Quasilinear elliptic problems with general growth and merely integrable, or measure, data

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Abstract

Boundary value problems for a class of quasilinear elliptic equations, with an Orlicz type growth and L^1 right-hand side are considered. Both Dirichlet and Neumann problems are contemplated. Existence and uniqueness of generalized solutions, as well as their regularity, are established. The case of measure right-hand sides is also analyzed.

1 Introduction

We are concerned with boundary value problems for quasilinear elliptic equations whose right-hand side is just an integrable function, or even a signed mesaure with finite total variation. Problems of this kind, involving elliptic operators modeled upon the p-Laplacian, have been systematically investigated in the literature, starting with the papers [BG1, BG2], where measure right-hand sides are also taken into account. Contributions on this topic include [ACMM, AFT, AM, AMST, BG, BBGGPV, BGM, Da, DM, DLR, De, DHM3, Dr, DV, FS, KuMi1, LM, Mi1, Mi2, Mu1, Mu2, Po, Pr, Ra1, Ra2]. Uniqueness of bounded solutions under boundary conditions prescribed outside exceptional sets had earlier been established in [Ma3] – see also [Ma5, Section 15.8.4]. The analysis of linear problems goes back to [Ma2, Ma4] and [St].

The present paper focuses on a class of elliptic operators whose nonlinearity is not necessarily of power type. Specifically, we mainly deal with existence, uniqueness and regularity of solutions to Dirichlet problems of the form

$$(1.1)\qquad \begin{cases} -\mathrm{div}(\mathcal{A}(x,\nabla u)) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

Mathematics Subject Classifications: 35J25, 35J60, 35B65.
Keywords: Quasilinear elliptic equations, general growth, Orlicz spaces, L^1-data, approximable solutions, Dirichlet problems, Neumann problems.

and parallel Neumann problems

$$\begin{cases} -\mathrm{div}(\mathcal{A}(x,\nabla u)) = f(x) & \text{in } \Omega \\ \mathcal{A}(x,\nabla u)\cdot\nu = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1.2}$$

Here, and throughout the paper, Ω is a domain - i.e. a connected open set - in $\mathbb{R}^n$, with finite Lebesgue measure $|\Omega|$. Moreover, ν denotes the outward unit vector on $\partial\Omega$, the dot " $\cdot$ " stands for scalar product, and $\mathcal{A}:\Omega\times\mathbb{R}^n\to\mathbb{R}^n$ is a Carathéodory function. The datum f is just assumed to belong to $L^1(\Omega)$. Obviously, the additional compatibility condition

$$\int_\Omega f(x)\,dx = 0 \tag{1.3}$$

must be imposed when the Neumann problem (1.2) is taken into account. Suitable regularity on Ω has also to be required in this case.

The more general problems where f is replaced in (1.1) and (1.2) by a signed Radon measure, with finite total variation over Ω, will also be discussed.

As hinted above, a critical trait of problems (1.1) and (1.2) is that the role played by the function t^p in problems governed by power type nonlinearities is performed by a more general Young function $B(t)$, namely a convex function from $[0,\infty)$ into $[0,\infty]$, vanishing at 0. Any function of this kind can obviously be written in the form

$$B(t) = \int_0^t b(\tau)\,d\tau \qquad \text{for } t\geq 0, \tag{1.4}$$

for some (nontrivial) non-decreasing function $b:[0,\infty)\to[0,\infty]$.
Our hypotheses on the function $\mathcal{A}$ amount to the ellipiticty condition

$$\mathcal{A}(x,\xi)\cdot\xi \geq B(|\xi|) \qquad \text{for a.e. } x\in\Omega, \text{ and for } \xi\in\mathbb{R}^n, \tag{1.5}$$

where the dot " $\cdot$ " denotes scalar product in $\mathbb{R}^n$, the growth condition

$$|\mathcal{A}(x,\xi)| \leq C(b(|\xi|)+g(x)) \qquad \text{for a.e. } x\in\Omega, \text{ and for } \xi\in\mathbb{R}^n, \tag{1.6}$$

for some function $g\in L^{\widetilde{B}}(\Omega)$, where $\widetilde{B}$ stands for the Young conjugate of B, and the strict monotonicity condition

$$\big(\mathcal{A}(x,\xi)-\mathcal{A}(x,\eta)\big)\cdot(\xi-\eta) > 0 \quad \text{for a.e. } x\in\Omega, \text{ and for } \xi,\eta\in\mathbb{R}^n \text{ with } \xi\neq\eta. \tag{1.7}$$

The function B is assumed to be finite-valued, and to fulfill the condition

$$1 < i_B \leq s_B < \infty, \tag{1.8}$$

where

$$i_B = \inf_{t>0}\frac{tb(t)}{B(t)} \qquad \text{and} \qquad s_B = \sup_{t>0}\frac{tb(t)}{B(t)}\,. \tag{1.9}$$

Assumption (1.8) is a counterpart of the assumption that $1<p<\infty$ in the case of p-Laplacian type equations, in which case $B(t)=t^p$, and $i_B=s_B=p$. In particular, the standard p-Laplace operator corresponds to the choice $\mathcal{A}(x,\xi)=|\xi|^{p-2}\xi$.

An appropriate functional framework in the situation at hand is provided by the Orlicz-Sobolev spaces associated wtih the function B. Assumption (1.8) entails that these spaces are reflexive. Hence, when the right-hand side f is regular enough, a classical theory of monotone operators applies to the Dirichlet problem (1.1) and to the Neumann problem (1.2), and yields the existence and uniqueness (up to additive constants in case of the Neumann problem) of a weak solution u. Membership in suitable duals of the natural Orlicz-Sobolev energy spaces associated with problems (1.1) and (1.2) is a minimal assumption on f for weak solutions to be well defined, and for this theory to apply.

The point is that, as in the case of the plain p-Laplace equation, if f is only supposed to be in $L^1(\Omega)$, then it need not belong to the relevant dual spaces. This is of course the interesting case. Weak solutions to problems (1.1) and (1.2) are not even well defined in this case. Nevertheless, we shall show that a unique solution, in a suitably generalized sense, does exist both to the Dirichlet problem (1.1), and, under some mild regularity assumption on Ω, to the Neumann problem (1.2). The regularity of the relevant solution will also be described. This is indeed the main purpose of our work, that also gives grounds for further regularity results under additional structure assumptions on the differential operator appearing in problems (1.1) and (1.2) – see [CM] with this regard.

The notion of solution that will be adopted is that of approximable solution, namely a function which is the limit of a sequence of weak solutions to problems whose right-hand sides are smooth and converge to f. An approximable solution need not be even weakly differentiable. However, it is associated with a vector-valued function on Ω, which plays the role of a substitute for its gradient in the definition of distributional solution to (1.1) or (1.2). With some abuse of notation, it will still be denoted by ∇u. This notion of solution extends to the present generalized setting that introduced for operators with power type growth in [BG1, Da].

A sharp analysis of the regularity of the approximable solution to problems (1.1) and (1.2) calls for the use of optimal embedding theorems for Orlicz-Sobolev spaces. In particular, the regularity of the solution to the Neumann problem (1.2) turns out to depend on the degree of regularity of the domain Ω. This is intrinsic in the problem, as demonstrated, for instance, by the examples in [Ma4], and is related to the fact that the target spaces in the Orlicz-Sobolev embeddings in question, for functions that need not vanish on $\partial\Omega$, depend on the regularity of Ω.

The existence and regularity results to be established for approximable solutions to problems (1.1) and (1.2) continue to hold in the case when the right-hand side is a signed measure with finite total variation over Ω, with a completely analogous approach. On the other hand, the proof of the uniqueness of solutions does not carry over to this more general setting. In fact, as far as we know, the problem of uniqueness of solutions to boundary value problems for equations with a measure on the right-hand side is still open even in the basic case of differential operators with power type nonlinearities. Partial results dealing with measures that are not too much concentrated, or with differential operators fulfilling additional assumptions, can be found in [BGO, DHM3, DMOP, KX].

Let us mention that results on boundary value problems for quasilinear elliptic equations with Orlicz type growth and L^1 or measure right-hand side are available in the literature – see e.g. [BB] and subsequent papers by the same authors and their collaborators. However, different notions of generalized solutions are employed in these contributions, and, importantly, their precise regularity properties are not discussed. Pointwise gradient bounds for local solutions to equations with Uhlenbeck type structure and Orlicz growth, with measure-valued right-hand side, are the subject of [Ba].

We add that assumption (1.8) could probably be weakened, as to also include certain borderline situations. The results would, however, take a more technical form, and we prefer not to address this issue, not being our primary interest here.

Let us conclude this section by recalling that elliptic systems with a right-hand side in L^1, or in the space of finite measures, have also been considered in the literature. The references [KuMi2] for local solutions, and [DHM1, DHM2, DHM3] for Dirichlet boundary problems, are relevant in this connection. The results of the latter contributions are likely to admit an extension to the generalized framework of the present paper. However, their proofs would entail substantial modifications, and the function spaces coming into play would take a somewhat different form, due to the failure of standard truncation techniques for vector-valued functions. This is of course an issue of interest, but goes beyond the scope of this work.

2 Function spaces

2.1 Spaces of measurable functions

The decreasing rearrangement $u^* : [0,\infty) \to [0,\infty]$ of a measurable function $u : \Omega \to \mathbb{R}$ is the unique right-continuous, non-increasing function in $[0,\infty)$ equidistributed with u. In formulas,

$$u^*(s) = \inf\{t \geq 0 : |\{|u| > t\}| \leq s\} \qquad \text{for } s \in [0,\infty). \tag{2.1}$$

Clearly, $u^*(s) = 0$ if $s \geq |\Omega|$. A basic property of rearrangements tells us that

$$\int_E |u(x)|\, dx \leq \int_0^{|E|} u^*(s)\, ds \tag{2.2}$$

for every measurable set $E \subset \Omega$.

Let $\varphi : (0,|\Omega|) \to (0,\infty)$ be a continuous increasing function. We denote by $L^{\varphi,\infty}(\Omega)$ the Marcinkiewicz type space of those measurable functions u in Ω such that

$$\sup_{s\in(0,|\Omega|)} \frac{u^*(s)}{\varphi^{-1}(\lambda/s)} < \infty$$

for some $\lambda > 0$. Note that $L^{\varphi,\infty}(\Omega)$ is not always a normed space. Special choices of the function φ recover standard spaces of weak type. For instance, if $\varphi(t) = t^q$ for some $q > 0$, then $L^{\varphi,\infty}(\Omega) = L^{q,\infty}(\Omega)$, the customary weak-$L^q(\Omega)$ space. When $\varphi(t)$ behaves like $t^q(\log t)^\beta$ near infinity for some $q > 0$ and $\beta \in \mathbb{R}$, we shall adopt the notation $L^{q,\infty}(\log L)^\beta(\Omega)$ for $L^{\varphi,\infty}(\Omega)$. The meaning of the notation $L^{q,\infty}(\log L)^\beta(\log\log L)^{-1}(\Omega)$ is analogous.

The Orlicz spaces extend the Lebesgue spaces in the sense that the role of powers in the definition of the norms is instead played by Young functions, i.e. functions B of the form (1.4). The Young conjugate of a Young function B is the Young function $\widetilde{B}$ defined as

$$\widetilde{B}(t) = \sup\{st - B(s) : s \geq 0\} \qquad \text{for } t \geq 0.$$

One has that

$$\widetilde{B}(t) = \int_0^t b^{-1}(\tau)\, d\tau \quad \text{for } t \geq 0, \tag{2.3}$$

where b^{-1} denotes the (generalized) left-continuous inverse of the function b appearing in (1.4). As a consequence of the monotonicity of the function b, one has that

$$\frac{t}{2} b(t/2) \leq B(t) \leq t b(t) \qquad \text{for } t \geq 0. \tag{2.4}$$

An application of inequality (2.4), and of the same inequality with B replaced by $\widetilde{B}$, yield

$$\frac{1}{4}B(b^{-1}(t)) \le \widetilde{B}(t) \le B(2b^{-1}(t)) \quad \text{for } t \ge 0. \tag{2.5}$$

A Young function (and, more generally, an increasing function) B is said to belong to the class Δ_2 if there exists a constant c such that

$$B(2t) \le cB(t) \qquad \text{for } t \ge 0.$$

Note that, if $B \in \Delta_2$, then the first inequality in (2.4) ensures that

$$tb(t) \le cB(t) \quad \text{for } t \ge 0, \tag{2.6}$$

for some constant c, and the second inequality in (2.5) implies that

$$\widetilde{B}(b(t)) \le cB(t) \quad \text{for } t \ge 0, \tag{2.7}$$

for some constant $c > 0$.
The second inequality in (1.8) is equivalent to the fact that $B \in \Delta_2$; the first inequality is equivalent to $\widetilde{B} \in \Delta_2$. Under assumption (1.8),

$$\frac{B(t)}{t^{i_B}} \text{ is inon-decreasing, and } \frac{B(t)}{t^{s_B}} \text{ is non-increasing} \quad \text{for } t > 0; \tag{2.8}$$

furthermore,

$$\frac{\widetilde{B}(t)}{t^{s'_B}} \text{ is non-decreasing, and } \frac{\widetilde{B}(t)}{t^{i'_B}} \text{ is non-increasing} \quad \text{for } t > 0. \tag{2.9}$$

Here, and in what follows, prime stands for the Hölder conjugate, namely $i'_B = \frac{i_B}{i_B-1}$ and $s'_B = \frac{s_B}{s_B-1}$.
A Young function A is said to dominate another Young function B near infinity [resp. globally] if there exist constants $c > 0$ and $t_0 \ge 0$ such that

$$B(t) \le A(ct) \quad \text{for } t \ge t_0 \quad [t \ge 0]. \tag{2.10}$$

The functions A and B are called equivalent near infinity [globally] if they dominate each other near infinity [globally].

The Orlicz space $L^B(\Omega)$ built upon a Young function B is the Banach function space of those real-valued measurable functions u on Ω whose Luxemburg norm

$$\|u\|_{L^B(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega B\left(\frac{|u(x)|}{\lambda}\right) dx \le 1\right\}$$

is finite. The choice $B(t) = t^p$, for some $p \ge 1$, yields $L^B(\Omega) = L^p(\Omega)$. The Orlicz spaces associated with Young functions of the form $B(t) = e^{t^\beta} - 1$ for some $\beta > 0$, and are denoted by $\exp L^\beta(\Omega)$. The notation $\exp\exp L(\Omega)$ is adopted for Orlicz spaces associated with the Young function $B(t) = e^{e^t} - e$.
The Hölder type inequality

$$\int_\Omega |u(x)v(x)|\, dx \le 2\|u\|_{L^B(\Omega)}\|v\|_{L^{\tilde{B}}(\Omega)} \tag{2.11}$$

holds for every $u \in L^B(\Omega)$ and $v \in L^{\widetilde{B}}(\Omega)$. Given two Young functions A and B, one has that

$$L^A(\Omega) \to L^B(\Omega) \text{ if and only if } A \text{ dominates } B \text{ near infinity.} \tag{2.12}$$

We refer to the monograph [RR] for a comprehensive treatment of Young functions and Orlicz spaces.

2.2 Spaces of weakly differentiable functions

Given a Young function B, we denote by $W^{1,B}(\Omega)$ the Orlicz-Sobolev space defined as

$$W^{1,B}(\Omega) = \{u \in L^B(\Omega) : \text{is weakly differentiable in } \Omega \text{ and } |\nabla u| \in L^B(\Omega)\}\,, \tag{2.13}$$

equipped with the norm

$$\|u\|_{W^{1,B}(\Omega)} = \|u\|_{L^B(\Omega)} + \|\nabla u\|_{L^B(\Omega)}.$$

The notation $W^{1,B}_0(\Omega)$ is employed for the subspace of $W^{1,B}(\Omega)$ given by

$$W^{1,B}_0(\Omega) = \{u \in W^{1,B}(\Omega) : \text{the continuation of } u \text{ by } 0 \text{ outside } \Omega \text{ is weakly differentiable in } \mathbb{R}^n\}\,.$$

Thanks to a Poincaré type inequalty in Orlicz spaces – see [Ta2] – the functional $\|\nabla u\|_{L^B(\Omega)}$ defines a norm on $W^{1,B}_0(\Omega)$ equivalent to $\|u\|_{W^{1,B}(\Omega)}$.
The duals of $W^{1,B}(\Omega)$ and $W^{1,B}_0(\Omega)$ will be denoted by $(W^{1,B}(\Omega))'$ and $(W^{1,B}_0(\Omega))'$, respectively. If $B \in \Delta_2$ and $\widetilde{B} \in \Delta_2$, then the spaces $W^{1,B}(\Omega)$ and $W^{1,B}_0(\Omega)$ are separable and reflexive.

Our results involve certain function spaces that consist of those functions whose truncations are Orlicz-Sobolev functions. They extend the spaces introduced in [BBGGPV] in the standard case when $B(t) = t^p$ for some $p \geq 1$. Specifically, given any $t > 0$, let $T_t : \mathbb{R} \to \mathbb{R}$ be the function defined as

$$T_t(s) = \begin{cases} s & \text{if } \ |s| \leq t \\ t\,\text{sign}(s) & \text{if } \ |s| > t\,. \end{cases} \tag{2.14}$$

We set

$$\mathcal{T}^{1,B}(\Omega) = \left\{u \,\text{is measurable in } \Omega : T_t(u) \in W^{1,B}(\Omega) \text{ for every } t > 0\right\}\,. \tag{2.15}$$

The subspace $\mathcal{T}^{1,B}_0(\Omega)$ is defined accordingly, on replacing $W^{1,B}(\Omega)$ by $W^{1,B}_0(\Omega)$ on the right-hand side of (2.15). Note that $u \in \mathcal{T}^{1,B}(\Omega)$ if and only if $u + c \in \mathcal{T}^{1,B}(\Omega)$ for every $c \in \mathbb{R}$.
If $u \in \mathcal{T}^{1,B}(\Omega)$, then there exists a (unique) measurable function $Z_u : \Omega \to \mathbb{R}^n$ such that

$$\nabla T_t(u) = \chi_{\{|u|<t\}} Z_u \qquad \text{a.e. in } \Omega \tag{2.16}$$

for every $t > 0$. This is a consequence of [BBGGPV, Lemma 2.1]. Here χ_E denotes the characteristic function of the set E. One has that $u \in W^{1,B}(\Omega)$ if and only if $u \in \mathcal{T}^{1,B}(\Omega) \cap L^B(\Omega)$ and $|Z_u| \in L^B(\Omega)$; in this case, $Z_u = \nabla u$ a.e. in Ω. An analogous property holds provided that $W^{1,B}(\Omega)$ and $\mathcal{T}^{1,B}(\Omega)$ are replaced with $W^{1,B}_0(\Omega)$ and $\mathcal{T}^{1,B}_0(\Omega)$, respectively . With abuse of notation, for every $u \in \mathcal{T}^{1,B}(\Omega)$ we denote Z_u simply by ∇u.

A sharp embedding theorem for Orlicz-Sobolev spaces is critical in our results. As in standard embeddings of Sobolev type, such an embedding depends on the regularity of the domain Ω. The regularity of Ω can effectively be descirbed in terms of its isoperimetric function, or, equivalently, in terms of a relative isoperimetric inequality. Recall that, given $\sigma \in [n, \infty)$, the domain $\Omega \subset \mathbb{R}^n$ is said to satisfy a relative isoperimetric inequality with exponent $1/\sigma'$ if there exists a positive constants C such that

$$C \min\{|E|, |\Omega \setminus E|\}^{\frac{1}{\sigma'}} \leq P(E;\Omega) \tag{2.17}$$

for every measurable set $E \subset \Omega$. Here, $P(E;\Omega)$ denotes the perimeter of E relative to Ω, in the sense of geometric measure theory. Recall that

$$P(E;\Omega) = \mathcal{H}^{n-1}(\partial E \cap \Omega)$$

whenever $\partial E \cap \Omega$ is sufficiently smooth. Here, $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure. The assumption that $\sigma \in [n, \infty)$ is due to the fact that inequality (2.17) cannot hold if $\sigma < n$, whatever is Ω. This can be shown on testing inequality (2.17) when E is a ball, and letting its radius tend to 0.
We denote by $\mathcal{G}_{1/\sigma'}$ the class of domains in $\mathbb{R}^n$ satisfing a relative isoperimetric inequality with exponent $1/\sigma'$. These classes were introduced in [Ma1], where membership of a domain Ω in $\mathcal{G}_{1/\sigma'}$ was shown to be equivalent to the Sobolev embedding $W^{1,1}(\Omega) \to L^{\sigma'}(\Omega)$.
Any bounded Lipschitz domain belongs to $\mathcal{G}_{1/n'}$. Moreover, the constant C in (2.17), with $\sigma = n$, admits a lower bound depending on Ω only via its diameter and its Lipschitz constant.
More generally, if Ω is a bounded domain in $\mathbb{R}^n$ whose boundary is locally the graph of an Hölder continuous function with exponent $\alpha \in (0, 1]$, then

$$\Omega \in \mathcal{G}_{\frac{n-1}{n-1+\alpha}}. \tag{2.18}$$

This follows on coupling [La, Theorem] with [Ma5, Corollary 5.2.3] – see also [Ci1, Theorem 1] for an earlier direct proof when $n = 2$.
Another customary class of open sets is that of John domains. Any John domain belongs to $\mathcal{G}_{1/n'}$. The class of John domains is a special instance, corresponding to the choice $\gamma = 1$, of the family of the so called γ-John domains, with$\gamma \geq 1$. The latter consists of all bounded open sets Ω in $\mathbb{R}^n$ with the property that there exist a constant $c \in (0, 1)$ and a point $x_0 \in \Omega$ such that for every $x \in \Omega$ there exists a rectifiable curve $\varpi : [0, l] \to \Omega$, parametrized by arclenght, such that $\varpi(0) = x$, $\varpi(l) = x_0$, and

$$\operatorname{dist}(\varpi(r), \partial\Omega) \geq c r^{\gamma} \qquad \text{for } r \in [0, l].$$

The isoperimetric inequality established in the proof of [HK, Corollary 5] tells us that, if Ω is a γ-John domain, with $1 \leq \gamma < n'$, then

$$\Omega \in \mathcal{G}_{\frac{\gamma}{n'}}. \tag{2.19}$$

Now, given $\sigma \in [n, \infty)$, let B be a Young function such that

$$\int_0 \left(\frac{t}{B(t)}\right)^{\frac{1}{\sigma-1}} dt < \infty. \tag{2.20}$$

Define the function $H_\sigma : [0, \infty) \to [0, \infty)$ as

$$H_\sigma(s) = \left(\int_0^s \left(\frac{t}{B(t)}\right)^{\frac{1}{\sigma-1}} dt\right)^{\frac{1}{\sigma'}} \qquad \text{for } s \geq 0, \tag{2.21}$$

and let B_σ be the Young function given by

$$B_\sigma(t) = B(H_\sigma^{-1}(t)) \qquad \text{for } t \geq 0, \tag{2.22}$$

where $H_\sigma^{-1} : [0, \infty) \to [0, \infty]$ stands for the generalized left-continuous inverse of H_σ, and we adopt the convention that $B(\infty) = \infty$. Assume that $\Omega \in \mathcal{G}_{1/\sigma'}$. Then

$$W^{1,B}(\Omega) \to L^{B_\sigma}(\Omega)\,, \tag{2.23}$$

where the arrow " $\to$ " stands for continuous embedding. Moreover, $L^{B_\sigma}(\Omega)$ can be shown to be optimal, namely smallest possible, among all Orlicz spaces. Embedding (2.23) is proved in [Ci3] with B_σ replaced by an equivalent Young function; the present version follows via a variant in the proof as in [Ci4]. Embedding (2.23) is equivalent to a Sobolev-Poincaré inequality in integral form, which asserts that

$$\int_\Omega B_\sigma\left(\frac{|u-\mathrm{med}(u)|}{c\big(\int_\Omega B(|\nabla u)|)dy\big)^{1/\sigma}}\right)dx \le \int_\Omega B(|\nabla u|)dx \tag{2.24}$$

for some constant $c = c(\Omega)$ and for every $u \in W^{1,B}(\Omega)$. Here,

$$\mathrm{med}(u) = \inf\{t \in \mathbb{R} : |\{u > t\}| \le |\Omega|/2\}\,, \tag{2.25}$$

the median of u over Ω. Let us observe that the norm of embedding (2.23) and the constant c in inequality (2.24) admit a bound from above in terms of the constants C and δ involved in inequality (2.17).

One can verify that $\lim_{s\to\infty} \frac{B_\sigma^{-1}(s)}{B^{-1}(s)} = 0$. Hence, combining embedding (2.23) with a standard property of Orlicz spaces (e.g. as in the proof of [Ci3, Theorem 3]) tells us that

$$W^{1,B}(\Omega) \to L^B(\Omega)\,. \tag{2.26}$$

Moreover, the Poincaré inequality

$$\|u - \mathrm{med}(u)\|_{L^B(\Omega)} \le C\|\nabla u\|_{L^B(\Omega)} \tag{2.27}$$

holds for some constant $C = C(\Omega)$ and for every $u \in W^{1,B}(\Omega)$. Notice that, If

$$\int^\infty \left(\frac{t}{B(t)}\right)^{\frac{1}{\sigma-1}} dt = \infty, \tag{2.28}$$

then the function H_σ is classically invertible, and the fuction B_σ is finite-valued. On the other-hand, in the case when B grows so fast near infinity that

$$\int^\infty \left(\frac{t}{B(t)}\right)^{\frac{1}{\sigma-1}} dt < \infty, \tag{2.29}$$

the function H_σ^{-1}, and hence B_σ, equals infinity for large values of its argument. In particular, if (2.29) is in force, embedding (2.23) amounts to

$$W^{1,B}(\Omega) \to L^\infty(\Omega). \tag{2.30}$$

Embedding (2.30) can be formulated in a form, which is suitable for our purposes. Define the function $F_\sigma : (0,\infty) \to [0,\infty)$ as

$$F_\sigma(t) = t^{\sigma'} \int_t^\infty \frac{\widetilde{B}(s)}{s^{1+\sigma'}}\,ds \quad \text{for } t > 0, \tag{2.31}$$

and the function $G_\sigma : [0,\infty) \to [0,\infty)$ as

$$G_\sigma(s) = \frac{s}{F_\sigma^{-1}(s)} \qquad \text{for } s > 0, \tag{2.32}$$

and $G_\sigma(0) = 0$. Then

$$\|u - \mathrm{med}(u)\|_{L^\infty(\Omega)} \le c\, G_\sigma\bigg(\int_\Omega B(|\nabla u|)\,dx\bigg) \tag{2.33}$$

for some constant $c = c(n)$ and for every $u \in W^{1,B}(\Omega)$ – see [AC]. Let us notice that condition (2.29) is equivalent to the convergence of the integral in the definition of the function F_σ [C5, Lemma 4.1], and that this function is increasing – a Young function, in fact.
In particular, if Ω is a bounded Lipschitz domain, then $\Omega \in \mathcal{G}_{1/n'}$, and the results recalled above hold with $\sigma = n$. Moreover, the norm of embedding (2.23), as well as the constant c in inequalities (2.24) and (2.33) admit an upper bound depending on Ω only through its diameter and its Lipschitz constant.

The function B_n, given by (2.22) with $\sigma = n$, comes into play whenever embeddings for the space $W_0^{1,B}(\Omega)$ are in question. Actually, one has that

$$W_0^{1,B}(\Omega) \to L^{B_n}(\Omega). \tag{2.34}$$

Furthermore, the inequality

$$\int_\Omega B_n\left(\frac{|u|}{c\big(\int_\Omega B(|\nabla u)|)dy\big)^{1/n}}\right) dx \le \int_\Omega B(|\nabla u|)dx \tag{2.35}$$

holds for some constant $c = c(n)$ and for every $u \in W_0^{1,B}(\Omega)$. Also, the embedding

$$W_0^{1,B}(\Omega) \to L^B(\Omega) \tag{2.36}$$

is compact. In particular, if condition (2.29) holds with $\sigma = n$, then

$$\|u\|_{L^\infty(\Omega)} \le c\, G_n\left(\int_\Omega B(|\nabla u|)\, dx\right) \tag{2.37}$$

for some constant $c = c(n)$ and for every $u \in W_0^{1,B}(\Omega)$. Let us stress that no regularity on Ω is needed for (2.34) – (2.37) to hold.

3 Main results

Assume that $f \in L^1(\Omega) \cap (W_0^{1,B}(\Omega))'$. A function $u \in W_0^{1,B}(\Omega)$ is called a weak solution to the Dirichlet problem (1.1) if

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla\varphi\, dx = \int_\Omega f\varphi\, dx \tag{3.1}$$

for every $\varphi \in W_0^{1,B}(\Omega)$. In particular, the growth condition (1.6) ensures that the integral on the left-hand side of (3.1) is convergent. Under the present assumption on f, the existence of a unique weak solution to problem (1.1) is a consequence of the Browder-Minthy theory of monotone operators. This follows along the same lines as in [Ze, Proposition 26.12 and Corollary 26.13].

When f solely belongs to $L^1(\Omega)$, the right-hand side of equation (3.1) is not well defined for every $\varphi \in W_0^{1,B}(\Omega)$, unless condition (2.29) holds with $\sigma = n$, in which case, owing to (2.37), any such function $\varphi \in L^\infty(\Omega)$. One might require that equation (3.1) be just fulfilled for every test function $\varphi \in C_0^\infty(\Omega)$, namely consider distributional solutions u to problem (1.1). Simple examples for the Laplace equation in a ball show that this kind of solution cannot be expected to belong to the natural energy spaces. Moreover, uniqueness may fail in the class of merely distributional solutions. Indeed, a classical example of [Se] demonstrates that, besides customary solutions, pathological distributional solutions may show up even in linear problems.

One effective way to overcome these drawbacks is to define a solution u to the Dirichlet problem (1.1) as a limit of solutions to a family of approximating problems with regular right-hand sides.

Definition 3.1 Assume that $f \in L^1(\Omega)$. A measurable function $u : \Omega \to \mathbb{R}$ is called an approximable solution to the Dirichlet problem (1.1) if there exists a sequence $\{f_k\} \subset L^1(\Omega) \cap (W^{1,B}_0(\Omega))'$ such that $f_k \to f$ in $L^1(\Omega)$, and the sequence of weak solutions $\{u_k\} \subset W^{1,B}_0(\Omega)$ to problems (1.1), with f replaced by f_k, satisfies $u_k \to u$ a.e. in Ω.

Approximable solutions turn out to be unique. An approximable solution u to (1.1) does not belong to $W^{1,B}_0(\Omega)$ in general, and is not necessarily weakly differentiable. However, u belongs to the space $\mathcal{T}^{1,B}_0(\Omega)$ introduced in Section 2. Moreover, if ∇u is interpreted as the function Z_u appearing in equation (2.16), it also fulfills the definition of distributional solution to (1.1). These facts, and additional integrability properties of u and ∇u are the content of our first main result. The relevant integrability properties are suitably described in terms of membership in spaces of Marcinkiewicz type, defined by the functions $\Phi_\sigma, \Psi_\sigma : (0,\infty) \to [0,\infty)$ associated with a number $\sigma \in [n,\infty)$ and a Young function B as follows. Let $\phi_\sigma : [0,\infty) \to [0,\infty)$ be the function given by

$$\phi_\sigma(s) = \int_0^s \left(\frac{t}{B(t)}\right)^{\frac{1}{\sigma-1}} dt \quad \text{for } s \geq 0, \tag{3.2}$$

with B modified (if necessary) near zero in such a way that (2.20) holds. Then we set

$$\Phi_\sigma(t) = \frac{B(\phi_\sigma^{-1}(t))}{t} \quad \text{for } t > 0, \tag{3.3}$$

and

$$\Psi_\sigma(t) = \frac{B(t)}{\phi_\sigma(t)} \quad \text{for } t > 0. \tag{3.4}$$

Let us notice that the behavior of Φ_σ and Ψ_σ near infinity – the only piece of information that will be needed on these functions – does not depend (up to equivalence) on the behavior of B near zero.

Theorem 3.2 *Let Ω be any domain in $\mathbb{R}^n$ such that $|\Omega| < \infty$. Assume that the function $\mathcal{A}$ satisfies assumptions* (1.5)–(1.7)*, for some finite-valued Young function B fulfilling* (1.8)*. Let $f \in L^1(\Omega)$. Then there exists a unique approximable solution u to the Dirichlet problem* (1.1)*. Moreover, $u \in \mathcal{T}^{1,B}_0(\Omega)$,*

$$\int_\Omega b(|\nabla u|)\, dx \leq C \int_\Omega |f|\, dx \tag{3.5}$$

for some constant $C = C(n, |\Omega|, i_B, s_B)$, and

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi\, dx = \int_\Omega f \varphi\, dx \tag{3.6}$$

for every $\varphi \in C^\infty_0(\Omega)$. Also,

$$u \in \begin{cases} L^{\Phi_n,\infty}(\Omega) & \textit{if } (2.28) \textit{ holds with } \sigma = n, \\ L^\infty(\Omega) & \textit{otherwise,} \end{cases} \tag{3.7}$$

and

$$\nabla u \in L^{\Psi_n,\infty}(\Omega), \tag{3.8}$$

where Φ_n and Ψ_n are the functions defined as in (3.3) *and* (3.4)*, with $\sigma = n$.*
If $\{f_k\}$ is any sequence as in Definition 3.1, and $\{u_k\}$ is the associated sequence of weak solutions, then

$$u_k \to u \quad \text{and} \quad \nabla u_k \to \nabla u \quad \text{a.e. in } \Omega, \tag{3.9}$$

up to subsequences.

Remark 3.3 The uniqueness of the approximable solution to problem (1.1) ensures that, if $f \in L^1(\Omega) \cap (W_0^{1,B}(\Omega))'$, then this solution agrees with its weak solution, which is trivially also an approximable solution.

Example 3.4 Assume that the function $\mathcal{A}$ fulfills conditions (1.5)–(1.7) with B satsisfying (1.8) and such that $B(t) \approx t^p(\log t)^\beta$ near infinity, for some $p > 1$ and $\beta > 0$. Then, by Theorem 3.2, there exists a unique approximable solution to the Dirichlet problem (1.1), and

$$u \in \begin{cases} L^{\frac{n(p-1)}{n-p},\infty}(\log L)^{\frac{\beta p}{n-p}}(\Omega) & \text{if } 1 < p < n, \\ \exp L^{\frac{n-1}{n-1-\beta}}(\Omega) & \text{if } p = n,\ \beta < n-1, \\ \exp\exp L(\Omega) & \text{if } p = n,\ \beta = n-1, \\ L^\infty(\Omega) & \text{if either } p > n, \text{ or } p = n \text{ and } \beta > n-1. \end{cases} \tag{3.10}$$

Moreover,

$$\nabla u \in \begin{cases} L^{\frac{n(p-1)}{n-1},\infty}(\log L)^{\frac{\beta}{n-1}}(\Omega) & \text{if } 1 < p < n, \\ L^{n,\infty}(\log L)^{\frac{\beta n}{n-1}-1}(\Omega) & \text{if } p = n,\ \beta < n-1, \\ L^{n,\infty}(\log L)^{n-1}(\log\log L)^{-1}(\Omega) & \text{if } p = n,\ \beta = n-1, \\ L^{n,\infty}(\log L)^{\beta}(\Omega) & \text{if either } p > n, \text{ or } p = n \text{ and } \beta > n-1. \end{cases} \tag{3.11}$$

In particular, if $\beta = 0$, then the equation in (1.1) is of p-Laplacian type, and equation (3.10) recovers available results in the literature. Equation (3.11) also reproduces known results, with the exception of the borderline case when $p = n$. Indeed, a theorem from [DHM3] yields the somewhat stronger piece of information that $\nabla u \in L^{n,\infty}(\Omega)$ in this case. The proof of this result relies upon sophisticated techniques, exploiting special features of the n-Laplace operator, and does not seem to carry over to the present general setting. Let us however mention that, whereas our conclusions in (3.11) hold for any domain Ω with finite measure, the result of [DHM3] requires some regularity assumption on the Ω.

Remark 3.5 As mentioned is Section 1, pointwise gradient estimates for solutions to a class of equations with Orlicz type growth were established in [Ba] in terms of a Riesz potential type operator applied to an integrable, or even just measure-valued, right-hand side. The equations considered in [Ba] have the special Uhlenbeck structure

$$\mathcal{A}(x,\xi) = \frac{b(|\xi|)}{|\xi|}\xi \tag{3.12}$$

for $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Moreover, in [Ba] the first inequality in (1.8) is replaced by the stronger requirement that

$$2 \le i_B\,, \tag{3.13}$$

namely that the differential operator has a "superquadratic" growth. [Ba, Theorem 2.1], combined with the boundedness of the Riesz potential oprator of order one from $L^1(\mathbb{R}^n)$ into $L^{n',\infty}(\mathbb{R}^n)$, implies that

$$\nabla u \in M^{\Theta,\infty}_{\rm loc}(\Omega) \tag{3.14}$$

whenever the right-hand side $f \in L^1_{\rm loc}(\Omega)$, where

$$\Theta(t) = b(t)^{n'} \quad \text{for } t \geq 0. \tag{3.15}$$

Here, the subscript "loc" attached to the notation of a function space on Ω indicates the collection of functions that belong to the relevant space on any compact subset of Ω. Since the function $B(t)/t$ is non-decreasing, one has that $\Psi_n(t) \leq \Theta(t)$ for $t \geq 0$. Thus, assertion (3.14) is (locally) at least as strong as (3.8). Properties (3.14) and (3.8) are in fact (locally) equivalent in the non-borderline case when

$$s_B < n, \tag{3.16}$$

since, under (3.16), the functions Ψ_n and Θ can be shown to be bounded by each other (up to multiplicative constants), thanks to (the second assertion) in (2.8). However, (3.15) can actually yield somewhat stronger local information than (3.8) if (3.16) fails. This is the case, for instance, when $B(t) = t^n$, since (3.15) tells us that $\nabla u \in L^{n',\infty}_{\rm loc}(\Omega)$, thus providing a local version of the result from [DHM3] recalled in Example 3.4.

Consider now the more general case when the function f in the equation in (1.1) is replaced by a signed Radon measure μ with finite total variation $\|\mu\|(\Omega)$ on Ω. Approximable solutions to the corresponding Dirichlet problem

$$\begin{cases} -\mathrm{div}(\mathcal{A}(x, \nabla u)) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.17}$$

can be introduced as in Definition 3.1, on replacing the convergence of f_k to f in $L^1(\Omega)$ by the weak convergence in measure, namely that

$$\lim_{k\to\infty} \int_\Omega \varphi f_k \, dx = \int_\Omega \varphi \, d\mu \tag{3.18}$$

for every $\varphi \in C_0(\Omega)$. Here, $C_0(\Omega)$ denotes the space of continuous functions with compact support in Ω.

Theorem 3.6 *Let Ω, $\mathcal{A}$ and B be as in Theorem 3.2. Let μ be a signed Radon measure with finite total variation $\|\mu\|(\Omega)$. Then there exists an approximable solution u to the Dirichlet problem* (3.17). *Moreover, $u \in \mathcal{T}^{1,B}_0(\Omega)$,*

$$\int_\Omega b(|\nabla u|) \, dx \leq C\|\mu\|(\Omega) \tag{3.19}$$

for some constant $C = C(n, |\Omega|, i_B, s_B)$, and

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla\varphi \, dx = \int_\Omega \varphi \, d\mu(x) \tag{3.20}$$

for every $\varphi \in C^\infty_0(\Omega)$. Also, the function u fulfills properties (3.7) *and* (3.8).

Let us next turn our attention to Neumann problems. The definition of weak solution $u \in W^{1,B}(\Omega)$ to problem (1.2), with $f \in L^1(\Omega) \cap (W^{1,B}(\Omega))'$ fulfilling (1.3), reads:

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx \tag{3.21}$$

for every $\varphi \in W^{1,B}(\Omega)$. Analogously to the Dirichlet problem, the existence and uniqueness of a weak solution to problem (1.2) can be established via the theory of monotone operators.

When $f \in L^1(\Omega)$ only, a notion of approximable solution can be introduced via a variant of Definition 3.1.

Definition 3.7 Assume that $f \in L^1(\Omega)$. A measurable function u is called an approximable solution to the Nuemann problem (1.2) if there exists a sequence $\{f_k\} \subset L^1(\Omega) \cap (W^{1,B}(\Omega))'$ such that $\int_\Omega f_k(x)\, dx = 0$ for $k \in \mathbb{N}$, $f_k \to f$ in $L^1(\Omega)$, and the sequence of (suitably normalized by additive constants) weak solutions $\{u_k\} \subset W^{1,B}(\Omega)$ to problems (1.2), with f replaced by f_k, satisfies $u_k \to u$ a.e. in Ω.

Our existence, uniqueness, and regularity result on the Neumann problem (1.2) is stated in the next theorem.

Theorem 3.8 *Let Ω be domain in $\mathbb{R}^n$ such that $\Omega \in \mathcal{G}_{1/\sigma'}$ for some $\sigma \in [n, \infty)$. Assume that the function $\mathcal{A}$ satisfies conditions* (1.5)–(1.7), *for some finite-valued Young function B fulfilling* (1.8). *Let $f \in L^1(\Omega)$ be a function satisfying* (1.3). *Then there exists a unique (up to additive constants) approximable solution u to the Neumann problem* (1.2). *Moreover, $u \in \mathcal{T}^{1,B}(\Omega)$,*

$$\int_\Omega b(|\nabla u|)\, dx \le C \int_\Omega |f|\, dx \tag{3.22}$$

for some constant $C = C(n, \Omega, i_B, s_B)$, and

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx \tag{3.23}$$

for every $\varphi \in W^{1,\infty}(\Omega)$. Also,

$$u \in \begin{cases} L^{\Phi_\sigma, \infty}(\Omega) & \textit{if } (2.28) \textit{ holds}, \\ L^\infty(\Omega) & \textit{otherwise}, \end{cases} \tag{3.24}$$

and

$$\nabla u \in L^{\Psi_\sigma, \infty}(\Omega), \tag{3.25}$$

where Φ_σ and Ψ_σ are the functions defined as in (3.3) *and* (3.4).
If $\{f_k\}$ is any sequence as in Definition 3.7, and $\{u_k\}$ is the associated sequence of (suitably normalized) weak solutions, then

$$u_k \to u \quad \textit{and} \quad \nabla u_k \to \nabla u \quad \textit{a.e. in } \Omega, \tag{3.26}$$

up to subsequences.

A remark analogous to Remark 3.3 applies about the coincidence of the approximable solution with the weak solution to the Neumann problem (1.2) when $f \in L^1(\Omega) \cap (W^{1,B}(\Omega))'$.

In the case when Ω is a bounded Lipschitz domain, Theorem 3.8 provides us with the following piece of information.

Corollary 3.9 *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. Let $\mathcal{A}$, B and f be as in Theorem 3.8. Then equations* (3.24) *and* (3.25) *hold with $\sigma = n$, and the constant C in* (3.32) *admits an upper bound depending on Ω only through its diameter and its Lipschitz constant.*

Example 3.10 Assume that the functions $\mathcal{A}$ and B are as in Example 3.4. In particular, $B(t) \approx t^p(\log t)^\beta$ near infinity. Let $\Omega \in \mathcal{G}_{1/\sigma'}$. Then Theorem 3.8 tells us that there exists a unique (up to additive constants) approximable solution to the Neumann problem (1.2), and

$$u \in \begin{cases} L^{\frac{\sigma(p-1)}{\sigma-p},\infty}(\log L)^{\frac{\beta p}{\sigma-p}}(\Omega) & \text{if } 1 < p < \sigma, \\ \exp L^{\frac{\sigma-1}{\sigma-1-\beta}}(\Omega) & \text{if } p = \sigma,\ \beta < \sigma - 1, \\ \exp\exp L(\Omega) & \text{if } p = \sigma,\ \beta = \sigma - 1, \\ L^\infty(\Omega) & \text{if either } p > \sigma, \text{ or } p = \sigma \text{ and } \beta > \sigma - 1. \end{cases} \tag{3.27}$$

Moreover,

$$\nabla u \in \begin{cases} L^{\frac{\sigma(p-1)}{\sigma-1},\infty}(\log L)^{\frac{\beta}{\sigma-1}}(\Omega) & \text{if } 1 < p < \sigma, \\ L^{\sigma,\infty}(\log L)^{\frac{\beta\sigma}{\sigma-1}-1}(\Omega) & \text{if } p = \sigma,\ \beta < \sigma - 1, \\ L^{\sigma,\infty}(\log L)^{\sigma-1}(\log\log L)^{-1}(\Omega) & \text{if } p = \sigma,\ \beta = \sigma - 1, \\ L^{\sigma,\infty}(\log L)^{\beta}(\Omega) & \text{if either } p > \sigma, \text{ or } p = \sigma \text{ and } \beta > \sigma - 1. \end{cases} \tag{3.28}$$

In particular, if Ω is a bounded Hölder domain with exponent $\alpha \in (0, 1]$, then from (2.18) we infer that equations (3.27) and (3.28) hold with $\sigma = \frac{n-1+\alpha}{\alpha}$.
In the case when Ω is a γ-John domain for some $\gamma \in [1, n')$, equations (3.27) and (3.28) follow from (2.19) with $\sigma = \frac{n}{n-\gamma(n-1)}$.

We conclude this section by mentioning that approximable solutions to the Neumann problem

$$\begin{cases} -\mathrm{div}(\mathcal{A}(x, \nabla u)) = \mu & \text{in } \Omega \\ A(x, \nabla u) \cdot \nu = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{3.29}$$

can be also be defined when μ is a signed Radon measure with $\|\mu\|(\Omega) < \infty$. The counterpart of the compatibility condition (1.3) is now

$$\mu(\Omega) = 0\,. \tag{3.30}$$

Definition 3.7 can be adjusted by replacing the convergence of f_k to f in $L^1(\Omega)$ by the assumption that

$$\lim_{k\to\infty} \int_\Omega \varphi f_k\, dx = \int_\Omega \varphi\, d\mu \tag{3.31}$$

for every $\varphi \in L^\infty(\Omega) \cap C(\Omega)$. Here, $C(\Omega)$ denotes the space of continuous functions in Ω.

Theorem 3.11 *Let Ω, $\mathcal{A}$ and B be as in Theorem 3.8. Let μ be a signed Radon measure with finite total variation $\|\mu\|(\Omega)$, fulfilling* (3.30). *Then there exists an approximable solution u to the Neumann problem* (3.29). *Moreover, $u \in \mathcal{T}^{1,B}(\Omega)$,*

$$\int_\Omega b(|\nabla u|)\, dx \le C\|\mu\|(\Omega) \tag{3.32}$$

for some constant $C = C(n, \Omega, i_B, s_B)$, and

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla\varphi\, dx = \int_\Omega \varphi\, d\mu(x) \tag{3.33}$$

for every $\varphi \in W^{1,\infty}(\Omega)$. Also, the function u fulfills properties (3.24) *and* (3.25).

4 Preliminary estimates

Although we are adopting a notion of solutions different from that of [BBGGPV], our approach to Theorems 3.2 and 3.8 rests upon a priori estimates for weak solutions to the problems approximating (1.1) and (1.2), whose proof follows the outline of [BBGGPV]. However, ad hoc Orlicz space techniques and results, such as the Sobolev type embeddings stated in Subsection 2.2, have to be exploited in the present situation. Some key steps in this connection are accomplished in this section, which is devoted to precise weak type estimates for Orlicz-Sobolev functions satisfying a decay condition on an integral of the gradient over their level sets. This condition will be shown to be satisfied by the approximating solutions for problems (1.1) and (1.2). The ensuing estimates are crucial in the proof of the convergence of the sequence of these solutions, and of regularity properties of their limit.

Lemma 4.1 *Let Ω be a domain in $\mathbb{R}^n$ such that $|\Omega| < \infty$, and let $\sigma \in [n, \infty)$. Assume that B is a Young function fulfilling* (2.20)*, and let B_σ be defined as in* (2.22)*.*
(i) *Let $u \in W_0^{1,B}(\Omega)$, and assume that there exist constants $M > 0$ and $t_0 \geq 0$ such that*

$$\int_{\{|u|<t\}} B(|\nabla u|)\, dx \leq Mt \qquad \text{for } t \geq t_0. \tag{4.1}$$

(a) *If* (2.28) *holds with $\sigma = n$, then there exists a constant $c = c(n)$ such that*

$$|\{|u| > t\}| \leq \frac{Mt}{B_n(ct^{\frac{1}{n'}}/M^{\frac{1}{n}})} \qquad \text{for } t \geq t_0. \tag{4.2}$$

(b) *If* (2.29) *holds with $\sigma = n$, then there exists a constant $t_1 = t_1(t_0, n, M)$ such that*

$$|\{|u| > t\}| = 0 \qquad \text{for } t \geq t_1. \tag{4.3}$$

(ii) *Suppose, in addition, that $\Omega \in \mathcal{G}_{1/\sigma'}$. Let $u \in W^{1,B}(\Omega)$, and assume that there exist constants $M > 0$ and $t_0 \geq 0$ such that*

$$\int_{\{|u-\mathrm{med}(u)|<t\}} B(|\nabla u|)\, dx \leq Mt \qquad \text{for } t \geq t_0. \tag{4.4}$$

(a) *If* (2.28) *holds, then there exists a constant $c = c(\Omega)$ such that*

$$|\{|u - \mathrm{med}(u)| > t\}| \leq \frac{Mt}{B_\sigma(ct^{\frac{1}{\sigma'}}/M^{\frac{1}{\sigma}})} \qquad \text{for } t \geq t_0. \tag{4.5}$$

(b) *If* (2.29) *holds, then there exists a constant $t_1 = t_1(t_0, \Omega, M)$ such that*

$$|\{|u - \mathrm{med}(u)| > t\}| = 0 \qquad \text{for } t \geq t_1. \tag{4.6}$$

If condition (2.20) *is not satisfied, then an analogous statement holds, provided that B_σ is defined as in* (2.22)*, with B modified near zero in such a way that* (2.20) *is fulfilled. In this case, the constant c in* (4.2) *and* (4.5) *also depends on B. Moreover, in* (4.2) *and* (4.5) *the constant t_0 has to be replaced by another constant also depending on B, and M has to be replaced by another constant depending on M, B and $|\Omega|$. Finally, the constant t_1 also depends on B and $|\Omega|$ in* (4.3) *and* (4.6)*.*

Remark 4.2 One can verify that that the behavior of B_σ near infinity is independent (up to equivalence of Young functions) of the behavior of B near 0. Thus, inequalities (4.2) and (4.5) are invariant (up to different choices of the involved constants, depending on the quantities described in the last part of the statement of Lemma 4.1) under possibly different modifications of B near zero in the definition of B_σ.

Remark 4.3 A close inspection of the proof of Lemma 4.1 will reveal that the same conclusions apply under the weaker assumption that $u \in \mathcal{T}^{1,B}_0(\Omega)$ in part (i), and that $u \in \mathcal{T}^{1,B}(\Omega)$ in part (ii). Of course, in this case ∇u has to be interpreted as the vector-valued function Z_u appearing in equation (2.16).

Corollary 4.4 (i) *Under the assumptions of Lemma 4.1, part (i), for every $\varepsilon > 0$, there exists $\bar{t}$ depending on ε, M, t_0, B, n such that*

$$|\{|u| > t\}| < \varepsilon \quad \textit{for } t \geq \bar{t}. \tag{4.7}$$

(ii) *Under the assumptions of Lemma 4.1, part (ii), for every $\varepsilon > 0$, there exists $\bar{t}$ depending on ε, M, t_0, B, Ω such that*

$$|\{|u - \mathrm{med}(u)| > t\}| < \varepsilon \quad \textit{for } t \geq \bar{t}. \tag{4.8}$$

Proof. Equation (4.7) holds trivially if the assumptions of Lemma 4.1, part (i), case (b) are in force. Under the assumptions of case (a), equation (4.7) follows from (4.2). Indeed, owing to (2.8), $B(t) \geq ct^{i_B}$, with $i_B > 1$, for some positive constant c and sufficiently large t (depending on B), and hence

$$\lim_{t\to\infty} \frac{B_n(t)}{t^{n'}} = \infty. \tag{4.9}$$

The proof of equation (4.8) is completely analogous. Now, one has to make use of the fact that the limit (4.9) also holds if B_n is replaced by B_σ, and $t^{n'}$ by $t^{\sigma'}$. □

Proof of Lemma 4.1. Let us focus on part (ii). Assume, for the time being, that condition (2.20) is fulfilled. Consider first case (a). Set

$$v = u - \mathrm{med}(u). \tag{4.10}$$

Clearly, $v \in W^{1,B}(\Omega)$, and hence $T_t(v) \in W^{1,B}(\Omega)$ for $t > 0$. Moreover, $\mathrm{med}(v) = 0$, whence $\mathrm{med}(T_t(v)) = 0$, and $\nabla v = \nabla u$ in Ω. By the Orlicz-Sobolev inequality (2.24) applied to the function $T_t(v)$,

$$\int_\Omega B_\sigma\left(\frac{|T_t(v)|}{C\big(\int_\Omega B(|\nabla T_t(v)|)dy\big)^{1/\sigma}}\right) dx \leq \int_\Omega B(|\nabla(T_t(v))|)dx. \tag{4.11}$$

One has that

$$\int_\Omega B(|\nabla T_t(v)|)dx = \int_{\{|v|<t\}} B(|\nabla v|)dx \quad \text{for } t > 0, \tag{4.12}$$

and

$$\{|T_t(v)| > t\} = \{|v| > t\} \quad \text{for } t > 0. \tag{4.13}$$

Thus,

$$(4.14)\qquad |\{|v|>t\}|B_\sigma\bigg(\frac{t}{C(\int_{\{|v|<t\}}B(|\nabla v|)dy)^{\frac{1}{\sigma}}}\bigg)\leq \int_{\{|v|>t\}}B_\sigma\bigg(\frac{|v|}{C\big(\int_{\{|v|<t\}}B(|\nabla v|)dy\big)^{1/\sigma}}\bigg)dx$$
$$\leq \int_{\{|v|<t\}}B(|\nabla v|)dx$$

for $t>0$. Hence, by (4.4),

$$(4.15)\qquad |\{|v|>t\}|B_\sigma\bigg(\frac{t}{C(tM)^{\frac{1}{\sigma}}}\bigg)\leq Mt \qquad \text{for } t\geq t_0,$$

namely (4.5), with $c=C$.
As far as case (b) is concerned, an application of inequality (2.33) to the function $T_t(v)$, and the use of (4.4) yield

$$(4.16)\qquad \|T_t(v)\|_{L^\infty(\Omega)}\leq c\,G_\sigma\bigg(\int_\Omega B(|\nabla T_t(v)|)\,dx\bigg)$$
$$= c\,G_\sigma\bigg(\int_{\{|v|<t\}}B(|\nabla v|)\,dx\bigg)\leq c\,G_\sigma(Mt)\qquad \text{for } t\geq t_0.$$

By (4.13) and (4.16),

$$(4.17)\qquad |\{|v|>t\}|=0 \qquad \text{if } t\geq c\,G_\sigma(Mt) \text{ and } t\geq t_0.$$

Hence, equation (4.6) follows, with $t_1=\max\big\{\frac{F_\sigma(cM)}{M},t_0\big\}$, where F_σ is deinded by (2.31).
Assume now that condition (2.20) does not hold. Consider the Young function $\overline{B}$ defined as

$$(4.18)\qquad \overline{B}(t)=\begin{cases} tB(1) & \text{if } 0\leq t\leq 1,\\ B(t) & \text{if } t>1,\end{cases}$$

and let $\overline{b}:[0,\infty)\to[0,\infty)$ be the left-continuous derivative of $\overline{B}$. Thus,

$$\overline{B}(t)=\int_0^t \overline{b}(s)\,ds \quad \text{for } t\geq 0.$$

Clearly, $B(t)\leq\overline{B}(t)$ for $t\geq 0$, and condition (2.20) is fulfilled with B replaced by $\overline{B}$. Let $\overline{B}_\sigma$ be the function defined as in (2.22), with B replaced by $\overline{B}$. If u satisfies (4.4), then

$$(4.19)\qquad \int_{\{|v|<t\}}\overline{B}(|\nabla v|)\,dx\leq \int_{\{|v|<t,|\nabla v|>1\}}B(|\nabla v|)\,dx+\int_{\{|v|<t,|\nabla v|\leq 1\}}\overline{B}(|\nabla v|)\,dx$$
$$\leq \int_{\{|v|<t\}}B(|\nabla v|)\,dx+B(1)|\{|v|<t\}|\leq t(M+B(1)|\Omega|)$$

for $t\geq\max\{t_0,1\}$. Thereby, u satisfies assumption (4.4) with B replaced by $\overline{B}$, M replaced by $M+B(1)|\Omega|$, and t_0 replaced by $\max\{t_0,1\}$. Hence, equation (4.5) holds with B_σ replaced by $\overline{B}_\sigma$, M replaced by $M+B(1)|\Omega|$, and t_0 replaced by $\max\{t_0,1\}$.
By the same argument, under assumption (2.29), the function u still fulfills (4.6). This concludes the proof of part (ii).
The proof of part (i) is completely analgous (and even simpler). One has to argue just on u instead of v, and to make use of inequalities (2.35) and (2.37) instead of (2.24) and (2.33). □

The next lemma provides us with a uniform integrability result for functions satisfying the assumptions of Lemma 4.1. It will enable us to pass to the limit in the definition of distributional solution to the approximating problems.

Lemma 4.5 *Let Ω be a domain in $\mathbb{R}^n$ such that $|\Omega| < \infty$. Let B be a finite-valued Young function satisfying* (1.8).
(i) *Assume that the function $u \in W_0^{1,B}(\Omega)$ fulfills assumption* (4.1) *for some $t_0 \geq 0$. Then there exists a function $\zeta : [0, |\Omega|] \to [0, \infty)$, depending on n, $|\Omega|$, t_0, B and on the constant M appearing in* (4.1)*, such that*

$$\lim_{s \to 0^+} \eta(s) = 0, \tag{4.20}$$

and

$$\int_E b(|\nabla u|)\, dx \leq \zeta(|E|) \quad \text{for every measurable set } E \subset \Omega. \tag{4.21}$$

(ii) *Suppose, in addition, that $\Omega \in \mathcal{G}_{1/\sigma'}$ for some $\sigma \in [n, \infty)$. Assume that the function $u \in W^{1,B}(\Omega)$ fulfills assumption* (4.4) *for some $t_0 \geq 0$. Then there exists a function $\zeta : [0, |\Omega|] \to [0, \infty)$, depending on n, Ω, t_0, B and on the constant M appearing in* (4.4)*, such that* (4.20) *and* (4.21) *hold.*

Proof. As in the proof of Lemma 4.1, we limit ourselves to proving part (ii). Let v be defined as in (4.10). Suppose, for the time being, that the function B satisfies condition (2.20).
Assume first that condition (2.28) holds. By assumption (4.4),

$$|\{B(|\nabla v|) > s, |v| \leq t\}| \leq \frac{1}{s} \int_{\{B(|\nabla v|)>s, |v|\leq t\}} B(|\nabla v|)\, dx \leq \frac{Mt}{s} \quad \text{for } t \geq t_0 \text{ and } s > 0. \tag{4.22}$$

Hence, via inequality (4.5), we deduce that

$$\begin{aligned} |\{B(|\nabla v|) > s\}| &\leq |\{|v| > t\}| + |\{B(|\nabla v|) > s, |v| \leq t\}| \\ &\leq \frac{Mt}{B_\sigma(ct^{\frac{1}{\sigma'}}/M^{\frac{1}{\sigma}})} + \frac{Mt}{s} \quad \text{for } t \geq t_0 \text{ and } s > 0. \end{aligned} \tag{4.23}$$

Set $t = \big(\frac{1}{c} M^{1/\sigma} B_\sigma^{-1}(s)\big)^{\sigma'}$ in (4.23), with $s \geq B_\sigma\big(ct_0^{1/\sigma'} M^{-1/\sigma}\big)$, so that $t \geq t_0$. Hence,

$$|\{B(|\nabla u|) > s\}| \leq \frac{2M^{\sigma'}}{c^{\sigma'}} \frac{B_\sigma^{-1}(s)^{\sigma'}}{s} \quad \text{for } s \geq B_\sigma\big(ct_0^{1/\sigma'} M^{-1/\sigma}\big). \tag{4.24}$$

Next, set $\tau_0 = b\Big(B^{-1}\Big(B_\sigma\big(ct_0^{1/\sigma'} M^{-1/\sigma}\big)\Big)\Big)$, choose $s = B(b^{-1}(\tau))$ in (4.24) with $\tau \geq \tau_0$, and make use of (2.22) to obtain that

$$|\{b(|\nabla u|) > \tau\}| \leq \frac{2M^{\sigma'}}{c^{\sigma'}} \frac{H_\sigma(b^{-1}(\tau))^{\sigma'}}{B(b^{-1}(\tau))} \qquad \text{for } \tau \geq \tau_0. \tag{4.25}$$

Now, define the function $I_\sigma : (0, \infty) \to [0, \infty)$ as

$$I_\sigma(\tau) = \frac{\tau b^{-1}(\tau)}{H_\sigma(b^{-1}(\tau))^{\sigma'}} \qquad \text{for } \tau > 0.$$

Since $\frac{B(t)}{t}$ is an increasing function, the function $\frac{I_\sigma(\tau)}{\tau}$ is increasing as well. Hence, in particular, the function I_σ is increasing. Furthermore, by inequality (2.6), there exists a constant $c>0$ such that

$$\frac{H_\sigma(b^{-1}(\tau))^{\sigma'}}{B(b^{-1}(\tau))} I_\sigma(\tau) \le c \qquad \text{for } t \ge 0. \tag{4.26}$$

Coupling (4.25) with (4.26) tells us that

$$|\{b(|\nabla u|) > \tau\}| I_\sigma(\tau) \le c M^{\sigma'} \qquad \text{for } \tau \ge \tau_0, \tag{4.27}$$

for some constant $c>0$. Consequently,

$$b(|\nabla u|)^*(s) \le I_\sigma^{-1}\Big(\frac{cM^{\sigma'}}{s}\Big) \qquad \text{if } 0 < s < |\{b(|\nabla u|) > \tau_0\}|, \tag{4.28}$$

whence

$$b(|\nabla u|)^*(s) \le \max\Big\{I_\sigma^{-1}\Big(\frac{cM^{\sigma'}}{s}\Big), \tau_0\Big\} \qquad \text{for } s \in (0, |\Omega|). \tag{4.29}$$

From (4.29) and (2.2), inequality (4.21) will follow with $\zeta(s) = \int_0^s \max\{I_\sigma^{-1}(cM^{\sigma'}/r), \tau_0\}\, dr$, if we show that

$$\int_0^{|\Omega|} I_\sigma^{-1}\Big(\frac{1}{s}\Big) ds < \infty\,. \tag{4.30}$$

By a change of variable and Fubini's theorem,

$$\int_0^{|\Omega|} I_\sigma^{-1}\Big(\frac{1}{s}\Big) ds = \int_{1/|\Omega|}^\infty \frac{I_\sigma^{-1}(t)}{t^2}\, dt = |\Omega| I_\sigma^{-1}(1/|\Omega|) + \int_{1/|\Omega|}^\infty \frac{dr}{I_\sigma(r)}. \tag{4.31}$$

Owing to Fubini's theorem again,

(4.32)
$$\begin{aligned}\int_{1/|\Omega|}^\infty \frac{dr}{I_\sigma(r)} &= \int_{1/|\Omega|}^\infty \frac{1}{rb^{-1}(r)} \int_0^{b^{-1}(r)} \Big(\frac{s}{B(s)}\Big)^{\frac{1}{\sigma-1}} ds\, dr \\ &= \int_0^{b^{-1}(1/|\Omega|)} \Big(\frac{s}{B(s)}\Big)^{\frac{1}{\sigma-1}} ds \int_{1/|\Omega|}^\infty \frac{dr}{rb^{-1}(r)} + \int_{b^{-1}(1/|\Omega|)}^\infty \Big(\frac{s}{B(s)}\Big)^{\frac{1}{\sigma-1}} \int_{b(s)}^\infty \frac{dr}{rb^{-1}(r)}\, ds.\end{aligned}$$

Note that, by (2.4) and (2.6) applied with B replaced by $\widetilde{B}$, and by (2.9), there exists a constant c such that

$$\int_t^\infty \frac{dr}{rb^{-1}(r)} \le \int_t^\infty \frac{dr}{\widetilde{B}(r)} \le \frac{t^{s'_B}}{\widetilde{B}(t)} \int_t^\infty \frac{dr}{r^{s'_B}} = \frac{t}{(s'_B - 1)\widetilde{B}(t)} \le \frac{c}{b^{-1}(t)} \tag{4.33}$$

for $t>0$. On the other hand, owing to (2.8), one has that $B(t) \ge ct^{i_B}$ for some positive constant c and sufficiently large t. Thus, thanks to (1.8), the right-hand side of (4.32) is finite, and inequality (4.30) is established in this case.

Assume next that condition (2.29) is in force. Then, by inequality (4.6),

$$|\{B(|\nabla v|) > s\}| \le |\{|v| > t\}| + |\{B(|\nabla v|) > s, |v| \le t\}| \le \frac{Mt}{s} \quad \text{for } t \ge t_1 \text{ and } s > 0. \tag{4.34}$$

An application of inequality (4.34) with $t = t_1$ and $s = B(b^{-1}(\tau))$ implies that

$$|\{b(|\nabla v|) > \tau\}|\widetilde{B}(\tau) \le \frac{Mt_1\widetilde{B}(\tau)}{B(b^{-1}(\tau))} \quad \text{for } \tau > 0. \tag{4.35}$$

From inequalities (4.35) and (2.7), we deduce that

$$|\{b(|\nabla v|) > \tau\}|\widetilde{B}(\tau) \le cMt_1 \quad \text{for } \tau > 0, \tag{4.36}$$

for some constant c. Therefore,

$$b(|\nabla v|)^*(s) \le \widetilde{B}^{-1}\Big(\frac{cMt_1}{s}\Big) \quad \text{for } s \in (0, |\Omega|). \tag{4.37}$$

Inequality (4.21) now follows from the convergence of the integral

$$\int_0^{|\Omega|} \widetilde{B}^{-1}\Big(\frac{1}{s}\Big)\, ds,$$

which is in turn a consequence of (2.9).
It remains to remove the temporary assumption (2.20). If (2.20) fails, one can argue as above, with B and B_σ replaced with the functions $\overline{B}$ and $\overline{B}_\sigma$ defined as in the proof of Lemma 4.1, and the constants M, c, t_0 and t_1 appearing in inequalities (4.4)–(4.6) modified accordingly. □

5 Proof of the main results

We begin with an estimate of the form (3.19) for the weak solution to the Dirichlet problem (1.1) with regular right-hand side f, and a parallel estimate for the weak solution to the Neumann problem (1.2). They follow from a slight extensions of results of [Ta1] and [Ci2], respectively (see also of [Ma4] for linear equations).

Proposition 5.1 *Assume that conditions* (1.5)–(1.8) *are fulfilled. Let Ω be a domain in $\mathbb{R}^n$ with $|\Omega| < \infty$.*
(i) *Assume that $f \in L^1(\Omega) \cap (W_0^{1,B}(\Omega))'$. Let u be the weak solution to the Dirichlet problem* (1.1). *Then there exists a constant $C = C(n, i_B, s_B)$ such that*

$$\int_\Omega b(|\nabla u|)\, dx \le C|\Omega|^{\frac{1}{n}} \int_\Omega |f|\, dx. \tag{5.1}$$

(ii) *Suppose, in addition, that $\Omega \in \mathcal{G}_{1/\sigma'}$ for some $\sigma \in [n, \infty)$. Assume that $f \in L^1(\Omega) \cap (W^{1,B}(\Omega))'$ and satifies condition* (1.3) *. Let u be a weak solution to the Neumann problem* (1.2). *Then there exists a constant $C = C(n, \Omega, i_B, s_B)$ such that*

$$\int_\Omega b(|\nabla u|)\, dx \le C \int_\Omega |f|\, dx. \tag{5.2}$$

In particular, if Ω is a bounded Lipschitz domain, then the constant C in (5.2) *depends only Ω only through its diameter and its Lipschitz constant of Ω.*

Proof, sketched. Part (i). If the function $B(b^{-1}(t))$ is convex, then an application of [Ta1, Theorem 1, part (vi)], with the choice $M(t) = b(t)$ (in the notation of that theorem) entails that there exists a constant $C = C(n)$ such that

$$\int_\Omega b(|\nabla u|)\,dx \le C\int_0^{|\Omega|}\frac{1}{s^{1/n'}}\int_0^s f^*(r)\,dr\,ds. \tag{5.3}$$

Hence, inequality (5.1) follows, since

$$\int_0^s f^*(r)\,dr \le \|f\|_{L^1(\Omega)} \qquad \text{for } s\in[0,|\Omega|].$$

In general, the function $B(b^{-1}(t))$ is only equivalent to a convex function. Indeed, owing to inequalities (2.5) and (2.7), there exist constants c_1 and c_2, depending on i_B and s_B, such that

$$c_1\widetilde{B}(t) \le B(b^{-1}(t)) \le c_2\widetilde{B}(t) \quad \text{for } t>0. \tag{5.4}$$

Following the lines of the proof of [Ta1, Theorem 1, part (vi)], with the choice $K(t) = \widetilde{B}(t)$ (again in the notation of that proof), and making use of (5.4), tell us that inequality (5.3) continues to hold, with C depending also on i_B and s_B. Thus, inequality (5.1) holds also in this case.
Part (ii) The proof of inequality (5.2) relies upon a counterpart of the result of [Ta1] for the Neumann problem (1.2), contained in [Ci2, Theorem 3.1]. A variant of the proof of that theorem as hinted above, exploiting (5.4) again, implies that

$$\int_\Omega b(|\nabla u|)\,dx \le C\int_0^{|\Omega|}\frac{1}{s^{1/\sigma'}}\int_0^s f^*(r)\,dr\,ds \le C\|f\|_{L^1(\Omega)}\int_0^{|\Omega|}\frac{1}{s^{1/\sigma'}}\,ds = \sigma C|\Omega|^{1/\sigma}\|f\|_{L^1(\Omega)}\,, \tag{5.5}$$

where the constant C also depends on the constant appearing in the relative isoperimetric inequality (2.17). This yields (5.2).
The assertion about bounded Lipschitz domains rests upon the fact that, for these domains, inequality (2.17) holds, with $\sigma = n$, and a constant C depending on Ω only via its diameter and its Lipschitz constant. □

We are now ready to prove our main results. We limit ourselves to provide a full proof of Theorems 3.8, and a sketch of the (completely analogous) proof of Theorem 3.11. The proofs of Theorems 3.2 and 3.6 follow along the same lines, with some slight simplification. One has basically to replace the use of parts (ii) of the preliminary results established above and in Section 4 with the corresponding parts (i), whenever they come into play. The details will be omitted, for brevity.

Proof of Theorem 3.8. For ease of presentation, we split this proof in steps.
Step 1 *There exists a sequence of problems, approximating* (1.2)*, with regular right-hand sides, and a corresponding sequence* $\{u_k\}$ *of weak solutions.*
Let $\{f_k\}\subset C_0^\infty(\Omega)$ be a sequence of functions such that $\int_\Omega f_k\,dx = 0$ for $k\in\mathbb{N}$, and

$$f_k \to f \qquad \text{in } L^1(\Omega). \tag{5.6}$$

We may also clearly assume that

$$\|f_k\|_{L^1(\Omega)} \le 2\|f\|_{L^1(\Omega)}, \qquad \text{for } k\in\mathbb{N}. \tag{5.7}$$

As observed in Section 3, under our assumptions on f_k for each $k\in\mathbb{N}$ there exists a unique weak solution $u_k\in W^{1,B}(\Omega)$ to the Neumann problem

$$\begin{cases} -\mathrm{div}(\mathcal{A}(x,\nabla u_k)) = f_k(x) & \text{in } \Omega\\ \mathcal{A}(x,\nabla u_k)\cdot\nu = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{5.8}$$

normalized in such a way that

$$\operatorname{med}(u_k) = 0. \tag{5.9}$$

Hence,

$$\int_\Omega \mathcal{A}(x, \nabla u_k) \cdot \nabla \varphi \, dx = \int_\Omega f_k \, \varphi \, dx \tag{5.10}$$

for every $\varphi \in W^{1,B}(\Omega)$.

Step 2. *There exists a measurable function $u : \Omega \to \mathbb{R}$ such that*

$$u_k \to u \qquad \text{a.e. in } \Omega, \tag{5.11}$$

up to subsequences. Hence, u is an approximable solution to problem (1.2).
Given any $t, \tau > 0$, one has that

$$|\{|u_k - u_m| > \tau\}| \le |\{|u_k| > t\}| + |\{|u_m| > t\}| + |\{|T_t(u_k) - T_t(u_m)| > \tau\}|, \tag{5.12}$$

for $k, m \in \mathbb{N}$. On choosing $\varphi = T_t(u_k)$ in (5.10) and making use of assumptions (1.5) and (5.7) one obtains that

$$\begin{aligned} \int_\Omega B(|\nabla T_t(u_k)|) \, dx &= \int_{\{|u_k|<t\}} B(|\nabla u_k|) \, dx \le \int_{\{|u_k|<t\}} \mathcal{A}(x, \nabla u_k) \cdot \nabla u_k \, dx \\ &= \int_\Omega \mathcal{A}(x, \nabla u_k) \cdot \nabla T_t(u_k) \, dx = \int_\Omega f_k \, T_t(u_k) \, dx \le 2t \|f\|_{L^1(\Omega)}, \end{aligned} \tag{5.13}$$

for $k \in \mathbb{N}$. By (5.13) and Corollary 4.4, part (ii), fixed any $\varepsilon > 0$, the number t can be chosen so large that

$$|\{|u_k| > t\}| < \varepsilon \quad \text{and} \quad |\{|u_m| > t\}| < \varepsilon \tag{5.14}$$

for every $k, m \in \mathbb{N}$. Thanks to condition (5.9), one has that $\operatorname{med}(T_t(u_k)) = 0$. Thus, owing to inequalities (2.27) and (5.13), the sequence $\{T_t(u_k)\}$ is bounded in $W^{1,B}(\Omega)$. By the compactness of embedding (2.36), the sequence $\{T_t(u_k)\}$ converges (up to subsequences) to some function in $L^B(\Omega)$. In particular, $\{T_t(u_k)\}$ is a Cauchy sequence in measure in Ω. Thus,

$$|\{|T_t(u_k) - T_t(u_m)| > \tau\}| \le \varepsilon \tag{5.15}$$

provided that k and m are sufficiently large. By (5.12), (5.14) and (5.15), $\{u_k\}$ is (up to subsequences) a Cauchy sequence in measure in Ω, and hence there exists a measurable function $u : \Omega \to \mathbb{R}$ such that (5.11) holds.

Step 3.

$$\{\nabla u_k\} \textit{ is a Cauchy sequence in measure.} \tag{5.16}$$

Let $t > 0$. Fix any $\varepsilon > 0$. Given any $\tau, \delta > 0$, we have that

$$\begin{aligned} |\{|\nabla u_k - \nabla u_m| > t\}| &\le |\{|\nabla u_k| > \tau\}| + |\{|\nabla u_m| > \tau\}| + |\{|u_k - u_m| > \delta\}| \\ &\quad + |\{|u_k - u_m| \le \delta, |\nabla u_k| \le \tau, |\nabla u_m| \le \tau, |\nabla u_k - \nabla u_m| > t\}|, \end{aligned} \tag{5.17}$$

for $k, m \in \mathbb{N}$. By (5.2) and (5.7),

$$\int_\Omega b(|\nabla u_k|)\,dx \le C\|f\|_{L^1(\Omega)}\,, \tag{5.18}$$

for some constant $C = C(n, \Omega, i_B, s_B)$. Hence,

$$|\{|\nabla u_k| > \tau\}| \le b^{-1}\Big(\frac{C}{\tau}\|f\|_{L^1(\Omega)}\Big), \tag{5.19}$$

for $k \in \mathbb{N}$ and for some constant C independent of k. Thus τ can be chosen so large that

$$|\{|\nabla u_k| > \tau\}| < \varepsilon \qquad \text{for } k \in \mathbb{N}. \tag{5.20}$$

For such a choice of τ, set

$$G = \{|u_k - u_m| \le \delta, |\nabla u_k| \le \tau, |\nabla u_m| \le \tau, |\nabla u_k - \nabla u_m| \ge t\}. \tag{5.21}$$

We claim that there exists $\delta > 0$ such that

$$|G| < \varepsilon. \tag{5.22}$$

To verify our claim, observe that, if we define

$$S = \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n : |\xi| \le \tau, |\eta| \le \tau, |\xi - \eta| \ge t\},$$

and $\psi : \Omega \to [0, \infty)$ as

$$\psi(x) = \inf\big\{\big(\mathcal{A}(x,\xi) - \mathcal{A}(x,\eta)\big)\cdot(\xi - \eta) : (\xi,\eta) \in S\big\},$$

then $\psi(x) \ge 0$ and

$$|\{\psi(x) = 0\}| = 0. \tag{5.23}$$

This is a consequence of inequality (1.7) and of the fact that the set S is compact and the function $\mathcal{A}(x,\xi)$ is continuous in ξ for every x outside a subset of Ω of Lebesgue measure zero. Therefore,

$$\begin{aligned}\int_G \psi(x)\,dx &\le \int_G \big(\mathcal{A}(x,\nabla u_k) - \mathcal{A}(x,\nabla u_m)\big)\cdot(\nabla u_k - \nabla u_m)\,dx \\ &\le \int_{\{|u_k - u_m| \le \delta\}} \big(\mathcal{A}(x,\nabla u_k) - \mathcal{A}(x,\nabla u_m)\big)\cdot(\nabla u_k - \nabla u_m)\,dx \\ &= \int_\Omega \big(\mathcal{A}(x,\nabla u_k) - \mathcal{A}(x,\nabla u_m)\big)\cdot\nabla(T_\delta(u_k - u_m))\,dx \\ &= \int_\Omega (f_k - f_m)T_\delta(u_k - u_m)dx \le 4\delta\|f\|_{L^1(\Omega)},\end{aligned} \tag{5.24}$$

where the last equality follows on making use of $T_\delta(u_k - u_m)$ as test function in (5.10), and in the same equation with k replaced with m, and subtracting the resulting equations. Inequality (5.22) is a consequence of (5.24).
Finally, since, by Step 1, $\{u_k\}$ is a Cauchy sequence in measure in Ω,

$$|\{|u_k - u_m| > \delta\}| < \varepsilon\,, \tag{5.25}$$

if k and m are sufficiently large. Combining (5.17), (5.20), (5.22) and (5.25) yields

$$|\{|\nabla u_k - \nabla u_m| > t\}| < 4\varepsilon\,,$$

for sufficiently large k and m. Owing to the arbitrariness of t, property (5.16) is thus established.

Step 4. $u \in \mathcal{T}^{1,B}(\Omega)$, *and*

$$\nabla u_k \to \nabla u \qquad \text{a.e. in } \Omega\,, \tag{5.26}$$

up to subsequences. Here, ∇u *is the generalized gradient of* u *in the sense of the function* Z_u *appearing in* (2.16).

Since $\{\nabla u_k\}$ is a Cauchy sequence in measure, there exists a measurable function $W : \Omega \to \mathbb{R}^n$ such that

$$\nabla u_k \to W \qquad \text{a.e. in } \Omega \tag{5.27}$$

(up to subsequences). As observed in Step 2, the sequence $\{T_t(u_k)\}$ is bounded in $W^{1,B}(\Omega)$. Since the Sobolev space $W^{1,B}(\Omega)$ is reflexive, there exists a function $\overline{u}_t \in W^{1,B}(\Omega)$ such that

$$T_t(u_k) \rightharpoonup \overline{u}_t \qquad \text{weakly in } W^{1,B}(\Omega) \tag{5.28}$$

(up to subsequences). By Step 2, $T_t(u_k) \to T_t(u)$ a.e. in Ω, and hence

$$\overline{u}_t = T_t(u) \qquad \text{a.e. in } \Omega. \tag{5.29}$$

Thereby, $T_t(u) \in W^{1,B}(\Omega)$, and

$$T_t(u_k) \rightharpoonup T_t(u) \qquad \text{weakly in } W^{1,B}(\Omega). \tag{5.30}$$

By the arbitrariness of t, one has that $u \in \mathcal{T}^{1,B}(\Omega)$, and

$$\nabla T_t(u) = \chi_{\{|u|<t\}} \nabla u \qquad \text{a.e. in } \Omega \tag{5.31}$$

for $t > 0$. Owing to (5.11) and (5.27),

$$\nabla T_t(u_k) = \chi_{\{|u_k|<t\}} \nabla u_k \to \chi_{\{|u|<t\}} W \qquad \text{a.e. in } \Omega$$

for $t > 0$. Hence, by (5.30)

$$\nabla T_t(u) = \chi_{\{|u|<t\}} W \qquad \text{a.e. in } \Omega \tag{5.32}$$

for $t > 0$. Coupling (5.31) with (5.32) yields

$$W = \nabla u \qquad \text{a.e. in } \Omega. \tag{5.33}$$

Equation (5.26) is a consequence of (5.27) and (5.33).

Step 5. *u fulfills inequality* (3.32) *and equation* (3.33).

From (5.26) and (5.18), via Fatou's lemma, we deduce that

$$\int_\Omega b(|\nabla u|)\, dx \le C\|f\|_{L^1(\Omega)},$$

namely (3.32).

As far as equation (3.33) is concerned, observe that, by (5.26),

$$\mathcal{A}(x,\nabla u_k)\to\mathcal{A}(x,\nabla u)\qquad\text{for a.e. } x\in\Omega\,. \tag{5.34}$$

Now, fix any function $\varphi\in W^{1,\infty}(\Omega)$ and any measurable set $E\subset\Omega$. Owing to inequality (1.6) and Lemma 4.5, part (ii) (whose assumptions are fulfilled thanks to (5.13)),

$$\begin{aligned}\int_E|\mathcal{A}(x,\nabla u_k)\cdot\nabla\varphi|\,dx&\le C\|\nabla\varphi\|_{L^\infty(\Omega)}\bigg(\int_E b\big(|\nabla u_k|\big)\,dx+\int_E g(x)\,dx\bigg)\\&\le C\|\nabla\varphi\|_{L^\infty(\Omega)}\bigg(\zeta(|E|)+\int_E g(x)\,dx\bigg)\end{aligned}\tag{5.35}$$

for some function $\zeta:[0,|\Omega|\to[0,\infty)$ such that $\lim_{s\to0^+}\zeta(s)=0$. From (5.34) and (5.35), via Vitali's convergence theorem, we deduce that the left-hand side of (5.10) converges to the left-hand side of (3.33) as $k\to\infty$. The right-hand side of (5.10) trivially converges to the right-hand side of (3.33), by (5.6). Hence, equation (3.33) follows.

Step 6. *The solution u is unique (up to additive constants).*
Assume that u and $\overline{u}$ are approximable solutions to problem (1.1). Then, there exist sequences $\{f_k\}$ and $\{\overline{f}_k\}$ in $L^1(\Omega)\cap(W^{1,B}(\Omega))'$ such that $\int_\Omega f_k\,dx=\int_\Omega\overline{f}_k\,dx=0$ for $k\in\mathbb{N}$, $f_k\to f$ and $\overline{f}_k\to f$ in $L^1(\Omega)$, the weak solutions u_k to problem (5.8) fulfill $u_k\to u$ a.e. in Ω, and the weak solutions $\overline{u}_k$ to problem (5.8) with f_k replaced by $\overline{f}_k$ fulfill $\overline{u}_k\to\overline{u}$ a.e. in Ω. Fix any $t>0$, and choose the test function $\varphi=T_t(u_k-\overline{u}_k)$ in (5.10), and in the same equation with u_k and f_k replaced by $\overline{u}_k$ and $\overline{f}_k$, respectively. Subtracting the resulting equations yields

$$\int_\Omega\chi_{\{|u_k-\overline{u}_k|\le t\}}\big(\mathcal{A}(x,\nabla u_k)-\mathcal{A}(x,\nabla\overline{u}_k)\big)\cdot(\nabla u_k-\nabla\overline{u}_k)\,dx=\int_\Omega(f_k-\overline{f}_k)T_t(u_k-\overline{u}_k)\,dx \tag{5.36}$$

for $k\in\mathbb{N}$. Since $|T_t(u_k-\overline{u}_k)|\le t$ in Ω and $f_k-\overline{f}_k\to0$ in $L^1(\Omega)$, the right-hand side of (5.36) converges to 0 as $k\to\infty$. On the other hand, the arguments of Steps 3 and 4 tell us that $\nabla u_k\to\nabla u$ and $\nabla\overline{u}_k\to\nabla\overline{u}$ a.e. in Ω (up to subsequences). Hence, by (1.7) and Fatou's lemma, passing to the limit as $k\to\infty$ in equation (5.36) tells us that

$$\int_{\{|u-\overline{u}|\le t\}}\big(\mathcal{A}(x,\nabla u)-\mathcal{A}(x,\nabla\overline{u})\big)\cdot(\nabla u-\nabla\overline{u})\,dx=0.$$

Thus, owing to (1.7), we have that $\nabla u=\nabla\overline{u}$ a.e. in $\{|u-\overline{u}|\le t\}$ for every $t>0$, whence

$$\nabla u=\nabla\overline{u}\qquad\text{a.e. in }\Omega. \tag{5.37}$$

By a version of the Poincaré inequality for domains in the class $\mathcal{G}_{1/\sigma'}$ [Ma5, Corollary 5.2.3] (this is also a special case of inequality (2.24)), there exists a constant $C=C(\Omega)$ such that

$$\bigg(\int_\Omega|v-\operatorname{med}(v)|^{\sigma'}\,dx\bigg)^{\frac{1}{\sigma'}}\le C\int_\Omega|\nabla v|\,dx, \tag{5.38}$$

for every $v\in W^{1,1}(\Omega)$. Fix any $t,\tau>0$. An application of (5.38) with $v=T_\tau(u-T_t(\overline{u}))$, and the use of (5.37) entail that

$$\bigg(\int_\Omega|T_\tau(u-T_t(\overline{u}))-\operatorname{med}(T_\tau(u-T_t(\overline{u})))|^{\sigma'}\,dx\bigg)^{\frac{1}{\sigma'}} \tag{5.39}$$

$$\leq C\left(\int_{\{t<|u|<t+\tau\}}|\nabla u|\,dx+\int_{\{t-\tau<|u|<t\}}|\nabla u|\,dx\right).$$

We claim that, for each $\tau>0$, the right-hand side of (5.39) converges to 0 as $t\to\infty$. To verify this claim, choose the test function $\varphi=T_\tau(u_k-T_t(u_k))$ in (5.10) to deduce that

$$\int_{\{t<|u_k|<t+\tau\}}B(|\nabla u_k|)\,dx\leq\int_{\{t<|u_k|<t+\tau\}}\mathcal{A}(x,\nabla u_k)\cdot\nabla u_k\,dx\leq\tau\int_{\{|u_k|>t\}}|f_k|\,dx. \tag{5.40}$$

Passing to the limit as $k\to\infty$ in (5.40) yields, by Fatou's lemma,

$$\int_{\{t<|u|<t+\tau\}}B(|\nabla u|)\,dx\leq\tau\int_{\{|u|>t\}}|f|\,dx. \tag{5.41}$$

Hence, since the function B is convex, by Jensen's inequality the first integral on the right-hand side of (5.39) approaches 0 as $t\to\infty$. An analogous argument shows that also the last integral in (5.39) tends to 0 as $t\to\infty$. Since

$$\lim_{t\to\infty}T_\tau(u-T_t(\overline{u}))-\operatorname{med}(T_\tau(u-T_t(\overline{u})))=T_\tau(u-\overline{u})-\operatorname{med}(T_\tau(u-\overline{u}))\quad\text{a.e. in }\Omega,$$

from (5.39), via Fatou's lemma, we obtain that

$$\int_\Omega|T_\tau(u-\overline{u})-\operatorname{med}(T_\tau(u-\overline{u}))|^{\sigma'}\,dx=0 \tag{5.42}$$

for $\tau>0$. Thus, the integrand in (5.42) vanishes a.e. in Ω for every $\tau>0$, and hence also its limit as $\tau\to\infty$ vanishes a.e. in Ω. Therefore, the function $u-\overline{u}$ is constant a.e. in Ω.

Step 7. *Equations* (3.24) *and* (3.25) *hold.*
Passing to the limit as $k\to\infty$ in inequality (5.13), and making use of (5.11) and (5.26), tell us that u satisfies the assumptions of Lemma 4.1, part (ii). Hence, either inequality (4.5) or inequality (4.6) holds, for a suitable constant M and sufficiently large t, depending on whether B fulfills (2.28) or (2.29). In the latter case, $u\in L^\infty(\Omega)$. In the former case, in view of definition (3.3), inequality (4.5) yields

$$|\{|u-\operatorname{med}(u)|>t\}|\Phi_\sigma(t/c)\leq c \tag{5.43}$$

for a suitable positive constant c and sufficiently large t. Hence, $u\in L^{\Phi_\sigma,\infty}(\Omega)$.
As for ∇u, if (2.28) is in force, then, by (4.5), inequality (4.24) holds for a suitable M and sufficicently large s. Therefore,

$$|\{|\nabla u|>t\}|\Psi_\sigma(t)\leq c$$

for a suitable constant c and sufficiently large t. This implies that $\nabla u\in L^{\Psi_\sigma,\infty}(\Omega)$. The same conclusion follows under assumption (2.29), on choosing $t=t_1$ and $s=B(t)$ in (4.34), since $\lim_{t\to\infty}\phi_\sigma(t)<\infty$ in this case, and hence Ψ_σ is equivalent to B near infinity.

Step 8. *Equation* (3.26) *holds for any sequence* $\{f_k\}$ *as in Definition 3.7.*
This assertion follows via the arguments of Steps 2, 3 and 4, applied to the sequence $\{f_k\}$. □

Proof of Theorem 3.11, sketched. The proof proceeds along the same lineas as in Steps 1-5 and 7 of the proof of Theorem 3.8. One has just to start by choosing a sequence $\{f_k\}\subset L^1(\Omega)\cap(W^{1,B}(\Omega))'$ fulfilling equation (3.31), and such that $\int_\Omega f_k\,dx=0$ and

$$\|f_k\|_{L^1(\Omega)}\leq 2\|\mu\|(\Omega)$$

for $k \in \mathbb{N}$. The function f_k can be defined, for $k \in \mathbb{N}$, as $f_k = g_k - \frac{1}{|\Omega|}\int_\Omega g_k(y)\,dy$, where $g_k : \Omega \to \mathbb{R}$ is the function given by

$$g_k(x) = \int_\Omega k^n \varrho(|y-x|k)\,d\mu(y) \quad \text{for } x \in \Omega,$$

and $\varrho : \mathbb{R}^n \to [0,\infty]$ is a standard mollifier, namely a smooth function, compactly supported in the unit ball of $\mathbb{R}^n$, such that $\int_{\mathbb{R}^n} \varrho(x)\,dx = 1$.
All the equations appearing in the relevant steps then continue to hold, provided that $\|f\|_{L^1(\Omega)}$ is replaced by $\|\mu\|(\Omega)$. In particular, note that, if $\varphi \in W^{1,\infty}(\Omega)$, then φ is locally Lipschitz, whence $\varphi \in C(\Omega)$. Moreover, $\varphi \in L^\infty(\Omega)$, as a consequence embedding (2.30), with $B(t) = \infty$ for large values of t. Thus, any function $\varphi \in W^{1,\infty}(\Omega)$ is admissible in (3.31), and hence, for any such function φ, the right-hand side of (5.10) actually converges to $\int_\Omega \varphi\,d\mu(x)$ as $k \to \infty$. $\square$

Acknowledgments. We wish to thank the referee for his careful reading of the manuscript and for his valuable remarks and suggestions.
This research was partly supported by the Research Project of the Italian Ministry of University and Research (MIUR) Prin 2015 n.2015HY8JCC "Partial differential equations an related analytic-geometric inequalities", by GNAMPA of the Italian INdAM (National Institute of High Mathematics), and by the Ministry of Education and Science of the Russian Federation, agreement n. 02.a03.21.0008.

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