Bounds for eigenfunctions of the Laplacian on noncompact Riemannian manifolds

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Abstract

We deal with eigenvalue problems for the Laplacian on noncompact Riemannian manifolds M of finite volume. Sharp conditions ensuring $L^q(M)$ and $L^\infty(M)$ bounds for eigenfunctions are exhibited in terms of either the isoperimetric function or the isocapacitary function of M.

1 Introduction

We are concerned with a class of eigenvalue problems for the Laplacian on n-dimensional Riemannian manifolds M whose weak formulation is:

$$\int_M \langle \nabla u\,,\nabla v\rangle\, d\mathcal{H}^n(x) = \gamma \int_M u\, v\, d\mathcal{H}^n(x) \tag{1.1}$$

for every test function v in the Sobolev space $W^{1,2}(M)$. Here, $u \in W^{1,2}(M)$ is an eigenfunction associated with the eigenvalue $\gamma \in \mathbb{R}$, ∇ is the gradient operator, $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure on M, i.e. the volume measure on M induced by its Riemannian metric, and $\langle\cdot\,,\cdot\rangle$ stands for the associated scalar product.
Note that various special instances are included in this framework. For example, if M is a complete Riemannian manifold, then (1.1) is equivalent to a weak form of the equation

$$\Delta u + \gamma u = 0 \qquad \text{on } M. \tag{1.2}$$

In the case when M is an open subset of a Riemannian manifold, and in particular of the Euclidean space $\mathbb{R}^n$, equation (1.1) is a weak form of the eigenvalue problem obtained on coupling equation (1.2) with homogeneous Neumann boundary conditions.

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It is well known that quantitative information on eigenvalues and eigenfunctions for elliptic operators in open subsets of Euclidean space $\mathbb{R}^n$ can be derived in terms of geometric quantities associated with the domain. The quantitative analysis of spectral problems, especially for the Laplacian, on Riemannian manifolds is also very classical. A great deal of contributions to this topic regard compact manifolds. We do not even attempt to provide an exhaustive bibliography on contributions to this matter; let us just mention the reference monographs [Cha, BGM], and the papers [Bou, Br, BD, Che, CGY, DS, Do1, Do2, Es, Ga, Gr2, HSS, JMS, Na, SS, So, SZ, Ya].

The present paper focuses the case when

$$M \text{ need not be compact,}$$

although

$$\mathcal{H}^n(M) < \infty\,,$$

an assumption which will be kept in force throughout. We shall also assume that M is connected.

We are concerned with estimates for Lebesgue norms of eigenfunctions of the Laplacian on M. When M is compact, one easily infers, via local regularity results for elliptic equations, that any eigenfunction u of the Laplacian belongs to $L^\infty(M)$. Explicit bounds, with sharp dependence on the eigenvalue γ, are also available [SS, SZ], and require sophisticated tools from differential geometry and harmonic analysis. If the compactness assumption is dropped, then the membership of u in $W^{1,2}(M)$ only (trivially) entails that $u \in L^2(M)$. Higher integrability of eigenfunctions is not guaranteed anymore.

Our aim is to exhibit minimal assumptions on M ensuring $L^q(M)$ bounds for all $q < \infty$, or even $L^\infty(M)$ bounds for eigenfunctions of the Laplacian on M. The results that will be presented can easily be extended to linear uniformly elliptic differential operators, in divergence form, with merely measurable coefficients on M. However, we emphasize that our estimates are new even for the Neumann Laplacian on open subsets of $\mathbb{R}^n$ of finite volume.

The geometry of the manifold M will come into play through either the isocapacitary function ν_M, or the isoperimetric function λ_M of M. They are the largest functions of the measure of subsets of M which can be estimated by the capacity, or by the perimeter of the relevant subsets. Loosely speaking, the asymptotic behavior of ν_M and λ_M at 0 accounts for the regularity of the geometry of the noncompact manifold M: decreasing this regularity causes $\nu_M(s)$ and $\lambda_M(s)$ to decay faster to 0 as s goes to 0.

The inequalities associated with ν_M and λ_M are called the isocapacitary inequality and the isoperimetric inequality on M, respectively. Thus, the isoperimetric inequality on M reads

$$\lambda_M(\mathcal{H}^n(E)) \le P(E) \tag{1.3}$$

for every measurable set $E \subset M$ with $\mathcal{H}^n(E) \le \mathcal{H}^n(M)/2$, where $\lambda_M : [0, \mathcal{H}^n(M)/2] \to [0, \infty)$. In the isocapacitary inequality that we are going to exploit, the perimeter on the right-hand side of (1.3) is replaced by the condenser capacity of E with respect to any subset $G \supset E$. The resulting inequality has the form

$$\nu_M(\mathcal{H}^n(E)) \le C(E, G) \tag{1.4}$$

for every measurable sets $E \subset G \subset M$ with $\mathcal{H}^n(G) \le \mathcal{H}^n(M)/2$. Here, $C(E, G)$ denotes the capacity of the condenser $(E; G)$, and $\nu_M : [0, \mathcal{H}^n(M)/2] \to [0, \infty]$ (see Section 3 for precise definitions).

Introduced in [Ma1], the isoperimetric function λ_M has been employed to provide necessary and sufficient conditions for embeddings of the Sobolev space $W^{1,1}(M)$ when M is a domain in $\mathbb{R}^n$ [Ma1], and in a priori estimates for solutions to elliptic boundary value problems [Ma2, Ma6]. Isocapacitary functions were introduced and used in [Ma1, Ma3, Ma4, Ma5, Ma7] in the characterization of Sobolev embeddings for $W^{1,p}(M)$, with $p > 1$, when M is a domain in $\mathbb{R}^n$. Extensions to the case of Riemannian manifolds can be found in [Gr1, Gr2].

Both the conditions in terms of ν_M, and those in terms of λ_M, for eigenfunction estimates in $L^q(M)$ or $L^\infty(M)$ that will be presented are sharp in the class of manifolds M with prescribed asymptotic behavior of ν_M and λ_M at 0. Each one of these two approaches has its own advantages. The isoperimetric function λ_M has a transparent geometric character, and it is usually easier to investigate. The isocapacitary function can be less simple to compute; however its use is in a sense more appropriate in the present framework since it not only implies the results involving λ_M, but leads to finer conclusions in general. Typically, this is the case when manifolds with complicated geometric configurations are taken into account.

As for the proofs, let us just mention here that crucial use is made of the isocapacitary inequality (1.4) applied when E is any level set of an eigenfunction u. Note that customary methods, such as Moser iteration technique, which can be exploited to derive eigenfunction estimates in classical situations (see e.g. [Sa]), are of no utility in the present framework. In fact, Moser technique would require a Sobolev embedding theorem for $W^{1,2}(M)$ into some Lebesgue space smaller than $L^2(M)$, and this will not be guaranteed under the assumptions of our results.

The paper is organized as follows. The main results are stated in the next section. The subsequent Section 3 contains some basic definitions and properties concerning perimeter and capacity which enter in our discussion. In Section 4 we analyze a class of manifolds of revolution, which are used as model manifolds in the proof of the optimality of our results and in some examples. In particular, the behavior of their isoperimetric and isocapacitary functions is investigated. Proofs of our bounds in $L^q(M)$ and in $L^\infty(M)$ are the object of Section 5 and Section 6, respectively, where explicit estimates depending on eigenvalues are also provided. The final Section 7 deals with applications of our results to two special instances: a family of manifolds of revolution whose profile has a borderline exponential behavior, and a family of manifolds with a sequence of clustering mushroom-shaped submanifolds. In particular, the latter example demonstrates that the use of ν_M instead of λ_M can actually lead to stronger results when the regularity of eigenfunctions of the Laplacian is in question.

2 Main results

Our results will involve the manifold M only through the asymptotic behavior of either ν_M, or λ_M at 0. They are stated in Subsections 2.1 and 2.2, respectively.

Although the criteria involving λ_M admit independent proofs, along the same lines as those involving ν_M, the former will be deduced from the latter via the inequality:

$$\frac{1}{\nu_M(s)} \leq \int_s^{\mathcal{H}^n(M)/2} \frac{dr}{\lambda_M(r)^2} \qquad \text{for } s \in (0, \mathcal{H}^n(M)/2), \tag{2.1}$$

which holds for any manifold M (see the proof of [Ma7, Proposition 4.3.4/1]). Let us notice that a reverse inequality in (2.1) does not hold in general, even up to a multiplicative constant.

The proofs of the sharpness of the criteria for λ_M and ν_M require essentially the same construction. We shall again focus on the latter, and we shall explain how the relevant construction also applies to the former.

2.1 Eigenvalue estimates via the isocapacitary function of M

We begin with an optimal condition on the decay of ν_M at 0 ensuring $L^q(M)$ estimates for eigenfunctions of the Laplacian on M for $q \in (2, \infty)$. Interestingly enough, such a condition is independent of q.

Theorem 2.1 [**L^q bounds for eigenfunctions via** ν_M] *Assume that*

$$\lim_{s \to 0} \frac{s}{\nu_M(s)} = 0\,. \tag{2.2}$$

Then for any $q \in (2, \infty)$ and for any eigenvalue γ, there exists a constant $C = C(\nu_M, q, \gamma)$ such that

$$\|u\|_{L^q(M)} \leq C\|u\|_{L^2(M)} \tag{2.3}$$

for every eigenfunction u of the Laplacian on M associated with γ.

An estimate for the constant C in inequality (2.3) can also be provided – see Proposition 5.1, Section 5.

Let us note that condition (2.2) turns out to be equivalent to the compactness of the embedding $W^{1,2}(M) \to L^2(M)$ [CM]. Hence, in particular, the variational characterization of the eigenvalues of the Laplacian on M entails that they certainly exist under (2.2).

Incidentally, let us also mention that, when M is a complete manifold, condition (2.2) is also equivalent to the discreteness of the spectrum of the Laplacian on M [CM].

The next result shows that assumption (2.2) is essentially minimal in Theorem 2.1, in the sense that $L^q(M)$ regularity of eigenfunctions may fail under the mere assumption that

$$\nu_M(s) \approx s \qquad \text{near } 0.$$

Here, and in what follows, the notation

$$f \approx g \tag{2.4}$$

for functions $f, g : (0, \infty) \to [0, \infty)$ means that there exist positive constants c_1 and c_2 such that

$$c_1 g(c_1 s) \leq f(s) \leq c_2 g(c_2 s) \qquad \text{for } s > 0. \tag{2.5}$$

Condition (2.5) is said to hold near 0, or near infinity, if there exists a constant $s_0 > 0$ such that (2.5) holds for $0 < s \leq s_0$ or for $s \geq s_0$, respectively.
As in (2.2), all criteria that will be presented are invariant under replacement of ν_M or λ_M with functions $\approx$ near 0.

Theorem 2.2 [**Sharpness of condition** (2.2)] *For any $n \geq 2$ and $q \in (2, \infty]$, there exists an n-dimensional Riemannian manifold M such that*

$$\nu_M(s) \approx s \qquad \textit{near } 0, \tag{2.6}$$

and the Laplacian on M has an eigenfunction $u \notin L^q(M)$.

The important case when $q = \infty$, corresponding to the problem of the boundedness of eigenfunctions, is not included in Theorem 2.1. This is the object of the following result, where a slight strengthening of assumption (2.2) is shown to yield $L^\infty(M)$ estimates for eigenfunctions of the Laplacian on M.

Theorem 2.3 **[Boundedness of eigenfunctions via ν_M]** *Assume that*

$$\int_0 \frac{ds}{\nu_M(s)} < \infty \,. \tag{2.7}$$

Then for any eigenvalue γ, there exists a constant $C = C(\nu_M, \gamma)$ such that

$$\|u\|_{L^\infty(M)} \leq C\|u\|_{L^2(M)} \tag{2.8}$$

for every eigenfunction u of the Laplacian on M associated with γ.

An estimate for the constant C in inequality (2.8) is given in Proposition 6.1, Section 6.

Condition (2.7) in Theorem 2.3 is essentially sharp for the boundedness of eigenfunctions of the Laplacian on M. In particular, it cannot be relaxed to (2.2), although the latter ensures $L^q(M)$ estimates for every $q < \infty$. Indeed, under some mild qualification, Theorem 2.4 below asserts that given (up to equivalence) any isocapacitary function fulfilling (2.2) but not (2.7), there exists a manifold M with the prescribed isocapacitary function on which the Laplacian has an unbounded eigenfunction.
A precise statement of this result involves the notion of function of class Δ_2. Recall that a non-decreasing function $f : (0, \infty) \to [0, \infty)$ is said to belong to the class Δ_2 near 0 if there exist constants c and s_0 such that

$$f(2s) \leq cf(s) \qquad \text{if } 0 < s \leq s_0. \tag{2.9}$$

Theorem 2.4 **[Sharpness of condition** (2.7) **]** *Let ν be a non-decreasing function, vanishing only at 0, such that*

$$\lim_{s\to 0} \frac{s}{\nu(s)} = 0\,,$$

but

$$\int_0 \frac{ds}{\nu(s)} = \infty\,.$$

Assume in addition that $\nu \in \Delta_2$ near 0, and that either $n \geq 3$ and

$$\frac{\nu(s)}{s^{\frac{n-2}{n}}} \approx \textit{ a non-decreasing function near } 0, \tag{2.10}$$

or $n = 2$ and there exists $\alpha > 0$ such that

$$\frac{\nu(s)}{s^\alpha} \approx \textit{ a non-decreasing function near } 0. \tag{2.11}$$

Then, there exists an n-dimensional Riemannian manifold M fulfilling

$$\nu_M(s) \approx \nu(s) \qquad \textit{near } 0, \tag{2.12}$$

and such that the Laplacian on M has an unbounded eigenfunction.

Assumption (2.10) or (2.11) in Theorem 2.4 is explained by the fact that, if M is compact, then

$$\nu_M(s) \approx \begin{cases} s^{\frac{n-2}{n}} & \text{if } n \geq 3, \\ \left(\log \frac{1}{s}\right)^{-1} & \text{if } n = 2, \end{cases} \tag{2.13}$$

near 0, and that $\nu_M(s)$ cannot decay more slowly to 0 as $s \to 0$ in general. The assumption that $\nu \in \Delta_2$ near 0 is due to technical reasons.

2.2 Eigenvalue estimates via the isoperimetric function

The following criterion for $L^q(M)$ bounds of eigenfunctions in terms of the isoperimetric function λ_M can be derived via Theorem 2.1 and inequality (2.1).

Theorem 2.5 [**L^q bounds for eigenfunctions via** λ_M] *Assume that*

$$\lim_{s\to 0} \frac{s}{\lambda_M(s)} = 0\,. \tag{2.14}$$

Then for any $q \in (2, \infty)$ and any eigenvalue γ, there exists a constant $C = C(\lambda_M, q, \gamma)$ such that

$$\|u\|_{L^q(M)} \le C\|u\|_{L^2(M)} \tag{2.15}$$

for every eigenfunction u of the Laplacian on M associated with γ.

An analogue of Theorem 2.2 on the minimality of assumption (2.14) in Theorem 2.5 is contained in the the next result, showing that, for every $q > 2$, eigenfunctions which do not belong to $L^q(M)$ may actually exist when

$$\lambda_M(s) \approx s \qquad \text{near } 0.$$

Theorem 2.6 [**Sharpness of condition** (2.14)] *For any $n \ge 2$ and $q \in (2, \infty]$, there exists an n-dimensional Riemannian manifold M such that*

$$\lambda_M(s) \approx s \qquad \textit{near } 0, \tag{2.16}$$

and the Laplacian on M has an eigenfunction $u \notin L^q(M)$.

A condition on λ_M, parallel to (2.7), ensuring the boundedness of eigenfunctions of the Laplacian on M follows from Theorem 2.3 and inequality (2.1).

Theorem 2.7 [**Boundedness of eigenfunctions via** λ_M] *Assume that*

$$\int_0 \frac{s}{\lambda_M(s)^2}\, ds < \infty\,. \tag{2.17}$$

Then for any eigenvalue γ, there exists a constant $C = C(\lambda_M, \gamma)$ such that

$$\|u\|_{L^\infty(M)} \le C\|u\|_{L^2(M)} \tag{2.18}$$

for every eigenfunction u of the Laplacian on M associated with γ.

Our last result tell us that the gap between condition (2.17), ensuring $L^\infty(M)$ bounds for eigenfunctions, and condition (2.14), yielding $L^q(M)$ bounds for any $q < \infty$, cannot be essentially filled.

Theorem 2.8 [**Sharpness of condition** (2.17)] *Let λ be a non-decreasing function, vanishing only at 0, such that*

$$\lim_{s\to 0} \frac{s}{\lambda(s)} = 0\,,$$

but

$$\int_0 \frac{s}{\lambda(s)^2}\, ds = \infty \, .$$

Assume in addition that

$$\frac{\lambda(s)}{s^{\frac{n-1}{n}}} \approx \text{ a non-decreasing function near } 0. \tag{2.19}$$

Then, there exists an n-dimensional Riemannian manifold M fulfilling

$$\lambda_M(s) \approx \lambda(s) \qquad \text{near } 0, \tag{2.20}$$

and such that the Laplacian on M has an unbounded eigenfunction.

Assumption (2.19) in Theorem 2.8 is required in the light of the fact that

$$\lambda_M(s) \approx s^{\frac{n-1}{n}} \qquad \text{near } 0 \tag{2.21}$$

for any compact manifold M, and that $\lambda_M(s)$ cannot decay more slowly to 0 as $s \to 0$ in the noncompact case.

3 Background and preliminaries

Let E be a measurable subset of M. The perimeter $P(E)$ of E is defined as

$$P(E) = \mathcal{H}^{n-1}(\partial^* E)\, ,$$

where $\partial^* E$ stands for the essential boundary of E in the sense of geometric measure theory, and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure on M induced by its Riemannian metric ([AFP, Ma7]). Recall that $\partial^* E$ agrees with the topological boundary ∂E of E when E is sufficiently regular, for instance an open subset of M with a smooth boundary. In the special case when $M = \Omega$, an open subset of $\mathbb{R}^n$, and $E \subset \Omega$, we have that $P(E) = \mathcal{H}^{n-1}(\partial^*_{\mathbb{R}^n} E \cap \Omega)$, where $\partial^*_{\mathbb{R}^n} E$ denotes the essential boundary of E in $\mathbb{R}^n$.
The isoperimetric function λ_M of M is defined as

$$\lambda_M(s) = \inf\{P(E) : s \leq \mathcal{H}^n(E) \leq \mathcal{H}^n(M)/2\} \qquad \text{for } s \in [0, \mathcal{H}^n(M)/2]\, . \tag{3.1}$$

The isoperimetric inequality (1.3) is just a rephrasing of definition (3.1). The point is thus to derive information about the function λ_M, which is explicitly known only for Euclidean balls and spheres [BuZa, Ci2, Ma7], convex cones [LP], and manifolds in special classes [BC, CF, CGL, GP, MHH, Kl, MJ, Pi, Ri]. Various qualitative and quantitative properties of λ_M are however available – see e.g. [BuZa, Ci1, HK, KM, La, Ma7]. In particular, since we are assuming that M is connected,

$$\lambda_M(s) > 0 \qquad \text{for } s \in (0, \mathcal{H}^n(M)/2], \tag{3.2}$$

as an analogous argument as in [Ma7, Lemma 3.2.4] shows.

The Sobolev space $W^{1,p}(M)$ is defined, for $p \in [1, \infty]$, as

$$W^{1,p}(M) = \{u \in L^p(M) : \ u \text{ is weakly differentiable on } M \text{ and } |\nabla u| \in L^p(M) \ \}\, .$$

We denote by $W^{1,p}_0(M)$ the closure in $W^{1,p}(M)$ of the set of smooth compactly supported functions on M.

The standard p-capacity of a set $E \subset M$ can be defined as

$$C_p(E) = \inf\left\{\int_M |\nabla u|^p \, dx : u \in W^{1,p}_0(M), u \geq 1 \text{ in some neighbourhood of } E\right\}. \tag{3.3}$$

A property concerning the pointwise behavior of functions is said to hold C_p-quasi everywhere in M, C_p-q.e. for short, if it is fulfilled outside a set of p-capacity zero.
Each function $u \in W^{1,p}(M)$ has a representative $\widetilde{u}$, called the precise representative, which is C_p-quasi continuous, in the sense that for every $\varepsilon > 0$, there exists a set $A \subset M$, with $C_p(A) < \varepsilon$, such that $f_{|M\setminus A}$ is continuous in $M\setminus A$. The function $\widetilde{u}$ is unique, up to subsets of p-capacity zero. In what follows, we assume that any function $u \in W^{1,p}(M)$ agrees with its precise representative.

In the light of a classical result in the theory of capacity ([Da, Proposition 12.4], [MZ, Corollary 2.25]), we adopt the following definition of capacity of a condenser. Given sets $E \subset G \subset M$, the capacity $C_p(E, G)$ of the condenser (E, G) relative to Ω is defined as

$$C_p(E,G) = \inf\left\{\int_M |\nabla u|^p \, dx : u \in W^{1,p}(M), u \geq 1 \ C_p\text{-q.e. in } E \text{ and } u \leq 0 \ C_p\text{-q.e. in } M \setminus G\right\}. \tag{3.4}$$

Accordingly, the p-isocapacitary function $\nu_{M,p} : [0, \mathcal{H}^n(M)/2] \to [0, \infty]$ of M is given by

$$\begin{aligned}\nu_{M,p}(s) = \inf\{C_p(E,G) : \ & E \text{ and } G \text{ are measurable subsets of } M \text{ such that} \\ & E \subset G \subset M \text{ and } s \leq \mathcal{H}^n(E) \leq \mathcal{H}^n(G) \leq \mathcal{H}^n(M)/2\} \qquad \text{for } s \in [0, \mathcal{H}^n(M)/2].\end{aligned} \tag{3.5}$$

The function $\nu_{M,p}$ is clearly non-decreasing. In what follows, we shall always deal with the left-continuous representative of $\nu_{M,p}$, which, owing to the monotonicity of $\nu_{M,p}$, is pointwise dominated by the right-hand side of (3.5). Note that

$$\nu_{M,1} = \lambda_M \tag{3.6}$$

as shown by an analogous argument as in [Ma7, Lemma 2.2.5].

When $p = 2$, the case of main interest in the present paper, we drop the index p in $C_p(E, G)$ and $\nu_{M,p}$, and simply set

$$C(E,G) = C_2(E,G)\,,$$

and

$$\nu_M = \nu_{M,2}.$$

By (3.2) and (2.1), one has that

$$\nu_M(s) > 0 \qquad \text{for } s \in [0, \mathcal{H}^n(M)/2]. \tag{3.7}$$

For any measurable function u on M, we define its distribution function $\mu_u : \mathbb{R} \to [0, \infty)$ as

$$\mu_u(t) = \mathcal{H}^n(\{x \in M : u(x) \geq t\}) \qquad \text{for } t \in \mathbb{R}.$$

Note that here μ_u is defined in terms of u, and not of $|u|$ as customary. The signed decreasing rearrangement $u^\circ : [0, \mathcal{H}^n(M)] \to [-\infty, \infty]$ of u is given by

$$u^\circ(s) = \sup\{t : \mu_u(t) \geq s\} \qquad \text{for } s \in [0, \mathcal{H}^n(M)].$$

The median of u is defined by

$$\mathrm{med}(u) = u^\circ(\mathcal{H}^n(M)/2)\,. \tag{3.8}$$

Since u and u° are equimeasurable functions, one has that

$$\|u^\circ\|_{L^q(0,\mathcal{H}^n(M))} = \|u\|_{L^q(M)} \tag{3.9}$$

for every $q \in [1,\infty]$. Moreover, by an analogous argument as in [CEG, Lemma 6.6], if $u \in W^{1,p}(M)$ for some $p \in [1,\infty]$, then

$$u^\circ \text{ is locally absolutely continuous in } (0,\mathcal{H}^n(M)). \tag{3.10}$$

Given $u \in W^{1,2}(M)$, we define the function $\psi_u : \mathbb{R} \to [0,\infty)$ as

$$\psi_u(t) = \int_0^t \frac{d\tau}{\int_{\{u=\tau\}} |\nabla u|\, d\mathcal{H}^{n-1}(x)} \qquad \text{for } t \in \mathbb{R}\,. \tag{3.11}$$

On making use of (a version on manifolds) of [Ma7, Lemma 2.2.2/1], one can easily show that if

$$\mathrm{med}(u) = 0\,, \tag{3.12}$$

then

$$\nu_M(\mathcal{H}^n(\{u \geq t\})) \leq \frac{1}{\psi_u(t)} \quad \text{for } t > 0, \tag{3.13}$$

and

$$\nu_M(s) \leq \frac{1}{\psi_u(u^\circ(s))} \qquad \text{for } s \in (0,\mathcal{H}^n(M)/2). \tag{3.14}$$

4 Manifolds of revolution

In this section we focus on a class of manifolds of revolution to be employed in our proofs of Theorems 2.2 and 2.4. Specifically, we investigate on their isoperimetric and isocapacitary functions.

Let $L \in (0,\infty]$, and let $\varphi : [0,L) \to [0,\infty)$ be a function in $C^1([0,L))$, such that

$$\varphi(r) > 0 \quad \text{for } r \in (0,L), \tag{4.1}$$

$$\varphi(0) = 0\,, \qquad \text{and} \qquad \varphi'(0) = 1\,. \tag{4.2}$$

Here, φ' denotes the derivative of φ. For $n \geq 2$, we call n-dimensional manifold of revolution M associated with φ the ball in $\mathbb{R}^n$ given, in polar coordinates, by $\{(r,\omega) : r \in [0,L), \omega \in \mathbb{S}^{n-1}\}$ and endowed with the Riemannian metric

$$ds^2 = dr^2 + \varphi(r)^2 d\omega^2\,. \tag{4.3}$$

Here, $d\omega^2$ stands for the standard metric on $\mathbb{S}^{n-1}$. Owing to our assumptions on φ, the metric (4.3) is of class $C^1(M)$. Note that, in particular,

$$\int_M u\, d\mathcal{H}^n = \int_{\mathbb{S}^{n-1}} \int_0^L u\,\varphi(r)^{n-1}\, dr\, d\sigma_{n-1}\,, \tag{4.4}$$

for any integrable function $u : M \to \mathbb{R}$. Here, σ_{n-1} denotes the $(n-1)$-dimensional Hausdorff measure on $\mathbb{S}^{n-1}$.

The length of the gradient of a function $u : M \to \mathbb{R}$ is defined by $|\nabla u| = \sqrt{\langle \nabla u\,, \nabla u \rangle}$, and takes the form

$$|\nabla u| = \sqrt{\Big(\frac{\partial u}{\partial r}\Big)^2 + \frac{1}{\varphi(r)^2}|\nabla_{\mathbb{S}^{n-1}} u|^2}, \tag{4.5}$$

where $\nabla_{\mathbb{S}^{n-1}}$ denotes the gradient operator on $\mathbb{S}^{n-1}$. Moreover, if u depends only on r, then

$$\Delta u = \frac{1}{\varphi(r)^{n-1}}\frac{d}{dr}\left(\varphi(r)^{n-1}\frac{du}{dr}\right). \tag{4.6}$$

Thus, for functions u depending only on r, equation (1.2) reduces to the ordinary differential equation

$$\frac{d}{dr}\left(\varphi(r)^{n-1}\frac{du}{dr}\right) + \gamma\varphi(r)^{n-1}u = 0 \qquad \text{for } r \in (0, L). \tag{4.7}$$

The membership of u in $W^{1,2}(M)$ reads

$$\int_0^L \left(u^2 + \left(\frac{du}{dr}\right)^2\right)\varphi(r)^{n-1}dr < \infty\,. \tag{4.8}$$

Now, fix any $r_0 \in (0, L)$, set

$$s_0 = \int_{r_0}^L \frac{d\rho}{\varphi(\rho)^{n-1}}, \tag{4.9}$$

and define $\psi : (0, L) \to \mathbb{R}$ as

$$\psi(r) = \int_{r_0}^r \frac{d\rho}{\varphi(\rho)^{n-1}} \qquad \text{for } r \in (0, L). \tag{4.10}$$

Under the change of variables

$$s = \psi(r),$$

$$v(s) = u(\psi^{-1}(s)),$$

and

$$p(s) = \varphi\big(\psi^{-1}(s)\big)^{2(n-1)},$$

equations (4.7) and (4.8) turn into

$$\frac{d^2v}{ds^2} + \gamma p(s)v = 0 \qquad \text{for } s \in (-\infty, s_0), \tag{4.11}$$

and

$$\int_{-\infty}^{s_0} \left(v^2p(s) + \left(\frac{dv}{ds}\right)^2\right)ds < \infty\,, \tag{4.12}$$

respectively.

We finally note that if $r \in (0,L)$ and $B(r) = \{(\rho,\omega) : \rho \in [0,r), \omega \in \mathbb{S}^{n-1}\}$, a ball on M centered at 0, then

$$\mathcal{H}^{n-1}(\partial(M \setminus B(r))) = \mathcal{H}^{n-1}(\partial B(r)) = \omega_{n-1}\varphi(r)^{n-1}\,, \tag{4.13}$$

and

$$\mathcal{H}^{n}(M \setminus B(r)) = \omega_{n-1}\int_r^L \varphi(\rho)^{n-1}d\rho\,, \tag{4.14}$$

where $\omega_{n-1} = \mathcal{H}^{n-1}(\mathbb{S}^{n-1})$.

The main result of this section is contained in the following theorem, which provides us (up to equivalence) with the functions λ_M and ν_M for a manifold of revolution M as above. In what follows, we set $n' = \frac{n}{n-1}$, the Hölder conjugate of n.

Theorem 4.1 *Let $L \in (0,\infty]$ and let $\varphi : [0,L) \to [0,\infty)$ be a function in $C^1([0,L))$ fulfilling (4.1) and (4.2) and such that:*
(i) $\lim_{r\to L}\varphi(r) = 0$;
(ii) there exists $L_0 \in (0,L)$ such that φ is decreasing and convex in (L_0,L);
(iii) $\int_0^L \varphi(\rho)^{n-1}\,d\rho < \infty$.
Then the metric of the n-dimensional manifold of revolution M built upon φ is of class $C^1(M)$, and $\mathcal{H}^n(M) < \infty$. Moreover, let λ_0 be the function implicitly defined by

$$\lambda_0\left(\omega_{n-1}\int_r^L \varphi(\rho)^{n-1}d\rho\right) = \omega_{n-1}\varphi(r)^{n-1} \qquad \text{for } r \in (L_0,L)\,, \tag{4.15}$$

and such that $\lambda_0(s) = \lambda_0\big(\omega_{n-1}\int_{L_0}^L \varphi(\rho)^{n-1}d\rho\big)$ for $s \in \big(0, \omega_{n-1}\int_{L_0}^L \varphi(r)^{n-1}dr\big)$. Then

$$\lambda_M(s) \approx \lambda_0(s) \qquad \text{near } 0\,, \tag{4.16}$$

and

$$\nu_M(s) \approx \frac{1}{\int_s^{\mathcal{H}^n(M)/2}\frac{dr}{\lambda_0(r)^2}} \qquad \text{near } 0. \tag{4.17}$$

Proof The fact that M is a Riemannian manifold of class C^1 follows from assumptions (4.1) and (4.2). Furthermore, by (iii),

$$\mathcal{H}^n(M) = \omega_{n-1}\int_0^L \varphi(\rho)^{n-1}d\rho < \infty.$$

As for (4.16) and (4.17), let us begin by observing that, since φ is decreasing in (L_0,L), the function λ_0 is increasing in $\big(0,\omega_{n-1}\int_{L_0}^L\varphi(r)^{n-1}dr\big)$. Moreover, there exists a constant C such that

$$\lambda_0(s) \le Cs^{1/n'} \tag{4.18}$$

for $s \in \big(0,\omega_{n-1}\int_{L_0}^L\varphi(r)^{n-1}dr\big)$. Indeed, since $\lim_{r\to L}\varphi(r) = 0$ and $-\varphi'(r)$ is a nonnegative non-increasing function in (L_0,L), one has that

$$\begin{aligned} -\varphi'(L_0)\int_r^L \varphi(\varrho)^{n-1}d\varrho &\ge -\varphi'(r)\int_r^L \varphi(\varrho)^{n-1}d\varrho \\ &\ge \int_r^L -\varphi'(\varrho)\varphi(\varrho)^{n-1}d\varrho = \frac{1}{n}\,\varphi(r)^n \qquad \text{for } r \in (L_0,L), \end{aligned} \tag{4.19}$$

whence (4.18) follows, owing to (4.15).
Define the map $\Phi : M \setminus \overline{B(L_0)} \to \mathbb{R}^n$ as

$$\Phi(r,\omega) = (\varphi(r), \omega) \qquad \text{for } (r,\omega) \in (L_0, L) \times \mathbb{S}^{n-1}. \tag{4.20}$$

Clearly, Φ is a diffeomorphism between $M \setminus \overline{B(L_0)}$ and $\Phi\big(M \setminus \overline{B(L_0)}\big)$.
Given any smooth function $v : M \setminus \overline{B(L_0)} \to \mathbb{R}$, we have that

(4.21)

$$\begin{aligned}
\int_{\Phi(M\setminus\overline{B(L_0)})} |\nabla(v\circ\Phi^{-1})|dx &= \int_{\mathbb{S}^{n-1}}\int_0^{\varphi(L_0)} \sqrt{(v\circ\Phi^{-1})_\varrho^2 + \frac{1}{\varrho^2}|\nabla_{\mathbb{S}^{n-1}}(v\circ\Phi^{-1})|^2}\,\varrho^{n-1}d\varrho d\sigma_{n-1}\\
&= \int_{\mathbb{S}^{n-1}}\int_{L_0}^{L} \sqrt{\frac{1}{\varphi'(r)^2}\Big(\frac{\partial v}{\partial r}\Big)^2 + \frac{1}{\varphi(r)^2}|\nabla_{\mathbb{S}^{n-1}}v|^2}\,\varphi(r)^{n-1}|\varphi'(r)|drd\sigma_{n-1}\\
&\le 2(1+\sup_{r\in[L_0,L)}|\varphi'(r)|)\int_{\mathbb{S}^{n-1}}\int_{L_0}^{L}\sqrt{\Big(\frac{\partial v}{\partial r}\Big)^2 + \frac{1}{\varphi(r)^2}|\nabla_{\mathbb{S}^{n-1}}v|^2}\,\varphi(r)^{n-1}drd\sigma_{n-1}\\
&= 2(1-\varphi'(L_0))\int_{M\setminus\overline{B(L_0)}}|\nabla v|d\mathcal{H}^n\,.
\end{aligned}$$

By approximation, the inequality between the leftmost side and the rightmost side of (4.21) continues to hold for any function of bounded variation v, provided that the integrals of the gradients are replaced by the total variations. In particular, on applying the resulting inequality to characteristic function of sets, we obtain that

$$\mathcal{H}^{n-1}(\partial(\Phi(E))) \le C\mathcal{H}^{n-1}(\partial E) \tag{4.22}$$

for every smooth set $E \subset M \setminus \overline{B(L_0)}$, where $C = 2(1-\varphi'(L_0))$. Given any such set E, the classical isoperimetric inequality in $\mathbb{R}^n$ tells us that

$$n^{1/n'}\omega_{n-1}^{1/n}\mathcal{L}^n(\Phi(E))^{1/n'} \le \mathcal{H}^{n-1}(\partial(\Phi(E)))\,, \tag{4.23}$$

where $\mathcal{L}^n$ denotes the Lebesgue measure in $\mathbb{R}^n$. On the other hand,

$$\mathcal{L}^n(\Phi(E)) = \int_{\mathbb{S}^{n-1}}\int_0^{\varphi(L_0)}\chi_{\Phi(E)}\varrho^{n-1}d\varrho d\sigma_{n-1} = \int_{\mathbb{S}^{n-1}}\int_{L_0}^{L}\chi_E\varphi(r)^{n-1}|\varphi'(r)|drd\sigma_{n-1}\,, \tag{4.24}$$

where χ_E and $\chi_{\Phi(E)}$ stand for the characteristic functions of the sets E and $\Phi(E)$, respectively.
Define $\Lambda : [0,L) \to [0,\mathcal{H}^n(M)]$ as

$$\Lambda(r) = \omega_{n-1}\int_r^L \varphi(\rho)^{n-1}d\rho \qquad \text{for } r\in[0,L),$$

whence, by (4.14), $\Lambda(r) = \mathcal{H}^n(M\setminus B(r)$ for $r\in[0,L)$. Since $|\varphi'| = -\varphi'$ in (L_0,L), and $-\varphi'$ is a non-increasing function in (L_0,L),

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}}\int_{L_0}^{L}\chi_E\varphi(r)^{n-1}|\varphi'(r)|drd\sigma_{n-1} &\ge \int_{\mathbb{S}^{n-1}}\int_{\Lambda^{-1}(\mathcal{H}^n(E))}^{L}\varphi(r)^{n-1}|\varphi'(r)|drd\sigma_{n-1}\\
&= \omega_{n-1}\int_{\Lambda^{-1}(\mathcal{H}^n(E))}^{L}\varphi(r)^{n-1}(-\varphi'(r))dr = \frac{\omega_{n-1}}{n}\varphi(\Lambda^{-1}(\mathcal{H}^n(E)))^n\,.
\end{aligned} \tag{4.25}$$

Combining (4.22)-(4.25) yields

$$C\varphi(\Lambda^{-1}(\mathcal{H}^n(E)))^{n-1} \le \mathcal{H}^{n-1}(\partial E) \tag{4.26}$$

for some positive constant C. By (4.15),

$$\omega_n \varphi(\Lambda^{-1}(\mathcal{H}^n(E)))^{n-1} = \lambda_0\Big(\omega_{n-1}\int_{\Lambda^{-1}(\mathcal{H}^n(E))}^{L} \varphi(\rho)^{n-1} d\rho\Big) = \lambda_0(\mathcal{H}^n(E)) \,. \tag{4.27}$$

From (4.26) and (4.27) we obtain that

$$C\lambda_0(\mathcal{H}^n(E)) \le \mathcal{H}^{n-1}(\partial E) \,, \tag{4.28}$$

from some constant C and any smooth set $E \subset M \setminus \overline{B(L_0)}$.

Now, let $L_1 \in (L_0, L)$ be such that $\mathcal{H}^n(B(L_1)) > \mathcal{H}^n(M)/2$. Observe that $\overline{B(L_1)}$ is a smooth compact Riemannian submanifold of M with boundary $\partial B(L_1)$ diffeormorphic to a closed ball in $\mathbb{R}^n$. Thus, an isoperimetric inequality of the form

$$C\mathcal{H}^n(E)^{1/n'} \le \mathcal{H}^{n-1}(\partial E) \tag{4.29}$$

holds for some constant C and for any set of finite perimeter $E \subset \overline{B(L_1)}$. Moreover, there exists a positive constant C such that

$$\mathcal{H}^{n-1}(E \cap \partial B(L_1)) \le C\mathcal{H}^{n-1}(\partial E \cap B(L_1)) \tag{4.30}$$

for any smooth set $E \subset \overline{B(L_1)}$ such that $\mathcal{H}^n(E) \le \mathcal{H}^n(M)/2$ ($< \mathcal{H}^n\big(\overline{B(L_1)}\big)$, by our choice of L_1).

Owing to (4.28)-(4.30), for any smooth set $E \subset M$ such that $\mathcal{H}^n(E) \le \mathcal{H}^n(M)/2$

$$\begin{aligned}\mathcal{H}^{n-1}(\partial E) &= \mathcal{H}^{n-1}(\partial(E \cap \overline{B(L_1)})) + \mathcal{H}^{n-1}(\partial(E \cap (M \setminus B(L_1)))) - 2\mathcal{H}^{n-1}(E \cap \partial B(L_1)) \\ &\ge C\mathcal{H}^n(E \cap \overline{B(L_1)})^{1/n'} + C\lambda_0(\mathcal{H}^n(E \cap (M \setminus B(L_1))) - C\mathcal{H}^{n-1}(B(L_1) \cap \partial E) \\ &\ge C\mathcal{H}^n(E \cap \overline{B(L_1)})^{1/n'} + C\lambda_0(\mathcal{H}^n(E \cap (M \setminus B(L_1))) - C\mathcal{H}^{n-1}(\partial E) \,,\end{aligned} \tag{4.31}$$

for some positive constant C. Consequently, there exists a constant C such that

$$C\mathcal{H}^{n-1}(\partial E) \ge \mathcal{H}^n(E \cap \overline{B(L_1)})^{1/n'} + \lambda_0(\mathcal{H}^n(E \cap (M \setminus B(L_1))) \tag{4.32}$$

for any smooth set $E \subset M$ such that $\mathcal{H}^n(E) \le \mathcal{H}^n(M)/2$. Now, we claim that there exists a constant C such that such that

$$s^{1/n'} + \lambda_0(\sigma) \ge C\lambda_0\Big(\frac{s+\sigma}{2}\Big) \qquad \text{for } s, \sigma \in [0, \mathcal{H}^n(M)/2]. \tag{4.33}$$

Indeed, if $\sigma \le s$, then, by (4.18),

$$s^{1/n'} \ge C\lambda_0(s) \ge C\lambda_0\Big(\frac{s+\sigma}{2}\Big) \tag{4.34}$$

for some positive constant C, whereas, if $s \le \sigma$, then trivially

$$\lambda_0(\sigma) \ge \lambda_0\Big(\frac{s+\sigma}{2}\Big) \,. \tag{4.35}$$

Coupling (4.32) and (4.33) yields

$$C\lambda_0(\mathcal{H}^n(E)/2) \le \mathcal{H}^{n-1}(\partial E) \tag{4.36}$$

for some positive constant C and for every smooth set $E \subset M$ such that $\mathcal{H}^n(E) \le \mathcal{H}^n(M)/2$. By approximation, inequality (4.36) continues to hold for every set of finite perimeter $E \subset M$ such that $\mathcal{H}^n(E) \le \mathcal{H}^n(M)/2$. Since λ_0 is non-decreasing, inequality (4.36) ensures that

$$\lambda_M(s) \ge C\lambda_0(s/2) \qquad \text{for } s \in [0, \mathcal{H}^n(M)/2]. \tag{4.37}$$

On the other hand, equation (4.15) entails that $\lambda_M(s) \le \lambda_0(s)$ for small s, and hence there exists a constant C such that

$$\lambda_M(s) \le C\lambda_0(s) \qquad \text{for } s \in [0, \mathcal{H}^n(M)/2]. \tag{4.38}$$

Equation (4.16) is fully proved.

As far as (4.17) is concerned, by (2.1) and (4.37),

$$\begin{aligned} \frac{1}{\nu_M(s)} &\le \int_s^{\mathcal{H}^n(M)/2} \frac{dr}{\lambda_M(r)^2} \le \frac{1}{C^2}\int_s^{\mathcal{H}^n(M)/2} \frac{dr}{\lambda_0(r/2)^2} \\ &\le \frac{2}{C^2}\int_{s/2}^{\mathcal{H}^n(M)/2} \frac{dr}{\lambda_0(r)^2} \qquad \text{for } s \in (0, \mathcal{H}^n(M)/2]. \end{aligned} \tag{4.39}$$

In order to prove a reverse inequality, set $R = \max\{L_0, \Lambda^{-1}(\mathcal{H}^n(M)/2)\}$. Moreover, given $s \in (0, \mathcal{H}^n(M)/2)$, let $r \in (R, L)$ be such that

$$s = \mathcal{H}^n(M \setminus B(r)) = \omega_{n-1}\int_r^L \varphi(\tau)^{n-1}\, d\tau\,. \tag{4.40}$$

Let $u = u(\rho)$ be the function given by

$$u(\rho) = \begin{cases} 0 & \text{if } \rho \in (0, R], \\ \dfrac{\int_R^\rho \frac{d\tau}{\varphi(\tau)^{n-1}}}{\int_R^r \frac{d\tau}{\varphi(\tau)^{n-1}}} & \text{if } \rho \in (R, r), \\ 1 & \text{if } \rho \in [r, L), \end{cases}$$

and let

$$E = M \setminus B(r) \qquad \text{and} \qquad G = M \setminus B(R).$$

Hence,

$$\begin{aligned} \nu_M(s) \le C(E,G) \le \int_M |\nabla u|^2 d\mathcal{H}^n(x) = \frac{\omega_{n-1}}{\int_R^r \frac{d\tau}{\varphi(\tau)^{n-1}}} \\ = \frac{1}{\int_s^{\Lambda(R)} \frac{d\rho}{\lambda_0(\rho)^2}} \le \frac{C}{\int_s^{\mathcal{H}^n(M)/2} \frac{d\rho}{\lambda_0(\rho)^2}} \qquad \text{for } s \in (0, \mathcal{H}^n(M)/2], \end{aligned} \tag{4.41}$$

for some constant C. Note that the second equality is a consequence of (4.15), owing to a change of variable. Equation (4.17) follows from (4.39) and (4.41). $\square$

From (4.16) and (4.17), it is easily verified that conditions (2.2) and (2.14) are equivalent for manifolds of revolution as in Theorem 4.1. Moreover, these conditions can be characterized in terms of the function φ appearing in its statement. The same observation applies to (2.7) and (2.17). These observations are summarized in the following statement.

Corollary 4.2 *Let φ be as in Theorem 4.1, and let M be the n-dimensional manifold of revolution built upon φ. Then:*
(i) Conditions (2.2), (2.14), *and*

$$\lim_{r\to L}\left(\int_R^r \frac{d\varrho}{\varphi(\varrho)^{n-1}}\right)\left(\int_r^L \varphi(\varrho)^{n-1}d\varrho\right)=0$$

for any $R\in(0,L)$ are equivalent.
(ii) Conditions (2.7), (2.17), *and*

$$\int^L \left(\frac{1}{\varphi(r)^{n-1}}\int_r^L \varphi(\rho)^{n-1}d\rho\right)dr<\infty$$

are equivalent.

The remaining part of this section is devoted to showing that, given functions ν and λ as in the statements of Theorems 2.4 and 2.8, respectively, there do exist a manifold of revolution M fulfilling $\nu_M\approx\nu$ and a manifold of revolution M fulfilling $\lambda_M\approx\lambda$. This is accomplished in the following Proposition 4.3, dealing with λ, and in Proposition 4.5, dealing with ν.

Proposition 4.3 *Let $n\geq 2$, and let $\lambda:[0,\infty)\to[0,\infty)$ be such that*

$$\frac{\lambda(s)}{s^{1/n'}}\approx \text{ a non-decreasing function near } 0. \tag{4.42}$$

Then there exist $L\in(0,\infty]$ and $\varphi:[0,L)\to[0,\infty)$ as in the statement of Theorem 4.1 such that:
(i) the n-dimensional manifold of revolution M associated with φ fulfills (4.15) *for some function λ_0 such that*

$$\lambda_0\approx\lambda \qquad \text{near } 0; \tag{4.43}$$

(ii) the isoperimetric function λ_M of M fulfills

$$\lambda_M\approx\lambda \qquad \text{near } 0. \tag{4.44}$$

Moreover, $L=\infty$ if and only if

$$\int_0 \frac{dr}{\lambda(r)}=\infty\,. \tag{4.45}$$

Remark 4.4 If

$$\int_0 \frac{r}{\lambda(r)^2}dr=\infty\,, \tag{4.46}$$

then (4.45) holds. Indeed, if (4.45) fails, namely if

$$\int_0 \frac{dr}{\lambda(r)}<\infty\,, \tag{4.47}$$

then $\lim_{s\to 0}\frac{s}{\lambda(s)}=0$, and this limit, combined with (4.47), implies the convergence of the integral in (4.46).

Proof of Proposition 4.3. Let V be a positive number such that (4.42) holds in $(0,V)$; namely, there exists a non-decreasing function ϑ such that

$$\frac{\lambda(s)^{n'}}{s} \approx \vartheta(s) \qquad \text{for } s \in (0,V).$$

Thus, the function

$$\lambda_1(s) = (s\vartheta(s))^{1/n'}$$

satisfies

$$\lambda_1 \approx \lambda \qquad \text{in } (0,V),$$

and

$$\frac{\lambda_1(s)^{n'}}{s} \qquad \text{is non-decreasing in } (0,V). \tag{4.48}$$

Assumption (4.48) in turn ensures that, on defining

$$\lambda_2(s) = \left(\int_0^s \frac{\lambda_1(r)^{n'}}{r}dr\right)^{1/n'} \qquad \text{for } s \in (0,V),$$

we have that $\lambda_2 \in C^0(0,V)$, $\lambda_2^{n'}$ is convex in $(0,V)$, and $\lambda_2 \approx \lambda_1 \approx \lambda$ in $(0,V)$. Similarly, on setting

$$\lambda_3(s) = \left(\int_0^s \frac{\lambda_2(r)^{n'}}{r}dr\right)^{1/n'} \qquad \text{for } s \in (0,V),$$

we have that $\lambda_3 \in C^1(0,V)$, $\lambda_3^{n'}$ is convex in $(0,V)$, and $\lambda_3 \approx \lambda_2 \approx \lambda_1 \approx \lambda$ in $(0,V)$.
Thus, in what follows we may assume, on replacing if necessary λ by λ_3 near 0, that λ is a non-decreasing function in $[0,L)$ such that $\lambda \in C^1(0,V)$, $\lambda(0)=0$, and

$$\lambda^{n'} \qquad \text{is convex in } (0,V). \tag{4.49}$$

Define

$$L = 2\int_0^{V/2} \frac{dr}{\lambda(r)}\,, \tag{4.50}$$

and note that $L=\infty$ if and only if (4.45) is in force. Next, set

$$R = \begin{cases} L/2 & \text{if } L<\infty,\\ 1 & \text{if } L=\infty. \end{cases}$$

Let $N : [R,L) \to [0,V/2]$ be the function implicitly defined by

$$\int_{N(r)}^{V/2} \frac{dr}{\lambda(r)} = r - R \qquad \text{for } r \in [R,L)\,. \tag{4.51}$$

Clearly, $N \in C^1(R,L)$ and N decreases monotonically from $V/2$ to 0. Define $\varphi : [R,L) \to [0,\infty)$ as

$$\varphi(r) = \left(\frac{\lambda(N(r))}{\omega_{n-1}}\right)^{\frac{1}{n-1}} \qquad \text{for } r \in (R,L)\,, \tag{4.52}$$

and observe that $\varphi \in C^1(R, L)$. Since,

$$\lambda(N(r)) = -N'(r) \qquad \text{for } r \in [R, L)\,, \tag{4.53}$$

and $N(L) = 0$, one has that

$$\int_r^L \lambda(N(\rho))d\rho = N(r) \qquad \text{for } r \in [R, L)\,, \tag{4.54}$$

whence

$$\lambda\Big(\int_r^L \lambda(N(\rho))d\rho\Big) = \lambda(N(r)) \qquad \text{for } r \in [R, L)\,, \tag{4.55}$$

and finally, by (4.52),

$$\lambda\bigg(\omega_{n-1}\int_r^L \varphi(\rho)^{n-1}d\rho\bigg) = \omega_{n-1}\varphi(r)^{n-1} \qquad \text{for } r \in [R, L)\,, \tag{4.56}$$

namely (4.15) with λ_0 replaced by λ.
Now, observe that the function φ is decreasing in (R, L) and $\lim_{r\to L}\varphi(r) = 0$. Furthermore, φ is convex owing to (4.49). Indeed, by (4.53),

$$\begin{aligned} \omega_{n-1}^{\frac{1}{n-1}}\varphi'(r) = \big(\lambda(N(r))^{\frac{1}{n-1}}\big)' &= \frac{1}{n-1}\lambda'(N(r))\lambda(N(r))^{\frac{1}{n-1}-1}N'(r) \\ &= -\frac{1}{n-1}\lambda'(N(r))\lambda(N(r))^{\frac{1}{n-1}} \quad \text{for } r \in (R, L). \end{aligned} \tag{4.57}$$

Thus, since $N(r)$ is decreasing, $\varphi'(r)$ is increasing if and only if $-\lambda'(s)\lambda(s)^{\frac{1}{n-1}}$ is decreasing, namely if and only if $\lambda'(s)\lambda(s)^{\frac{1}{n-1}}$ is increasing, and this is in turn equivalent to the convexity of $\lambda(s)^{n'}$.
Finally, let us continue φ smoothly to the whole of $[0, L)$ in such a way that (4.1) and (4.2) are fulfilled, and that

$$\omega_{n-1}\int_0^R \varphi(r)^{n-1}\,dr = \omega_{n-1}\int_R^L \varphi(r)^{n-1}\,dr = N(R) = \frac{V}{2}.$$

The resulting function φ fulfils the assumptions of Theorem 4.1. Hence, the conclusion follows. □

Proposition 4.5 *Let $n \geq 2$, and let $\nu : [0, \infty) \to [0, \infty)$ be a function such that $\nu \in \Delta_2$ near 0, and either $n \geq 3$ and*

$$\frac{\nu(s)}{s^{\frac{n-2}{n}}} \qquad \textit{is equivalent to a non-decreasing function near } 0, \tag{4.58}$$

or $n = 2$ and there exists $\alpha > 0$ such that

$$\frac{\nu(s)}{s^{\alpha}} \qquad \textit{is equivalent to a non-decreasing function near } 0. \tag{4.59}$$

Then there exist $L \in (0, \infty]$ and $\varphi : [0, L) \to [0, \infty)$ as in the statement of Theorem 4.1, such that:

(i) the n-dimensional manifold of revolution M built upon φ fulfills (4.15) *for some function λ_0 such that $\lambda_0 \approx \lambda_M$ near* 0*;*
(ii)

$$\nu(s) \approx \nu_M(s) \approx \frac{1}{\int_s^{\mathcal{H}^n(M)/2} \frac{dr}{\lambda_M(r)^2}} \qquad \text{near } 0\,. \tag{4.60}$$

Moreover, $L = \infty$ if and only if

$$\int_0 \frac{dr}{\sqrt{r\nu(r)}} = \infty\,. \tag{4.61}$$

Remark 4.6 If

$$\int_0 \frac{dr}{\nu(r)} = \infty\,, \tag{4.62}$$

then (4.61) holds. This is a consequence of the fact that there exists an absolute constant C such that

$$\left(\int_0^1 f(s)^2 ds\right)^{1/2} \leq C \int_0^1 f(s) \frac{ds}{\sqrt{s}}$$

for every non-increasing function $f : (0,1) \to [0,\infty)$.

Proof of Proposition 4.5. Let us assume that $n \geq 3$, the case when $n = 2$ being analogous. Let V be a positive number such that $\nu \in \Delta_2$ in $(0,V)$ and (4.58) holds in $(0,V)$. An analogous argument as at the beginning of the proof of Proposition 4.3 tells us that on replacing ν, if necessary, by an equivalent function, we may assume that $\nu \in C^1(0,V)$, $\nu'(s) > 0$ for $s \in (0,V)$, and

$$s\nu'(s) \approx \nu(s) \qquad \text{for } s \in (0,V). \tag{4.63}$$

Define $\lambda : (0,V) \to (0,\infty)$ by

$$\lambda(s) = \frac{\nu(s)}{\sqrt{\nu'(s)}} \qquad \text{for } s \in (0,V). \tag{4.64}$$

Given any $a \in (0,V)$, we thus have that

$$\frac{1}{\nu(s)} - \frac{1}{\nu(a)} = \int_s^a \frac{dr}{\lambda(r)^2} \qquad \text{for } s \in (0,V), \tag{4.65}$$

and hence there exists $\overline{s}$ such that

$$\frac{1}{2\nu(s)} \leq \int_s^a \frac{dr}{\lambda(r)^2} \leq \frac{1}{\nu(s)} \qquad \text{if } 0 < s < \overline{s}. \tag{4.66}$$

Moreover,

$$\frac{\lambda(s)}{s^{1/n'}} \quad \text{is non-decreasing in } (0,V). \tag{4.67}$$

Indeed, by (4.63) and by the Δ_2 condition for ν in $(0,V)$,

$$\frac{\lambda(s)^2}{s^{\frac{2}{n'}}} = \frac{\nu(s)^2}{\nu'(s)s^{\frac{2}{n'}}} \approx \frac{\nu(s)}{s^{\frac{n-2}{n}}} \qquad \text{for } s \in (0,V)\,. \tag{4.68}$$

Owing to (4.68) and (4.58), the function λ fulfills the assumptions of Proposition 4.3. Let M be the n dimensional manifold of revolution associated with λ as in Proposition 4.3. By (4.43), (4.44), (4.16) and (4.17),

$$\frac{1}{\nu_M(s)} \approx \int_s^{\mathcal{H}^n(M)/2} \frac{dr}{\lambda(r)^2} \qquad \text{near } 0. \tag{4.69}$$

On the other hand, by (4.66),

$$\int_s^{\mathcal{H}^n(M)/2} \frac{dr}{\lambda(r)^2} \approx \frac{1}{\nu(s)} \qquad \text{near } 0. \tag{4.70}$$

Equation (4.60) follows from (4.69) and (4.70).

As for the assertion concerning (4.61), by Proposition 4.3 one has that $L = \infty$ if and only if the function λ given by (4.64) fulfils (4.45). One has that

$$\int_0^V \frac{ds}{\lambda(s)} = \int_0^V \frac{\sqrt{\nu'(s)}}{\nu(s)} \geq C \int_0^V \frac{\sqrt{\nu(Cs)}}{\sqrt{s}\nu(s)} ds \geq C' \int_0^V \frac{ds}{\sqrt{s\nu(s)}} \tag{4.71}$$

for suitable constants C and C', where the first inequality holds by (4.63) and the last one by the Δ_2 condition for ν. An analogous chain yields

$$\int_0^V \frac{ds}{\lambda(s)} \leq C \int_0^V \frac{ds}{\sqrt{s\nu(s)}}$$

for a suitable positive constant C. Hence, equation (4.61) is equivalent to $L = \infty$. □

5 L^q bounds for eigenfunctions

We begin with the proof of Theorem 2.1 on $L^q(M)$ bounds for eigenfunctions.

Proof of Theorem 2.1. Given $s \in (0, \mathcal{H}^n(M))$ and $h > 0$, choose the test function v defined as

$$v(x) = \begin{cases} 0 & \text{if } u(x) < u^\circ(s+h) \\ u(x) - u^\circ(s+h) & \text{if } u^\circ(s+h) \leq u(x) \leq u^\circ(s) \\ u^\circ(s) - u^\circ(s+h) & \text{if } u^\circ(s) < u(x)\,, \end{cases} \tag{5.1}$$

in equation (1.1). Notice that $v \in W^{1,2}(M)$ by standard results on truncations of Sobolev functions. We thus obtain that

$$\begin{aligned} &\int_{\{u^\circ(s+h)<u<u^\circ(s)\}} |\nabla u|^2 \, d\mathcal{H}^n(x) \qquad (5.2)\\ &= \gamma \int_{\{u^\circ(s+h)<u\leq u^\circ(s)\}} u(x)\big(u(x) - u^\circ(s+h)\big)\, d\mathcal{H}^n(x) + \big(u^\circ(s) - u^\circ(s+h)\big)\gamma \int_{\{u>u^\circ(s)\}} u(x)\, d\mathcal{H}^n(x)\,. \end{aligned}$$

Consider the function $U : (0, \mathcal{H}^n(M)) \to [0, \infty)$ given by

$$U(s) = \int_{\{u\leq u^\circ(s)\}} |\nabla u|^2 \, d\mathcal{H}^n(x) \qquad \text{for } s \in (0, \mathcal{H}^n(M)). \tag{5.3}$$

By (3.10), the function u° is locally absolutely continuous (a.c., for short) in $(0, \mathcal{H}^n(M))$. The function

$$(0,\infty) \ni t \mapsto \int_{\{u\leq t\}} |\nabla u|^2 \, d\mathcal{H}^n(x)$$

is also locally a.c., inasmuch as, by the coarea formula,

$$\int_{\{u\leq t\}} |\nabla u|^2 \, d\mathcal{H}^n(x) = \int_{-\infty}^{t} \int_{\{u=\tau\}} |\nabla u| d\mathcal{H}^{n-1}(x) d\tau \qquad \text{for } t \in \mathbb{R}. \tag{5.4}$$

Consequently, U is locally a.c., for it is the composition of monotone locally a.c. functions, and

$$U'(s) = -u^{\circ\prime}(s) \int_{\{u=u^\circ(s)\}} |\nabla u| d\mathcal{H}^{n-1}(x) \quad \text{for a.e. } s \in (0, \mathcal{H}^n(M)). \tag{5.5}$$

Thus, dividing through by h in (5.2), and passing to the limit as $h \to 0^+$ yield

$$-u^{\circ\prime}(s) \int_{\{u=u^\circ(s)\}} |\nabla u| d\mathcal{H}^{n-1}(x) = \gamma(-u^{\circ\prime}(s)) \int_{\{u>u^\circ(s)\}} u \, d\mathcal{H}^n(x) \qquad \text{for a.e. } s \in (0, \mathcal{H}^n(M)). \tag{5.6}$$

On the other hand, it is easily verified via the definition of signed rearrangement that

$$(-u^{\circ\prime}(s)) \int_{\{u>u^\circ(s)\}} u(x)\, dx = (-u^{\circ\prime}(s)) \int_0^s u^\circ(r)\, dr \qquad \text{for a.e. } s \in (0, \mathcal{H}^n(M)). \tag{5.7}$$

Coupling (5.6) and (5.7) tells us that

$$- u^{\circ\prime}(r) = \frac{-u^{\circ\prime}(r)\, \gamma}{\int_{\{u=u^\circ(r)\}} |\nabla u| \, d\mathcal{H}^{n-1}(x)} \int_0^r u^\circ(\varrho)\, d\varrho \qquad \text{for a.e. } r \in (0, \mathcal{H}^n(M)). \tag{5.8}$$

Let $0 < s \leq \varepsilon \leq \mathcal{H}^n(M)/2$. On integrating both sides of (5.8) over the interval (s, ε), we obtain that

$$u^\circ(s) - \gamma \int_s^\varepsilon \left(\int_0^r u^\circ(\varrho) d\varrho \right) \big(- \psi_u(u^\circ(r))\big)' \, dr = u^\circ(\varepsilon) \qquad \text{for } s \in (0, \varepsilon), \tag{5.9}$$

where ψ_u is the function defined as in (3.11). Define the operator T as

$$Tf(s) = \int_s^\varepsilon \left(\int_0^r f(\varrho) d\varrho \right) \big(- \psi_u(u^\circ(r))\big)' \, dr \qquad \text{for } s \in (0, \varepsilon), \tag{5.10}$$

for any integrable function f on $(0, \varepsilon)$. Equation (5.9) thus reads

$$(I - \gamma T)(u^\circ) = u^\circ(\varepsilon)\, . \tag{5.11}$$

We want now to show that, if (2.2) holds, then

$$T : L^q(0, \varepsilon) \to L^q(0, \varepsilon), \tag{5.12}$$

and there exist an absolute constant C such that

$$\|T\|_{(L^q(0,\varepsilon) \to L^q(0,\varepsilon))} \leq C\Theta(\varepsilon)\, , \tag{5.13}$$

where $\Theta : (0, \mathcal{H}^n(M)/2] \to [0, \infty)$ is the function defined as

$$\Theta(s) = \sup_{r \in (0,s)} \frac{r}{\nu_M(r)} \qquad \text{for } s \in (0, \mathcal{H}^n(M)/2]. \tag{5.14}$$

Set

$$v = u - \operatorname{med}(u), \tag{5.15}$$

and observe that

$$\operatorname{med}(v) = 0\,, \tag{5.16}$$

$$v^\circ = u^\circ - \operatorname{med}(u)\,,$$

and

$$\big(\psi_u(u^\circ(s))\big)' = \big(\psi_v(v^\circ(s))\big)' \qquad \text{for } s \in (0, \mathcal{H}^n(M)). \tag{5.17}$$

Moreover,

$$v^\circ(s) \geq 0 \qquad \text{if } s \in (0, \mathcal{H}^n(M)/2). \tag{5.18}$$

Given any $q \in [2, \infty)$, $f \in L^q(0, \varepsilon)$ and $0 < s \leq \varepsilon \leq \mathcal{H}^n(M)/2$, the following chain holds:

$$\begin{aligned}
|Tf(s)| &= \left| \int_s^\varepsilon \left(\int_0^r f(\varrho) d\varrho \right) \big(- \psi_u(u^\circ(r))\big)' \, dr \right| \\
&= \left| \int_s^\varepsilon \left(\int_0^r f(\varrho) d\varrho \right) \big(- \psi_v(v^\circ(r))\big)' \, dr \right| \\
&\hspace{6cm} \text{(by (5.17))} \\
&\leq \int_s^\varepsilon \left(\int_0^r |f(\varrho)| d\varrho \right) \frac{d}{dr} \left(- \int_0^{v^\circ(r)} \frac{d\tau}{\int_{\{v=\tau\}} |\nabla v| \, d\mathcal{H}^{n-1}(x)} \right) dr \\
&= \left(\int_s^\varepsilon \frac{d}{dr} \left(- \int_0^{v^\circ(r)} \frac{d\tau}{\int_{\{v=\tau\}} |\nabla v| \, d\mathcal{H}^{n-1}(x)} \right) dr \right) \int_0^s |f(\varrho)| d\varrho \\
&\quad + \int_s^\varepsilon \left(\int_\varrho^\varepsilon \frac{d}{dr} \left(- \int_0^{v^\circ(r)} \frac{d\tau}{\int_{\{v=\tau\}} |\nabla v| \, d\mathcal{H}^{n-1}(x)} \right) dr \right) |f(\varrho)| d\varrho \\
&\hspace{6cm} \text{(by Fubini's theorem)} \\
&= \left(\int_{v^\circ(\varepsilon)}^{v^\circ(s)} \frac{d\tau}{\int_{\{v=\tau\}} |\nabla v| \, d\mathcal{H}^{n-1}(x)} \right) \int_0^s |f(\varrho)| d\varrho \\
&\quad + \int_s^\varepsilon \int_{v^\circ(\varepsilon)}^{v^\circ(\varrho)} \frac{d\tau}{\int_{\{v=\tau\}} |\nabla v| \, d\mathcal{H}^{n-1}(x)} |f(\varrho)| d\varrho \\
&\leq \left(\int_0^{v^\circ(s)} \frac{d\tau}{\int_{\{v=\tau\}} |\nabla v| \, d\mathcal{H}^{n-1}(x)} \right) \int_0^s |f(\varrho)| d\varrho \\
&\quad + \int_s^\varepsilon \int_0^{v^\circ(\varrho)} \frac{d\tau}{\int_{\{v=\tau\}} |\nabla v| \, d\mathcal{H}^{n-1}(x)} |f(\varrho)| d\varrho
\end{aligned} \tag{5.19}$$

$$(v^\circ(\varepsilon) \geq 0 \text{ by } (5.18))$$

$$\begin{aligned}
&= \psi_v(v^\circ(s)) \int_0^s |f(\varrho)| d\varrho + \int_s^\varepsilon \psi_v(v^\circ(\varrho)) |f(\varrho)| d\varrho \\
&\leq \frac{1}{\nu_M(s)} \int_0^s |f(\varrho)| d\varrho + \int_s^\varepsilon \frac{1}{\nu_M(\varrho)} |f(\varrho)| d\varrho
\end{aligned}$$

$$(\text{by } (3.14) \text{ with } u \text{ replaced by } v).$$

Thus, if we show that there exist constants C_1 and C_2 such that

$$\left(\int_0^\varepsilon \left(\frac{1}{\nu_M(s)} \int_0^s |f(r)| dr \right)^q ds \right)^{1/q} \leq C_1 \left(\int_0^\varepsilon |f(s)|^q ds \right)^{1/q} \tag{5.20}$$

and

$$\left(\int_0^\varepsilon \left(\int_s^\varepsilon \frac{1}{\nu_M(r)} |f(r)|\, dr \right)^q ds \right)^{1/q} \leq C_2 \left(\int_0^\varepsilon |f(s)|^q ds \right)^{1/q} \tag{5.21}$$

for every $f \in L^q(0, \varepsilon)$, then we obtain that

$$\|Tf\|_{L^q(0,\varepsilon)} \leq (C_1 + C_2) \|f\|_{L^q(0,\varepsilon)} \tag{5.22}$$

for every $f \in L^q(0, \varepsilon)$ By standard weighted Hardy inequalities (see e.g. [Ma7, Section 1.3.1]), inequalities (5.20) and (5.21) hold if and only if

$$\sup_{s \in (0,\varepsilon)} s^{1/q'} \|1/\nu_M\|_{L^q(s,\varepsilon)} < \infty \tag{5.23}$$

and

$$\sup_{s \in (0,\varepsilon)} s^{1/q} \|1/\nu_M\|_{L^{q'}(s,\varepsilon)} < \infty\,, \tag{5.24}$$

respectively. Furthermore, the constants C_1 and C_2 in (5.20) and (5.21) are equivalent (up to absolute multiplicative constants) to the left-hand sides of (5.23) and (5.24), respectively.
The left-hand sides of (5.23) and (5.24) agree if $q = 2$. We claim that, if $q \in (2, \infty)$, then the left-hand side of (5.23) does not exceed the left-hand side of (5.24), up to an absolute multiplicative constant. Actually, since ν_M is non-decreasing,

$$\begin{aligned}
\sup_{r \in (0,s)} r^{1/q} \|1/\nu_M\|_{L^{q'}(r,s)} &\geq (s/2)^{1/q} \|1/\nu_M\|_{L^{q'}(s/2,s)} \\
&\geq \frac{s}{2\nu_M(s)} \qquad \text{for } s \in (0, \mathcal{H}^n(M)/2).
\end{aligned} \tag{5.25}$$

Thus,

$$
\begin{aligned}
(5.26)\qquad s^{\frac{1}{q'}}\,\|1/\nu_M\|_{L^q(s,\varepsilon)} &\le s^{\frac{1}{q'}}\left(\frac{1}{\nu_M(s)}\right)^{1-\frac{q'}{q}}\|1/\nu_M\|^{\frac{q'}{q}}_{L^{q'}(s,\varepsilon)}\\
&=\left(\left(\frac{s}{\nu_M(s)}\right)^{q-2} s^{1/q}\,\|1/\nu_M\|_{L^{q'}(s,\varepsilon)}\right)^{\frac{1}{q-1}}\\
&\le 2^{\frac{q-2}{q-1}}\sup_{r\in(0,\varepsilon)} r^{\frac{1}{q}}\,\|1/\nu_M\|_{L^{q'}(r,\varepsilon)}\\
&\le 2\sup_{r\in(0,\varepsilon)} r^{\frac{1}{q}}\,\|1/\nu_M\|_{L^{q'}(r,\varepsilon)} \qquad \text{for } s\in(0,\varepsilon),
\end{aligned}
$$

where the last but one inequality holds owing to (5.25). Thus, our claim follows. On the other hand,

$$
\begin{aligned}
(5.27)\qquad \sup_{s\in(0,\varepsilon)} s^{1/q}\,\|1/\nu_M\|_{L^{q'}(s,\varepsilon)} &= \sup_{s\in(0,\varepsilon)} s^{1/q}\bigg(\int_s^\varepsilon \left(\frac{r}{\nu_M(r)}\right)^{q'}\frac{dr}{r^{q'}}\bigg)^{1/q'}\\
&\le\left(\sup_{s\in(0,\varepsilon)}\frac{s}{\nu_M(s)}\right)\sup_{s\in(0,\varepsilon)} s^{1/q}\bigg(\int_s^\varepsilon\frac{dr}{r^{q'}}\bigg)^{1/q'} \le \frac{1}{(q'-1)^{1/q'}}\Theta(\varepsilon)\,.
\end{aligned}
$$

Hence, (5.13) follows as well.
On denoting by $(I-\gamma T)_q$ the restriction of $I-\gamma T$ to $L^q(0,\varepsilon)$, we deduce via a classical result of functional analysis that the operator

$$(5.28)\qquad (I-\gamma T)_q : L^q(0,\varepsilon)\to L^q(0,\varepsilon)$$

is invertible, with a bounded inverse, provided that ε is so small that

$$(5.29)\qquad C\gamma\Theta(\varepsilon)<1\,,$$

where C is the constant appearing in (5.13). Moreover,

$$(5.30)\qquad \|(I-\gamma T)_q^{-1}\|\le\frac{1}{1-C\gamma\Theta(\varepsilon)}\,.$$

The next step consists in showing that there exists an absolute constant C' such that also the restriction $(I-\gamma T)_2$ of $I-\gamma T$ to $L^2(0,\varepsilon)$ is invertible, with a bounded inverse, provided that

$$(5.31)\qquad C'\gamma\Theta(\varepsilon)<1\,.$$

An analogous chain as in (5.26) tells us that

$$(5.32)\qquad s^{1/2}\,\|1/\nu_M\|_{L^2(s,\varepsilon)}\le 2\sup_{r\in(0,\varepsilon)} r^{1/q}\,\|1/\nu_M\|_{L^{q'}(r,\varepsilon)} \qquad \text{for } s\in(0,\varepsilon).$$

Consequently,

$$(5.33)\qquad \sup_{s\in(0,\varepsilon)} s^{1/2}\,\|1/\nu_M\|_{L^2(s,\varepsilon)}\le 2\Theta(\varepsilon)\,.$$

Hence, the invertibility of $(I-\gamma T)_2$ under (5.31) follows via the same argument as above. Owing to assumption (2.2), both (5.29) and (5.31) hold provided that ε is sufficiently small. In particular, one can choose

$$\varepsilon = \Theta^{-1}\big(1/(2\gamma C'')\big)\,, \tag{5.34}$$

where Θ^{-1} is the generalized left-continuous inverse of Θ, and $C'' = \max\{C, C'\}$.
We have that $u^\circ \in L^2(0,\varepsilon)$, for $u \in L^2(M)$. Thus, since the constant function $u^\circ(\varepsilon)$ trivially belongs to $L^q(0,\varepsilon) \subset L^2(0,\varepsilon)$, from (5.11) we deduce that

$$u^\circ = (I-\gamma T)_2^{-1}(u^\circ(\varepsilon)) = (I-\gamma T)_q^{-1}(u^\circ(\varepsilon))\,. \tag{5.35}$$

Hence, $u^\circ \in L^q(0,\varepsilon)$, and, by (5.30) and (5.34),

$$\|u^\circ\|_{L^q(0,\varepsilon)} \le \frac{\|u^\circ(\varepsilon)\|_{L^q(0,\varepsilon)}}{1-C\gamma\Theta(\varepsilon)} = \frac{\varepsilon^{1/q}|u^\circ(\varepsilon)|}{1-C\gamma\Theta(\varepsilon)} \le 2\varepsilon^{1/q}|u^\circ(\varepsilon)|\,. \tag{5.36}$$

Since $\varepsilon \le \mathcal{H}^n(M)/2$, one can easily verify that

$$\|u\|_{L^2(M)} \ge \varepsilon^{1/2}|u^\circ(\varepsilon)|\,. \tag{5.37}$$

From (5.36) and (5.37) one has that

$$\|u^\circ\|_{L^q(0,\varepsilon)} \le \frac{2\|u\|_{L^2(M)}}{\varepsilon^{\frac{1}{2}-\frac{1}{q}}}\,. \tag{5.38}$$

Next, observe that

$$|\mathrm{med}(u)| \le (2/\mathcal{H}^n(M))^{1/2}\|u\|_{L^2(\Omega)}. \tag{5.39}$$

By (5.37) and (5.39), there exists a constant $C = C(\mathcal{H}^n(M))$ such that

$$\begin{aligned}\|u^\circ - \mathrm{med}(u)\|_{L^q(\varepsilon,\mathcal{H}^n(M)/2)} &\le (u^\circ(\varepsilon) - \mathrm{med}(u))^{\frac{q-2}{q}}\|u^\circ - \mathrm{med}(u)\|_{L^2(\varepsilon,\mathcal{H}^n(M)/2)}^{\frac{2}{q}}\\ &\le C(\varepsilon^{\frac{1}{2}-\frac{1}{q}}+1)\|u\|_{L^2(M)}.\end{aligned} \tag{5.40}$$

From (5.37), (5.39) and (5.40) we deduce that

$$\begin{aligned}\|u^\circ\|_{L^q(0,\mathcal{H}^n(M)/2)} &\le \|u^\circ\|_{L^q(0,\varepsilon)} + \|u^\circ - \mathrm{med}(u)\|_{L^q(\varepsilon,\mathcal{H}^n(M)/2)} + \|\mathrm{med}(u)\|_{L^q(\varepsilon,\mathcal{H}^n(M)/2)}\\ &\le (\varepsilon^{\frac{1}{2}-\frac{1}{q}}+1)\|u\|_{L^2(M)}\end{aligned} \tag{5.41}$$

for some constant $C = C(\mathcal{H}^n(M))$. Hence, there exists a constant $C = C(\mathcal{H}^n(M))$ such that

$$\|u^\circ\|_{L^q(0,\mathcal{H}^n(M)/2)} \le \frac{C\|u\|_{L^2(M)}}{\big(\Theta^{-1}\big(1/(\gamma C)\big)\big)^{\frac{1}{2}-\frac{1}{q}}}\,. \tag{5.42}$$

A combination of (5.41) with an analogous estimate for $\|u^\circ\|_{L^q(\mathcal{H}^n(M)/2,\mathcal{H}^n(M))}$ obtained via the same argument applied to $-u$, yields (2.3), since $(-u)^\circ(s) = -u^\circ(\mathcal{H}^n(M)-s)$ for $s \in (0,\mathcal{H}^n(M))$. □

An inspection of the proof of Theorem 2.1 reveals that, in fact, the following estimate for the constant appearing in (2.3) holds.

Proposition 5.1 *Define the function* $\Theta : (0, \mathcal{H}^n(M)/2] \to [0, \infty)$ *as*

$$\Theta(s) = \sup_{r \in (0,s)} \frac{r}{\nu_M(r)} \qquad \text{for } s \in (0, \mathcal{H}^n(M)/2].$$

Then inequality (2.3) *holds with*

$$C(\nu_M, q, \gamma) = \frac{C_1}{\left(\Theta^{-1}(C_2/\gamma)\right)^{\frac{1}{2}-\frac{1}{q}}} ,$$

where $C_1 = C_1(q, \mathcal{H}^n(M))$ *and* $C_2 = C_2(q, \mathcal{H}^n(M))$ *are suitable constants, and* Θ^{-1} *is the generalized left-continuous inverse of* Θ*.*

Example 5.2 Assume that there exists $\beta \in [(n-2)/n, 1)$ such that the manifold M fulfils $\nu_M(s) \geq Cs^\beta$ for some positive constant C and for small s. Then (2.2) holds, and, by Proposition 5.1, for every $q \in (2, \infty)$ there exists an constant $C = C(q, \mathcal{H}^n(M))$ such that

$$\|u\|_{L^q(M)} \leq C\gamma^{\frac{q-2}{2q(1-\beta)}} \|u\|_{L^2(M)}$$

for every eigenfunction u of the Laplacian on M associated with the eigenvalue γ.

We now prove Theorem 2.2.

Proof of Theorem 2.2 Given $q > 2$ and $n \geq 2$, we shall construct an n-dimensional manifold of revolution M as in Theorem 4.1, fulfilling (2.6) and such that the Laplacian on M has an eigenfunction $u \notin L^q(M)$.

In the light of the discussion preceding Theorem 4.1, in order to exhibit such eigenfunction it suffices to produce a positive number γ and a smooth function $p : \mathbb{R} \to (0, \infty)$ such that

$$\int_{-\infty}^{s} \sqrt{p(\varrho)} d\varrho < \infty \qquad \text{for } s \in \mathbb{R},$$

and

$$\int_{\mathbb{R}} \sqrt{p(\varrho)} d\varrho = \infty ,$$

and a function $v : \mathbb{R} \to \mathbb{R}$ fulfilling (4.11) and (4.12), and such that

$$\int_{\mathbb{R}} v(s)^q p(s) ds = \infty . \tag{5.43}$$

The function φ in the definition of M is recovered from p by

$$\varphi(r) = p(F^{-1}(r))^{\frac{1}{2(n-1)}} \qquad \text{for } r > 0, \tag{5.44}$$

and $\varphi(0) = 0$, where $F : \mathbb{R} \to [0, \infty)$ is given by

$$F(s) = \int_{-\infty}^{s} \sqrt{p(\varrho)} d\varrho \qquad \text{for } s \in \mathbb{R}. \tag{5.45}$$

We define the function p piecewise as follows. Let $s_1 \le -1 \le 1 \le s_2$ to be fixed later, and set

$$p(s) = \frac{1}{s^2} \quad \text{if } s \ge s_2. \tag{5.46}$$

Let

$$0 < \gamma < \frac{1}{4},$$

and

$$\alpha = \frac{1 - \sqrt{1 - 4\gamma}}{2}.$$

With this choice of α, the function

$$v(s) = s^{\alpha} \tag{5.47}$$

solves equation (4.11) in $[s_2, \infty)$. On the other hand, if p is defined in $(-\infty, s_1]$ by

$$p(s) = \begin{cases} \frac{4e^{2s}}{\gamma(1-e^{2s})} & \text{if } n = 2, \\ \frac{(-s)^{\frac{2n-2}{2-n}}}{(n-2)^{\frac{2n-2}{n-2}} - \frac{\gamma(n-2)^2}{2n}(-s)^{\frac{2}{2-n}}} & \text{if } n > 2, \end{cases} \tag{5.48}$$

then the function given by

$$v(s) = \begin{cases} 1 - e^{2s} & \text{if } n = 2, \\ (n-2)^{\frac{2n-2}{n-2}} - \frac{\gamma(n-2)^2}{2n}(-s)^{\frac{2}{2-n}} & \text{if } n > 2, \end{cases} \tag{5.49}$$

solves equation (4.11) in $(-\infty, s_1]$. Next, given $\beta > 0$ and neighborhoods I_{-1} and I_1 of -1 and 1, respectively, let p be defined in $I_1 \cup I_1$ as

$$p(s) = \begin{cases} \frac{6}{\gamma(\beta-(s-1)^2)} & \text{for } s \in I_1, \\ \frac{6}{\gamma(\beta-(s+1)^2)} & \text{for } s \in I_{-1}. \end{cases}$$

Then the function v given by

$$v(s) = \begin{cases} \beta(s-1) - (s-1)^3 & \text{for } s \in I_1, \\ -\beta(s+1) + (s+1)^3 & \text{for } s \in I_{-1}, \end{cases}$$

is a solution to (4.11) in $I_1 \cup I_1$. Moreover, v is convex in a left neighborhood of 1 and in a right neighborhood of -1, whereas it is concave in a right neighborhood of 1 and in a left neighborhood of -1. It is easily seen that, if β is sufficiently large, s_2 and $-s_1$ are sufficiently large depending on β, and I_1 and I_{-1} have a sufficiently small radius, then v can be continued to the whole of $\mathbb{R}$ in such a way that:

$$v \in C^2(\mathbb{R});$$

$$v'' \le -C \text{ and } v \ge C \text{ in } \mathbb{R} \setminus (I_{-1} \cup (-1,1) \cup I_1), \text{ for some positive constant C;}$$

$$v'' \ge C \text{ and } v \le -C \text{ in } (-1,1) \setminus (I_{-1} \cup I_1), \text{ for some positive constant C.}$$

Thus, p can be continued to the whole of $\mathbb{R}$ as a positive function in $C^2(\mathbb{R})$ in such a way that v is a solution to equation (4.11) in $\mathbb{R}$. Also, the function v fulfils (4.12) for every $\gamma \in (0, \frac{1}{4})$, and (5.43) provided that γ is sufficiently close to $\frac{1}{4}$.

The manifold of revolution M built upon the function φ given by (5.44) satisfies the assumtpions of Theorem 4.1. Indeed, $\varphi(r) > 0$ if $r > 0$, and (4.2) holds, as a consequence of the fact that

$$\lim_{s\to-\infty} \frac{p'(s)}{p(s)^{\frac{3n-4}{2n-2}}} = 2(n-1)\,. \tag{5.50}$$

Assumptions (i)–(iii) of Theorem 4.1 are also fulfilled, since there exists $b > 0$ such that

$$\varphi(r) = be^{-\frac{r}{n-1}} \qquad \text{for large } r. \tag{5.51}$$

Finally, if λ_0 denotes the function given by (4.15), then we obtain via (5.51) that

$$\lim_{s\to0} \frac{s}{\lambda_0(s)} \approx \lim_{r\to\infty} \frac{1}{\varphi(r)^{n-1}} \int_r^\infty \varphi(\rho)^{n-1} d\rho = 1. \tag{5.52}$$

Hence, by (4.17),

$$\lim_{s\to0} \frac{s}{\nu_M(s)} \approx \lim_{s\to0} s \int_s^{\mathcal{H}^n(M)/2} \frac{d\varrho}{\lambda_0(\varrho)^2} \approx \lim_{s\to0} \frac{s^2}{\lambda_0(s)^2} \approx 1.$$

Hence (2.6) follows. □

We conclude this section by sketching the proofs of the results dealing with $L^q(M)$ estimates in terms of λ_M.

Proof of Theorem 2.5 By (2.14), given $\varepsilon > 0$, there exists s_ε such that $\frac{s^2}{\lambda_M(s)^2} < \varepsilon$ if $0 < s < s_\varepsilon$. By inequality (2.1), if $0 < s < s_\varepsilon$, then

$$\begin{aligned} \frac{s}{\nu_M(s)} \le s \int_s^{\mathcal{H}^n(M)/2} \frac{dr}{\lambda_M(r)^2} &\le \varepsilon s \int_s^{s_\varepsilon} \frac{dr}{r^2} + s \int_{s_\varepsilon}^{\mathcal{H}^n(M)/2} \frac{dr}{\lambda_M(r)^2} \\ &\le \varepsilon + s \int_{s_\varepsilon}^{\mathcal{H}^n(M)/2} \frac{dr}{\lambda_M(r)^2}. \end{aligned} \tag{5.53}$$

The rightmost side of (5.53) tends to ε as $s \to 0$. Hence (2.2) follows, owing to the arbitrariness of ε. Inequality (2.15) in thus a consequence of Theorem 2.1. □

Proof of Theorem 2.6 The proof is the same as that of Theorem 2.2. One has just to notice that (5.52) and (4.16) imply (2.16). □

6 Boundedness of eigenfunctions

Our main task in this section is to prove Theorem 2.3, which provides a condition on ν_M for the boundedness of the eigenvalues of the Laplacian, and Theorem 2.4, showing the sharpness of such condition. The proofs of the parallel results of Theorems 2.7 and 2.8 involving λ_M are sketched at the end of the section.

Proof of Theorem 2.3 We start as in the proof of Theorem 2.1, define the operator T as in (5.10), and, for $\varepsilon \in (0, \mathcal{H}^n(M)/2)$, we write again equation (5.9) as

$$(I - \gamma T)(u^\circ) = u^\circ(\varepsilon)\,. \tag{6.1}$$

We claim that, if (2.7) is satisfied, then

$$T : L^\infty(0,\varepsilon) \to L^\infty(0,\varepsilon), \tag{6.2}$$

and

$$\|T\|_{(L^\infty(0,\varepsilon)\to L^\infty(0,\varepsilon))} \le \int_0^\varepsilon \frac{dr}{\nu_M(r)}\,. \tag{6.3}$$

To verify our claim, define v as in (5.15), recall (5.17), and note that

$$\begin{aligned}
\|Tf\|_{L^\infty(0,\varepsilon)} &\le \|f\|_{L^\infty(0,\varepsilon)} \int_0^\varepsilon \left(\int_0^r d\varrho\right) \frac{d}{dr}\left(-\int_0^{v^\circ(r)} \frac{d\tau}{\int_{\{v=\tau\}} |\nabla v|\, d\mathcal{H}^{n-1}(x)}\right) dr \\
&= \|f\|_{L^\infty(0,\varepsilon)} \int_0^\varepsilon \int_\varrho^\varepsilon \frac{d}{dr}\left(-\int_0^{v^\circ(r)} \frac{d\tau}{\int_{\{v=\tau\}} |\nabla v|\, d\mathcal{H}^{n-1}(x)}\right) dr\, d\varrho \\
&= \|f\|_{L^\infty(0,\varepsilon)} \int_0^\varepsilon \int_{v^\circ(\varepsilon)}^{v^\circ(\varrho)} \frac{d\tau}{\int_{\{v=\tau\}} |\nabla v|\, d\mathcal{H}^{n-1}(x)} d\varrho \\
&\le \|f\|_{L^\infty(0,\varepsilon)} \int_0^\varepsilon \int_0^{v^\circ(\varrho)} \frac{d\tau}{\int_{\{v=\tau\}} |\nabla v|\, d\mathcal{H}^{n-1}(x)} d\varrho \\
&= \|f\|_{L^\infty(0,\varepsilon)} \int_0^\varepsilon \psi_v(v^\circ(\varrho))\, d\varrho \\
&\le \|f\|_{L^\infty(0,\varepsilon)} \int_0^\varepsilon \frac{d\varrho}{\nu_M(\varrho)}.
\end{aligned} \tag{6.4}$$

Observe that we have made use of the inequality $v(\varepsilon) \ge 0$, due to (5.18), in the last but one inequality, and of (3.14) (with u replaced by v) in the last inequality. Now, define the function $\Xi : (0, \mathcal{H}^n(M)/2] \to [0,\infty)$ as

$$\Xi(s) = \int_0^s \frac{dr}{\nu_M(r)} \qquad \text{for } s \in (0, \mathcal{H}^n(M)/2]. \tag{6.5}$$

If ε is sufficiently small, in particular

$$\varepsilon \le \Xi^{-1}\big(1/(2\gamma)\big),$$

where Ξ^{-1} is the generalized left-continuous inverse of Ξ, then we deduce from (6.3) that the restriction $(I-\gamma T)_\infty$ of $I-\gamma T$ to $L^\infty(0,\varepsilon)$,

$$(I-\gamma T)_\infty : L^\infty(0,\varepsilon) \to L^\infty(0,\varepsilon) \tag{6.6}$$

is invertible, with a bounded inverse, and

$$\|(I-\gamma T)_\infty^{-1}\| \le \frac{1}{1-\gamma \int_0^\varepsilon \frac{d\varrho}{\nu_M(\varrho)}} \le 2\,. \tag{6.7}$$

Since $\Xi(\varepsilon) \ge \Theta(\varepsilon)$, where Θ is the function defined by (5.14), we infer from (5.33) and from the proof of Theorem 2.1 that there exists an absolute constant C such that the restriction

$$(I-\gamma T)_2 : L^2(0,\varepsilon) \to L^2(0,\varepsilon) \tag{6.8}$$

of $I - \gamma T$ to $L^2(0,\varepsilon)$ is also invertible, with a bounded inverse, provided that $C\Xi(\varepsilon) < 1$. Set $C' = \max\{2, C\}$, and choose

$$\varepsilon = \Xi^{-1}\big(1/(\gamma C')\big).$$

Since $u^\circ \in L^2(0,\varepsilon)$ and $u^\circ(\varepsilon) \in L^\infty(0,\varepsilon) \subset L^2(0,\varepsilon)$, from (6.1) we deduce that

$$u^\circ = (I - \gamma T)_2^{-1}(u^\circ(\varepsilon)) = (I - \gamma T)_\infty^{-1}(u^\circ(\varepsilon))\,. \tag{6.9}$$

Hence, $u^\circ \in L^\infty(0,\varepsilon)$, and

$$\|u^\circ\|_{L^\infty(0,\varepsilon)} \leq \frac{|u^\circ(\varepsilon)|}{1 - \gamma \int_0^\varepsilon \frac{d\varrho}{\nu_M(\varrho)}} \leq 2|u^\circ(\varepsilon)|\,. \tag{6.10}$$

Since

$$\|u\|_{L^2(M)} \geq \varepsilon^{1/2}|u^\circ(\varepsilon)| = \big(\Xi^{-1}\big(1/(\gamma C')\big)\big)^{1/2}|u^\circ(\varepsilon)|\,, \tag{6.11}$$

we have that

$$u^\circ(0) \leq \|u^\circ\|_{L^\infty(0,\varepsilon)} \leq \frac{2\|u\|_{L^2(M)}}{\big(\Xi^{-1}\big(1/(\gamma C')\big)\big)^{1/2}}\,. \tag{6.12}$$

The same argument, applied to $-u$, yields the same estimate for $-u^\circ(\mathcal{H}^n(M))$. Since

$$\|u\|_{L^\infty(M)} = \max\{u^\circ(0), -u^\circ(\mathcal{H}^n(M))\},$$

inequality (2.8) follows. □

The following estimate for the constant in (2.8) is provided in the proof of Theorem 2.3.

Proposition 6.1 *Assume that* (2.7) *is in force. Define the function* $\Xi : (0, \mathcal{H}^n(M)/2] \to [0,\infty)$ *as*

$$\Xi(s) = \int_0^s \frac{dr}{\nu_M(r)} \qquad \text{for } s \in (0, \mathcal{H}^n(M)/2].$$

Then inequality (2.8) *holds with*

$$C(\nu_M, \gamma) = \frac{C_1}{\big(\Xi^{-1}(C_2/\gamma)\big)^{\frac{1}{2}}}\,,$$

where C_1 *and* C_2 *are suitable absolute constants, and* Ξ^{-1} *is the generalized left-continuous inverse of* Ξ.

Example 6.2 Assume that there exists $\beta \in [(n-2)/n, 1)$ such that the manifold M fulfils $\nu_M(s) \geq Cs^\beta$ for some positive constant C and for small s. Then (2.7) holds, and, by Proposition 6.1, there exists an absolute constant C such that

$$\|u\|_{L^\infty(M)} \leq C\gamma^{\frac{1}{2(1-\beta)}}\|u\|_{L^2(M)}$$

for every eigenfunction u of the Laplacian on M associated with the eigenvalue γ.

Next, we give a proof of Theorem 2.4.

Proof of Theorem 2.4 By Proposition 4.5, if ν is as in the statement, then there exists a function φ such that the associated n-dimensional manifold of revolution M (as in Section 4) fulfils (2.12), and hence

$$\lim_{s\to 0}\frac{s}{\nu_M(s)}=\lim_{s\to 0}\frac{s}{\nu(s)}=0 \tag{6.13}$$

and

$$\int_0\frac{ds}{\nu_M(s)}=\int_0\frac{ds}{\nu(s)}=\infty\,. \tag{6.14}$$

In particular, the latter equation entails, via Remark 4.6, that (4.61) holds, and hence that $L=\infty$ in Proposition 4.5. Thus, $\varphi:[0,\infty)\to[0,\infty)$. Now, recall that the function φ given by Proposition 4.5 is defined in such a way that (4.2) holds. Hence,

$$\int_0^1\frac{dr}{\varphi(r)^{n-1}}=\infty\,, \tag{6.15}$$

and

$$\lim_{r\to 0}\left(\int_r^1\frac{d\rho}{\varphi(\rho)^{n-1}}\right)\left(\int_0^r\varphi(\rho)^{n-1}d\rho\right)=0\,. \tag{6.16}$$

Moreover,

$$\int_1^\infty\frac{dr}{\varphi(r)^{n-1}}=\infty\,, \tag{6.17}$$

since $\lim_{r\to\infty}\varphi(r)=0$ by property (i) of Theorem 4.1.
Owing to Corollary 4.2, condition (6.13) is equivalent to

$$\lim_{r\to \infty}\left(\int_1^r\frac{d\varrho}{\varphi(\varrho)^{n-1}}\right)\left(\int_r^\infty\varphi(\varrho)^{n-1}d\varrho\right)=0\,, \tag{6.18}$$

and condition (6.14) is equivalent to

$$\int^\infty\left(\frac{1}{\varphi(r)^{n-1}}\int_r^\infty\varphi(\rho)^{n-1}d\rho\right)dr=\infty\,. \tag{6.19}$$

By the discussion preceding Theorem 4.1, the conclusion will follow if we exhibit a number $\gamma>0$ and an unbounded solution $v:\mathbb{R}\to\mathbb{R}$ to equation (4.11) fulfilling (4.12). Note that $s_0=\infty$ in (4.9) owing to (6.17). Conditions (6.16) and (6.18) are equvalent to

$$\lim_{s\to -\infty}s\int_{-\infty}^s p(t)dt=0\,, \tag{6.20}$$

and

$$\lim_{s\to \infty}s\int_s^\infty p(t)dt=0\,, \tag{6.21}$$

respectively. Condition (6.19) amounts to

$$\int^\infty s\,p(s)\,ds=\infty\,. \tag{6.22}$$

Assumptions (6.20) and (6.21) ensure that the the embedding

$$W^{1,2}(\mathbb{R}) \to L^2(\mathbb{R}, p(s)ds) \tag{6.23}$$

is compact – see e.g. [KK]. Here, $L^2(\mathbb{R}, p(s)ds)$ denotes the space L^2 on $\mathbb{R}$ equipped with the measure $p(s)ds$. Consider the functional

$$J(v) = \frac{\int_{\mathbb{R}} \left(\frac{dv}{ds}\right)^2 ds}{\int_{\mathbb{R}} v^2\, p(s)ds}\,. \tag{6.24}$$

By the compactness of embedding (6.23), there exists $\min J(v)$ among all (non trivial) functions $v \in W^{1,2}(\mathbb{R})$ such that $\int_{\mathbb{R}} v\, p(s)ds = 0$. Moreover, the minimizer v is a solution to equation (4.11) with $\gamma = \min J$.

By Hille's theorem [Hi], condition (6.21) entails that equation (4.11) is nonoscillatory at infinity, and hence that every solution has constant sign at infinity. Thus, we may assume that $v(s) > 0$ for large s. Consequently,

$$\frac{d^2v}{ds^2} < 0 \qquad \text{for large } s,$$

and hence v is concave near ∞. Now, assume by contradiction that v is bounded. Then there exists $\lim_{s\to\infty} v(s)$, and, on denoting by $v(\infty)$ this limit, one has that $v(\infty) \in (0, \infty)$. Moreover,

$$\lim_{s\to\infty} \frac{dv}{ds} = 0\,.$$

Integration of (4.11) and the last equation yield

$$\frac{dv}{ds} = \gamma \int_s^\infty p(t)v(t)dt \qquad \text{for large } s.$$

Hence, there exists $\widehat{s}$ such that

$$\frac{dv}{ds} \geq \gamma \frac{v(\infty)}{2} \int_s^\infty p(t)dt \qquad \text{for } s \geq \widehat{s}.$$

By a further integration, we obtain that

$$v(\infty) - v(\widehat{s}) \geq \gamma \frac{v(\infty)}{2} \int_{\widehat{s}}^\infty \int_s^\infty p(t)dt\, ds = \gamma \frac{v(\infty)}{2} \left(\int_{\widehat{s}}^\infty tp(t)dt - \widehat{s} \int_{\widehat{s}}^\infty p(t)dt \right),$$

thus contradicting (6.22). □

Proof of Theorem 2.7 Assumption (2.17) implies (2.7), via Fubini's theorem. Hence, the conclusion follows via Theorem 2.3. □

Proof of Theorem 2.8 By Corollary 4.2, the same argument as in the proof of Theorem 2.4 provides a manifold M, fulfilling (2.14), but not (2.17), on which the Laplacian has an unbounded eigenfunction. □

7 Applications

We conclude with applications of our results to two special instances. Both of them involve families of noncompact manifolds. However, the former is less pathological, and can be handled either by isoperimetric or by isocapacitary methods, with the same output. That isocapacitary inequalities can actually yield sharper conclusions than those obtained via isoperimetric inequalities is demonstrated by the latter example, which deals with a class of manifolds with a more complicated geometry.

7.1 A family of manifolds of revolution with borderline decay

Consider a one-parameter family of manifolds of revolution M as in Section 4, whose profile $\varphi : [0, \infty) \to [0, \infty)$ is such that

$$\varphi(r) = e^{-r^\alpha} \qquad \text{for large } r, \tag{7.1}$$

and fulfills the assumptions of Theorem 4.1 (Figure 1). This theorem tells us that

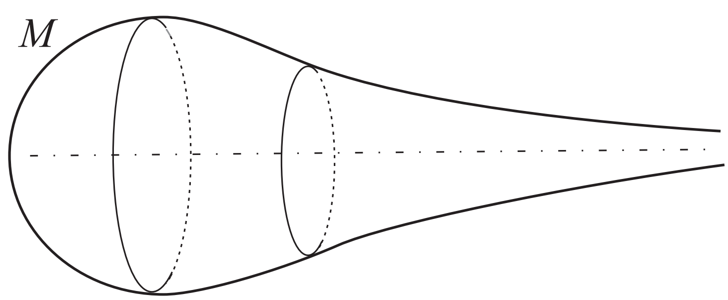

Figure 1: A manifold of revolution

$$\lambda_M(s) \approx s\big(\log(1/s)\big)^{1-1/\alpha} \qquad \text{near } 0, \tag{7.2}$$

and

$$\nu_M(s) \approx \left(\int_s^{\mathcal{H}^n(M)/2} \frac{dr}{\lambda_M(r)^2}\right)^{-1} \approx s\big(\log(1/s)\big)^{2-2/\alpha} \qquad \text{near } 0. \tag{7.3}$$

An application of Theorem 2.1 ensures, via (7.3), that all eigenfunctions of the Laplacian on M belong to $L^q(M)$, provided that

$$\alpha > 1. \tag{7.4}$$

On the other hand, from Theorem 2.3 and equation (7.3) one can infer that the relevant eigenfunctions are bounded under the more stringent assumption that

$$\alpha > 2. \tag{7.5}$$

The same conclusions can be derived via Theorems 2.5 and 2.7, respectively. Thus, as for any other manifold of revolution of the kind considered in Theorem 4.1 (see Corollary 4.2), isoperimetric and isocapacitary methods lead to equivalent results for this family of noncompact manifolds.

Note that, if $\alpha > 1$, then, by Proposition 5.1, there exist constants $C_1 = C_1(q)$ and $C_2 = C_2(q)$ such that

$$\|u\|_{L^q(M)} \leq C_1 e^{C_2 \gamma^{\frac{\alpha}{2\alpha-2}}} \|u\|_{L^2(M)}$$

for any eigenfunction u of the Laplacian associated with the eigenvalue γ. Moreover, if $\alpha > 2$, then by Proposition 6.1,

$$\|u\|_{L^\infty(M)} \leq C_1 e^{C_2 \gamma^{\frac{\alpha}{\alpha-2}}} \|u\|_{L^2(M)}$$

for some absolute constants C_1 and C_2 and for every for any eigenfunction u associated with γ. In both cases, the existence of eigenfunctions of the Laplacian is guaranteed by condition (2.2) – see the comments following Theorem 2.1.

7.2 A family of manifolds with clustering submanifolds

Here, we are concerned with a class of non compact surfaces M in $\mathbb{R}^3$, which are patterned on an example appearing in [CH] and dealing with a planar domain. Their main feature is that they contain a sequence of mushroom-shaped submanifolds $\{N^k\}$ clustering at some point (Figure 2). Let us emphasize that the submanifolds $\{N^k\}$ are not obtained just by dilation of each other.

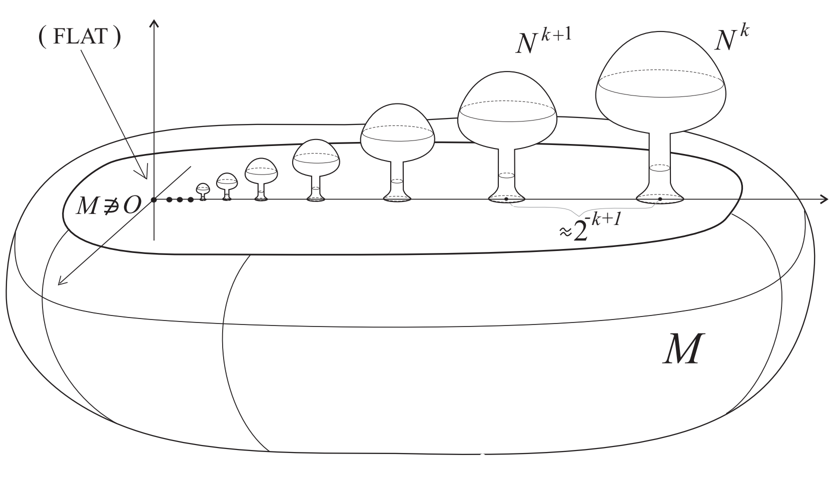

Figure 2: A manifold with a family of clustering submanifolds

Roughly speaking, the diameter of the head and the length of the neck of N^k decay to 0 as 2^{-k} when $k \to \infty$, whereas the width of the neck of N^k decays to 0 as $\sigma(2^{-k})$, where σ is a function such that

$$\lim_{s \to 0} \frac{\sigma(s)}{s} = 0.$$

The isoperimetric and isocapacitary functions of M depend on the behavior of σ at 0 in a way described in the next result (Proposition 7.1). Qualitatively, a faster decay to 0 of the function

$\sigma(s)$ as $s \to 0$ results in a faster decay to 0 of $\lambda_M(s)$ and $\nu_M(s)$, and hence in a manifold M with a more irregular geometry. Proposition 7.1 is a special case of Proposition 7.2, dealing with the isocapacitary function $\nu_{M,p}$ of M for arbitrary $p \in [1, 2]$, stated and proved below. We also refer to the proof of Proposition 7.2 for a more precise definition of the manifold M.

Proposition 7.1 *Let M be the 2-dimensional manifold in Figure 2. Assume that $\sigma : [0, \infty) \to [0, \infty)$ is an increasing function of class Δ_2 such that*

$$\frac{s^{\beta+1}}{\sigma(s)} \quad \text{is non-increasing} \tag{7.6}$$

for some $\beta > 0$.
(i) If

$$\frac{s^2}{\sigma(s)} \quad \text{is non-decreasing,}$$

then

$$\lambda_M(s) \approx \sigma(s^{1/2}) \qquad \text{near } 0. \tag{7.7}$$

(ii) If

$$\frac{s^3}{\sigma(s)} \quad \text{is non-decreasing,}$$

then

$$\nu_M(s) \approx \sigma(s^{1/2})s^{-\frac{1}{2}} \qquad \text{near } 0. \tag{7.8}$$

Owing to equation (7.8), one can derive the following conclusions from Theorems 2.1 and 2.3, involving the isocapacitary function ν_M. Assume that

$$\lim_{s\to 0} \frac{s^3}{\sigma(s)} = 0\,. \tag{7.9}$$

Then any eigenfunction of the Laplacian on M belongs to $L^q(M)$ for any $q < \infty$. If (7.9) is strengthened to

$$\int_0 \frac{s^2}{\sigma(s)}\, ds < \infty\,, \tag{7.10}$$

then any eigenfunction of the Laplacian on M is in fact bounded.
Conditions (7.9) and (7.10) are weaker than parallel conditions which are obtained from an application of Theorems 2.5 and 2.7 and (7.7), and read

$$\lim_{s\to 0} \frac{s^2}{\sigma(s)} = 0\,, \tag{7.11}$$

and

$$\int_0 \frac{s^3}{\sigma(s)^2}\, ds < \infty\,, \tag{7.12}$$

respectively. For instance, if $b > 1$ and

$$\sigma(s) = s^b \qquad \text{for } s > 0,$$

then (7.9) and (7.10) amount to $b < 3$, whereas (7.11) and (7.12) are equivalent to the more stringent condition that $b < 2$.
Since, by (7.8), $\nu_M(s) \approx s^{\frac{b-1}{2}}$, from Examples 5.2 and 6.2 we deduce that there exists a constant $C = C(q)$ such that

$$\|u\|_{L^q(M)} \leq C\gamma^{\frac{q-2}{q(3-b)}} \|u\|_{L^2(M)}$$

for every $q \in (2, \infty]$ and for any eigenfunction u of the Laplacian associated with the eigenvalue γ. The existence of such eigenfunction follows from condition (2.2), as explained in the comments following Theorem 2.1.

Proposition 7.2 *Let M be the 2-dimensional manifold in Figure 2. Let $1 \leq p \leq 2$, and let $\sigma : [0, \infty) \to [0, \infty)$ be an increasing function of class Δ_2. Then there exist a constant C such that*

$$\nu_{M,p}(s) \leq C\sigma(s^{1/2})s^{-\frac{p-1}{2}} \qquad \text{near } 0. \tag{7.13}$$

Assume, in addition, that

$$\frac{s^{\beta+1}}{\sigma(s)} \quad \text{is non-increasing} \tag{7.14}$$

for some $\beta > 0$, and

$$\frac{s^{p+1}}{\sigma(s)} \quad \text{is non-decreasing.} \tag{7.15}$$

Then

$$\nu_{M,p}(s) \approx \sigma(s^{1/2})s^{-\frac{p-1}{2}} \qquad \text{near } 0. \tag{7.16}$$

Note that equation (7.7) of Proposition 7.1 follows from (7.16), owing to property (3.6), whereas equation (7.8) agrees with (7.16) for $p = 2$.

One step in the proof of Proposition 7.2 makes use of Orlicz spaces. Recall that given a non-atomic, σ-finite measure space $(\mathcal{R}, m)$ and a Young function A, namely a convex function from $[0, \infty)$ into $[0, \infty)$ vanishing at 0, the Orlicz space $L^A(\mathcal{R})$ is the Banach space of those real-valued m-measurable functions f on $\mathcal{R}$ whose Luxemburg norm

$$\|f\|_{L^A(\mathcal{R})} = \inf\left\{\lambda > 0 : \int_{\mathcal{R}} A\left(\frac{|f|}{\lambda}\right) dm \leq 1\right\},$$

is finite. A generalized Hölder type inequality in Orlicz spaces tells us that if A_i, $i = 1, 2, 3$, are Young functions such that $A_1^{-1}(r)A_2^{-1}(r) \leq CA_3^{-1}(r)$, then there exists a constant C' such that

$$\|fg\|_{L^{A_3}(\mathcal{R})} \leq C'\|f\|_{L^{A_1}(\mathcal{R})}\|g\|_{L^{A_2}(\mathcal{R})} \tag{7.17}$$

for every $f \in L^{A_1}(\mathcal{R})$ and $g \in L^{A_2}(\mathcal{R})$ [On].

Proof of Proposition 7.2.
Part I. Here we show that, if (7.14) and (7.15) are in force, then there exists a constant C such that

$$\nu_{M,p}(s) \geq C\sigma(s^{1/2})s^{-\frac{p-1}{2}} \qquad \text{for } s \in (0, \mathcal{H}^n(M)/2). \tag{7.18}$$

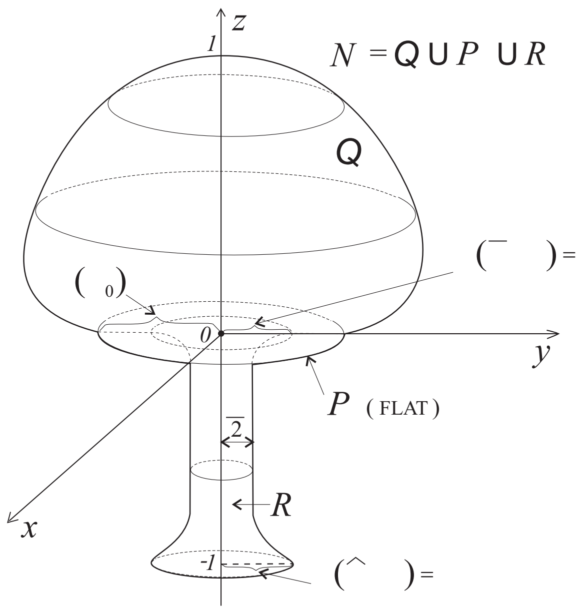

Figure 3: An auxiliary submanifold

We split the proof of (7.18) is steps.
Step 1. Fixed $\varepsilon_0 > 0$ and $\varepsilon \in (0, \varepsilon_0)$, let $N_\varepsilon = Q \cup P_\varepsilon \cup R_\varepsilon$ be the auxiliary surface of revolution in $\mathbb{R}^3$ depicted in Figure 3.

Let $q = \frac{2p}{2-p}$ if $p < 2$, and let q be a sufficiently large number, to be chosen later, if $p = 2$. We shall show that

$$\left(\int_{Q \cup P_\varepsilon} |u|^q d\mathcal{H}^2 \right)^{\frac{p}{q}} \leq \frac{C}{\varepsilon} \left(\int_{N_\varepsilon} |\nabla u|^p d\mathcal{H}^2 + \int_{\partial N_\varepsilon} |u|^p d\mathcal{H}^1 \right) \tag{7.19}$$

for every $u \in W^{1,p}(N_\varepsilon)$, and for some constant C independent of ε and u.
Let $(\rho, \vartheta) \in [0, \hat{\rho} - \varepsilon) \times [0, 2\pi]$ be geodesic coordinates on N_ε with respect to the point $(0, 0, 1)$, and

$$\begin{cases} x & = \phi(\rho) \cos \vartheta \\ y & = \phi(\rho) \sin \vartheta \\ z & = \psi(\rho) \end{cases}$$

be a parametrization of N_ε for some given smooth functions $\phi, \psi : [0, \hat{\rho} - \varepsilon] \to [0, \infty)$. In particular, $\phi'(\rho)^2 + \psi'(\rho)^2 = 1$ for $\rho \in [0, \hat{\rho} - \varepsilon)$. The functions ϕ and ψ are independent of ε in $[0, \rho_0]$ and (up to a translation, of lenght ε, in the variable ρ) in $[\overline{\rho} - \varepsilon, \hat{\rho} - \varepsilon]$; on the other hand, since P_ε is a flat annulus, we have that $\psi(\rho) = 0$ for $\rho \in [\rho_0, \overline{\rho} - \varepsilon]$ and that ϕ is an affine function in the same interval.
Thus, the metric on M is given by

$$ds^2 = d\rho^2 + \phi(\rho)^2 d\vartheta^2 \,.$$

In particular,

$$\int_{N_\varepsilon} f d\mathcal{H}^2 = \int_0^{2\pi} \int_0^{\hat{\rho} - \varepsilon} f \, \phi(\rho) \, d\rho d\vartheta$$

for any integrable function f on M. Moreover, if $u \in W^{1,p}(N_\varepsilon)$,

$$|\nabla u| = \sqrt{u_\rho^2 + \frac{u_\vartheta^2}{\phi(\rho)^2}} \qquad \text{a.e. in } N_\varepsilon\,. \tag{7.20}$$

Define

$$\overline{u}(\vartheta) = \frac{1}{\int_0^{\overline{\rho}-\varepsilon}\phi(\rho)d\rho}\int_0^{\overline{\rho}-\varepsilon} u(\rho,\vartheta)\phi(\rho)d\rho \qquad \text{for a.e. } \vartheta \in [0,2\pi]\,. \tag{7.21}$$

One has that

$$\Big(\int_{Q\cup P_\varepsilon}|u|^q d\mathcal{H}^2\Big)^{\frac{p}{q}} \le 2^{p-1}\Big[\Big(\int_{Q\cup P_\varepsilon}|u-\overline{u}|^q d\mathcal{H}^2\Big)^{\frac{p}{q}} + \Big(\int_{Q\cup P_\varepsilon}|\overline{u}|^q d\mathcal{H}^2\Big)^{\frac{p}{q}}\Big], \tag{7.22}$$

where $\overline{u}$ is regarded as a function defined on N_ε. It is easily verified that the function $u - \overline{u}$ has mean value 0 on $Q \cup P_\varepsilon$. Thus, by a standard Poincaré inequality,

$$\Big(\int_{Q\cup P_\varepsilon}|u-\overline{u}|^q d\mathcal{H}^2\Big)^{\frac{p}{q}} \le C\int_{Q\cup P_\varepsilon}|\nabla u|^p d\mathcal{H}^2\,, \tag{7.23}$$

for some constant C independent of ε and u. This is a consequence of the fact that Q is independent of ε, and P_ε is an open subset of $\mathbb{R}^2$ (an annulus) enjoying the cone property for some cone independent of ε.
Next, the following chain holds:

$$\begin{aligned}
\Big(\int_{Q\cup P_\varepsilon}|\overline{u}|^q d\mathcal{H}^2\Big)^{\frac{p}{q}} &= \Big(\int_0^{2\pi}\int_0^{\overline{\varrho}-\varepsilon}\Big|\frac{1}{\int_0^{\overline{\rho}-\varepsilon}\phi(r)dr}\int_0^{\overline{\rho}-\varepsilon}u(r,\vartheta)\phi(r)\,dr\Big|^q\phi(\rho)d\rho d\vartheta\Big)^{\frac{p}{q}} \\
&= \Big(\int_0^{\overline{\rho}-\varepsilon}\phi(\rho)d\rho\Big)^{\frac{p}{q}}\Big(\int_0^{2\pi}\Big|\frac{1}{\int_0^{\overline{\rho}-\varepsilon}\phi(r)dr}\int_0^{\overline{\rho}-\varepsilon}u(r,\vartheta)\phi(r)dr\Big|^q d\vartheta\Big)^{\frac{p}{q}} \\
&\le \Big(\int_0^{\overline{\rho}-\varepsilon}\phi(\rho)d\rho\Big)^{\frac{p}{q}}(2\pi)^{\frac{p}{q}}\sup_{\vartheta\in[0,2\pi]}\Big|\frac{1}{\int_0^{\overline{\rho}-\varepsilon}\phi(r)dr}\int_0^{\overline{\rho}-\varepsilon}u(r,\vartheta)\phi(r)dr\Big|^p \\
&\le C\Big(\int_0^{\overline{\rho}-\varepsilon}\phi(\rho)d\rho\Big)^{\frac{p}{q}}\Big[\Big(\int_0^{2\pi}\frac{1}{\int_0^{\overline{\rho}-\varepsilon}\phi(r)dr}\int_0^{\overline{\rho}-\varepsilon}|u_\vartheta(r,\vartheta)|\phi(r)drd\vartheta\Big)^p \\
&\quad + \Big(\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\int_0^{\overline{\rho}-\varepsilon}\phi(r)dr}\int_0^{\overline{\rho}-\varepsilon}|u(r,\vartheta)|\phi(r)drd\vartheta\Big)^p\Big] \\
&\le C\Big(\int_0^{\overline{\rho}-\varepsilon}\phi(\rho)d\rho\Big)^{\frac{p}{q}}\Big[\Big(\int_0^{2\pi}\frac{1}{\int_0^{\overline{\rho}-\varepsilon}\phi(r)dr}\int_0^{\overline{\rho}-\varepsilon}|\nabla u|\phi(r)^2drd\vartheta\Big)^p \\
&\quad + \Big(\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\int_0^{\overline{\rho}-\varepsilon}\phi(r)dr}\int_0^{\overline{\rho}-\varepsilon}|u(r,\vartheta)|\phi(r)drd\vartheta\Big)^p\Big] \\
&\le C\Big(\int_0^{\overline{\rho}-\varepsilon}\phi(\rho)d\rho\Big)^{\frac{p}{q}}\Big[\max|\phi|^p\Big(\int_0^{2\pi}\frac{1}{\int_0^{\overline{\rho}-\varepsilon}\phi(r)dr}\int_0^{\overline{\rho}-\varepsilon}|\nabla u|\phi(r)drd\vartheta\Big)^p \\
&\quad + \Big(\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\int_0^{\overline{\rho}-\varepsilon}\phi(r)dr}\int_0^{\overline{\rho}-\varepsilon}|u(r,\vartheta)|\phi(r)drd\vartheta\Big)^p\Big]
\end{aligned} \tag{7.24}$$

$$
\begin{aligned}
&\leq C\Big(\int_0^{\overline{\rho}-\varepsilon}\phi(\rho)d\rho\Big)^{\frac{p}{q}}\Big[\max|\phi|^p\Big(\int_0^{2\pi}\Big(\frac{1}{\int_0^{\overline{\rho}-\varepsilon}\phi(r)dr}\int_0^{\overline{\rho}-\varepsilon}|\nabla u|^p\phi(r)dr\Big)^{1/p}d\vartheta\Big)^p \\
&\quad+\Big(\frac{1}{2\pi}\int_0^{2\pi}\Big(\frac{1}{\int_0^{\overline{\rho}-\varepsilon}\phi(r)dr}\int_0^{\overline{\rho}-\varepsilon}|u(r,\vartheta)|^p\phi(r)dr\Big)^{1/p}d\vartheta\Big)^p\Big] \\
&\leq C\Big(\int_0^{\overline{\rho}-\varepsilon}\phi(\rho)d\rho\Big)^{\frac{p}{q}}\Big[\frac{\max|\phi|^p(2\pi)^{p-1}}{\int_0^{\overline{\rho}-\varepsilon}\phi(r)dr}\int_0^{2\pi}\int_0^{\overline{\rho}-\varepsilon}|\nabla u|^p\phi(r)drd\vartheta \\
&\quad+\frac{1}{2\pi\int_0^{\overline{\rho}-\varepsilon}\phi(r)dr}\int_0^{2\pi}\int_0^{\overline{\rho}-\varepsilon}|u(r,\vartheta)|^p\phi(r)drd\vartheta\Big] \\
&\leq C'\Big[\int_{Q\cup P_\varepsilon}|\nabla u|^p d\mathcal{H}^2+\int_0^{2\pi}\int_0^{\overline{\rho}-\varepsilon}|u(r,\vartheta)|^p\phi(r)drd\vartheta\Big],
\end{aligned}
$$

for some constants C and C' independent of ε and u. Note that a rigorous derivation of the inequality between the leftmost and rightmost sides of (7.24) requires an approximation argument of u by smooth functions. Since, for a.e. $\vartheta\in[0,2\pi]$,

$$
u(\rho,\vartheta)=u(\hat{\rho}-\varepsilon,\vartheta)-\int_\rho^{\hat{\rho}-\varepsilon}u_\rho(r,\vartheta)dr \qquad \text{for } \rho\in(0,\hat{\rho}-\varepsilon), \tag{7.25}
$$

we have that

$$
|u(\rho,\vartheta)|^p\leq C|u(\hat{\rho}-\varepsilon,\vartheta)|^p+C\int_\rho^{\hat{\rho}-\varepsilon}|u_\rho(r,\vartheta)|^p dr \qquad \text{for a.e. } (\rho,\vartheta)\in(0,\widehat{\rho}-\varepsilon)\times(0,2\pi)\,, \tag{7.26}
$$

for some constant C independent of ε and u. Thus,

$$
\begin{aligned}
&\int_0^{2\pi}\int_0^{\overline{\rho}-\varepsilon}|u(\rho,\vartheta)|^p\phi(\rho)d\rho d\vartheta \\
&\quad\leq C\int_0^{2\pi}\int_0^{\hat{\rho}-\varepsilon}\Big(\int_\rho^{\hat{\rho}-\varepsilon}|u_\rho(r,\vartheta)|^p dr\Big)\phi(\rho)d\rho d\vartheta+C\int_0^{2\pi}\int_0^{\hat{\rho}-\varepsilon}|u(\hat{\rho}-\varepsilon,\vartheta)|^p\phi(\rho)d\rho d\vartheta \\
&\quad\leq C\int_0^{2\pi}\int_0^{\hat{\rho}-\varepsilon}\Big(\int_\rho^{\hat{\rho}-\varepsilon}|\nabla u(r,\vartheta)|^p dr\Big)\phi(\rho)d\rho d\vartheta+C\int_0^{2\pi}\int_0^{\hat{\rho}-\varepsilon}|u(\hat{\rho}-\varepsilon,\vartheta)|^p\phi(\rho)d\rho d\vartheta \\
&\quad= C\int_0^{2\pi}\int_0^{\hat{\rho}-\varepsilon}|\nabla u(r,\vartheta)|^p\Big(\int_0^r\phi(\rho)d\rho\Big)drd\vartheta+C\Big(\int_0^{\hat{\rho}-\varepsilon}\phi(\rho)d\rho\Big)\Big(\int_0^{2\pi}|u(\hat{\rho}-\varepsilon,\vartheta)|^p d\vartheta\Big) \\
&\quad\leq C\Big(\sup_{r\in(0,\hat{\rho}-\varepsilon)}\frac{\int_0^r\phi(\rho)d\rho}{\phi(r)}\Big)\int_0^{2\pi}\int_0^{\hat{\rho}-\varepsilon}|\nabla u(r,\vartheta)|^p\phi(r)drd\vartheta \\
&\qquad+C\Big(\int_0^{\hat{\rho}-\varepsilon}\phi(\rho)d\rho\Big)\Big(\int_0^{2\pi}|u(\hat{\rho}-\varepsilon,\vartheta)|^p d\vartheta\Big) \\
&\quad\leq C'\frac{1}{\phi((\hat{\rho}-\overline{\rho})/2))}\int_0^{2\pi}\int_0^{\hat{\rho}-\varepsilon}|\nabla u(r,\vartheta)|^p\phi(r)drd\vartheta+C\Big(\int_0^{\hat{\rho}-\varepsilon}\phi(\rho)d\rho\Big)\Big(\int_0^{2\pi}|u(\hat{\rho}-\varepsilon,\vartheta)|^p d\vartheta\Big) \\
&\quad= C'\frac{1}{\varepsilon/2}\int_{N_\varepsilon}|\nabla u|^p d\mathcal{H}^2+\frac{C''}{\varepsilon}\int_{\partial N_\varepsilon}|u|^p d\mathcal{H}^1\,,
\end{aligned} \tag{7.27}
$$

for some constants C, C' and C'' independent of ε and u. Note that the last inequality holds since ϕ is increasing in a neighborhood of 0.

Inequality (7.19) follows from (7.22), (7.23), (7.24) and (7.27).
Step 2. Let $N_{\varepsilon,\delta}$ be the manifold obtained on scaling N_ε by a factor δ. Thus, $N_{\varepsilon,\delta}$ is parametrized by $(x,y,z) = (\delta\phi(\rho)\cos\vartheta, \delta\phi(\rho)\sin\vartheta, \delta\psi(\rho))$.
Let $u \in W^{1,p}(N_{\varepsilon,\delta})$. From (7.19) we obtain that

$$\delta^{-\frac{2p}{q}}\Big(\int_{Q_\delta\cup P_{\varepsilon,\delta}} |u|^q d\mathcal{H}^2\Big)^{\frac{p}{q}} \le \frac{C}{\varepsilon}\Big(\delta^{p-2}\int_{N_{\varepsilon,\delta}} |\nabla u|^p d\mathcal{H}^2 + \delta^{-1}\int_{\partial N_{\varepsilon,\delta}} |u|^p d\mathcal{H}^1\Big), \tag{7.28}$$

where Q_δ and $P_{\varepsilon,\delta}$ denote the subsets of $N_{\varepsilon,\delta}$ obtained on scaling Q and P_ε, respectively, by a factor δ. Now, let A be a Young function whose inverse satisfies

$$A^{-1}(\delta^{-2}) \approx \frac{\delta^{p-1}}{\sigma(\delta)} \qquad \text{for } \delta > 0. \tag{7.29}$$

Notice that such a function A does exist. Indeed, the function $H : (0,\infty) \to [0,\infty)$ given by $H(t) = \frac{t^{-\frac{p-1}{2}}}{\sigma(t^{-\frac{1}{2}})}$ for $t > 0$ is increasing by (7.14), and the function $\frac{H(t)}{t}$ is non-increasing by (7.15). Thus, $\frac{H^{-1}(\tau)}{\tau}$ is a non-decreasing function, and, on taking

$$A(t) = \int_0^t \frac{H^{-1}(\tau)}{\tau}\, d\tau \qquad \text{for } t \ge 0,$$

equation (7.29) holds, inasmuch as $A(t) \approx H^{-1}(t)$ for $t \ge 0$. Next, we claim that a Young function E exists whose inverse fulfils

$$E^{-1}(\tau) \approx \frac{A^{-1}(\tau)}{\tau^{p/q}} \qquad \text{for } \tau > 0. \tag{7.30}$$

To see this, note that the function $J(\tau) = \frac{A^{-1}(\tau)}{\tau^{p/q}}$ is equivalent to an increasing function $F(\tau)$ (for sufficiently large q, depending on β, if $p = 2$) by (7.14), and that the function $\frac{J(\tau)}{\tau} = \frac{A^{-1}(\tau)}{\tau^{1+p/q}}$ is trivially decreasing. Set $J_1(\tau) = \frac{J(\tau)}{\tau}$. Thus, $\frac{F(\tau)}{\tau} \approx J_1(\tau)$ for $\tau > 0$. As a consequence, one can show that $\frac{F^{-1}(t)}{t} \approx \frac{1}{J_1(F^{-1}(t))}$, an increasing function. Thus the function E given by

$$E(t) = \int_0^t \frac{d\tau}{J_1(F^{-1}(\tau))} \qquad \text{for } t \ge 0,$$

is a Young function, and since $E(t) \approx \frac{t}{J_1(F^{-1}(t))} \approx F^{-1}(t)$, one has that $E^{-1}(\tau) \approx F(\tau) \approx J(\tau) = \frac{A^{-1}(\tau)}{t^{p/q}}$, whence (7.30) follows.
Owing to (7.30), inequality (7.17) ensures that

$$\begin{aligned}\||u|^p\|_{L^A(Q_\delta\cup P_{\varepsilon,\delta})} &\le C\||u|^p\|_{L^{q/p}(Q_\delta\cup P_{\varepsilon,\delta})}\|1\|_{L^E(Q_\delta\cup P_{\varepsilon,\delta})} \\ &= C\|u\|^p_{L^q(Q_\delta\cup P_{\varepsilon,\delta})}\frac{1}{E^{-1}(1/\mathcal{H}^2(Q_\delta\cup P_{\varepsilon,\delta}))} \le C\|u\|^p_{L^q(Q_\delta\cup P_{\varepsilon,\delta})}\frac{1}{E^{-1}(C'/\delta^2)}\,,\end{aligned} \tag{7.31}$$

for some constants C and C' independent of ε, δ and u. Combining (7.28)–(7.31) yields

$$\||u|^p\|_{L^A(Q_\delta\cup P_{\varepsilon,\delta})} \le \frac{C\sigma(\delta)}{\varepsilon\delta}\int_{N_{\varepsilon,\delta}} |\nabla u|^p d\mathcal{H}^2 + \frac{C\sigma(\delta)}{\varepsilon\delta^p}\int_{\partial N_{\varepsilon,\delta}} |u|^p d\mathcal{H}^1\,. \tag{7.32}$$

Now, choose

$$\varepsilon = \frac{\sigma(\delta)}{\delta}\,,$$

and obtain from (7.32)

$$\| |u|^p \|_{L^A(Q_\delta \cup P_{\frac{\sigma(\delta)}{\delta},\delta})} \leq C \int_{N_{\frac{\sigma(\delta)}{\delta},\delta}} |\nabla u|^p d\mathcal{H}^2 + C\delta^{1-p} \int_{\partial N_{\frac{\sigma(\delta)}{\delta},\delta}} |u|^p d\mathcal{H}^1\,. \tag{7.33}$$

Note also that

$$\mathcal{H}^1(\partial N_{\frac{\sigma(\delta)}{\delta},\delta}) = 2\pi\sigma(\delta), \tag{7.34}$$

a fact that will be tacitly used in what follows.
Step 3. Choose $\delta_k = 2^{-k}$ for $k \in \mathbb{N}$, and denote

$$Q^k = Q_{\delta_k}, \quad P^k = P_{\frac{\sigma(\delta_k)}{\delta_k},\delta_k}, \quad N^k = N_{\frac{\sigma(\delta_k)}{\delta_k},\delta_k}.$$

Define the manifold M in such a way that the distance between the centers of the circumferences ∂N^k and ∂N^{k+1} equals 2^{-k+1}. Given $u \in W^{1,p}(M)$, one has that

$$\| |u|^p \|_{L^A(\cup_k (Q^k \cup P^k)} \leq \sum_k \| |u|^p \|_{L^A(Q^k \cup P^k)}\,, \tag{7.35}$$

and

$$\int_{\cup_k N^k} |\nabla u|^p d\mathcal{H}^2 = \sum_{k\in\mathbb{N}} \int_{N^k} |\nabla u|^p d\mathcal{H}^2\,. \tag{7.36}$$

Now, notice that the manifold M is flat in a neighborhood of $\cup_k N^k$. For $k \in \mathbb{N}$, let us denote by Ω_k the open set on M bounded by the circumference ∂N^k (having radius $\sigma(\delta_k)$) and by the boundary of the square on M, with sides parallel to the coordinate axes, whose side-length is $3\sigma(\delta_k)$, and whose center agrees with the center of ∂N^k. Hence, in particular,

$$\mathcal{H}^2(\Omega_k) \leq 9\sigma(\delta_k)^2. \tag{7.37}$$

Observe that

$$\int_{\partial N^k} |u|^p \, d\mathcal{H}^1 \leq C\sigma(\delta_k)^{p-1} \int_{\Omega_k} |\nabla u|^p d\mathcal{H}^2 + C\sigma(\delta_k)^{-1} \int_{\Omega_k} |u|^p d\mathcal{H}^2 \tag{7.38}$$

for some constant C independent of k and u. Inequality (7.38) can be derived via a scaling argument applied to a standard trace inequality for subsets of $\mathbb{R}^n$ with a Lipschitz boundary. Thus,

$$\begin{aligned} &\sum_{k\in\mathbb{N}} \delta_k^{1-p} \int_{\partial N^k} |u|^p \, d\mathcal{H}^1 \\ &\quad \leq C \sum_{k\in\mathbb{N}} \delta_k^{1-p} \sigma(\delta_k)^{p-1} \int_{\Omega_k} |\nabla u|^p d\mathcal{H}^2 + C \sum_{k\in\mathbb{N}} \delta_k^{1-p} \sigma(\delta_k)^{-1} \int_{\Omega_k} |u|^p d\mathcal{H}^2\,. \end{aligned} \tag{7.39}$$

Assumption (7.14) ensures that

$$\lim_{\delta\to 0} \frac{\sigma(\delta)}{\delta} = 0\,,$$

and hence, in particular, there exists a constant C such that

$$\frac{\sigma(\delta)}{\delta} \leq C \qquad \text{if } 0 < \delta \leq 1. \tag{7.40}$$

Consequently,

$$\sum_{k\in\mathbb{N}} \delta_k^{1-p}\sigma(\delta_k)^{p-1}\int_{\Omega_k} |\nabla u|^p d\mathcal{H}^2 \leq C\int_M |\nabla u|^p d\mathcal{H}^2 \tag{7.41}$$

for some constant C. As far as the second addend on the right-hand side of (7.39) is concerned, if $1 \leq p < 2$ by Hölder's inequality and (7.37) one has that

$$\begin{aligned}\sum_{k\in\mathbb{N}} \delta_k^{1-p}\sigma(\delta_k)^{-1}\int_{\Omega_k} |u|^p d\mathcal{H}^2 &= \int_M \sum_{k\in\mathbb{N}} \chi_{\Omega_k}\delta_k^{1-p}\sigma(\delta_k)^{-1}|u|^p d\mathcal{H}^2 \\ &\leq \Big(\int_{\cup_k\Omega_k} |u|^{\frac{2p}{2-p}} d\mathcal{H}^2\Big)^{\frac{2-p}{2}} \Big(9\sum_{k\in\mathbb{N}} \sigma(\delta_k)^2\big(\delta_k^{1-p}\sigma(\delta_k)^{-1}\big)^{\frac{2}{p}}\Big)^{\frac{p}{2}} \\ &\leq C\Big(\int_{\cup_k\Omega_k} |u|^{\frac{2p}{2-p}} d\mathcal{H}^2\Big)^{\frac{2-p}{2}} \Big(\int_0^1 \frac{\sigma(\delta)^{2-\frac{2}{p}}}{\delta^{3-\frac{2}{p}}}\, d\delta\Big)^{\frac{p}{2}},\end{aligned} \tag{7.42}$$

for some constant C independent of k and u. If $p=2$, then given $a>1$ one similarly has that

$$\sum_{k\in\mathbb{N}} \delta_k^{-1}\sigma(\delta_k)^{-1}\int_{\Omega_k} |u|^2 d\mathcal{H}^2 \leq C\Big(\int_{\cup_k\Omega_k} |u|^{2a} d\mathcal{H}^2\Big)^{\frac{1}{a}} \Big(\int_0^1 \frac{\sigma(\delta)^{2-a'}}{\delta^{1+a'}}\, d\delta\Big)^{\frac{1}{a'}}. \tag{7.43}$$

Thanks to (7.14), $\int_0^1 \frac{\sigma(\delta)^{2-\frac{2}{p}}}{\delta^{3-\frac{2}{p}}}\, d\delta < \infty$ if $1 \leq p < 2$, and $\int_0^1 \frac{\sigma(\delta)^{2-a'}}{\delta^{1+a'}}\, d\delta < \infty$ if $p = 2$, provided that a is sufficiently large.
On the other hand, by our choice of δ_k and of the distance between the centers of ∂N^k and ∂N^{k+1}, any regular neighborhood of $\cup_k \partial N^k$ in M, containing $\cup_k\Omega_k$, is a planar domain having the cone property. Hence, by the Sobolev inequality, if $1 \leq p < 2$

$$\Big(\int_{\cup_k\Omega_k} |u|^{\frac{2p}{2-p}} d\mathcal{H}^2\Big)^{\frac{2-p}{2}} \leq C\int_M \big(|\nabla u|^p + |u|^p\big) d\mathcal{H}^2\,,$$

and, if $p=2$,

$$\Big(\int_{\cup_k\Omega_k} |u|^{2a} d\mathcal{H}^2\Big)^{\frac{1}{a}} \leq C\int_M \big(|\nabla u|^2 + |u|^2\big) d\mathcal{H}^2$$

for some constant C independent of u. Altogether, we infer that there exists a constant C such that

$$\sum_{k\in\mathbb{N}} \delta_k^{1-p}\sigma(\delta_k)^{-1}\int_{\partial N^k} |u|^p d\mathcal{H}^1 \leq C\Big(\int_M |\nabla u|^p dx + \int_M |u|^p dx\Big). \tag{7.44}$$

Combining (7.33), (7.35), (7.36) and (7.44) tells us that there exists a constant C such that

$$\||u|^p\|^{1/p}_{L^A(\cup_k(Q^k\cup P^k))} \leq C\big(\|\nabla u\|_{L^p(M)} + \|u\|_{L^p(M)}\big) \tag{7.45}$$

for every $u \in W^{1,p}(M)$.

Step 4. Denote by $R_{\frac{\sigma(\delta)}{\delta},\delta}$ the manifold obtained on scaling $R_{\frac{\sigma(\delta)}{\delta}}$ by the factor δ. We shall show that inequality (7.45) continues to hold if $\cup_k(Q^k \cup P^k)$ is replaced by $\cup_k R^k$, where

$$R^k = R_{\frac{\sigma(\delta_k)}{\delta_k},\delta_k}.$$

Let ρ_i, $i = 1, \dots, m$, be such that $\rho_1 = \overline{\rho} - \varepsilon$, $\rho_m = \hat{\rho} - \varepsilon$, the difference $\rho_{i+1} - \rho_i$ is independent of i for $i = 1, \dots, m-1$, and

$$1 \le \frac{\rho_{i+1} - \rho_i}{\sigma(\delta)} \le 2 \qquad \text{for } i = 1, \dots, m-1. \tag{7.46}$$

Let

$$R^i_\delta = \{(\rho, \vartheta) \in R_{\frac{\sigma(\delta)}{\delta},\delta} : \rho_i \le \rho \le \rho_{i+1}\} \qquad \text{for } i = 1, \dots, m-1.$$

Define

$$\hat{u}(\rho) = \frac{1}{2\pi}\int_0^{2\pi} u(\rho, \vartheta)d\vartheta \qquad \text{for a.e. } \rho \in (\overline{\rho} - \varepsilon, \hat{\rho} - \varepsilon).$$

We have that

$$\||u|^p\|_{L^A(R_{\frac{\sigma(\delta)}{\delta},\delta})} \le 2^{p-1}\||u - \hat{u}|^p\|_{L^A(R_{\frac{\sigma(\delta)}{\delta},\delta})} + 2^{p-1}\||\hat{u}|^p\|_{L^A(R_{\frac{\sigma(\delta)}{\delta},\delta})}\,, \tag{7.47}$$

where $\hat{u}$ is regarded as a function defined on $R_{\frac{\sigma(\delta)}{\delta},\delta}$, and A is the Young function introduced in Step 2. Furthermore,

$$\||u - \hat{u}|^p\|_{L^A(R_{\frac{\sigma(\delta)}{\delta},\delta})} \le \sum_{i=1}^{m-1} \||u - \hat{u}|^p\|_{L^A(R^i_\delta)}\,. \tag{7.48}$$

The mean value of $u - \hat{u}$ over each R^i_δ is 0. The manifolds $R^2_\delta, \cdots, R^{m-2}_\delta$ agree, up to translations, with the same cylinder. The manifolds R^1_δ and R^{m-1}_δ also coincide, up to isometries. Moreover,

$$\mathcal{H}^2(R^i_\delta) \approx \sigma(\delta)^2\,,$$

owing to assumption (7.46). An analogous scaling argument as in the proof of Step 2 tells us

$$\||u - \hat{u}|^p\|_{L^A(R^i_\delta)} \le C\frac{\sigma(\delta)^{p-2}}{A^{-1}(C\sigma(\delta)^{-2})}\|\nabla u\|^p_{L^p(R^i_\delta)}\,, \tag{7.49}$$

for some constant C. Next, since

$$|\hat{u}(\rho)|^p \le C\bigg(|\hat{u}(\hat{\rho} - \varepsilon)|^p + \bigg(\int_{\overline{\rho}-\varepsilon}^{\hat{\rho}-\varepsilon} |\hat{u}'(r)|dr\bigg)^p\bigg) \qquad \text{for a.e. } \rho \in (\overline{\rho} - \varepsilon, \hat{\rho} - \varepsilon),$$

one has that

(7.50) $\| |\hat{u}|^p \|_{L^A(R_{\frac{\sigma(\delta)}{\delta},\delta})}$

$$\le C_0 |\hat{u}(\hat\rho-\varepsilon)|^p \|1\|_{L^A(R_{\frac{\sigma(\delta)}{\delta},\delta})} + C_0 \left\| \left(\int_{\overline{\rho}-\varepsilon}^{\hat\rho-\varepsilon} \frac{1}{2\pi} \int_0^{2\pi} |u_\rho(\rho,\vartheta)| d\vartheta d\rho \right)^p \right\|_{L^A(R_{\frac{\sigma(\delta)}{\delta},\delta})}$$

$$\le \frac{C_0}{A^{-1}(1/\mathcal{H}^2(R_{\frac{\sigma(\delta)}{\delta},\delta}))} |\hat{u}(\hat\rho-\varepsilon)|^p + C_0 \left\| \left(\int_{\overline{\rho}-\varepsilon}^{\hat\rho-\varepsilon} \frac{1}{2\pi} \int_0^{2\pi} |\nabla u| d\vartheta d\rho \right)^p \right\|_{L^A(R_{\frac{\sigma(\delta)}{\delta},\delta})}$$

$$\le \frac{C_0}{A^{-1}(1/\mathcal{H}^2(R_{\frac{\sigma(\delta)}{\delta},\delta}))} |\hat{u}(\hat\rho-\varepsilon)|^p$$

$$+ \frac{C_0}{\min_{\rho\in(\overline{\rho}-\varepsilon,\hat\rho-\varepsilon)} \phi(\rho)^p} \left\| \left(\int_{\overline{\rho}-\varepsilon}^{\hat\rho-\varepsilon} \frac{1}{2\pi} \int_0^{2\pi} |\nabla u| \phi(\rho) d\vartheta d\rho \right)^p \right\|_{L^A(R_{\frac{\sigma(\delta)}{\delta},\delta})}$$

$$\le \frac{C_0}{A^{-1}(1/\mathcal{H}^2(R_{\frac{\sigma(\delta)}{\delta},\delta}))} |\hat{u}(\hat\rho-\varepsilon)|^p + \frac{C_1}{\sigma(\delta)^p} \left\| \left(\int_{R_{\frac{\sigma(\delta)}{\delta},\delta}} |\nabla u| d\mathcal{H}^2 \right)^p \right\|_{L^A(R_{\frac{\sigma(\delta)}{\delta},\delta})}$$

$$\le \frac{C_0}{A^{-1}(1/\mathcal{H}^2(R_{\frac{\sigma(\delta)}{\delta},\delta}))} |\hat{u}(\hat\rho-\varepsilon)|^p + \frac{C_1}{\sigma(\delta)^p} \mathcal{H}^2(R_{\frac{\sigma(\delta)}{\delta},\delta})^{p-1} \|\nabla u\|^p_{L^p(R_{\frac{\sigma(\delta)}{\delta},\delta})} \|1\|_{L^A(R_{\frac{\sigma(\delta)}{\delta},\delta})}$$

$$\le \frac{C_2}{A^{-1}(C_3/(\delta\sigma(\delta)))} \left(\frac{1}{\sigma(\delta)} \int_{\partial N_{\frac{\sigma(\delta)}{\delta},\delta}} |u| d\mathcal{H}^1 \right)^p + \frac{C_2 \delta^{p-1} \sigma(\delta)^{p-1}}{\sigma(\delta)^p A^{-1}(C_3/\delta\sigma(\delta))} \|\nabla u\|^p_{L^p(R_{\frac{\sigma(\delta)}{\delta},\delta})}$$

$$\le \frac{C_4}{\sigma(\delta) A^{-1}(C_3/(\delta\sigma(\delta)))} \int_{\partial N_{\frac{\sigma(\delta)}{\delta},\delta}} |u|^p d\mathcal{H}^1 + \frac{C_2 \delta^{p-1}}{\sigma(\delta) A^{-1}(C_3/(\delta\sigma(\delta)))} \|\nabla u\|^p_{L^p(R_{\frac{\sigma(\delta)}{\delta},\delta})},$$

for suitable constants C_i, $i = 0, \dots, 4$. Here, we have made use of the fact that

$$\mathcal{H}^2(R_{\frac{\sigma(\delta)}{\delta},\delta}) \approx \delta\sigma(\delta).$$

An approximation argument for u by smooth functions is also required. Owing to (7.40), for any $C > 0$, there exists a constant $C' > 0$ such that

$$\frac{\sigma(\delta)^{p-2}}{A^{-1}(C/\sigma(\delta)^2)} \le \frac{C' \delta^{p-1}}{\sigma(\delta) A^{-1}(C'/(\delta\sigma(\delta)))}. \tag{7.51}$$

Thus, from (7.47)–(7.51) one deduces that there exists a constant $C > 0$ such that

(7.52) $\| |u|^p \|_{L^A(R_{\frac{\sigma(\delta)}{\delta},\delta})}$

$$\le \frac{C}{\sigma(\delta) A^{-1}(C/(\delta\sigma(\delta)))} \int_{\partial N_{\frac{\sigma(\delta)}{\delta},\delta}} |u|^p d\mathcal{H}^1 + \frac{C\delta^{p-1}}{\sigma(\delta) A^{-1}(C/(\delta\sigma(\delta)))} \int_{R_{\frac{\sigma(\delta)}{\delta},\delta}} |\nabla u|^p d\mathcal{H}^2.$$

Consequently,

(7.53)

$$\| |u|^p \|_{L^A(\cup_k R^k)} \le \sum_{k\in\mathbb{N}} \| |u|^p \|_{L^A(R^k)}$$

$$\le \sum_{k\in\mathbb{N}} \frac{C}{\sigma(\delta_k) A^{-1}(C/(\delta_k\sigma(\delta_k)))} \int_{\partial N^k} |u|^p d\mathcal{H}^1 + \sum_{k\in\mathbb{N}} \frac{C\delta_k^{p-1}}{\sigma(\delta_k) A^{-1}(C/(\delta_k\sigma(\delta_k)))} \int_{R^k} |\nabla u|^p d\mathcal{H}^2.$$

By (7.40) and (7.29),

$$\frac{\delta_k^{p-1}}{\sigma(\delta_k)A^{-1}(C/(\delta_k\sigma(\delta_k)))} \leq \frac{\delta_k^{p-1}}{\sigma(\delta_k)A^{-1}(C'/\delta_k^2)} \leq C'', \tag{7.54}$$

for some positive constants C' and C''. Thus,

$$\||u|^p\|_{L^A(\cup_k R^k)} \leq \sum_{k\in\mathbb{N}} \frac{C}{\delta_k^{p-1}} \int_{\partial N^k} |u|^p d\mathcal{H}^1 + C\|\nabla u\|^p_{L^p(\cup_k R^k)}, \tag{7.55}$$

for some constant C. Hence, since $\sigma(\delta_k)$ is bounded for $k \in \mathbb{Z}$, we deduce from (7.44) that

$$\||u|^p\|^{1/p}_{L^A(\cup_k R^k)} \leq C\big(\|\nabla u\|_{L^p(M)} + \|u\|_{L^p(M)}\big). \tag{7.56}$$

Step 5. A variant of [Ma7, Theorem 2.3.2], with analogous proof, tells us that given a (2-dimensional) Riemannian manifold Z with $\mathcal{H}^2(Z) < \infty$, and a Young function B, the inequality

$$\||u|^p\|^{1/p}_{L^B(Z)} \leq C\big(\|\nabla u\|_{L^p(Z)} + \|u\|_{L^p(Z)}\big) \tag{7.57}$$

holds for some constant C and for every $u \in W^{1,p}(Z)$ if and only if

$$\frac{1}{B^{-1}(1/s)} \leq C'\nu_{Z,p}(s) \qquad \text{for } s \in (0, \mathcal{H}^2(Z)/2), \tag{7.58}$$

for some constant C'. In Step 3 we have observed that a regular neighbourhood of $\cup_k \partial N^k$ is a planar domain fulfilling the cone property. Hence, the standard Sobolev inequality holds on $M\setminus(\cup_k N^k)$, and, consequently, (7.58) holds with $Z = M\setminus(\cup_k N^k)$ and $B(t) = t^{\frac{2}{2-p}}$ if $1 \leq p < 2$, and with $B(t) = t^a$ for any $a \geq 1$ if $p = 2$. Thus, since the right-hand side of (7.30) is equivalent to a non-decreasing function, inequality (7.58) also holds with $B = A$. Hence, there exists a constant C such that

$$\||u|^p\|^{1/p}_{L^A(M\setminus(\cup_k N^k))} \leq C\big(\|\nabla u\|_{L^p(M\setminus(\cup_k N^k))} + \|u\|_{L^p(M\setminus(\cup_k N^k))}\big) \tag{7.59}$$

for $u \in W^{1,p}(M)$. Combining (7.45), (7.56) and (7.59) tells us that

$$\||u|^p\|^{1/p}_{L^A(M)} \leq C\big(\|\nabla u\|_{L^p(M)} + \|u\|_{L^p(M)}\big) \tag{7.60}$$

for some constant C and for every $u \in W^{1,p}(M)$. Hence,

$$\frac{1}{A^{-1}(1/s)} \leq C\nu_{M,p}(s) \qquad \text{for } s \in (0, \mathcal{H}^2(M)/2) \tag{7.61}$$

and (7.18) follows, owing to (7.29).

Part II. Here we show that, if $p \geq 1$ and σ is non-decreasing and is of class Δ_2 near 0, then inequality (7.13) holds. Consider the sequence of condensers $(Q^k \cup P^k, N^k)$. Let $\{u_k\}$ be the sequence of Lipschitz continuous functions given by $u_k = 1$ in $Q^k \cup P^k$, $u_k = 0$ in $M \setminus N^k$ and such that u_k depends only on ρ and is a linear function of ρ in R^k. For $k \in \mathbb{N}$, we have that

$$\mathcal{H}^2(Q^k \cup P^k) \approx \delta_k^2, \tag{7.62}$$

and

$$\int_M |\nabla u_k|^p d\mathcal{H}^2 \approx \frac{\mathcal{H}^2(R^k)}{\delta_k^p} \approx \frac{\sigma(\delta_k)}{\delta_k^{p-1}}\,. \tag{7.63}$$

Thus, there exist constants C and C' such that

$$\nu_{M,p}(C\delta_k^2) \le C_p(Q^k \cup P^k, N^k) \le \frac{C'\sigma(\delta_k)}{\delta_k^{p-1}}\,. \tag{7.64}$$

It is easily seen that (7.64) continues to hold with δ_k replaced by any $s \in (0, \mathcal{H}^n(M)/2)$. Hence (7.13) follows.
The proof is complete. □

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