

Second-order L^2 -regularity in nonlinear elliptic problems

Andrea Cianchi

Dipartimento di Matematica e Informatica “U. Dini”, Università di Firenze

Viale Morgagni 67/A, 50134 Firenze, Italy

e-mail: cianchi@unifi.it

Vladimir G. Maz’ya

Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden

and

Department of Mathematical Sciences, M&O Building

University of Liverpool, Liverpool L69 3BX, UK;

e-mail: vmaz@mai.liu.se

Abstract

A second-order regularity theory is developed for solutions to a class of quasilinear elliptic equations in divergence form, including the p -Laplace equation, with merely square-integrable right-hand side. Our results amount to the existence and square integrability of the weak derivatives of the nonlinear expression of the gradient under the divergence operator. This provides a nonlinear counterpart of the classical L^2 -coercivity theory for linear problems, which is missing in the existing literature. Both local and global estimates are established. The latter apply to solutions to either Dirichlet or Neumann boundary value problems. Minimal regularity on the boundary of the domain is required. If the domain is convex, no regularity of its boundary is needed at all.

1 Introduction

A prototypical result in the theory of elliptic equations asserts that, if Ω is a bounded open set in \mathbb{R}^n , $n \geq 2$, with $\partial\Omega \in C^2$, and u is the weak solution to the Dirichlet problem for the inhomogeneous Laplace equation whose right-hand side $f \in L^2(\Omega)$, then $u \in W^{2,2}(\Omega)$. Moreover, a two-sided coercivity estimate for $\|\nabla^2 u\|_{L^2(\Omega)}$ holds in terms of $\|f\|_{L^2(\Omega)}$, up to multiplicative constants. This can be traced back to [Be] for $n = 2$, and to [Sch] for $n \geq 3$. A comprehensive analysis of this topic can be found in [ADN], [Hö, Chapter 10], [LaUr, Chapter 3], [MazSh, Chapter 14].

The regularity theory for (possibly degenerate or singular) nonlinear equations in divergence form, extending the Laplace equation, whose prototype is the p -Laplace equation, has thoroughly been developed in the last fifty years. Regularity properties of solutions and of their first-order

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derivatives have been investigated in a number of contributions, including the classics [ChDi, Di, DiMa, Ev, Iw, KiMa, Le, Li, Si,L., To, Uh, Ur] and the more recent advances [BCDKS, BDS, CKP, CiMa2, BDS, DuMil, KuMi].

Despite the huge amount of work devoted to this kind of equations, the picture of second-order regularity for their solutions is apparently still quite incomplete. A result is available for p -harmonic functions, namely local solutions u to the homogenous equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } \Omega,$$

and asserts that the nonlinear expression of the gradient $|\nabla u|^{\frac{p-2}{2}}\nabla u \in W_{\text{loc}}^{1,2}(\Omega)$ – see [Uh] for $p \in (2, \infty)$, and [ChDi] for every $p \in (1, \infty)$. If $p \in (1, 2)$, coupling this property with the local boundedness of ∇u in Ω ensures that $u \in W_{\text{loc}}^{2,2}(\Omega)$. On the other hand, the existence of second-order weak derivatives of p -harmonic functions is an open problem for $p \in (2, \infty)$.

Information on this issue concerning inhomogeneous equations is even more limited. In fact, this case seems to be almost unexplored. With this regard, let us mention that (global) twice weak differentiability of solutions to Dirichlet problems for the inhomogeneous p -Laplace equation is proved in [BeCr] under the assumption that p is smaller than, and sufficiently close to 2, and relies upon the linear theory, via a perturbation argument. Fractional-order regularity for the gradient of solutions to a class of nonlinear inhomogeneous equations, modelled upon the p -Laplacian, is established in [Mi]. An earlier contribution in this direction is [Si,J.]

The present paper offers a second-order regularity principle for a class of quasilinear elliptic problems in divergence form, that encompasses the inhomogeneous p -Laplace equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(x) \quad \text{in } \Omega,$$

for any $p \in (1, \infty)$ and any right-hand side $f \in L^2(\Omega)$. In contrast with the customary results recalled above, our statements involve exactly the nonlinear function of ∇u appearing under divergence in the relevant elliptic operators. In the light of our conclusions, this turns out to be the correct expression to call into play, inasmuch as it admits a two-sided L^2 -estimate in terms of the datum on the right-hand side of the equation, and hence exhibits a regularity-preserving property.

Both local solutions, and solutions to Dirichlet and Neumann boundary value problems are addressed. A distinctive trait of our results is the minimal regularity imposed on $\partial\Omega$ when dealing with global bounds. In particular, if Ω is convex, no additional regularity has to be required on $\partial\Omega$. However, we stress that the results to be proved are new even for smooth domains.

An additional striking feature is that they apply to a very weak notion of solutions, which has to be adopted since the right-hand side of the equations is allowed to enjoy a low degree of integrability.

To conclude this preliminary overview, let us point out that the validity of second-order L^2 -estimates raises the natural question of a more general second-order theory in L^q for $q \neq 2$, or in other function spaces. This would amount to a second-order nonlinear Calderón-Zygmund theory, namely an analogue of the corpus of gradient estimates for p -Laplacian type equations, with right-hand side in divergence form, whose study was initiated in [Iw].

2 Main results

Although our main focus is on global estimates for solutions to boundary value problems, we begin our discussion with a local bound for local solutions, of independent interest. The equations

under consideration have the form

$$(2.1) \quad -\operatorname{div}(a(|\nabla u|)\nabla u) = f(x) \quad \text{in } \Omega$$

where Ω is any open set in \mathbb{R}^n , and $f \in L^2_{\text{loc}}(\Omega)$. The function $a : (0, \infty) \rightarrow (0, \infty)$ is of class $C^1(0, \infty)$, and such that

$$(2.2) \quad -1 < i_a \leq s_a < \infty,$$

where

$$(2.3) \quad i_a = \inf_{t>0} \frac{ta'(t)}{a(t)} \quad \text{and} \quad s_a = \sup_{t>0} \frac{ta'(t)}{a(t)},$$

and a' stands for the derivative of a . Assumption (2.2) ensures that the differential operator in (2.1) satisfies ellipticity and monotonicity conditions, not necessarily of power type [CiMa1, CiMa2]. Regularity for equations governed by generalized nonlinearities of this kind has also been extensively studied – see e.g. [Ba, BSV, Ci2, Ci3, DKS, DSV, Ko, Li, Mar, Ta]. Observe that the standard p -Laplace operator corresponds to the choice $a(t) = t^{p-2}$, with $p > 1$. Clearly, $i_a = s_a = p - 2$ in this case.

As already warned in Section 1, due to the mere square summability assumption on the function f , solutions to equation (2.1) may have to be understood in a suitable generalized sense, even in the case of the p -Laplacian. We shall further comment on this at the end of this section. Precise definitions can be found in Sections 4 and 5.

In what follows, $B_r(x)$ denotes the ball with radius $r > 0$, centered at $x \in \mathbb{R}^n$. The simplified notation B_r is employed when information on the center is irrelevant. In this case, balls with different radii appearing in the same formula (or proof) will be tacitly assumed to have the same center.

Theorem 2.1 [Local estimate] *Assume that the function $a \in C^1(0, \infty)$, and satisfies condition (2.2). Let Ω be any open set in \mathbb{R}^n , with $n \geq 2$, and let $f \in L^2_{\text{loc}}(\Omega)$. Let u be a generalized local solution to equation (2.1). Then*

$$(2.4) \quad a(|\nabla u|)\nabla u \in W^{1,2}_{\text{loc}}(\Omega),$$

and there exists a constant $C = C(n, i_a, s_a)$ such that

$$(2.5) \quad \|a(|\nabla u|)\nabla u\|_{W^{1,2}(B_R)} \leq C(\|f\|_{L^2(B_{2R})} + R^{-\frac{n}{2}}\|a(|\nabla u|)\nabla u\|_{L^1(B_{2R})})$$

for any ball $B_{2R} \subset\subset \Omega$.

Remark 2.2 Observe that the expression $a(|\nabla u|)\nabla u$ agrees with $|\nabla u|^{p-2}\nabla u$ when the differential operator in equation (2.1) is the p -Laplacian, and hence differs in the exponent of $|\nabla u|$ from the results recalled above about p -harmonic functions.

Our global results concern Dirichlet or Neumann problems, with homogeneous boundary data, associated with equation (2.1). Namely, Dirichlet problems of the form

$$(2.6) \quad \begin{cases} -\operatorname{div}(a(|\nabla u|)\nabla u) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and Neumann problems of the form

$$(2.7) \quad \begin{cases} -\operatorname{div}(a(|\nabla u|)\nabla u) = f(x) & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, Ω is a bounded open set in \mathbb{R}^n , ν denotes the outward unit vector on $\partial\Omega$, $f \in L^2(\Omega)$, and $a : (0, \infty) \rightarrow (0, \infty)$ is as above. Of course, the compatibility condition

$$(2.8) \quad \int_{\Omega} f(x) dx = 0$$

has to be required when dealing with (2.7).

A basic version of the global second-order estimates for the solutions to (2.6) and (2.7) holds in any bounded convex open set $\Omega \subset \mathbb{R}^n$.

Theorem 2.3 [Global estimate in convex domains] *Assume that the function $a \in C^1(0, \infty)$, and satisfies condition (2.2). Let Ω be any convex bounded open set in \mathbb{R}^n , with $n \geq 2$, and let $f \in L^2(\Omega)$. Let u be the generalized solution to either the Dirichlet problem (2.6), or the Neumann problem (2.7). Then*

$$(2.9) \quad a(|\nabla u|)\nabla u \in W^{1,2}(\Omega).$$

Moreover,

$$(2.10) \quad C_1 \|f\|_{L^2(\Omega)} \leq \|a(|\nabla u|)\nabla u\|_{W^{1,2}(\Omega)} \leq C_2 \|f\|_{L^2(\Omega)}$$

for some constants $C_1 = C_1(n, s_a)$ and $C_2 = C_2(\Omega, i_a, s_a)$.

Heuristically speaking, the validity of a global estimate in Theorem 2.3 is related to the fact that the second fundamental form on the boundary of a convex set is semidefinite. In the main result of this paper, the convexity assumption on Ω is abandoned. Dropping signature information on the (weak) second fundamental form on $\partial\Omega$ calls for an assumption on its summability. We assume that the domain Ω is locally the subgraph of a Lipschitz continuous function of $(n-1)$ variables, which is also twice weakly differentiable. The second-order derivatives of this function are required to belong to either the weak Lebesgue space L^{n-1} , called $L^{n-1,\infty}$, or the weak Zygmund space $L \log L$, called $L^{1,\infty} \log L$, according to whether $n \geq 3$ or $n = 2$. This will be denoted by $\partial\Omega \in L^{n-1,\infty}$, and $\partial\Omega \in L^{1,\infty} \log L$, respectively. As a consequence, the weak second fundamental form \mathcal{B} on $\partial\Omega$ belongs to the same weak type spaces with respect to the $(n-1)$ -dimensional Hausdorff measure \mathcal{H}^{n-1} on $\partial\Omega$. Our key summability assumption on \mathcal{B} amounts to:

$$(2.11) \quad \lim_{r \rightarrow 0^+} \left(\sup_{x \in \partial\Omega} \|\mathcal{B}\|_{L^{n-1,\infty}(\partial\Omega \cap B_r(x))} \right) < c \quad \text{if } n \geq 3,$$

or

$$(2.12) \quad \lim_{r \rightarrow 0^+} \left(\sup_{x \in \partial\Omega} \|\mathcal{B}\|_{L^{1,\infty} \log L(\partial\Omega \cap B_r(x))} \right) < c \quad \text{if } n = 2,$$

for a suitable constant $c = c(L_\Omega, d_\Omega, n, i_a, s_a)$. Here, L_Ω denotes the Lipschitz constant of Ω , and d_Ω its diameter. Let us emphasize that such an assumption is essentially sharp – see Remark 2.5 below.

Theorem 2.4 [Global estimate in minimally regular domains] *Assume that the function $a \in C^1(0, \infty)$, and satisfies condition (2.2). Let Ω be a Lipschitz bounded domain in \mathbb{R}^n , $n \geq 2$ such that $\partial\Omega \in W^2L^{n-1, \infty}$ if $n \geq 3$, or $\partial\Omega \in W^2L^{1, \infty} \log L$ if $n = 2$. Assume that $f \in L^2(\Omega)$, and let u be the generalized solution to either the Dirichlet problem (2.6), or the Neumann problem (2.7). There exists a constant $c = c(L_\Omega, d_\Omega, n, i_a, s_a)$ such that, if Ω fulfils (2.11) or (2.12) for such a constant c , then*

$$(2.13) \quad a(|\nabla u|)\nabla u \in W^{1,2}(\Omega).$$

Moreover,

$$(2.14) \quad C_1 \|f\|_{L^2(\Omega)} \leq \|a(|\nabla u|)\nabla u\|_{W^{1,2}(\Omega)} \leq C_2 \|f\|_{L^2(\Omega)}$$

for some positive constants $C_1 = C_1(n, s_a)$ and $C_2 = C_2(\Omega, i_a, s_a)$.

We conclude this section with some remarks on Theorems 2.1, 2.3 and 2.4.

Remark 2.5 Assumption (2.11), or (2.12), cannot be weakened in Theorem 2.4 for all equations of the form appearing in (2.6) and (2.7). This can be shown by taking into account the linear problem corresponding to the case when the function a is constant. Indeed, domains Ω can be exhibited such that $\partial\Omega \in W^2L^{n-1, \infty}$ if $n \geq 3$ [Maz4], or $\partial\Omega \in W^2L^{1, \infty} \log L$ if $n = 2$ [Maz3], but the limit in (2.11) or (2.12) exceeds some explicit threshold, and the corresponding solution u to the Dirichlet problem for the Laplace equation fails to belong to $W^{2,2}(\Omega)$ (see also [MazSh, Section 14.6.1] in this connection).

Remark 2.6 Condition (2.11) is certainly fulfilled if $\partial\Omega \in W^{2, n-1}$, and (2.12) is fulfilled if $\partial\Omega \in W^2L \log L$, or, a fortiori, if $\partial\Omega \in W^{2, q}$ for some $q > 1$. This follows from the embedding of L^{n-1} into $L^{n-1, \infty}$ and of $L \log L$ (or L^q) into $L^{1, \infty} \log L$ for $q > 1$, and from the absolute continuity of the norm in any Lebesgue and Zygmund space. Notice also that, since the Lorentz space $L^{n-1, 1} \not\subseteq L^{n-1}$, assumption (2.11) is, in particular, weaker than requiring that $\partial\Omega \in W^2L^{n-1, 1}$. The latter condition has been shown to ensure the global boundedness of the gradient of the solutions to problems (2.6) or (2.7), for $n \geq 3$, provided that f belongs to the Lorentz space $L^{n, 1}(\Omega)$ [CiMa1, CiMa2]. Note that hypothesis (2.11) does not imply that $\partial\Omega \in C^{1, 0}$, a property that is instead certainly fulfilled under the stronger condition that $\partial\Omega \in W^2L^{n-1, 1}$.

Remark 2.7 The global gradient bound mentioned in Remark 2.6 enables one to show, via a minor variant in the proof of Theorems 2.3–2.4, that the solutions to problems (2.6) and (2.7) are actually in $W^{2,2}(\Omega)$, provided that

$$(2.15) \quad \inf_{t \in [0, M]} a(t) > 0$$

for every $M > 0$, and f and Ω have the required regularity for the relevant gradient bound to hold. A parallel result holds for local solutions to the equation (2.1), thanks to a local gradient estimate from [Ba], extending [DuMi1]. To be more specific, if $f \in L_{\text{loc}}^{n, 1}(\Omega)$, and u is a generalized local solution to equation (2.1), then

$$(2.16) \quad u \in W_{\text{loc}}^{2,2}(\Omega).$$

Moreover, if $n \geq 3$, $f \in L^{n, 1}(\Omega)$, $\partial\Omega \in W^2L^{n-1, 1}$, and u is the generalized solution to either the Dirichlet problem (2.6), or the Neumann problem (2.7), then

$$(2.17) \quad u \in W^{2,2}(\Omega).$$

Equation (2.17) continues to hold if Ω is any bounded convex domain in \mathbb{R}^n , whatever $\partial\Omega$ is. Let us stress that these conclusion may fail if assumption (2.15) is dropped. This can be verified, for instance, on choosing $a(t) = t^{p-2}$, i.e. the p -Laplace operator, and considering functions of the form $u(x) = |x_1|^\beta$, where $x = (x_1, \dots, x_n)$ and $\beta > 1$. These functions are local solutions to (2.1) with $f \in L_{\text{loc}}^{n,1}(\mathbb{R}^n)$ (and even $f \in L_{\text{loc}}^\infty(\mathbb{R}^n)$) provided that p is large enough, but $u \notin W_{\text{loc}}^{2,2}(\mathbb{R}^n)$ if $\beta \leq \frac{3}{2}$. In fact, $u \notin W_{\text{loc}}^{2,q}(\mathbb{R}^n)$ for any given $q > 1$, if β is sufficiently close to 1.

Remark 2.8 Weak solutions to problems (2.6) or (2.7), namely distributional solutions belonging to the energy space associated with the relevant differential operator, need not exist if f is merely in $L^2(\Omega)$. This phenomenon is well-known to occur in the model case of the p -Laplace equation, if p is not large enough for $L^2(\Omega)$ to be contained in the dual of $W^{1,p}(\Omega)$. Yet, weaker definitions of solutions to boundary value problems for this equation, ensuring their uniqueness, which apply to any $p \in (1, \infty)$ and even to right-hand sides $f \in L^1(\Omega)$, are available in the literature[ACMM, BBGGPV, BoGa, DaA, DuMi1, LiMu, Maz5, Mu]. Among the diverse, but a posteriori equivalent, definitions, we shall adopt that (adjusted to the framework under consideration in this paper) of a solution which is the limit of a sequence of solutions to problems whose right-hand sides are smooth and converge to f [DaA]. This will be called a generalized solution throughout. A parallel notion of generalized local solution to (2.1) will be employed. A generalized solution need not be weakly differentiable. However, it is associated with a vector-valued function on Ω , which plays the role of a substitute for its gradient in the distributional definition of solution. With some abuse of notation, this is the meaning attributed to ∇u in the statements of Theorems 2.1, 2.3 and 2.4.

A definition of generalized solution to problem (2.6) and to problem (2.7) is given in Section 4, where an existence, uniqueness and first-order summability result from [CiMa3] is also recalled. Note that, owing to its uniqueness, this kind of generalized solution agrees with the weak solution whenever f is summable enough, depending on the nonlinearity of the differential operator, for a weak solution to exist. Generalized local solutions to equation (2.1) are defined in Section 5.

3 A differential inequality

The subject of this section is a lower bound for the square of the differential operator on the left-hand side of the equations in (2.6) and (2.7) in terms of an operator in divergence form, plus (a positive constant times) derivatives of $a(|\nabla u|)\nabla u$ squared. This is a critical step in the proof of our main results, and is the content of the following lemma.

Lemma 3.1 *Assume that $a \in C^1[0, \infty)$, and that the first inequality in (2.2) holds. Then there exists a positive constant $C = C(n, i_a)$ such that*

$$(3.1) \quad (\operatorname{div}(a(|\nabla u|)\nabla u))^2 \geq \sum_{j=1}^n (a(|\nabla u|)^2 u_{x_j} \Delta u)_{x_j} - \sum_{i=1}^n \left(a(|\nabla u|)^2 \sum_{j=1}^n u_{x_j} u_{x_i x_j} \right)_{x_i} + Ca(|\nabla u|)^2 |\nabla^2 u|^2$$

for every function $u \in C^3(\Omega)$. Here, $|\nabla^2 u| = \left(\sum_{i,j=1}^n u_{x_i x_j}^2 \right)^{\frac{1}{2}}$.

Proof. Let $u \in C^3(\Omega)$. Computations show that

$$\begin{aligned}
(3.2) \quad & (\operatorname{div}(a(|\nabla u|)\nabla u))^2 = (a(|\nabla u|)\Delta u + a'(|\nabla u|)\nabla|\nabla u| \cdot \nabla u)^2 \\
& = a(|\nabla u|)^2((\Delta u)^2 - |\nabla^2 u|^2) + a(|\nabla u|)^2|\nabla^2 u|^2 + \\
& \quad + a'(|\nabla u|)^2(\nabla|\nabla u| \cdot \nabla u)^2 + 2a(|\nabla u|)a'(|\nabla u|)\Delta u \nabla|\nabla u| \cdot \nabla u \\
& = a(|\nabla u|)^2\left(\sum_{j=1}^n (u_{x_j} \Delta u)_{x_j} - \sum_{i,j=1}^n (u_{x_j} u_{x_i x_j})_{x_i}\right) + a(|\nabla u|)^2|\nabla^2 u|^2 \\
& \quad + a'(|\nabla u|)^2(\nabla|\nabla u| \cdot \nabla u)^2 + 2a(|\nabla u|)a'(|\nabla u|)\Delta u \nabla|\nabla u| \cdot \nabla u \\
& = \sum_{j=1}^n (a(|\nabla u|)^2 u_{x_j} \Delta u)_{x_j} - \sum_{i,j=1}^n (a(|\nabla u|)^2 u_{x_j} u_{x_i x_j})_{x_i} \\
& \quad - 2a(|\nabla u|)a'(|\nabla u|)(\Delta u \nabla|\nabla u| \cdot \nabla u - \sum_{i,j=1}^n |\nabla u|_{x_i} u_{x_j} u_{x_i x_j}) \\
& \quad + a(|\nabla u|)^2|\nabla^2 u|^2 + a'(|\nabla u|)^2(\nabla|\nabla u| \cdot \nabla u)^2 + 2a(|\nabla u|)a'(|\nabla u|)\Delta u \nabla|\nabla u| \cdot \nabla u \\
& = \sum_{j=1}^n (a(|\nabla u|)^2 u_{x_j} \Delta u)_{x_j} - \sum_{i,j=1}^n (a(|\nabla u|)^2 u_{x_j} u_{x_i x_j})_{x_i} \\
& \quad + 2a(|\nabla u|)a'(|\nabla u|) \sum_{i,j=1}^n |\nabla u|_{x_i} u_{x_j} u_{x_i x_j} + a(|\nabla u|)^2|\nabla^2 u|^2 + a'(|\nabla u|)^2(\nabla|\nabla u| \cdot \nabla u)^2,
\end{aligned}$$

where “ \cdot ” stands for scalar product in \mathbb{R}^n . After relabeling the indices, one has that

$$\begin{aligned}
(3.3) \quad & a'(|\nabla u|)^2(\nabla|\nabla u| \cdot \nabla u)^2 + 2a(|\nabla u|)a'(|\nabla u|) \sum_{i,j=1}^n |\nabla u|_{x_i} u_{x_j} u_{x_i x_j} + a(|\nabla u|)^2|\nabla^2 u|^2 \\
& = a(|\nabla u|)^2 \left[\left(\frac{|\nabla u| a'(|\nabla u|)}{a(|\nabla u|)} \right)^2 \left(\sum_{i,k=1}^n \frac{u_{x_k} u_{x_i}}{|\nabla u|^2} u_{x_k x_i} \right)^2 \right. \\
& \quad \left. + 2 \sum_{i,j,k=1}^n \frac{|\nabla u| a'(|\nabla u|)}{a(|\nabla u|)} \frac{u_{x_k} u_{x_i}}{|\nabla u|^2} u_{x_k x_j} u_{x_i x_j} + \sum_{i,j=1}^n u_{x_i x_j}^2 \right].
\end{aligned}$$

Now, set

$$\omega_u = \frac{\nabla u}{|\nabla u|}, \quad \vartheta_u = \frac{|\nabla u| a'(|\nabla u|)}{a(|\nabla u|)}, \quad H_u = \nabla^2 u.$$

Observe that $\omega_u \in \mathbb{R}^n$, with $|\omega_u| = 1$, H_u is a symmetric matrix in $\mathbb{R}^{n \times n}$, and, by (2.2), $\vartheta_u \geq i_a$. With this notation in place, the expression in square brackets on the right-hand side of (3.3) takes the form

$$(3.4) \quad \vartheta_u^2 (H\omega_u \cdot \omega_u)^2 + 2\vartheta_u H\omega_u \cdot H\omega_u + \operatorname{tr}(H_u^2),$$

where “ tr ” denotes the trace of a matrix. The proof of inequality (3.1) is thus reduced to showing that

$$(3.5) \quad \vartheta_u^2 (H\omega_u \cdot \omega_u)^2 + 2\vartheta_u H\omega_u \cdot H\omega_u + \operatorname{tr}(H_u^2) \geq C \operatorname{tr}(H_u^2)$$

for some positive constant $C = C(n, i_a)$. To establish inequality (3.5), define the function $\psi : \mathbb{R} \times \mathbb{R}^n \times (\mathbb{R}^{n \times n} \setminus \{0\}) \rightarrow \mathbb{R}$ as

$$\psi(\vartheta, \omega, H) = \vartheta^2 \frac{(H\omega \cdot \omega)^2}{\operatorname{tr}(H^2)} + 2\vartheta \frac{H\omega \cdot H\omega}{\operatorname{tr}(H^2)} + 1$$

for $(\vartheta, \omega, H) \in \mathbb{R} \times \mathbb{R}^n \times (\mathbb{R}^{n \times n} \setminus \{0\})$, and note that (3.5) will follow if we show that there exists a positive constant C such that

$$(3.6) \quad \psi(\vartheta, \omega, H) \geq C$$

if $\vartheta \geq i_a$, $|\omega| = 1$ and H is any non-vanishing symmetric matrix H . For each fixed ω and H , the quadratic function $\vartheta \mapsto \psi(\vartheta, \omega, H)$ attains its minimum at $\vartheta = -\frac{H\omega \cdot H\omega}{(H\omega \cdot \omega)^2}$. We claim that

$$(3.7) \quad -\frac{H\omega \cdot H\omega}{(H\omega \cdot \omega)^2} \leq -1.$$

To verify equation (3.7), choose a basis in \mathbb{R}^n in which H has diagonal form $\text{diag}(\lambda_1, \dots, \lambda_n)$, and let $(\omega_1, \dots, \omega_n)$ denote the vector of the components of ω with respect to this basis. Then

$$H\omega \cdot H\omega = \sum_{i=1}^n \lambda_i^2 \omega_i^2, \quad H\omega \cdot \omega = \sum_{i=1}^n \lambda_i \omega_i^2,$$

whence (3.7) follows, since

$$(3.8) \quad \left(\sum_{i=1}^n \lambda_i \omega_i^2 \right)^2 \leq \left(\sum_{i=1}^n \lambda_i^2 \omega_i^2 \right) \left(\sum_{i=1}^n \omega_i^2 \right) = \left(\sum_{i=1}^n \lambda_i^2 \omega_i^2 \right),$$

by Schwarz' inequality. Note that the equality holds in (3.8) inasmuch as $\sum_{i=1}^n \omega_i^2 = 1$. Owing to (3.7), $\psi(\vartheta, \omega, H)$ is a strictly increasing function of ϑ for $\vartheta \geq -1$. Hence, by the first inequality in (2.2),

$$(3.9) \quad \psi(\vartheta, \omega, H) \geq \psi(i_a, \omega, H) > \psi(-1, \omega, H)$$

if $\vartheta \geq i_a$ and $|\omega| = 1$. Assume, for a moment, that we know that

$$(3.10) \quad \psi(-1, \omega, H) \geq 0$$

if $|\omega| = 1$ and H is any symmetric matrix. Since ψ is a continuous function, we deduce from (3.9) and (3.10) that

$$(3.11) \quad \psi(\vartheta, \omega, H) \geq \psi(i_a, \omega, H) \geq \inf_{|\omega|=1, H \text{ sym}} \psi(i_a, \omega, H) = \min_{|\omega|=1, H \text{ sym}, |H|=1} \psi(i_a, \omega, H) > 0$$

if $|\omega| = 1$ and H is symmetric and different from 0. Hence (3.6) follows. Observe that the equality holds in (3.11) since ψ is a homogenous function of degree 0 in H .

It remains to prove inequality (3.10), namely that

$$(3.12) \quad (H\omega \cdot \omega)^2 - 2H\omega \cdot H\omega + \text{tr}(H^2) \geq 0$$

if $|\omega| = 1$ and H is symmetric. After diagonalizing H as above, inequality (3.12) reads

$$(3.13) \quad \sum_{i=1}^n (\omega_i^2 - 1)^2 \lambda_i^2 + 2 \sum_{1 \leq i < j \leq n} \omega_i^2 \omega_j^2 \lambda_i \lambda_j \geq 0,$$

if $\sum_{i=1}^n \omega_i^2 = 1$ and $\lambda_i \in \mathbb{R}$ for $i = 1, \dots, n$. Inequality (3.13) is a consequence of the following lemma. \square

Lemma 3.2 Assume that $\eta_i \in \mathbb{R}$ are such that $\eta_i \geq 0$, $i = 1, \dots, n$, and $\sum_{i=1}^n \eta_i \leq 1$. Then

$$(3.14) \quad \sum_{i=1}^n (\eta_i - 1)^2 \lambda_i^2 + 2 \sum_{1 \leq i < j \leq n} \eta_i \eta_j \lambda_i \lambda_j \geq 0$$

for every $\lambda_i \in \mathbb{R}$, $i = 1, \dots, n$.

Proof. By Sylvester's criterion, it suffices to show that the determinants of the north-west minors of the $n \times n$ matrix

$$(3.15) \quad \begin{pmatrix} (\eta_1 - 1)^2 & \eta_1 \eta_2 & \dots & \eta_1 \eta_n \\ \eta_2 \eta_1 & (\eta_2 - 1)^2 & \dots & \eta_2 \eta_n \\ \vdots & \vdots & \ddots & \vdots \\ \eta_n \eta_1 & \eta_n \eta_2 & \dots & (\eta_n - 1)^2 \end{pmatrix},$$

associated with the quadratic form on the left-hand side of (3.14), are nonnegative for every $\eta_i \geq 0$, $i = 1, \dots, n$, with $\sum_{i=1}^n \eta_i \leq 1$. Since every minor of this kind has the same structure as the entire matrix, and $\sum_{i=1}^2 \eta_i \leq \sum_{i=1}^3 \eta_i \leq \dots \leq \sum_{i=1}^n \eta_i \leq 1$, it suffices to prove that just the determinant of the whole matrix in (3.15) is nonnegative. To this purpose, let us begin by showing that

$$(3.16) \quad \begin{aligned} & \det \begin{pmatrix} (\eta_1 - 1)^2 & \eta_1 \eta_2 & \dots & \eta_1 \eta_n \\ \eta_2 \eta_1 & (\eta_2 - 1)^2 & \dots & \eta_2 \eta_n \\ \vdots & \vdots & \ddots & \vdots \\ \eta_n \eta_1 & \eta_n \eta_2 & \dots & (\eta_n - 1)^2 \end{pmatrix} \\ &= \eta_1^2 (1 - 2\eta_2)(1 - 2\eta_3) \times \dots \times (1 - 2\eta_n) + \eta_2^2 (1 - 2\eta_1)(1 - 2\eta_3) \times \dots \times (1 - 2\eta_n) + \dots \\ & \quad \dots + \eta_n^2 (1 - 2\eta_1)(1 - 2\eta_2) \times \dots \times (1 - 2\eta_{n-1}) + (1 - 2\eta_1)(1 - 2\eta_2) \times \dots \times (1 - 2\eta_n). \end{aligned}$$

Equation (3.16) can be verified by induction on n . The case when $n = 2$ is trivial. Assume that (3.16) holds with n replaced by $n - 1$. We have that

$$(3.17) \quad \begin{aligned} & \det \begin{pmatrix} (\eta_1 - 1)^2 & \eta_1 \eta_2 & \dots & \eta_1 \eta_n \\ \eta_2 \eta_1 & (\eta_2 - 1)^2 & \dots & \eta_2 \eta_n \\ \vdots & \vdots & \ddots & \vdots \\ \eta_n \eta_1 & \eta_n \eta_2 & \dots & (\eta_n - 1)^2 \end{pmatrix} \\ &= \det \begin{pmatrix} \eta_1^2 & \eta_1 \eta_2 & \dots & \eta_1 \eta_n \\ \eta_2 \eta_1 & (\eta_2 - 1)^2 & \dots & \eta_2 \eta_n \\ \vdots & \vdots & \ddots & \vdots \\ \eta_n \eta_1 & \eta_n \eta_2 & \dots & (\eta_n - 1)^2 \end{pmatrix} + \det \begin{pmatrix} (1 - 2\eta_1) & \eta_1 \eta_2 & \dots & \eta_1 \eta_n \\ 0 & (\eta_2 - 1)^2 & \dots & \eta_2 \eta_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \eta_n \eta_2 & \dots & (\eta_n - 1)^2 \end{pmatrix} \end{aligned}$$

Our induction assumption tells us that

$$(3.18) \quad \det \begin{pmatrix} (1 - 2\eta_1) & \eta_1 \eta_2 & \dots & \eta_1 \eta_n \\ 0 & (\eta_2 - 1)^2 & \dots & \eta_2 \eta_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \eta_n \eta_2 & \dots & (\eta_n - 1)^2 \end{pmatrix}$$

$$\begin{aligned}
&= (1 - 2\eta_1) \det \begin{pmatrix} (\eta_2 - 1)^2 & \cdots & \eta_2 \eta_n \\ \vdots & \ddots & \vdots \\ \eta_n \eta_2 & \cdots & (\eta_n - 1)^2 \end{pmatrix} \\
&= \eta_2^2 (1 - 2\eta_1)(1 - 2\eta_3) \times \cdots \times (1 - 2\eta_n) + \cdots + \eta_n^2 (1 - 2\eta_1)(1 - 2\eta_2) \times \cdots \times (1 - 2\eta_{n-1}) \\
&\quad + (1 - 2\eta_1)(1 - 2\eta_2) \times \cdots \times (1 - 2\eta_n).
\end{aligned}$$

On the other hand, we claim that

$$(3.19) \quad \det \begin{pmatrix} \eta_1^2 & \eta_1 \eta_2 & \cdots & \eta_1 \eta_n \\ \eta_2 \eta_1 & (\eta_2 - 1)^2 & \cdots & \eta_2 \eta_n \\ \vdots & \vdots & \ddots & \vdots \\ \eta_n \eta_1 & \eta_n \eta_2 & \cdots & (\eta_n - 1)^2 \end{pmatrix} = \eta_1^2 (1 - 2\eta_2)(1 - 2\eta_3) \cdots (1 - 2\eta_n).$$

Equation (3.19) can be proved by induction again. If $n = 2$, this equation can be verified via a direct computation. Assume now that it holds with n replaced by $(n - 1)$. Then,

$$\begin{aligned}
(3.20) \quad & \det \begin{pmatrix} \eta_1^2 & \eta_1 \eta_2 & \eta_1 \eta_3 & \cdots & \eta_1 \eta_n \\ \eta_2 \eta_1 & (\eta_2 - 1)^2 & \eta_2 \eta_3 & \cdots & \eta_2 \eta_n \\ \eta_3 \eta_1 & \eta_3 \eta_2 & (\eta_3 - 1)^2 & \cdots & \eta_3 \eta_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \eta_n \eta_1 & \eta_n \eta_2 & \eta_n \eta_3 & \cdots & (\eta_n - 1)^2 \end{pmatrix} \\
&= \det \begin{pmatrix} \eta_1^2 & \eta_1 \eta_2 & \eta_1 \eta_3 & \cdots & \eta_1 \eta_n \\ \eta_2 \eta_1 & \eta_2^2 & \eta_2 \eta_3 & \cdots & \eta_2 \eta_n \\ \eta_3 \eta_1 & \eta_3 \eta_2 & (\eta_3 - 1)^2 & \cdots & \eta_3 \eta_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \eta_n \eta_1 & \eta_n \eta_2 & \eta_n \eta_3 & \cdots & (\eta_n - 1)^2 \end{pmatrix} \\
&\quad + \det \begin{pmatrix} \eta_1^2 & 0 & \eta_1 \eta_3 & \cdots & \eta_1 \eta_n \\ \eta_2 \eta_1 & 1 - 2\eta_2 & \eta_2 \eta_3 & \cdots & \eta_2 \eta_n \\ \eta_3 \eta_1 & 0 & (\eta_3 - 1)^2 & \cdots & \eta_3 \eta_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \eta_n \eta_1 & 0 & \eta_n \eta_3 & \cdots & (\eta_n - 1)^2 \end{pmatrix} \\
&= 0 + (1 - 2\eta_2) \det \begin{pmatrix} \eta_1^2 & \eta_1 \eta_3 & \cdots & \eta_1 \eta_n \\ \eta_3 \eta_1 & (\eta_3 - 1)^2 & \cdots & \eta_3 \eta_n \\ \vdots & \vdots & \ddots & \vdots \\ \eta_n \eta_1 & \eta_n \eta_3 & \cdots & (\eta_n - 1)^2 \end{pmatrix} \\
&= \eta_1^2 (1 - 2\eta_2)(1 - 2\eta_3) \cdots (1 - 2\eta_n).
\end{aligned}$$

Note that in the last equality we have made use of the induction assumption, and of the fact that the determinant of a matrix with a couple of linearly dependent columns vanishes. Equation (3.16) follows from (3.17), (3.18) and (3.19).

With equation (3.16) at disposal, let us define the function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ as

$$(3.21) \quad \begin{aligned} \phi(\eta) &= \eta_1^2 (1 - 2\eta_2) \times \cdots \times (1 - 2\eta_n) + \cdots + \eta_n^2 (1 - 2\eta_1) \times \cdots \times (1 - 2\eta_{n-1}) \\ &\quad + (1 - 2\eta_1)(1 - 2\eta_2) \times \cdots \times (1 - 2\eta_n) \end{aligned}$$

for $\eta \in \mathbb{R}^n$, where we have set $\eta = (\eta_1, \dots, \eta_n)$. Define

$$A = \left\{ \eta \in \mathbb{R}^n : \eta_i \geq 0, i = 1, \dots, n, \sum_{i=1}^n \eta_i \leq 1 \right\}.$$

We have to show that

$$(3.22) \quad \phi(\eta) \geq 0 \quad \text{for every } \eta \in A.$$

On performing the products on the right-hand side of (3.21), and rearranging the resulting terms, one can verify that

$$(3.23) \quad \begin{aligned} \phi(\eta) = & \eta_1^2 \left[1 + (-2) \sum_{i \neq 1} \eta_i + (-2)^2 \sum_{\substack{i_1 < i_2 \\ i_1, i_2 \neq 1}} \eta_{i_1} \eta_{i_2} + \dots \right. \\ & \dots + (-2)^k \sum_{\substack{i_1 < i_2 < \dots < i_k \\ i_1, \dots, i_k \neq 1}} \eta_{i_1} \eta_{i_2} \dots \eta_{i_k} + \dots + (-2)^{n-1} \eta_2 \dots \eta_n \left. \right] + \dots \\ & \vdots \\ & \dots + \eta_n^2 \left[1 + (-2) \sum_{i \neq n} \eta_i + (-2)^2 \sum_{\substack{i_1 < i_2 \\ i_1, i_2 \neq n}} \eta_{i_1} \eta_{i_2} + \dots \right. \\ & \dots + (-2)^k \sum_{\substack{i_1 < i_2 < \dots < i_k \\ i_1, \dots, i_k \neq n}} \eta_{i_1} \eta_{i_2} \dots \eta_{i_k} + \dots + (-2)^{n-1} \eta_1 \dots \eta_{n-1} \left. \right] \\ & + 1 + (-2) \sum_{i=1, \dots, n} \eta_i + (-2)^2 \sum_{i_1 < i_2} \eta_{i_1} \eta_{i_2} + (-2)^3 \sum_{i_1 < i_2 < i_3} \eta_{i_1} \eta_{i_2} \eta_{i_3} + \dots + (-2)^n \eta_1 \dots \eta_n. \end{aligned}$$

Let us denote by \mathcal{S}_k , for $k = 1, \dots, n$, the elementary symmetric functions of the n numbers η_1, \dots, η_n . Namely,

$$\mathcal{S}_k = \sum_{i_1 < i_2 < \dots < i_k} \eta_{i_1} \eta_{i_2} \dots \eta_{i_k}.$$

Observe that

$$(3.24) \quad (1 - \mathcal{S}_1)^2 = \left(1 - \sum_{i=1}^n \eta_i \right)^2 = 1 - 2 \sum_{i=1}^n \eta_i + 2 \sum_{i_1 < i_2} \eta_{i_1} \eta_{i_2} + \sum_{i=1}^n \eta_i^2.$$

Moreover,

$$(3.25) \quad \begin{aligned} \mathcal{S}_1 \mathcal{S}_k &= \sum_{i=1, \dots, n} \eta_i \sum_{i_1 < \dots < i_k} \eta_{i_1} \eta_{i_2} \dots \eta_{i_k} \\ &= \eta_1^2 \sum_{\substack{i_1 < i_2 < \dots < i_{k-1} \\ i_1, \dots, i_{k-1} \neq 1}} \eta_{i_1} \eta_{i_2} \dots \eta_{i_{k-1}} + \dots + \eta_n^2 \sum_{\substack{i_1 < i_2 < \dots < i_{k-1} \\ i_1, \dots, i_{k-1} \neq n}} \eta_{i_1} \eta_{i_2} \dots \eta_{i_{k-1}} \\ &\quad + (k+1) \sum_{i_1 < i_2 < \dots < i_{k+1}} \eta_{i_1} \eta_{i_2} \dots \eta_{i_{k+1}} \end{aligned}$$

$$= \eta_1^2 \sum_{\substack{i_1 < i_2 < \dots < i_{k-1} \\ i_1, \dots, i_{k-1} \neq 1}} \eta_{i_1} \eta_{i_2} \cdots \eta_{i_{k-1}} + \dots + \eta_n^2 \sum_{\substack{i_1 < i_2 < \dots < i_{k-1} \\ i_1, \dots, i_{k-1} \neq n}} \eta_{i_1} \eta_{i_2} \cdots \eta_{i_{k-1}} + (k+1)\mathcal{S}_{k+1}$$

for $k = 2, \dots, n-1$, and

$$(3.26) \quad \mathcal{S}_1 \mathcal{S}_n = \left(\sum_{i=1, \dots, n} \eta_i \right) \eta_1 \cdots \eta_n = \eta_1^2 \eta_2 \cdots \eta_n + \dots + \eta_n^2 \eta_1 \cdots \eta_{n-1}.$$

On making use of equations (3.24), (3.25) and (3.26), one can combine the terms on the right-hand side of equation (3.23) and infer that

$$(3.27) \quad \phi(\eta) = (1 - \mathcal{S}_1) \left[1 + \sum_{k=1}^n (-1)^k 2^{k-1} \mathcal{S}_k \right] + \sum_{k=3}^n (-1)^{k-1} (k-2) 2^{k-2} \mathcal{S}_k.$$

Since $\mathcal{S}_1 = \sum_{i=1}^n \eta_i$, we have that

$$(3.28) \quad 1 - \mathcal{S}_1 \geq 0 \quad \text{for } \eta \in A.$$

The sums on the right-hand side of equation (3.27) can be estimated from below via the inequality

$$(3.29) \quad \mathcal{S}_{k+1} \leq \frac{n-k}{n(k+1)} \mathcal{S}_k \mathcal{S}_1 \leq \frac{n-k}{n(k+1)} \mathcal{S}_k \quad \text{for } \eta \in A,$$

and $k = 1, \dots, n-1$. Note that the second inequality in (3.29) holds by (3.28), whereas the first one follows via an iterated use of Newton's inequality [HLP, Theorem 51]. We claim that

$$(3.30) \quad 1 + \sum_{k=1}^n (-1)^k 2^{k-1} \mathcal{S}_k = 1 - \mathcal{S}_1 + \sum_{k=2}^n (-1)^k 2^{k-1} \mathcal{S}_k \geq 0 \quad \text{for } \eta \in A.$$

Indeed, by (3.29),

$$(3.31) \quad 2^{2h-1} \mathcal{S}_{2h} - 2^{2h} \mathcal{S}_{2h+1} \geq 0,$$

if $1 \leq h \leq \frac{n-1}{2}$. When n is odd, the sum starting from $k=2$ in (3.30) is exhausted by differences of the form appearing in (3.31). When n is even, this sum contains an additional nonnegative term. Hence, inequality (3.30) follows. We next observe that

$$(3.32) \quad \sum_{k=3}^n (-1)^{k-1} (k-2) 2^{k-2} \mathcal{S}_k \geq 0 \quad \text{for } \eta \in A.$$

Actually, inequality (3.29) again ensures that

$$(3.33) \quad (2h-1) 2^{2h-1} \mathcal{S}_{2h+1} - 2h 2^{2h} \mathcal{S}_{2h+2} \geq 0,$$

if $1 \leq h \leq \frac{n-2}{2}$. When n is even, the sum in (3.32) is exhausted by differences of the form appearing in (3.33). When n is odd, this sum contains an additional nonnegative term. Inequality (3.32) is thus established. Inequality (3.22) follows from (3.27), via (3.28), (3.30) and (3.32). Note that, in fact,

$$\min_{\eta \in A} \phi(\eta) = 0,$$

inasmuch as $\phi(\eta) = 0$ whenever η is a vector all of whose components vanish, but just one, and the latter equals one. The proof is complete. \square

4 Global estimates

This section is devoted to proving Theorems 2.3 and 2.4. As a preliminary, we briefly discuss the notion of generalized solutions adopted in our results, and recall some of their basic properties.

When the function f appearing on the right-hand side of the equation in problems (2.6) or (2.7) has a sufficiently high degree of summability to belong to the dual of the Sobolev type space associated with the function a , weak solutions to the relevant problems are well defined. In particular, the existence and uniqueness of these solutions can be established via standard monotonicity methods. We are not going to give details in this connection, since they are not needed for our purposes, and refer the interested reader to [CiMa3] for an account on this issue. We rather focus on the case when f merely belongs to $L^q(\Omega)$ for any $q \geq 1$. A definition of generalized solution in this case involves the use of spaces that consist of functions whose truncations are weakly differentiable. Specifically, given any $t > 0$, let $T_t : \mathbb{R} \rightarrow \mathbb{R}$ denote the function defined as $T_t(s) = s$ if $|s| \leq t$, and $T_t(s) = t \operatorname{sign}(s)$ if $|s| > t$. We set

$$(4.1) \quad \mathcal{T}_{\text{loc}}^{1,1}(\Omega) = \left\{ u \text{ is measurable in } \Omega : T_t(u) \in W_{\text{loc}}^{1,1}(\Omega) \text{ for every } t > 0 \right\}.$$

The spaces $\mathcal{T}^{1,1}(\Omega)$ and $\mathcal{T}_0^{1,1}(\Omega)$ are defined accordingly, on replacing $W_{\text{loc}}^{1,1}(\Omega)$ with $W^{1,1}(\Omega)$ and $W_0^{1,1}(\Omega)$, respectively, on the right-hand side of (4.1).

If $u \in \mathcal{T}_{\text{loc}}^{1,1}(\Omega)$, there exists a (unique) measurable function $Z_u : \Omega \rightarrow \mathbb{R}^n$ such that

$$(4.2) \quad \nabla(T_t(u)) = \chi_{\{|u| < t\}} Z_u \quad \text{a.e. in } \Omega$$

for every $t > 0$ – see [BBGGPV, Lemma 2.1]. Here χ_E denotes the characteristic function of the set E . As already mentioned in Section 1, with abuse of notation, for every $u \in \mathcal{T}_{\text{loc}}^{1,1}(\Omega)$ we denote Z_u simply by ∇u .

Assume that $f \in L^q(\Omega)$ for some $q \geq 1$. A function $u \in \mathcal{T}_0^{1,1}(\Omega)$ will be called a generalized solution to the Dirichlet problem (2.6) if $a(|\nabla u|)\nabla u \in L^1(\Omega)$,

$$(4.3) \quad \int_{\Omega} a(|\nabla u|)\nabla u \cdot \nabla \varphi \, dx = \int_{\Omega} f \varphi \, dx$$

for every $\varphi \in C_0^\infty(\Omega)$, and there exists a sequence $\{f_k\} \subset C_0^\infty(\Omega)$ such that $f_k \rightarrow f$ in $L^q(\Omega)$ and the sequence of weak solutions $\{u_k\}$ to the problems (2.6) with f replaced by f_k satisfies

$$u_k \rightarrow u \quad \text{a.e. in } \Omega.$$

In (4.3), ∇u stands for the function Z_u fulfilling (4.2).

By [CiMa3], there exists a unique generalized solution u to problem (2.6), and

$$(4.4) \quad \|a(|\nabla u|)\nabla u\|_{L^1(\Omega)} \leq C \|f\|_{L^1(\Omega)}$$

for some constant $C = C(|\Omega|, n, i_a, s_a)$. Moreover, if $\{f_k\}$ is any sequence as above, and $\{u_k\}$ is the associated sequence of weak solutions, then

$$(4.5) \quad u_k \rightarrow u \quad \text{and} \quad \nabla u_k \rightarrow \nabla u \quad \text{a.e. in } \Omega,$$

up to subsequences.

The definition of generalized solutions to the Neumann problem (2.7) can be given analogously. Assume that $f \in L^q(\Omega)$ for some $q \geq 1$, and satisfies (2.8). A function $u \in \mathcal{T}^{1,1}(\Omega)$ will

be called a generalized solution to problem (2.7) if $a(|\nabla u|)\nabla u \in L^1(\Omega)$, equation (4.3) holds for every $\varphi \in C^\infty(\Omega) \cap W^{1,\infty}(\Omega)$, and there exists a sequence $\{f_k\} \subset C_0^\infty(\Omega)$, with $\int_\Omega f_k(x) dx = 0$ for $k \in \mathbb{N}$, such that $f_k \rightarrow f$ in $L^q(\Omega)$ and the sequence of (suitably normalized by additive constants) weak solutions $\{u_k\}$ to the problems (2.7) with f replaced by f_k satisfies

$$u_k \rightarrow u \quad \text{a.e. in } \Omega.$$

Owing to [CiMa3], if Ω is a bounded Lipschitz domain, then there exists a unique (up to additive constants) generalized solution u to problem (2.7), and

$$(4.6) \quad \|a(|\nabla u|)\nabla u\|_{L^1(\Omega)} \leq C\|f\|_{L^1(\Omega)}$$

for some constant $C = C(L_\Omega, d_\Omega, n, i_a, s_a)$. Moreover, $\{f_k\}$ is any sequence as above, and $\{u_k\}$ is the associated sequence of (normalized) weak solutions, then

$$(4.7) \quad u_k \rightarrow u \quad \text{and} \quad \nabla u_k \rightarrow \nabla u \quad \text{a.e. in } \Omega,$$

up to subsequences.

We conclude our background by recalling the definitions of Marcinkiewicz, and, more generally, Lorentz spaces that enter in our results. Let (\mathcal{R}, m) be a σ -finite non atomic measure space. Given $q \in [1, \infty]$, the Marcinkiewicz space $L^{q,\infty}(\mathcal{R}, m)$, also called weak $L^q(\mathcal{R}, m)$ space, is the Banach function space endowed with the norm defined as

$$(4.8) \quad \|\psi\|_{L^{q,\infty}(\mathcal{R}, m)} = \sup_{s \in (0, m(\mathcal{R}))} s^{\frac{1}{q}} \psi^{**}(s)$$

for a measurable function ψ on \mathcal{R} . Here, ψ^* denotes the decreasing rearrangement of ψ , and $\psi^{**}(s) = \int_0^s \psi^*(r) dr$ for $s > 0$. The space $L^{q,\infty}(\mathcal{R}, m)$ is borderline in the family of Lorentz spaces $L^{q,\sigma}(\mathcal{R})$, with $q \in [1, \infty]$ and $\sigma \in [1, \infty]$, that are equipped with the norm given by

$$(4.9) \quad \|\psi\|_{L^{q,\sigma}(\mathcal{R})} = \|s^{\frac{1}{q} - \frac{1}{\sigma}} \psi^{**}(s)\|_{L^\sigma(0, m(\mathcal{R}))}$$

for ψ as above. Indeed, one has that

$$(4.10) \quad L^{q,\sigma_1}(\mathcal{R}) \subsetneq L^{q,\sigma_2}(\mathcal{R}) \quad \text{if } q \in [1, \infty] \text{ and } 1 \leq \sigma_1 < \sigma_2 \leq \infty.$$

Also

$$L^{q,q}(\mathcal{R}) = L^q(\mathcal{R}) \quad \text{for } q \in (1, \infty],$$

up to equivalent norms. In the limiting case when $q = 1$, the Marcinkiewicz type space $L^{1,\infty} \log L(\mathcal{R}, m)$ comes into play in our results as a replacement for $L^{1,\infty}(\mathcal{R}, m)$, which agrees with $L^1(\mathcal{R}, m)$. A norm in $L^{1,\infty} \log L(\mathcal{R}, m)$ is defined as

$$(4.11) \quad \|\psi\|_{L^{1,\infty} \log L(\mathcal{R}, m)} = \sup_{s \in (0, m(\mathcal{R}))} s \log \left(1 + \frac{C}{s}\right) \psi^{**}(s),$$

for any constant $C > m(\mathcal{R})$. Different constants C result in equivalent norms in (4.11).

Proof of Theorem 2.4. We begin with a proof in the case when u is the generalized solution to the Dirichlet problem (2.6). The needed variants for the solution to the Neumann problem (2.7) are indicated at the end.

The proof is split in steps. In Step 1 we establish the result under some additional regularity assumptions on a , Ω and f . The remaining steps are devoted to removing the extra assumptions,

by approximation.

Step 1. Here, we assume that the following extra conditions are in force:

$$(4.12) \quad f \in C_0^\infty(\Omega);$$

$$(4.13) \quad \partial\Omega \in C^\infty;$$

$$(4.14) \quad a : [0, \infty) \rightarrow [0, \infty) \quad \text{and} \quad c_1 \leq a(t) \leq c_2 \quad \text{for } t \geq 0,$$

for some constants $c_2 > c_1 > 0$; the function $\mathcal{A} : \mathbb{R}^n \rightarrow [0, \infty)$, defined as $\mathcal{A}(\eta) = a(|\eta|)$ for $\eta \in \mathbb{R}^n$, is such that

$$(4.15) \quad \mathcal{A} \in C^\infty(\mathbb{R}^n).$$

Standard regularity results then ensure that the solution u to problem (2.6) is classical, and $u \in C^\infty(\bar{\Omega})$ (see e.g. [CiMa1, Proof of Theorem 1.1] for details). Let $\xi \in C_0^\infty(\mathbb{R}^n)$. Squaring both sides of the equation in (2.6), multiplying through the resulting equation by ξ^2 , integrating both sides over Ω , and making use of inequality (3.1) yield

$$(4.16) \quad \begin{aligned} \int_{\Omega} \xi^2 f^2 dx &= \int_{\Omega} \xi^2 (\operatorname{div}(a(|\nabla u|)\nabla u))^2 dx \\ &\geq \int_{\Omega} \xi^2 \left[\sum_{j=1}^n (a(|\nabla u|)^2 u_{x_j} \Delta u)_{x_j} - \sum_{i=1}^n \left(a(|\nabla u|)^2 \sum_{j=1}^n u_{x_j} u_{x_i x_j} \right)_{x_i} \right] dx \\ &\quad + C \int_{\Omega} \xi^2 a(|\nabla u|)^2 |\nabla^2 u|^2 dx \end{aligned}$$

for some constant $C = C(n, i_a)$. Now, [Gr, Equation (3,1,1,2)] tells us that

$$(4.17) \quad \begin{aligned} \Delta u \frac{\partial u}{\partial \nu} - \sum_{i,j=1}^n u_{x_i x_j} u_{x_i} \nu_j \\ = \operatorname{div}_T \left(\frac{\partial u}{\partial \nu} \nabla_T u \right) - \operatorname{tr} \mathcal{B} \left(\frac{\partial u}{\partial \nu} \right)^2 - \mathcal{B}(\nabla_T u, \nabla_T u) - 2 \nabla_T u \cdot \nabla_T \frac{\partial u}{\partial \nu} \end{aligned} \quad \text{on } \partial\Omega,$$

where \mathcal{B} is the second fundamental form on $\partial\Omega$, $\operatorname{tr} \mathcal{B}$ is its trace, div_T and ∇_T denote the divergence and the gradient operator on $\partial\Omega$, respectively, and ν_j stands for the j -th component of ν . From the divergence theorem and equation (4.17) we deduce that

$$(4.18) \quad \begin{aligned} \int_{\Omega} \xi^2 \left[\sum_{j=1}^n (a(|\nabla u|)^2 u_{x_j} \Delta u)_{x_j} - \sum_{i=1}^n (a(|\nabla u|)^2 \sum_{j=1}^n u_{x_j} u_{x_i x_j})_{x_i} \right] dx \\ = \int_{\partial\Omega} \xi^2 a(|\nabla u|)^2 \left[\Delta u \frac{\partial u}{\partial \nu} - \sum_{i,j=1}^n u_{x_i x_j} u_{x_i} \nu_j \right] d\mathcal{H}^{n-1}(x) \\ \quad - 2 \int_{\Omega} a(|\nabla u|)^2 \xi \nabla \xi \cdot \left[\Delta u \nabla u - \sum_{j=1}^n u_{x_j} \nabla u_{x_j} \right] dx \\ = \int_{\partial\Omega} \xi^2 a(|\nabla u|)^2 \left[\operatorname{div}_T \left(\frac{\partial u}{\partial \nu} \nabla_T u \right) - \operatorname{tr} \mathcal{B} \left(\frac{\partial u}{\partial \nu} \right)^2 \right] \end{aligned}$$

$$\begin{aligned}
& - \mathcal{B}(\nabla_T u, \nabla_T u) - 2 \nabla_T u \cdot \nabla_T \frac{\partial u}{\partial \nu} \Big] d\mathcal{H}^{n-1}(x) \\
& - 2 \int_{\Omega} a(|\nabla u|)^2 \xi \nabla \xi \cdot \left[\Delta u \nabla u - \sum_{j=1}^n u_{x_j} \nabla u_{x_j} \right] dx.
\end{aligned}$$

By Young's inequality, there exists a constant $C = C(n)$ such that

$$\begin{aligned}
(4.19) \quad & 2 \left| \int_{\Omega} a(|\nabla u|)^2 \xi \nabla \xi \cdot \left[\Delta u \nabla u - \sum_{j=1}^n u_{x_j} \nabla u_{x_j} \right] dx \right| \\
& \leq \varepsilon C \int_{\Omega} \xi^2 a(|\nabla u|)^2 |\nabla^2 u|^2 dx + \frac{C}{\varepsilon} \int_{\Omega} |\nabla \xi|^2 a(|\nabla u|)^2 |\nabla u|^2 dx
\end{aligned}$$

for every $\varepsilon > 0$. Equations (4.16), (4.18) and (4.19) ensure that there exist constants $C = C(n, i_a)$ and $C' = C'(n, i_a)$ such that

$$\begin{aligned}
(4.20) \quad & C(1 - \varepsilon) \int_{\Omega} \xi^2 a(|\nabla u|)^2 |\nabla^2 u|^2 dx \leq \int_{\Omega} \xi^2 f^2 dx + \frac{C'}{\varepsilon} \int_{\Omega} |\nabla \xi|^2 a(|\nabla u|)^2 |\nabla u|^2 dx \\
& + \left| \int_{\partial\Omega} \xi^2 a(|\nabla u|)^2 \left[\operatorname{div}_T \left(\frac{\partial u}{\partial \nu} \nabla_T u \right) - \operatorname{tr} \mathcal{B} \left(\frac{\partial u}{\partial \nu} \right)^2 \right. \right. \\
& \quad \left. \left. - \mathcal{B}(\nabla_T u, \nabla_T u) - 2 \nabla_T u \cdot \nabla_T \frac{\partial u}{\partial \nu} \right] d\mathcal{H}^{n-1}(x) \right|.
\end{aligned}$$

On the other hand, owing to the Dirichlet boundary condition, $\nabla_T u = 0$ on $\partial\Omega$, and hence

$$\begin{aligned}
(4.21) \quad & \left| \int_{\partial\Omega} \xi^2 a(|\nabla u|)^2 \left[\operatorname{div}_T \left(\frac{\partial u}{\partial \nu} \nabla_T u \right) - \operatorname{tr} \mathcal{B} \left(\frac{\partial u}{\partial \nu} \right)^2 \right. \right. \\
& \quad \left. \left. - \mathcal{B}(\nabla_T u, \nabla_T u) - 2 \nabla_T u \cdot \nabla_T \frac{\partial u}{\partial \nu} \right] d\mathcal{H}^{n-1}(x) \right| \\
& = \left| - \int_{\partial\Omega} \xi^2 a(|\nabla u|)^2 \operatorname{tr} \mathcal{B} \left(\frac{\partial u}{\partial \nu} \right)^2 d\mathcal{H}^{n-1}(x) \right| \\
& \leq C \int_{\partial\Omega} \xi^2 a(|\nabla u|)^2 |\nabla u|^2 |\mathcal{B}| d\mathcal{H}^{n-1}(x),
\end{aligned}$$

for some constant $C = C(n)$. Here, $|\mathcal{B}|$ denotes the norm of \mathcal{B} . Next, assume that

$$(4.22) \quad \xi \in C_0^\infty(B_r(x))$$

for some $x \in \overline{\Omega}$ and $r > 0$.

First, suppose that $x \in \partial\Omega$. Let us distinguish the cases when $n \geq 3$ or $n = 2$. When $n \geq 3$, set

$$(4.23) \quad Q(r) = \sup_{x \in \partial\Omega} \sup_{E \subset \partial\Omega \cap B_r(x)} \frac{\int_E |\mathcal{B}| d\mathcal{H}^{n-1}(y)}{\operatorname{cap}(E)} \quad \text{for } r > 0,$$

where $\operatorname{cap}(E)$ stands for the capacity of the set E given by

$$(4.24) \quad \operatorname{cap}(E) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla v|^2 dy : v \in C_0^1(\mathbb{R}^n), v \geq 1 \text{ on } E \right\}.$$

A weighted trace inequality on half-balls [Maz1, Maz2] (see also [Maz6, Section 2.5.2]), combined with a local flattening argument for Ω on a half-space, and with an even-extension argument from a half-space into \mathbb{R}^n , ensures that there exists a constant $C = C(L_\Omega, d_\Omega, n)$ such that

$$(4.25) \quad \int_{\partial\Omega \cap B_r(x)} v^2 |\mathcal{B}| d\mathcal{H}^{n-1}(y) \leq CQ(r) \int_{\Omega \cap B_r(x)} |\nabla v|^2 dy$$

for every $x \in \partial\Omega$, $r > 0$ and $v \in C_0^1(B_r(x))$. Furthermore, a standard trace inequality tells us that there exists a constant $C = C(L_\Omega, d_\Omega, n)$ such that

$$(4.26) \quad \left(\int_{\partial\Omega \cap B_r(x)} |v|^{\frac{2(n-1)}{n-2}} d\mathcal{H}^{n-1}(y) \right)^{\frac{n-2}{n-1}} \leq C \int_{\Omega \cap B_r(x)} |\nabla v|^2 dy$$

for every $x \in \partial\Omega$, $r > 0$ and $v \in C_0^1(B_r(x))$. By definition (4.24), choosing trial functions v in (4.26) such that $v \geq 1$ on E implies that

$$(4.27) \quad \mathcal{H}^{n-1}(E)^{\frac{n-2}{n-1}} \leq C \text{cap}(E)$$

for every set $E \subset \partial\Omega$. By a basic property of the decreasing rearrangement (with respect to \mathcal{H}^{n-1}) [BeSh, Chapter 2, Lemma 2.1], and (4.27),

$$(4.28) \quad \begin{aligned} Q(r) &\leq \sup_{x \in \partial\Omega} \sup_{E \subset \partial\Omega \cap B_r(x)} \frac{\int_0^{\mathcal{H}^{n-1}(E)} (|\mathcal{B}|_{|\partial\Omega \cap B_r(x)})^*(r) dr}{\text{cap}(E)} \\ &\leq C \sup_{x \in \partial\Omega} \sup_{s > 0} \frac{\int_0^s (|\mathcal{B}|_{|\partial\Omega \cap B_r(x)})^*(r) dr}{s^{\frac{n-2}{n-1}}} = C \sup_{x \in \partial\Omega} \|\mathcal{B}\|_{L^{n-1, \infty}(\partial\Omega \cap B_r(x))} \end{aligned}$$

for some constant $C = C(L_\Omega, d_\Omega, n)$, for every $x \in \partial\Omega$ and $r > 0$. An application of inequality (4.25) with $v = \xi a(|\nabla u|)u_{x_i}$, for $i = 1, \dots, n$, yields, via (4.28),

$$(4.29) \quad \begin{aligned} &\int_{\partial\Omega} \xi^2 a(|\nabla u|)^2 |\nabla u|^2 |\mathcal{B}| d\mathcal{H}^{n-1}(x) \\ &\leq C \sup_{x \in \partial\Omega} \|\mathcal{B}\|_{L^{n-1, \infty}(\partial\Omega \cap B_r(x))} \left(\int_{\Omega} \xi^2 a(|\nabla u|)^2 |\nabla^2 u|^2 dx + \int_{\Omega} |\nabla \xi|^2 a(|\nabla u|)^2 |\nabla u|^2 dx \right) \end{aligned}$$

for some constant $C = C(L_\Omega, d_\Omega, n, s_a)$. Note that here we have made use of the second inequality in (2.2) to infer that

$$(4.30) \quad |\nabla(a(|\nabla u|)u_{x_i})| \leq C a(|\nabla u|) |\nabla^2 u| \quad \text{in } \Omega,$$

for $i = 1, \dots, n$, and for some constant $C = C(n, s_a)$. Combining equations (4.20) and (4.29) tells us that

$$(4.31) \quad \begin{aligned} &\left[C_1(1 - \varepsilon) - C_2 \sup_{x \in \partial\Omega} \|\mathcal{B}\|_{L^{n-1, \infty}(\partial\Omega \cap B_r(x))} \right] \int_{\Omega} \xi^2 a(|\nabla u|)^2 |\nabla^2 u|^2 dx \\ &\leq \int_{\Omega} \xi^2 f^2 dx + \left[C_2 \sup_{x \in \partial\Omega} \|\mathcal{B}\|_{L^{n-1, \infty}(\partial\Omega \cap B_r(x))} + \frac{C_3}{\varepsilon} \right] \int_{\Omega} |\nabla \xi|^2 a(|\nabla u|)^2 |\nabla u|^2 dx \end{aligned}$$

for some constants $C_1 = C_1(n, i_a)$, $C_2 = C_2(L_\Omega, d_\Omega, n, s_a)$ and $C_3 = C_3(n)$. If condition (2.11) is fulfilled with $c = \frac{C_1}{C_2}$, then there exists $r_0 > 0$ such that

$$C_1(1 - \varepsilon) - C_2 \sup_{x \in \partial\Omega} \|\mathcal{B}\|_{L^{n-1, \infty}(\partial\Omega \cap B_r(x))} > 0$$

if $0 < r \leq r_0$ and ε is sufficiently small. Therefore, by inequality (4.31),

$$(4.32) \quad \int_{\Omega} \xi^2 a(|\nabla u|)^2 |\nabla^2 u|^2 dx \leq C \int_{\Omega} \xi^2 f^2 dx + C \int_{\Omega} |\nabla \xi|^2 a(|\nabla u|)^2 |\nabla u|^2 dx$$

for some constant $C = C(L_{\Omega}, d_{\Omega}, n, i_a, s_a)$, if $0 < r \leq r_0$ in (4.22).

In the case when $n = 2$, define

$$(4.33) \quad Q_1(r) = \sup_{x \in \partial\Omega} \sup_{E \subset \partial\Omega \cap B_r(x)} \frac{\int_E |\mathcal{B}| d\mathcal{H}^1(y)}{\text{cap}_{B_1(x)}(E)} \quad \text{for } r \in (0, 1),$$

where $\text{cap}_{B_1(x)}(E)$ stands for the capacity of the set E given by

$$(4.34) \quad \text{cap}_{B_1(x)}(E) = \inf \left\{ \int_{B_1(x)} |\nabla v|^2 dy : v \in C_0^1(B_1(x)), v \geq 1 \text{ on } E \right\}.$$

A counterpart of inequality (4.25) reads

$$(4.35) \quad \int_{\partial\Omega \cap B_r(x)} v^2 |\mathcal{B}| d\mathcal{H}^1(y) \leq C Q_1(r) \int_{\Omega \cap B_r(x)} |\nabla v|^2 dy$$

for every $x \in \partial\Omega$, $r \in (0, 1)$ and $v \in C_0^1(B_r(x))$, where $C = C(L_{\Omega}, d_{\Omega})$.

A borderline version of the trace inequality – see e.g. [AdHe, Section 7.6.4] – ensures that there exists a constant $C = C(L_{\Omega}, d_{\Omega}, n)$ such that

$$(4.36) \quad \sup_{E \subset \partial\Omega \cap B_1(x)} \frac{\left(\frac{1}{\mathcal{H}^1(E)} \int_E v d\mathcal{H}^1(y) \right)^2}{\log \left(1 + \frac{\mathcal{H}^1(\partial\Omega \cap B_1(x))}{\mathcal{H}^1(E)} \right)} \leq C \int_{\Omega \cap B_1(x)} |\nabla v|^2 dy$$

for every $x \in \partial\Omega$, and $v \in C_0^1(B_1(x))$. Notice that the left-hand side of (4.36) is equivalent to the norm in an Orlicz space associated with the Young function $e^{t^2} - 1$. The choice of trial functions v in (4.36) such that $v \geq 1$ on E yields, via definition (4.34),

$$(4.37) \quad \frac{1}{\log \left(1 + \frac{C}{\mathcal{H}^1(E)} \right)} \leq C \text{cap}_{B_1(x)}(E),$$

for some constant $C = C(L_{\Omega}, d_{\Omega})$, and for every set $E \subset \partial\Omega \cap B_1(x)$. Thanks to (4.37) and to the Hardy-Littlewood inequality again,

$$(4.38) \quad \begin{aligned} Q_1(r) &\leq \sup_{x \in \partial\Omega} \sup_{E \subset \partial\Omega \cap B_r(x)} \frac{\int_0^{\mathcal{H}^1(E)} (|\mathcal{B}|_{|\partial\Omega \cap B_r(x)}|^*)^*(r) dr}{\text{cap}_{B_1(x)}(E)} \\ &\leq C \sup_{x \in \partial\Omega} \sup_{s \in (0, \mathcal{H}^1(\partial\Omega \cap B_r(x)))} \log \left(1 + \frac{C}{s} \right) \int_0^s (|\mathcal{B}|_{|\partial\Omega \cap B_r(x)}|^*)^*(r) dr \\ &= C \sup_{x \in \partial\Omega} \|\mathcal{B}\|_{L^{1, \infty} \log L(\partial\Omega \cap B_r(x))} \end{aligned}$$

for some constant $C = C(L_{\Omega}, d_{\Omega})$, and for $r \in (0, 1)$. On exploiting (4.38) instead of (4.28), and arguing as in the case when $n \geq 3$, yield (4.32) also for $n = 2$.

When $B_r(x) \subset\subset \Omega$, the derivation of (4.32) is even simpler, and follows directly from (4.16), (4.18) and (4.19), since the boundary integral on the rightmost side of (4.18) vanishes in this case.

Now, let $\{B_{r_k}\}_{k \in K}$ be a finite covering of $\bar{\Omega}$ by balls B_{r_k} , with $r_k \leq r_0$, such that either B_{r_k} is centered on $\partial\Omega$, or $B_{r_k} \subset \subset \Omega$. Note that this covering can be chosen in such a way that the multiplicity of overlapping of the balls B_{r_k} only depends on n . Let $\{\xi_k\}_{k \in K}$ be a family of functions such that $\xi_k \in C_0^\infty(B_{r_k})$ and $\{\xi_k^2\}_{k \in K}$ is a partition of unity associated with the covering $\{B_{r_k}\}_{k \in K}$. Thus $\sum_{k \in K} \xi_k^2 = 1$ in $\bar{\Omega}$. On applying inequality (4.32) with $\xi = \xi_k$ for each k , and adding the resulting inequalities one obtains that

$$(4.39) \quad \int_{\Omega} a(|\nabla u|)^2 |\nabla^2 u|^2 dx \leq C \int_{\Omega} f^2 dx + C \int_{\Omega} a(|\nabla u|)^2 |\nabla u|^2 dx$$

for some constant $C = C(L_\Omega, d_\Omega, n, i_a, s_a)$.

A version of the Sobolev inequality entails that, for every $\sigma > 0$, there exists a constant $C = C(L_\Omega, d_\Omega, n, \sigma)$ such that

$$(4.40) \quad \int_{\Omega} v^2 dx \leq \sigma \int_{\Omega} |\nabla v|^2 dx + C \left(\int_{\Omega} |v| dx \right)^2$$

for every $v \in W^{1,2}(\Omega)$ (see e.g. [Maz6, Proof of Theorem 1.4.6/1]). Applying inequality (4.40) with $v = a(|\nabla u|)u_{x_i}$, $i = 1, \dots, n$, an recalling (4.30) tell us that

$$(4.41) \quad \int_{\Omega} a(|\nabla u|)^2 |\nabla u|^2 dx \leq \sigma C_1 \int_{\Omega} a(|\nabla u|)^2 |\nabla^2 u|^2 dx + C_2 \left(\int_{\Omega} a(|\nabla u|) |\nabla u| dx \right)^2$$

for some constant $C_1 = C_1(n, s_a)$ and $C_2 = C_2(L_\Omega, d_\Omega, n, s_a, \sigma)$. On choosing $\sigma = \frac{1}{2CC_1}$, where C is the constant appearing in (4.39), and combining inequalities (4.39), (4.41) and (4.4) we conclude that

$$(4.42) \quad \int_{\Omega} a(|\nabla u|)^2 |\nabla^2 u|^2 dx \leq C \int_{\Omega} f^2 dx$$

for some constant $C = C(L_\Omega, d_\Omega, n, i_a, s_a)$. Inequalities (4.41), (4.42) and (4.4) imply, via (4.30), that

$$(4.43) \quad \|a(|\nabla u|)\nabla u\|_{W^{1,2}(\Omega)} \leq C \|f\|_{L^2(\Omega)}$$

for some constant $C = C(L_\Omega, d_\Omega, n, i_a, s_a)$. In particular, the dependence of the constant C in (4.43) is in fact just through an upper bound for the quantities L_Ω, d_Ω, s_a , and through a lower bound for i_a . This is crucial in view of the next steps.

Step 2. Here we remove assumptions (4.14) and (4.15). To this purpose, we make use of a family of functions $\{a_\varepsilon\}_{\varepsilon \in (0,1)}$, with $a_\varepsilon : [0, \infty) \rightarrow (0, \infty)$, satisfying the following properties:

$$(4.44) \quad a_\varepsilon : [0, \infty) \rightarrow [0, \infty) \quad \text{and} \quad \varepsilon \leq a_\varepsilon(t) \leq \varepsilon^{-1} \quad \text{for } t \geq 0;$$

$$(4.45) \quad \min\{i_a, 0\} \leq i_{a_\varepsilon} \leq s_{a_\varepsilon} \leq \max\{s_a, 0\};$$

$$(4.46) \quad \lim_{\varepsilon \rightarrow 0} a_\varepsilon(|\xi|)\xi = a(|\xi|)\xi \quad \text{uniformly in } \{\xi \in \mathbb{R}^n : |\xi| \leq M\} \text{ for every } M > 0;$$

the function $\mathcal{A}_\varepsilon : \mathbb{R}^n \rightarrow [0, \infty)$, defined as $\mathcal{A}_\varepsilon(\eta) = a_\varepsilon(|\eta|)$ for $\eta \in \mathbb{R}^n$, is such that

$$(4.47) \quad \mathcal{A}_\varepsilon \in C^\infty(\mathbb{R}^n).$$

The construction of a family of functions enjoying these properties can be accomplished on combining [CiMa1, Lemma 3.3] and [CiMa2, Lemma 4.5]. Now, let u_ε be the solution to the problem

$$(4.48) \quad \begin{cases} -\operatorname{div}(a_\varepsilon(|\nabla u_\varepsilon|)\nabla u_\varepsilon) = f(x) & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Owing to (4.44) and (4.47), the assumptions of Step 1 are fulfilled by problem (4.48). Thus, as a consequence of (4.43), there exists a constant $C = C(L_\Omega, d_\Omega, n, i_a, s_a)$ such that

$$(4.49) \quad \|a_\varepsilon(|\nabla u_\varepsilon|)\nabla u_\varepsilon\|_{W^{1,2}(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

for $\varepsilon \in (0, 1)$. Observe that the constant C in (4.49) is actually independent of ε , thanks to (4.45). By (4.49), there exists a sequence $\{\varepsilon_k\}$ and a function $U : \Omega \rightarrow \mathbb{R}^n$ such that $U \in W^{1,2}(\Omega)$,

$$(4.50) \quad a_{\varepsilon_k}(|\nabla u_{\varepsilon_k}|)\nabla u_{\varepsilon_k} \rightarrow U \quad \text{in } L^2(\Omega) \quad \text{and} \quad a_{\varepsilon_k}(|\nabla u_{\varepsilon_k}|)\nabla u_{\varepsilon_k} \rightharpoonup U \quad \text{in } W^{1,2}(\Omega),$$

where the arrow “ \rightharpoonup ” stands for weak convergence. On the other hand, a global estimate for $\|u_{\varepsilon_k}\|_{L^\infty(\Omega)}$ following from a result of [Ta], coupled with a local gradient estimate of [Li, Theorem 1.7] ensures that $u_{\varepsilon_k} \in C_{\text{loc}}^{1,\alpha}(\Omega)$, and that for any open set $\Omega' \subset\subset \Omega$ there exists a constant C such that

$$(4.51) \quad \|u_{\varepsilon_k}\|_{C^{1,\alpha}(\Omega')} \leq C$$

for $k \in \mathbb{N}$. Thus, there exists a function $v \in C^1(\Omega)$ such that, on taking, if necessary, a subsequence,

$$(4.52) \quad u_{\varepsilon_k} \rightarrow v \quad \text{and} \quad \nabla u_{\varepsilon_k} \rightarrow \nabla v \quad \text{pointwise in } \Omega.$$

In particular,

$$(4.53) \quad a(|\nabla v|)\nabla v = U,$$

and hence

$$(4.54) \quad a(|\nabla u|)\nabla u \in W^{1,2}(\Omega).$$

Testing the equation in (4.48) with any function $\varphi \in C_0^\infty(\Omega)$ yields

$$(4.55) \quad \int_\Omega a_{\varepsilon_k}(|\nabla u_{\varepsilon_k}|)\nabla u_{\varepsilon_k} \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx.$$

Owing to (4.50) and (4.53), on passing to the limit in (4.55) as $k \rightarrow \infty$ one deduces that

$$(4.56) \quad \int_\Omega a(|\nabla v|)\nabla v \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx.$$

Thus $v = u$, the weak solution to problem (2.6). Furthermore, by (4.49), we obtain via (4.50) and (4.53) that

$$(4.57) \quad \|a(|\nabla u|)\nabla u\|_{W^{1,2}(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

for some constant $C = C(L_\Omega, d_\Omega, n, i_a, s_a)$.

Step 3. Here, we remove assumption (4.13). Via smooth approximation of the functions which locally describe $\partial\Omega$, one can construct a sequence $\{\Omega_m\}$ of open sets in \mathbb{R}^n such that $\partial\Omega_m \in C^\infty$, $\Omega \subset \Omega_m$, $\lim_{m \rightarrow \infty} |\Omega_m \setminus \Omega| = 0$, and the Hausdorff distance between Ω_m and Ω tends to 0 as $m \rightarrow \infty$. Also, there exists a constant $C = C(\Omega)$ such that

$$(4.58) \quad L_{\Omega_m} \leq CL_\Omega \quad \text{and} \quad d_{\Omega_m} \leq Cd_\Omega$$

for $m \in \mathbb{N}$. Moreover, although smooth functions are neither dense in $W^2L^{n-1,\infty}$ if $n \geq 3$, nor in $W^2L^{1,\infty} \log L$ if $n = 2$, one has that

$$\sup_{x \in \partial\Omega} \|\mathcal{B}_m\|_{L^{n-1,\infty}(\partial\Omega_m \cap B_r(x))} \leq C \sup_{x \in \partial\Omega} \|\mathcal{B}\|_{L^{n-1,\infty}(\partial\Omega \cap B_r(x))} \quad \text{if } n \geq 3,$$

or

$$\sup_{x \in \partial\Omega} \|\mathcal{B}_m\|_{L^{1,\infty} \log L(\partial\Omega_m \cap B_r(x))} \leq C \sup_{x \in \partial\Omega} \|\mathcal{B}\|_{L^{1,\infty} \log L(\partial\Omega \cap B_r(x))} \quad \text{if } n = 2,$$

for some constant $C = C(\Omega)$, where \mathcal{B}_m denotes the second fundamental form on $\partial\Omega_m$. Let u_m be the weak solution to the Dirichlet problem

$$(4.59) \quad \begin{cases} -\operatorname{div}(a(|\nabla u_m|)\nabla u_m) = f(x) & \text{in } \Omega_m \\ u_m = 0 & \text{on } \partial\Omega_m, \end{cases}$$

where f still fulfils (4.12), and is extended by 0 outside Ω . By inequality (4.57) of Step 2,

$$(4.60) \quad \|a(|\nabla u_m|)\nabla u_m\|_{W^{1,2}(\Omega_m)} \leq C\|f\|_{L^2(\Omega_m)} = C\|f\|_{L^2(\Omega)},$$

the constant C being independent of m , by the properties of Ω_m mentioned above.

Thanks to (4.60), the sequence $\{a(|\nabla u_m|)\nabla u_m\}$ is bounded in $W^{1,2}(\Omega)$, and hence there exists a subsequence, still denoted by $\{u_m\}$ and a function $U : \Omega \rightarrow \mathbb{R}^n$ such that $U \in W^{1,2}(\Omega)$,

$$(4.61) \quad a(|\nabla u_m|)\nabla u_m \rightarrow U \quad \text{in } L^2(\Omega) \quad \text{and} \quad a(|\nabla u_m|)\nabla u_m \rightharpoonup U \quad \text{in } W^{1,2}(\Omega).$$

By the local gradient estimate recalled in Step 2, there exists $\alpha \in (0, 1)$ such that $u_m \in C_{\text{loc}}^{1,\alpha}(\Omega)$, and for every open set $\Omega' \subset\subset \Omega$ there exists a constant C , independent of m , such that

$$(4.62) \quad \|u_m\|_{C^{1,\alpha}(\Omega')} \leq C.$$

Thus, on taking, if necessary, a further subsequence,

$$(4.63) \quad u_m \rightarrow v \quad \text{and} \quad \nabla u_m \rightarrow \nabla v \quad \text{pointwise in } \Omega,$$

for some function $v \in C^1(\Omega)$. In particular,

$$(4.64) \quad a(|\nabla u_m|)\nabla u_m \rightarrow a(|\nabla v|)\nabla v \quad \text{pointwise in } \Omega.$$

By (4.64) and (4.61),

$$(4.65) \quad a(|\nabla v|)\nabla v = U \in W^{1,2}(\Omega).$$

Given any function $\varphi \in C_0^\infty(\Omega)$, on passing to the limit as $m \rightarrow \infty$ in the weak formulation of problem (4.59), namely in the equation

$$(4.66) \quad \int_{\Omega_m} a(|\nabla u_m|)\nabla u_m \cdot \nabla \varphi \, dx = \int_{\Omega_m} f \varphi \, dx,$$

we infer from (4.61) and (4.65) that

$$\int_{\Omega} a(|\nabla v|) \nabla v \cdot \nabla \varphi \, dx = \int_{\Omega} f \varphi \, dx.$$

Therefore, $u = v$, the weak solution to problem (2.6). Furthermore, owing to (4.60), (4.61) and (4.30),

$$(4.67) \quad \|a(|\nabla u|) \nabla u\|_{W^{1,2}(\Omega)} \leq C \|f\|_{L^2(\Omega)}$$

for some constant $C = C(L_{\Omega}, d_{\Omega}, n, i_a, s_a)$.

Step 4. We conclude by removing the remaining additional assumption (4.12). Let $f \in L^2(\Omega)$. Owing to (4.5), given any sequence $\{f_k\} \subset C_0^{\infty}(\Omega)$ such that $f_k \rightarrow f$ in $L^2(\Omega)$, the sequence $\{u_k\}$ of the weak solutions to the Dirichlet problems

$$(4.68) \quad \begin{cases} -\operatorname{div}(a(|\nabla u_k|) \nabla u_k) = f_k & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega, \end{cases}$$

fulfills

$$(4.69) \quad u_k \rightarrow u \quad \text{and} \quad \nabla u_k \rightarrow \nabla u \quad \text{a.e. in } \Omega.$$

By inequality (4.67) of the previous step, we have that $a(|\nabla u_k|) \nabla u_k \in W^{1,2}(\Omega)$, and there exist constants C_1 and C_2 , independent of k , such that

$$(4.70) \quad \|a(|\nabla u_k|) \nabla u_k\|_{W^{1,2}(\Omega)} \leq C_1 \|f_k\|_{L^2(\Omega)} \leq C_2 \|f\|_{L^2(\Omega)}.$$

Hence, the sequence $\{a(|\nabla u_k|) \nabla u_k\}$ is uniformly bounded in $W^{1,2}(\Omega)$, and there exists a subsequence, still indexed by k , and a function $U : \Omega \rightarrow \mathbb{R}^n$ such that $U \in W^{1,2}(\Omega)$ and

$$(4.71) \quad a_k(|\nabla u_k|) \nabla u_k \rightarrow U \quad \text{in } L^2(\Omega) \quad \text{and} \quad a_k(|\nabla u_k|) \nabla u_k \rightharpoonup U \quad \text{in } W^{1,2}(\Omega).$$

From (4.69) we thus infer that $a(|\nabla u|) \nabla u = U \in W^{1,2}(\Omega)$, and the second inequality in (2.14) follows via (4.70) and (4.71). The first inequality is easily verified, via (4.30). The statement concerning the solution to the Dirichlet problem (2.6) is thus fully proved.

We point out hereafter the changes required for the solution to the Neumann problem (2.7).

Step 1. The additional assumption (2.8) has to be coupled with (4.12). Moreover, since $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, the middle term in the chain (4.21) is replaced with

$$\left| - \int_{\partial\Omega} \xi^2 a(|\nabla u|)^2 \mathcal{B}(\nabla_T u, \nabla_T u) \, d\mathcal{H}^{n-1}(x) \right|.$$

Step 2. The Dirichlet boundary condition in problem (4.48) must, of course, be replaced with the Neumann condition $\frac{\partial u_{\varepsilon}}{\partial \nu} = 0$. The solution of the resulting Neumann problem is only unique up to additive constants. A bound of the form $\|u_{\varepsilon_k} - c_k\|_{L^{\infty}(\Omega)} \leq C$ now holds for a suitable sequence $\{c_k\}$ with $c_k \in \mathbb{R}$ [Ci1]. Hence, u_{ε_k} has to be replaced with $u_{\varepsilon_k} - c_k$ in equations (4.51) and (4.52). Moreover, the test functions φ in equation (4.55) now belong to $W^{1,\infty}(\Omega)$.

Step 3. The Dirichlet problem (4.59) has to be replaced with the Neumann problem with boundary condition $\frac{\partial u_m}{\partial \nu} = 0$. Accordingly, the corresponding sequence of solutions $\{u_m\}$ has to be normalized by a suitable sequence of additive constants.

Passage to the limit as $m \rightarrow \infty$ in equation (4.66) can be justified as follows. Extend any test function $\varphi \in W^{1,\infty}(\Omega)$ to a function in $W^{1,\infty}(\mathbb{R}^n)$, still denoted by φ . The left-hand side of equation (4.66) can be split as

$$(4.72) \quad \int_{\Omega_m} a(|\nabla u_m|) \nabla u_m \cdot \nabla \varphi \, dx = \int_{\Omega} a(|\nabla u_m|) \nabla u_m \cdot \nabla \varphi \, dx + \int_{\Omega_m \setminus \Omega} a(|\nabla u_m|) \nabla u_m \cdot \nabla \varphi \, dx .$$

The first integral on the right-hand side of (4.72) converges to

$$\int_{\Omega} a(|\nabla v|) \nabla v \cdot \nabla \varphi \, dx$$

as $m \rightarrow \infty$, owing to (4.61) and (4.65). The second integral tends to 0, by (4.60) and the fact that $|\Omega_m \setminus \Omega| \rightarrow 0$.

Step 4. The sequence of approximating functions $\{f_k\}$ has to fulfill the additional compatibility condition $\int_{\Omega} f_k(x) \, dx = 0$ for $k \in \mathbb{N}$. Moreover, the Dirichlet boundary condition in problem (4.68) has to be replaced with the Neumann condition $\frac{\partial u_k}{\partial \nu} = 0$ on $\partial\Omega$. \square

Proof of Theorem 2.3. The proof parallels (and is even simpler than) that of Theorem 2.4. We limit ourselves to pointing out the variants and simplifications needed.

Step 1. Assume that Ω , a and f are as in Step 1 of the proof of Theorem 2.4 and that, in addition, Ω is convex. One can proceed as in that proof, and exploit the fact that the right-hand side of equation (4.17) is nonnegative owing to the convexity of Ω , since it reduces to either

$$-\operatorname{tr} \mathcal{B} \left(\frac{\partial u}{\partial \nu} \right)^2 \geq 0 \quad \text{or} \quad -\mathcal{B}(\nabla_T u, \nabla_T u) \geq 0 \quad \text{on } \partial\Omega ,$$

according to whether u is the solution to the Dirichlet problem (2.6), or to the Neumann problem (2.7). Therefore, inequality (4.20) can be replaced with the stronger inequality

$$(4.73) \quad C(1 - \varepsilon) \int_{\Omega} \xi^2 a(|\nabla u|)^2 |\nabla^2 u|^2 \, dx \leq \int_{\Omega} \xi^2 f^2 \, dx + \frac{C'}{\varepsilon} \int_{\Omega} |\nabla \xi|^2 a(|\nabla u|)^2 |\nabla u|^2 \, dx .$$

Starting from this inequality, instead of (4.20), estimate (4.65) follows analogously.

Step 2. The proof is the same as that of Theorem 2.4.

Step 3. The proof is analogous to that of Theorem 2.4, save that the approximating domains Ω_m have to be chosen in such a way that they are convex.

Step 4. The proof is the same as that of Theorem 2.4. \square

5 Local estimates

Here, we provide a proof of Theorem 2.1. The generalized local solutions to equation (2.1) considered in the statement can be defined as follows.

Assume that $f \in L^q_{\text{loc}}(\Omega)$ for some $q \geq 1$. A function $u \in \mathcal{T}^{1,1}_{\text{loc}}(\Omega)$ is called a generalized local solution to equation (2.1) if $a(|\nabla u|) \nabla u \in L^1_{\text{loc}}(\Omega)$, equation (4.3) holds for every $\varphi \in C_0^\infty(\Omega)$, and there exists a sequence $\{f_k\} \subset C_0^\infty(\Omega)$ and a corresponding sequence of local weak solutions $\{u_k\}$ to equation (2.1), with f replaced by f_k , such that $f_k \rightarrow f$ in $L^q(\Omega')$,

$$(5.1) \quad u_k \rightarrow u \quad \text{and} \quad \nabla u_k \rightarrow \nabla u \quad \text{a.e. in } \Omega,$$

and

$$(5.2) \quad \lim_{k \rightarrow \infty} \int_{\Omega'} a(|\nabla u_k|) |\nabla u_k| dx = \int_{\Omega'} a(|\nabla u|) |\nabla u| dx$$

for every open set $\Omega' \subset \subset \Omega$.

Note that, by the results from [CiMa3] recalled at the beginning of Section 4, the generalized solutions to the boundary value problems (2.6) and (2.7) are, in particular, generalized local solutions to equation (2.1).

Proof of Theorem 2.1. This proof follows the outline of that of Theorem 2.4. Some variants are however required, due to the local nature of the result. Of course, the step concerning the approximation of Ω by domains with a smooth boundary is not needed at all.

Step 1. Assume the additional conditions (4.12) on f , and (4.14) – (4.15) on a , and let u be a local weak solution to equation (2.1). Thanks to the current assumption on a and f , the function u is in fact a classical smooth solution. Let B_{2R} be any ball such that $B_{2R} \subset \subset \Omega$, and let $R \leq \sigma < \tau \leq 2R$. An application of inequality (4.20), with $\varepsilon = \frac{1}{2}$ and any function $\xi \in C_0^\infty(B_\tau)$ such that $\xi = 1$ in B_σ and $|\nabla \xi| \leq C/(\tau - \sigma)$ for some constant $C = C(n)$, tells us that

$$(5.3) \quad \int_{B_\sigma} a(|\nabla u|)^2 |\nabla^2 u|^2 dx \leq C \int_{B_{2R}} f^2 dx + \frac{C}{(\tau - \sigma)^2} \int_{B_\tau \setminus B_\sigma} a(|\nabla u|)^2 |\nabla u|^2 dx$$

for some constant $C = C(n, i_a, s_a)$. We claim that there exists a constant $C = C(n)$ such that

$$(5.4) \quad \int_{B_\tau \setminus B_\sigma} v^2 dx \leq \frac{\delta^2}{(\tau - \sigma)^2} \int_{B_\tau \setminus B_\sigma} |\nabla v|^2 dx + \frac{C(\tau - \sigma)R^{n-1}}{\delta^n} \left(\int_{B_\tau \setminus B_\sigma} |v| dx \right)^2$$

for every $\delta > 0$ and every $v \in W^{1,2}(B_\tau \setminus B_\sigma)$, provided that R, τ and σ are as above. This claim can be verified as follows. Denote by Q_r a cube of sidelength $r > 0$. The inequality

$$(5.5) \quad \int_{Q_1} v^2 dx \leq C_1 \int_{Q_1} |\nabla v|^2 dx + C_2 \left(\int_{Q_1} |v| dx \right)^2$$

holds for every $v \in W^{1,2}(Q_1)$, for suitable constants $C_1 = C_1(n)$ and $C_2(n)$. Given $\varepsilon > 0$, a scaling argument tells us that a parallel inequality holds in Q_ε , with C_1 replaced with $C_1 \varepsilon^2$ and C_2 replaced with $C_2 \varepsilon^{-n}$. A covering argument for Q_1 by cubes of sidelength ε then yields inequality (5.5) with C_1 and C_2 replaced by $C_1 \varepsilon^2$ and $C_2 \varepsilon^{-n}$, respectively. Another scaling argument, applied to the resulting inequality in Q_1 , provides us with the inequality

$$(5.6) \quad \int_{Q_\delta} v^2 dx \leq C_1 (\varepsilon \delta)^2 \int_{Q_\delta} |\nabla v|^2 dx + C_2 (\varepsilon \delta)^{-n} \left(\int_{Q_\delta} |v| dx \right)^2$$

for every $v \in W^{1,2}(Q_\delta)$. Via a covering argument for $B_2 \setminus B_1$ by (quasi)-cubes of suitable sidelength δ , one infers from (5.6) that

$$(5.7) \quad \int_{B_2 \setminus B_1} v^2 dx \leq C \varepsilon^2 \int_{B_2 \setminus B_1} |\nabla v|^2 dx + C \varepsilon^{-n} \left(\int_{B_2 \setminus B_1} |v| dx \right)^2$$

for a suitable constant $C = C(n)$. Inequality (5.4) can be derived from (5.7) on mapping $B_2 \setminus B_1$ into $B_\tau \setminus B_\sigma$ via the bijective map $\Phi : B_2 \setminus B_1 \rightarrow B_\tau \setminus B_\sigma$ defined as

$$\Phi(x) = \frac{x}{|x|} [\sigma + (|x| - 1)(\tau - \sigma)] \quad \text{for } x \in B_2 \setminus B_1,$$

and making use of the fact that

$$c_1(\tau - \sigma)R^{n-1} \leq |\det(\nabla\Phi(x))| \leq c_2(\tau - \sigma)R^{n-1} \quad \text{for } x \in B_2 \setminus B_1$$

and

$$|\nabla(\Phi^{-1})(y)| \geq c_1(\tau - \sigma) \quad \text{for } y \in B_\tau \setminus B_\sigma,$$

for suitable positive constants $c_1 = c_1(n)$ and $c_2 = c_2(n)$.

Choosing $\delta = (\tau - \sigma)^2$ in inequality (5.4), and applying the resulting inequality with $v = a(|\nabla u|)u_{x_i}$, for $i = 1 \dots, n$ yields

$$(5.8) \quad \frac{1}{(\tau - \sigma)^2} \int_{B_\tau \setminus B_\sigma} a(|\nabla u|)^2 |\nabla u|^2 dx \\ \leq C \int_{B_\tau \setminus B_\sigma} a(|\nabla u|)^2 |\nabla^2 u|^2 dx + \frac{CR^{n-1}}{(\tau - \sigma)^{2n-1}} \left(\int_{B_\tau \setminus B_\sigma} a(|\nabla u|) |\nabla u| dx \right)^2$$

for some constant $C = C(n, s_a)$. Observe that in (5.8) we have also made use of equation (4.30). Inequalities (5.3) and (5.8) imply that

$$(5.9) \quad \int_{B_\sigma} a(|\nabla u|)^2 |\nabla^2 u|^2 dx \leq C \int_{B_\tau \setminus B_\sigma} a(|\nabla u|)^2 |\nabla^2 u|^2 dx \\ + C \int_{B_{2R}} f^2 dx + \frac{CR^{n-1}}{(\tau - \sigma)^{2n-1}} \left(\int_{B_{2R}} a(|\nabla u|) |\nabla u| dx \right)^2$$

for some constant $C = C(n, i_a, s_a)$. Adding the quantity $C \int_{B_\sigma} a(|\nabla u|)^2 |\nabla^2 u|^2 dx$ to both sides of inequality (5.9), and dividing through the resulting inequality by $(1 + C)$ enable us to deduce that

$$(5.10) \quad \int_{B_\sigma} a(|\nabla u|)^2 |\nabla^2 u|^2 dx \leq \frac{C}{1 + C} \int_{B_\tau} a(|\nabla u|)^2 |\nabla^2 u|^2 dx \\ + C' \int_{B_{2R}} f^2 dx + \frac{C'R^{n-1}}{(\tau - \sigma)^{2n-1}} \left(\int_{B_{2R}} a(|\nabla u|) |\nabla u| dx \right)^2$$

for positive constants $C = C(n, i_a, s_a)$ and $C' = C'(n, i_a, s_a)$. Inequality (5.10), via a standard iteration argument (see e.g. [Gi, Lemma 3.1, Chapter 5]), entails that

$$(5.11) \quad \int_{B_R} a(|\nabla u|)^2 |\nabla^2 u|^2 dx \leq C \int_{B_{2R}} f^2 dx + \frac{C}{R^n} \left(\int_{B_{2R}} a(|\nabla u|) |\nabla u| dx \right)^2$$

for some constant $C = C(n, i_a, s_a)$. On the other hand, a scaling argument applied to the Sobolev inequality (4.40), with $\Omega = B_1$ and $\sigma = 1$, tells us that there exists a constant $C = C(n, s_a)$ such that

$$(5.12) \quad \int_{B_R} a(|\nabla u|)^2 |\nabla u|^2 dx \leq \int_{B_R} a(|\nabla u|)^2 |\nabla^2 u|^2 dx + \frac{C}{R^n} \left(\int_{B_R} a(|\nabla u|) |\nabla u| dx \right)^2.$$

Coupling inequality (5.11) with (5.12) yields

$$(5.13) \quad \|a(|\nabla u|) \nabla u\|_{W^{1,2}(B_R)} \leq C(\|f\|_{L^2(B_{2R})} + R^{-\frac{n}{2}} \|a(|\nabla u|) \nabla u\|_{L^1(B_{2R})})$$

for some constant $C = C(n, i_a, s_a)$.

Step 2. Assume that u is a local solution to equation (2.1), with a as in the statement, and f still fulfilling (4.12). One has that $u \in L_{\text{loc}}^\infty(\Omega)$. This follows from [Ko, Theorem 5.1], or from gradient regularity results of [Ba] or [DKS]. As a consequence, by [Li, Theorem 1.7], $u \in C_{\text{loc}}^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$. Next, consider a family of functions $\{a_\varepsilon\}_{\varepsilon \in (0,1)}$ satisfying properties (4.44) – (4.47). Denote by u_ε the solution to the problem

$$(5.14) \quad \begin{cases} -\operatorname{div}(a_\varepsilon(|\nabla u_\varepsilon|)\nabla u_\varepsilon) = f(x) & \text{in } B_{2R} \\ u_\varepsilon = u & \text{on } \partial B_{2R}. \end{cases}$$

Since $u \in C^{1,\alpha}(\overline{B_{2R}})$, by [Li, Theorem 1.7 and subsequent remarks]

$$(5.15) \quad \|u_\varepsilon\|_{C^{1,\beta}(\overline{B_{2R}})} \leq C$$

for some constant independent of ε . Hence, in particular,

$$(5.16) \quad \|a(|\nabla u_\varepsilon|)\nabla u_\varepsilon\|_{L^1(B_{2R})} \leq C$$

for some constant independent of ε . The functions a_ε satisfy the assumptions imposed on a in Step 1. Thus, by inequality (5.13),

$$(5.17) \quad \|a_\varepsilon(|\nabla u_\varepsilon|)\nabla u_\varepsilon\|_{W^{1,2}(B_R)} \leq C(\|f\|_{B_{2R}} + R^{-\frac{n}{2}}\|a_\varepsilon(|\nabla u_\varepsilon|)\nabla u_\varepsilon\|_{L^1(B_{2R})}),$$

where, owing to (4.45), the constant $C = C(n, i_a, s_a)$, and, in particular, is independent of ε . Inequalities (5.16) and (5.17) ensure that the sequence $\{a_\varepsilon(|\nabla u_\varepsilon|)\nabla u_\varepsilon\}$ is bounded in $W^{1,2}(B_R)$, and hence there exists a function $U : B_R \rightarrow \mathbb{R}^n$, with $U \in W^{1,2}(B_R)$, and a sequence $\{\varepsilon_k\}$ such that

$$(5.18) \quad a_{\varepsilon_k}(|\nabla u_{\varepsilon_k}|)\nabla u_{\varepsilon_k} \rightarrow U \quad \text{in } L^2(B_R) \quad \text{and} \quad a_{\varepsilon_k}(|\nabla u_{\varepsilon_k}|)\nabla u_{\varepsilon_k} \rightharpoonup U \quad \text{in } W^{1,2}(B_R).$$

Moreover, by (5.15), there exists a function $v \in C^1(\overline{B_{2R}})$ such that, up to subsequences,

$$(5.19) \quad u_{\varepsilon_k} \rightarrow v \quad \text{and} \quad \nabla u_{\varepsilon_k} \rightarrow \nabla v$$

pointwise in $\overline{B_{2R}}$. In particular,

$$(5.20) \quad v = u \quad \text{on } \partial B_{2R},$$

inasmuch as $u_{\varepsilon_k} = u$ on ∂B_{2R} for every $k \in \mathbb{R}$. Thanks to (5.18) and (5.19),

$$(5.21) \quad a(|\nabla v|)\nabla v = U \in W^{1,2}(B_R).$$

The weak formulation of problem (5.14) amounts to

$$(5.22) \quad \int_{B_{2R}} a_{\varepsilon_k}(|\nabla u_{\varepsilon_k}|)\nabla u_{\varepsilon_k} \cdot \nabla \varphi \, dx = \int_{B_{2R}} f \varphi \, dx$$

for every $\varphi \in C_0^\infty(B_{2R})$. By (5.18) and (5.21), passing to the limit in (5.22) as $k \rightarrow \infty$ results in

$$(5.23) \quad \int_{B_{2R}} a(|\nabla v|)\nabla v \cdot \nabla \varphi \, dx = \int_{B_{2R}} f \varphi \, dx.$$

Thus v is the weak solution to the problem

$$(5.24) \quad \begin{cases} -\operatorname{div}(a(|\nabla v|)\nabla v) = f(x) & \text{in } B_{2R} \\ v = u & \text{on } \partial B_{2R}. \end{cases}$$

Since u solves the same problem, $u = v$ in B_{2R} . Moreover, equations (5.17), (5.18) and (5.21) entail that $a(|\nabla u|)\nabla u \in W^{1,2}(B_R)$, and

$$(5.25) \quad \|a(|\nabla u|)\nabla u\|_{W^{1,2}(B_R)} \leq C(\|f\|_{L^2(B_{2R})} + R^{-\frac{n}{2}}\|a(|\nabla u|)\nabla u\|_{L^1(B_{2R})}).$$

Step 3. Let a and f be as in the statement, let u be a generalized local solution to equation (2.1), and let f_k and u_k be as in the definition of this kind of solution given at the beginning of the present section. An application of Step 2 to u_k tells us that $a(|\nabla u_k|)\nabla u_k \in W^{1,2}(B_R)$, and

$$(5.26) \quad \begin{aligned} \|a(|\nabla u_k|)\nabla u_k\|_{W^{1,2}(B_R)} &\leq C(\|f_k\|_{L^2(B_{2R})} + R^{-\frac{n}{2}}\|a(|\nabla u_k|)\nabla u_k\|_{L^1(B_{2R})}) \\ &\leq C(\|f_k\|_{L^2(B_{2R})} + R^{-\frac{n}{2}}\|a(|\nabla u|)\nabla u\|_{L^1(B_{2R})}), \end{aligned}$$

where the constant C is independent of k . Therefore, the sequence $\{a(|\nabla u_k|)\nabla u_k\}$ is bounded in $W^{1,2}(B_R)$, and hence there exists a function $U : B_R \rightarrow \mathbb{R}^n$, with $U \in W^{1,2}(B_R)$, and a subsequence, still indexed by k , such that

$$(5.27) \quad a(|\nabla u_k|)\nabla u_k \rightarrow U \quad \text{in } L^2(B_R) \quad \text{and} \quad a_k(|\nabla u_k|)\nabla u_k \rightharpoonup U \quad \text{in } W^{1,2}(B_R).$$

By assumption (5.1), $\nabla u_k \rightarrow \nabla u$ a.e. in Ω . Hence, owing to (5.27),

$$(5.28) \quad a(|\nabla u|)\nabla u = U \quad \text{in } B_R,$$

and

$$(5.29) \quad \liminf_{k \rightarrow \infty} \|a(|\nabla u_k|)\nabla u_k\|_{W^{1,2}(B_R)} \geq \|a(|\nabla u|)\nabla u\|_{W^{1,2}(B_R)}.$$

Inequality (2.5) follows from (5.26) and (5.29). \square

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