Neumann problems and isocapacitary inequalities

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Abstract

A priori bounds for solutions to (nonlinear) elliptic Neumann problems in open subsets Ω of $\mathbb{R}^n$ are established via inequalities relating the Lebesgue measure of subsets of Ω to their relative capacity. Both norm and capacitary estimates for solutions, and norm estimates for their gradients are derived which improve classical results even in the case of the Laplace equation.

1 Introduction

We are concerned with a priori estimates for solutions to nonlinear elliptic problems, subject to homogeneous Neumann boundary conditions, having the form

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u)) = f(x) & \text{in } \Omega \\ a(x,u,\nabla u)\cdot \mathbf{n} = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1.1}$$

Here, Ω is a connected open subset of $\mathbb{R}^n$, $n \geq 2$, having finite Lebesgue measure $|\Omega|$, $a : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function fulfilling, for some $p > 1$ and for a.e. $x \in \Omega$, the ellipticity condition

$$a(x,t,\xi)\cdot\xi \geq |\xi|^p \quad \text{for } (t,\xi) \in \mathbb{R}\times\mathbb{R}^n\,, \tag{1.2}$$

$f \in L^1(\Omega)$, " $\cdot$ " stands for inner product in $\mathbb{R}^n$, and $\mathbf{n}$ denotes the outward unit normal on $\partial\Omega$.

Problems of this kind have been extensively studied in the literature. In particular, bounds for their solutions are well known to depend not only on the exponent p and on the degree of integrability of the datum f, but also on the regularity of the ground domain Ω. As shown in [Ma2, Ma3], such a regularity can be prescribed in terms of isoperimetric inequalities, and, specifically, via the relative isoperimetric inequality in Ω, which reads

$$\lambda_\Omega(|E|) \leq P(E;\Omega) \qquad \text{for every measurable set } E \subset \Omega \text{ with } |E| \leq |\Omega|/2. \tag{1.3}$$

Here, $P(E;\Omega)$ denotes the perimeter of a measurable set E relative to Ω, and $\lambda_\Omega : [0, |\Omega|/2) \to [0,\infty)$ is the isoperimetric function of Ω. The method of [Ma3] relies upon truncation techniques and on estimates on the level sets of solutions. Roughly speaking, such estimates require, in turn, bounds for the measure of these level sets, whose boundary can overlap with $\partial\Omega$, in terms of the $(n-1)$-dimensional Hausdorff measure of the only part of their boundary which lies inside Ω. This accounts for the presence of the relative perimeter, instead of the whole perimeter, in inequality (1.3), which plays its role at this stage.

The use of isoperimetric inequalities in a priori estimates is quite effective when dealing with problems (1.1) in sufficiently regular domains Ω. This is even more apparent when the Neumann boundary condition is replaced by the (homogeneous) Dirichlet condition: in this case, the isoperimetric inequality to be employed is just the classical isoperimetric inequality involving the standard perimeter in $\mathbb{R}^n$, since the level sets of solutions cannot reach $\partial\Omega$ ([Ma3]), and the relevant a priori estimates take the form of symmetrization comparison principles [Ta1, Ta2]. (See also [We] for an earlier related result, and [Ke, Tr, Va] for accounts of the vast bibliography on developments on these topics.) However, isoperimetric methods need not yield the best possible results in general.

In the present paper we propose, instead, an approach to a priori estimates for solutions to problems (1.1), and for their gradient, relying upon isocapacitary inequalities. In a sense, the isocapacitary inequality in an open set Ω can be regarded as a strengthening of (1.3), when the relative perimeter of sets is replaced by their (condenser) p-capacity. The resulting inequality tells us that

$$\nu_{\Omega,p}(|E|) \le C_p(E,G) \quad \text{for every measurable set } E \subset G \subset \Omega \text{ with } |G| \le |\Omega|/2, \tag{1.4}$$

where $C_p(E,G)$ is the p-capacity of the condenser $(E;G)$ relative to Ω, and $\nu_{\Omega,p} : [0,|\Omega|/2) \to [0,\infty)$ is the isocapacitary function of Ω (we refer to Subsection 2.2 for basic material concerning perimeter and capacity).

The conclusions that will be presented are new, as far as we know, even in the simplest linear case when $p = 2$ and the differential operator in (1.1) is the Laplacian, and improve the available results in the literature under two respects.

First, we obtain estimates in terms of the isocapacitary function $\nu_{\Omega,p}$, which are sharper than corresponding estimates depending on the isoperimetric function λ_Ω. As a consequence, bounds are derived for norms of solutions and of their gradients that are essentially stronger then those which follow via standard techniques, at least when irregular domains are involved.

Second, our method enables us also to deduce capacitary estimates for solutions, which improve customary norm estimates even in regular domains, in that the role of the measure of level sets in the definition of norms is played instead by their capacity.

Our key estimates for solutions u to (1.1), whose precise definition is given in Subsection 2.1, have the form of both rearrangement and of pointwise capacitary inequalities, and are established in Section 3. These inequalities are the starting point for the norm and capacitary bounds for u in terms of norms of the datum f which are proved in Section 4. For example, one very special case of our conclusions, yet giving the flavor of their nature, ensures that if f is bounded and

$$\int_0 \left(\frac{s}{\nu_{\Omega,p}(s)}\right)^{\frac{1}{p-1}} \frac{ds}{s} < \infty\,,$$

then any solution u to (1.1) is bounded as well – see Theorem 4.1 (v), Section 4.
Gradient estimates, resting upon closely related techniques, are the content of Section 5. Finally, applications to special instances are exhibited in Section 6.

2 Preliminaries

2.1 Solutions

Since we are not going to assume a priori any extra integrabilty condition on the datum f and any regularity on the domain Ω, a generalized notion of solution to the Neumann problem (1.1) has to be adopted. The following definition of solution, patterned on that of entropy solution introduced in [BBGGPV], turns out to be well suited for our framework.

Given any $k_1, k_2 \in \mathbb{R}$, with $k_1 < k_2$, let $T_{k_1,k_2} : \mathbb{R} \to \mathbb{R}$ be the function defined as

$$T_{k_1,k_2}(s) = \begin{cases} k_1 & \text{if } \ s < k_1 \\ s & \text{if } \ k_1 \le s \le k_2 \\ k_2 & \text{if } \ k_2 < s \,. \end{cases} \tag{2.1}$$

Then, we set, for $p \ge 1$,

$$W^{1,p}_T(\Omega) = \{u : u \text{ is a measurable function in } \Omega \\ \text{such that } T_{-k,k}(u) \in W^{1,p}(\Omega) \text{ for every } k > 0\} \,. \tag{2.2}$$

Here, $W^{1,p}(\Omega)$ is the standard Sobolev space of weakly differentiable functions which belong to $L^p(\Omega)$ together with their first-order derivatives.
[BBGGPV, Lemma 2.1] ensures that, for every $u \in W^{1,p}_T(\Omega)$, a unique measurable function $V_u : \Omega \to \mathbb{R}^n$ exists such that

$$\nabla\big(T_{-k,k}(u)\big) = V_u \chi_{\{|u|<k\}} \qquad \text{a.e. in } \Omega \tag{2.3}$$

for every $k > 0$, where χ_E denotes the characteristic function of the set E. Furthermore, $u \in W^{1,p}(\Omega)$ if and only if $u \in L^p(\Omega)$ and $V_u \in L^p(\Omega, \mathbb{R}^n)$, and, in this case, $V_u = \nabla u$, the weak gradient of u. In what follows, with abuse of notation, for every $u \in W^{1,p}_T(\Omega)$ we denote V_u by ∇u.
A function $u \in W^{1,p}_T(\Omega)$ is called an entropy solution to (1.1) if

$$\int_\Omega a(x,u,\nabla u) \cdot \nabla\big(T_{k_1,k_2}(u-\varphi)\big)\,dx \le \int_\Omega f(x) T_{k_1,k_2}(u-\varphi)\,dx \tag{2.4}$$

for every $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and every $k_1 < k_2$.
Let us emphasize that, since we are only concerned with a priori estimates, no additional assumption on the function $a(x,t,\xi)$ is really needed. However, in order to make definition (2.4) meaningful for every $u \in W^{1,p}_T(\Omega)$, one can assume that there exist a function $g \in L^{p'}(\Omega)$, where $p' = \frac{p}{p-1}$, and a non-decreasing function $H : [0,\infty) \to [0,\infty)$ such that, for a.e. $x \in \Omega$,

$$|a(x,t,\xi)| \le H(|t|)\big(g(x) + |\xi|^{p-1}\big) \quad \text{for } \ (t,\xi) \in \mathbb{R} \times \mathbb{R}^n \,. \tag{2.5}$$

Actually, since $T_{k_1,k_2}(u-\varphi) \in L^\infty(\Omega)$, the integral on right-hand side of (2.4) is convergent whenever $f \in L^1(\Omega)$. Moreover, inasmuch as $\nabla T_{k_1,k_2}(u-\varphi)$ vanishes outside $\{k_1 + \varphi < u < k_2 + \varphi\}$, and u is essentially bounded in this set if $\varphi \in L^\infty(\Omega)$, also the integral on the left-hand side is convergent when (2.5) is in force.

2.2 Perimeter and capacity

The isoperimetric function $\lambda_\Omega : [0, |\Omega|/2) \to [0, \infty)$ of Ω is defined as

$$\lambda_\Omega(s) = \inf\{P(E, \Omega) : s \leq |E| \leq |\Omega|/2\} \qquad \text{for } s \in [0, |\Omega|/2)\,. \tag{2.6}$$

Here, $P(E; \Omega)$ is the perimeter of E relative to Ω (in the sense of geometric measure theory), which agrees with $\mathcal{H}^{n-1}(\partial^M E \cap \Omega)$, where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure, and $\partial^M E$ stands for the essential boundary of E (see e.g. [AFP, Ma4]).

The relative isoperimetric inequality (1.3) is a straightforward consequence of definition (2.6). The point is that the isoperimetric function λ_Ω is explicitly known only for very special domains, such as balls ([Ma4, BuZa]), half-spaces and convex cones ([LP]). However, the available qualitative and quantitative information on λ_Ω is sufficient for several applications, including Sobolev inequalities ([HK, Ma1, Ma4, MP]), eigenvalue estimates ([Ch, Ci2, Ga]), and the a priori bounds for solutions to Neumann problems mentioned in Section 1 ([Be, Ci2, Fe, MS1, MS2, Ma3]).
In particular, the function λ_Ω is known to be strictly positive in $(0, |\Omega|/2)$ when Ω is connected [Ma4, Lemma 3.2.4]. Moreover, the asymptotic behavior of $\lambda_\Omega(s)$ as $s \to 0^+$ is related to the regularity of the boundary of Ω. For instance, when Ω has a Lipschitz boundary,

$$\lambda_\Omega(s) = O(s^{1/n'}) \qquad \text{as } s \to 0^+ \tag{2.7}$$

([Ma4, Corollary 3.2.1/3]). A parallel result dealing with sets with an Hölder continuous boundary in the plane is contained in [Ci1]. More precise asymptotic estimates for λ_Ω can be derived under additional assumptions on $\partial\Omega$ (see e.g. [CY, Ci3]).

As mentioned in Section 1, the main novelty in this paper is in the use of estimates for the Lebesgue measure of subsets of Ω via their relative condenser capacity instead of their relative perimeter. In order to give a precise definition of this capacity, let us preliminarily recall a few basic facts from potential theory.

The standard p-capacity of a set $E \subset \Omega$ can be defined for $p \geq 1$ as

$$C_p(E) = \inf\left\{\int_\Omega |\nabla u|^p\,dx : u \in W^{1,p}_0(\Omega), u \geq 1 \text{ in some neighbourhood of } E\right\}, \tag{2.8}$$

where $W^{1,p}_0(\Omega)$ denotes the closure in $W^{1,p}(\Omega)$ of the set of smooth compactly supported functions in Ω. A property concerning the pointwise behavior of functions is said to hold C_p-quasi everywhere in Ω, C_p-q.e. for short, if it is fulfilled outside a set of p-capacity zero.
Every function $u \in W^{1,p}(\Omega)$ has a representative $\widetilde{u}$ - its precise representative - which is C_p-quasi continuous, in the sense that for every $\varepsilon > 0$, there exists a set $A \subset \Omega$, with $C_p(A) < \varepsilon$, such that $f_{|\Omega\setminus A}$ is continuous in $\Omega \setminus A$. The function $\widetilde{u}$ is unique, up to subsets of p-capacity zero. In what follows, we assume that any function $u \in W^{1,p}(\Omega)$ agrees with its precise representative.
One has, for every set $E \subset \Omega$,

$$C_p(E) = \inf\left\{\int_\Omega |\nabla u|^p\,dx : u \in W^{1,p}_0(\Omega), u \geq 1 \ C_p\text{-q.e. in } E\right\} \tag{2.9}$$

- see e.g. [MZ, Corollary 2.25].

The following definition is consistent with (2.9). Given sets $E \subset G \subset \Omega$, we define the capacity $C_p(E, G)$ of the condenser $(E; G)$ relative to Ω as
(2.10)

$$C_p(E, G) = \inf\left\{\int_\Omega |\nabla u|^p\,dx : u \in W^{1,p}(\Omega), u \geq 1\ C_p\text{-q.e. in } E \text{ and } u \leq 0\ C_p\text{-q.e. in } \Omega \setminus G\right\}.$$

The p-isocapacitary function $\nu_{\Omega,p} : [0, |\Omega|/2) \to [0, \infty)$ of Ω is then given by

$$(2.11)\quad \nu_{\Omega,p}(s) = \inf \{C_p(E,G) : E \text{ and } G \text{ are measurable subsets of } \Omega \text{ such that } E \subset G \subset \Omega,\ s \le |E| \text{ and } |G| \le |\Omega|/2\} \qquad \text{for } s \in [0, |\Omega|/2).$$

The function $\nu_{\Omega,p}$ is clearly non-decreasing. In what follows, we shall always deal with the left-continuous representative of $\nu_{\Omega,p}$, which, owing to the monotonicity of $\nu_{\Omega,p}$, is pointwise dominated by the right-hand side of (2.11).

With definition (2.11) in place, the isocapacitary inequality in Ω takes the form (1.4). Let us mention that inequality (1.4) can be used, for instance, to characterize Sobolev inequalities for arbitrary functions in $W^{1,p}(\Omega)$ ([Ma4, MP]).

If $p > 1$, the functions λ_Ω and $\nu_{\Omega,p}$ are related by

$$(2.12)\qquad \nu_{\Omega,p}(s) \ge \left(\int_s^{|\Omega|/2} \frac{dr}{\lambda_\Omega(r)^{p'}} \right)^{1-p} \qquad \text{for } s \in (0, |\Omega|/2)$$

([Ma4, Proposition 4.3.4/1]). In particular, $\nu_{\Omega,p}$ is strictly positive in $(0, |\Omega|/2)$ for every connected open set having finite measure. Moreover,

$$(2.13)\qquad \nu_{\Omega,p}(|\Omega|/2-) = \infty\,,$$

where the expression on the left-hand side of (2.13) stands for $\lim_{s \to |\Omega|/2^-} \nu_{\Omega,p}(s)$.

Let us emphasize that, in general, a reverse inequality (even up to a multiplicative constant) does not hold in (2.12). As will be clear in the next sections, this accounts for the fact that estimates for solutions to (1.1) depending on $\nu_{\Omega,p}$ are more accurate than those resting upon λ_Ω. However, the two sides of (2.12) are equivalent when Ω is sufficiently regular. This happens, for instance, if Ω has a Lipschitz boundary. In this case, if $1 < p < n$, then

$$(2.14)\qquad \nu_{\Omega,p}(s) = O(s^{\frac{n-p}{n}}) \qquad \text{as } s \to 0^+\,.$$

Equation (2.14) is a consequence of (2.7), (2.12) and of the fact that

$$C_p(\{y : |y - x| < r\}) = n\omega_n \Big(\frac{n-p}{p-1}\Big)^{p-1} r^{n-p}$$

for every $x \in \mathbb{R}^n$ and $r > 0$, where ω_n denotes the measure of the unit ball in $\mathbb{R}^n$.

The following relations between the condenser capacity of level sets of functions from $W^{1,p}_T(\Omega)$ and integrals of their gradient over their level surfaces will be crucial in our approach.
The same argument as in the proof of [Ma4, Lemma 2.2.2/1] tells us that, if $E \subset G \subset \Omega$, then

$$(2.15)\quad C_p(E,G) = \inf \left\{ \left(\int_0^1 \frac{dt}{\big(\int_{\{u=t\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}(x)\big)^{1/(p-1)}} \right)^{1-p} : u \in W^{1,p}(\Omega),\ u \ge 1\ C_p\text{-q.e. in } E \text{ and } u \le 0\ C_p\text{-q.e. in } \Omega \setminus G \right\}.$$

As a consequence, for every $u \in W^{1,p}_T(\Omega)$, one has that

$$(2.16)\quad C_p(\{u \ge t\}, \{u > 0\}) \le \left(\int_0^t \frac{d\tau}{\big(\int_{\{u=\tau\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}(x)\big)^{1/(p-1)}} \right)^{1-p} \qquad \text{for } t > 0.$$

Actually, if $u \in W^{1,p}_T(\Omega)$, $t > 0$, $E = \{u \geq t\}$, and $G = \{u > 0\}$, then $\frac{1}{t}T_{k_1,k_2}(u)$ is an admissible trial function on the right-hand side of (2.15), provided that $k_1 < 0$ and $k_2 > t$.

Now, given $u \in W^{1,p}_T(\Omega)$, define $\psi_u : [0,\infty) \to [0,\infty)$ as

$$\psi_u(t) = \int_0^t \frac{d\tau}{\left(\int_{\{u=\tau\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}(x)\right)^{1/(p-1)}} \qquad \text{for } t \geq 0\,. \tag{2.17}$$

Moreover, set

$$\mathrm{med}(u) = \sup\{t \in \mathbb{R} : |\{u > t\}| \geq |\Omega|/2\}\,, \tag{2.18}$$

the median of u, and observe that, if

$$\mathrm{med}(u) = 0\,, \tag{2.19}$$

then

$$|\{u > 0\}| \leq |\Omega|/2 \qquad \text{and} \qquad |\{u < 0\}| \leq |\Omega|/2\,. \tag{2.20}$$

Thus, given any $u \in W^{1,p}_T(\Omega)$ fulfilling (2.19), from (1.4) applied with $E = \{u_\pm \geq t\}$ and $G = \{u_\pm > 0\}$, and from (2.16) applied with u replaced by u_+ and u_-, we deduce that

$$\nu_{\Omega,p}(|\{u_\pm \geq t\}|) \leq \psi_{u_\pm}(t)^{1-p} \qquad \text{for } t > 0. \tag{2.21}$$

Here, $u_+ = \frac{|u|+u}{2}$ and $u_- = \frac{|u|-u}{2}$, the positive and the negative part of u, respectively.

3 Rearrangement and pointwise capacitary inequalities

The present section is devoted to fundamental estimates for the decreasing rearrangement of solutions to problem (1.1), and for the capacity of their level sets, in terms of the decreasing rearrangement of the datum f. These estimates are the object of Theorems 3.1 and 3.4, respectively.

In what follows, the distribution function of a measurable function $u : \Omega \to \mathbb{R}$ is denoted by $\mu_u : [0,\infty) \to [0,\infty)$, and defined as

$$\mu_u(t) = |\{x \in \Omega : |u(x)| \geq t\}| \qquad \text{for } t \geq 0.$$

The decreasing rearrangement $u^* : (0, |\Omega|) \to [0,\infty)$ of u is given by

$$u^*(s) = \sup\{t \geq 0 : \mu_u(t) \geq s\} \qquad \text{for } s \in (0, |\Omega|).$$

Theorem 3.1 *Let Ω be a connected open subset of $\mathbb{R}^n$ having finite measure, and let $\overline{u}$ be an entropy solution to problem* (1.1)*. Let*

$$u = \overline{u} - \mathrm{med}(\overline{u})\,. \tag{3.1}$$

Then,

$$u^*_\pm(s) \leq \int_s^{|\Omega|/2} \left(\int_0^r f^*_\pm(\rho)d\rho\right)^{\frac{1}{p-1}} d(-D\nu_{\Omega,p}^{\frac{1}{1-p}})(r) \qquad \textit{for } s \in (0, |\Omega|/2)\,. \tag{3.2}$$

Here, $D\nu_{\Omega,p}^{\frac{1}{1-p}}$ *denotes the derivative in the sense of measures of the non-increasing function* $\nu_{\Omega,p}^{\frac{1}{1-p}}$*, and* f_+ *and* f_- *denote the positive part and the negative part of* f*, respectively.*
Inequality (3.2) continues to hold even if $\nu_{\Omega,p}$ *is replaced by any non-decreasing, left-continuous function* $\nu : [0, |\Omega|/2) \to [0, \infty)$ *fulfilling*

$$\nu(s) \leq \nu_{\Omega,p}(s) \qquad \text{for } s \in [0, |\Omega|/2), \tag{3.3}$$

provided that the expression $\nu(|\Omega|/2-)^{\frac{1}{1-p}}\big(\int_0^{|\Omega|/2} f_\pm^*(r)dr\big)^{\frac{1}{p-1}}$ *is added to the right-hand side in case* $\nu(|\Omega|/2-) < \infty$*.*

Theorem 3.1 should be compared with a parallel result, to which we alluded above, relying upon the relative isoperimetric inequality (1.3) and yielding, under the same assumptions,

$$u_\pm^*(s) \leq \int_s^{|\Omega|/2} \left(\int_0^r f_\pm^*(\rho)d\rho\right)^{\frac{1}{p-1}} \frac{dr}{\lambda_\Omega(r)^{p'}} \qquad \text{for } s \in (0, |\Omega|/2) \tag{3.4}$$

– see [Ma3] for the linear case, and [Ci2, MS2] for the nonlinear case. The next Proposition shows that the right-hand side of (3.2) never exceeds the right-hand side of (3.4), thus demonstrating that estimate (3.2) is always at least as sharp as (3.4). As anticipated in Section 1, the former is actually essentially stronger than the latter for certain domains Ω - see e.g. Example 3, Section 6.

Proposition 3.2 *Let* Ω *be a connected open subset of* $\mathbb{R}^n$ *having finite measure, let* $p > 1$*, and let* $\nu_{\Omega,p}$ *and* λ_Ω *be the isocapacitary and the isoperimetric functions of* Ω*, respectively. Assume that* $f \in L^1(\Omega)$*. Then*

$$\int_s^{|\Omega|/2} \left(\int_0^r f_\pm^*(\rho)d\rho\right)^{\frac{1}{p-1}} d(-D\nu_{\Omega,p}^{\frac{1}{1-p}})(r) \leq \int_s^{|\Omega|/2} \left(\int_0^r f_\pm^*(\rho)d\rho\right)^{\frac{1}{p-1}} \frac{dr}{\lambda_\Omega(r)^{p'}} \tag{3.5}$$

for $s \in (0, |\Omega|/2)$*.*

Proof. Fix any $s \in (0, |\Omega|/2)$. On calling $\phi : (0, |\Omega|/2) \to [0, \infty)$ the non-decreasing function given by

$$\phi(r) = \chi_{(s,|\Omega|/2)}(r)\left(\int_0^r f_\pm^*(\rho)d\rho\right)^{\frac{1}{p-1}} \qquad \text{for } r \in (0, |\Omega|/2),$$

inequality (3.5) reads

$$\int_0^{|\Omega|/2} \phi(r)\, d(-D\nu_{\Omega,p}^{\frac{1}{1-p}})(r) \leq \int_0^{|\Omega|/2} \phi(r)\, \frac{dr}{\lambda_\Omega(r)^{p'}}\,. \tag{3.6}$$

Since $\phi(r) = \int_{[0,r)} d(D\phi)(\rho)$ for $r \in (0, |\Omega|/2)$, and $D\phi$ is a nonnegative measure, inequality (3.6) is easily seen to hold, via Fubini's theorem, if

$$\int_\rho^{|\Omega|/2} d(-D\nu_{\Omega,p}^{\frac{1}{1-p}})(r) \leq \int_\rho^{|\Omega|/2} \frac{dr}{\lambda_\Omega(r)^{p'}} \qquad \text{for } \rho \in (0, |\Omega|/2)\,,$$

namely, if

$$\nu_{\Omega,p}^{\frac{1}{1-p}}(\rho) \leq \int_\rho^{|\Omega|/2} \frac{dr}{\lambda_\Omega(r)^{p'}} \qquad \text{for } \rho \in (0, |\Omega|/2)\,. \tag{3.7}$$

Inequality (3.7) is nothing but (2.12). $\square$

Remark 3.3 An argument analogous to that in the proof of Proposition 3.2 shows that, if the domain Ω is regular enough for $\nu_{\Omega,p}$ and the right-hand side of (2.12) to be equivalent, namely if a constant C exists such that

$$\nu_{\Omega,p}(s) \le C\left(\int_s^{|\Omega|/2} \frac{dr}{\lambda_\Omega(r)^{p'}}\right)^{1-p} \qquad \text{for } s \in (0, |\Omega|/2), \tag{3.8}$$

then a reverse inequality holds in (3.5) (up to a multiplicative constant) as well. As a consequence, for this kind of domains, estimates (3.2) and (3.4) enable one to derive exactly the same a priori bounds for u.

Let us now turn to a pointwise capacitary estimate for solutions to problem (1.1), which is the object of the next theorem. In what follows, we use the abridged notation

$$C_{p,u_\pm}(t) = C_p(\{u_\pm \ge t\}, \{u_\pm > 0\}) \qquad \text{for } t > 0. \tag{3.9}$$

Theorem 3.4 *Under the same assumptions as in Theorem 3.1, let $F_\pm : [0,\infty) \to [0,\infty)$ be the functions defined as*

$$F_\pm(\tau) = \int_0^\tau \left(\int_0^{\nu_{\Omega,p}^{-1}(r^{1-p})} f_\pm^*(\rho)d\rho\right)^{\frac{1}{p-1}} dr \qquad \textit{for } \tau \in [0,\infty), \tag{3.10}$$

where $\nu_{\Omega,p}^{-1} : [0,\infty) \to [0, |\Omega|/2)$ denotes the generalized right-continuous inverse of $\nu_{\Omega,p}$. Then,

$$C_{p,u_\pm}(t) \le \left(F_\pm{}^{-1}(t)\right)^{1-p} \qquad \textit{for } t \in (0, \operatorname{esssup} u_\pm). \tag{3.11}$$

The same statement continues to hold if $\nu_{\Omega,p}$ is replaced in (3.10) *by any non-decreasing, left-continuous function $\nu : [0, |\Omega|/2) \to [0,\infty)$ fulfilling* (3.3).

We shall first establish Theorem 3.4, and we shall then make use of (3.11) in the proof of Theorem 3.1. The starting point in our derivation of (3.11) is a basic inequality contained in the following lemma. The relevant inequality is a counterpart for entropy solutions to (1.1) of analogous estimates for weak solutions to Dirichlet problems ([Ta2]; see also [Ma3] for the linear case).

Lemma 3.5 *Under the same assumptions as in Theorem 3.1,*

$$\int_{\{u_\pm = t\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}(x) \le \int_0^{\mu_{u_\pm}(t)} f_\pm^*(r)\, dr \qquad \textit{for a.e. } t > 0. \tag{3.12}$$

Proof. Since $u \in W_T^{1,p}(\Omega)$, given any $t, h > 0$, we may choose $\varphi = T_{-t,t}(u)$ and $k_1 = 0$, $k_2 = h$ in definition (2.4). Inasmuch as

$$T_{0,h}(u - T_{-t,t}(u)) = \begin{cases} 0 & \text{if } \ u < t \\ u - t & \text{if } \ t \le u \le t+h \\ h & \text{if } \ t+h < u\,, \end{cases} \tag{3.13}$$

we get

$$\int_{\{t<u<t+h\}} a(x, \overline{u}, \nabla u) \cdot \nabla u\, dx \le \int_{\{t<u\le t+h\}} f(x)(u(x) - t)\, dx + h \int_{\{u>t+h\}} f(x)\, dx\,. \tag{3.14}$$

Dividing through by h in (3.14) and making use of ellipticity condition (1.2) yield

$$\frac{1}{h}\int_{\{t<u<t+h\}} |\nabla u|^p\,dx \le \frac{1}{h}\int_{\{t<u\le t+h\}} f(x)(u(x)-t)\,dx + \int_{\{u>t+h\}} f(x)\,dx\,. \tag{3.15}$$

Hence, on passing to the limit as $h \to 0^+$, and making use of the coarea formula for Sobolev functions ([BrZi]) applied to truncations of u, we get

$$\int_{\{u=t\}} |\nabla u|^p\,d\mathcal{H}^{n-1}(x) \le \int_{\{u>t\}} f(x)\,dx \qquad \text{for a.e. } t>0. \tag{3.16}$$

By the Hardy-Littlewood inequality ([BS, Theorem 2.2, Chap. 2]),

$$\int_{\{u>t\}} f(x)\,dx \le \int_{\{u>t\}} f_+(x)\,dx \le \int_0^{\mu_{u_+}(t)} f_+^*(r)\,dr \qquad \text{for } t>0. \tag{3.17}$$

Inequality (3.12) for u_+ follows from (3.16) and (3.17).
Replacing $T_{0,h}$ by $T_{-h,0}$ in (3.13) leads to inequality (3.12) for u_-, via an analogous argument.
□

Proof of Theorem 3.4. We prove inequality (3.11) for u_+, the proof for u_- being analogous. Moreover, we deal with the general case where $\nu_{\Omega,p}$ is replaced by any function ν as in (3.3). Inequalities (2.21) and (3.3) entail that

$$\nu(\mu_{u_+}(t)) \le \psi_{u_+}(t)^{1-p} \qquad \text{for } t\in(0,\operatorname{esssup} u). \tag{3.18}$$

The generalized right-continuous inverse ν^{-1} of ν fulfills

$$s \le \nu^{-1}(\nu(s)) \qquad \text{for } s\in[0,|\Omega|/2). \tag{3.19}$$

From (3.18) and (3.19) one gets that

$$\mu_{u_+}(t) \le \nu^{-1}\big(\psi_{u_+}(t)^{1-p}\big) \qquad \text{for } t\in(0,\operatorname{esssup} u). \tag{3.20}$$

The function ψ_{u_+} is obviously locally absolutely continuous, and

$$\psi'_{u_+}(t) = \frac{1}{\big(\int_{\{u=t\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}(x)\big)^{1/(p-1)}} \qquad \text{for a.e. } t\in(0,\operatorname{esssup} u)\,. \tag{3.21}$$

Combining (3.12), (3.20) and (3.21) yields

$$1 \le \psi'_{u_+}(t)^{p-1}\int_0^{\nu^{-1}(\psi_{u_+}(t)^{1-p})} f_+^*(r)\,dr \qquad \text{for a.e. } t\in(0,\operatorname{esssup} u)\,. \tag{3.22}$$

Hence,

$$\begin{aligned} t &\le \int_0^t \psi'_{u_+}(\tau)\bigg(\int_0^{\nu^{-1}(\psi_{u_+}(\tau)^{1-p})} f_+^*(r)\,dr\bigg)^{\frac{1}{p-1}} d\tau \\ &= \int_0^{\psi_{u_+}(t)} \bigg(\int_0^{\nu^{-1}(\rho^{1-p})} f_+^*(r)\,dr\bigg)^{\frac{1}{p-1}} d\rho = F_+(\psi_{u_+}(t)) \qquad \text{for } t\in(0,\operatorname{esssup} u)\,. \end{aligned} \tag{3.23}$$

Note that the first equality holds in (3.23) since $F_+\circ\psi_{u_+}$ is a locally absolutely continuous function, being the composition of the locally absolutely continuous function ψ_{u_+} and of the Lipschitz continuous function F_+. From (3.23) one gets

$$F_+{}^{-1}(t) \le \psi_{u_+}(t) \qquad \text{for } t\in(0,\operatorname{esssup} u)\,, \tag{3.24}$$

whence (3.11) follows, owing to (2.16). □

We conclude this section with the proof of Theorem 3.1.

Proof of Theorem 3.1. As in the preceding proof, we focus on u_+, and deal with the general case where $\nu_{\Omega,p}$ is replaced by any function ν satisfying (3.3).
Upon choosing $t = u_+^*(s)$ in (3.23), one gets

$$u_+^*(s) \leq \int_0^{\psi_{u_+}(u_+^*(s))} \left(\int_0^{\nu^{-1}(r^{1-p})} f_+^*(\rho)\, d\rho\right)^{\frac{1}{p-1}} dr \qquad \text{for } s \in (0, |\Omega|/2)\,. \tag{3.25}$$

By (3.18),

$$\nu\big(\mu_{u_+}(u_+^*(s))\big) \leq \psi_{u_+}(u_+^*(s))^{1-p} \qquad \text{for } s \in (0, |\Omega|/2). \tag{3.26}$$

On the other hand,

$$s \leq \mu_{u_+}(u_+^*(s)) \qquad \text{for } s \in (0, |\Omega|/2). \tag{3.27}$$

Inequalities (3.26) and (3.27) entail that

$$\psi_{u_+}(u_+^*(s)) \leq \nu(s)^{\frac{1}{1-p}} \qquad \text{for } s \in (0, |\Omega|/2). \tag{3.28}$$

Coupling (3.25) and (3.28) yields

$$u_+^*(s) \leq \int_0^{\nu(s)^{\frac{1}{1-p}}} \left(\int_0^{\nu^{-1}(r^{1-p})} f_+^*(\rho)\, d\rho\right)^{\frac{1}{p-1}} dr \qquad \text{for } s \in (0, |\Omega|/2)\,. \tag{3.29}$$

In order to complete the proof, our task is to show that

$$\begin{aligned} &\int_0^{\nu(s)^{\frac{1}{1-p}}} \left(\int_0^{\nu^{-1}(r^{1-p})} f_+^*(\rho)\, d\rho\right)^{\frac{1}{p-1}} dr \\ &\quad = \int_s^{|\Omega|/2} \left(\int_0^r f_\pm^*(\rho) d\rho\right)^{\frac{1}{p-1}} d(-D\nu_{\Omega,p}^{\frac{1}{1-p}})(r) + \nu(|\Omega|/2-)^{\frac{1}{1-p}} \left(\int_0^{|\Omega|/2} f_\pm^*(r) dr\right)^{\frac{1}{p-1}} \end{aligned} \tag{3.30}$$

for $s \in (0, |\Omega|/2)$. Formally, equation (3.30) follows via a change of variable. However, a rigorous proof requires some care, since the function ν need neither be absolutely continuous, nor strictly increasing.
The left-hand side of (3.30) agrees with $F_+\big(\nu^{\frac{1}{1-p}}(s)\big)$. The function $\nu^{\frac{1}{1-p}}$ is monotone, and hence of (locally) bounded variation in $(0, |\Omega|/2)$, whereas the function F_+ is monotone and Lipschitz continuous. Hence, $F_+ \circ \nu^{\frac{1}{1-p}}$ is of locally bounded variation in $(0, |\Omega|/2)$, and the chain rule for functions of bounded variation (see e.g. [AFP, Theorem 3.99]) tells us that

$$D\big(F_+ \circ \nu^{\frac{1}{1-p}}\big) = F_+{}'\big(\nu^{\frac{1}{1-p}}\big)\big[\big(\nu^{\frac{1}{1-p}}\big)'\mathcal{L}^1 + D^c\nu^{\frac{1}{1-p}}\big] + \big[F_+\big(\nu_+^{\frac{1}{1-p}}\big) - F_+\big(\nu_-^{\frac{1}{1-p}}\big)\big]\mathcal{H}^0 \lfloor J_\nu\,. \tag{3.31}$$

Here, $\mathcal{L}^1$ denotes the one-dimensional Lebesgue measure, $D^c\nu^{\frac{1}{1-p}}$ stands for the Cantor part of the measure $D\nu^{\frac{1}{1-p}}$, $\mathcal{H}^0$ is the 0-dimensional Hausdorff measure, i.e. the counting measure, J_ν is the jump set of ν, namely

$$J_\nu = \big\{s \in (0, |\Omega|/2) :\, \nu(s-) \neq \nu(s+)\big\},$$

and

$$\nu_\pm(s) = \nu(s\pm) \qquad \text{for } s \in (0, |\Omega|/2)\,. \tag{3.32}$$

Consider the first addendum on the right-hand side of (3.31). Let us set

$$\widetilde{D}\nu^{\frac{1}{1-p}} = \big(\nu^{\frac{1}{1-p}}\big)'\mathcal{L}^1 + D^c\nu^{\frac{1}{1-p}}, \tag{3.33}$$

the diffuse part of $\nu^{\frac{1}{1-p}}$. We have

$$F_+{}'\big(\nu^{\frac{1}{1-p}}\big)\widetilde{D}\nu^{\frac{1}{1-p}} = \bigg(\int_0^{\nu^{-1}(\nu)} f_+^*(r)\,dr\bigg)^{\frac{1}{p-1}}\widetilde{D}\nu^{\frac{1}{1-p}}. \tag{3.34}$$

Since $\nu^{-1}(\nu(s)) = s$ if s does not belong to any open interval where ν is constant, and since the support of $\widetilde{D}\nu^{\frac{1}{1-p}}$ is disjoint from the union of all such intervals, one infers from (3.34) that

$$F_+{}'\big(\nu^{\frac{1}{1-p}}(\cdot)\big)\widetilde{D}\nu^{\frac{1}{1-p}} = \bigg(\int_0^{\cdot} f_+^*(r)\,dr\bigg)^{\frac{1}{p-1}}\widetilde{D}\nu^{\frac{1}{1-p}}. \tag{3.35}$$

As for the second addendum on the right-hand side of (3.31), we have, for each $s \in J_\nu$,

$$\begin{aligned}
&F_+\big(\nu_+^{\frac{1}{1-p}}(s)\big) - F_+\big(\nu_-^{\frac{1}{1-p}}(s)\big)\\
&= \int_0^{\nu_+^{\frac{1}{1-p}}(s)}\bigg(\int_0^{\nu^{-1}(r^{1-p})} f_+^*(\rho)\,d\rho\bigg)^{\frac{1}{p-1}}dr - \int_0^{\nu_-^{\frac{1}{1-p}}(s)}\bigg(\int_0^{\nu^{-1}(r^{1-p})} f_+^*(\rho)\,d\rho\bigg)^{\frac{1}{p-1}}dr\\
&= -\int_{\nu_+^{\frac{1}{1-p}}(s)}^{\nu_-^{\frac{1}{1-p}}(s)}\bigg(\int_0^{\nu^{-1}(r^{1-p})} f_+^*(\rho)\,d\rho\bigg)^{\frac{1}{p-1}}dr\,.
\end{aligned} \tag{3.36}$$

Since $\nu^{-1}(t) = s$ if $s \in J_\nu$ and $t \in (\nu_-(s), \nu_+(s))$,

$$\int_0^{\nu^{-1}(r^{1-p})} f_+^*(\rho)\,d\rho = \int_0^s f_+^*(\rho)\,d\rho \qquad \text{if } r \in \big(\nu_+^{\frac{1}{1-p}}(s), \nu_-^{\frac{1}{1-p}}(s)\big). \tag{3.37}$$

Thus, (3.36) and (3.37) tell us that

$$\begin{aligned}
\big[F_+\big(\nu_+^{\frac{1}{1-p}}(\cdot)\big) - F_+\big(\nu_-^{\frac{1}{1-p}}(\cdot)\big)\big]\mathcal{H}^0\lfloor J_\nu &= \bigg(\int_0^{\cdot} f_+^*(r)\,dr\bigg)^{\frac{1}{p-1}}\big[\nu_+^{\frac{1}{1-p}}(\cdot) - \nu_-^{\frac{1}{1-p}}(\cdot)\big]\mathcal{H}^0\lfloor J_\nu\\
&= \bigg(\int_0^{\cdot} f_+^*(r)\,dr\bigg)^{\frac{1}{p-1}} D^J\nu^{\frac{1}{1-p}}\,,
\end{aligned} \tag{3.38}$$

where $D^J\nu^{\frac{1}{1-p}}$ denotes the jump part of $D\nu^{\frac{1}{1-p}}$. Combining (3.31), (3.33), (3.34) and (3.38), and recalling that $D = \widetilde{D} + D^J$, imply that

$$D\big(F_+\circ\nu^{\frac{1}{1-p}}\big) = \bigg(\int_0^{\cdot} f_+^*(r)\,dr\bigg)^{\frac{1}{p-1}} D\nu^{\frac{1}{1-p}}\,. \tag{3.39}$$

Consequently,

$$\begin{aligned}(3.40)\qquad & \int_0^{\nu^{\frac{1}{1-p}}(s)} \Big(\int_0^{\nu^{-1}(r^{1-p})} f_+^*(\rho)\,d\rho\Big)^{\frac{1}{p-1}} dr \\ & = F_+\big(\nu^{\frac{1}{1-p}}(s)\big) = -D\big(F_+ \circ \nu^{\frac{1}{1-p}}\big)\big([s,|\Omega|/2)\big) + F_+\big(\nu^{\frac{1}{1-p}}(|\Omega|/2-)\big) \\ & = \int_{[s,|\Omega|/2)} \Big(\int_0^r f_+^*(\rho)d\rho\Big)^{\frac{1}{p-1}} d(-D\nu^{\frac{1}{1-p}})(r) + \nu(|\Omega|/2-)^{\frac{1}{1-p}} \Big(\int_0^{|\Omega|/2} f_\pm^*(r)dr\Big)^{\frac{1}{p-1}}\end{aligned}$$

for $s \in (0, |\Omega|/2)$. The conclusion (3.2) follows from (3.29) and (3.40). Notice that the term $\nu(|\Omega|/2-)^{\frac{1}{1-p}}\big(\int_0^{|\Omega|/2} f_\pm^*(r)dr\big)^{\frac{1}{p-1}}$ is actually missing in inequality (3.2) if $\nu = \nu_{\Omega,p}$, or, more generally, if $\nu(|\Omega|/2-) = \infty$, since $F_+(0) = 0$. □

4 Norm and capacitary estimates

Our purpose here is to show how Theorems 3.1 and 3.4 can be used to derive norm and capacitary estimates, respectively, for solutions to problem (1.1) via norms of f. We shall exhibit a few possible applications. The general advantage in the use of Theorems 3.1 and 3.4, which can also be employed in other instances, is in that they reduce the original n-dimensional a priori estimates for solutions to (1.1) to considerably simpler one-dimensional Hardy type inequalities.

We shall be concerned with estimates involving rearrangement invariant (r.i., for short) norms or quasi-norms. Recall that a r.i. norm in Ω is a Banach function norm $\|\cdot\|_{X(\Omega)}$ such that

$$(4.1)\qquad \|u_1\|_{X(\Omega)} = \|u_2\|_{X(\Omega)} \quad \text{whenever} \quad {u_1}^* = {u_2}^*\,.$$

A r.i. quasi-norm is defined analogously, save that it fulfills the triangle inequality only up to a multiplicative constant.

The simplest instances of r.i. norms and quasi-norms are those in the Lebesgue spaces $L^q(\Omega)$ with $q \in [1, \infty]$ and $q \in (0, 1)$, respectively. The Luxemburg norms in the Orlicz spaces provide an extension of the Lebesgue norms. Given a Young function $Y : [0, \infty) \to [0, \infty]$, namely a convex function vanishing at 0, the Luxemburg norm $\|\cdot\|_{L^Y(\Omega)}$ is defined as

$$(4.2)\qquad \|u\|_{L^Y(\Omega)} = \inf\left\{c > 0 : \int_\Omega Y\Big(\frac{|u(x)|}{c}\Big)dx \le 1\right\}.$$

Besides Lebesgue norms, corresponding to the choice $Y(t) = t^q$ with $q \ge 1$, customary examples of Luxemburg norms include the exponential norms $\|u\|_{\mathrm{Exp}L^\alpha(\Omega)}$, with $\alpha > 0$, associated with the Young functions $Y(t) = e^{t^\alpha} - 1$.

Another useful class of r.i. quasi-norms (which are, in fact, norms in many instances) is that of the so-called classical Lorentz spaces. Given a number $k \in (0, \infty)$ and a locally integrable function $\omega : (0, \infty) \to (0, \infty)$, we set

$$(4.3)\qquad \|u\|_{\Lambda^k_\omega(\Omega)} = \Big(\int_0^{|\Omega|} u^*(s)^k\,\omega(s)\,ds\Big)^{\frac{1}{k}}$$

for any measurable function u in Ω. A variant of (4.3) is obtained on replacing u^* by the function u^{**} defined as

$$u^{**}(s) = \frac{1}{s}\int_0^s u^*(r)\,dr \qquad \text{for } s > 0.$$

Observe that u^{**} is a decreasing function, and $u^*(s) \le u^{**}(s)$ for $s > 0$. The resulting quantity is denoted by $\|u\|_{\Gamma^k_\omega(\Omega)}$. Namely, we set

$$(4.4) \qquad \|u\|_{\Gamma^k_\omega(\Omega)} = \left(\int_0^{|\Omega|} u^{**}(s)^k \, \omega(s) \, ds \right)^{\frac{1}{k}}$$

for any measurable function u in Ω.
The special choice $\omega(s) = s^{\frac{k}{q}-1}$, with $q, k > 0$ in (4.3) and (4.4) yields the standard Lorentz quasi-norms

$$(4.5) \qquad \|u\|_{L^{q,k}(\Omega)} = \left(\int_0^{|\Omega|} \left(s^{\frac{1}{q}} u^*(s)\right)^k \frac{ds}{s} \right)^{\frac{1}{k}}$$

and

$$(4.6) \qquad \|u\|_{L^{(q,k)}(\Omega)} = \left(\int_0^{|\Omega|} \left(s^{\frac{1}{q}} u^{**}(s)\right)^k \frac{ds}{s} \right)^{\frac{1}{k}} .$$

Note that $\| \cdot \|_{L^{q,k}(\Omega)}$ and $\| \cdot \|_{L^{(q,k)}(\Omega)}$ are equivalent (up to multiplicative constants) whenever $q > 1$. Note also that, if $q > 1$ and $0 < k_1 < k_2 \le \infty$, a constant $C = C(q, k_1, k_2)$ exists such that

$$(4.7) \qquad \|u\|_{L^{q,k_2}(\Omega)} \le C \|u\|_{L^{q,k_1}(\Omega)}$$

for every measurable function u in Ω.
Thanks to the equimeasurability of u and u^*, one has

$$(4.8) \qquad \|u\|_{L^q(\Omega)} = \|u^*\|_{L^q(0,|\Omega|)}$$

for $q > 0$, whence, in particular, $\| \cdot \|_{L^{q,q}(\Omega)} = \| \cdot \|_{L^q(\Omega)}$. Moreover, if $q > 1$, then

$$(4.9) \qquad \|u^{**}\|_{L^q(0,|\Omega|)} \le C \|u^*\|_{L^q(0,|\Omega|)}$$

for some constant $C = C(q)$ and for every measurable function u in Ω. An analogue of (4.8) continues to hold for the Luxemburg norms, i.e.

$$(4.10) \qquad \|u\|_{L^Y(\Omega)} = \|u^*\|_{L^Y(0,|\Omega|)}$$

for every Toung function Y and for every measurable function u in Ω. The very definitions (4.3)-(4.6) yield a corresponding property also for $\| \cdot \|_{\Lambda^k_\omega(\Omega)}$, $\| \cdot \|_{\Gamma^k_\omega(\Omega)}$, $\| \cdot \|_{L^{q,k}(\Omega)}$ and $\| \cdot \|_{L^{(q,k)}(\Omega)}$. For further details on r.i. spaces we refer to [BS].

Theorem 4.1 *Let Ω, p, ν, f and u be as in Theorem 3.1. Let $1 \le q \le \infty$ and $0 < \sigma \le \infty$. Assume that $f \in L^q(\Omega)$. Then a constant C exists such that*

$$(4.11) \qquad \|u\|_{L^\sigma(\Omega)} \le C \|f\|_{L^q(\Omega)}^{\frac{1}{p-1}}$$

if either
(i) $1 < q < \infty$, $q(p-1) \le \sigma < \infty$ *and*

$$(4.12) \qquad \sup_{0<s<\frac{|\Omega|}{2}} \frac{s^{\frac{p-1}{\sigma} + \frac{1}{q'}}}{\nu(s)} < \infty ,$$

or
(ii) $1 < q < \infty$, $0 < \sigma < q(p-1)$ *and*

$$\int_0^{|\Omega|/2} \left(\frac{s}{\nu(s)}\right)^{\frac{\sigma q}{q(p-1)-\sigma}} ds < \infty \,, \tag{4.13}$$

or
(iii) $\sigma = \infty$, $q(p-1) \le 1$ *and*

$$\sup_{0<s<\frac{|\Omega|}{2}} \frac{s^{\frac{1}{q'}}}{\nu(s)} < \infty \,, \tag{4.14}$$

or
(iv) $\sigma = \infty$, $1 < q(p-1) < \infty$ *and*

$$\int_0^{|\Omega|/2} \left(\frac{s^{2-p}}{\nu(s)}\right)^{\frac{q}{q(p-1)-1}} ds < \infty \,, \tag{4.15}$$

or
(v) $\sigma = \infty$, $q = \infty$ *and*

$$\int_0^{|\Omega|/2} \left(\frac{s}{\nu(s)}\right)^{\frac{1}{p-1}} \frac{ds}{s} < \infty \,, \tag{4.16}$$

or
(vi) $\sigma < \infty$, $q = \infty$ *and*

$$\int_0^{|\Omega|/2} \left(\left(\frac{s}{\nu(s)}\right)^{\frac{1}{p-1}} + \int_s^{|\Omega|/2} \left(\frac{r}{\nu(r)}\right)^{\frac{1}{p-1}} \frac{dr}{r}\right)^{\sigma} ds < \infty \,, \tag{4.17}$$

or
(vii) $\sigma < \infty$, $q = 1$ *and*

$$\int_0^{|\Omega|/2} \frac{1}{\nu(s)^{\frac{\sigma}{p-1}}} ds < \infty \,. \tag{4.18}$$

Moreover, in each one of cases (i)-(vii), the constant C in (4.11) *depends only on the quantity on the left-hand side of* (4.12)-(4.18)*, respectively, and on p, q, σ, $|\Omega|$ and $\nu^{\frac{1}{1-p}}(|\Omega|/2-)$.*

Proof. Our starting point is Theorem 3.1. Let us assume, throughout the proof, that $\nu(|\Omega|/2-) = \infty$. If this is not the case, the extra term appearing on the right-hand side of (3.2) in that theorem can be easily estimated by Hölder's inequality.
Assume first that $0 < \sigma < \infty$ and $1 < q < \infty$, and hence that we are dealing with cases (i)-(ii). Owing to (4.8) and (4.9), inequality (4.11) will follow if we show that

$$\left(\int_0^{|\Omega|/2} u_\pm^*(s)^\sigma ds\right)^{\frac{1}{\sigma}} \le C\left(\int_0^{|\Omega|/2} {f_\pm}^{**}(s)^q ds\right)^{\frac{1}{q(p-1)}} \tag{4.19}$$

for some constant C depending on the quantities specified in the statement, and for every $f \in L^q(\Omega)$. In particular, the constant in (4.11) will depend only on the constant C in (4.19) and on q. By Theorem 3.1, inequality (4.19) is in turn reduced to showing that
(4.20)

$$\left(\int_0^{|\Omega|/2} \left(\int_s^{|\Omega|/2} {f_\pm}^{**}(r)^{\frac{1}{p-1}} r^{\frac{1}{p-1}} \, d(-D\nu^{\frac{1}{1-p}})(r)\right)^{\sigma} ds\right)^{\frac{1}{\sigma}} \le C\left(\int_0^{|\Omega|/2} {f_\pm}^{**}(s)^q ds\right)^{\frac{1}{q(p-1)}}$$

for every $f \in L^q(\Omega)$. Thus, upon setting $\phi = (f_{\pm}{}^{**})^{\frac{1}{p-1}}$, we are led to prove that

$$\Big(\int_0^{|\Omega|/2}\Big(\int_s^{|\Omega|/2}\phi(r)r^{\frac{1}{p-1}}\,d(-D\nu^{\frac{1}{1-p}})(r)\Big)^{\sigma}ds\Big)^{\frac{1}{\sigma}} \leq C\Big(\int_0^{|\Omega|/2}\phi(s)^{q(p-1)}ds\Big)^{\frac{1}{q(p-1)}} \tag{4.21}$$

for every non-increasing function $\phi : (0, |\Omega|/2) \to [0, \infty)$. Inequality (4.21) can be handled by [Go, Theorem 1.1]. Assume first that

$$q(p-1) > 1. \tag{4.22}$$

Then [Go, Theorem 1.1] tells us that inequality (4.21) holds provided that the quantities H and K given by

$$H = \begin{cases} \sup\limits_{0<\rho<\frac{|\Omega|}{2}} \dfrac{\big(\int_0^{\rho}\big(\int_s^{\rho} r^{\frac{1}{p-1}}\,d(-D\nu^{\frac{1}{1-p}})(r)\big)^{\sigma}ds\big)^{\frac{1}{\sigma}}}{\rho^{\frac{1}{q(p-1)}}} & \text{if } \ q(p-1) \leq \sigma \\ \Big(\int_0^{|\Omega|/2}\Big(\int_0^{\rho}\Big(\int_s^{\rho} r^{\frac{1}{p-1}}\,d(-D\nu^{\frac{1}{1-p}})(r)\Big)^{\sigma}ds\Big)^{\frac{q(p-1)}{q(p-1)-\sigma}}\rho^{\frac{\sigma}{\sigma-q(p-1)}-1}d\rho\Big)^{\frac{q(p-1)-\sigma}{q\sigma(p-1)}} & \text{if } \ q(p-1) > \sigma\,, \end{cases} \tag{4.23}$$

and

$$K = \begin{cases} \sup\limits_{0<s<\frac{|\Omega|}{2}} s^{\frac{1}{\sigma}}\Big(\int_s^{|\Omega|/2}\Big(\int_s^{\rho} r^{\frac{1}{p-1}}\,d(-D\nu^{\frac{1}{1-p}})(r)\Big)^{\frac{q(p-1)}{q(p-1)-1}}\rho^{\frac{1}{1-q(p-1)}-1}d\rho\Big)^{\frac{q(p-1)-1}{q(p-1)}} & \\ & \text{if } \ q(p-1) \leq \sigma\,, \\ \Big(\int_0^{|\Omega|/2}\Big(\int_s^{|\Omega|/2}\Big(\int_s^{\rho} r^{\frac{1}{p-1}}\,d(-D\nu^{\frac{1}{1-p}})(r)\Big)^{\frac{q(p-1)}{q(p-1)-1}}\rho^{\frac{1}{1-q(p-1)}-1}d\rho\Big)^{\frac{\sigma[q(p-1)-1]}{q(p-1)-\sigma}} s^{\frac{q(p-1)}{q(p-1)-\sigma}-1}ds\Big)^{\frac{q(p-1)-\sigma}{q(p-1)\sigma}} & \\ & \text{if } \ q(p-1) > \sigma\,, \end{cases} \tag{4.24}$$

are finite. Moreover, the constant C in (4.21) depends only on H,K, p, q, σ and $|\Omega|$. Note that, in fact, the result of [Go] deals with more general inequalities in the whole of $(0, \infty)$; however, as far as (4.21) is concerned, the conditions of [Go, Theorem 1.1] can be easily shown to reduce to the finiteness of H and K.
Consider case (i); namely, assume that

$$q(p-1) \leq \sigma. \tag{4.25}$$

Observe that

$$\int_s^{\rho} r^{\frac{1}{p-1}}\,d(-D\nu^{\frac{1}{1-p}})(r) \leq s^{\frac{1}{p-1}}\nu^{\frac{1}{1-p}}(s) + \frac{1}{p-1}\int_s^{\rho} r^{\frac{1}{p-1}-1}\,\nu^{\frac{1}{1-p}}(r)\,dr \qquad \text{if } 0 < s \leq \rho < |\Omega|/2. \tag{4.26}$$

Thus,

$$H \leq \sup_{0<\rho<\frac{|\Omega|}{2}} \frac{\big(\int_0^{\rho} s^{\frac{\sigma}{p-1}}\nu^{\frac{\sigma}{1-p}}(s)ds\big)^{\frac{1}{\sigma}} + \frac{1}{p-1}\big(\int_0^{\rho}\big(\int_s^{\rho} r^{\frac{1}{p-1}-1}\,\nu^{\frac{1}{1-p}}(r)\big)^{\sigma}ds\big)^{\frac{1}{\sigma}}}{\rho^{\frac{1}{q(p-1)}}}\,, \tag{4.27}$$

and

$$\begin{aligned} K \leq C &\sup_{0<s<\frac{|\Omega|}{2}} s^{\frac{1}{\sigma}+\frac{1}{p-1}}\nu^{\frac{1}{1-p}}(s)s^{-\frac{1}{q(p-1)}} \\ &+ C\sup_{0<s<\frac{|\Omega|}{2}} s^{\frac{1}{\sigma}}\Big(\int_s^{|\Omega|/2}\Big(\int_s^{\rho} r^{\frac{1}{p-1}-1}\,\nu^{\frac{1}{1-p}}(r)\,dr\Big)^{\frac{q(p-1)}{q(p-1)-1}}\rho^{\frac{1}{1-q(p-1)}-1}d\rho\Big)^{\frac{q(p-1)-1}{q(p-1)}} \end{aligned} \tag{4.28}$$

for some constant $C = C(p, q, \sigma)$. Denote by K_1 and K_2 the addenda on the right-hand side of (4.28). Assumption (4.12) is equivalent to

$$K_1 < \infty\,, \tag{4.29}$$

and can be written as

$$s^{\frac{1}{p-1}}\nu^{\frac{1}{1-p}}(s) \le C_1 s^{\frac{1}{q(p-1)}-\frac{1}{\sigma}} \qquad \text{for } s \in (0, |\Omega|/2), \tag{4.30}$$

where C_1 agrees with the left-hand side of (4.12). Furthermore,

$$K_2 \le C \sup_{0<s<\frac{|\Omega|}{2}} s^{\frac{1}{\sigma}}\Big(\int_s^{|\Omega|/2}\Big(\int_s^\rho r^{\frac{1}{q(p-1)}-\frac{1}{\sigma}-1}\,dr\Big)^{\frac{q(p-1)}{q(p-1)-1}}\rho^{\frac{1}{1-q(p-1)}-1}d\rho\Big)^{\frac{q(p-1)-1}{q(p-1)}} < \infty\,, \tag{4.31}$$

for some constant $C = C(C_1, p, q, \sigma)$. Inequality (4.30) also entails that

$$H \le C \sup_{0<\rho<\frac{|\Omega|}{2}} \frac{\big(\int_0^\rho s^{\frac{\sigma}{q(p-1)}-1}ds\big)^{\frac{1}{\sigma}} + \big(\int_0^\rho\big(\int_s^\rho r^{\frac{1}{q(p-1)}-\frac{1}{\sigma}-1}\,dr\big)^\sigma ds\big)^{\frac{1}{\sigma}}}{\rho^{\frac{1}{q(p-1)}}} < \infty\,, \tag{4.32}$$

for some constant $C = C(C_1, p, q, \sigma)$. Owing to (4.29), (4.31) and (4.32), inequality (4.21) follows, and hence (4.11) is established when assumption (4.22) is in force. Under the same assumption, let us consider case (ii), namely suppose that

$$q(p-1) > \sigma\,. \tag{4.33}$$

By (4.26), we have now

$$\begin{aligned} H \le\; & C\Big(\int_0^{|\Omega|/2}\Big(\int_0^\rho s^{\frac{\sigma}{p-1}}\nu(s)^{\frac{\sigma}{1-p}}ds\Big)^{\frac{q(p-1)}{q(p-1)-\sigma}}\rho^{\frac{\sigma}{\sigma-q(p-1)}-1}d\rho\Big)^{\frac{q(p-1)-\sigma}{q\sigma(p-1)}} \\ & + C\Big(\int_0^{|\Omega|/2}\Big(\int_0^\rho\Big(\int_s^\rho r^{\frac{1}{p-1}-1}\,\nu^{\frac{1}{1-p}}(r)\,dr\Big)^\sigma ds\Big)^{\frac{q(p-1)}{q(p-1)-\sigma}}\rho^{\frac{\sigma}{\sigma-q(p-1)}-1}d\rho\Big)^{\frac{q(p-1)-\sigma}{q\sigma(p-1)}}, \end{aligned} \tag{4.34}$$

and

(4.35)
$$\begin{aligned} K \le\; & C\Big(\int_0^{|\Omega|/2}\Big(\big(s^{\frac{1}{p-1}}\nu^{\frac{1}{1-p}}(s)\big)^{\frac{q(p-1)}{q(p-1)-1}}\int_s^{|\Omega|/2}\rho^{\frac{1}{1-q(p-1)}-1}d\rho\Big)^{\frac{\sigma[q(p-1)-1]}{q(p-1)-\sigma}} s^{\frac{q(p-1)}{q(p-1)-\sigma}-1}ds\Big)^{\frac{q(p-1)-\sigma}{q(p-1)\sigma}} \\ & + C\Big(\int_0^{|\Omega|/2}\Big(\int_s^{|\Omega|/2}\Big(\int_s^\rho r^{\frac{1}{p-1}-1}\,\nu^{\frac{1}{1-p}}(r)\,dr\Big)^{\frac{q(p-1)}{q(p-1)-1}}\rho^{\frac{1}{1-q(p-1)}-1}d\rho\Big)^{\frac{\sigma[q(p-1)-1]}{q(p-1)-\sigma}} s^{\frac{q(p-1)}{q(p-1)-\sigma}-1}ds\Big)^{\frac{q(p-1)-\sigma}{q(p-1)\sigma}} \end{aligned}$$

for some constant $C = C(p, q, \sigma)$. Let us denote by H_1' and H_2' and by K_1' and by K_2' the addenda on the right-hand sides of (4.34) and (4.35), respectively. By (4.13),

$$K_1' \le C\Big(\int_0^{|\Omega|/2}\Big(\frac{s}{\nu(s)}\Big)^{\frac{\sigma q}{q(p-1)-\sigma}}ds\Big)^{\frac{q(p-1)-\sigma}{q(p-1)\sigma}} < \infty\,, \tag{4.36}$$

for some constant $C = C(p, q, \sigma)$. Let us next estimate K_2'. By Minkowski integral inequality,

(4.37)
$$\begin{aligned}
&\bigg(\int_s^{|\Omega|/2} \bigg(\int_s^{\rho} r^{\frac{1}{p-1}-1}\, \nu^{\frac{1}{1-p}}(r)\, dr\bigg)^{\frac{q(p-1)}{q(p-1)-1}} \rho^{\frac{1}{1-q(p-1)}-1} d\rho\bigg)^{\frac{q(p-1)-1}{q(p-1)}} \\
&\le \int_0^{|\Omega|/2} \bigg(\int_0^{|\Omega|/2} \chi_{(s,|\Omega|/2)}(\rho)\chi_{(s,\rho)}(r)\big(r^{\frac{1}{p-1}-1}\, \nu^{\frac{1}{1-p}}(r)\big)^{\frac{q(p-1)}{q(p-1)-1}} \rho^{\frac{1}{1-q(p-1)}-1} d\rho\bigg)^{\frac{q(p-1)-1}{q(p-1)}} dr \\
&\le \int_s^{|\Omega|/2} \Big(\frac{r}{\nu(r)}\Big)^{\frac{1}{p-1}} \bigg(\int_r^{|\Omega|/2} \rho^{\frac{1}{1-q(p-1)}-1} d\rho\bigg)^{\frac{q(p-1)-1}{q(p-1)}} \frac{dr}{r} \le C \int_s^{|\Omega|/2} \Big(\frac{r}{\nu(r)}\Big)^{\frac{1}{p-1}} r^{-\frac{1}{q(p-1)}-1} dr
\end{aligned}$$

for some constant $C = C(p, q)$. Thus,

$$K_2' \le C\bigg(\int_0^{|\Omega|/2} \bigg(\int_s^{|\Omega|/2} \Big(\frac{r}{\nu(r)}\Big)^{\frac{1}{p-1}} r^{-\frac{1}{q(p-1)}-1} dr\bigg)^{\frac{\sigma q(p-1)}{q(p-1)-\sigma}} s^{\frac{q(p-1)}{q(p-1)-\sigma}-1} ds\bigg)^{\frac{q(p-1)-\sigma}{\sigma q(p-1)}} \tag{4.38}$$

for some constant $C = C(p, q, \sigma)$. We claim that

(4.39)
$$\int_0^{|\Omega|/2} \bigg(\int_s^{|\Omega|/2} \Big(\frac{r}{\nu(r)}\Big)^{\frac{1}{p-1}} r^{-\frac{1}{q(p-1)}-1} dr\bigg)^{\frac{\sigma q(p-1)}{q(p-1)-\sigma}} s^{\frac{q(p-1)}{q(p-1)-\sigma}-1} ds \le C \int_0^{|\Omega|/2} \Big(\frac{s}{\nu(s)}\Big)^{\frac{\sigma q}{q(p-1)-\sigma}} ds$$

for some constant $C = C(p, q, \sigma)$. Actually, inequality (4.39) is a Hardy type inequality for the (decreasing) function $\nu^{\frac{1}{1-p}}$, which follows via standard criteria when $\frac{\sigma q(p-1)}{q(p-1)-\sigma} > 1$ (see e.g. [Ma4, Section 1.3]), and via [Go, Theorem 1.1] when $\frac{\sigma q(p-1)}{q(p-1)-\sigma} \le 1$. By (4.38), (4.39) and (4.13), we have that $K_2' < \infty$, a piece of information which, combined with (4.36), yields

$$K < \infty\,. \tag{4.40}$$

We now estimate the quantities H_1' and H_2'. Since $\frac{q(p-1)}{q(p-1)-\sigma} > 1$, by a Hardy inequality ([Ma4, Section 1.3]), one has that

$$\int_0^{|\Omega|/2} \bigg(\int_0^{\rho} \Big(\frac{s}{\nu(s)}\Big)^{\frac{\sigma}{p-1}} ds\bigg)^{\frac{q(p-1)}{q(p-1)-\sigma}} \rho^{\frac{\sigma}{\sigma-q(p-1)}-1} d\rho \le C \int_0^{|\Omega|/2} \Big(\frac{s}{\nu(s)}\Big)^{\frac{\sigma q}{q(p-1)-\sigma}} ds \tag{4.41}$$

for some constant $C = C(p, q, \sigma)$. Thus, owing to (4.41) and (4.13),

$$H_1' < \infty\,. \tag{4.42}$$

As far as H_2' is concerned, inasmuch as $\frac{q(p-1)}{q(p-1)-\sigma} > 1$ we have that

$$\begin{aligned}
H_2' &\le \int_0^{|\Omega|/2} \bigg(\frac{1}{\rho}\int_0^{\rho} \bigg(\int_s^{|\Omega|/2} r^{\frac{1}{p-1}-1}\nu^{\frac{1}{1-p}}(r)dr\bigg)^{\sigma} ds\bigg)^{\frac{q(p-1)}{q(p-1)-\sigma}} d\rho \\
&\le C\int_0^{|\Omega|/2} \bigg(\int_\rho^{|\Omega|/2} r^{\frac{1}{p-1}-1}\nu^{\frac{1}{1-p}}(r)dr\bigg)^{\frac{\sigma q(p-1)}{q(p-1)-\sigma}} d\rho\,,
\end{aligned} \tag{4.43}$$

for some constant $C = C(p, q, \sigma)$. Observe that the last inequality holds by the plain Hardy inequality. Moreover,

$$\int_0^{|\Omega|/2} \bigg(\int_\rho^{|\Omega|/2} r^{\frac{1}{p-1}-1}\nu^{\frac{1}{1-p}}(r)dr\bigg)^{\frac{\sigma q(p-1)}{q(p-1)-\sigma}} d\rho \le C \int_0^{|\Omega|/2} \Big(\frac{r}{\nu(r)}\Big)^{\frac{\sigma q}{q(p-1)-\sigma}} dr \tag{4.44}$$

for some constant $C = C(p,q,\sigma)$, as a consequence of a standard Hardy type inequality if $\frac{\sigma q(p-1)}{q(p-1)-\sigma} > 1$ ([Ma4, Section 1.3]), and of a Hardy type inequality for non-increasing functions [Go, Theorem 1.1] if $\frac{\sigma q(p-1)}{q(p-1)-\sigma} \leq 1$. By (4.43), (4.44) and (4.13), one has that $H_2' < \infty$, whence, owing also to (4.42),

$$H < \infty \,. \tag{4.45}$$

Thanks to (4.40) and (4.45), inequality (4.21) follows. Hence, inequality (4.11) is proved also in case (ii) under assumption (4.22).
When (4.22) is not fulfilled, namely when

$$q(p-1) \leq 1 \,, \tag{4.46}$$

[Go, Theorem 1.1] tells us that (4.21) holds if both the quantity H given by (4.23) and the quantity M defined as
(4.47)

$$M = \begin{cases} \displaystyle\sup_{0<s<\frac{|\Omega|}{2}} \bigg(s^{\frac{1}{\sigma}} \sup_{s<\rho<\frac{|\Omega|}{2}} \frac{\int_s^\rho r^{\frac{1}{p-1}}\, d(-D\nu^{\frac{1}{1-p}})(r)}{\rho^{\frac{1}{q(p-1)}}} \bigg) & \text{if } q(p-1) \leq \sigma \,, \\ \displaystyle\bigg(\int_0^{|\Omega|/2} \bigg(\sup_{s<\rho<\frac{|\Omega|}{2}} \frac{\int_s^\rho r^{\frac{1}{p-1}}\, d(-D\nu^{\frac{1}{1-p}})(r)}{\rho^{\frac{1}{q(p-1)}}} \bigg)^{\frac{\sigma q(p-1)}{q(p-1)-\sigma}} s^{\frac{q(p-1)}{q(p-1)-\sigma}-1} ds \bigg)^{\frac{q(p-1)-\sigma}{\sigma q(p-1)}} & \text{if } q(p-1) > \sigma \,, \end{cases}$$

are finite. (As in the case where (4.22) is in force, this assertion requires a simple additional argument, since the characterization of [Go] concerns inequalities in $(0,\infty)$.) The proof of the fact that H is finite is exactly the same as in the case when $q(p-1) > 1$. As for M, assume first that (4.25) is fulfilled. Then, by (4.26),

$$M \leq \sup_{0<s<\frac{|\Omega|}{2}} s^{\frac{1}{p-1}+\frac{1}{\sigma}-\frac{1}{q(p-1)}} \nu^{\frac{1}{1-p}}(s) + \frac{1}{p-1} \sup_{0<s<\frac{|\Omega|}{2}} \bigg(s^{\frac{1}{\sigma}} \sup_{s<\rho<\frac{|\Omega|}{2}} \frac{\int_s^\rho r^{\frac{1}{p-1}-1} \nu^{\frac{1}{1-p}}(r)dr}{\rho^{\frac{1}{q(p-1)}}} \bigg) . \tag{4.48}$$

The right-hand side of (4.48) is bounded by a constant depending only on p and on the left-hand side of (4.12). Thus (4.21), and hence (4.11), follow in case (i).
If (4.33) is in force, then, by (4.26),

$$\begin{aligned} M \leq{}& C\bigg(\int_0^{|\Omega|/2} \Big(\frac{s}{\nu(s)} \Big)^{\frac{\sigma q}{q(p-1)-\sigma}} ds \bigg)^{\frac{q(p-1)-\sigma}{\sigma q(p-1)}} \\ &+ C\bigg(\int_0^{|\Omega|/2} \bigg(\sup_{s<\rho<\frac{|\Omega|}{2}} \frac{\int_s^\rho r^{\frac{1}{p-1}-1} \nu^{\frac{1}{1-p}}(r)dr}{\rho^{\frac{1}{q(p-1)}}} \bigg)^{\frac{\sigma q(p-1)}{q(p-1)-\sigma}} s^{\frac{\sigma}{q(p-1)-\sigma}} ds \bigg)^{\frac{q(p-1)-\sigma}{\sigma q(p-1)}} \end{aligned} \tag{4.49}$$

for some constant $C = C(p,q,\sigma)$. The former addendum on the right-hand side of (4.49) is finite, by (4.13). The latter does not exceed

$$C\bigg(\int_0^{|\Omega|/2} \Big(\int_s^{|\Omega|/2} r^{\frac{1}{p-1}-1} \nu^{\frac{1}{1-p}}(r)dr \Big)^{\frac{\sigma q(p-1)}{q(p-1)-\sigma}} ds \bigg)^{\frac{q(p-1)-\sigma}{\sigma q(p-1)}} , \tag{4.50}$$

an expression which, by (4.44), can be bounded by the left-hand side of (4.13). Therefore, we have shown that both H and M are finite; consequently, (4.21), and hence (4.11), follow also in this case. The proof of cases (i) and (ii) is complete.

We now take into account the case when $\sigma = \infty$ and $1 < q < \infty$. By Theorem 3.1, inequality (4.11) is reduced to showing that

$$\Big(\int_0^{|\Omega|/2}\Big(\int_0^s f_\pm^*(r)dr\Big)^{\frac{1}{p-1}} d(-D\nu^{\frac{1}{1-p}})(s)\Big) \leq C\Big(\int_0^{|\Omega|/2} f_\pm^*(s)^q ds\Big)^{\frac{1}{q(p-1)}} \tag{4.51}$$

for some constant C. Inequality (4.51) follows from either (4.14) or (4.15), according to whether $q(p-1) \leq 1$ or $q(p-1) > 1$, by a weighted Hardy type inequality ([Ma4, Section 1.3]). This establishes cases (iii) and (iv).

In case (v), namely if both $\sigma = \infty$ and $q = \infty$, one has

$$\begin{aligned}\|u_\pm\|_{L^\infty(\Omega)} &\leq \int_0^{|\Omega|/2}\Big(\int_0^s f_\pm^*(r)dr\Big)^{\frac{1}{p-1}} d(-D\nu^{\frac{1}{1-p}})(s)\\ &\leq \|f_\pm^*\|_{L^\infty(0,|\Omega|/2)}^{\frac{1}{p-1}}\int_0^{|\Omega|/2} s^{\frac{1}{p-1}}\, d(-D\nu^{\frac{1}{1-p}})(s) \leq \|f_\pm\|_{L^\infty(\Omega)}^{\frac{1}{p-1}}\frac{1}{p-1}\int_0^{|\Omega|/2}\Big(\frac{s}{\nu(s)}\Big)^{\frac{1}{p-1}}\frac{ds}{s},\end{aligned}$$

where the first inequality holds owing to Theorem 3.1, and the last one follows via Fubini's theorem. Thus, (4.11) is a consequence of (4.16).

Next, if $\sigma < \infty$ and $q = \infty$,

$$\begin{aligned}\|u_\pm\|_{L^\sigma(\Omega)} = \|u_\pm^*\|_{L^\sigma(0,|\Omega|/2)} &\leq \Big(\int_0^{|\Omega|/2}\Big(\int_s^{|\Omega|/2} f_\pm{}^{**}(r)^{\frac{1}{p-1}} r^{\frac{1}{p-1}}\, d(-D\nu^{\frac{1}{1-p}})(r)\Big)^\sigma ds\Big)^{\frac{1}{\sigma}}\\ &\leq \|f_\pm^*\|_{L^\infty(0,|\Omega|/2)}^{\frac{1}{p-1}}\Big(\int_0^{|\Omega|/2}\Big(\int_s^{|\Omega|/2} r^{\frac{1}{p-1}}\, d(-D\nu^{\frac{1}{1-p}})(r)\Big)^\sigma ds\Big)^{\frac{1}{\sigma}}\\ &\leq \|f_\pm\|_{L^\infty(\Omega)}^{\frac{1}{p-1}}\Big(\int_0^{|\Omega|/2}\Big(\Big(\frac{s}{\nu(s)}\Big)^{\frac{1}{p-1}} + \frac{1}{p-1}\int_s^{|\Omega|/2}\Big(\frac{r}{\nu(r)}\Big)^{\frac{1}{p-1}}\frac{dr}{r}\Big)^\sigma ds\Big)^{\frac{1}{\sigma}},\end{aligned}$$

where the last inequality rests upon (4.26). Therefore, (4.11) holds thanks to (4.17). Case (vi) is thus established.

Let us finally consider case (vii), namely, assume that $\sigma < \infty$ and $q = 1$. Since

$$f_\pm{}^{**}(s) \leq \frac{1}{s}\|f_\pm\|_{L^1(\Omega)} \qquad \text{for } s \in (0, |\Omega|/2), \tag{4.52}$$

we have

$$\begin{aligned}\|u_\pm\|_{L^\sigma(\Omega)} = \|u_\pm^*\|_{L^\sigma(0,|\Omega|/2)} &\leq \Big(\int_0^{|\Omega|/2}\Big(\int_s^{|\Omega|/2} f_\pm{}^{**}(r)^{\frac{1}{p-1}} r^{\frac{1}{p-1}}\, d(-D\nu^{\frac{1}{1-p}})(r)\Big)^\sigma ds\Big)^{\frac{1}{\sigma}}\\ &\leq \|f_\pm\|_{L^1(\Omega)}^{\frac{1}{p-1}}\Big(\int_0^{|\Omega|/2}\Big(\int_s^{|\Omega|/2} d(-D\nu^{\frac{1}{1-p}})(r)\Big)^\sigma ds\Big)^{\frac{1}{\sigma}}\\ &\leq \|f_\pm\|_{L^1(\Omega)}^{\frac{1}{p-1}}\Big(\int_0^{|\Omega|/2} \nu^{\frac{\sigma}{1-p}}(s)ds\Big)^{\frac{1}{\sigma}}.\end{aligned}$$

Inequality (4.11) follows from (4.18) □

The next result deals with a inequality between classical Lorentz norms of u and f.

Proposition 4.2 *Let Ω, p, ν, f and u be as in Theorem 3.1. Let $k \in (0,\infty)$, and let $\omega, \vartheta : (0,|\Omega|) \to [0,\infty)$ be measurable functions. Assume that $f \in \Gamma_\omega^{\frac{k}{p-1}}(\Omega)$. Then a constant C exists such that*

$$\|u\|_{\Lambda_\vartheta^k(\Omega)} \le C \|f\|^{\frac{1}{p-1}}_{\Gamma_\omega^{\frac{k}{p-1}}(\Omega)} \tag{4.53}$$

if

$$\sup_{0<\rho<\frac{|\Omega|}{2}} \frac{\left(\int_0^\rho s^{\frac{k}{p-1}} \nu^{\frac{k}{1-p}} \vartheta(s)ds\right)^{\frac{1}{k}} + \left(\int_0^\rho \left(\int_s^\rho r^{\frac{1}{p-1}-1} \nu^{\frac{1}{1-p}}(r)\,dr\right)^k \vartheta(s)ds\right)^{\frac{1}{k}}}{\left(\int_0^\rho \omega(s)ds\right)^{\frac{1}{k}}} < \infty\,, \tag{4.54}$$

and either $0<k\le 1$ and

$$\sup_{0<s<\frac{|\Omega|}{2}} \left[\left(\int_0^s \vartheta(\rho)d\rho\right)^{\frac{1}{k}} \sup_{s<\rho<\frac{|\Omega|}{2}} \frac{\int_s^\rho r^{\frac{1}{p-1}-1} \nu^{\frac{1}{1-p}}(r)\,dr}{\left(\int_0^\rho \omega(r)dr\right)^{\frac{1}{k}}}\right] < \infty\,, \tag{4.55}$$

or $k>1$ and

$$\sup_{0<s<\frac{|\Omega|}{2}} \left[\left(\int_0^s \vartheta(\rho)d\rho\right)^{\frac{1}{k}} \left(\frac{s^{\frac{1}{p-1}} \nu^{\frac{1}{1-p}}(s)}{\left(\int_0^s \omega(\rho)d\rho\right)^{\frac{1}{k}}} + \left(\int_s^{|\Omega|/2} \left(\int_s^\rho \left(\frac{r}{\nu(r)}\right)^{\frac{1}{p-1}} \frac{dr}{r}\right)^{k'} \frac{\omega(\rho)}{\left(\int_0^\rho \omega(r)dr\right)^{k'}} d\rho\right)^{\frac{1}{k'}}\right)\right] < \infty\,. \tag{4.56}$$

Moreover, the constant C in (4.53) depends on p, k, $|\Omega|$, $\nu^{\frac{1}{1-p}}(|\Omega|/2-)$ and the quantities on the left-hand sides of (4.54) and either (4.55) or (4.56).

Proof, sketched. By Theorem 3.1, inequality (4.53) is reduced to showing that

$$\left(\int_0^{|\Omega|/2} \left(\int_s^{|\Omega|/2} f_\pm{}^{**}(r)^{\frac{1}{p-1}} r^{\frac{1}{p-1}}\, d(-D\nu^{\frac{1}{1-p}})(r)\right)^k \vartheta(s)ds\right)^{\frac{1}{k}} \le C\left(\int_0^{|\Omega|/2} f_\pm{}^{**}(s)^{\frac{k}{p-1}} \omega(s)ds\right)^{\frac{1}{k}} \tag{4.57}$$

for every $f \in \Gamma_\omega^{\frac{k}{p-1}}(\Omega)$ (we are assuming, as in the proof of Theorem 4.1, that $\nu(|\Omega|/2-) = \infty$). Inequality (4.57) is a weighted Hardy inequality for non increasing functions $(f_\pm{}^{**})^{\frac{1}{p-1}}$. The conclusion follows via an application of [Go, Theorem 1.1] and of inequality (4.26). □

A specialization of Proposition 4.2 to the case of standard Lorentz spaces yields the following corollary.

Corollary 4.3 *Let Ω, p, ν, f and u be as in Theorem 3.1. Let $k,q,\sigma \in (0,\infty)$. Assume that $f \in L^{(q,\frac{k}{p-1})}(\Omega)$. If condition (4.12) is fulfilled, then a constant C, depending on k, q, σ, $|\Omega|$, $\nu^{\frac{1}{1-p}}(|\Omega|/2-)$ and the left-hand side of (4.12), exists such that*

$$\|u\|_{L^{\sigma,k}(\Omega)} \le C\|f\|^{\frac{1}{p-1}}_{L^{(q,\frac{k}{p-1})}(\Omega)}\,. \tag{4.58}$$

Proof. By Proposition 4.2, it suffices to show that, under (4.12), condition (4.54), and either (4.55) or (4.56), according to whether $0<k\le 1$ or $k>1$, hold, with $\vartheta(s) = s^{\frac{k}{\sigma}-1}$ and $\omega(s) = s^{\frac{k}{q(p-1)}-1}$. This can be verified by an easy computation. □

We now focus on capacitary estimates. These can be derived via the following result, which rests upon Theorem 3.4.

Theorem 4.4 *Let Ω, p, ν, f and u be as in Theorem 3.1. Let $A, B : [0,\infty) \to [0,\infty)$ be locally Lipschitz continuous increasing functions vanishing at 0; namely,*

$$A(t) = \int_0^t a(\tau)d\tau \qquad \text{for } t \geq 0 \tag{4.59}$$

$$B(t) = \int_0^t b(\tau)d\tau \qquad \text{for } t \geq 0\,, \tag{4.60}$$

for some locally bounded measurable functions $a, b : (0,\infty) \to (0,\infty)$ which are integrable at 0. Then

$$\begin{aligned}
&\int_0^\infty B\big(C_{p,u_\pm}(t)\big)a(t)dt \\
&\leq \int_0^{|\Omega|/2} A\left(\int_s^{|\Omega|/2}\left(\int_0^r f_\pm^*(\rho)d\rho\right)^{\frac{1}{p-1}} d(-D\nu^{\frac{1}{1-p}})(r) + \nu(|\Omega|/2-)^{\frac{1}{1-p}}\left(\int_0^{|\Omega|/2} f_\pm^*(r)dr\right)^{\frac{1}{p-1}}\right) d\big(D(B\circ\nu)\big)(s) \\
&\qquad + \int_{\nu(|\Omega|/2-)}^\infty A\left(t^{\frac{1}{1-p}}\left(\int_0^{|\Omega|/2} f_\pm^*(r)dr\right)^{\frac{1}{p-1}}\right)b(t)dt\,.
\end{aligned} \tag{4.61}$$

If $\nu(|\Omega|/2-) = \infty$ (and hence, in particular, if $\nu = \nu_{\Omega,p}$), then the term involving $\nu(|\Omega|/2-)^{\frac{1}{1-p}}$ and the last integral on the right-hand side of (4.61) are missing.

Proof. As in the proof of Theorems 3.1 and 3.4, we focus on u_+. Let us set

$$Q = \nu(|\Omega|/2-)^{\frac{1}{1-p}}\left(\int_0^{|\Omega|/2} f_+^*(r)dr\right)^{\frac{1}{p-1}}.$$

By Theorem 3.4,

$$\int_0^\infty B\big(C_{p,u_+}(t)\big)a(t)dt \leq \int_0^{F_+(\infty)} B\big({F_+}^{-1}(t)^{1-p}\big)a(t)dt\,, \tag{4.62}$$

where $F_+(\infty) = \lim_{\tau\to\infty} F_+(\tau)$. Since F_+ is absolutely continuous (Lipschitz continuous, in fact) and ${F_+}'$ is non-increasing and strictly positive in any bounded interval in $[0,\infty)$, the function ${F_+}^{-1} : [0, F_+(\infty)) \to [0,\infty)$ is also locally absolutely continuous. Hence,

$$\begin{aligned}
&\int_0^{F_+(\infty)} B\big({F_+}^{-1}(t)^{1-p}\big)a(t)dt \\
&= \int_0^{F_+(\infty)}\left(-\int_t^{F_+(\infty)}\big[B\big({F_+}^{-1}(\tau)^{1-p}\big)\big]' d\tau\right)a(t)dt \\
&= \int_0^{F_+(\infty)} -\big[B\big({F_+}^{-1}(\tau)^{1-p}\big)\big]' \int_0^\tau a(t)dt\, d\tau \\
&= \int_0^{F_+(\infty)} -\big[B\big({F_+}^{-1}(\tau)^{1-p}\big)\big]' A(\tau)d\tau \\
&= \int_0^\infty A\big(F_+(t^{\frac{1}{1-p}})\big)b(t)\, dt
\end{aligned} \tag{4.63}$$

$$
\begin{aligned}
&\le \int_0^{\nu(|\Omega|/2-)} A\bigg(\int_0^{\nu(\nu^{-1}(t))^{\frac{1}{1-p}}} \bigg(\int_0^{\nu^{-1}(\tau^{1-p})} f_+^*(\rho)d\rho\bigg)^{\frac{1}{p-1}} d\tau\bigg) b(t)\,dt \\
&\quad + \int_{\nu(|\Omega|/2-)}^{\infty} A\bigg(\int_0^{t^{\frac{1}{1-p}}} \bigg(\int_0^{\nu^{-1}(\tau^{1-p})} f_+^*(\rho)d\rho\bigg)^{\frac{1}{p-1}} d\tau\bigg) b(t)\,dt \\
&= \int_0^{\nu(|\Omega|/2-)} A\bigg(\int_{\nu^{-1}(t)}^{|\Omega|/2} \bigg(\int_0^{r} f_+^*(\rho)d\rho\bigg)^{\frac{1}{p-1}} d(-D\nu^{\frac{1}{1-p}})(r) + Q\bigg) b(t)dt \\
&\quad + \int_{\nu(|\Omega|/2-)}^{\infty} A\bigg(t^{\frac{1}{1-p}}\bigg(\int_0^{|\Omega|/2} f_+^*(r)dr\bigg)^{\frac{1}{p-1}}\bigg) b(t)dt\,,
\end{aligned}
$$

where the inequality holds since ν is left-continuous, and hence $\nu(\nu^{-1}(t)) \le t$ for $t > 0$, and the last equality is a consequence of (3.40).

Now, consider the function $\Psi : [0, \nu(|\Omega|/2-)) \to [0, \infty)$ given by

(4.64)

$$
\Psi(t) = \int_t^{\nu(|\Omega|/2-)} A\bigg(\int_{\nu^{-1}(\tau)}^{|\Omega|/2} \bigg(\int_0^{r} f_+^*(\rho)d\rho\bigg)^{\frac{1}{p-1}} d\big(-D\nu^{\frac{1}{1-p}}\big)(r) + Q\bigg) b(\tau)\,d\tau \qquad \text{for } t \in [0, \infty).
$$

Such a function is non-increasing and locally Lipschitz continuous in $(0, \infty)$. The chain rule for functions of bounded variation [AFP, Theorem 3.99] yields

(4.65)

$$
D\big(\Psi\circ\nu\big) = -A\bigg(\int_{\nu^{-1}(\nu)}^{|\Omega|/2} \bigg(\int_0^{r} f_+^*(\rho)d\rho\bigg)^{\frac{1}{p-1}} d\big(-D\nu^{\frac{1}{1-p}}\big)(r) + Q\bigg) b(\nu)\widetilde{D}\nu + \big[\Psi(\nu_+) - \Psi(\nu_-)\big]\mathcal{H}^0 \lfloor J_\nu\,,
$$

where $\widetilde{D}\nu = \nu'\mathcal{L}^1 + D^c\nu$, the diffuse part of ν, and $\nu_\pm$ are defined by (3.32). Since ν^{-1} is the right-continuous inverse of ν,

$$
s \le \nu^{-1}(\nu(s)) \qquad \text{for } s \in (0, |\Omega|/2). \tag{4.66}
$$

Thus,

$$
\begin{aligned}
&A\bigg(\int_{\nu^{-1}(\nu(s))}^{|\Omega|/2} \bigg(\int_0^{r} f_+^*(\rho)d\rho\bigg)^{\frac{1}{p-1}} d\big(-D\nu^{\frac{1}{1-p}}\big)(r) + Q\bigg) b(\nu(s)) \\
&\le A\bigg(\int_{s}^{|\Omega|/2} \bigg(\int_0^{r} f_+^*(\rho)d\rho\bigg)^{\frac{1}{p-1}} d\big(-D\nu^{\frac{1}{1-p}}\big)(r) + Q\bigg) b(\nu(s)) \quad \text{for } s \in (0, |\Omega|/2).
\end{aligned} \tag{4.67}
$$

On the other hand, if $s \in J_\nu$,

$$
\begin{aligned}
&\Psi(\nu_+(s)) - \Psi(\nu_-(s)) \\
&= -\int_{\nu_-(s)}^{\nu_+(s)} A\bigg(\int_{\nu^{-1}(\tau)}^{|\Omega|/2} \bigg(\int_0^{r} f_+^*(\rho)d\rho\bigg)^{\frac{1}{p-1}} d\big(-D\nu^{\frac{1}{1-p}}\big)(r) + Q\bigg) b(\tau)\,d\tau \\
&= -A\bigg(\int_{s}^{|\Omega|/2} \bigg(\int_0^{r} f_+^*(\rho)d\rho\bigg)^{\frac{1}{p-1}} d\big(-D\nu^{\frac{1}{1-p}}\big)(r) + Q\bigg) \int_{\nu_-(s)}^{\nu_+(s)} b(\tau)d\tau \\
&= -A\bigg(\int_{s}^{|\Omega|/2} \bigg(\int_0^{r} f_+^*(\rho)d\rho\bigg)^{\frac{1}{p-1}} d\big(-D\nu^{\frac{1}{1-p}}\big)(r) + Q\bigg) \big(B(\nu_+(s)) - B(\nu_-(s))\big)\,,
\end{aligned} \tag{4.68}
$$

where the second equality holds since $\nu^{-1}(t) = s$ if $\nu_-(s) < t < \nu_+(s)$. Combining (4.65), (4.67) and (4.68) yields

$$-D\big(\Psi\circ\nu\big) \le A\Big(\int_{\cdot}^{|\Omega|/2}\Big(\int_0^r f_+^*(\rho)d\rho\Big)^{\frac{1}{p-1}} d\big(-D\nu^{\frac{1}{1-p}}\big)(r) + Q\Big) D(B\circ\nu)\,. \tag{4.69}$$

Observe that here we have exploited the fact that, by the chain rule again, $D(B\circ\nu) = b(\nu)\widetilde{D}\nu + \big[B(\nu_+) - B(\nu_-)\big]\mathcal{H}^0\lfloor J_\nu$. Consequently,

$$\begin{aligned}\int_0^{\nu(|\Omega|/2-)} & A\Big(\int_{\nu^{-1}(t)}^{|\Omega|/2}\Big(\int_0^r f_+^*(\rho)d\rho\Big)^{\frac{1}{p-1}} d\big(-D\nu^{\frac{1}{1-p}}\big)(r) + Q\Big) b(t)\,dt \\ &= \Psi(\nu(0)) = -D\big(\Psi\circ\nu\big)\big([0,|\Omega|/2)\big) \\ &\le \int_0^{|\Omega|/2} A\Big(\int_s^{|\Omega|/2}\Big(\int_0^r f_+^*(\tau)d\tau\Big)^{\frac{1}{p-1}} d(-D\nu^{\frac{1}{1-p}})(r) + Q\Big) d\big(D(B\circ\nu)\big)(s)\,,\end{aligned} \tag{4.70}$$

where the second equality holds since $\Psi(\nu(|\Omega|/2-)) = 0$. Inequality (4.61) follows from (4.63) and (4.70). $\square$

Theorem 4.4 can be used, for instance, to deduce the following capacitary inequalities for solutions u to (1.1) in terms of Lebesgue norms of f, when the isocapacitary function $\nu_{\Omega,p}$ admits a power type lower bound near 0.

Proposition 4.5 *Let Ω, p, f and u be as in Theorem 3.1. Assume that there exist $\gamma \in (0,1)$ and $C_0 > 0$ such that*

$$\nu_{\Omega,p}(s) \ge C_0 s^\gamma \qquad \textit{for } s \in (0,|\Omega|/2). \tag{4.71}$$

Let $f \in L^q(\Omega)$ for some $q \ge 1$.
(i) *If $1 < q < \frac{1}{1-\gamma}$, then a constant $C = C(C_0,\gamma,p,q,|\Omega|)$ exists such that*

$$\Big(\int_0^\infty C_{p,u_\pm}(t)^{\frac{1-q(1-\gamma)}{\gamma}} d(t^{q(p-1)})\Big)^{\frac{1}{q}} \le C\|f\|_{L^q(\Omega)}\,. \tag{4.72}$$

(ii) *If $q = \frac{1}{1-\gamma}$ and $\frac{p-1}{1-\gamma} > 1$, then a constant $C = C(C_0,\gamma,p,|\Omega|)$ exists such that*

$$\Big(\int_0^\infty \Big[1 + \log_+\Big(\frac{C_0^{\frac{1}{\gamma}}|\Omega|}{C_{p,u_\pm}(t)^{\frac{1}{\gamma}}}\Big)\Big]^{\frac{2-p-\gamma}{1-\gamma}} d(t^{\frac{p-1}{1-\gamma}})\Big)^{1-\gamma} \le C\|f\|_{L^{\frac{1}{1-\gamma}}(\Omega)}\,. \tag{4.73}$$

Here, $\log_+(s) = \max\{\log(s),0\}$.
iii) *If $q = 1$ and B is any function as in* (4.60) *fulfilling*

$$\int_0^\infty \frac{b(t)}{t}\,dt < \infty\,, \tag{4.74}$$

then a constant $C = C(C_0,\gamma,p,|\Omega|,B)$ exists such that

$$\int_0^\infty B\big(C_{p,u_\pm}(t)\big)d(t^{p-1}) \le C\|f\|_{L^1(\Omega)}\,. \tag{4.75}$$

Proof, sketched. Assume first that $1 < q < \frac{1}{1-\gamma}$. Owing to Theorem 4.4 and to (4.71), inequality (4.72) can be reduced to showing that

$$(4.76)\quad \bigg(\int_0^{|\Omega|/2}\bigg(\int_s^{|\Omega|/2} f_\pm{}^{**}(r)^{\frac{1}{p-1}} r^{\frac{1-\gamma}{p-1}-1}dr\bigg)^{q(p-1)} s^{-q(1-\gamma)}ds\bigg)^{\frac{1}{q}} \leq C\bigg(\int_0^{|\Omega|/2} f_\pm{}^{**}(s)^q ds\bigg)^{\frac{1}{q}}$$

for some constant $C = C(\gamma, p, q, |\Omega|)$ (the extra terms appearing on the right-hand side of (4.61) are easily estimated by Hölder's inequality). Inequality (4.76) follows via [Go, Theorem 1.1].

Similarly, when $q = \frac{1}{1-\gamma}$ and $\frac{p-1}{1-\gamma} > 1$, again Theorem 4.4 and (4.71) can be used to show that (4.73) follows from

$$(4.77)\quad \bigg(\int_0^{|\Omega|/2}\bigg(\int_s^{|\Omega|/2} f_\pm{}^{**}(r)^{\frac{1}{p-1}} r^{\frac{1-\gamma}{p-1}-1}dr\bigg)^{\frac{p-1}{1-\gamma}}\Big(1+\log\Big(\frac{|\Omega|}{s}\Big)\Big)^{\frac{1-p}{1-\gamma}}\frac{ds}{s}\bigg)^{1-\gamma} \leq C\bigg(\int_0^{|\Omega|/2} f_\pm{}^{**}(s)^{\frac{1}{1-\gamma}}ds\bigg)^{1-\gamma}$$

for some constant $C = C(\gamma, \beta, p, |\Omega|)$. Inequality (4.77) can be derived from [Go, Theorem 1.1].

Finally, if $q = 1$, inequality (4.75) follows via Theorem 4.4 and inequalities (4.71) and (4.52). □

As a consequence of Proposition 4.5 and of the isocapacitary inequality (1.4) (and of Theorem 4.1), we have the following corollary.

Corollary 4.6 *Let Ω, p, γ, C_0 and u be as in Proposition 4.5. Assume that $f \in L^q(\Omega)$ for some $q > 1$.*
(i) *If $1 < q < \frac{1}{1-\gamma}$, then a constant $C = C(C_0, \gamma, p, q, |\Omega|)$ exists such that*

$$(4.78)\quad \|u\|_{L^{\frac{q(p-1)}{1-q(1-\gamma)}}(\Omega)} \leq C\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}}.$$

(ii) *If $q = \frac{1}{1-\gamma}$ and $\frac{p-1}{1-\gamma} > 1$, then a constant $C = C(C_0, \gamma, p, |\Omega|)$ exists such that*

$$(4.79)\quad \|u\|_{\mathrm{Exp}L^{\frac{p-1}{p+\gamma-2}}(\Omega)} \leq C\|f\|_{L^{\frac{1}{1-\gamma}}(\Omega)}^{\frac{1}{p-1}}.$$

(iii) *If either $q = \frac{1}{1-\gamma}$ and $\frac{p-1}{1-\gamma} \leq 1$, or $q > \frac{1}{1-\gamma}$, then a constant $C = C(C_0, \gamma, p, q, |\Omega|)$ exists such that*

$$(4.80)\quad \|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}}.$$

Proof. Assume that $q < \frac{1}{1-\gamma}$. Inequalities (4.72), (1.4) and (4.71) entail that

$$(4.81)\quad \bigg(\int_0^\infty \mu_{u_\pm}(t)^{1-q(1-\gamma)}d(t^{q(p-1)})\bigg)^{\frac{1}{q(p-1)}} \leq C\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}}$$

for some constant $C = C(C_0, \gamma, p, q, |\Omega|)$. The quantity on the left-hand side of (4.81) is equivalent to $\|u\|_{L^{\frac{q(p-1)}{1-q(1-\gamma)},q(p-1)}(\Omega)}$, and by (4.7) the latter dominates $\|u\|_{L^{\frac{q(p-1)}{1-q(1-\gamma)}}(\Omega)}$ (up to a multiplicative constant), since $\frac{q(p-1)}{1-q(1-\gamma)} > q(p-1)$. Inequality (4.78) follows.

When $q = \frac{1}{1-\gamma}$ and $\frac{p-1}{1-\gamma} > 1$, we infer from (4.73), via (1.4) and (4.71), that

$$\left(\int_0^\infty \left[1+\log\left(\frac{|\Omega|}{\mu_{u_\pm}(t)}\right)\right]^{\frac{2-p-\gamma}{1-\gamma}} d(t^{\frac{p-1}{1-\gamma}})\right)^{\frac{1-\gamma}{p-1}} \leq C\|f\|_{L^{\frac{1}{1-\gamma}}(\Omega)}^{\frac{1}{p-1}}. \tag{4.82}$$

Clearly,

(4.83)

$$\left(\int_0^\infty \left[1+\log\left(\frac{|\Omega|}{\mu_{u_\pm}(t)}\right)\right]^{\frac{2-p-\gamma}{1-\gamma}} d(t^{\frac{p-1}{1-\gamma}})\right)^{\frac{1-\gamma}{p-1}} \geq \left(\int_0^\tau \left[1+\log\left(\frac{|\Omega|}{\mu_{u_\pm}(t)}\right)\right]^{\frac{2-p-\gamma}{1-\gamma}} d(t^{\frac{p-1}{1-\gamma}})\right)^{\frac{1-\gamma}{p-1}}$$
$$\geq \left[1+\log\left(\frac{|\Omega|}{\mu_{u_\pm}(\tau)}\right)\right]^{\frac{2-p-\gamma}{p-1}} \left(\int_0^\tau d(t^{\frac{p-1}{1-\gamma}})\right)^{\frac{1-\gamma}{p-1}} = \tau\left[1+\log\left(\frac{|\Omega|}{\mu_{u_\pm}(\tau)}\right)\right]^{\frac{2-p-\gamma}{p-1}} \quad \text{for } \tau > 0.$$

Inequalities (4.82) and (4.83) entail that

$$u_\pm^*(s) \leq C\|f\|_{L^{\frac{1}{1-\gamma}}(\Omega)}^{\frac{1}{p-1}} \left[1+\log\left(\frac{|\Omega|}{s}\right)\right]^{\frac{p+\gamma-2}{p-1}} \qquad \text{for } s \in (0, |\Omega|/2), \tag{4.84}$$

whence (4.79) follows.
Finally, if $q > \frac{1}{1-\gamma}$, inequality (4.80) is a consequence of Theorem 4.1 (iii) and (iv). □

Remark 4.7 An inspection of the proof of Corollary 4.6 reveals that, in fact, the Lebesgue norm of u in (4.78) can be replaced by the stronger Lorentz norm appearing on the left-hand side of (4.81), and that the exponential norm of u in (4.79) can be replaced by the stronger Lorentz-Zygmund norm appearing on the left-hand side of (4.82) (in the spirit of the limiting Sobolev embedding of [BW, Ha, Ma4]). Moreover, when $q = 1$ and b fulfills (4.74), then one can infer from (4.75) that

$$\|u\|_{\Lambda_\omega^{p-1}(\Omega)} \leq C\|f\|_{L^1(\Omega)}^{\frac{1}{p-1}}$$

for some constant $C = C(C_0, \gamma, p, |\Omega|, B)$, where $\omega(s) = b(s^\gamma)s^{\gamma-1}$.

5 Gradient bounds

The approach via capacitary inequalities is exploited in this section to derive estimates for Lebesgue norms of the gradient of solutions to (1.1).

Theorem 5.1 *Let Ω, p, ν, f and u be as in Theorem 3.1. Let $1 \leq q \leq \infty$ and let $0 < \varrho \leq p$. Assume that $f \in L^q(\Omega)$. Then a constant C exists such that*

$$\|\nabla u\|_{L^\varrho(\Omega)} \leq C\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}}, \tag{5.1}$$

if either
(i) *$q > 1$, $q(p-1) \leq \varrho$ and*

$$\sup_{0<s<\frac{|\Omega|}{2}} \frac{s^{1+\frac{p(p-1)}{\varrho}-\frac{p}{q}}}{\nu(s)} < \infty, \tag{5.2}$$

or

(ii) $1 < q < \infty$, $0 < \varrho < q(p-1)$ *and*

$$\int_0^{|\Omega|/2} \left(\frac{s}{\nu(s)}\right)^{\frac{\varrho q}{p[q(p-1)-\varrho]}} ds < \infty\,, \tag{5.3}$$

or

(iii) $q = \infty$ *and*

$$\int_0^{|\Omega|/2} \left(\frac{s}{\nu(s)}\right)^{\frac{\varrho}{p(p-1)}} ds < \infty\,, \tag{5.4}$$

or

(iv) $q = 1$ *and*

$$\frac{p}{\varrho} > \inf\left\{\kappa : \int_0^{|\Omega|/2} \frac{s^{\kappa-2}}{\nu(s)^{\frac{1}{p-1}}} ds < \infty\right\}. \tag{5.5}$$

Moreover, in cases (i), (ii), (iii) the constant C *in* (5.1) *depends on* p, q, ϱ *and the quantity on the left-hand side of* (5.2), (5.3), (5.4), *respectively. In case (iv), the constant* C *in* (5.1) *depends on* p, ϱ *and* $\inf_{1<\kappa<\frac{p}{\varrho}} \left[(|\Omega|/2)^{\frac{p}{\varrho}-k}(\frac{p}{\varrho}-\kappa)^{-(\frac{p}{\varrho}-1)}(\kappa-1)\int_0^{|\Omega|/2} s^{\kappa-2}\nu(s)^{\frac{1}{1-p}} ds\right]$.

Remark 5.2 When $q = 1$, condition (5.5) can be replaced, with analogous proof, by the somewhat more general assumption that a measurable function $h : (0, |\Omega|/2) \to (0, \infty)$ exists such that

$$\int_0^{|\Omega|/2} \frac{h(s)}{\nu(s)^{\frac{1}{p-1}}} ds < \infty \qquad \text{and} \qquad \int_0^{|\Omega|/2} \frac{ds}{\left(\int_0^s h(r)dr\right)^{\frac{p}{p-\varrho}}} < \infty\,.$$

In this case, the constant C in (5.1) depends on p, ϱ and on the last two integrals.

Proof of Theorem 5.1. Hölder's inequality entails that

$$\begin{aligned} &\int_{\{u_+=t\}} |\nabla u|^{\varrho-1} d\mathcal{H}^{n-1}(x) \\ &\qquad \le \left(\int_{\{u_+=t\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}(x)\right)^{\frac{\varrho}{p}} \left(\int_{\{u_+=t\}} \frac{d\mathcal{H}^{n-1}(x)}{|\nabla u|}\right)^{1-\frac{\varrho}{p}} \quad \text{for a.e. } t > 0. \end{aligned} \tag{5.6}$$

Inasmuch as $\mathcal{H}^{n-1}(\{u = t\}) = P(\{u > t\}; \Omega)$ for a.e. $t > 0$ ([BrZi]), from inequality (5.6) with $\varrho = 1$ and the isoperimetric inequality (1.3) we infer that

$$\int_{\{u_+=t\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}(x) > 0 \qquad \text{for a.e. } t > 0. \tag{5.7}$$

Since $u \in W^{1,p}_T(\Omega)$, the function u_+^* is locally absolutely continuous in $(0, |\Omega|/2)$ [CEG, Lemma 6.6]. Hence, $u_+^{*\prime}$ vanishes a.e. in the inverse image through u_+^* of any Borel subset of $[0, \infty)$ having one-dimensional Lebesgue measure zero. Thus, owing to (5.7), for a.e. $s \in (0, |\Omega|/2)$,

$$\text{either} \quad \int_{\{u_+=u_+^*(s)\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}(x) > 0 \quad \text{or} \quad u_+^{*\prime}(s) \neq 0\,. \tag{5.8}$$

Now, recall that a set $S \subset (0, |\Omega|/2)$ exists such that

$$(0, |\Omega|/2) \setminus S = \cup_{i\in I}(a_i, b_i) \qquad \text{for some countable set } I, \tag{5.9}$$

$$(a_i, b_i) \cap (a_j, b_j) = \emptyset \qquad \text{if } i \neq j, \tag{5.10}$$

$$u_+^* \text{ is constant on each } (a_i, b_i)\,, \tag{5.11}$$

$$\mu_{u_+}(u_+^*(s)) = s \qquad \text{if } s \in S \tag{5.12}$$

(see e.g. [BrZi]). Owing to Lemma 3.5, we have
(5.13)

$$-u_+^{*'}(s)\bigg(\int_{\{u_+=u_+^*(s)\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}(x)\bigg)^{\frac{\varrho}{p-1}} \le -u_+^{*'}(s)\bigg(\int_0^s f_+^*(r)\,dr\bigg)^{\frac{\varrho}{p-1}} \quad \text{for a.e. } s \in (0, |\Omega|/2).$$

In writing (5.13), we have made use of (5.12) and of the fact that, by (5.11), $u_+^{*'}$ vanishes in $\cup_{i\in I}(a_i, b_i)$; we have also exploited the fact that $u_+^{*'}$ also vanishes a.e. in the inverse image through u_+^* of the set, having one-dimensional Lebesgue measure zero, of those values of t for which (3.12) does not hold. Inequality (5.13) entails that

$$\begin{aligned} & -u_+^{*'}(s)\int_{\{u_+=u_+^*(s)\}} |\nabla u|^{\varrho-1} d\mathcal{H}^{n-1}(x) \\ & \le -u_+^{*'}(s)\frac{\big(\int_0^s f_+^*(r)\,dr\big)^{\frac{\varrho}{p-1}} \int_{\{u_+=u_+^*(s)\}} |\nabla u|^{\varrho-1} d\mathcal{H}^{n-1}(x)}{\big(\int_{\{u_+=u_+^*(s)\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}(x)\big)^{\frac{\varrho}{p-1}}} \qquad \text{for a.e. } s \in (0, |\Omega|/2). \end{aligned} \tag{5.14}$$

Notice that the right-hand side of (5.14) is meaningful, thanks to (5.8). Equation (5.8) also ensures that the function

$$w : (0, |\Omega|/2) \to [0, \infty),$$

given by

$$w(s) = -u_+^{*'}(s)\frac{\int_{\{u_+=u_+^*(s)\}} |\nabla u|^{\varrho-1} d\mathcal{H}^{n-1}(x)}{\big(\int_{\{u_+=u_+^*(s)\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}(x)\big)^{\frac{\varrho}{p-1}}}\,, \tag{5.15}$$

is well-defined a.e. in $(0, |\Omega|/2)$. For each fixed $t_0 > 0$, the function

$$[0, \infty) \ni t \longmapsto \int_{\{t<u_+<t_0\}} |\nabla u|^{\varrho} dx$$

is absolutely continuous, by the coarea formula applied to truncations of u. Thus, also the function

$$(0, |\Omega|/2) \ni s \longmapsto \int_{\{u_+^*(s)<u_+<t_0\}} |\nabla u|^{\varrho} dx$$

is locally absolutely continuous, being the composition of locally absolutely continuous monotone functions, and, by the chain rule and the coarea formula,

$$-\frac{d}{ds}\int_{\{u_+^*(s)<u_+<t_0\}} |\nabla u|^{\varrho} dx = -u_+^{*'}(s)\int_{\{u_+=u_+^*(s)\}} |\nabla u|^{\varrho-1} d\mathcal{H}^{n-1}(x) \tag{5.16}$$

for a.e. $s \in (0, |\Omega|/2)$ such that $u_+^*(s) < t_0$. Owing to the arbitrariness of t_0, from (5.14)-(5.16) one can deduce that

$$\int_{\{u_+>0\}} |\nabla u|^\varrho dx \le \int_0^{|\Omega|/2} w(s)\Big(\int_0^s f_+^*(r)\,dr\Big)^{\frac{\varrho}{p-1}} ds\,. \tag{5.17}$$

We assume, hereafter, that

$$\varrho < p \tag{5.18}$$

the case where $\varrho = p$ being analogous, and even simpler. We claim that

$$w(s) \le \big(-u_+^{*'}(s)\big)^{\frac{\varrho}{p}}\Big(\int_{\{u_+=u_+^*(s)\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}(x)\Big)^{-\frac{\varrho}{p(p-1)}} \quad \text{for a.e. } s \in (0, |\Omega|/2). \tag{5.19}$$

To verify (5.19), observe that, by the coarea formula,

$$\begin{aligned}\mu_{u_+}(t_2) - \mu_{u_+}(t_1) &= |\{\nabla u = 0\} \cap \{t_2 \le u_+ < t_1\}| + \int_{t_2}^{t_1}\int_{\{u_+=t\}} \frac{d\mathcal{H}^{n-1}(x)}{|\nabla u|}\,dt \\ &\ge \int_{t_2}^{t_1}\int_{\{u_+=t\}} \frac{d\mathcal{H}^{n-1}(x)}{|\nabla u|}\,dt \qquad \text{if } 0 < t_2 < t_1\,.\end{aligned} \tag{5.20}$$

Let S be the set defined above, and let $s_1, s_2 \in S$, with $s_1 < s_2$. An application of (5.20) with $t_i = u_+^*(s_i)$, $i = 1, 2$, and the use of (5.12) tell us that

$$s_2 - s_1 \ge \int_{u_+^*(s_2)}^{u_+^*(s_1)}\int_{\{u_+=t\}} \frac{d\mathcal{H}^{n-1}(x)}{|\nabla u|}\,dt\,. \tag{5.21}$$

Furthermore, if $s_1 \in S$, $s_2 \in (a_i, b_i)$ for some $i \in I$, and $s_2 > s_1$, then

$$s_2 - s_1 \ge a_i - s_1 \ge \int_{u_+^*(a_i)}^{u_+^*(s_1)}\int_{\{u_+=t\}} \frac{d\mathcal{H}^{n-1}(x)}{|\nabla u|}\,dt = \int_{u_+^*(s_2)}^{u_+^*(s_1)}\int_{\{u_+=t\}} \frac{d\mathcal{H}^{n-1}(x)}{|\nabla u|}\,dt\,. \tag{5.22}$$

Consequently, inequality (5.21) holds for every $s_1 \in S$ and $s_2 \in (s_1, |\Omega|/2)$. Since $u_+^*(s_1) > u_+^*(s_2)$ in this case,

$$\frac{s_2 - s_1}{u_+^*(s_1) - u_+^*(s_2)} \le \frac{1}{u_+^*(s_1) - u_+^*(s_2)}\int_{u_+^*(s_2)}^{u_+^*(s_1)}\int_{\{u_+=t\}} \frac{d\mathcal{H}^{n-1}(x)}{|\nabla u|}\,dt\,. \tag{5.23}$$

On passing to the limit as $s_2 \to s_1^+$ in (5.23), we deduce that

$$\int_{\{u_+=u_+^*(s)\}} \frac{d\mathcal{H}^{n-1}(x)}{|\nabla u|}\,dt \le \frac{1}{-u_+^{*'}(s)} \qquad \text{for a.e. } s \in S. \tag{5.24}$$

By (5.6) (and the same reasoning as in the derivation of (5.13)),

$$\begin{aligned}&\int_{\{u_+=u_+^*(s)\}} |\nabla u|^{\varrho-1} d\mathcal{H}^{n-1}(x) \\ &\qquad \le \Big(\int_{\{u_+=u_+^*(s)\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}(x)\Big)^{\frac{\varrho}{p}}\Big(\int_{\{u_+=u_+^*(s)\}} \frac{d\mathcal{H}^{n-1}(x)}{|\nabla u|}\Big)^{1-\frac{\varrho}{p}}\end{aligned} \tag{5.25}$$

for a.e. $s \in (0, |\Omega|/2)$ such that $u_+^{*'}(s) \neq 0$. Thus, inequality (5.19) holds for a.e. $s \in S$ such that $u_+^{*'}(s) \neq 0$, thanks to (5.24), (5.15) and (5.25). Since (5.19) trivially holds when $u_+^{*'}(s) = 0$, and the latter equality is certainly true if $s \notin S$, inequality (5.19) holds, in fact, for a.e. $s \in (0, |\Omega|/2)$.

Owing to (5.17) and (5.19), the inequality

$$
\Big(\int_{\{u_+ > 0\}} |\nabla u|^\varrho dx \Big)^{\frac{p-1}{\varrho}} \leq C \Big(\int_\Omega f_+(x)^q \, dx \Big)^{\frac{1}{q}}, \tag{5.26}
$$

will follow if we show that

$$
\Big(\int_0^{|\Omega|/2} v(s) \Big(\int_0^s f_+^*(r) dr \Big)^{\frac{\varrho}{p-1}} ds \Big)^{\frac{p-1}{\varrho}} \leq C \Big(\int_0^{|\Omega|/2} f_+^*(s)^q ds \Big)^{\frac{1}{q}} \tag{5.27}
$$

(with the usual modification when $q = \infty$), where the function $v : (0, |\Omega|/2) \to [0, \infty)$ is defined as

$$
v(s) = \left[\frac{-u_+^{*'}(s)}{\big(\int_{\{u_+ = u_+^*(s)\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}(x) \big)^{\frac{1}{p-1}}} \right]^{\frac{\varrho}{p}} \qquad \text{for a.e. } s \in (0, |\Omega|/2). \tag{5.28}
$$

Let us now distinguish the cases (i)–(iv). Assume first that

$$
q > 1 \ \text{ and } \ q(p-1) \leq \varrho \,. \tag{5.29}
$$

By a weighted Hardy inequality ([Ma4, Section 1.3]), inequality (5.27) is equivalent to

$$
\sup_{0 < s < \frac{|\Omega|}{2}} \Big(\int_s^{|\Omega|/2} v(r) dr \Big)^{\frac{p-1}{\varrho}} s^{\frac{1}{q'}} < \infty \,. \tag{5.30}
$$

Fix any number α such that

$$
\frac{\alpha p}{p - \varrho} > 1 \,. \tag{5.31}
$$

By Hölder's inequality, we have that

$$
\begin{aligned}
\Big(\int_s^{|\Omega|/2} & v(r) dr \Big)^{\frac{p-1}{\varrho}} \\
& \leq \Big(\int_s^{|\Omega|/2} \frac{-u_+^{*'}(r) r^{\frac{\alpha p}{\varrho}}}{\big(\int_{\{u_+ = u_+^*(r)\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}(x) \big)^{\frac{1}{p-1}}} dr \Big)^{\frac{1}{p'}} \Big(\int_s^{|\Omega|/2} r^{-\frac{\alpha p}{p-\varrho}} dr \Big)^{\frac{p-\varrho}{\varrho p'}} \\
& \leq \big(\frac{\alpha p}{p - \varrho} - 1 \big)^{\frac{\varrho - p}{\varrho p'}} \Big(\int_s^{|\Omega|/2} \big(- \psi_{u_+}(u_+^*(r)) \big)' r^{\frac{\alpha p}{\varrho}} dr \Big)^{\frac{1}{p'}} s^{(1 - \frac{\alpha p}{p-\varrho}) \frac{p-\varrho}{\varrho p'}}
\end{aligned} \tag{5.32}
$$

for $s \in (0, |\Omega|/2)$, where ψ_{u_+} is defined as in (2.17). Notice that here we have made use of the fact that the function $\psi_{u_+} \circ u_+^*$ is locally absolutely continuous, being the composition of monotone locally absolutely continuous functions, and that

$$
\big(- \psi_{u_+}(u_+^*(s)) \big)' = \frac{-u_+^{*'}(s)}{\big(\int_{\{u_+ = u_+^*(s)\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}(x) \big)^{\frac{1}{p-1}}} \qquad \text{for a.e. } s \in (0, |\Omega|/2). \tag{5.33}
$$

Set

$$\eta = 1 + \frac{p(p-1)}{\varrho} - \frac{p}{q}\,, \tag{5.34}$$

the exponent appearing in (5.2). By (3.28) and (5.2),

$$s^{\eta} \le C\psi_{u_+}(u_+^*(s))^{1-p} \qquad \text{for } s \in (0, |\Omega|/2), \tag{5.35}$$

where C agrees with the left-hand side of (5.2). Hence,

$$\begin{aligned} \int_s^{|\Omega|/2} \big(-\psi_{u_+}(u_+^*(r))\big)' r^{\frac{\alpha p}{\varrho}}\, dr &\le C^{\frac{\alpha p}{\varrho\eta}} \int_s^{|\Omega|/2} \big(-\psi_{u_+}(u_+^*(r))\big)' \psi_{u_+}(u_+^*(r))^{-\frac{\alpha p(p-1)}{\varrho\eta}}\, dr \\ &= \frac{C^{\frac{\alpha p}{\varrho\eta}}}{1-\frac{\alpha p(p-1)}{\varrho\eta}} \psi_{u_+}(u_+^*(s))^{1-\frac{\alpha p(p-1)}{\varrho\eta}} \qquad \text{for } s \in (0, |\Omega|/2), \end{aligned} \tag{5.36}$$

provided that

$$\frac{\alpha p(p-1)}{\varrho\eta} < 1\,. \tag{5.37}$$

Note that the equality holds in (5.35) since $u_+^*(|\Omega|/2) = 0$, as a consequence of (2.20). Note also that conditions (5.31) and (5.37) are compatible. From (5.32), (5.36) and (5.35) one infers that (5.30) holds for some constant C depending only on the left-hand side of (5.2) and on p, q and ϱ. Thus, (5.27), and hence (5.26), follow in case (i).

Assume next that

$$1 < q < \infty \ \text{ and } \ 0 < \varrho < q(p-1)\,. \tag{5.38}$$

Then inequality (5.27) holds if

$$\int_0^{|\Omega|/2} \left(s^{\frac{\varrho}{p-1}-1} \int_s^{|\Omega|/2} v(r)dr \right)^{\frac{q(p-1)}{q(p-1)-\varrho}} ds < \infty \tag{5.39}$$

– see [SS]. By Proposition 5.3 below, applied with $\alpha = \frac{p}{\varrho}$, $\beta = \frac{\varrho}{p-1}$ and $\gamma = \frac{q(p-1)}{q(p-1)-\varrho}$, a constant $C = C(p, q, \varrho)$ exists such that

$$\begin{aligned} \int_0^{|\Omega|/2} s^{\frac{q(\varrho-p+1)}{q(p-1)-\varrho}} \left(\int_s^{|\Omega|/2} v(r)dr \right)^{\frac{q(p-1)}{q(p-1)-\varrho}} ds & \\ \le C \int_0^{|\Omega|/2} s^{\frac{q\varrho}{p(q(p-1)-\varrho)}} \left(\int_s^{|\Omega|/2} v(r)^{\frac{p}{\varrho}} dr \right)^{\frac{q\varrho(p-1)}{p(q(p-1)-\varrho)}} ds\,. \end{aligned} \tag{5.40}$$

On the other hand,

$$\begin{aligned} &\int_0^{|\Omega|/2} s^{\frac{q\varrho}{p(q(p-1)-\varrho)}} \left(\int_s^{|\Omega|/2} v(r)^{\frac{p}{\varrho}} dr \right)^{\frac{q\varrho(p-1)}{p(q(p-1)-\varrho)}} ds \\ &\quad = \int_0^{|\Omega|/2} s^{\frac{q\varrho}{p(q(p-1)-\varrho)}} \left(\int_s^{|\Omega|/2} \big(-\psi_{u_+}(u_+^*(r))\big)' dr \right)^{\frac{q\varrho(p-1)}{p(q(p-1)-\varrho)}} ds \\ &\quad = \int_0^{|\Omega|/2} \big(s\,\psi_{u_+}(u_+^*(s))^{p-1}\big)^{\frac{q\varrho}{p(q(p-1)-\varrho)}} ds \\ &\quad \le \int_0^{|\Omega|/2} \left(\frac{s}{\nu(s)}\right)^{\frac{\varrho q}{p(q(p-1)-\varrho)}} ds\,, \end{aligned} \tag{5.41}$$

where the inequality holds owing to (3.28). Equation (5.39) follows from (5.40), (5.41) and (5.3). Inequality (5.27), and hence also (5.26), are thus established in case (ii).

Now, consider the case when $q = \infty$. One has

$$\begin{aligned}\bigg(\int_0^{|\Omega|/2} v(s)\Big(\int_0^s f_+^*(r)dr\Big)^{\frac{\varrho}{p-1}} ds\bigg)^{\frac{p-1}{\varrho}} &\leq \|f_+^*\|_{L^\infty(0,|\Omega|/2)}\bigg(\int_0^{|\Omega|/2} v(s)s^{\frac{\varrho}{p-1}} ds\bigg)^{\frac{p-1}{\varrho}} \\ &= \|f_+\|_{L^\infty(\Omega)}\bigg(\frac{\varrho}{p-1}\int_0^{|\Omega|/2} s^{\frac{\varrho}{p-1}-1}\int_s^{|\Omega|/2} v(r)dr\, ds\bigg)^{\frac{p-1}{\varrho}}.\end{aligned}$$

Thus, a completely analogous argument as in the proof of case (ii) tells us that (5.27) holds, provided that (5.4) is in force. This establishes (5.26) in case (iii).

Finally, assume that $q = 1$. We have

$$\bigg(\int_0^{|\Omega|/2} v(s)\Big(\int_0^s f_+^*(r)dr\Big)^{\frac{\varrho}{p-1}} ds\bigg)^{\frac{p-1}{\varrho}} \leq \|f_+^*\|_{L^1(0,|\Omega|/2)}\bigg(\int_0^{|\Omega|/2} v(s)ds\bigg)^{\frac{p-1}{\varrho}}. \tag{5.42}$$

Condition (5.5) entails that there exists a number $\alpha \in (0, \frac{p-\varrho}{p})$ such that

$$\int_0^{|\Omega|/2} \frac{s^{\frac{\alpha p}{\varrho}-1}}{\nu^{\frac{1}{p-1}}}\, ds < \infty.$$

Thus, equation (5.33), Hölder's inequality, and inequality (3.28) yield

$$\begin{aligned}\int_0^{|\Omega|/2} v(s)\, ds &= \int_0^{|\Omega|/2}\bigg(\frac{-u_+^{*\prime}(s)}{\big(\int_{\{u_+=u_+^*(s)\}}|\nabla u|^{p-1}d\mathcal{H}^{n-1}(x)\big)^{\frac{1}{p-1}}}\bigg)^{\frac{\varrho}{p}} ds \\ &\leq \bigg(\int_0^{|\Omega|/2} s^{-\frac{\alpha p}{p-\varrho}} ds\bigg)^{\frac{p-\varrho}{p}}\bigg(\int_0^{|\Omega|/2}\big(-\psi_{u_+}(u_+^*(s))\big)' s^{\frac{\alpha p}{\varrho}} ds\bigg)^{\frac{\varrho}{p}} \\ &= \bigg(\frac{(|\Omega|/2)^{1-\frac{\alpha p}{p-\varrho}}}{1-\frac{\alpha p}{p-\varrho}}\bigg)^{\frac{p-\varrho}{p}}\bigg(\frac{\alpha p}{\varrho}\int_0^{|\Omega|/2}\psi_{u_+}(u_+^*(s))s^{\frac{\alpha p}{\varrho}-1} ds\bigg)^{\frac{\varrho}{p}} \\ &\leq \bigg(\frac{(|\Omega|/2)^{1-\frac{\alpha p}{p-\varrho}}}{1-\frac{\alpha p}{p-\varrho}}\bigg)^{\frac{p-\varrho}{p}}\bigg(\frac{\alpha p}{\varrho}\int_0^{|\Omega|/2}\frac{s^{\frac{\alpha p}{\varrho}-1}}{\nu(s)^{\frac{1}{p-1}}} ds\bigg)^{\frac{\varrho}{p}} < \infty.\end{aligned} \tag{5.43}$$

Inequality (5.27) is a consequence of (5.42) and (5.43). Thus, (5.26) is proved also in case (iv).

The same arguments as above yield an inequality analogous to (5.26), with $\{u_+ > 0\}$ replaced by $\{u_- > 0\}$ on the left-hand side, and f_+ replaced by f_- on the right-hand side. Hence, estimate (5.1) follows. □

Proposition 5.3 *Let $\alpha \in (1, \infty)$ and $\beta, \gamma \in (0, \infty)$. Then a constant $C = C(\alpha, \beta, \gamma)$ exists such that*

$$\int_0^\infty \Big(s^{\beta-1}\int_s^\infty \phi(r)dr\Big)^\gamma ds \leq C\int_0^\infty \Big(s^{\alpha\beta-1}\int_s^\infty \phi(r)^\alpha dr\Big)^{\frac{\gamma}{\alpha}} ds \tag{5.44}$$

for every measurable function $\phi : [0, \infty) \to [0, \infty)$.

Proof. We may clearly assume that $\int_s^\infty \phi(r)^\alpha dr < \infty$ for every $s > 0$, otherwise inequality (5.44) is trivially true. By Hölder's inequality, given any θ such that

$$\theta\alpha' > 1\,, \tag{5.45}$$

we have

$$\begin{aligned}\int_0^\infty \Big(s^{\beta-1}\int_s^\infty \phi(r)dr\Big)^\gamma ds &\le \int_0^\infty s^{\gamma(\beta-1)}\Big(\int_s^\infty \phi(r)^\alpha r^{\theta\alpha}dr\Big)^{\frac{\gamma}{\alpha}}\Big(\int_s^\infty r^{-\theta\alpha'}dr\Big)^{\frac{\gamma}{\alpha'}}ds \\ &= (\theta\alpha'-1)^{-\frac{\gamma}{\alpha'}}\int_0^\infty s^{\gamma(\beta-\frac{1}{\alpha}-\theta)}\Big(\int_s^\infty \phi(r)^\alpha r^{\theta\alpha}dr\Big)^{\frac{\gamma}{\alpha}}ds\,.\end{aligned} \tag{5.46}$$

Define $\Phi : (0,\infty) \to (0,\infty)$ as

$$\Phi(s) = \int_s^\infty \phi(r)^\alpha dr \qquad \text{for } s \in (0,\infty), \tag{5.47}$$

a non-increasing function. Thus, integration by parts yields

$$\begin{aligned}&\int_0^\infty s^{\gamma(\beta-\frac{1}{\alpha}-\theta)}\Big(\int_s^\infty \phi(r)^\alpha r^{\theta\alpha}dr\Big)^{\frac{\gamma}{\alpha}}ds \\ &\qquad\le C\int_0^\infty s^{\gamma(\beta-\frac{1}{\alpha})}\Phi(s)^{\frac{\gamma}{\alpha}}ds + C\int_0^\infty s^{\gamma(\beta-\frac{1}{\alpha}-\theta)}\Big(\int_s^\infty \Phi(r)r^{\theta\alpha-1}dr\Big)^{\frac{\gamma}{\alpha}}ds\,,\end{aligned} \tag{5.48}$$

for some constant $C = C(\alpha,\gamma,\theta)$. The first addendum on the right-hand side of (5.48) agrees with the integral on the right-hand side of (5.44). Thus, inequality (5.44) is reduced to showing that a constant $C = C(\alpha,\beta,\gamma)$ exists such that

$$\int_0^\infty s^{\gamma(\beta-\frac{1}{\alpha}-\theta)}\Big(\int_s^\infty \Phi(r)r^{\theta\alpha-1}dr\Big)^{\frac{\gamma}{\alpha}}ds \le C\int_0^\infty s^{\gamma(\beta-\frac{1}{\alpha})}\Phi(s)^{\frac{\gamma}{\alpha}}ds \tag{5.49}$$

for every non-increasing function $\Phi : (0,\infty) \to (0,\infty)$. Inequality (5.49) holds thanks to [Go, Theorem 1.1]. □

6 Applications and examples

We collect in this section a few applications of the above results to special domains, or classes of domains Ω.

Example 1. (Lipschitz domains).
Assume that Ω is a bounded and connected open set having a Lipschitz boundary, and let $1 < p < n$. By (2.14), Theorem 3.1 yields the estimate

$$u_\pm{}^*(s) \le C\int_s^{|\Omega|/2} r^{\frac{p(1-n)}{n(p-1)}}\Big(\int_0^r f_\pm^*(\tau)d\tau\Big)^{\frac{1}{p-1}}dr + C\Big(\int_0^{|\Omega|/2} f_\pm^*(r)dr\Big)^{\frac{1}{p-1}} \qquad \text{for } s \in (0,|\Omega|/2)\,,$$

for some constant $C = C(\Omega)$. By (2.7), such an estimate is equivalent to (3.4), up to the constant C and to the additional term $C\big(\int_0^{|\Omega|/2} f_\pm^*(r)dr\big)^{\frac{1}{p-1}}$.

Corollary 4.6 yields

$$\|u\|_{L^{\frac{q(p-1)n}{n-qp}}(\Omega)} \le C\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}} \tag{6.1}$$

if $1 < q < \frac{n}{p}$, and

$$\|u\|_{L^\infty(\Omega)} \le C\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}} \tag{6.2}$$

if either $q = \frac{n}{p}$ and $p' \ge n$, or $q > \frac{n}{p}$, for some constant C. When $q = \frac{n}{p}$ and $p' < n$, the same corollary tells us that

$$\|u\|_{\mathrm{Exp}L^{\frac{n}{n-p'}}(\Omega)} \le C\|f\|_{L^{\frac{n}{p}}(\Omega)}^{\frac{1}{p-1}}. \tag{6.3}$$

If $q = 1$, one has

$$\|u\|_{\Lambda_\omega^{p-1}(\Omega)} \le C\|f\|_{L^1(\Omega)}^{\frac{1}{p-1}}, \tag{6.4}$$

where $\omega(s) = b(s^{\frac{n-p}{n}})s^{-\frac{p}{n}}$, and b is any function fulfilling (4.74) – see Remark 4.7.
The capacitary counterparts of (6.1), (6.3) and (6.4) are given by Proposition 4.5, and tell us that

$$\left(\int_0^\infty C_{p,u_\pm}(t)^{\frac{n-pq}{n-p}} d(t^{q(p-1)})\right)^{\frac{1}{q}} \le C\|f\|_{L^q(\Omega)} \tag{6.5}$$

if $1 < q < \frac{n}{p}$,

$$\left(\int_0^\infty \left[1 + \log_+\left(\frac{C}{C_{p,u_\pm}(t)}\right)\right]^{1-\frac{n}{p'}} d(t^{\frac{n}{p'}})\right)^{\frac{p}{n}} \le C\|f\|_{L^{\frac{n}{p}}(\Omega)} \tag{6.6}$$

if $q = \frac{n}{p}$ and $p' < n$, and

$$\int_0^\infty B\big(C_{p,u_\pm}(t)\big)d(t^{p-1}) \le C\|f\|_{L^1(\Omega)},$$

if $q = 1$, and B is any function as in (4.60) fulfilling (4.74).
Finally, Theorem 5.1, cases (i) and (iv), gives the gradient estimate

$$\|\nabla u\|_{L^{\frac{q(p-1)n}{n-q}}(\Omega)} \le C\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}},$$

if $1 < q \le \frac{np}{np+p-n}$, and

$$\|\nabla u\|_{L^\varrho(\Omega)} \le C\|f\|_{L^1(\Omega)}^{\frac{1}{p-1}},$$

if $q = 1$ and $0 < \varrho < n'(p-1)$.

Example 2. (A domain with a cusp).
Given any $L > 0$ and any convex function $\vartheta : [0,\infty) \to [0,\infty)$ such that $\vartheta(0) = 0$, consider the set

$$\Omega = \{x \in \mathbb{R}^n : |x'| < \vartheta(x_n), 0 < x_n < L\}$$

(see Figure 1). Here, $x = (x', x_n)$, where $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. Define $\Theta : [0, \infty) \to [0, \infty)$ as

$$\Theta(\rho) = n\omega_n \int_0^\rho \vartheta(r)^{n-1} dr \qquad \text{for } \rho > 0.$$

Then one has

$$C\vartheta(\Theta^{-1}(s))^{n-1} \le \lambda_\Omega(s) \le \vartheta(\Theta^{-1}(s))^{n-1} \qquad \text{for } s \in (0, |\Omega|/2), \tag{6.7}$$

for some constant $C \in (0,1)$ - see [Ma4, 3.3.3, Example 1]. On the other hand, by [Ma4, 4.3.5/1],

$$C\Big(\int_{\Theta^{-1}(s)}^{\Theta^{-1}(|\Omega|/2)} \vartheta(r)^{\frac{1-n}{p-1}} dr \Big)^{1-p} \le \nu_{\Omega,p}(s) \le \Big(\int_{\Theta^{-1}(s)}^{\Theta^{-1}(|\Omega|/2)} \vartheta(r)^{\frac{1-n}{p-1}} dr \Big)^{1-p} \quad \text{for } s \in (0, |\Omega|/2), \tag{6.8}$$

for some constant $C \in (0,1)$. Thus, although Ω is not a Lipschitz domain when $\vartheta'(0) = 0$,

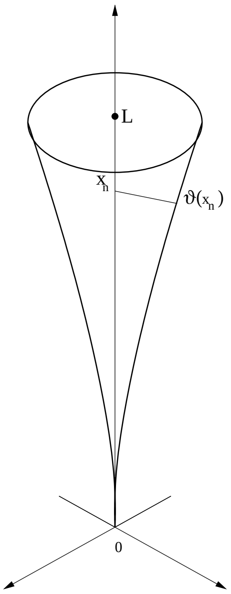

Figure 1: a domain with a cusp

estimate (3.8) is fulfilled also in the present example.
In particular, if there exists $\delta > \frac{p-1}{n-1}$ such that

$$\vartheta(r) = r^\delta \qquad \text{for } r \ge 0\,,$$

then

$$\nu_{\Omega,p}(s) = O\big(s^{\frac{\delta(n-1)+1-p}{\delta(n-1)+1}}\big) \qquad \text{as } s \to 0^+,$$

where $\frac{\delta(n-1)+1-p}{\delta(n-1)+1} \in (0,1)$. Thus, norm and capacitary estimates for solutions u to (1.1) can be derived from Corollary 4.6 and Proposition 4.5, respectively. Moreover, Theorem 5.1, cases (i) and (iv), yields the gradient bound

$$\|\nabla u\|_{L^{\frac{q(p-1)[\delta(n-1)+1]}{\delta(n-1)+1-q}}(\Omega)} \le C\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}}\,,$$

if $1 < q \le \frac{p[\delta(n-1)+1]}{(p-1)[\delta(n-1)+1]+p}$, and

$$\|\nabla u\|_{L^{\varrho}(\Omega)} \le C\|f\|_{L^1(\Omega)}^{\frac{1}{p-1}},$$

if $q = 1$ and $0 < \varrho < \frac{(p-1)[\delta(n-1)+1]}{\delta(n-1)}$.

Example 3 (Nikodým)
We conclude with the highly irregular domain $\Omega \subset \mathbb{R}^2$ depicted in Figure 2, which was introduced by Nikodým in his study of Sobolev embeddings.

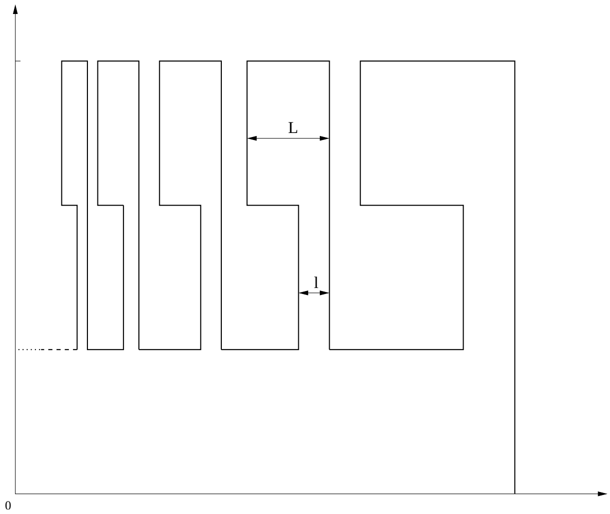

Figure 2: Nikodým example

Here, $L = 2^{-k}$ and $l = 2^{-\delta k}$, where $\delta > 1$ and $k \in \mathbb{N}$. One has

$$\lambda_\Omega(s) = O(s^\delta) \qquad \text{as } s \to 0^+, \tag{6.9}$$

and, if $1 < p < 2$,

$$\nu_{\Omega,p}(s) = O(s^\delta) \qquad \text{as } s \to 0^+ \tag{6.10}$$

([Ma4, Section 4.5]). Thus, since

$$\left(\int_s^{|\Omega|/2} \frac{dr}{\lambda_\Omega(r)^{p'}}\right)^{1-p} = O(s^{p(\delta-1)+1}) \qquad \text{as } s \to 0^+, \tag{6.11}$$

the isocapacitary function $\nu_{\Omega,p}(s)$ is not equivalent to $\left(\int_s^{|\Omega|/2} \frac{dr}{\lambda_\Omega(r)^{p'}}\right)^{1-p}$ for such a domain Ω. In fact, the estimates for solutions to (1.1) which can be derived via isocapacitary inequality are stronger than those obtained by isoperimetric inequalities.
To see this, note that Theorem 4.1, cases (ii) and (vii), yields

$$\|u\|_{L^\sigma(\Omega)} \le C\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}} \tag{6.12}$$

for every $q \geq 1$ and

$$\sigma < \frac{q(p-1)}{q(\delta-1)+1}. \tag{6.13}$$

A slightly stronger conclusion follows from Corollary 4.3, which tells us that

$$\|u\|_{L^{\frac{q(p-1)}{q(\delta-1)+1},q(p-1)}(\Omega)} \leq C\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}}. \tag{6.14}$$

On the other hand, an estimate like (6.12) can be deduced via inequality (3.4) for those q and σ such that

$$\left(\int_0^{|\Omega|/2}\left(\int_s^{|\Omega|/2}\left(\int_0^r \phi(\rho)(\rho)d\rho\right)^{\frac{1}{p-1}}\frac{dr}{\lambda_\Omega(r)^{p'}}\right)^\sigma ds\right)^{\frac{1}{\sigma}} \leq C\left(\int_0^{|\Omega|/2}\phi(s)^q ds\right)^{\frac{1}{q(p-1)}} \tag{6.15}$$

for some constant C and for every nonnegative, non-increasing function $\phi \in L^q(0,|\Omega|/2)$. Choosing $\phi = \chi_{(0,R)}$ as trial function in (6.15) and letting R vary in $(0,|\Omega|/2)$ shows that a necessary condition for (6.15) to hold is that

$$\sigma \leq \frac{q(p-1)}{pq(\delta-1)+1}, \tag{6.16}$$

a more stringent assumption than (6.13). (Incidentally, note that inequality (6.15) actually holds provided that the inequality in (6.16) is strict.)

As far as gradient estimates are concerned, Theorem 5.1, case (ii) and (iv), implies that

$$\|\nabla u\|_{L^\varrho(\Omega)} \leq C\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}}, \tag{6.17}$$

provided that

$$\varrho < \frac{pq(p-1)}{q(\delta-1)+p}. \tag{6.18}$$

Again, such a conclusion is stronger than an analogous result, which follows via the relative isoperimetric and tells us that

$$\left(\int_\Omega |\nabla u_\pm|^\varrho dx\right)^{\frac{1}{\varrho}} \leq \left(\int_0^{|\Omega|/2}\left(\frac{1}{\lambda_\Omega(s)}\int_0^s f_\pm^*(r)dr\right)^{\frac{\varrho}{p-1}} ds\right)^{\frac{1}{\varrho}} \tag{6.19}$$

([Ci2, Ma3]). Indeed, inequality (6.19) implies (6.17) for every q and ϱ such that

$$\left(\int_0^{|\Omega|/2}\left(\frac{1}{\lambda_\Omega(s)}\int_0^s \phi(r)dr\right)^{\frac{\varrho}{p-1}} ds\right)^{\frac{1}{\varrho}} \leq C\left(\int_0^{|\Omega|/2}\phi(s)^q ds\right)^{\frac{1}{q(p-1)}} \tag{6.20}$$

for some constant C and for every nonnegative and non-increasing function $\phi \in L^q(0,|\Omega|/2)$. The choice $\phi = \chi_{[0,R)}$ for arbitrary $R \in (0,|\Omega|/2)$ entails that a necessary condition for (6.20) to hold is

$$\varrho \leq \frac{q(p-1)}{q(\delta-1)+1},$$

which is stronger than (6.18).

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