An elementary proof of the Brezis and Mironescu theorem on the composition operator in fractional Sobolev spaces

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Abstract

An elementary proof of the Brezis and Mironescu theorem on the boundedness and continuity of the composition operator: $W^{s,p}(\mathbf{R}^n) \cap W^{1,sp}(\mathbf{R}^n) \to W^{s,p}(\mathbf{R}^n)$ is given. The proof includes the case $p = 1$.

1 Introduction

Let s be noninteger, $1 < s < \infty$, with $[s]$ and $\{s\}$ standing for the integer and fractional parts of s. We introduce the function

$$(D_{p,s}u)(x) = \Big(\int_{\mathbf{R}^n} |\nabla_{[s]} u(x+h) - \nabla_{[s]} u(x)|^p \frac{dh}{|h|^{n+p\{s\}}} \Big)^{1/p}, \tag{1}$$

where $p \geq 1$ and ∇_k is the collection of all partial derivatives ∂^α, $|\alpha| = k$. The fractional Sobolev space with the norm

$$||D_{p,s}u||_{L^p(\mathbf{R}^n)} + ||u||_{L^p(\mathbf{R}^n)}$$

will be denoted by $W^{s,p}(\mathbf{R}^n)$.

In the present article we prove

Theorem. *Let $p \geq 1$ and let s be noninteger, $1 < s < \infty$. For every complex-valued function f defined on $\mathbf{R}$ and such that $f(0) = 0$ and $f', \dots, f^{([s]+1)} \in L^\infty(\mathbf{R})$ there holds*

$$||f(u)||_{W^{s,p}(\mathbf{R}^n)} \leq c \sum_{l=1}^{[s]+1} ||f^{(l)}||_{L^\infty(\mathbf{R})} (||u||_{W^{s,p}(\mathbf{R}^n)} + ||\nabla u||^s_{L^{ps}(\mathbf{R}^n)}). \tag{2}$$

Here and elsewhere $\nabla = \nabla_1$, and c is a constant independent of f and u.

If additionally, $f^{([s]+1)} \in C(\mathbf{R})$ then the map

$$W^{s,p}(\mathbf{R}^n) \cap W^{1,sp}(\mathbf{R}^n) \ni u \to f(u) \in W^{s,p}(\mathbf{R}^n) \tag{3}$$

is continuous.

Inequality (2), with Bessel potential space $H^{s,p}(\mathbf{R}^n)$, $p \in (1,\infty)$ and $s > 0$, instead of $W^{s,p}(\mathbf{R}^n)$, was obtained by D.R.Adams and M.Frazier [AF].

For $p > 1$, a theorem on the boundedness and continuity of the operator (3) was recently proved by H.Brezis and P.Mironescu [BM]. As they say, their proof is quite involved being based on a 'microscopic' improvement of the Gagliardo-Nirenberg inequality, in Triebel-Lizorkin scale, namely $W^{s,p} \cap L^\infty \subset F^\sigma_{q,\nu}$ for every ν and on an important estimate on products of functions in the Triebel-Lizorkin spaces, due to T.Runst and W.Sickel [RS]. To this Brezis and Mironescu add: "It would be interesting to find a more elementary argument which avoids this excursion into $F^s_{p,q}$ scale".

In the present paper we give such an elementary argument which includes $p = 1$ and relies, in particular, upon the following pointwise interpolation inequality.

Lemma. *Suppose $\alpha \in (0,1)$, $p \geq 1$, and $u \in W^{1,p}_{loc}(\mathbf{R}^n)$. Then for almost all $x \in \mathbf{R}^n$*

$$(D_{p,\alpha}u)(x) \leq c((\mathcal{M}|u - u(x)|^p)(x))^{(1-\alpha)/p}((\mathcal{M}|\nabla u|^p)(x))^{\alpha/p}, \tag{4}$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal operator.

Proof. Let $B_\delta(x) = \{y \in \mathbf{R}^n : |y - x| < \delta\}$. By Hardy's inequality

$$\int_{B_\delta(x)} \frac{|u(y) - u(x)|^p}{|y - x|^{n+p\alpha}} dy \leq c \int_{B_\delta(x)} \frac{|\nabla u(y)|^p}{|y - x|^{n+p(\alpha-1)}} dy. \tag{5}$$

Arguing in the same way as Hedberg [H] we find

$$\int_{B_\delta(x)} \frac{|\nabla u(y)|^p}{|y - x|^{n+p(\alpha-1)}} dy \leq c \sum_{k=0}^{\infty} (\delta 2^{-k})^{p(1-\alpha)-n} \int_{B_{\delta 2^{-k}}(x) \setminus B_{\delta 2^{-k-1}}(x)} |\nabla u(y)|^p dy$$

$$\leq c\delta^{p(1-\alpha)}(\mathcal{M}|\nabla u|^p)(x) \tag{6}$$

and

$$\int_{\mathbf{R}^n \setminus B_\delta(x)} \frac{|u(y) - u(x)|^p}{|y - x|^{n+p\alpha}} dy \leq \sum_{k=0}^{\infty} (\delta 2^k)^{-p\alpha-n} \int_{B_{\delta 2^{k+1}}(x) \setminus B_{\delta 2^k}(x)} |u(y) - u(x)|^p dy$$

$$\leq c\delta^{-p\alpha}(\mathcal{M}|u - u(x)|^p)(x). \tag{7}$$

From (5)-(7) it follows that

$$(D_{p,s}u)(x) \leq c(\delta^{p(1-\alpha)}(\mathcal{M}|\nabla u|^p)(x) + \delta^{-p\alpha}(\mathcal{M}|u - u(x)|^p)(x)).$$

Minimizing the right-hand side over all $\delta > 0$ we complete the proof.

Another ingredient in the proof of Theorem is the Gagliardo-Nirenberg type inequality

$$||\nabla_k u||_{L^{ps/k}(\mathbf{R}^n)} \leq c||\nabla u||^{\frac{s-k}{s-1}}_{L^{ps}(\mathbf{R}^n)} \, ||D_{p,s}u||^{\frac{k-1}{s-1}}_{L^p(\mathbf{R}^n)}, \tag{8}$$

where $1 \leq k < s$, s is noninteger, and $p \geq 1$.

A short argument leading to (8) is by the pointwise inequality

$$|\nabla_k u(x)| \leq c((\mathcal{M}|\nabla u|)(x))^{\frac{s-k}{s-1}}((D_{p,s}u)(x))^{\frac{k-1}{s-1}}. \tag{9}$$

In fact, one uses (9) to majorize the $L^{ps/k}$-norm of $|\nabla_k u|$ and applies Hölder's inequality with exponents

$$(s-1)k/(s-k) \quad \text{and} \quad (s-1)k/(k-1)s$$

together with the boundedness of $\mathcal{M}$ in $L^{ps}(\mathbf{R}^n)$. Inequality (9) was derived in [MSh]. For readers' convenience, we reproduce its proof in Appendix.

2 Proof of Theorem

Inequality (2). Since $f(0)=0$, we have

$$\|f(u)\|_{L^p(\mathbf{R}^n)} \leq \|f'\|_{L^\infty(\mathbf{R})}\|u\|_{L^p(\mathbf{R}^n)}. \tag{10}$$

By the Leibniz rule,

$$||D_{p,s}f(u)||_{L^p(\mathbf{R}^n)} \leq c\sum_{l=1}^{[s]} \sum_{\substack{|\alpha_1|+\ldots+|\alpha_l|=[s]\\ |\alpha_i|\geq 1}} \Big\| D_{p,\{s\}}(f^{(l)}(u)\prod_{i=1}^{l}\partial^{\alpha_i}u)\Big\|_{L^p(\mathbf{R}^n)}. \tag{11}$$

We continue by putting

$$v=f^{(l)}(u) \quad \text{and} \quad w=\prod_{i=1}^{l}\partial^{\alpha_i}u$$

in the obvious inequality

$$||D_{p,\{s\}}(vw)||_{L^p(\mathbf{R}^n)} \leq ||vD_{p,\{s\}}w||_{L^p(\mathbf{R}^n)} + ||wD_{p,\{s\}}v||_{L^p(\mathbf{R}^n)}, \tag{12}$$

and arrive at

$$||D_{p,s}f(u)||_{L^p(\mathbf{R}^n)} \leq c\sum_{l=1}^{[s]} \sum_{\substack{|\alpha_1|+\ldots+|\alpha_l|=[s]\\ |\alpha_i|\geq 1}} \Big(\Big\|\prod_{i=1}^{l}\partial^{\alpha_i}u\cdot D_{p,\{s\}}f^{(l)}(u)\Big\|_{L^p(\mathbf{R}^n)}$$

$$+||f^{(l)}||_{L^\infty(\mathbf{R})}\Big\|D_{p,\{s\}}(\prod_{i=1}^{l}\partial^{\alpha_i}u)\Big\|_{L^p(\mathbf{R})^n)}\Big). \tag{13}$$

We set

$$\mathcal{I}_l := \Big\|\prod_{i=1}^{l}\partial^{\alpha_i}u\cdot D_{p,\{s\}}f^{(l)}(u)\Big\|_{L^p(\mathbf{R}^n)}. \tag{14}$$

By Lemma with $f^{(l)}(u)$ in place of u,

$$D_{p,\{s\}}f^{(l)}(u)(x) \leq c||f^{(l)}||^{1-\{s\}}_{L^\infty(\mathbf{R})}((\mathcal{M}|\nabla f^{(l)}(u)|^p)(x))^{\{s\}/p}$$

$$\leq c||f^{(l)}||^{1-\{s\}}_{L^\infty(\mathbf{R})}||f^{(l+1)}||^{\{s\}}_{L^\infty(\mathbf{R})}((\mathcal{M}|\nabla u|^p)(x))^{\{s\}/p}.$$

Hence, using Hölder's inequality with exponents $s/\{s\}$, $s/|\alpha_i|$, $i = 1, \ldots, l$, we find

$$\mathcal{I}_l \le c||f^{(l)}||^{1-\{s\}}_{L^\infty(\mathbf{R})}||f^{(l+1)}||^{\{s\}}_{L^\infty(\mathbf{R})} \prod_{i=1}^{l} \|\partial^{\alpha_i} u\|_{L^{\frac{ps}{|\alpha_i|}}(\mathbf{R}^n)} \|\mathcal{M}|\nabla u|^p\|^{\{s\}/p}_{L^s(\mathbf{R}^n)}. \tag{15}$$

Since $\mathcal{M}$ is bounded in $L^s(\mathbf{R}^n)$, the last factor on the right is majorized by $c||\nabla u||^{\{s\}}_{L^{ps}(\mathbf{R}^n)}$. By (8) the product $\prod_{i=1}^{l}$ in (15) does not exceed

$$c \prod_{i=1}^{l} \|\nabla u\|^{\frac{s-|\alpha_i|}{s-1}}_{L^{ps}(\mathbf{R}^n)} \|D_{p,s} u\|^{\frac{|\alpha_i|-1}{s-1}}_{L^p(\mathbf{R}^n)} = c\|\nabla u\|^{\frac{sl-[s]}{s-1}}_{L^{ps}(\mathbf{R}^n)} \|D_{p,s} u\|^{\frac{[s]-l}{s-1}}_{L^p(\mathbf{R}^n)}. \tag{16}$$

Hence and by (15)

$$\mathcal{I}_l \le c||f^{(l)}||^{1-\{s\}}_{L^\infty(\mathbf{R})}||f^{(l+1)}||^{\{s\}}_{L^\infty(\mathbf{R})}\|\nabla u\|^{\frac{s(l-1+\{s\})}{s-1}}_{L^{ps}(\mathbf{R}^n)} \|D_{p,s} u\|^{\frac{[s]-l}{s-1}}_{L^p(\mathbf{R}^n)}. \tag{17}$$

Let

$$\mathcal{J}_l := \Big\| D_{p,\{s\}} (\prod_{i=1}^{l} \partial^{\alpha_i} u) \Big\|_{L^p(\mathbf{R}^n)} \tag{18}$$

Clearly,

$$\mathcal{J}_1 \le ||D_{p,s} u||_{L^p(\mathbf{R}^n)}. \tag{19}$$

Now let $l > 1$. By (12),

$$\mathcal{J}_l \le \Big\| \prod_{i=2}^{l} \partial^{\alpha_i} u \cdot D_{p,\{s\}} \partial^{\alpha_1} u \Big\|_{L^p(\mathbf{R}^n)} + \Big\| \partial^{\alpha_1} u \cdot D_{p,\{s\}} (\prod_{i=2}^{l} \partial^{\alpha_i} u) \Big\|_{L^p(\mathbf{R}^n)}. \tag{20}$$

Applying first Lemma and then Hölder's inequality with exponents

$$s/|\alpha_i|, \;\; 2 \le i \le l, \;\; s/|\alpha_1|(1-\{s\}), \;\; \text{and} \;\; s/(1+|\alpha_1|)\{s\},$$

we find

$$\Big\| \prod_{i=2}^{l} \partial^{\alpha_i} u \cdot D_{p,\{s\}} \partial^{\alpha_1} u \Big\|_{L^p(\mathbf{R}^n)}$$

$$\le c \Big\| \prod_{i=2}^{l} \partial^{\alpha_i} u \cdot (\mathcal{M}|\partial^{\alpha_1} u|^p)^{(1-\{s\})/p} (\mathcal{M}|\nabla \partial^{|\alpha_1} u|^p)^{\{s\}/p} \Big\|_{L^p(\mathbf{R}^n)}$$

$$\le c \prod_{i=2}^{l} ||\partial^{\alpha_i} u||_{L^{\frac{ps}{|\alpha_i|}}(\mathbf{R}^n)} ||\mathcal{M}|\partial^{\alpha_1} u|^p||^{\frac{1-\{s\}}{p}}_{L^{\frac{s}{|\alpha_1|}}(\mathbf{R}^n)} ||\mathcal{M}|\nabla \partial^{\alpha_1} u|^p||^{\frac{\{s\}}{p}}_{L^{\frac{s}{1+|\alpha_1|}}(\mathbf{R}^n)}.$$

Hence, noting that $s > 1 + |\alpha_1|$ and using the boundedness of $\mathcal{M}$ in $L_q(\mathbf{R}^n)$ with $q > 1$, we obtain that the first term in the right-hand side of (20) is dominated by

$$c \prod_{i=2}^{l} ||\partial^{\alpha_i} u||_{L^{\frac{ps}{|\alpha_i|}}(\mathbf{R}^n)} ||\partial^{\alpha_1} u||^{1-\{s\}}_{L^{\frac{ps}{|\alpha_1|}}(\mathbf{R}^n)} ||\nabla \partial^{\alpha_1} u||^{\{s\}}_{L^{\frac{ps}{1+|\alpha_1|}}(\mathbf{R}^n)}. \tag{21}$$

By (8) this does not exceed

$$c\,||\nabla u||^{\frac{s(l-1)}{s-1}}_{L^{ps}(\mathbf{R}^n)}||D_{p,s}u||^{\frac{s-l}{s-1}}_{L^{p}(\mathbf{R}^n)}. \tag{22}$$

We estimate the second term in the right-hand side of (20). By Lemma,

$$\Big\|\partial^{\alpha_1}u\cdot D_{p,\{s\}}(\prod_{i=2}^{l}\partial^{\alpha_i}u)\Big\|_{L^p(\mathbf{R}^n)}$$

$$\le c\,\Big\|\partial^{\alpha_1}u\;(\mathcal{M}|\prod_{i=2}^{l}\partial^{\alpha_i}u|^p)^{(1-\{s\})/p}\;(\mathcal{M}|\nabla(\prod_{i=2}^{l}\partial^{\alpha_i}u)|^p)^{\{s\}/p}\Big\|_{L^p(\mathbf{R}^n)}$$

which by Hölder's inequality with exponents

$$s/|\alpha_1|,\quad s/([s]-|\alpha_1|)(1-\{s\}),\ \text{ and }\ s/(1+[s]-|\alpha_1|)\{s\},$$

is majorized by

$$c\,||\partial^{\alpha_1}u||_{L^{\frac{ps}{|\alpha_1|}}(\mathbf{R}^n)}\Big\|\mathcal{M}\prod_{i=2}^{l}|\partial^{\alpha_i}u|^p\Big\|^{(1-\{s\})/p}_{L^{\frac{s}{[s]-|\alpha_1|}}(\mathbf{R}^n)}\;\Big\|\mathcal{M}|\nabla\prod_{i=2}^{l}\partial^{\alpha_i}u|^p\Big\|^{\{s\}/p}_{L^{\frac{s}{1+[s]-|\alpha_1|}}(\mathbf{R}^n)}.$$

Using the L_q-boundedness of $\mathcal{M}$ with

$$q=s/([s]-|\alpha_1|)\quad\text{and}\quad q=s/(1+[s]-|\alpha_1|)$$

we conclude that

$$\Big\|\partial^{\alpha_1}u\cdot D_{p,\{s\}}(\prod_{i=2}^{l}\partial^{\alpha_i}u)\Big\|_{L^p(\mathbf{R}^n)}$$

$$\le c\,||\partial^{\alpha_1}u||_{L^{\frac{ps}{|\alpha_1|}}(\mathbf{R}^n)}\;\Big\|\prod_{i=2}^{l}\partial^{\alpha_i}u\Big\|^{1-\{s\}}_{L^{\frac{ps}{[s]-|\alpha_1|}}(\mathbf{R}^n)}\;\Big\|\nabla\prod_{i=2}^{l}\partial^{\alpha_i}u\Big\|^{\{s\}}_{L^{\frac{ps}{1+[s]-|\alpha_1|}}(\mathbf{R}^n)}. \tag{23}$$

By Hölder's inequality with exponents $([s]-|\alpha_1|)/|\alpha_i|$, $2\le i\le l$, the second norm in the right-hand side of (23) does not exceed

$$c\prod_{i=2}^{l}\|\partial^{\alpha_i}u\|_{L^{\frac{ps}{|\alpha_i|}}(\mathbf{R}^n)}. \tag{24}$$

Again by Hölder's inequality, now with exponents

$$(1+[s]-|\alpha_1|)/|\alpha_j|,\quad 2\le j\le l,\ j\ne i,\ \text{ and }\ (1+[s]-|\alpha_1|)/(1+|\alpha_i|),$$

the third norm in (23) is dominated by

$$\sum_{i=2}^{l}\prod_{\substack{j=2\\ j\ne i}}^{l}\|\partial^{\alpha_j}u\|_{L^{\frac{ps}{|\alpha_j|}}(\mathbf{R}^n)}\;\|\nabla\partial^{\alpha_i}u\|_{L^{\frac{ps}{1+|\alpha_i|}}(\mathbf{R}^n)}. \tag{25}$$

Combining (23)-(25) with (8) we find that the left-hand side of (23) does not exceed (22). Thus, (22) is a majorant for the second term in (20). Hence and by (19),

$$\mathcal{J}_l \le c\, \|\nabla u\|^{\frac{s(l-1)}{s-1}}_{L^{ps}(\mathbf{R}^n)}\, \|D_{p,s}u\|^{\frac{s-l}{s-1}}_{L^p(\mathbf{R}^n)}, \quad 1 \le l \le [s]. \tag{26}$$

Inserting (17) and (26) into (13) we arrive at

$$\|D_{p,s}f(u)\|_{L^p(\mathbf{R}^n)} \le c \sum_{l=1}^{[s]+1} \|f^{(l)}\|_{L^\infty(\mathbf{R})} (\|D_{p,s}u\|_{L^p(\mathbf{R}^n)} + \|\nabla u\|^s_{L^{ps}(\mathbf{R}^n)}).$$

Hence and by (10) the proof of (2) is complete.

Continuity of the map (3). Let $u_\nu \to u$ in $W^{s,p}(\mathbf{R}^n) \cap W^{1,sp}(\mathbf{R}^n)$. Since

$$||f^{(l)}(u_\nu) - f^{(l)}(u)||_{L^p(\mathbf{R}^n)} \le c||f^{(l+1)}||_{L^\infty(\mathbf{R})}||u_\nu - u||_{L^p(\mathbf{R}^n)}$$

for $l = 0, \dots, [s]$, we have

$$f^{(l)}(u_\nu) \to f^{(l)}(u) \quad \text{in} \quad L^p(\mathbf{R}^n). \tag{27}$$

We shall prove that

$$||D_{p,s}(f(u_\nu) - f(u))||_{L^p(\mathbf{R}^n)} \to 0. \tag{28}$$

Let α be a multiindex of order $|\alpha| = [s]$. By the Leibniz rule,

$$\partial^\alpha(f(u) - f(u_\nu)) = \sum_{l=1}^{[s]} \sum c(l, \alpha_1, \dots, \alpha_l) \Big(f^{(l)}(u) \prod_{i=1}^{l} \partial^{\alpha_i} u - f^{(l)}(u_\nu) \prod_{i=1}^{l} \partial^{\alpha_i} u_\nu \Big)$$

where the second sum is taken over all l-tuples of multi-indices $\{\alpha_1, \dots, \alpha_l\}$ such that $\alpha_1 + \ldots + \alpha_l = \alpha$ and $|\alpha_i| \ge 1$. We rewrite the difference

$$f^{(l)}(u) \prod_{i=1}^{l} \partial^{\alpha_i} u - f^{(l)}(u_\nu) \prod_{i=1}^{l} \partial^{\alpha_i} u_\nu \tag{29}$$

using the identity

$$\prod_{i=0}^{l} a_i - \prod_{i=0}^{l} b_i = \sum_{k=0}^{l} b_0 \ldots b_{k-1}(a_k - b_k) a_{k+1} \ldots a_l, \tag{30}$$

where the products of either b_i or a_i are missing if $k = 0$ or $k = l$, respectively. Setting

$$a_0 = f^{(l)}(u), \;\; a_i = \partial^{\alpha_i} u, \;\; b_0 = f^{(l)}(u_\nu), \;\; b_i = \partial^{\alpha_i} u_\nu, \;\; 1 \le i \le l$$

in (30), we find that (29) is equal to

$$(f^{(l)}(u) - f^{(l)}(u_\nu)) \prod_{i=1}^{l} \partial^{\alpha_i} u + f^{(l)}(u_\nu) \sum_{k=1}^{l} \prod_{i=1}^{k-1} \partial^{\alpha_i} u_\nu \, \partial^{\alpha_k}(u - u_\nu) \prod_{i=k+1}^{l} \partial^{\alpha_i} u.$$

Consequently,

$$\|D_{p,\{s\}}(\nabla_{[s]}(f(u)-f(u_\nu)))\|_{L^p(\mathbf{R}^n)}$$

$$\leq c\sum_{l=1}^{[s]}\sum_{\substack{|\alpha_1|+\ldots+|\alpha_l|=[s]\\ |\alpha_i|\geq 1}}\Big(\Big\|D_{p,\{s\}}((f^{(l)}(u)-f^{(l)}(u_\nu))\prod_{i=1}^{l}\partial^{\alpha_i}u)\Big\|_{L^p(\mathbf{R}^n)}$$

$$+\sum_{k=1}^{l}\Big\|D_{p,\{s\}}(f^{(l)}(u_\nu)\prod_{i=1}^{k-1}\partial^{\alpha_i}u_\nu\,\partial^{\alpha_k}(u-u_\nu)\prod_{i=k+1}^{l}\partial^{\alpha_i}u)\Big\|_{L^p(\mathbf{R}^n)}\Big). \tag{31}$$

By (26), (27) and the boundedness of derivatives of f we can apply the Lebesgue dominated convergence theorem to conclude that

$$\Big\|(f^{(l)}(u)-f^{(l)}(u_\nu))D_{p,\{s\}}(\prod_{i=1}^{l}\partial^{\alpha_i}u)\Big\|_{L^p(\mathbf{R}^n)}\to 0. \tag{32}$$

Using Lemma with u replaced by $f^{(l)}(u)-f^{(l)}(u_\nu)$ and employing Hölder's inequality with exponents $s/|\alpha_i|$, $1\leq i\leq l$, and $s/\{s\}$, we obtain

$$\Big\|\prod_{i=1}^{l}\partial^{\alpha_i}u\cdot D_{p,\{s\}}(f^{(l)}(u)-f^{(l)}(u_\nu))\Big\|_{L^p(\mathbf{R}^n)}$$

$$\leq c\Big\|\prod_{i=1}^{l}\partial^{\alpha_i}u\cdot(\mathcal{M}|f^{(l)}(u)-f^{(l)}(u_\nu)|^p)^{\frac{1-\{s\}}{p}}(\mathcal{M}|\nabla(f^{(l)}(u)-f^{(l)}(u_\nu))|^p)^{\frac{\{s\}}{p}}\Big\|_{L^p(\mathbf{R}^n)}$$

$$\leq c||f^{(l)}||^{1-\{s\}}_{L^\infty(\mathbf{R})}\prod_{i=1}^{l}\|\partial^{\alpha_i}u\|_{L^{\frac{ps}{|\alpha_i|}}(\mathbf{R}^n)}\|\mathcal{M}|\nabla(f^{(l)}(u)-f^{(l)}(u_\nu))|^p\|^{\frac{\{s\}}{p}}_{L^s(\mathbf{R}^n)}. \tag{33}$$

The boundedness of $\mathcal{M}$ in $L^s(\mathbf{R}^n)$ implies that the left-hand side of (33) is dominated by

$$c||f^{(l)}||^{1-\{s\}}_{L^\infty(\mathbf{R})}\prod_{i=1}^{l}\|\partial^{\alpha_i}u\|_{L^{\frac{ps}{|\alpha_i|}}(\mathbf{R}^n)}\|\nabla(f^{(l)}(u)-f^{(l)}(u_\nu))\|^{\{s\}}_{L^{ps}(\mathbf{R}^n)}. \tag{34}$$

By (8), the product $\prod_{i=1}^{l}$ has the majorant (16). Obviously,

$$||\nabla(f^{(l)}(u)-f^{(l)}(u_\nu))||_{L^{ps}(\mathbf{R}^n)}\leq c(||f^{(l)}||_{L^\infty(\mathbf{R})}||\nabla(u-u_\nu)||_{L^{ps}(\mathbf{R}^n)}$$

$$+||(f^{(l+1)}(u)-f^{(l+1)}(u_\nu))\nabla u||_{L^{ps}(\mathbf{R}^n)}).$$

Hence and by Lebesgue's dominated convergence theorem

$$\Big\|\prod_{i=1}^{l}\partial^{\alpha_i}u\cdot D_{p,\{s\}}(f^{(l)}(u)-f^{(l)}(u_\nu))\Big\|_{L^p(\mathbf{R}^n)}\to 0.$$

This together with (32) implies that the first term in brackets in the right-hand side of (31) tends to zero.

We now show that

$$\Big\| f^{(l)}(u_\nu) D_{p,\{s\}} \Big(\prod_{i=1}^{k-1} \partial^{\alpha_i} u_\nu \cdot \partial^{\alpha_k}(u_\nu - u) \cdot \prod_{j=k+1}^{l} \partial^{\alpha_j} u \Big) \Big\|_{L^p(\mathbf{R}^n)} \to 0 \tag{35}$$

for any $k = 1, \dots l$. Here the products $\prod_{i=1}^{k-1} \partial^{\alpha_i} u_\nu$ and $\prod_{j=k+1}^{l} \partial^{\alpha_j} u$ are missing for $k = 1$ and $k = l$ respectively. Clearly, (35) holds for $l = 1$. Let $l > 1$. By (12), the left-hand side in (35) is majorized by

$$||f^{(l)}||_{L^\infty(\mathbf{R})} \Big(\Big\| \prod_{i=1}^{k-1} \partial^{\alpha_i} u_\nu \cdot \prod_{j=k+1}^{l} \partial^{\alpha_j} u \cdot D_{p,\{s\}}(\partial^{\alpha_k}(u_\nu - u)) \Big\|_{L^p(\mathbf{R}^n)}$$

$$+ \Big\| \partial^{\alpha_k}(u_\nu - u) \cdot D_{p,\{s\}}(\prod_{i=1}^{k-1} \partial^{\alpha_i} u_\nu \cdot \prod_{j=k+1}^{l} \partial^{\alpha_j} u) \Big\|_{L^p(\mathbf{R}^n)} \Big). \tag{36}$$

Applying Lemma we find that the first term in brackets in (36) does not exceed

$$c \Big\| \prod_{i=1}^{k-1} \partial^{\alpha_i} u_\nu \cdot \prod_{j=k+1}^{l} \partial^{\alpha_j} u \cdot (\mathcal{M} |\partial^{\alpha_k}(u_\nu - u)|^p)^{\frac{1-\{s\}}{p}} (\mathcal{M} |\nabla \partial^{\alpha_k}(u_\nu - u)|^p)^{\frac{\{s\}}{p}} \Big\|_{L^p(\mathbf{R}^n)}. \tag{37}$$

Hölder's inequality with exponents

$$\frac{s}{|\alpha_i|},\ 1 \le i \le k-1,\ \ \frac{s}{|\alpha_j|},\ k+1 \le j \le l,\ \ \frac{s}{|\alpha_k|(1 - \{s\})},\ \frac{s}{(1 + |\alpha_k|)\{s\}}$$

as well as the boundedness of $\mathcal{M}$ in $L^q(\mathbf{R}^n)$ with $q = s/|\alpha_k|$ and $q = s/(1 + |\alpha_k|)$ yield that (37) is dominated by

$$c \prod_{i=1}^{k-1} \|\partial^{\alpha_i} u_\nu\|_{L^{\frac{ps}{|\alpha_i|}}(\mathbf{R}^n)} \prod_{j=k+1}^{l} \|\partial^{\alpha_j} u\|_{L^{\frac{ps}{|\alpha_j|}}(\mathbf{R}^n)}$$

$$\times \|\partial^{\alpha_k}(u_\nu - u)\|^{1-\{s\}}_{L^{\frac{ps}{|\alpha_k|}}(\mathbf{R}^n)} \|\nabla \partial^{\alpha_k}(u_\nu - u)\|^{\{s\}}_{L^{\frac{ps}{1+|\alpha_k|}}(\mathbf{R}^n)}. \tag{38}$$

By (8) applied to each factor we see that (38) and therefore the first term in brackets in (36) tend to zero.

Making use of Lemma once more we obtain that the second term in brackets in (36) is majorized by

$$c \Big\| \partial^{\alpha_k}(u_\nu - u) \Big(\mathcal{M} | \prod_{i=1}^{k-1} \partial^{\alpha_i} u_\nu \cdot \prod_{j=k+1}^{l} \partial^{\alpha_j} u|^p \Big)^{\frac{1-\{s\}}{p}}$$

$$\times \Big(\mathcal{M} |\nabla (\prod_{i=1}^{k-1} \partial^{\alpha_i} u_\nu \cdot \prod_{j=k+1}^{l} \partial^{\alpha_j} u|^p \Big)^{\frac{\{s\}}{p}} \Big\|_{L^p(\mathbf{R}^n)}. \tag{39}$$

Applying Hölder's inequality with exponents

$$\frac{s}{|\alpha_k|}, \quad \frac{s}{([s] - |\alpha_k|)(1 - \{s\})}, \quad \frac{s}{(1 + [s] - |\alpha_k|)\{s\}}$$

and using the L^q-boundedness of $\mathcal{M}$ for $q = s/([s] - |\alpha_k|)$ and $q = s/(1 + [s] - |\alpha_k|)$ we find that (39) does not exceed

$$cN_1^{1-\{s\}} N_2^{\{s\}} \|\partial^{\alpha_k}(u_\nu - u)\|_{L^{\frac{ps}{|\alpha_k|}}(\mathbf{R}^n)}, \tag{40}$$

where

$$N_1 := \Big\| \prod_{i=1}^{k-1} \partial^{\alpha_i} u_\nu \cdot \prod_{j=k+1}^{l} \partial^{\alpha_j} u \Big\|_{L^{\frac{ps}{[s]-|\alpha_k|}}(\mathbf{R}^n)}$$

and

$$N_2 := \Big\| \nabla(\prod_{i=1}^{k-1} \partial^{\alpha_i} u_\nu \cdot \prod_{j=k+1}^{l} \partial^{\alpha_j} u) \Big\|_{L^{\frac{ps}{1+[s]-|\alpha_k|}}(\mathbf{R}^n)}.$$

By (8),

$$\|\partial^{\alpha_k}(u_\nu - u)\|_{L^{\frac{ps}{|\alpha_k|}}(\mathbf{R}^n)} \to 0.$$

It remains to show the boundedness of N_1 and N_2. Using Hölder's inequality with exponents $([s] - |\alpha_k|)/|\alpha_i|$, $1 \le i \le k-1$, and $([s] - |\alpha_k|)/|\alpha_j|$, $k+1 \le j \le l$ we find

$$N_1 \le c \prod_{i=1}^{k-1} \|\partial^{\alpha_i} u_\nu\|_{L^{\frac{ps}{|\alpha_i|}}(\mathbf{R}^n)} \prod_{j=k+1}^{l} \|\partial^{\alpha_j} u\|_{L^{\frac{ps}{|\alpha_j|}}(\mathbf{R}^n)}$$

which is bounded owing to (8). Again by Hölder's inequality, now with exponents

$$\frac{1 + [s] - |\alpha_k|}{|\alpha_i|}, \quad 1 \le i \le k-1, \ i \ne r, \quad \frac{1 + [s] - |\alpha_k|}{1 + |\alpha_r|}$$

and

$$\frac{1 + [s] - |\alpha_k|}{|\alpha_j|}, \quad k+1 \le j \le l, \ j \ne r, \quad \frac{1 + [s] - |\alpha_k|}{1 + |\alpha_r|},$$

we find that

$$N_2 \le c\Big(\sum_{r=1}^{k-1} \prod_{\substack{i=1 \\ i \ne r}}^{k-1} \|\partial^{\alpha_i} u_\nu\|_{L^{\frac{ps}{|\alpha_i|}}(\mathbf{R}^n)} \|\nabla \partial^{\alpha_r} u_\nu\|_{L^{\frac{ps}{1+|\alpha_r|}}(\mathbf{R}^n)} \prod_{j=k+1}^{l} \|\partial^{\alpha_j} u\|_{L^{\frac{ps}{|\alpha_j|}}(\mathbf{R}^n)}$$

$$+ \sum_{r=k+1}^{l} \prod_{i=1}^{k-1} \|\partial^{\alpha_i} u_\nu\|_{L^{\frac{ps}{|\alpha_i|}}(\mathbf{R}^n)} \prod_{\substack{j=k+1 \\ j \ne r}}^{l} \|\partial^{\alpha_j} u\|_{L^{\frac{ps}{|\alpha_j|}}(\mathbf{R}^n)} \|\nabla \partial^{\alpha_r} u\|_{L^{\frac{ps}{1+|\alpha_r|}}(\mathbf{R}^n)} \Big).$$

By (8) every norm on the right is bounded. The proof is complete.

3 Appendix. Proof of inequality (9)

Clearly, it is enough to prove the inequality

$$|\nabla_l u(0)| \le c\, ((\mathcal{M}u)(0))^{1-l/\lambda} ((D_{p,\lambda} u)(0))^{l/\lambda}, \tag{41}$$

where λ is noninteger and $0 < l < \lambda$. Let η be a function in the ball $B_1(0)$ with uniformly Lipschitz derivatives up to order $[\lambda]-2$ which vanishes on $\partial B_1(0)$ together with all these derivatives. Also let

$$\int_{B_1(0)} \eta(y)dy = 1.$$

Let $l < [\lambda]$. We use Sobolev's integral representation

$$v(0) = \sum_{|\beta|<[\lambda]-l} \delta^{-n} \int_{B_\delta(0)} \frac{(-y)^\beta}{\beta!} \partial^\beta v(y) \eta(\frac{y}{\delta}) dy$$

$$+(-1)^{[\lambda]-l}([\lambda]-l) \sum_{|\alpha|=[\lambda]-l} \int_{B_\delta(0)} \frac{y^\alpha}{\alpha!} \partial^\alpha v(y) \int_{|y|/\delta}^\infty \eta\Big(\rho \frac{y}{|y|}\Big) \rho^{n-1} d\rho \frac{dy}{|y|^n}$$

(see [MP], Sect. 1.5.1). By setting here $v = \partial^\gamma u$ with an arbitrary multi-index γ of order l and integrating by parts in the first integral, we arrive at the identity

$$\partial^\gamma u(0) = (-1)^l \delta^{-n} \int_{B_\delta(0)} u(y) \sum_{|\beta|<[\lambda]-l} \frac{1}{\beta!} \partial^{\beta+\gamma} \Big(y^\beta \eta(\frac{y}{\delta}) \Big) dy$$

$$+ \sum_{|\alpha|=[\lambda]-l} (-1)^{[\lambda]-l}([\lambda]-l) \int_{B_\delta(0)} \frac{y^\alpha}{\alpha!} \partial^{\alpha+\gamma} u(y) \int_{|y|/\delta}^\infty \eta\Big(\rho \frac{y}{|y|}\Big) \rho^{n-1} d\rho \frac{dy}{|y|^n} \tag{42}$$

Hence, for $l < [\lambda]$

$$|\nabla_l u(0)| \le c\Big(\delta^{-l} \mathcal{M} u(0) + \delta^{[\lambda]-l} |\nabla_{[\lambda]} u(0)| + \int_{B_\delta(0)} \frac{|\nabla_{[\lambda]} u(y) - \nabla_{[\lambda]} u(0)|}{|y|^{n-[\lambda]+l}} dy\Big). \tag{43}$$

By Hölder's inequality,

$$\int_{B_\delta(0)} \frac{|\nabla_{[\lambda]} u(y) - \nabla_{[\lambda]} u(0)|}{|y|^{n-[\lambda]+l}} dy \le c\, \delta^{\lambda-l} D_{p,\lambda} u(0). \tag{44}$$

Let γ be an arbitrary multi-index of order $[\lambda]$. The identity

$$\partial^\gamma u(0) = \delta^{-n} \int_{B_\delta(0)} \eta\Big(\frac{y}{\delta}\Big) \partial^\gamma u(y) dy + \delta^{-n} \int_{B_\delta(0)} \eta\Big(\frac{y}{\delta}\Big) (\partial^\gamma u(0) - \partial^\gamma u(y)) dy$$

implies

$$|\nabla_{[\lambda]} u(0)| \le \delta^{-n-[\lambda]} \Big| \int_{B_\delta(0)} u(y) (\nabla_{[\lambda]} \eta) \Big(\frac{y}{\delta}\Big) dy \Big|$$

$$+\delta^{\{\lambda\}} \Big(\int_{B_\delta(0)} |\eta(y)|^q |y|^{(\frac{n}{p}+\{\lambda\})q} dy \Big)^{1/q} D_{p,\lambda} u(0), \tag{45}$$

where $1/p + 1/q = 1$. Combining (43)-(45) we arrive at

$$|\nabla_l u(0)| \le c\,(\delta^{-l} \mathcal{M} u(0) + \delta^{\lambda-l} D_{p,\lambda} u(0)), \quad \text{for} \quad l \le [\lambda].$$

Minimizing the right-hand side in δ we complete the proof.

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