Boundedness of the Hessian of a biharmonic function in a convex domain

Svitlana Mayboroda and Vladimir Maz'ya*

Abstract

We consider the Dirichlet problem for the biharmonic equation on an arbitrary convex domain and prove that the second derivatives of the variational solution are bounded in all dimensions.

1 Introduction

The properties of solutions of the Dirichlet problem for the Laplacian on convex domains are nowadays well understood. It is a classical fact that the gradient of a solution is bounded and in the last decade a number of results in L^p, Sobolev and Hardy spaces have been developed (see [1], [2], [11], [7], [8]). However, much less is known about the behavior of solutions to higher order elliptic equations. The aim of this paper is to establish the boundedness of the second derivatives of a biharmonic function in all dimensions.

To be more precise, given a bounded domain $\Omega \subset \mathbb{R}^n$ denote by $\mathring{W}_2^2(\Omega)$ the completion of $C_0^\infty(\Omega)$ in the norm of the Sobolev space of functions with second distributional derivatives in L^2. We consider the variational solution of the boundary value problem

$$\Delta^2 u = f \text{ in } \Omega, \quad f \in C_0^\infty(\Omega), \quad u \in \mathring{W}_2^2(\Omega), \tag{1.1}$$

that is a function $u \in \mathring{W}_2^2(\Omega)$ such that

$$\int_\Omega \Delta u \Delta v \, dx = \int_\Omega f v \, dx \quad \text{for every} \quad v \in \mathring{W}_2^2(\Omega). \tag{1.2}$$

The main result of this paper is the following.

Theorem 1.1 *Let Ω be a convex domain in $\mathbb{R}^n$, $O \in \partial\Omega$, and fix some $R \in (0, \operatorname{diam}(\Omega)/10)$. Suppose u is a solution of the Dirichlet problem (1.1) with $f \in C_0^\infty(\Omega \setminus B_{10R})$. Then*

$$|\nabla^2 u(x)| \leq \frac{C}{R^2} \left(\fint_{C_{R/2,\,5R} \cap \Omega} |u(x)|^2 \, dx \right)^{1/2} \quad \textit{for every} \quad x \in B_{R/5} \cap \Omega, \tag{1.3}$$

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where $\nabla^2 u$ *is the Hessian matrix of* u,

$$C_{R/2,\,5R} = \{x \in \mathbb{R}^n :\ R/2 \le |x| \le 5R\}, \quad B_{R/5} = \{x \in \mathbb{R}^n :\ |x| < R/5\}, \tag{1.4}$$

and the constant C *depends on the dimension only.*

In particular,

$$|\nabla^2 u| \in L^\infty(\Omega). \tag{1.5}$$

We would like to mention that the properties of solutions to boundary value problems for the biharmonic equation on general domains, such as pointwise estimates and an analogue of the Wiener criterion, have been studied in [12], [13]. In the context of Lipschitz domains, following the work of B. Dahlberg, C. Kenig and G. Verchota in [5] on the well-posedness of the Dirichlet problem with boundary data in L^2, J. Pipher and G. Verchota established the so-called boundary Gårding inequality ([18]), L^p estimates ([16]) and the Miranda-Agmon maximum principle in low dimensions ([17]). The more recent advances include the work of Z. Shen ([19], [20]) and regularity results by V. Adolfsson and J. Pipher in ([3]).

Evidently, every bounded convex domain is Lipschitz. However, the result of Theorem 1.1 may fail on a Lipschitz domain. Even in the three-dimensional case the solution in the exterior of a thin cone is only $C^{1,\alpha}$ for some $\alpha > 0$ (see [10]) and there is a four-dimensional domain for which the gradient of the solution is not bounded (see [14] and [16] for counterexamples).

Our methods are different from the methods of the Lipshitz theory as well as from those used in the case of Laplacian. The key to our approach lies in obtaining new integral identities. While they are valid for all domains, it is exactly the convexity that allows us to establish positivity (more precisely, a suitable positive lower bound) of the following expression:

$$\frac{n^2+n-2}{n} \int_\Omega \Delta u \Delta \left(\frac{ug}{|x|^n} \right) dx - \frac{n^2-2}{2n} \int_\Omega \Delta^2 u \left(\frac{(x \cdot \nabla u)\, g}{|x|^n} \right) dx. \tag{1.6}$$

One of the significant novelties of our argument is a subtle choice of the radial function g in the expression above. The role of convexity manifests itself in the positivity of the arising boundary integral.

The estimates for (1.6) are further used to control the local L^2 behavior of biharmonic functions near the boundary of the domain. Specifically, we show that under the assumptions of Theorem 1.1

$$\frac{1}{\rho^4} \fint_{C_{\rho/2,\rho} \cap \Omega} |u(x)|^2\, dx \le \frac{C}{R^4} \fint_{C_{R/2,5R} \cap \Omega} |u(x)|^2\, dx \quad \text{for every} \quad \rho < R/2. \tag{1.7}$$

This inequality, interesting in its own right, is a crucial ingredient of the proof of the main result.

Finally, it has to be noted that the boundedness of the Hessian of a biharmonic function for $n = 2$ was obtained from the asymptotic formulas for the solution to the Dirichlet problem on a planar convex domain in [9].

2 Global estimates: part I

Let Ω be an arbitrary domain in $\mathbb{R}^n$, $n \geq 2$. We assume that the origin belongs to the complement of Ω and $r = |x|$, $\omega = x/|x|$ are the spherical coordinates centered at the origin. In fact, we will mostly use the coordinate system (t, ω), where $t = \log r^{-1}$, and the mapping $\varkappa$ defined by

$$\mathbb{R}^n \ni x \xrightarrow{\varkappa} (t, \omega) \in \mathbb{R} \times S^{n-1}. \tag{2.1}$$

Here, and throughout the paper, S^{n-1} denotes the unit sphere in $\mathbb{R}^n$,

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}, \tag{2.2}$$

and $S^{n-1}_+ = S^{n-1} \cap \mathbb{R}^n_+$.

Next, given an open set $\Gamma \subset S^{n-1}$, the space $\mathring{W}^1_2(\Gamma)$ is the completion of $C_0^\infty(\Gamma)$ in the norm

$$\|\psi\|_{\mathring{W}^1_2(\Gamma)} = \left(- \int_\Gamma \bar{\psi}\, \delta_\omega \psi \, d\omega \right)^{1/2} = \left(\int_\Gamma |\nabla_\omega \psi|^2 \, d\omega \right)^{1/2}, \tag{2.3}$$

where δ_ω and ∇_ω stand for the Laplace-Beltrami operator and gradient on the unit sphere, respectively, and $\mathring{W}^2_2(\Gamma)$ is the completion of $C_0^\infty(\Gamma)$ with respect to the norm

$$\|\psi\|_{\mathring{W}^2_2(\Gamma)} = \left(\int_\Gamma |\delta_\omega \psi|^2 \, d\omega \right)^{1/2}. \tag{2.4}$$

If $\Gamma \subset S^{n-1}_+$ and $\psi \in \mathring{W}^k_2(\Gamma)$, $k = 1, 2$, we sometimes write that ψ belongs to $\mathring{W}^k_2(S^{n-1}_+)$, with the understanding that ψ is extended by zero outside of Γ, and similarly the functions originally defined on the subsets of $\mathbb{R}^n$ will be extended by zero and treated as functions on $\mathbb{R}^n$ whenever appropriate.

Lemma 2.1 *Let Ω be an arbitrary bounded domain in $\mathbb{R}^n$, $O \in \mathbb{R}^n \setminus \Omega$ and*

$$u \in C^2(\bar{\Omega}), \quad u\Big|_{\partial\Omega} = 0, \quad \nabla u\Big|_{\partial\Omega} = 0, \quad v = e^{2t}(u \circ \varkappa^{-1}). \tag{2.5}$$

Then

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Delta u(x) \Delta \left(\frac{u(x) G(\log |x|^{-1})}{|x|^n} \right) dx \\
&\quad = \int_{\mathbb{R}} \int_{S^{n-1}} \Big[(\delta_\omega v)^2 G + 2(\partial_t \nabla_\omega v)^2 G + (\partial_t^2 v)^2 G \\
&\quad -(\nabla_\omega v)^2 \Big(\partial_t^2 G + n\, \partial_t G + 2n\, G \Big) - (\partial_t v)^2 \Big(2\partial_t^2 G + 3n\, \partial_t G + (n^2 + 2n - 4)\, G \Big) \\
&\quad + \frac{1}{2} v^2 \Big(\partial_t^4 G + 2n\, \partial_t^3 G + (n^2 + 2n - 4)\, \partial_t^2 G + 2n(n-2)\, \partial_t G \Big) \Big] \, d\omega dt.
\end{aligned} \tag{2.6}$$

for every function G on $\mathbb{R}$ such that both sides of (2.6) are well-defined.

Proof. In the coordinates (t,ω) the n-dimensional Laplacian can be represented as

$$\Delta = e^{2t}\Lambda(\partial_t,\delta_\omega), \quad \text{where} \quad \Lambda(\partial_t,\delta_\omega) = \partial_t^2 - (n-2)\partial_t + \delta_\omega. \tag{2.7}$$

Then

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Delta u(x)\Delta\left(\frac{u(x)G(\log|x|^{-1})}{|x|^n}\right)dx \\
&= \int_{\mathbb{R}}\int_{S^{n-1}} \Lambda(\partial_t-2,\delta_\omega)v\,\Lambda(\partial_t+n-2,\delta_\omega)(vG)\,d\omega dt \\
&= \int_{\mathbb{R}}\int_{S^{n-1}} \big(\partial_t^2 v-(n+2)\partial_t v+2nv+\delta_\omega v\big)\,\big(\partial_t^2(vG)+(n-2)\partial_t(vG)+G\,\delta_\omega v\big)\,d\omega dt \\
&= \int_{\mathbb{R}}\int_{S^{n-1}} \big(\partial_t^2 v-(n+2)\partial_t v+2nv+\delta_\omega v\big) \\
&\quad\times\big(G\,\delta_\omega v+G\,\partial_t^2 v+(2\partial_t G+(n-2)G)\,\partial_t v+(\partial_t^2 G+(n-2)\partial_t G)\,v\big)\,d\omega dt.
\end{aligned} \tag{2.8}$$

Expanding the expression above and reassembling the terms, we write the integral in (2.8) as

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{S^{n-1}}\Big(G\,(\delta_\omega v)^2+2G\,\delta_\omega v\partial_t^2 v+G\,(\partial_t^2 v)^2 \\
&\quad+v\delta_\omega v\,\big(\partial_t^2 G+(n-2)\partial_t G+2nG\big)+\delta_\omega v\partial_t v\,(2\partial_t G-4G) \\
&\quad+\partial_t^2 v\partial_t v\,(2\partial_t G-4G)+v\partial_t^2 v\,\big(\partial_t^2 G+(n-2)\partial_t G+2nG\big) \\
&\quad+(\partial_t v)^2\,\big(-2(n+2)\partial_t G-(n^2-4)G\big) \\
&\quad+v\partial_t v\,\big(-(n+2)\partial_t^2 G-(n^2-4n-4)\partial_t G+2n(n-2)G\big) \\
&\quad+v^2\,\big(2n\partial_t^2 G+2n(n-2)\partial_t G\big)\Big)\,d\omega dt.
\end{aligned} \tag{2.9}$$

Since G does not depend on ω, after integration by parts the latter integral becomes

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{S^{n-1}}\Big(G\,(\delta_\omega v)^2-2G\,\delta_\omega\partial_t v\partial_t v+G\,(\partial_t^2 v)^2 \\
&\quad+(\nabla_\omega v)^2\,\big(-\partial_t^2 G-\partial_t^2 G-(n-2)\partial_t G-2nG+\partial_t^2 G-2\partial_t G\big) \\
&\quad+(\partial_t v)^2\,\big(-\partial_t^2 G+2\partial_t G-\big(\partial_t^2 G+(n-2)\partial_t G+2nG\big)-2(n+2)\partial_t G-(n^2-4)G\big) \\
&\quad+v\partial_t v\,\big(-\partial_t^3 G-(n-2)\partial_t^2 G-2n\partial_t G-(n+2)\partial_t^2 G-(n^2-4n-4)\partial_t G+2n(n-2)G\big) \\
&\quad+v^2\,\big(2n\partial_t^2 G+2n(n-2)\partial_t G\big)\Big)\,d\omega dt.
\end{aligned} \tag{2.10}$$

Finally, integrating by parts once again and collecting the terms, we arrive at (2.6). □

To proceed further we need some auxiliary results. By δ we denote the Dirac delta function.

Lemma 2.2 *A bounded solution of the equation*

$$\frac{d^4 g}{dt^4} + 2n\frac{d^3 g}{dt^3} + \left(n^2 - 2\right)\frac{d^2 g}{dt^2} - 2n\frac{dg}{dt} = \delta \tag{2.11}$$

subject to the restriction

$$g(t) \to 0 \text{ as } t \to +\infty, \tag{2.12}$$

is the function

$$g(t) = -\frac{1}{2n\sqrt{n^2+8}} \begin{cases} n\, e^{-1/2(n-\sqrt{n^2+8})t} - \sqrt{n^2+8}, & t < 0, \\ n\, e^{-1/2(n+\sqrt{n^2+8})t} - \sqrt{n^2+8}\, e^{-nt}, & t > 0. \end{cases} \tag{2.13}$$

Proof. The equation (2.11) can be written as

$$\frac{d}{dt}\left(\frac{d}{dt} + n\right)\left(\frac{d}{dt} + \frac{1}{2}\left(n + \sqrt{n^2+8}\right)\right)\left(\frac{d}{dt} + \frac{1}{2}\left(n - \sqrt{n^2+8}\right)\right) g = \delta. \tag{2.14}$$

Since we seek a bounded solution of (2.11) satisfying (2.12), g must have the form

$$g(t) = \begin{cases} a\, e^{-1/2(n-\sqrt{n^2+8})t} + b, & t < 0, \\ c\, e^{-1/2(n+\sqrt{n^2+8})t} + d\, e^{-nt}, & t > 0, \end{cases} \tag{2.15}$$

for some constants a, b, c, d. Once this is established, we find the system of coefficients so that $\partial_t^k g$ is continuous for $k = 0, 1, 2$ and $\lim_{t\to 0^+} \partial_t^3 g(t) - \lim_{t\to 0^-} \partial_t^3 g(t) = 1$. $\square$

Next, let us consider some estimates based on the spectral properties of the Laplace-Beltrami operator on the half-sphere. When $n \geq 3$ we will use the coordinates $\omega = (\theta, \varphi)$ on the unit sphere S^{n-1}, where $\theta \in [0, \pi]$, $\varphi \in S^{n-2}$. In the two-dimensional case $\omega = \theta \in [0, 2\pi)$.

Lemma 2.3 *For every* $v \in \mathring{W}_2^2(S_+^{n-1})$,

$$\int_{S_+^{n-1}} |\delta_\omega v|^2 \, d\omega \geq 2n \int_{S_+^{n-1}} |\nabla_\omega v|^2 \, d\omega. \tag{2.16}$$

The equality is achieved when $v = \cos^2\theta$.

Proof. Since $v \in \mathring{W}_2^2(S_+^{n-1})$,

$$\int_{S_+^{n-1}} |\nabla_\omega v|^2 \, d\omega = \int_{S_+^{n-1}} \left|\nabla_\omega (v - (v)_{S_+^{n-1}})\right|^2 d\omega \leq \|v - (v)_{S_+^{n-1}}\|_{L^2(S_+^{n-1})} \|\delta_\omega v\|_{L^2(S_+^{n-1})}, \tag{2.17}$$

where

$$(v)_{S_+^{n-1}} = \fint_{S_+^{n-1}} v\, d\omega. \tag{2.18}$$

If $n \geq 3$, let us denote

$$z(\theta) := \fint_{S^{n-2}} v(\theta,\varphi)\, d\varphi, \qquad y(\theta,\varphi) := v(\theta,\varphi) - z(\theta). \tag{2.19}$$

Then $(y)_{S_+^{n-1}} = 0$ and

$$\|v - (v)_{S_+^{n-1}}\|^2_{L^2(S_+^{n-1})} = \|y\|^2_{L^2(S_+^{n-1})} + \|z - (z)_{S_+^{n-1}}\|^2_{L^2(S_+^{n-1})}, \tag{2.20}$$

$$\|\nabla_\omega v\|^2_{L^2(S_+^{n-1})} = \|\nabla_\omega y\|^2_{L^2(S_+^{n-1})} + \|\nabla_\omega z\|^2_{L^2(S_+^{n-1})}. \tag{2.21}$$

By the definition (2.19) the function $y \in \overset{\circ}{W}{}^1_2(S_+^{n-1})$ is orthogonal to $\cos\theta$ on S_+^{n-1}. Therefore, y is orthogonal to the first eigenfunction of the Dirichlet problem for $-\delta_\omega$ on S_+^{n-1}. Since the second eigenvalue of $-\delta_\omega$ is $2n$, this yields

$$\int_{S_+^{n-1}} |\nabla_\omega y|^2\, d\omega \geq 2n \int_{S_+^{n-1}} |y|^2\, d\omega. \tag{2.22}$$

Turning to the estimates on z, we observe that

$$\Lambda = \inf \left\{ \frac{-\int_{S_+^{n-1}} \bar{\xi}\, \delta_\omega \xi\, d\omega}{\int_{S_+^{n-1}} |\xi|^2\, d\omega} : \quad \xi \in W_2^1(S_+^{n-1}),\ \int_{S_+^{n-1}} \xi\, d\omega = 0,\ \xi = \xi(\theta) \right\}, \tag{2.23}$$

is the first positive eigenvalue of the Neumann problem for $-\delta_\omega$ in the space of axisymmetric functions. Hence, $\Lambda = 2n$ and the corresponding eigenfunction is $n\cos^2\theta - 1$. Therefore,

$$\int_{S_+^{n-1}} |\nabla_\omega z|^2\, d\omega \geq 2n \int_{S_+^{n-1}} |z - (z)_{S_+^{n-1}}|^2\, d\omega. \tag{2.24}$$

Combined with (2.20)–(2.22), the formula above implies that

$$\int_{S_+^{n-1}} |\nabla_\omega v|^2\, d\omega \geq 2n \int_{S_+^{n-1}} |v - (v)_{S_+^{n-1}}|^2\, d\omega, \tag{2.25}$$

and by (2.17) this finishes the proof for $n \geq 3$.

In the case $n = 2$ there is no need to introduce functions z and y. One can work directly with v and prove (2.25) following the argument for (2.24) above. $\square$

Lemma 2.4 *Let Ω be a bounded convex domain in $\mathbb{R}^n$ and $O \in \mathbb{R}^n \setminus \Omega$. Suppose that*

$$u \in C^2(\bar{\Omega}), \quad u\Big|_{\partial\Omega} = 0, \quad \nabla u\Big|_{\partial\Omega} = 0, \quad v = e^{2t}(u \circ \varkappa^{-1}), \tag{2.26}$$

and g is given by (2.13). Then

$$\int_{\mathbb{R}^n} \Delta u(x)\Delta\left(\frac{u(x)g(\log(|\xi|/|x|))}{|x|^n}\right)\,dx$$
$$\geq -\int_{\mathbb{R}}\int_{S^{n-1}}\Big(2\partial_t^2 g(t-\tau)+3n\partial_t g(t-\tau)+\left(n^2-2\right)g(t-\tau)\Big)(\partial_t v(t,\omega))^2\,d\omega dt$$
$$+\frac{1}{2}\int_{S^{n-1}} v^2(\tau,\omega)\,d\omega, \tag{2.27}$$

for every $\xi\in\Omega$, $\tau=\log|\xi|^{-1}$.

Proof. Rearranging the terms in (2.6), we write

$$\int_{\mathbb{R}^n} \Delta u(x)\Delta\left(\frac{u(x)G(\log|x|^{-1})}{|x|^n}\right)\,dx$$
$$=\int_{\mathbb{R}}\int_{S^{n-1}}\Big[G\Big((\delta_\omega v)^2-2n(\nabla_\omega v)^2+(\partial_t^2 v)^2+2(\partial_t\nabla_\omega v)^2-(n^2+2n-4)(\partial_t v)^2\Big)$$
$$-\Big(2\partial_t^2 G+3n\partial_t G\Big)(\partial_t v)^2-\Big(\partial_t^2 G+n\partial_t G\Big)(\nabla_\omega v)^2$$
$$+\frac{1}{2}\Big(\partial_t^4 G+2n\partial_t^3 G+(n^2+2n-4)\partial_t^2 G+2n(n-2)\partial_t G\Big)v^2\Big]\,d\omega dt. \tag{2.28}$$

Let $G(t)=g(t-\tau)$, $t\in\mathbb{R}$, with g defined in (2.13). First of all observe that for such a choice of G the conclusion of Lemma 2.1 (and hence the equality (2.28)) remains valid.

Going further, $g\geq 0$ and

$$\partial_t^2 g(t)+n\partial_t g(t)\leq 0,\quad \text{for every}\quad t\in\mathbb{R}. \tag{2.29}$$

Indeed,

$$\partial_t\, g(t)=\frac{1}{2\sqrt{n^2+8}}\begin{cases}\frac{1}{2}\left(n-\sqrt{n^2+8}\right)e^{-1/2(n-\sqrt{n^2+8})t}, & t<0,\\ -\sqrt{n^2+8}\,e^{-nt}+\frac{1}{2}\left(n+\sqrt{n^2+8}\right)e^{-1/2(n+\sqrt{n^2+8})t}, & t>0,\end{cases} \tag{2.30}$$

and

$$\partial_t^2\, g(t)=\frac{1}{2\sqrt{n^2+8}}\begin{cases}-\frac{1}{4}\left(n-\sqrt{n^2+8}\right)^2 e^{-1/2(n-\sqrt{n^2+8})t}, & t<0,\\ n\sqrt{n^2+8}\,e^{-nt}-\frac{1}{4}\left(n+\sqrt{n^2+8}\right)^2 e^{-1/2(n+\sqrt{n^2+8})t}, & t>0.\end{cases} \tag{2.31}$$

Therefore,

$$\partial_t^2 g(t)+n\partial_t g(t)=-\frac{1}{\sqrt{n^2+8}}\begin{cases}e^{-1/2(n-\sqrt{n^2+8})t}, & t<0,\\ e^{-1/2(n+\sqrt{n^2+8})t}, & t>0,\end{cases} \tag{2.32}$$

is non-positive.

Recall that the first eigenvalue of the operator $-\delta_\omega$ on the half-sphere is $n-1$. Since Ω is a convex domain, for every fixed $t \in \mathbb{R}$

$$\int_{S^{n-1}} |\nabla_\omega v(t,\omega)|^2 \, d\omega \geq (n-1) \int_{S^{n-1}} |v(t,\omega)|^2 \, d\omega, \tag{2.33}$$

and the same estimate holds with v replaced by $\partial_t v$. Together with (2.28) and (2.29) this gives

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Delta u(x) \Delta \left(\frac{u(x) g(\log(|\xi|/|x|))}{|x|^n} \right) dx \\
&\quad \geq \int_{\mathbb{R}} \int_{S^{n-1}} \Big[g(t-\tau) \Big((\delta_\omega v)^2 - 2n (\nabla_\omega v)^2 \Big) \\
&\qquad - \Big(2\partial_t^2 g(t-\tau) + 3n \partial_t g(t-\tau) + \big(n^2 - 2\big)\, g(t-\tau) \Big) (\partial_t v)^2 \\
&\quad + \frac{1}{2} \Big(\partial_t^4 g(t-\tau) + 2n \partial_t^3 g(t-\tau) + (n^2-2) \partial_t^2 g(t-\tau) - 2n \partial_t g(t-\tau) \Big) v^2 \Big] \, d\omega dt.
\end{aligned} \tag{2.34}$$

On the other hand, for every $t \in \mathbb{R}$

$$\int_{S^{n-1}} \left((\delta_\omega v(t,\omega))^2 - 2n (\nabla_\omega v(t,\omega))^2 \right) \, d\omega \geq 0 \tag{2.35}$$

by Lemma 2.3, so it remains to estimate the terms in the last line of (2.34). However, according to Lemma 2.2, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{S^{n-1}} \Big(\partial_t^4 g(t-\tau) + 2n \partial_t^3 g(t-\tau) + (n^2-2) \partial_t^2 g(t-\tau) - 2n \partial_t g(t-\tau) \Big) v^2 \Big] \, d\omega dt \\
&\qquad = \int_{S^{n-1}} v^2(\tau, \omega) \, d\omega,
\end{aligned} \tag{2.36}$$

which completes the proof. $\square$

3 Global estimates: part II

Lemma 3.1 *Suppose Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $O \in \mathbb{R}^n \setminus \Omega$, and*

$$u \in C^4(\bar{\Omega}), \quad u\Big|_{\partial\Omega} = 0, \quad \nabla u\Big|_{\partial\Omega} = 0, \quad v = e^{2t}(u \circ \varkappa^{-1}). \tag{3.1}$$

Then

$$2 \int_{\mathbb{R}^n} \Delta u(x) \Delta \left(\frac{u(x) G(\log|x|^{-1})}{|x|^n} \right) dx - \int_{\mathbb{R}^n} \Delta^2 u(x) \, \left(\frac{(x \cdot \nabla u(x))\, G(\log|x|^{-1})}{|x|^n} \right) dx$$

$$= \int_{\mathbb{R}} \int_{S^{n-1}} \Big(-\tfrac{1}{2} (\delta_\omega v)^2 \partial_t G + (\partial_t \nabla_\omega v)^2 \left(\partial_t G + 2nG\right)$$

$$+ (\partial_t^2 v)^2 \left(\tfrac{3}{2}\, \partial_t G + 2nG\right) + n (\nabla_\omega v)^2 \partial_t G$$

$$+ (\partial_t v)^2 \left(-\tfrac{1}{2} \partial_t^3 G - n \partial_t^2 G - \tfrac{1}{2}(n^2+2n-4)\partial_t G - 2n(n-2)G \right) \Big)\, d\omega dt$$

$$-\frac{1}{2} \int_{\varkappa(\partial\Omega)} \left((\delta_\omega v)^2 + 2(\partial_t \nabla_\omega v)^2 + (\partial_t^2 v)^2 \right) \, G \, \cos(\nu, t)\, d\sigma_{\omega,t}, \tag{3.2}$$

where ν stands for an outward unit normal to Ω and G is a function on $\mathbb{R}$ for which both sides of (3.2) are well-defined.

Proof. Passing to the coordinates (t,ω) and using (2.7), one can see that

$$\int_{\mathbb{R}^n} \Delta^2 u(x) \left(\frac{(x\cdot \nabla u(x))\, G(\log |x|^{-1})}{|x|^n} \right) dx$$

$$= \int_{\mathbb{R}} \int_{S^{n-1}} \Lambda(\partial_t, \delta_\omega) \Lambda(\partial_t - 2, \delta_\omega) v \,(2v - \partial_t v)\, G\, d\omega dt, \tag{3.3}$$

and therefore,

$$2\int_{\mathbb{R}^n} \Delta u(x) \Delta \left(\frac{u(x) G(\log |x|^{-1})}{|x|^n} \right) dx - \int_{\mathbb{R}^n} \Delta^2 u(x) \left(\frac{(x\cdot \nabla u(x))\, G(\log |x|^{-1})}{|x|^n} \right) dx$$

$$= \int_{\mathbb{R}} \int_{S^{n-1}} \Lambda(\partial_t, \delta_\omega) \Lambda(\partial_t - 2, \delta_\omega) v \, \partial_t v \, G\, d\omega dt$$

$$= \int_{\mathbb{R}} \int_{S^{n-1}} \left(\partial_t^2 - (n-2)\partial_t + \delta_\omega\right) \left(\partial_t^2 - (n+2)\partial_t + 2n + \delta_\omega\right) v\, \partial_t v\, G\, d\omega dt. \tag{3.4}$$

This, in turn, is equal to

$$\int_{\mathbb{R}} \int_{S^{n-1}} \Big(\delta_\omega^2 v + 2\delta_\omega \partial_t^2 v + \partial_t^4 v - 2n \partial_t \delta_\omega v - 2n \partial_t^3 v - 2n(n-2)\partial_t v$$

$$+ 2n \delta_\omega v + (n^2+2n-4)\partial_t^2 v \Big)\, \partial_t v \, G\, d\omega dt$$

$$= \int_{\mathbb{R}} \int_{S^{n-1}} \Big(\delta_\omega^2 v \partial_t v G - 2\partial_t^2 \nabla_\omega v \cdot \partial_t \nabla_\omega v G - \partial_t^3 v \partial_t^2 v G - \partial_t^3 v \partial_t v \partial_t G$$

$$+ 2n(\partial_t \nabla_\omega v)^2 G + 2n (\partial_t^2 v)^2 G + 2n \partial_t^2 v \partial_t v \partial_t G - 2n(n-2)(\partial_t v)^2 G$$

$$- 2n \partial_t \nabla_\omega v \cdot \nabla_\omega v G + (n^2+2n-4)\partial_t^2 v \partial_t v G \Big)\, d\omega dt. \tag{3.5}$$

Let us consider the first term in (3.5). Since the first derivatives of v vanish on the boundary,

$$\int_{\mathbb{R}}\int_{S^{n-1}} \delta_\omega^2 v \partial_t v\, G\, d\omega dt = -\int_{\mathbb{R}}\int_{S^{n-1}} \nabla_\omega \delta_\omega v \cdot \nabla_\omega \partial_t v\, G\, d\omega dt$$
$$= \int_{\mathbb{R}}\int_{S^{n-1}} (-\delta_\omega(\nabla_\omega v \cdot \partial_t \nabla_\omega v) + \nabla_\omega v \cdot \partial_t \delta_\omega \nabla_\omega v + 2\delta_\omega v \partial_t \delta_\omega v)\, G\, d\omega dt$$
$$= \frac{1}{2}\int_{\mathbb{R}}\int_{S^{n-1}} \left(-\delta_\omega \partial_t (\nabla_\omega v)^2 + \partial_t(\delta_\omega v)^2\right) G\, d\omega dt. \tag{3.6}$$

Integrating by parts in t, we can rewrite the last expression as

$$\frac{1}{2}\int_{\mathbb{R}}\int_{S^{n-1}} \left(\delta_\omega(\nabla_\omega v)^2 - (\delta_\omega v)^2\right) \partial_t G\, d\omega dt - \frac{1}{2}\int_{\varkappa(\partial\Omega)} \left(\delta_\omega(\nabla_\omega v)^2 - (\delta_\omega v)^2\right) G\, \cos(\nu, t)\, d\sigma_{\omega,t}$$
$$= \frac{1}{2}\int_{\mathbb{R}}\int_{S^{n-1}} \left(2\delta_\omega \nabla_\omega v \cdot \nabla_\omega v + (\delta_\omega v)^2\right) \partial_t G\, d\omega dt - \frac{1}{2}\int_{\varkappa(\partial\Omega)} (\delta_\omega v)^2 G\, \cos(\nu, t)\, d\sigma_{\omega,t}$$
$$= -\frac{1}{2}\int_{\mathbb{R}}\int_{S^{n-1}} (\delta_\omega v)^2 \partial_t G\, d\omega dt - \frac{1}{2}\int_{\varkappa(\partial\Omega)} (\delta_\omega v)^2 G\, \cos(\nu, t)\, d\sigma_{\omega,t}. \tag{3.7}$$

Now (3.2) follows directly from (3.5) by integration by parts and (3.6)–(3.7). $\square$

Lemma 3.2 *Suppose Ω is a bounded convex domain in $\mathbb{R}^n$, $O \in \partial\Omega$,*

$$u \in C^4(\bar{\Omega}), \quad u\Big|_{\partial\Omega} = 0, \quad \nabla u\Big|_{\partial\Omega} = 0, \quad v = e^{2t}(u \circ \varkappa^{-1}), \tag{3.8}$$

and g is defined by (2.13). Then

$$2\int_{\mathbb{R}^n} \Delta u(x) \Delta \left(\frac{u(x) g(\log|\xi|/|x|)}{|x|^n}\right) dx - \int_{\mathbb{R}^n} \Delta^2 u(x) \left(\frac{(x \cdot \nabla u(x))\, g(\log|\xi|/|x|)}{|x|^n}\right) dx$$
$$\geq -\frac{1}{2}\int_{\mathbb{R}}\int_{S^{n-1}} \Big(\partial_t^3 g(t-\tau) + 2n\, \partial_t^2 g(t-\tau)$$
$$+ (n^2-2)\, \partial_t g(t-\tau) - 4n g(t-\tau)\Big) (\partial_t v(t,\omega))^2\, d\omega dt, \tag{3.9}$$

for every $\xi \in \Omega$, $\tau = \log|\xi|^{-1}$.

Proof. Observe that $g \geq 0$ and for every convex domain $\cos(\nu, t) \leq 0$ so that the boundary integral in (3.2) is non-positive.

Going further, the formula (2.30) shows that $\partial_t g \leq 0$. For $t < 0$ it is obvious and when $t > 0$ this function can change sign at most once, while at 0 and in the neighborhood of $+\infty$ it is negative. In combination with Lemma 2.3 this yields

$$\int_{\mathbb{R}}\int_{S^{n-1}} \left(-\tfrac{1}{2}(\delta_\omega v(t,\omega))^2 \partial_t g(t-\tau) + n(\nabla_\omega v(t,\omega))^2 \partial_t g(t-\tau)\right) d\omega dt \geq 0. \tag{3.10}$$

Using (2.13), (2.30) we can also show that the coefficients of $(\partial_t^2 v)^2$ and $(\partial_t \nabla_\omega v)^2$ are nonnegative. Indeed, we compute

$$\frac{3}{2}\,\partial_t\, g(t) + 2n\, g(t)$$
$$= \frac{1}{2\sqrt{n^2+8}} \begin{cases} \left(-\frac{5}{4}n - \frac{3}{4}\sqrt{n^2+8}\right) e^{-1/2(n-\sqrt{n^2+8})t} + 2\sqrt{n^2+8}, & t<0, \\ \frac{1}{2}\sqrt{n^2+8}\, e^{-nt} + \left(-\frac{5}{4}n + \frac{3}{4}\sqrt{n^2+8}\right) e^{-1/2(n+\sqrt{n^2+8})t}, & t>0. \end{cases}$$

The positivity of this function can be proved in a way similar to the argument for $\partial_t g$: this time, the function is positive at 0 and in the neighborhood of $\pm\infty$. Since $\partial_t g \le 0$, we also have that $\partial_t g + 2ng \ge 0$.

Finally, since Ω is a convex domain, we can apply (2.33) for the function $\partial_t v(t,\cdot)$, $t \in \mathbb{R}$, and obtain the estimate (3.9). $\square$

Lemma 3.3 *Suppose Ω is a bounded convex domain in $\mathbb{R}^n$, $O \in \partial\Omega$,*

$$u \in C^4(\bar{\Omega}), \quad u\Big|_{\partial\Omega} = 0, \quad \nabla u\Big|_{\partial\Omega} = 0, \quad v = e^{\lambda t}(u \circ \varkappa^{-1}). \tag{3.11}$$

and g is given by (2.13). Then

$$\begin{aligned} \frac{1}{2}\int_{S^{n-1}} v^2(\tau,\omega)\, d\omega \;\le\; & \frac{n^2+n-2}{n}\int_{\mathbb{R}^n} \Delta u(x) \Delta\left(\frac{u(x) g(\log|\xi|/|x|)}{|x|^n}\right) dx \\ & -\frac{n^2-2}{2n}\int_{\mathbb{R}^n} \Delta^2 u(x) \left(\frac{(x\cdot\nabla u(x))\, g(\log|\xi|/|x|)}{|x|^n}\right) dx \end{aligned} \tag{3.12}$$

for every $\xi \in \Omega$, $\tau = \log|\xi|^{-1}$.

Proof. First of all, (2.27) combined with (3.9) implies

$$\begin{aligned} &(n^2-2)\left(2\int_{\mathbb{R}^n} \Delta u(x)\Delta\left(\frac{u(x)g(\log|\xi|/|x|)}{|x|^n}\right) dx - \int_{\mathbb{R}^n}\Delta^2 u(x)\frac{(x\cdot\nabla u(x))\, g(\log|\xi|/|x|)}{|x|^n}\, dx\right) \\ &\qquad + 2n\int_{\mathbb{R}^n}\Delta u(x)\Delta\left(\frac{u(x)g(\log|\xi|/|x|)}{|x|^n}\right) dx \\ &\ge n\int_{S^{n-1}} v^2(\tau,\omega)\, d\omega - \int_{\mathbb{R}}\int_{S^{n-1}} \Big(\left(n^2/2-1\right)\partial_t^3 g(t-\tau) \\ &\qquad + (n^3+2n)\partial_t^2 g(t-\tau) + \left(n^4/2+4n^2+2\right)\partial_t g(t-\tau)\Big)(\partial_t v)^2\, d\omega dt \end{aligned} \tag{3.13}$$

Using the formulas (2.13), (2.30), (2.31) and

$$
\partial_t^3\, g(t) = \frac{1}{2\sqrt{n^2+8}} \begin{cases} \frac{1}{8}\left(n-\sqrt{n^2+8}\right)^3 e^{-1/2(n-\sqrt{n^2+8})t}, & t<0, \\ -n^2\sqrt{n^2+8}\, e^{-nt} + \frac{1}{8}\left(n+\sqrt{n^2+8}\right)^3 e^{-1/2(n+\sqrt{n^2+8})t}, & t>0. \end{cases} \tag{3.14}
$$

we compute

$$
\begin{aligned}
&\left(n^2/2-1\right)\partial_t^3 g(t) + (n^3+2n)\partial_t^2 g(t) + \left(n^4/2+4n^2+2\right)\partial_t g(t) = -\frac{1}{2\sqrt{n^2+8}} \times \\
&\times \begin{cases} n\left(n\sqrt{n^2+8}+6\right) e^{-1/2(n-\sqrt{n^2+8})t}, & t<0, \\ (2+n^2)\sqrt{n^2+8}\, e^{-nt} + n\left(-n\sqrt{n^2+8}+6\right) e^{-1/2(n+\sqrt{n^2+8})t}, & t>0. \end{cases}
\end{aligned} \tag{3.15}
$$

The function above is non-positive, which in combination with (3.13) implies (3.12). □

4 Local estimates for the solution of the Dirichlet problem

Throughout this section we will adopt the following notation:

$$
\begin{aligned}
S_r(Q) &:= \{x\in\mathbb{R}^n:\ |x-Q|=r\}, \qquad S_r := S_r(O), \\
B_r(Q) &:= \{x\in\mathbb{R}^n:\ |x-Q|<r\}, \qquad B_r := B_r(O), \\
C_{r,R}(Q) &:= \{x\in\mathbb{R}^n:\ r\le|x-Q|\le R\},\ C_{r,R} := C_{r,R}(O),
\end{aligned}
$$

where $Q\in\mathbb{R}^n$ and $0<r<R<\infty$.

Having this at hand, let us start with a suitable version of the Caccioppoli inequality for the biharmonic equation.

Lemma 4.1 *Let Ω be an arbitrary domain on $\mathbb{R}^n$, $Q\in\partial\Omega$ and $R\in(0,\operatorname{diam}(\Omega)/5)$. Suppose*

$$
\Delta^2 u = f \ \text{ in } \Omega, \quad f\in C_0^\infty(\Omega\setminus B_{5R}(Q)), \quad u\in \mathring{W}_2^2(\Omega). \tag{4.1}
$$

Then

$$
\int_{B_\rho(Q)\cap\Omega} |\nabla^2 u|^2\, dx + \frac{1}{\rho^2}\int_{B_\rho(Q)\cap\Omega} |\nabla u|^2\, dx \le \frac{C}{\rho^4}\int_{C_{\rho,2\rho}(Q)\cap\Omega} |u|^2\, dx \tag{4.2}
$$

for every $\rho<4R$.

We now proceed with the local estimates for solutions near a boundary point of the domain.

Theorem 4.2 *Let Ω be a bounded smooth convex domain in $\mathbb{R}^n$, $Q \in \partial\Omega$, and $R \in (0, \operatorname{diam}(\Omega)/5)$. Suppose*

$$\Delta^2 u = f \text{ in } \Omega, \quad f \in C_0^\infty(\Omega \setminus B_{5R}(Q)), \quad u \in \mathring{W}_2^2(\Omega). \tag{4.3}$$

Then

$$\frac{1}{\rho^4} \fint_{S_\rho(Q)\cap\Omega} |u(x)|^2 \, d\sigma_x \leq \frac{C}{R^4} \fint_{C_{R,4R}(Q)\cap\Omega} |u(x)|^2 \, dx \quad \text{for every} \quad \rho < R, \tag{4.4}$$

where the constant C depends on the dimension only.

Proof. Assume for the moment that $Q = O$. For a smooth domain Ω the solution to the boundary value problem (4.3) belongs to the class $C^4(\bar{\Omega})$ by the standard elliptic theory.

Going further, take some $\eta \in C_0^\infty(\mathbb{R})$ such that $0 \leq \eta \leq 1$ and

$$\eta = 0 \text{ for } t \leq \log(2R)^{-1}, \quad \eta = 1 \text{ for } t \geq \log R^{-1}, \quad |\partial_t^k \eta| \leq C, \quad k \leq 4. \tag{4.5}$$

Since $u \in C^4(\bar{\Omega})$ we can apply Lemma 3.3 with $(\eta \circ \varkappa)u$ in place of u. The function u is biharmonic on the support of $\eta \circ \varkappa$, so that

$$\left[\Delta^2, \eta \circ \varkappa\right] u = \Delta^2((\eta \circ \varkappa)\, u) - (\eta \circ \varkappa)\Delta^2 u = \Delta^2((\eta \circ \varkappa)u) \tag{4.6}$$

and hence by (3.12),

$$\begin{aligned}
&\frac{1}{2}\,\eta^2(\tau) \int_{S^{n-1}} v^2(\tau,\omega)\, d\omega \\
&\quad \leq \frac{n^2+n-2}{n} \int_{\mathbb{R}^n} \left[\Delta^2, \eta(\log|x|^{-1})\right] u(x) \left(\frac{u(x)\eta(\log|x|^{-1}) g(\log(|\xi|/|x|))}{|x|^n} \right) dx \\
&\quad - \frac{n^2-2}{2n} \int_{\mathbb{R}^n} \left[\Delta^2, \eta(\log|x|^{-1})\right] u(x) \left(\frac{x\cdot\nabla(u(x)\eta(\log|x|^{-1}))\, g(\log(|\xi|/|x|))}{|x|^n} \right) dx \\
&\quad \leq \sum_{i=1}^{2} \sum_{j,k=0}^{2} C_{i,j,k} \int_{\mathbb{R}} \int_{S^{n-1}} (\partial_t^k \nabla_\omega^j v)^2 \left(\partial_t^i \eta\right)^2 \, d\omega dt,
\end{aligned} \tag{4.7}$$

where $C_{i,j,k}$ are some constants depending on the dimension only. Here for the last inequality we used integration by parts, Cauchy-Schwartz inequality and boundedness of the function g and its derivatives. Observe that $\left[\Delta^2, \eta \circ \varkappa\right] u$ contains only the derivatives of u of order less than or equal to 3, for that reason there are no boundary terms coming from the integration by parts and the order of derivatives in the final expression does not exceed 2.

Observe also that each term on the right hand side of (4.7) contains some derivative of η. However,

$$\operatorname{supp} \partial_t^k \eta \subset \left(\log(2R)^{-1}, \log R^{-1}\right), \qquad \text{for} \quad k \geq 1, \tag{4.8}$$

so that (4.7) entails the estimate

$$\int_{S^{n-1}} v^2(\tau,\omega)\,d\omega \le C \sum_{j,k=0}^{2} \int_{\log(2R)^{-1}}^{\log R^{-1}} \int_{S^{n-1}} (\partial_t^k \nabla_\omega^j v)^2\,d\omega dt, \tag{4.9}$$

for $\tau \ge \log R^{-1}$. In the Cartesian coordinates this amounts to

$$\frac{1}{\rho^4} \fint_{S_\rho} |u(x)|^2\,d\sigma_x \le C \sum_{k=0}^{2} \frac{1}{R^{4-2k}} \fint_{C_{R,2R}} |\nabla^k u(x)|^2\,dx, \tag{4.10}$$

for every $\rho < R$. Next, invoking Lemma 4.1, we deduce the estimate (4.4) for $Q = O$. Then (4.4) follows from it in full generality since C is a constant depending solely on the dimension. $\square$

Given Theorem 4.2 it is a matter of approximation to prove the following result.

Corollary 4.3 *Let Ω be a bounded convex domain in $\mathbb{R}^n$, $Q \in \partial\Omega$, and $R \in (0, \operatorname{diam}(\Omega)/10)$. Suppose*

$$\Delta^2 u = f \ \text{ in } \Omega, \quad f \in C_0^\infty(\Omega \setminus B_{10R}(Q)), \quad u \in \mathring{W}_2^2(\Omega). \tag{4.11}$$

Then

$$\frac{1}{\rho^4} \fint_{C_{\rho/2,\rho}(Q)\cap\Omega} |u(x)|^2\,dx \le \frac{C}{R^4} \fint_{C_{R/2,5R}(Q)\cap\Omega} |u(x)|^2\,dx \quad \text{for every} \quad \rho < R/2, \tag{4.12}$$

where the constant C depends on the dimension only.

Proof. Let us start approximating Ω by a sequence of smooth convex domains $\{\Omega_n\}_{n=1}^\infty$ such that

$$\bigcup_{n=1}^{\infty} \Omega_n = \Omega, \quad \overline{\Omega}_n \subset \Omega_{n+1} \quad \text{for every} \quad n \in \mathbb{N}. \tag{4.13}$$

Choose $N_0 \in \mathbb{N}$ such that $\operatorname{supp} f \subset \Omega_n$ for every $n \ge N_0$ and denote by u_n, the solution of the Dirichlet problem

$$\Delta^2 u_n = f \quad \text{in} \quad \Omega_n, \quad f \in C_0^\infty(\Omega_n), \quad u_n \in \mathring{W}_2^2(\Omega_n), \quad n \ge N_0. \tag{4.14}$$

Finally, let $Q_n \in \partial\Omega_n$ be a sequence of points converging to $Q \in \partial\Omega$.

Now fix some ρ as in (4.12) and let $\alpha = 2^{-100}$. Then there exists $N = N(\rho) \ge N_0$ such that $|Q - Q_n| < \alpha\rho$ for every $n > N$. In particular,

$$C_{\rho/2,\,\rho}(Q) \subset C_{(1/2-\alpha)\rho,\,(1+\alpha)\rho}(Q_n) \quad \text{and} \quad C_{R,4R}(Q_n) \subset C_{R-\alpha\rho,\,4R+\alpha\rho}(Q). \tag{4.15}$$

Therefore, for every ρ such that $(1+\alpha)\rho < R$

$$\frac{1}{\rho^2}\left(\fint_{C_{\rho/2,\,\rho}(Q)}|u(x)|^2\,dx\right)^{1/2}$$
$$\le \frac{1}{\rho^2}\left(\fint_{C_{\rho/2,\,\rho}(Q)}|u(x)-u_n(x)|^2\,dx\right)^{1/2}+\frac{1}{\rho^2}\left(\fint_{C_{(1/2-\alpha)\rho,\,(1+\alpha)\rho}(Q_n)}|u_n(x)|^2\,dx\right)^{1/2}$$
$$\le \frac{1}{\rho^2}\left(\fint_{C_{\rho/2,\rho}(Q)}|u(x)-u_n(x)|^2\,dx\right)^{1/2}+\frac{C}{R^2}\left(\fint_{C_{R,4R}(Q_n)}|u_n(x)|^2\,dx\right)^{1/2}, \tag{4.16}$$

where we have used (4.15) and Theorem 4.2. Similarly, the last term above can be further estimated by

$$\frac{C}{R^2}\left(\fint_{C_{R,4R}(Q_n)}|u_n(x)-u(x)|^2\,dx\right)^{1/2}+\frac{C}{R^2}\left(\fint_{C_{R-\alpha\rho,\,4R+\alpha\rho}(Q)}|u(x)|^2\,dx\right)^{1/2}. \tag{4.17}$$

Now recall that the solutions u_n, being extended by zero outside of Ω_n and treated as elements of $\mathring{W}_2^2(\Omega)$, strongly converge to u in $\mathring{W}_2^2(\Omega)$ (see, e.g., [15]). Then passing to the limit as $n\to+\infty$ we conclude that

$$\frac{1}{\rho^2}\left(\fint_{C_{\rho/2,\rho}(Q)}|u(x)|^2\,dx\right)^{1/2}\le\frac{C}{R^2}\left(\fint_{C_{R-\alpha\rho,\,4R+\alpha\rho}(Q)}|u(x)|^2\,dx\right)^{1/2}, \tag{4.18}$$

for every $\rho<R/(1+\alpha)$ and α sufficiently small, for instance, $\alpha=2^{-100}$, and finish the argument. □

Finally, we are ready for the

Proof of Theorem 1.1. The interior estimates for solutions of elliptic equations (see [4]) imply that for $x\in B_{R/5}$

$$|\nabla^2 u(x)|^2\le C\fint_{B_{d(x)/2}(x)}|\nabla^2 u(y)|^2\,dy \tag{4.19}$$

where $d(x)$ denotes the distance from x to $\partial\Omega$. Denote by x_0 a point on the boundary of Ω such that $d(x)=|x-x_0|$. Since $x\in B_{R/5}=B_{R/5}(O)$, we have $x\in B_{R/5}(x_0)$, and therefore for $\alpha=2^{-100}$ we obtain

$$\fint_{B_{d(x)/2}(x)}|\nabla^2 u(y)|^2\,dy\le\frac{C}{d(x)^4}\fint_{C_{d(x)/2,3d(x)}(x_0)}|u(y)|^2\,dy\le\frac{C}{R^4}\fint_{C_{(1-\alpha)R,(4+\alpha)R}(x_0)}|u(y)|^2\,dy,$$

using Lemma 4.1 for the first estimate and (4.18) with $Q=x_0$ for the second one. Since

$$x\in B_{R/5}(O),\quad x\in B_{R/5}(x_0),\quad y\in C_{(1-\alpha)R,(4+\alpha)R}(x_0)\quad\Longrightarrow\quad y\in C_{R/2,\,5R}(O), \tag{4.20}$$

combining (4.19)– (4.20) we finish the proof. □

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Svitlana Mayboroda
Department of Mathematics, The Ohio State University,
231 W 18th Av., Columbus, OH, 43210, USA
svitlana@math.ohio-state.edu

Vladimir Maz'ya
Department of Mathematics, The Ohio State University,
231 W 18th Av., Columbus, OH, 43210, USA

Department of Mathematical Sciences, M&O Building,
University of Liverpool, Liverpool L69 3BX, UK

Department of Mathematics, Linköping University,
SE-581 83 Linköping, Sweden
vlmaz@math.ohio-state.edu, vlmaz@mai.liu.se