0. \tag{3.18}$$

However, inequalities (3.14) and (3.16) are equivalent in the case when the quasi-norm in $X(\Omega)$ fulfils

$$\left\| \frac{1}{s}\int_0^s \phi(\rho)d\rho \right\|_{\overline{X}(0,|\Omega|)} \leq C\|\phi\|_{\overline{X}(0,|\Omega|)} \tag{3.19}$$

for some constant C and for every $\phi \in \overline{X}(0,|\Omega|)$. Thus, if $X(\Omega)$ satisfies (3.19), then the sole inequality (3.16) implies the gradient estimate (3.15). The rearrangement-invariant Banach function spaces $X(\Omega)$ making inequality (3.19) true can be characterized in terms of their upper Boyd index $I(X)$. The definition of $I(X)$ relies upon that of dilation operator. The dilation operator $D_\delta : \overline{X}(0,|\Omega|) \to \overline{X}(0,|\Omega|)$ is defined for $\delta > 0$ and $\phi \in \overline{X}(0,|\Omega|)$ as

$$D_\delta\phi(s) = \begin{cases} \phi(s\delta) & \text{if } s\delta \in (0,|\Omega|) \\ 0 & \text{otherwise,} \end{cases}$$

and is bounded whenever $X(\Omega)$ is a rearrangement-invariant Banach function space [BS, Chapter 3, Prop. 5.11]. Its norm is denoted by $\|D_\delta\|$. The Boyd index $I(X)$ of $X(\Omega)$ is given by

$$I(X) = \lim_{\delta\to 0} \frac{\log\|D_\delta\|}{\log(1/\delta)}\,.$$

One has that $I(X) \in [0,1]$ for every rearrangement-invariant Banach function space $X(\Omega)$. Moreover, inequality (3.19) holds if and only if $I(X) < 1$ [BS, Theorem 5.15].

We conclude by stating explicit gradient bounds for Lebesgue, Lorentz, Lorentz-Zygmund, and Orlicz norms for solutions to either the Dirichlet problem (3.1) or the Neumann problem (3.2).
Our first result concerns gradient estimates in classical Lebesgue spaces. In the statements below, C denotes a constant independent of u and f.

Theorem 3.11 *Let Ω, p and u be as in Theorem 3.8. Assume that $f \in L^q(\Omega)$.*
(i) If $q = 1$, then, for every $r < \frac{n(p-1)}{n-1}$,

$$\|\nabla u\|_{L^r(\Omega)} \leq C\|f\|_{L^1(\Omega)}^{\frac{1}{p-1}}. \tag{3.20}$$

(ii) If $1 < q < n$, then

$$\|\nabla u\|_{L^{\frac{qn(p-1)}{n-q}}(\Omega)} \leq C\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}}. \tag{3.21}$$

(iii) If $q = n$, then,for every $r < \infty$,

$$\|\nabla u\|_{L^r(\Omega)} \leq C\|f\|_{L^n}^{\frac{1}{p-1}}. \tag{3.22}$$

(v) If $q > n$, then

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}}. \tag{3.23}$$

Theorem 3.11 overlaps with various contributions, including [AM, BBGGPV, BG1, DMOP, Ma2, Ma3, Li3, Ta1, Ta2].
More general and sharper estimates in Lorentz and Lorentz-Zygmund spaces are contained in the next theorem. The Lorentz-Zygmund spaces extend the Lorentz spaces, and come into play in certain borderline situations. If either $q \in (1, \infty]$, $k \in (0, \infty]$, $\beta \in \mathbb{R}$, or $q = 1$, $k \in (0, 1]$, $\beta \in [0, \infty)$, the Lorentz-Zygmund space $L^{q,k;\beta}(\Omega)$ is defined as the set of all measurable functions g on Ω making the expression

$$\|g\|_{L^{q,k;\beta}(\Omega)} = \|s^{\frac{1}{q}-\frac{1}{k}}(1 + \log(|\Omega|/s))^{\beta} g^*(s)\|_{L^k(0,|\Omega|)} \tag{3.24}$$

finite. If $k \geq 1$ and the weight multiplying $g^*(s)$ on the right-hand side of (3.24) is non-increasing, then the functional $\|g\|_{L^{q,k;\beta}(\Omega)}$ is actually a norm, and $L^{q,k;\beta}(\Omega)$ is a rearrangement-invariant Banach function space equipped with this norm. Otherwise, this functional is only a quasi-norm. For certain values of the parameters q, k and β, it is however equivalent to a rearrangement-invariant norm obtained on replacing g^* by g^{**} in the definition. A comprehensive analysis of Lorentz-Zygmund spaces is the subject of [OP].

Theorem 3.12 *Let Ω, p and u be as in Theorem 3.8 . Assume that $f \in L^{q,k}(\Omega)$.*
(i) If $q = 1$ and $0 < k \leq 1$, then

$$\|\nabla u\|_{L^{\frac{n(p-1)}{n-1},\infty}(\Omega)} \leq C\|f\|_{L^{1,k}(\Omega)}^{\frac{1}{p-1}}.$$

(ii) If $1 < q < n$ and $0 < k \leq \infty$, then

$$\|\nabla u\|_{L^{\frac{qn(p-1)}{n-q},k(p-1)}(\Omega)} \leq C\|f\|_{L^{q,k}(\Omega)}^{\frac{1}{p-1}}.$$

(iii) If $q = n$ and $k > 1$, then

$$\|\nabla u\|_{L^{\infty,k(p-1);-\frac{1}{p-1}}(\Omega)} \leq C\|f\|_{L^{n,k}(\Omega)}^{\frac{1}{p-1}}.$$

(iv) If either $q = n$ and $k \leq 1$, or $q > n$ and $0 < k \leq \infty$, then

$$\|\nabla u\|_{L^{\infty}(\Omega)} \leq C\|f\|_{L^{q,k}(\Omega)}^{\frac{1}{p-1}}.$$

Various cases of Theorem 3.12 are known, possibly under stronger assumption on Ω – see e.g. [ACMM, AFT, AM, BBGGPV].

Our last application concerns gradient estimates in Orlicz spaces. Let $A : [0, \infty) \to [0, \infty]$ be a Young function, namely a convex function, vanishing at 0, which is neither identically equal to 0, nor to ∞. The Orlicz space $L^A(\Omega)$ associated with A is the rearrangement-invariant space of those measurable functions g on Ω such that the Luxemburg norm

$$\|g\|_{L^A(\Omega)} = \inf\left\{\lambda > 0 : \int_{\Omega} A\Big(\frac{|g(x)|}{\lambda}\Big)\, dx \leq 1\right\}$$

is finite. Since we are assuming that $|\Omega| < \infty$, the Orlicz spaces $L^A(\Omega)$ and $L^B(\Omega)$ agree, up to equivalent norms, if and only if the Young functions A and B are equivalent near infinity, in the sense that there exist positive constants c and t_0 such that $B(t/c) \leq A(t) \leq B(ct)$ for $t \geq t_0$.

The Young conjugate of A is the Young function $\widetilde{A}$ given by

$$\widetilde{A}(t) = \sup\{st - A(s) : s \geq 0\} \qquad \text{for } t \geq 0.$$

The Sobolev conjugate, introduced in [C4, C5], of a Young function A such that

$$\int_0 \left(\frac{t}{A(t)}\right)^{\frac{1}{n-1}} dt < \infty\,, \tag{3.25}$$

is the Young function A_n defined as

$$A_n(t) = A\big(H^{-1}(t)\big) \qquad \text{for } t \geq 0, \tag{3.26}$$

where $H : [0, \infty) \to [0, \infty)$ is given by

$$H(s) = \left(\int_0^s \left(\frac{t}{A(t)}\right)^{\frac{1}{n-1}} dt\right)^{1/n'} \qquad \text{for } s \geq 0, \tag{3.27}$$

and H^{-1} is the generalized left-continuous inverse of H. Accordingly, given a Young function B such that

$$\int_0 \left(\frac{t}{\widetilde{B}(t)}\right)^{\frac{1}{n-1}} dt < \infty\,, \tag{3.28}$$

we denote by $\big(\widetilde{B}\big)_n$ the Sobolev conjugate of $\widetilde{B}$, obtained as in (3.26)–(3.27), on replacing A with $\widetilde{B}$.

Theorem 3.13 *Let Ω, p and u be as in Theorem 3.8. Let A and B be Young functions fulfilling* (3.25) *and* (3.28)*, respectively. Assume that $f \in L^A(\Omega)$, and that there exist $c > 0$ and $t_0 > 0$ such that*

$$B(t) \leq A_n(ct) \quad \text{and} \quad \widetilde{A}(t) \leq \big(\widetilde{B}\big)_n(ct) \quad \text{for } t \geq t_0. \tag{3.29}$$

Let E be the Young function given by

$$E(t) = B(t^{p-1}) \quad \text{for } t \geq 0.$$

Then

$$\|\nabla u\|_{L^E(\Omega)} \leq C\|f\|_{L^A(\Omega)}^{\frac{1}{p-1}}. \tag{3.30}$$

Remark 3.14 Assumptions (3.25) and (3.28) are, in fact, irrelevant in the statement of Theorem 3.13. Indeed, the functions A and B can be replaced, if necessary, by Young functions equivalent near infinity, which fulfil (3.25) and (3.28). Such a replacement leaves the spaces $L^A(\Omega)$ and $L^B(\Omega)$ unchanged, up to equivalent norms.

Theorem 3.13 can be easily specialized to the case when $L^A(\Omega)$ is a Zygmund space, namely

$$A(t) \text{ is equivalent to } t^q \log^\alpha(1+t) \text{ near infinity,}$$

where either $q > 1$ and $\alpha \in \mathbb{R}$, or $q = 1$ and $\alpha \geq 0$. In this case, the space $L^A(\Omega)$ is denoted by $L^q(\log L)^\alpha(\Omega)$. Let us notice that the Zugmund spaces are indeed a subclass of the Lorentz-Zygmund spaces, since $L^q(\log L)^\alpha(\Omega) = L^{q,q;\frac{\alpha}{q}}(\Omega)$, up to equivalent norms. If

$$A(t) \text{ is equivalent to } e^{t^\beta} - 1 \text{ near infinity},$$

for some $\beta > 0$, we denote $L^A(\Omega)$ by $\exp L^\beta(\Omega)$. Similarly, we use the notation $\exp\big(\exp L^\beta\big)(\Omega)$ for the Orlicz space associated with a Young function

$$A(t) \text{ equivalent to } e^{e^{t^\beta}} - e \text{ near infinity}.$$

Theorem 3.15 *Let Ω, p and u be as in Theorem 3.8. Let $f \in L^q(\log L)^\alpha(\Omega)$.*
(i) If $q = 1$ and $\alpha > 0$, then

$$\|\nabla u\|_{L^{\frac{n(p-1)}{n-1}}(\log L)^{\frac{n\alpha}{n-1}-1}(\Omega)} \leq C\|f\|_{L(\log L)^\alpha(\Omega)}^{\frac{1}{p-1}}. \tag{3.31}$$

(ii) If $1 < q < n$ and $\alpha \in \mathbb{R}$, then

$$\|\nabla u\|_{L^{\frac{nq(p-1)}{n-q}}(\log L)^{\frac{n\alpha}{n-q}}(\Omega)} \leq C\|f\|_{L^q(\log L)^\alpha(\Omega)}^{\frac{1}{p-1}}. \tag{3.32}$$

(iii) If $q = n$ and $\alpha < n-1$, then

$$\|\nabla u\|_{\exp L^{\frac{n(p-1)}{n-1-\alpha}}(\Omega)} \leq C\|f\|_{L^n(\log L)^\alpha(\Omega)}^{\frac{1}{p-1}}. \tag{3.33}$$

(iv) If $q = n$ and $\alpha = n-1$, then

$$\|\nabla u\|_{\exp\left(\exp L^{\frac{n(p-1)}{n-1}}\right)(\Omega)} \leq C\|f\|_{L^n(\log L)^{n-1}(\Omega)}^{\frac{1}{p-1}}. \tag{3.34}$$

(v) If either $q = n$ and $\alpha > n-1$, or $q > n$ and $\alpha \in \mathbb{R}$, then

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^q(\log L)^\alpha(\Omega)}^{\frac{1}{p-1}}. \tag{3.35}$$

Special cases of Theorem 3.15 are known. In particular, some instances of case (i) can be found in [BBGGPV, De].

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