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Weighted L_p estimates of solutions to boundary value problems for second order elliptic systems in polyhedral domains

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In this paper we are concerned with boundary value problems for general second order elliptic equations and systems in a polyhedral domain. We consider solutions in weighted L_p Sobolev spaces. A special section is dedicated to weak solutions. We prove solvability theorems and regularity assertions for the solutions.

1 Introduction

This paper is a continuation of [16], where we obtained point estimates for Green's matrices of boundary value problems to second order systems in a polyhedral cone. These estimates are used in the present paper for the proof of theorems on the solvability in weighted L_p Sobolev spaces and of regularity assertions for the solutions.

Elliptic boundary value problems in domains with edges have been studied in many works. For general elliptic problems we refer here to the papers by Maz'ya and Plamenevskiĭ [11,12], Maz'ya and Rossmann [15], the books of Dauge [3], Nazarov and Plamenevskiĭ [19], and for pseudodifferential boundary value problems to the papers by Schrohe and Schulze [22,23]. Moreover, there are many publications on boundary value problems for special differential equations or systems. But most of them do not include the Neumann problem. Concerning this problem we mention the papers of Zajaczkowski and Solonnikov [25] (Neumann problem to the Laplace equation), Nazarov [17,18], Rossmann [20] and the book of Nazarov and Plamenevskiĭ [19] (more general boundary value problems), where solvability theorems for the Neumann problem in weighted L_2 Sobolev spaces as well as regularity assertions for the solutions are given. Grisvard [6] considered solutions of the Neumann problems to the Laplace equation and the Lamé system in L_2 Sobolev spaces without weight. Dauge [2,4] proved the solvability of the Neumann problem to the Laplace equation in L_p Sobolev spaces. In contrast to the present paper, the above mentioned works do not make use of Green's function. Estimates of Green's function were first employed by Maz'ya and Plamenevskiĭ [13] in order to prove the solvability of elliptic boundary value problems in weighted Sobolev and Hölder spaces for domains with edges. The results in [13] are applicable, e.g., to the Dirichlet but not to the Neumann problem. Using estimates of Green's function, Solonnikov [24], Grachev and Maz'ya [5] proved the solvability of the Neumann problem to the Laplace equation in weighted Sobolev and Hölder spaces.

The present paper generalizes results of the preprints [5] and [24] to boundary value problems for general elliptic second order systems. Moreover, we deal with weak solutions in weighted L_p Sobolev spaces. We describe here the main results of the paper. Let

$$\mathcal{K} = \{x \in \mathbb{R}^3 : \omega = x/|x| \in \Omega\} \tag{1.1}$$

be a polyhedral cone with faces $\Gamma_j = \{x : x/|x| \in \gamma_j\}$ and edges M_j, $j = 1, \dots, n$. Here Ω is a curvilinear polygon on the unit sphere bounded by the arcs $\gamma_1, \dots, \gamma_n$. Suppose that $\mathcal{K}$ coincides with a dihedral angle $\mathcal{D}_j$ in a neighborhood of an arbitrary edge point $x \in M_j$. We consider the boundary value problem

$$L(\partial_x)\, u = -\sum_{i,j=1}^{3} A_{i,j}\, \partial_{x_i} \partial_{x_j} u = f \;\text{ in } \mathcal{K}, \tag{1.2}$$

$$u = g_j \text{ on } \Gamma_j \text{ for } j \in J_0, \tag{1.3}$$

$$B(\partial_x)\, u = \sum_{i,j=1}^{3} A_{i,j}\, n_j\, \partial_{x_i} u = g_k \;\text{ on } \Gamma_k \text{ for } k \in J_1, \tag{1.4}$$

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where $A_{i,j}$ are constant $\ell \times \ell$ matrices such that $A_{i,j} = A^*_{j,i}$, $J_0 \cup J_1 = J = \{1, \dots, n\}$, $J_0 \cap J_1 = \emptyset$, u, f, g are vector-valued functions, and (n_1, n_2, n_3) denotes the exterior normal to Γ_k. We denote by $\mathcal{H}$ the closure of the set $\{u \in C_0^\infty(\overline{\mathcal{K}})^\ell :\ u = 0$ on Γ_j for $j \in J_0\}$ with respect to the norm

$$\|u\|_{\mathcal{H}} = \Big(\int_{\mathcal{K}} \sum_{j=1}^{3} |\partial_{x_j} u|^2\, dx \Big)^{1/2}.$$

($C_0^\infty(G)$ is the set of all infinitely differentiable functions u such that supp u is compact and contained in G.) Throughout the paper, it is assumed that the sesquilinear form

$$b_{\mathcal{K}}(u, v) = \int_{\mathcal{K}} \sum_{i,j=1}^{3} A_{i,j} \partial_{x_i} u \cdot \partial_{x_j} \overline{v}\, dx \tag{1.5}$$

is $\mathcal{H}$-coercive, i.e.,

$$b_{\mathcal{K}}(u, u) \ge c\, \|u\|^2_{\mathcal{H}} \text{ for all } u \in \mathcal{H}. \tag{1.6}$$

In Sect. 2 we introduce weighted Sobolev spaces $\mathcal{W}^{l,p}_{\beta,\vec{\delta}}(\mathcal{K}; \tilde{J})$, where $\tilde{J}$ is a subset of J, l is an integer, $l \ge 0$, p and β are real numbers, $p > 1$, and $\vec{\delta} = (\delta_1, \dots, \delta_n) \in \mathbb{R}^n$. For $\tilde{J} = \emptyset$ (the case of the Neumann problem) this space is denoted by $W^{l,p}_{\beta,\vec{\delta}}(\mathcal{K})$ and is equipped with the norm

$$\|u\|_{W^{l,p}_{\beta,\vec{\delta}}(\mathcal{K})} = \Big(\int_{\mathcal{K}} \sum_{|\alpha| \le l} |x|^{p(\beta - l + |\alpha|)} \prod_{j=1}^{n} \Big(\frac{r_j(x)}{|x|} \Big)^{p\delta_j}\ \big|\partial_x^\alpha u(x)\big|^p\, dx \Big)^{1/p}, \quad \text{where } r_j(x) = \text{dist}(x, M_j).$$

Sect. 3 contains some auxiliary results for the problem in a dihedron. We give here a regularity assertion for the solution and study inhomogeneous boundary conditions. In Sect. 4 we prove the existence and uniqueness of solutions of problem (1.2)–(1.4) in the space $\mathcal{W}^{2,p}_{\beta,\vec{\delta}}(\mathcal{K}; \tilde{J})^\ell$ if the line $\text{Re}\, \lambda = 2 - \beta - 3/p$ is free of eigenvalues of an operator pencil $\mathfrak{A}(\lambda)$ and the components δ_j of $\vec{\delta}$ satisfy certain inequalities. Here $\mathfrak{A}(\lambda)$ is the operator of the parameter-dependent boundary value problem

$$\mathcal{L}(\lambda) u = f \ \text{ in } \Omega, \quad u = g_j \ \text{ on } \gamma_j \text{ for } j \in J_0, \quad \mathcal{B}(\lambda) u = g_k \ \text{ on } \gamma_k \text{ for } k \in J_1,$$

with differential operators $\mathcal{L}(\lambda)$ and $\mathcal{B}(\lambda)$ defined by

$$\mathcal{L}(\lambda) u = \rho^{2-\lambda}\, L(\partial_x) \big(\rho^\lambda u(\omega)\big), \quad \mathcal{B}(\lambda) u = \rho^{1-\lambda}\, B(\partial_x) \big(\rho^\lambda u(\omega)\big), \tag{1.7}$$

$\rho = |x|,\ \omega = x/|x|$. The bounds for δ_j depend on the eigenvalues of certain operator pencils $A_j(\lambda)$ which are generated by boundary value problems in a plane angle. For example, the Neumann problem for the Laplace equation is uniquely solvable in $W^{2,p}_{\beta,\vec{\delta}}(\mathcal{K})$ for arbitrary $f \in W^{0,p}_{\beta,\vec{\delta}}(\mathcal{K})$, $g_k \in W^{1-1/p,p}_{\beta,\vec{\delta}}(\Gamma_k)$ if the line $\text{Re}\, \lambda = 2 - \beta - 3/p$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$ and the components of $\vec{\delta}$ satisfy the inequality

$$\max\big(2 - \frac{\pi}{\theta_j}, 0\big) < \delta_j + 2/p < 2. \tag{1.8}$$

Here θ_j denotes the angle at the edge M_j, π/θ_j is the smallest positive eigenvalue of the pencil $A_j(\lambda)$. An analogous assertion holds for the Neumann problem to the Lamé system. Here the conditions for the components of $\vec{\delta}$ are

$$\max\big(2 - \frac{\pi}{\theta_j}, 0\big) < \delta_j + \frac{2}{p} < 2 \quad \text{if } \theta_j < \pi, \quad 2 - \frac{\xi_+(\theta_j)}{\theta_j} < \delta_j + \frac{2}{p} < 2 \quad \text{if } \theta_j > \pi, \tag{1.9}$$

where $\xi_+(\theta)$ is the smallest positive solution of the equation $\xi^{-1} \sin\xi + \theta^{-1} \sin\theta = 0$. Condition (1.8) means, in particular, that $2 - \delta_j - 2/p$ is less than the smallest positive eigenvalue of the pencil $A_j(\lambda)$. A feature of the Neumann problem for the Lamé system is that $\lambda = 1$ is always an eigenvalue of this pencil $A_j(\lambda)$. For $\theta_j < \pi$ this is the smallest positive eigenvalue. Condition (1.9) allows that the number $2 - \delta_j - 2/p$ exceeds the eigenvalue $\lambda = 1$ if $\theta_j < \pi$. However, then the boundary data must satisfy a compatibility condition on the edge M_j (in the case $p = 2$, $\delta_j = 0$ see also [6, Th. 4.4.1]).

Sect. 5 deals with weak solutions of problem (1.2)–(1.4). In particular, we prove that for an arbitrary linear and continuous functional F on $W^{1,q}_{-\beta,-\vec{\delta}}(\mathcal{K})^\ell$ there exists a unique $u \in W^{1,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$, $p^{-1} + q^{-1} = 1$, satisfying

$$b_{\mathcal{K}}(u, v) = \overline{F(v)} \quad \text{for all } v \in W^{1,q}_{-\beta,-\vec{\delta}}(\mathcal{K})^\ell$$

provided the line $\operatorname{Re}\lambda = 1-\beta-3/p$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$ and the components of $\vec{\delta}$ satisfy the inequalities $\max(1-\mu_j,0) < \delta_j + 2/p < 1$ for $j=1,\dots,n$, where μ_j is the real part of the first eigenvalue of the pencil $A_j(\lambda)$ on the right of the imaginary axis. Furthermore, we study the smoothness of weak solutions. For example, we obtain the following result for the weak solution $u \in \mathcal{H} = W^{1,2}_{0,\vec{0}}(\mathcal{K})^3$ of the Neumann problem to the Lamé system: If the functional $F \in \mathcal{H}^*$ has the representation

$$F(v) = \int_{\mathcal{K}} \bar{f}\cdot v\,dx \quad \text{for all } v \in C_0^\infty(\overline{\mathcal{K}}\backslash\{0\})^3,$$

where $f \in W^{l-2,p}_{\beta,\vec{\delta}}(\mathcal{K})^3$, the strip $-1/2 \le \operatorname{Re}\lambda \le l-\beta-3/p$ contains at most the eigenvalue $\lambda=0$ of the pencil $\mathfrak{A}(\lambda)$, and the components of $\vec{\delta}$ satisfy the inequalities

$$\max\Big(l-\frac{\pi}{\theta_j},0\Big) < \delta_j + \frac{2}{p} < l \quad \text{if } \theta_j < \pi, \qquad l - \frac{\xi_+(\theta_j)}{\theta_j} < \delta_j + \frac{2}{p} < l \quad \text{if } \theta_j > \pi,$$

then there exists a constant vector c such that $u - c \in W^{l,p}_{\beta,\vec{\delta}}(\mathcal{K})^3$. In particular, we can conclude from this result that the second derivatives of u are square summable in a neighborhood of an arbitrary edge point if $f \in L_2(\mathcal{K})^3$ and the angle at this edge is less than π. The same is true in the case of inhomogeneous boundary conditions if the boundary data $g_j \in W^{1/2,2}(\Gamma_j)$ satisfy certain compatibility conditions on the edges of $\mathcal{K}$. For the square summability of the second derivatives of the solution in a neighborhood of the vertex of the cone it is necessary that additionally the strip $0 < \operatorname{Re}\lambda \le 1/2$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$. Note that, by a result of Dauge [4], the last condition is satisfied for the Neumann problem to the Laplace equation if the cone is convex.

In the last section we consider the problem with variable coefficients in a bounded domain of polyhedral type. By means of the results of Sect. 5, we prove a regularity assertion for weak solutions and show that the operator of the boundary value problem is Fredholm.

2 Weighted Sobolev spaces

2.1 Weighted Sobolev spaces in a dihedron

Let $\mathcal{D}$ be the dihedron

$$\mathcal{D} = \{x = (x', x_3) :\ x' \in K,\ x_3 \in \mathbb{R}\}, \tag{2.1}$$

where K is an infinite angle which has the form $\{x' = (x_1,x_2) \in \mathbb{R}^2 :\ 0 < r < \infty,\ 0 < \varphi < \theta\}$ in polar coordinates r,φ. The boundary of $\mathcal{D}$ consists of two half-planes Γ^+ and Γ^- and the edge M. We denote by $V^{l,p}_\delta(\mathcal{D})$ and $W^{l,p}_\delta(\mathcal{D})$ the weighted Sobolev spaces with the norms

$$\|u\|_{V^{l,p}_\delta(\mathcal{D})} = \bigg(\int_{\mathcal{D}} \sum_{|\alpha|\le l} |x'|^{p(\delta-l+|\alpha|)}\,\big|\partial_x^\alpha u\big|^p\,dx\bigg)^{1/p}, \qquad \|u\|_{W^{l,p}_\delta(\mathcal{D})} = \bigg(\int_{\mathcal{D}} \sum_{|\alpha|\le l} |x'|^{p\delta}\,\big|\partial_x^\alpha u\big|^p\,dx\bigg)^{1/p}.$$

Analogously, the spaces $V^{l,p}_\delta(K)$ and $W^{l,p}_\delta(K)$ are defined (here in the above norms $\mathcal{D}$ has to be replaced by K and dx by dx'). By Hardy's inequality, every function $u \in C_0^\infty(\overline{\mathcal{D}})$ satisfies the inequality

$$\int_{\mathcal{D}} r^{p(\delta-1)}|u|^2\,dx \le c\int_{\mathcal{D}} r^{p\delta}\,|\nabla u|^p\,dx$$

for $\delta > 1-2/p$ with a constant c depending only on p and δ. Consequently, the space $W^{l,p}_\delta(\mathcal{D})$ is continuously imbedded into $W^{l-1,p}_{\delta-1}(\mathcal{D})$ if $\delta > 1-2/p$. If $\delta > l-2/p$, then $W^{l,p}_\delta(\mathcal{D}) \subset V^{l,p}_\delta(\mathcal{D})$.

The following result will be used in Sect. 4.6.

Lemma 2.1. *If $\partial^j_{x_3}u \in V^{2,p}_\delta(\mathcal{D})$, $1<p<2$, for $j=0,1,2$, then $u \in V^{0,2}_{\delta-3+2/p}(\mathcal{D})$.*

P r o o f. Let $\mathcal{D}_j = \{(x',x_3) \in \mathcal{D} :\ 2^{-j} < |x'| < 2^{-j+1}\}$. By the continuity of the imbedding $W^{2,p}(\mathcal{D}_0) \subset L_2(\mathcal{D}_0)$, there is the estimate

$$\int_{\mathcal{D}_0} |u(x)|^2\,dx \le c\bigg(\sum_{|\alpha|+k\le 2}\int_{\mathcal{D}_0} \big|\partial^\alpha_{x'}\partial^k_{x_3}u(x)\big|^p\,dx\bigg)^{2/p}.$$

Multiplying this inequality by $2^{-2j(\delta-2+2/p)}$ and substituting $x' = 2^j y',\ x_3 = y_3$, we obtain

$$\int_{\mathcal{D}_j} |y'|^{2(\delta-3+2/p)}\, |v(y)|^2\, dy \le c \Big(\sum_{|\alpha|+k\le 2} \int_{\mathcal{D}_j} |y'|^{p(\delta-2+|\alpha|)} \big|\partial_{y'}^\alpha \partial_{y_3}^k v(y)\big|^p\, dy\Big)^{2/p},$$

where $v(y) = u(2^j y', y_3)$. Here the constant c is independent of u and j. Summing up over all integer j, we arrive at the inequality

$$\|v\|^2_{V^{0,2}_{\delta-3+2/p}(\mathcal{D})} \le c \sum_{k=0}^{2} \|\partial_{y_3}^k v\|^2_{V^{2,p}_\delta(\mathcal{D})}\,.$$

This proves the lemma. □

Corollary 2.1. *If $\partial_{x_3}^j u \in W^{3,p}_\delta(\mathcal{D})$ for $j = 0,1,2$, where $1 < p < 2$ and $\delta > 2 - 2/p$, then $u \in W^{1,2}_{\delta-3+2/p}(\mathcal{D})$.*

Proof. By Lemma 2.1, the inclusion $\partial_{x_3}^j u \in W^{2,p}_\delta(\mathcal{D}) = V^{2,p}_\delta(\mathcal{D})$ for $j \le 2$ implies $u \in V^{0,2}_{\delta-3+2/p}(\mathcal{D})$. Furthermore, by our assumptions, $\partial_{x_3}^j \nabla u \in V^{2,p}_\delta(\mathcal{D})^3$ for $j = 0,1,2$ and, therefore, $\nabla u \in V^{0,2}_{\delta-3+2/p}(\mathcal{D})^3$. The result follows. □

Let $V^{l-1/p,p}_\delta(\Gamma^\pm)$ and $W^{l-1/p,p}_\delta(\Gamma^\pm)$ be the trace spaces corresponding to $V^{l,p}_\delta(\mathcal{D})$ and $W^{l,p}_\delta(\mathcal{D})$, respectively. The trace spaces for $V^{l,p}_\delta(K)$ and $W^{l,p}_\delta(K)$ on the sides $\gamma^\pm$ of K are denoted by $V^{l-1/p,p}_\delta(\gamma^\pm)$ and $W^{l-1/p,p}_\delta(\gamma^\pm)$, respectively. From the representation of the norm in $V^{l-1/p,p}_\delta(\Gamma^\pm)$ given in [12] we obtain, in particular, the following result.

Lemma 2.2. *The function $g \in W^{1-1/p,p}_\delta(\Gamma^+)$ belongs to $V^{1-1/p,p}_\delta(\Gamma^+)$ if and only if*

$$\int_{\Gamma^+} r^{p(\delta-1)+1} \big|g(r,x_3)\big|^p\, dr\, dx_3 < \infty. \tag{2.2}$$

Note that, due to the continuity of the imbedding $W^{1,p}_\delta(\mathcal{D}) \subset V^{1,p}_\delta(\mathcal{D})$, condition (2.2) is satisfied for every $g \in W^{1-1/p,p}_\delta(\Gamma^+)$ if $\delta > 1 - 2/p$. In the case $\delta < 1 - 2/p$ condition (2.2) is equivalent to $g|_M = 0$, where M is the edge of $\mathcal{D}$.

2.2 Weighted Sobolev spaces in a cone

Let $\mathcal{K}$ be the same cone (1.1) as in the introduction. We denote by $\mathcal{S}$ the set $M_1 \cup \ldots \cup M_n \cup \{0\}$ of all singular boundary points. Furthermore, for an arbitrary point $x \in \mathcal{K}$ we denote by $\rho(x) = |x|$ the distance to the vertex of the cone, by $r_j(x)$ the distance to the edge M_j, and by $r(x)$ the regularized distance to $\mathcal{S}$, i.e. an infinitely differentiable function in $\mathcal{K}$ which satisfies $c_1 \text{dist}\,(x,M) < r(x) < c_2 \text{dist}\,(x,M)$ and $|\partial_x^\alpha r(x)| \le c_\alpha \text{dist}\,(x,M)^{1-|\alpha|}$.

Let $\tilde J$ be a subset of $J = \{1,2,\ldots,n\}$, l a nonnegative integer, $1 < p < \infty$, $\beta \in \mathbb{R}$, $\vec\delta = (\delta_1,\ldots,\delta_n) \in \mathbb{R}^n$, $\delta_j > -2/p$ for $j \in J\backslash \tilde J$. Then $\mathcal{W}^{l,p}_{\beta,\vec\delta}(\mathcal{K};\tilde J)$ is defined as the weighted Sobolev space with the norm

$$\|u\|_{\mathcal{W}^{l,p}_{\beta,\vec\delta}(\mathcal{K};\tilde J)} = \Big(\int_{\mathcal{K}} \sum_{|\alpha|\le l} \rho^{p(\beta-l+|\alpha|)}\, |\partial_x^\alpha u|^p \prod_{j\in\tilde J} \big(\frac{r_j}{\rho}\big)^{p(\delta_j-l+|\alpha|)} \prod_{j\in J\backslash\tilde J} \big(\frac{r_j}{\rho}\big)^{p\delta_j}\, dx\Big)^{1/p}.$$

Furthermore, we define $V^{l,p}_{\beta,\vec\delta}(\mathcal{K}) = \mathcal{W}^{l,p}_{\beta,\vec\delta}(\mathcal{K};J)$ and $W^{l,p}_{\beta,\vec\delta}(\mathcal{K}) = \mathcal{W}^{l,p}_{\beta,\vec\delta}(\mathcal{K};\emptyset)$.

Passing to spherical coordinates ρ,ω, one obtains the following equivalent norm in $\mathcal{W}^{l,p}_{\beta,\vec\delta}(\mathcal{K};\tilde J)$:

$$\|u\| = \Big(\int_0^\infty \rho^{p(\beta-l)+2} \sum_{k=0}^{l} \|(\rho\partial_\rho)^k u(\rho,\cdot)\|^p_{\mathcal{W}^{l-k,p}_{\vec\delta}(\Omega;\tilde J)}\, d\rho\Big)^{1/p},$$

where the norm in $\mathcal{W}^{l,p}_{\vec\delta}(\Omega;\tilde J)$ is given by

$$\|v\|_{\mathcal{W}^{l,p}_{\vec\delta}(\Omega;\tilde J)} = \Big(\int\limits_{\substack{\mathcal{K}\\ 1<|x|<2}} \sum_{|\alpha|\le l} \big|\partial_x^\alpha v(x)\big|^p \prod_{j\in\tilde J} r_j^{p(\delta_j-l+|\alpha|)} \prod_{j\in J\backslash\tilde J} r_j^{p\delta_j}\, dx\Big)^{1/p}$$

(here the function v on Ω is extended by $v(x) = v\big(x/|x|\big)$ to the cone $\mathcal{K}$). Obviously,

$$V^{l,p}_{\beta,\vec\delta}(\mathcal{K}) \subset \mathcal{W}^{l,p}_{\beta,\vec\delta}(\mathcal{K};\tilde J) \subset W^{l,p}_{\beta,\vec\delta}(\mathcal{K}).$$

By Hardy's inequality, the space $\mathcal{W}^{l+1,p}_{\beta+1,\vec\delta'}(\Omega;\tilde J)$ is continuously imbedded into $\mathcal{W}^{l,p}_{\beta,\vec\delta}(\Omega;\tilde J)$ if $\vec\delta = (\delta_1,\dots,\delta_n)$, $\vec\delta' = (\delta'_1,\dots,\delta'_n)$ are such that $\delta'_j - \delta_j \le 1$ for $j = 1,\dots,n$ and $\delta_j > -2/p$, $\delta'_j > -2/p$ for $j \notin \tilde J$. This implies that, under the above assumptions on $\vec\delta$ and $\vec\delta'$, there is the imbedding

$$\mathcal{W}^{l+1,p}_{\beta+1,\vec\delta'}(\mathcal{K};\tilde J) \subset \mathcal{W}^{l,p}_{\beta,\vec\delta}(\mathcal{K};\tilde J).$$

In particular, we have $V^{l,p}_{\beta,\vec\delta}(\mathcal{K}) = \mathcal{W}^{l,p}_{\beta,\vec\delta}(\mathcal{K};\tilde J)$ if $\delta_j > l - 2/p$ for all $j \notin \tilde J$.

Let ζ_k be smooth functions depending only on $\rho = |x|$ such that

$$\operatorname{supp}\zeta_k \subset (2^{k-1}, 2^{k+1}), \qquad \sum_{k=-\infty}^{+\infty} \zeta_k = 1, \quad |(\rho\partial_\rho)^j \zeta_k(\rho)| \le c_j \tag{2.3}$$

with constants c_j independent of k and ρ. It can be easily shown (cf. [8, Le. 6.1.1]) that the norm in $\mathcal{W}^{l,p}_{\beta,\vec\delta}(\mathcal{K};\tilde J)$ is equivalent to

$$\|u\| = \Big(\sum_{k=-\infty}^{+\infty} \|\zeta_k u\|^p_{\mathcal{W}^{l,p}_{\beta,\vec\delta}(\mathcal{K};\tilde J)} \Big)^{1/p}. \tag{2.4}$$

We denote the trace spaces for $V^{l,p}_{\beta,\vec\delta}(\mathcal{K})$, $W^{l,p}_{\beta,\vec\delta}(\mathcal{K})$ and $\mathcal{W}^{l,p}_{\beta,\vec\delta}(\mathcal{K};\tilde J)$, $l \ge 1$, on Γ_j by $V^{l-1/p,p}_{\beta,\vec\delta}(\Gamma_j)$, $W^{l-1/p,p}_{\beta,\vec\delta}(\Gamma_j)$ and $\mathcal{W}^{l-1/p,p}_{\beta,\vec\delta}(\Gamma_j;\tilde J)$, respectively.

3 The boundary value problem in a dihedron

In this section we consider the boundary value problem

$$L(\partial_x)u = f \ \text{ in } \mathcal{D}, \quad d^\pm u + (1-d^\pm)\, B(\partial_x)u = g^\pm \ \text{ on } \Gamma^\pm, \tag{3.1}$$

where $\mathcal{D} = K \times \mathbb{R}$ is the dihedron (2.1), $\Gamma^\pm = \gamma^\pm \times \mathbb{R}$ are the sides of $\mathcal{D}$, L and B are the same differential operators as in (1.2), (1.4), and $d^\pm \in \{0,\, 1\}$.

3.1 Regularity assertions for solutions of the boundary value problem in a dihedron

Lemma 3.1. *Let u be a solution of problem* (3.1) *such that $\partial_x^\alpha u \in L_p(\mathcal{C})$ for every compact subset $\mathcal{C} \subset \overline{\mathcal{D}}\backslash M$ and $|\alpha| \le l$. Furthermore, let ϕ, ψ be infinitely differentiable functions with compact supports on $\overline{\mathcal{D}}$ such that $\psi = 1$ in a neighborhood of* supp ϕ.

1) *If $\psi u \in V^{k,p}_{\delta-l+k}(\mathcal{D})^\ell$, $\psi f \in V^{l-2,p}_\delta(\mathcal{D})^\ell$, and $\psi g^\pm \in V^{l+d^\pm-1-1/p,p}_\delta(\Gamma^\pm)^\ell$, $k \ge 0$, $l \ge 2$, then $\phi u \in V^{l,p}_\delta(\mathcal{D})^\ell$ and*

$$\|\phi u\|_{V^{l,p}_\delta(\mathcal{D})^\ell} \le c \left(\|\psi u\|_{V^k_{\delta-l+k}(\mathcal{D})^\ell} + \|\psi f\|_{V^{l-2,p}_\delta(\mathcal{D})^\ell} + \|\psi g^\pm\|_{V^{l+d^\pm-1-1/p,p}_\delta(\Gamma^\pm)^\ell} \right). \tag{3.2}$$

2) *If $\psi u \in W^{k,p}_{\delta-l+k}(\mathcal{D})^\ell$, $\psi f \in W^{l-2,p}_\delta(\mathcal{D})^\ell$, and $\psi g^\pm \in W^{l+d^\pm-1-1/p,p}_\delta(\Gamma^\pm)^\ell$, $l \ge k+1 \ge 2$, $\delta > l-k-2/p$, then $\phi u \in W^{l,p}_\delta(\mathcal{D})^\ell$ and an estimate analogous to* (3.2) *holds for the norm of ϕu.*

P r o o f. For the first part we refer to [12, Th. 10.2]. We prove the second part for $l = k+1$. By [15, Le. 1.3] (for integer $\delta + 2/p$ see [21, Cor. 2, Rem. 2]), the vector function $\phi u \in W^{l-1,p}_{\delta-1}(\mathcal{D})^\ell$ admits the representation $\phi u = v + w$, where v, w have compact supports, $v \in V^{l-1,p}_{\delta-1}(\mathcal{D})^\ell$, $w \in W^{l,p}_\delta(\mathcal{D})^\ell$. Thus, $Lv = \phi f + [L,\phi]u - Lw \in W^{l-2,p}_\delta(\mathcal{D})^\ell \cap V^{l-3,p}_{\delta-1}(\mathcal{D})^\ell \subset V^{l-2,p}_\delta(\mathcal{D})^\ell$ (here $[L,\phi] = L\phi - \phi L$ denotes the commutator of L and ϕ) and, analogously, $d^\pm v + (1-d^\pm)Bv \in V^{l+d^\pm-1-1/p,p}_\delta(\Gamma^\pm)^\ell$. Consequently, by [12, Th. 4.1], we obtain $v \in V^{l,p}_\delta(\mathcal{D})^\ell$ and, therefore, $\phi u \in W^{l,p}_\delta(\mathcal{D})^\ell$. This proves the lemma for $l = k+1$. Repeating this argument and using the imbeddings $W^{l-2,p}_\delta(\mathcal{D}) \subset W^{l-3,p}_{\delta-1}(\mathcal{D}) \subset \cdots \subset W^{k-1}_{\delta-l+k+1}(\mathcal{D})$ and $W^{l+d^\pm-1-1/p,p}_\delta(\Gamma^\pm) \subset W^{l+d^\pm-2-1/p,p}_{\delta-1}(\Gamma^\pm) \subset \cdots \subset W^{k+d^\pm-1/p,p}_{\delta-l+k+1}(\Gamma^\pm)^\ell$, we obtain the assertion for $l = k+2, k+3, \dots$. □

Problem (3.1) is connected with the following operator pencil $A(\lambda)$. Let

$$L(\partial_{x'},0) = -\sum_{i,j=1}^{2} A_{i,j}\, \partial_{x_i}\partial_{x_j}\,, \qquad B^\pm(\partial_{x'},0) = \sum_{i,j=1}^{2} A_{i,j}\, n_j^\pm \partial_{x_i}.$$

Here $n_j^\pm$ are the components of the exterior normal to $\Gamma^\pm$. We define the differential operators $\mathcal{L}(\lambda)$ and $\mathcal{B}^\pm(\lambda)$ depending on the complex parameter λ by

$$\mathcal{L}(\lambda)\, u(\varphi) = r^{2-\lambda}\, L(\partial_{x'},0)\,\big(r^\lambda u(\varphi)\big), \quad \mathcal{B}^\pm(\lambda)\, u(\varphi) = d^\pm u(\varphi) + (1-d^\pm)\, r^{1-\lambda}\, B^\pm(\partial_{x'},0)\,\big(r^\lambda u(\varphi)\big),$$

where again r, φ are the polar coordinates in the (x_1,x_2)-plane. Then $A(\lambda)$ denotes the operator

$$W^2(0,\theta)^\ell \ni u \to \Big(\mathcal{L}(\lambda)u\,,\, \mathcal{B}^+(\lambda)\, u(\varphi)\big|_{\varphi=0}\,, \mathcal{B}^-(\lambda)\, u(\varphi)\big|_{\varphi=\theta}\Big) \in L_2(0,\theta)^\ell \times \mathbb{C}^\ell \times \mathbb{C}^\ell.$$

Lemma 3.2. *Let ϕ, ψ be infinitely differentiable functions on $\overline{\mathcal{D}}$ with compact supports, $\psi = 1$ in a neighborhood of* supp ϕ*, and let u be a solution of problem* (3.1) *such that $\psi u \in V_\delta^{l,p}(\mathcal{D})^\ell$, $\psi\partial_{x_3} u \in V_\delta^{l,p}(\mathcal{D})^\ell$, $\psi f \in V_\delta^{l-1,p}(\mathcal{D})^\ell$, $\psi g^\pm \in V_\delta^{l+d^\pm-1/p,p}(\Gamma^\pm)^\ell$. If there are no eigenvalues of the pencil $A(\lambda)$ in the strip $l-\delta-2/p \le \operatorname{Re}\lambda \le l+1-\delta-2/p$, then $\phi u \in V_\delta^{l+1,p}(\mathcal{D})^\ell$ and*

$$\|\phi u\|_{V_\delta^{l+1,p}(\mathcal{D})^\ell} \le c\left(\sum_{j=0}^{1} \|\psi\partial_{x_3}^j u\|_{V_\delta^{l,p}(\mathcal{D})^\ell} + \|\psi f\|_{V_\delta^{l-1,p}(\mathcal{D})^\ell} + \sum_\pm \|\psi g^\pm\|_{V_\delta^{l+d^\pm-1/p,p}(\Gamma^\pm)^\ell}\right).$$

P r o o f. Obviously,

$$L(\partial_{x'},0)\,(\phi u) = F \overset{\text{def}}{=} \phi f + \phi L_1 \partial_{x_3} u + [L(\partial_{x'},0),\phi]\, u,$$

where L_1 is a differential operator of first order and $[L(\partial_{x'},0),\phi] = L(\partial_{x'},0)\phi - \phi L(\partial_{x'},0)$ denotes the commutator of $L(\partial_{x'},0)$ and ϕ. By our assumptions on u and f, the function $F(\cdot,x_3)$ belongs to $V_\delta^{l-1,p}(K)^\ell$ for arbitrary fixed x_3. Analogously, we have $d^\pm u + (1-d^\pm)\, B^\pm(\partial_{x'},0)\,(\phi u) = G^\pm$, where $G^\pm(\cdot,x_3)|_{\gamma^\pm} \in V_\delta^{l+d^\pm-1/p,p}(\gamma^\pm)^\ell$ for fixed x_3. Consequently, by [14] (in the case $p=2$ see also [7] and [8, Th. 6.1.4]), we obtain $(\phi u)(\cdot,x_3) \in V_\delta^{l+1,p}(K)^\ell$ and

$$\|(\phi u)(\cdot,x_3)\|^p_{V_\delta^{l+1,p}(K)^\ell} \le c\left(\|F(\cdot,x_3)\|^p_{V_\delta^{l-1,p}(K)^\ell} + \sum_\pm \|G^\pm(\cdot,x_3)\|^p_{V_\delta^{l+d^\pm-1/p,p}(\Gamma^\pm)^\ell}\right)$$

with a constant c independent of x_3. Integrating this inequality and using the assumption that $\partial_{x_3} u \in V_\delta^{l,p}(\mathcal{D})^\ell$, we obtain $u \in V_\delta^{l+1,p}(\mathcal{D})^\ell$. □

We prove an analogous result for the Neumann problem and the class of the spaces $W_\delta^{l,p}$. For this end, we study first the Neumann problem in the plane angle K.

Lemma 3.3. *Let $u \in W_\delta^{l,p}(K)^\ell$ be a solution of the problem*

$$L(\partial_{x'},0)\, u = f \ \text{ in } K, \quad B^\pm(\partial_{x'},0)\, u = g^\pm \ \text{ on } \gamma^\pm,$$

where $f \in W_\delta^{l-1,p}(K)^\ell$, $g^\pm \in W_\delta^{l-1/p,p}(\gamma^\pm)$, $\delta > -2/p$. If the strip $l-\delta-2/p \le \operatorname{Re}\lambda \le l+1-\delta-2/p$ is free of eigenvalues of the pencil $A(\lambda)$, then $u \in W_\delta^{l+1,p}(K)$.

P r o o f. If $\delta > l-2/p$, then $W_\delta^{l,p}(K)^\ell \subset V_\delta^{l,p}(K)^\ell$, $W_\delta^{l-1,p}(K)^\ell \subset V_\delta^{l-1,p}(K)^\ell$, and $W_\delta^{l-1/p,p}(\gamma^\pm)^\ell \subset V_\delta^{l-1/p,p}(\gamma^\pm)^\ell$. Therefore, it follows from [14] that $u \in V_\delta^{l+1,p}(K)^\ell \cap W_\delta^{l,p}(K)^\ell \subset W_\delta^{l+1,p}(K)^\ell$.

We suppose that $\delta \le l-2/p$. Then u has continuous derivatives up to order $m = \langle l-\delta-2/p\rangle$ at $x=0$, where $\langle s\rangle$ is the greatest integer less than s. Let first $\delta + 2/p$ be not integer, and let ζ be a smooth cut-off function on $\overline{K}$ equal to one near the vertex $x=0$. We denote by $p_m(x')$ the Taylor polynomial of degree m of u and set $v = u - \zeta p_m$. By [10] (for $p=2$ see also [8, Th. 7.1.1]), we have $v \in V_\delta^{l,p}(K)^\ell$. Furthermore,

$$L(\partial_{x'},0)\, v = f - L(\partial_{x'},0)\,(\zeta p_m) \in W_\delta^{l-1,p}(K)^\ell \cap V_\delta^{l-2,p}(K)^\ell.$$

Analogously, $B^\pm(\partial_{x'},0)\, v|_{\gamma^\pm} \in W_\delta^{l-1/p,p}(\gamma^\pm)^\ell \cap V_\delta^{l-1-1/p,p}(\gamma^\pm)^\ell$. Consequently, there are the representations

$$L(\partial_{x'},0)\, v = \zeta\, p^\circ_{m-1} + F, \quad B^\pm(\partial_{x'},0)\, v = \zeta\, q^\pm_m + G^\pm,$$

where p°_{m-1}, $q^\pm_m$ are homogeneous polynomials of degrees $m-1$ and m, respectively, $F \in V_\delta^{l-1,p}(K)^\ell$, and $G^\pm|_{\gamma^\pm} \in V_\delta^{l-1/p,p}(\gamma^\pm)^\ell$. Since $\lambda = m+1$ is not an eigenvalue of the pencil $A(\lambda)$, there exists a homogeneous polynomial p°_{m+1} of degree $m+1$ such that

$$L(\partial_{x'},0)\, p^\circ_{m+1} = p^\circ_{m-1} \ \text{ in } K, \quad B^\pm(\partial_{x'},0)\, p^\circ_{m+1} = q^\pm_m \ \text{ on } \gamma^\pm$$

(see [16, Le. 2.4]). Hence,

$$L(\partial_{x'},0)\,(v-\zeta p^\circ_{m+1})\in V^{l-1,p}_\delta(K)^\ell, \quad \text{and } B(\partial_{x'},0)\,(v-\zeta p^\circ_{m+1})\in V^{l-1/p,p}_\delta(\gamma^\pm)^\ell.$$

Since $v-\zeta p^\circ_{m+1}\in V^{l,p}_\delta(K)^\ell$ and the strip $l-\delta-2/p\le \mathrm{Re}\,\lambda\le l+1-\delta-2/p$ is free of eigenvalues of the pencil $A(\lambda)$, we conclude that $v-\zeta p^\circ_{m+1}\in V^{l+1,p}_\delta(K)^\ell$ and, therefore, $u=v+\zeta p_m\in W^{l+1,p}_\delta(K)^\ell$.

We consider the case when $\delta+2/p$ is integer. Since $\zeta u\in W^{l,p}_{\delta+\varepsilon}(K)^\ell$, $L(\partial_{x'},0)\,(\zeta u)\in W^{l-1,p}_{\delta+\varepsilon}(K)^\ell$, and $B(\partial_{x'},0)\,(\zeta u)\in W^{l-1/p,p}_{\delta+\varepsilon}(\gamma^\pm)^\ell$ with arbitrary positive ε, we obtain, by the first part of the proof, that $\zeta u\in W^{l+1,p}_{\delta+\varepsilon}(K)^\ell$. Consequently, the vector function $v=\zeta(u-p_m)$, where $m=l-\delta-2/p$, belongs to $V^{l+1,p}_{\delta+\varepsilon}(K)^\ell$. Furthermore, analogously to the first part of the proof, we have

$$L(\partial_{x'},0)\,v\in W^{l-1,p}_\delta(K)^\ell\cap V^{l-1,p}_{\delta+\varepsilon}(K)^\ell, \quad B^\pm(\partial_{x'},0)\,v|_{\gamma^\pm}\in W^{l-1/p,p}_\delta(\gamma^\pm)^\ell\cap V^{l-1/p,p}_{\delta+\varepsilon}(K)^\ell.$$

Consequently, there are the representations

$$L(\partial_{x'},0)\,v=\sum_{i+j=m-1} f_{i,j}(r)\,x_1^i\,x_2^j+F, \quad B^\pm(\partial_{x'},0)\,v|_{\gamma^\pm}=g^\pm_m(r)\,r^m+G^\pm, \tag{3.3}$$

where $F\in V^{l-1,p}_\delta(K)^\ell$, $G^\pm\in W^{l-1/p,p}_\delta(\gamma^\pm)^\ell$, $f_{i,j},\ g^\pm_m$ are functions in $W^{1/p,p}((0,\infty))^\ell$ with support in $[0,1)$ such that

$$\int_0^1 r^{ps-1}|\partial_r^s f_{i,j}(r)|^p\,dr\le c_s\,\|L(\partial_{x'},0)\,v\|_{W^{l-1,p}_\delta(K)^\ell}, \tag{3.4}$$

$$\int_0^1 r^{ps-1}|\partial_r^s g^\pm_m(r)|^p\,dr\le c_s\,\|B^\pm(\partial_{x'},0)\,v\|_{W^{l-1/p,p}_\delta(\gamma^\pm)^\ell} \tag{3.5}$$

for $s=1,2,\dots$ (cf. [8, Th. 7.3.2]). Since $\lambda=m+1$ is not an eigenvalue of the pencil $A(\lambda)$, there exist homogeneous matrix-valued polynomials $p_{i,j}$, $q^\pm_{m+1}$ of degree $m+1$ such that

$$L(\partial_{x'},0)\,p_{i,j}=x_1^i x_2^j\,I_\ell \text{ in } K, \quad B^\pm(\partial_{x'},0)\,p_{i,j}=0 \text{ on } \gamma^\pm,\ i+j=m-1,$$

$$L(\partial_{x'},0)\,q^\pm_{m+1}=0 \text{ in } K, \quad B^\pm(\partial_{x'},0)\,q^\pm_{m+1}=r^m\,I_\ell \text{ on } \gamma^\pm,\ B^\mp(\partial_{x'},0)\,q^\pm_{m+1}=0 \text{ on } \gamma^\mp,$$

where I_ℓ denotes the $\ell\times\ell$ identity matrix. We set

$$w=\sum_{i+j=m-1} p_{i,j}(x')\,f_{i,j}(r)+\sum_\pm q^\pm_{m+1}(x')\,g_{m+1}(r).$$

From (3.4), (3.5) it follows that $w\in W^{l+1,p}_\delta(K)^\ell\cap V^{l+1,p}_{\delta+\varepsilon}(K)^\ell$. Furthermore, according to (3.3)–(3.5), we have

$$L(\partial_{x'},0)\,(v-w)\in V^{l-1,p}_\delta(K)^\ell, \quad B(\partial_{x'},0)\,(v-w)\big|_{\gamma^\pm}\in V^{l-1/p,p}_\delta(\gamma^\pm)^\ell.$$

By [14], this implies that $v-w\in V^{l+1,p}_\delta(K)^\ell$ and therefore, $u\in W^{l+1,p}_\delta(K)^\ell$. The proof is complete. □

Now the following lemma can be proved analogously to Lemma 3.2 by means of Lemma 3.3.

Lemma 3.4. *Let ϕ, ψ be as in Lemma 3.2, and let u be a solution of problem (3.1) with $d^+=d^-=0$ such that $\psi u\in W^{l,p}_\delta(\mathcal{D})^\ell$, $\psi\partial_{x_3}u\in W^{l,p}_\delta(\mathcal{D})^\ell$, $\psi f\in W^{l-1,p}_\delta(\mathcal{D})^\ell$, $\psi g^\pm\in W^{l-1/p,p}_\delta(\Gamma^\pm)^\ell$. If there are no eigenvalues of the pencil $A(\lambda)$ in the strip $l-\delta-2/p\le\mathrm{Re}\,\lambda\le l+1-\delta-2/p$, then $\phi u\in W^{l+1,p}_\delta(\mathcal{D})^\ell$ and*

$$\|\phi u\|_{W^{l+1,p}_\delta(\mathcal{D})^\ell}\le c\left(\sum_{j=0}^1\|\psi\partial^j_{x_3}u\|_{W^{l,p}_\delta(\mathcal{D})^\ell}+\|\psi f\|_{W^{l-1,p}_\delta(\mathcal{D})^\ell}+\sum_\pm\|\psi g^\pm\|_{W^{l-1/p,p}_\delta(\Gamma^\pm)^\ell}\right).$$

3.2 Boundary conditions on the sides of a dihedron

For the following lemma we refer to [12, Le. 3.1].

Lemma 3.5. *For arbitrary $g^\pm\in V^{l+d^\pm-1-1/p,p}_\delta(\Gamma^\pm)^\ell$, $l+\min(d^+,d^-)>1$, there exists a vector function $u\in V^{l,p}_\delta(\mathcal{D})^\ell$ such that $d^\pm u+(1-d^\pm)\,Bu=g^\pm$ on $\Gamma^\pm$ and*

$$\|u\|_{V^{l,p}_\delta(\mathcal{D})^\ell}\le c\sum_\pm\|g^\pm\|_{V^{l+d^\pm-1-1/p,p}_\delta(\Gamma^\pm)^\ell} \tag{3.6}$$

with c independent of g^+,g^-.

We need an analogous result for the Neumann problem in the class of the spaces $W_\delta^{l,p}$.

Lemma 3.6. *Let $d^+ = d^- = 0$, $g^\pm \in W_\delta^{1-1/p,p}(\Gamma^\pm)^\ell$, $\delta > -2/p$, $g^\pm(x) = 0$ for $|x'| > 1$. In the case $-2/p < \delta \le 1-2/p$ we suppose further that there is no non-zero linear vector function $p(x') = c_1 x_1 + c_2 x_2$, $c_1, c_2 \in \mathbb{C}^\ell$, such that $B^\pm(\partial_{x'}, 0)p(x') = 0$. Then there exists a vector function $u \in W_\delta^{2,p}(\mathcal{D})^\ell$ satisfying $Bu = g^\pm$ on $\Gamma^\pm$ and an estimate analogous to* (3.6).

Proof. If $\delta > 1-2/p$, then $W_\delta^{1-1/p,p}(\Gamma^\pm) \subset V_\delta^{1-1/p,p}(\Gamma^\pm)$ and the assertion follows from Lemma 3.5.

Suppose that $-2/p < \delta < 1-2/p$. It is known that the trace of any function $g^\pm \in W_\delta^{1-1/p,p}(\Gamma^\pm)$ on the edge M belongs to the Sobolev-Slobodetskiĭ space $W^{1-\delta-2/p,p}(M)$. Conversely, any function $\phi \in W^{1-\delta-2/p,p}(M)$ can be extended to a function in $W_\delta^{1-1/p,p}(\Gamma^\pm)$ by

$$(E\phi)(r,x_3) = \chi(r) \int_{-1}^{1} \phi(x_3 + tr)\, \psi(t)\, dt, \tag{3.7}$$

where χ and ψ are infinitely differentiable functions with supports in $[0,1]$ and $(-1,+1)$, respectively, $\chi(r) = 1$ for $0 \le r < 1/2$, $\int \psi(t)\,dt = 1$. Note that in (3.7) the function $E\phi$ can be also considered as a function on $\mathcal{D}$ depending only on $r = |x'|$ and x_3. Then the operator E realizes a continuous mapping $W^{1-\delta-2/p,p}(M) \to W_\delta^{1,p}(\mathcal{D})$. Furthermore,

$$\int_{\mathcal{D}} r^{p(\delta-1+|\alpha|)} \left|\partial_x^\alpha E\phi\right|^p dx \le c\, \|\phi\|^p_{W^{1-\delta-2/p,p}(M)} \quad \text{for } \alpha \ne 0, \tag{3.8}$$

$$\int_{\mathcal{D}} r^{p(\delta-1} \left|(E\phi)(x',x_3) - \phi(x_3)\right|^p dx \le c\, \|\phi\|^p_{W^{1-\delta-2/p,p}(M)} \tag{3.9}$$

(see [15, Le. 1.2]). By [15, Le. 1.3], there is the representation

$$g^\pm = Eg_0^\pm + G^\pm, \quad \text{where } g_0^\pm = g^\pm\big|_M \quad \text{and } G^\pm \in V_\delta^{1-1/p,p}(\Gamma^\pm)^\ell.$$

We denote by $a_j^\pm$ the constant vectors $B(\partial_{x'}, 0)\, x_j$, $j = 1,\, 2$. From the conditions of the lemma it follows that the system of the linear equations

$$B^\pm(\partial_{x'}, 0)\, (v_1 x_1 + v_2 x_2) = a_1^\pm v_1 + a_2^\pm v_2 = g_0^\pm(x_3)$$

has a unique solution $v_1 = v_1(x_3)$, $v_2 = v_2(x_3)$, and the functions v_1, v_2 belong to $W^{1-\delta-2/p,p}(M)^\ell$. We set $v = x_1\, Ev_1 + x_2\, Ev_2$. Then

$$B^\pm(\partial_x)\, v = B^\pm(\partial_{x'}, 0)\, v + \left(A_{3,1} n_1^\pm + A_{3,2} n_2^\pm\right) \partial_{x_3} v,$$

where $\partial_{x_3} v = x_1 \partial_{x_3} Ev_1 + x_2 \partial_{x_3} Ev_2 \in V_\delta^{1,p}(\mathcal{D})^\ell$ and

$$B^\pm(\partial_{x'}, 0) v\big|_{\Gamma^\pm} - g^\pm = \sum_{j=1}^2 x_j\, B^\pm(\partial_{x'}, 0)\, Ev_j\big|_{\Gamma^\pm} + \sum_{j=1}^2 a_j^\pm\, Ev_j\big|_{\Gamma^\pm} - Eg_0^\pm - G^\pm.$$

From (3.8), (3.9) it follows that $x_j\, B^\pm(\partial_{x'}, 0)\, Ev_j \in V_\delta^{1,p}(\mathcal{D})^\ell$ and

$$\sum_{j=1}^2 a_j^\pm\, Ev_j - Eg_0^\pm = \sum_{j=1}^2 a_j^\pm\, (Ev_j - v_j) - (Eg_0^\pm - g_0^\pm) \in V_\delta^{1,p}(\mathcal{D})^\ell.$$

This implies $B^\pm(\partial_{x'}, 0) v\big|_{\Gamma^\pm} - g^\pm \in V_\delta^{1-1/p,p}(\Gamma^\pm)^\ell$. Applying Lemma 3.5, we obtain the assertion of the lemma in the case $-2/p < \delta < 1-2/p$. In the case $\delta = 1-2/p$ the lemma can be proved analogously using the relation between the spaces $V_\delta^{1-1/p,p}(\Gamma^\pm)$ and $W_\delta^{1-1/p,p}(\Gamma^\pm)$ given in [21]. □

Remark 3.1. The condition of the non-existence of a non-zero linear vector function $p(x') = c_1 x_1 + c_2 x_2$ with $B(\partial_{x'}, 0)\, p(x') = 0$ in the last lemma means that $\lambda = 1$ is not an eigenvalue of the pencil $A(\lambda)$ or $\lambda = 1$ is an eigenvalue, but the corresponding eigenfunctions are not restrictions of linear functions to the unit circle. Otherwise, for the existence of a vector function $u \in W_\delta^{2,p}(\mathcal{D})^\ell$, $-2/p < \delta \le 1-2/p$, with $Bu = g^\pm$ on $\Gamma^\pm$ it is necessary that g^+ and g^- satisfy certain compatibility conditions on the edge M. For example, in the case of the Neumann problem to the Lamé system g^+ and g^- have to satisfy the condition

$$n^- \cdot g^+\big|_M = n^+ \cdot g^-\big|_M \quad \text{if } \delta < 1-2/p,$$

$$\int_0^1 \int_{\mathbb{R}} r^{-1} \left(n^- \cdot g^+(r,x_3) - n^+ \cdot g^-(r,x_3)\right) dx_3\, dr < \infty \quad \text{if } \delta = 1-2/p$$

(see [6, Ch.4], [16, §2.5], [21]).

4 Solvability of the boundary value problem in a polyhedral cone

We consider problem (1.2)–(1.4) in the cone (1.1). In this and in the following section it is assumed that condition (1.6) is satisfied for the sesquilinear form (1.5). We denote by $\tilde{J}$ the set all $j \in J$ such that the Dirichlet condition in problem (1.2)–(1.4) is given on at least one face Γ_k adjacent to the edge M_j, i.e. $M_j \subset \overline{\Gamma}_k$ for at least one $k \in J_0$. Furthermore, we set

$$d_j = 1 \text{ for } j \in J_0, \quad d_j = 0 \text{ for } j \in J_1.$$

The main results of this section are given in Sects. 4.5–4.7. In Sects. 4.5 and 4.6 we restrict ourselves to the Neumann problem, i.e., $J_0 = \tilde{J} = \emptyset$. Solvability theorems and regularity assertions for the solutions to the Dirichlet and mixed problems can be proved analogously. In the case of the Dirichlet problem or mixed problem with $\tilde{J} = J$ the proofs are even easier, since then we have to deal with solutions in the weighted Sobolev spaces $V^{l,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ with homogeneous norms.

4.1 Operator pencils generated by the boundary value problem

We introduce the following operator pencils $\mathfrak{A}$ and A_j.

1. Let $\mathcal{H}_\Omega = \{u \in W^{1,2}(\Omega)^\ell : \, u = 0 \text{ on } \gamma_j \text{ for } j \in J_0\}$ and

$$a(u,v;\lambda) = \frac{1}{\log 2} \int\limits_{\substack{\mathcal{K}\\ 1<|x|<2}} \sum_{i,j=1}^{3} A_{i,j} \partial_{x_i} U \cdot \partial_{x_j} \overline{V}\, dx,$$

where $U(x) = \rho^\lambda(\omega)$, $V(x) = \rho^{-1-\overline{\lambda}} v(\omega)$, $u, v \in \mathcal{H}_\Omega$, and $\lambda \in \mathbb{C}$. Then the operator $\mathfrak{A}(\lambda) : \, \mathcal{H}_\Omega \to \mathcal{H}^*_\Omega$ is defined by

$$\big(\mathfrak{A}(\lambda)u, v\big)_\Omega = a(u,v;\lambda), \quad u, v \in \mathcal{H}_\Omega\,.$$

Here $(\cdot,\cdot)_\Omega$ denotes the extension of the L_2 scalar product to $\mathcal{H}^*_\Omega \times \mathcal{H}_\Omega$.

2. Let $\Gamma_{j_+}, \Gamma_{j_-}$ be the faces of $\mathcal{K}$ adjacent to the edge M_j. We introduce new Cartesian coordinates $y = (y_1, y_2, y_3)$ such that M_j coincides with the positive y_3-axis and $\Gamma_{j_+}, \Gamma_{j_-}$ are contained in the half-planes $\{y \in \mathbb{R}^3 : \, \varphi = 0\}$ and $\{y \in \mathbb{R}^3 : \, \varphi = \theta_j\}$, respectively, where r, φ are the polar coordinates in the (y_1, y_2)-plane. Furthermore, we define the operators $\mathcal{L}_j(\lambda)$ and $\mathcal{B}_{j_\pm}(\lambda)$ on the Sobolev space $W^{2,2}(0,\theta_j)^\ell$ by

$$\mathcal{L}_j(\lambda)\, u(\varphi) = r^{2-\lambda}\, L\big(r^\lambda u(\varphi)\big), \quad \mathcal{B}_{j_\pm}(\lambda)\, u(\varphi) = \begin{cases} u(\varphi) & \text{if } j_\pm \in J_0, \\ r^{1-\lambda}\, B\big(r^\lambda u(\varphi)\big) & \text{if } j_\pm \in J_1. \end{cases}$$

By $A_j(\lambda)$ we denote the operator

$$W^2(0,\theta_j)^\ell \ni u \to \Big(\mathcal{L}_j(\lambda) u\,,\, \mathcal{B}_{j_+}(\lambda)\, u(\varphi)\,\big|_{\varphi=0}\,,\, \mathcal{B}_{j_-}(\lambda)\, u(\varphi)\,\big|_{\varphi=\theta_j}\Big) \in L_2(0,\theta_j)^\ell \times \mathbb{C}^\ell \times \mathbb{C}^\ell.$$

As is known, the spectra of the pencils $\mathfrak{A}$ and A_j consist of isolated points, the eigenvalues. We denote by $\lambda_1^{(j)}$ the eigenvalue of the pencil A_j with smallest positive real part and set $\mu_j = \text{Re}\, \lambda_1^{(j)}$.

4.2 Reduction to homogeneous boundary conditions

The proof of the following lemma is given in [16, Le. 4.2] for the case $p = 2$. The proof for $p \neq 2$ proceeds analogously.

Lemma 4.1. *Let $g_j \in V^{l+d_j-1-1/p,p}_{\beta,\vec{\delta}}(\Gamma_j)^\ell$ for $j = 1,\ldots,n$, where $l \geq 2$ if $J_1 \neq \emptyset$ and $l \geq 1$ else. Then there exists a vector function $u \in V^{l,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ such that $u = g_j$ on Γ_j for $j \in J_0$, $Bu = g_j$ on Γ_j for $j \in J_1$, and*

$$\|u\|_{V^{l,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell} \leq c \sum_{j=1}^{n} \|g_j\|_{V^{l+d_j-1-1/p,p}_{\beta,\vec{\delta}}(\Gamma_j)^\ell} \tag{4.1}$$

with a constant c independent of g_j, $j = 1,\ldots,n$.

By means of Lemma 3.6, we can prove an analogous result in the space $\mathcal{W}^{2,p}_{\beta,\vec{\delta}}(\mathcal{K};\tilde{J})^\ell$.

Lemma 4.2. *Let $g_j \in V^{2-1/p,p}_{\beta,\vec{\delta}}(\Gamma_j)^\ell$ for $j \in J_0$, $g_j \in \mathcal{W}^{1-1/p,p}_{\beta,\vec{\delta}}(\Gamma_j;\tilde{J})^\ell$ for $j \in J_1$. For $j \in J\backslash\tilde{J}$ we assume that $\delta_j > -2/p$ and that $\lambda = 1$ is not an eigenvalue of the pencil $A_j(\lambda)$ if $\delta_j \leq 1 - 2/p$. Then there exists a vector function $u \in \mathcal{W}^{2,p}_{\beta,\vec{\delta}}(\mathcal{K};\tilde{J})^\ell$ such that $u = g_j$ on Γ_j for $j \in J_0$, $Bu = g_j$ on Γ_j for $j \in J_1$, and*

$$\|u\|_{\mathcal{W}^{2,p}_{\beta,\vec{\delta}}(\mathcal{K};\tilde{J})^\ell} \leq c \Big(\sum_{j\in J_0} \|g_j\|_{V^{2-1/p,p}_{\beta,\vec{\delta}}(\Gamma_j)^\ell} + \sum_{j \in J_1} \|g_j\|_{\mathcal{W}^{1-1/p,p}_{\beta,\vec{\delta}}(\Gamma_j;\tilde{J})^\ell} \Big).$$

Proof. Let ζ_k be smooth functions on $\overline{\mathcal{K}}$ depending only on $\rho = |x|$ and satisfying (2.3). We set $h_{k,j}(x) = \zeta_k(2^k x)\, g_j(2^k x)$ for $j \in J_0$ and $h_{k,j}(x) = 2^k\, \zeta_k(2^k x)\, g_j(2^k x)$ for $j \in J_1$. The support of $h_{k,j}$ is contained in $\{x : \frac{1}{2} < |x| < 2\}$. Consequently, by Lemmas 3.5 and 3.6, there exists a vector function $w_k \in \mathcal{W}^{2,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ such that $w_k(x) = 0$ for $|x| < 1/4$ and $|x| > 4$, $w_k = h_{k,j}$ on Γ_j for $j \in J_0$, $Bw_k = h_{k,j}$ on Γ_j for $j \in J_1$,

$$\|w_k\|_{\mathcal{W}^{2,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell} \le c \left(\sum_{j\in J_0} \|h_{k,j}\|_{V^{2-1/p,p}_{\beta,\vec{\delta}}(\Gamma_j)^\ell} + \sum_{j\in J_1} \|h_{k,j}\|_{\mathcal{W}^{1-1/p,p}_{\beta,\vec{\delta}}(\Gamma_j;\tilde{J})^\ell} \right), \tag{4.2}$$

where c is independent of k. From this we conclude that the function $u_k(x) = w_k(2^{-k}x)$ satisfies $u_k = \zeta_k g_j$ on Γ_j for $j \in J_0$, $Bu_k = \zeta_k g_j$ for $j \in J_1$ and the estimate (4.2) with $\zeta_k g_j$ instead of $h_{k,j}$. Thus, $u = \sum u_k$ has the desired properties. □

4.3 Two regularity assertions

By means of Lemma 3.1, we can prove the following two lemmas. Here $\vec{1}$ denotes the tuple $(1, 1, \dots, 1)$.

Lemma 4.3. *Let $u \in V^{0,p}_{\beta-l,\vec{\delta}-l\vec{1}}(\mathcal{K})^\ell$ be a solution of problem* (1.2)–(1.4) *with $f \in V^{l-2,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ and boundary data $g_j \in V^{l+d_j-1-1/p,p}_{\beta,\vec{\delta}}(\Gamma_j)^\ell$, $l \ge 2$. Then $u \in V^{l,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ and*

$$\|u\|_{V^{l,p}_{\beta,\vec{\delta}}(\mathcal{K};\tilde{J})^\ell} \le c \left(\|u\|_{V^{0,p}_{\beta-l,\vec{\delta}-l\vec{1}}(\mathcal{K};\tilde{J})^\ell} + \|f\|_{V^{l-2,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell} + \sum_{j=1}^n \|g_j\|_{V^{l+d_j-1-1/p,p}_{\beta,\vec{\delta}}(\Gamma_j)^\ell} \right).$$

Proof. Due to Lemma 4.1, we may assume, without loss of generality that $g_j = 0$ for $j = 1, \dots, n$. Let ζ_k be smooth functions on $\overline{\mathcal{K}}$ depending only $\rho = |x|$ and satisfying (2.3). We set $\eta_k = \zeta_{k-1} + \zeta_k + \zeta_{k+1}$, $\tilde{\zeta}_k(x) = \zeta_k(2^k x)$, $\tilde{\eta}_k(x) = \eta_k(2^k x)$, and $v(x) = u(2^k x)$. The support of $\tilde{\zeta}_k$ is contained in $\{x : \frac{1}{2} < |x| < 2\}$, and the derivatives $\partial_x^\alpha \tilde{\zeta}_k$ are bounded by constants c_α independent of k. Consequently, by the first part of Lemma 3.1, $\tilde{\zeta}_k v \in V^l_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ and

$$\|\tilde{\zeta}_k v\|^p_{V^l_{\beta,\vec{\delta}}(\mathcal{K})^\ell} \le c \left(\|\tilde{\eta}_k v\|^p_{V^{0,p}_{\beta-l,\vec{\delta}-l\vec{1}}(\mathcal{K})^\ell} + \|\tilde{\eta}_k L v\|^p_{V^{l-2,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell} \right),$$

where c is independent of k. Multiplying this inequality by $2^{kp(\beta-l)+3k}$ and substituting $2^k x = y$, we obtain the same inequality with ζ_k, η_k instead of $\tilde{\zeta}_k, \tilde{\eta}_k$ for the vector function u. Now the lemma follows from the equivalence of the norm in $\mathcal{W}^{l,p}_{\beta,\vec{\delta}}(\mathcal{K};\tilde{J})^\ell$ with the norm (2.4). □

The proof of the following lemma proceeds analogously.

Lemma 4.4. *Let $u \in \mathcal{W}^{k,p}_{\beta-l+k,\vec{\delta}-(l-k)\vec{1}}(\mathcal{K};\tilde{J})^\ell$, where $l \ge k \ge 0$, $l \ge 2$, and $\delta_j > l - k - 2/p$ for $j \notin \tilde{J}$. If u is a solution of problem* (1.2)–(1.4) *with $f \in \mathcal{W}^{l-2,p}_{\beta,\vec{\delta}}(\mathcal{K};\tilde{J})^\ell$, $g_j \in \mathcal{W}^{l+d_j-1-1/p,p}_{\beta,\vec{\delta}}(\Gamma_j;\tilde{J})^\ell$, then $u \in \mathcal{W}^{l,p}_{\beta,\vec{\delta}}(\mathcal{K};\tilde{J})^\ell$ and*

$$\|u\|_{\mathcal{W}^{l,p}_{\beta,\vec{\delta}}(\mathcal{K};\tilde{J})^\ell} \le c \left(\|u\|_{\mathcal{W}^{k,p}_{\beta-l+k,\vec{\delta}-(l-k)\vec{1}}(\mathcal{K};\tilde{J})^\ell} + \|f\|_{\mathcal{W}^{l-2,p}_{\beta,\vec{\delta}}(\mathcal{K};\tilde{J})^\ell} + \sum_{j=1}^n \|g_j\|_{\mathcal{W}^{l+d_j-1-1/p,p}_{\beta,\vec{\delta}}(\Gamma_j;\tilde{J})^\ell} \right).$$

4.4 Estimates of Green's matrix

Let κ be a fixed real number such that the line $\mathrm{Re}\,\lambda = -\kappa - 1/2$ is free of eigenvalues of the pencil $\mathfrak{A}$. Then, according to [16], there exists a unique solution $G(x,\xi)$ of the problem

$$L(\partial_x)\, G(x,\xi) = \delta(x-\xi)\, I_\ell, \quad x,\xi \in \mathcal{K}, \tag{4.3}$$

$$G(x,\xi) = 0, \quad x \in \Gamma_j,\ \xi \in \mathcal{K},\ j \in J_0, \tag{4.4}$$

$$B(\partial_x)\, G(x,\xi) = 0, \quad x \in \Gamma_j,\ \xi \in \mathcal{K},\ j \in J_1 \tag{4.5}$$

(I_ℓ denotes the $\ell \times \ell$ identity matrix) such that the function $x \to \zeta\left(\frac{|x-\xi|}{r(\xi)}\right) G(x,\xi)$ belongs to the space $W^{1,2}_{\kappa,0}(\mathcal{K})^{\ell\times\ell}$ for every fixed $\xi \in \mathcal{K}$ and for every smooth function ζ on $(0,\infty)$, $\zeta(t) = 0$ for $t < \frac{1}{2}$, $\zeta(t) = 1$ for $t > 1$. We denote by $\Lambda_- < \mathrm{Re}\,\lambda < \Lambda_+$ the widest strip in the complex plane which contains the line $\mathrm{Re}\,\lambda = -\kappa - 1/2$ and is free of eigenvalues of the pencil $\mathfrak{A}$. By [16], Green's function $G(x,\xi)$ satisfies the following estimates:

$$\left|\partial_x^\alpha \partial_\xi^\gamma G(x,\xi)\right| \le c\,|x-\xi|^{-1-|\alpha|-|\gamma|} \tag{4.6}$$

$$\text{if} \quad |\xi|/2 < |x| < 2|\xi|,\ |x-\xi| < \min(r(x), r(\xi)),$$

$$\left|\partial_x^\alpha \partial_\xi^\gamma G(x,\xi)\right| \le c\,|x-\xi|^{-1-|\alpha|-|\gamma|} \prod_{j=1}^n \left(\frac{r_j(x)}{|x-\xi|}\right)^{\delta_{j,\alpha}} \prod_{j=1}^n \left(\frac{r_j(\xi)}{|x-\xi|}\right)^{\delta_{j,\gamma}} \tag{4.7}$$

$$\text{if} \quad |\xi|/2 < |x| < 2|\xi|,\ |x-\xi| > \min(r(x), r(\xi)),$$

$$\left|\partial_x^\alpha \partial_\xi^\gamma G(x,\xi)\right| \le c\,|x|^{\Lambda_+-|\alpha|-\varepsilon}\,|\xi|^{-1-\Lambda_+-|\gamma|+\varepsilon} \prod_{j=1}^n \left(\frac{r_j(x)}{|x|}\right)^{\delta_{j,\alpha}} \prod_{j=1}^n \left(\frac{r_j(\xi)}{|\xi|}\right)^{\delta_{j,\gamma}} \tag{4.8}$$

$$\text{if} \quad |x| < |\xi|/2,$$

$$\left|\partial_x^\alpha \partial_\xi^\gamma G(x,\xi)\right| \le c\,|x|^{\Lambda_--|\alpha|+\varepsilon}\,|\xi|^{-1-\Lambda_--|\gamma|-\varepsilon} \prod_{j=1}^n \left(\frac{r_j(x)}{|x|}\right)^{\delta_{j,\alpha}} \prod_{j=1}^n \left(\frac{r_j(\xi)}{|\xi|}\right)^{\delta_{j,\gamma}} \tag{4.9}$$

$$\text{if} \quad |x| > 2|\xi|.$$

Here $\delta_{j,\alpha} = \mu_j - |\alpha| - \varepsilon$ for $j \in \tilde J$ and $\delta_{j,\alpha} = \min(0, \mu_j - |\alpha| - \varepsilon)$ for $j \notin \tilde J$ (ε is an arbitrarily small positive number).

Remark 4.2. In some cases, when $\mu_j = 1$, estimates (4.7)–(4.9) can be improved (see [16, Rem. 4.3]). Let the following conditions be satisfied for a certain index j:

(i) The strip $0 < \operatorname{Re}\lambda < 1$ does not contain eigenvalues of the pencil $A_j(\lambda)$ and $\lambda = 1$ is the only eigenvalue on the line $\operatorname{Re}\lambda = 1$.
(ii) The eigenvectors of $A_j(\lambda)$ corresponding to the eigenvalue $\lambda = 1$ are restrictions of linear vector functions to the unit circle, while generalized eigenvectors corresponding to this eigenvalue do not exist.
(iii) The ranks of the matrices $\mathcal{N}\mathcal{A}$ and $\mathcal{N}\mathcal{A}\mathcal{N}^T$, where

$$\mathcal{A} = \begin{pmatrix} A_{1,1} & A_{2,1} & A_{3,1} \\ A_{1,2} & A_{2,2} & A_{3,2} \\ A_{1,3} & A_{2,3} & A_{3,3} \end{pmatrix} \quad \text{and} \quad \mathcal{N} = \begin{pmatrix} n_1^+ I_\ell & n_2^+ I_\ell & n_3^+ I_\ell \\ n_1^- I_\ell & n_2^- I_\ell & n_3^- I_\ell \end{pmatrix}$$

(n^+, n^- are the normal vectors to the faces Γ_{j_+} and Γ_{j_-} adjacent to the edge M_j, I_ℓ denotes the $\ell \times \ell$ identity matrix, and $\mathcal{N}^T$ denotes the transposed matrix of $\mathcal{N}$), coincide.

Then the number $\mu_j = 1$ can be replaced by the real part $\mu_j^{(2)}$ of the first eigenvalue of the pencil $A_j(\lambda)$ on the right of the line $\operatorname{Re}\lambda = 1$.

Note that the rank of the matrix $\mathcal{N}\mathcal{A}$ determines the number of necessary compatibility conditions for the boundary data g^+ and g^- on the faces Γ_{j_+} and Γ_{j_-}, respectively, if the solution is assumed to be smooth. Indeed, the boundary conditions on these faces can be written in the form $(n_1^\pm I_\ell, n_2^\pm I_\ell, n_3^\pm I_\ell)\,\mathcal{A}\,\nabla u = g^\pm$. Here ∇u is considered as a column vector containing the vectors $\partial_{x_k} u$, $k = 1, 2, 3$. If u is sufficiently smooth, then the traces of ∇u and $g^\pm$ on M_j exist and we obtain the algebraic system

$$\mathcal{N}\mathcal{A}\,\nabla u|_{M_j} = \begin{pmatrix} g^+ \\ g^- \end{pmatrix}\Big|_{M_j}$$

for $\nabla u|_{M_j}$. Hence the vector $(g^+(x), g^-(x))^T$ must belong to the range of $\mathcal{N}\mathcal{A}$ for every $x \in M_j$. Furthermore, for the existence of an eigenvector of the pencil $A_j(\lambda)$ corresponding to $\lambda = 1$ which is a restriction of a linear vector function to the unit sphere it is necessary and sufficient that there exists a linear vector function

$$u = (n^+ \cdot x)\,c + (n^- \cdot x)\,d, \quad c, d \in \mathbb{C}^\ell$$

(i.e., a linear vector function vanishing on M_j) satisfying the homogeneous boundary conditions $B^\pm u = 0$ on $\Gamma_{j_\pm}$. This means that (c, d) is a solution of the algebraic system

$$\mathcal{N}\mathcal{A}\mathcal{N}^T \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore, under condition (ii), the number of eigenvectors corresponding to the eigenvalue $\lambda = 1$ is $2\ell - r$, where r is the rank of the matrix $\mathcal{N}\mathcal{A}\mathcal{N}^T$.

Let us mention that conditions (i)–(iii) are satisfied, e.g., for the Neumann problem to the Lamé system and in anisotropic elasticity if the angle θ_j at the edge M_j is less than π. Here the matrices $\mathcal{N}\mathcal{A}$ and $\mathcal{N}\mathcal{A}\mathcal{N}^T$ have rank 5.

4.5 Existence of solutions to the Neumann problem

In this subsection we prove the existence of solutions to the Neumann problem in the space $W^{2,p}_{\beta,\vec\delta}(\mathcal{K})^\ell$, where β is a real number such that the line $\text{Re}\,\lambda = -\beta+2-3/p$ is free of eigenvalues of the pencil $\mathfrak{A}$ and the components δ_j of $\vec\delta$ satisfy the inequalities

$$\max(2-\mu_j,0) < \delta_j + 2/p < 2, \quad j=1,\dots,n. \tag{4.10}$$

Lemmas 4.1 and 4.2 allow us to restrict ourselves to homogeneous boundary conditions.

We introduce the operator S which is defined on $W^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell$ by

$$(Sf)(x) = \int_{\mathcal{K}} G(x,\xi)\cdot f(\xi)\,d\xi,$$

where $G(x,\xi)$ is the Green matrix introduced in the foregoing subsection with $\kappa = \beta + \frac{3}{p} - \frac{5}{2}$. We show that S continuously maps the space $W^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell$ into $W^{2,p}_{\beta,\vec\delta}(\mathcal{K})^\ell$.

Lemma 4.5. *Let ζ_k, $k=0,\pm1,\dots$, be smooth functions on $\mathcal{K}$ depending only on $\rho=|x|$ such that* (2.3) *is satisfied. Suppose that the line* $\text{Re}\,\lambda = -\beta+2-3/p$ *does not contain eigenvalues of the pencil $\mathfrak{A}$ and that for the components of $\vec\delta$ the inequalities* (4.10) *are valid. Then for arbitrary $f \in W^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell$ and $|k-l|\ge 3$ we have*

$$\|\zeta_k S\zeta_l f\|_{W^{2,p}_{\beta,\vec\delta}(\mathcal{K})^\ell} \le c\,2^{-|k-l|\sigma}\,\|\zeta_l f\|_{W^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell}$$

with positive constants c and σ independent of k, l, and f.

P r o o f. We have to show that

$$\|(\partial_x^\gamma\zeta_k)\partial_x^\alpha S\zeta_l f\|_{W^{0,p}_{\beta-2+|\alpha|+|\gamma|,\vec\delta}(\mathcal{K})^\ell} \le c\,2^{-|k-l|\sigma}\,\|\zeta_l f\|_{W^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell} \tag{4.11}$$

for $|\alpha|+|\gamma|\le 2$, $|k-l|\ge 3$. Since $|\partial_x^\gamma\zeta_k| \le 2^{-k|\gamma|}$, we get, by means of Hölder's inequality,

$$\|(\partial_x^\gamma\zeta_k)\partial_x^\alpha S\zeta_l f\|^p_{W^{0,p}_{\beta-2+|\alpha|+|\gamma|,\vec\delta}(\mathcal{K})^\ell}$$

$$\le c \int\limits_{\substack{\mathcal{K}\\ 2^{k-1}<|x|<2^{k+1}}} |x|^{p(\beta-2+|\alpha|)} \prod\Big(\frac{r_j(x)}{|x|}\Big)^{p\delta_j} \Big|\int_{\mathcal{K}} \partial_x^\alpha G(x,\xi)\,\zeta_l(\xi)\,f(\xi)\,d\xi\Big|^p\,dx$$

$$\le c\|\zeta_l f\|^p_{W^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell} \int\limits_{\substack{\mathcal{K}\\ 2^{k-1}<|x|<2^{k+1}}} |x|^{p(\beta-2+|\alpha|)} \prod\Big(\frac{r_j(x)}{|x|}\Big)^{p\delta_j}$$

$$\times\Bigg(\int\limits_{\substack{\mathcal{K}\\ 2^{l-1}<|\xi|<2^{l+1}}} |\xi|^{-q\beta}\prod\Big(\frac{r_j(\xi)}{|\xi|}\Big)^{-q\delta_j}\,\big|\partial_x^\alpha G(x,\xi)\big|^q\,d\xi\Bigg)^{p/q} dx,$$

where $q=p/(p-1)$. Let $k\ge l+3$. Then $|\xi|<|x|/2$ for $x\in\operatorname{supp}\zeta_k$, $\xi\in\operatorname{supp}\zeta_l$ and (4.9) implies

$$\|(\partial_x^\gamma\zeta_k)\partial_x^\alpha S\zeta_l f\|^p_{W^{0,p}_{\beta-2+|\alpha|+|\gamma|,\vec\delta}(\mathcal{K})^\ell}$$

$$\le c\,\|\zeta_l f\|^p_{W^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell} \int\limits_{\substack{\mathcal{K}\\ 2^{k-1}<|x|<2^{k+1}}} |x|^{p(\beta-2+\Lambda_-+\varepsilon)} \prod\Big(\frac{r_j(x)}{|x|}\Big)^{p(\delta_j+\delta_{j,\alpha})}\,dx$$

$$\times\Bigg(\int\limits_{\substack{\mathcal{K}\\ 2^{l-1}<|\xi|<2^{l+1}}} |\xi|^{-q(\beta+1+\Lambda_-+\varepsilon)}\prod\Big(\frac{r_j(\xi)}{|\xi|}\Big)^{-q\delta_j}\,d\xi\Bigg)^{p/q},$$

where $\delta_{j,\alpha}=\min(0,\mu_j-|\alpha|-\varepsilon)$. Under our assumptions on $\vec\delta$, we have $p(\delta_j+\delta_{j,\alpha})>-2$ for $|\alpha|\le 2$ and $-q\delta_j>-2$. Consequently,

$$\|(\partial_x^\gamma\zeta_k)\partial_x^\alpha S\zeta_l f\|^p_{W^{0,p}_{\beta-2+|\alpha|+|\gamma|,\vec\delta}(\mathcal{K})^\ell} \le 2^{(k-l)p(\beta-2+3/p+\Lambda_-+\varepsilon)}\,\|\zeta_l f\|^p_{W^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell},$$

where $\beta-2+3/p+\Lambda_-<0$. This proves the lemma for $k\ge l+3$. Analogously, it can be proved for $k\le l-3$. □

For the proof of an analogous result in the case $|k-l| \le 2$, we need the following lemma.

Lemma 4.6. *Let $\mathcal{D}$ be the dihedron* (2.1)*, and let $r(x)$ denote the distance of x to the edge. If $\alpha+\beta > 3$ and $\beta < 2$, then*

$$\int\limits_{\substack{\mathcal{D}\\|\xi-x|>r(x)/3}} |\xi-x|^{-\alpha}\, r(\xi)^{-\beta}\, d\xi \le c\, r(x)^{3-\alpha-\beta}$$

with a constant c independent of x.

P r o o f. The substitution $y = x/r(x)$, $\eta = \xi/r(x)$ yields

$$\int\limits_{\substack{\mathcal{D}\\|\xi-x|>r(x)/3}} |\xi-x|^{-\alpha}\, r(\xi)^{-\beta}\, d\xi = r(x)^{3-\alpha-\beta} \int\limits_{\substack{\mathcal{D}\\|\eta-y|>1/3}} |\eta-y|^{-\alpha}\, r(\eta)^{-\beta}\, d\eta.$$

Since $r(y) = 1$, the integral on the right is majorized by a finite constant c. This proves the lemma. □

Corollary 4.1. *Let $c_1, c_2, \alpha, \beta_j, \gamma_j, \delta_j$ be nonnegative real numbers, $\beta_j+\gamma_j < 2$ and $3-\alpha-\gamma_j+\delta_j < 0$ for $j = 1,\dots,n$. Furthermore, let $\mathcal{K}_x = \big\{\xi \in \mathcal{K} :\ c_1|x| < |\xi| < c_2|x|,\ |\xi-x| > r(x)/3\big\}$. Then*

$$\int\limits_{\mathcal{K}_x} |x-\xi|^{-\alpha} \prod_{j=1}^n \left(\frac{r_j(\xi)}{|x-\xi|}\right)^{-\beta_j} \prod_{j=1}^n \left(\frac{r_j(\xi)}{|\xi|}\right)^{-\gamma_j} \prod_{j=1}^n \left(\frac{r_j(x)}{|x-\xi|}\right)^{-\delta_j} d\xi$$

$$\le c\,|x|^{3-\alpha} \prod_{j=1}^n \left(\frac{r_j(x)}{|x|}\right)^{3-\alpha-\gamma_j} \tag{4.12}$$

with c independent of x.

P r o o f. The left-hand side of (4.12) is equal to

$$|x|^{3-\alpha} \int\limits_{\mathcal{K}_y} |y-\eta|^{-\alpha} \prod_{j=1}^n \left(\frac{r_j(\eta)}{|y-\eta|}\right)^{-\beta_j} \prod_{j=1}^n \big(r_j(\eta)\big)^{-\gamma_j} \prod_{j=1}^n \left(\frac{r_j(y)}{|y-\eta|}\right)^{-\delta_j} d\eta, \tag{4.13}$$

where $y = x/|x|$, $\eta = \xi/|x|$. Without loss of generality, we may assume that M_1 is the nearest edge to x and y. If η lies in a neighborhood of another edge M_j, then the integrand in (4.13) is majorized by $c\, r_j(\eta)^{-\beta_j-\gamma_j}\, r_1(y)^{-\delta_1}$, where $-\delta_1 > 3-\alpha-\gamma_1$. If M_1 is the nearest edge to η, then an upper bound for the integrand is

$$c\,|y-\eta|^{-\alpha+\beta_1+\delta_1}\, r_1(\eta)^{-\beta_1-\gamma_1}\, r_1(y)^{-\delta_1}.$$

Applying Lemma 4.6, we obtain (4.12). □

Lemma 4.7. *Let ζ_k, β, and $\vec\delta$ be as in Lemma* 4.5*. Then for arbitrary $f \in W^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell$ and $|k-l| \le 2$ we have*

$$\|\zeta_k S \zeta_l f\|_{W^{2,p}_{\beta,\vec\delta}(\mathcal{K})^\ell} \le c\, \|\zeta_l f\|_{W^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell} \tag{4.14}$$

with a constant c independent of k, l, and f.

P r o o f. Let χ be a smooth function on $[0,\infty)$, $\chi(t) = 1$ for $0 \le t \le 1/2$, $\chi(t) = 0$ for $t \ge 3/4$. We set $\chi^+(x,\xi) = \chi\big(|x-\xi|/r(x)\big)$, $\chi^-(x,\xi) = 1-\chi^+(x,\xi)$, and

$$u_l^\pm(x) = \int_{\mathcal{K}} \chi^\pm(x,\xi)\, G(x,\xi)\, \zeta_l(\xi)\, f(\xi)\, d\xi.$$

Then $S\zeta_l f = u_l^+ + u_l^-$. Note that $|\partial_x^\alpha \chi^\pm(x,\xi)| \le c\,|x-\xi|^{-|\alpha|}$ with a constant c independent of x and ξ.

We show first that

$$\|\zeta_k u_l^+\|_{V^{0,p}_{\beta-2,\vec\delta-\vec2}(\mathcal{K})^\ell} \le c\, \|\zeta_l f\|_{V^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell}, \tag{4.15}$$

where $\vec 2 = (2,\dots,2)$. From Hölder's inequality and (4.6) it follows that

$$\Big| \int_{\mathcal{K}} \chi^+(x,\xi)\, G(x,\xi)\, \zeta_l(\xi)\, f(\xi)\, d\xi \Big|^p \le c \int_{\mathcal{K}} |x-\xi|^{-1} \big|\chi^+(x,\xi)\,\zeta_l(\xi)\, f(\xi)\big|^p\, d\xi \Big(\int\limits_{|x-\xi|<r(x)} |x-\xi|^{-1}\, d\xi \Big)^{p-1}$$
$$\le c\, r(x)^{2(p-1)} \int_{\mathcal{K}} |x-\xi|^{-1} \big|\chi^+(x,\xi)\,\zeta_l(\xi)\, f(\xi)\big|^p\, d\xi$$

for $x \in \operatorname{supp}\zeta_k$. Since $2^{-4}|x| \le |\xi| \le 2^4|x|$ and $c_1\, r_j(x) \le r_j(\xi) \le c_2 r_j(x)$ for $x \in \operatorname{supp}\zeta_k$, $\xi \in \operatorname{supp}\zeta_l \cap \operatorname{supp}\chi^+(x,\cdot)$, the last inequality implies

$$\|\zeta_k u_l^+\|^p_{V^{0,p}_{\beta-2,\vec{\delta}-\vec{2}}(\mathcal{K})^\ell}$$
$$\le c \int_{\mathcal{K}} |x|^{p(\beta-2)} \prod_{j=1}^n \Big(\frac{r_j(x)}{|x|}\Big)^{p(\delta_j-2)} |\zeta_k(x)|^p\, r(x)^{2(p-1)} \Big(\int_{\mathcal{K}} |x-\xi|^{-1} \big|\chi^+(x,\xi)\,\zeta_l(\xi)\, f(\xi)\big|^p\, d\xi \Big)\, dx$$
$$\le c \int_{\mathcal{K}} |\xi|^{p(\beta-2)} \prod_{j=1}^n \Big(\frac{r_j(\xi)}{|\xi|}\Big)^{p(\delta_j-2)} r(\xi)^{2(p-1)} \big|\zeta_l(\xi)\, f(\xi)\big|^p \Big(\int\limits_{|x-\xi|<3r(\xi)} |x-\xi|^{-1}\, dx \Big)\, d\xi$$
$$\le c \int_{\mathcal{K}} |\xi|^{p(\beta-2)} \prod_{j=1}^n \Big(\frac{r_j(\xi)}{|\xi|}\Big)^{p(\delta_j-2)} r(\xi)^{2p} \big|\zeta_l(\xi)\, f(\xi)\big|^p\, d\xi \le c\, \|\zeta_l f\|^p_{W^{0,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell}.$$

Here we used the inequality $r(x) \le c\,|x| \prod_j r_j(x/|x|)$. Analogously to (4.15), we can prove that

$$\int_{\Gamma_\nu} |x|^{p(\beta-1)+1} \prod_{j=1}^n \Big(\frac{r_j(x)}{|x|}\Big)^{p(\delta_j-1)+1} \big|\zeta_k \partial_{x_i} u_l^+\big|^p\, dx \le c\, \|\zeta_l f\|^p_{V^{0,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell} \tag{4.16}$$

for $\nu = 1, \dots, n$. Indeed, for $-1+3/p < \alpha < 2/p$, $x \in \operatorname{supp}\zeta_k$, we have

$$\Big| \int_{\mathcal{K}_x^+} \partial_{x_i}\chi^+(x,\xi)\, G(x,\xi)\, \zeta_l(\xi)\, f(\xi)\, d\xi \Big|^p \le \Big(\int\limits_{\substack{\mathcal{K} \\ 4|x-\xi|<3r(x)}} |x-\xi|^{-2} \big|\zeta_l(\xi)\, f(\xi)\big|\, d\xi \Big)^p$$
$$\le \int\limits_{\substack{\mathcal{K} \\ 4|x-\xi|<3r(x)}} |x-\xi|^{-p\alpha} \big|\zeta_l(\xi)\, f(\xi)\big|\, d\xi \Big(\int\limits_{4|x-\xi|<3r(x)} |x-\xi|^{-q(2-\alpha)}\, d\xi \Big)^{p-1}$$
$$\le c\, r(x)^{p(1+\alpha)-3} \int\limits_{\substack{\mathcal{K} \\ 4|x-\xi|<3r(x)}} |x-\xi|^{-p\alpha} \big|\zeta_l(\xi)\, f(\xi)\big|\, d\xi,$$

where $q = p/(p-1)$. Consequently,

$$\int_{\Gamma_\nu} |x|^{p(\beta-1)+1} \prod_{j=1}^n \Big(\frac{r_j(x)}{|x|}\Big)^{p(\delta_j-1)+1} \big|\zeta_k \partial_{x_i} u_l^+\big|^p\, dx$$
$$\le c \int_{\mathcal{K}} |\xi|^{p(\beta-1)+1} \prod_{j=1}^n \Big(\frac{r_j(\xi)}{|\xi|}\Big)^{p(\delta_j-1)+1} r(\xi)^{p(1+\alpha)-3} \big|\zeta_l(\xi)\, f(\xi)\big|^p \Big(\int\limits_{\substack{\Gamma_\nu \\ |x-\xi|<3r(\xi)}} |x-\xi|^{-p\alpha}\, dx \Big)\, d\xi$$
$$\le c \int_{\mathcal{K}} |\xi|^{p(\beta-1)+1} \prod_{j=1}^n \Big(\frac{r_j(\xi)}{|\xi|}\Big)^{p(\delta_j-1)+1} r(\xi)^{p-1} \big|\zeta_l(\xi)\, f(\xi)\big|^p\, d\xi \le c\, \|\zeta_l f\|^p_{V^{0,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell}.$$

Next we show that

$$\|\zeta_k u_l^-\|_{W^{2,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell} \le c\, \|\zeta_l f\|_{W^{0,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell}. \tag{4.17}$$

For this end, we consider the norm of $(\partial_x^\gamma \zeta_k)\, \partial_x^\alpha u_l^-$ in $W^{0,p}_{\beta-2+|\alpha|+|\gamma|,\vec{\delta}}(\mathcal{K})^\ell$ for $|\alpha|+|\gamma| \le 2$. For $|\alpha| = 0$ we have

$$\big|(\partial_x^\gamma \zeta_k)\, u_l^-\big|^p \le c\, 2^{-kp|\gamma|} \Big(\int_{\mathcal{K}} |x-\xi|^{-1} \big|\chi^-(x,\xi)\, \zeta_l(\xi)\, f(\xi)\big|\, d\xi \Big)^p$$

$$\leq c\, 2^{-kp|\gamma|} \int_{\mathcal{K}} |x-\xi|^{-1} \prod_j \left(\frac{r_j(\xi)}{|\xi|}\right)^{p\delta_j} \big|\zeta_l(\xi)\, f(\xi)\big|^p \, d\xi \Bigg(\int\limits_{\substack{\mathcal{K}\\ 2^{l-1}<|\xi|<2^{l+1}}} |x-\xi|^{-1} \prod_j \left(\frac{r_j(\xi)}{|\xi|}\right)^{-q\delta_j} d\xi \Bigg)^{p-1}.$$

Substituting $x/|x| = y$, $\xi/|x| = \eta$, we obtain

$$\int\limits_{\substack{\mathcal{K}\\ 2^{l-1}<|\xi|<2^{l+1}}} |x-\xi|^{-1} \prod_j \left(\frac{r_j(\xi)}{|\xi|}\right)^{-q\delta_j} d\xi \leq |x|^2 \int\limits_{\substack{\mathcal{K}\\ 2^{-4}<|\eta|<2^{4}}} |y-\eta|^{-1} \prod_j \left(\frac{r_j(\eta)}{|\eta|}\right)^{-q\delta_j} d\eta \leq c\, 2^{2k}$$

for $x \in \operatorname{supp} \zeta_k$. Therefore,

$$\begin{aligned}
&\|(\partial_x^\gamma \zeta_k)\, u_l^-\|^p_{W^{0,p}_{\beta-2+|\gamma|,\vec{\delta}}(\mathcal{K})^\ell} \\
&\leq c\, 2^{k(p\beta-2)} \int\limits_{\substack{\mathcal{K}\\ 2^{k-1}<|x|<2^{k+1}}} \prod_j \left(\frac{r_j(x)}{|x|}\right)^{p\delta_j} \left(\int_{\mathcal{K}} |x-\xi|^{-1} \prod_j \left(\frac{r_j(\xi)}{|\xi|}\right)^{p\delta_j} \big|\zeta_l(\xi)\, f(\xi)\big|^p \, d\xi \right) dx \\
&\leq c\, 2^{-2k} \int_{\mathcal{K}} |\xi|^{p\beta} \prod_j \left(\frac{r_j(\xi}{|\xi|}\right)^{p\delta_j} \big|\zeta_l(\xi)\, f(\xi)\big|^p \Bigg(\int\limits_{\substack{\mathcal{K}\\ 2^{k-1}<|x|<2^{k+1}}} |x-\xi|^{-1} \prod_j \left(\frac{r_j(x)}{|x|}\right)^{p\delta_j} dx \Bigg) d\xi \\
&\leq c\, \|\zeta_l f\|^p_{W^{0,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell}.
\end{aligned}$$

We consider $(\partial_x^\gamma \zeta_k) \partial_x^\alpha u_l^-$ for $1 \leq |\alpha| \leq 2$. By our assumptions on $\vec{\delta}$, there exist real numbers s_j and t_j such that

$$\begin{aligned}
&\max\left(\delta_j\,,\ \frac{2-|\alpha|-\delta_{j,\alpha}}{q}\right) < s_j < \min\left(\delta_j + 2 - |\alpha| + \frac{|\alpha|+\delta_{j,\alpha}}{p}\,,\ \frac{2}{q}\,,\right), \\
&0 \leq t_j \leq 2 - |\alpha| \quad \text{and} \quad s_j - \delta_j - \frac{|\alpha|+\delta_{j,\alpha}}{p} < t_j < s_j - \delta_j,
\end{aligned}$$

where $\delta_{j,\alpha} = \min(0, \mu_j - |\alpha| - \varepsilon)$, $q = p/(p-1)$. Using (4.7), Hölder's inequality and (4.12), we obtain

$$\begin{aligned}
&\big|(\partial_x^\gamma \zeta_k)\partial_x^\alpha u_l^-(x)\big|^p \leq c\, 2^{-kp|\gamma|} \Bigg(\int\limits_{\substack{\mathcal{K}\\ |x-\xi|>r(x)/2}} |x-\xi|^{-1-|\alpha|} \prod_j \left(\frac{r_j(x)}{|x-\xi|}\right)^{\delta_{j,\alpha}} \big|\zeta_l(\xi)\, f(\xi)\big|^p \, d\xi \Bigg)^p \\
&\leq c\, 2^{-kp|\gamma|} \int\limits_{\substack{\mathcal{K}\\ |x-\xi|>r(x)/2}} |x-\xi|^{-1-|\alpha|} \prod_j \left(\frac{r_j(x)}{|x-\xi|}\right)^{\delta_{j,\alpha}} \prod_j \left(\frac{r_j(\xi)}{|\xi|}\right)^{ps_j} \big|\zeta_l(\xi)\, f(\xi)\big|^p \, d\xi \\
&\quad \times \Bigg(\int\limits_{\substack{|x|/16<|\xi|<16|x|\\ |x-\xi|>r(x)/2}} |x-\xi|^{-1-|\alpha|} \prod_j \left(\frac{r_j(x)}{|x-\xi|}\right)^{\delta_{j,\alpha}} \prod_j \left(\frac{r_j(\xi)}{|\xi|}\right)^{-qs_j} d\xi \Bigg)^{p-1} \\
&\leq c\, 2^{-kp|\gamma|} \left(|x|^{2-|\alpha|} \prod_j \left(\frac{r_j(x)}{|x|}\right)^{2-|\alpha|-qs_j} \right)^{p-1} \\
&\quad \times \int\limits_{\substack{\mathcal{K}\\ |x-\xi|>r(x)/2}} |x-\xi|^{-1-|\alpha|} \prod_j \left(\frac{r_j(x)}{|x-\xi|}\right)^{\delta_{j,\alpha}} \prod_j \left(\frac{r_j(\xi)}{|\xi|}\right)^{ps_j} \big|\zeta_l(\xi)\, f(\xi)\big|^p \, d\xi.
\end{aligned}$$

Consequently,

$$\|(\partial_x^\gamma \zeta_k)\partial_x^\alpha u_l^-\|^p_{W^{0,p}_{\beta-2+|\alpha|+|\gamma|,\vec\delta}(\mathcal{K})^\ell} \le c\, 2^{k(p\beta+|\alpha|-2)} \int\limits_{\substack{\mathcal{K}\\ 2^{k-1}<|x|<2^{k+1}}} \prod_j \Big(\frac{r_j(x)}{|x|}\Big)^{p(\delta_j-s_j)+(p-1)(2-|\alpha|)}$$

$$\times \Bigg(\int\limits_{\substack{\mathcal{K}\\ |x-\xi|>r(x)/2}} |x-\xi|^{-1-|\alpha|} \prod_j \Big(\frac{r_j(x)}{|x-\xi|}\Big)^{\delta_{j,\alpha}} \prod_j \Big(\frac{r_j(\xi)}{|\xi|}\Big)^{ps_j} \big|\zeta_l(\xi)\, f(\xi)\big|^p\, d\xi\Bigg)\, dx$$

$$\le c\, 2^{k(|\alpha|-2)} \int_{\mathcal{K}} |\xi|^{p\beta} \prod_j \Big(\frac{r_j(\xi)}{|\xi|}\Big)^{ps_j} \big|\zeta_l(\xi)\, f(\xi)\big|^p$$

$$\times \Bigg(\int\limits_{\substack{2^{k-1}<|x|<2^{k+1}\\ |x-\xi|>r(\xi)/3}} |x-\xi|^{-1-|\alpha|} \prod_j \Big(\frac{r_j(x)}{|x|}\Big)^{p(\delta_j-s_j+t_j)-2+|\alpha|} \prod_j \Big(\frac{r_j(x)}{|x-\xi|}\Big)^{\delta_{j,\alpha}} dx\Bigg)\, d\xi$$

$$\le c \int_{\mathcal{K}} |\xi|^{p\beta} \prod_j \Big(\frac{r_j(\xi)}{|\xi|}\Big)^{p(\delta_j+t_j)} \big|\zeta_l(\xi)\, f(\xi)\big|^p\, dx \le c\, \|\zeta_l f\|^p_{W^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell}.$$

This proves (4.17). Let $\eta_k = \zeta_{k-1}+\zeta_k+\zeta_{k+1}$. Since $L(u_l^+ + u_l^-) = \zeta_l f$ we have $\eta_k L u_l^+ = \eta_k\zeta_l f - \eta_k L u_l^- \in V^{0,p}_{\beta,\delta}(\mathcal{K})^\ell$ and

$$\|\eta_k L u_l^+\|_{V^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell} \le c\, \|\zeta_l f\|_{V^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell}. \tag{4.18}$$

Furthermore, $\eta_k B u_l^+ = -\eta_k B u_l^- \in W^{1-1/p,p}_{\beta,\delta}(\Gamma_j)^\ell$. This together with (4.16) yields $\eta_k B u_l^+ \in V^{1-1/p,p}_{\beta,\delta}(\Gamma_j)^\ell$ and

$$\|\eta_k B u_l^+\|_{V^{1-1/p,p}_{\beta,\delta}(\Gamma_j)^\ell} \le c\, \|\zeta_l f\|_{V^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell} \tag{4.19}$$

(see Lemma 2.2). From (4.15), (4.18), (4.19), and Lemma 3.1 we obtain

$$\|\zeta_k u_l^+\|_{V^{2,p}_{\beta,\vec\delta}(\mathcal{K})^\ell} \le c\, \|\zeta_l f\|_{V^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell}.$$

Here the constant c is independent of k (cf. proof of Lemma 4.3). The last inequality and (4.17) imply (4.14). □

The following lemma is proved in [14].

Lemma 4.8. *Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces of functions on $\mathcal{K}$ in each of them the multiplication with a scalar function from $C_0^\infty(\overline{\mathcal{K}}\backslash\{0\})$ is defined. We suppose that the inequalities*

$$\|f\|_{\mathcal{X}} \ge c_1 \Big(\sum_{k=-\infty}^{+\infty} \|\zeta_k f\|^p_{\mathcal{X}}\Big)^{1/p}, \qquad \|u\|_{\mathcal{Y}} \le c_2 \Bigg(\sum_{k=-\infty}^{+\infty} \|\zeta_k u\|^p_{\mathcal{Y}}\Bigg)^{1/p}$$

are satisfied for all $f \in \mathcal{X}$, $u \in \mathcal{Y}$. Furthermore, let $\mathcal{O}$ be a linear operator from $\mathcal{X}$ into $\mathcal{Y}$ defined on functions with compact support in $\overline{\mathcal{K}}\backslash\{0\}$ such that

$$\|\zeta_k \mathcal{O}\zeta_l f\|_{\mathcal{Y}} \le c_3\, 2^{-\sigma|k-l|}\, \|\zeta_l f\|_{\mathcal{X}}$$

with positive constants c, σ independent of k, l, and f. Then

$$\|\mathcal{O} f\|_{\mathcal{Y}} \le c\, \|f\|_{\mathcal{X}}$$

for all $f \in \mathcal{X}$ with compact support in $\overline{\mathcal{K}}\backslash\{0\}$.

As a consequence of Lemmas 4.5–4.8, we get the following statement.

Theorem 4.1. *Suppose that the line* $\operatorname{Re}\lambda = -\beta + 2 - 3/p$ *does not contain eigenvalues of the pencil $\mathfrak{A}$ and that the components δ_j of $\vec\delta$ satisfy condition* (4.10). *Then for arbitrary $f \in W^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell$ and $g_j \in W^{1-1/p,p}_{\beta,\vec\delta}(\Gamma_j)^\ell$ there exists a solution $u \in W^{2,p}_{\beta,\vec\delta}(\mathcal{K})^\ell$ of problem* (1.2), (1.4) *satisfying the estimate*

$$\|u\|_{W^{2,p}_{\beta,\vec\delta}(\mathcal{K})^\ell} \le c \Bigg(\|f\|_{W^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell} + \sum_{j=1}^n \|g_j\|_{W^{1-1/p,p}_{\beta,\vec\delta}(\Gamma_j)^\ell}\Bigg)$$

with a constant c independent of f and g.

P r o o f. From (4.10) it follows that $\delta_j > 1 - 2/p$ or $\mu_j > 1$. Hence, by Lemma 4.2, there exists a vector function $v \in W^{2,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ such that $Bv = g_j$ on Γ_j. Thus, we may assume, without loss of generality, that $g_j = 0$. Then $u = Sf$ is a solution of problem (1.2), (1.4). Since the assumptions on $\mathcal{X}$ and $\mathcal{Y}$ in Lemma 4.8 are satisfied for $\mathcal{X} = W^{0,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ and $\mathcal{Y} = W^{2,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$, it follows from Lemmas 4.5 and 4.7 that the operator S continuously maps $W^{0,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ into $W^{2,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$. This proves the theorem. □

4.6 Uniqueness of the solution to the Neumann problem

In the case $p \le 2$ we will prove the uniqueness of the solution in Theorem 4.1 by means of Corollary 2.1. For this end, we pass to the coordinates t, ω, where $t = \log\rho = \log|x|$ and $\omega = x/|x|$. We denote by $W^{l,p}_{\vec{\delta}}(\mathbb{R}\times\Omega)$ the weighted Sobolev space with the norm

$$\|u\|_{W^{l,p}_{\vec{\delta}}(\mathbb{R}\times\Omega)} = \Big(\int_{\mathbb{R}} \sum_{j=0}^{l} \|\partial_t^j u(t,\cdot)\|^p_{W^{l-j,p}_{\vec{\delta}}(\Omega)}\, dt\Big)^{1/p}.$$

For an arbitrary function $v \in W^{l,p}_{\vec{\delta}}(\mathbb{R}\times\Omega)$ we define by v_ε the mollification with respect to the variable t of v, i.e.,

$$v_\varepsilon(t,\omega) = \int_{\mathbb{R}} v(\tau,\omega)\, h_\varepsilon(t-\tau)\, d\tau,$$

where $h_\varepsilon(t) = \varepsilon^{-1}h(t/\varepsilon)$ and h is a smooth function with compact support, $\int h(t)\,dt = 1$. Since

$$\partial_\omega^\alpha \partial_t^{j+k} v_\varepsilon(\omega,t) = \int_{\mathbb{R}} (\partial_\omega^\alpha \partial_t^k v)(\omega,\tau)\, h_\varepsilon^{(j)}(t-\tau)\, d\tau,$$

it follows that $\partial_t^j v_\varepsilon \in W^{l,p}_{\vec{\delta}}(\mathbb{R}\times\Omega)$ for $v \in W^{l,p}_{\vec{\delta}}(\mathbb{R}\times\Omega)$, $\varepsilon > 0$, $j = 0, 1, \ldots$.

Lemma 4.9. *Let* $1 < p \le 2$, *and let the conditions of Theorem* 4.1 *be satisfied. Then the homogeneous boundary value problem* (1.2), (1.4) *has only the trivial solution* $u = 0$ *in* $W^{2,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$.

P r o o f. Since $W^{2,p}_{\beta,\vec{\delta}}(\mathcal{K}) \subset W^{2,p}_{\beta,\vec{\delta}'}(\mathcal{K})$ if $\delta_j \le \delta'_j$ for $j = 1, \ldots, n$, it suffices to prove the lemma for the case when $\max(2-\mu_j, 1) < \delta_j + 2/p < 2$.

Let $u \in W^{2,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ be a solution of the homogeneous problem (1.2), (1.4). From Lemma 4.4 it follows that $u \in W^{3,p}_{\beta+1,\vec{\delta}+\vec{1}}(\mathcal{K})^\ell$. We set $v = \rho^{\beta-2+3/p}u$. Then, in the coordinates (t,ω), where $t = \log\rho$, $\rho = |x|$, $\omega = x/|x|$, we have $v \in W^{3,p}_{\vec{\delta}+\vec{1}}(\mathbb{R}\times\Omega)^\ell$ and, therefore, $\partial_t^j v_\varepsilon \in W^{3,p}_{\vec{\delta}+\vec{1}}(\mathbb{R}\times\Omega)^\ell$ for $j = 0, 1, 2, \ldots$. Furthermore, both v and v_ε are solutions of the problem

$$\mathcal{L}(\omega, \partial_\omega, \partial_t - \beta + 2 - 3/p)\, v = 0, \quad t \in \mathbb{R},\ \omega \in \Omega,$$
$$\mathcal{B}(\omega, \partial_\omega, \partial_t - \beta + 2 - 3/p)\, v = 0, \quad t \in \mathbb{R},\ \omega \in \gamma_j,\ j = 1, \ldots, n,$$

where $\mathcal{L}$, $\mathcal{B}$ are defined by (1.7). Using Corollary 2.1, we get $v_\varepsilon \in W^{1,2}_{\vec{\delta}'}(\mathbb{R}\times\Omega)^\ell$, where $\delta'_j = \delta_j - 2 + 2/p < 0$, i.e., $v_\varepsilon \in W^{1,2}_{\vec{0}}(\mathbb{R}\times\Omega)^\ell$. Consequently, the function $u_\varepsilon = \rho^{-\beta+2-3/p}v_\varepsilon$ belongs to $W^1_{\beta-1+3/p-3/2,\vec{0}}(\mathcal{K})^\ell$ (in Cartesian coordinates). Since u_ε is also a solution of the homogeneous problem (1.2), (1.4), we conclude from [16, Th. 4.3] that $u_\varepsilon = 0$ for all $\varepsilon > 0$. This implies $u = 0$. □

Theorem 4.2. *Suppose that the line* $\operatorname{Re}\lambda = -\beta + 2 - 3/p$ *does not contain eigenvalues of the pencil* $\mathfrak{A}$, *and the components* δ_j *of* $\vec{\delta}$ *satisfy condition* (4.10). *Then problem* (1.2), (1.4) *is uniquely solvable in* $W^{2,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ *for arbitrary* $f \in W^{0,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ *and* $g_j \in W^{1-1/p,p}_{\beta,\vec{\delta}}(\Gamma_j)^\ell$.

P r o o f. For $1 < p \le 2$ the assertion follows immediately from Theorem 4.1 and Lemma 4.9. Let $p > 2$ and let $u \in W^{2,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ be a solution of the homogeneous problem (1.2), (1.4). By ϕ we denote a smooth cut-off function on $\overline{\mathcal{K}}$ equal to one for $|x| < 1$ and to zero for $|x| > 2$. Furthermore, we set $\beta' = \beta - \frac{3}{2} + \frac{3}{p}$ and $\delta'_j = \delta_j - 1 + \frac{2}{p}$ for $j = 1, \ldots, n$. Then, by Hölder's inequality,

$$\int_{\mathcal{K}} \rho^{2(\beta'+\varepsilon-2+|\alpha|)} \prod \Big(\frac{r_j}{\rho}\Big)^{2(\delta'_j+\varepsilon)} |\partial_x^\alpha(\phi u)|^2\, dx$$
$$\le \Big(\int_{\mathcal{K}} \rho^{p(\beta-2+|\alpha|)} \prod \Big(\frac{r_j}{\rho}\Big)^{p\delta_j} |\partial_x^\alpha(\phi u)|^p\, dx\Big)^{2/p} \Big(\int\limits_{\substack{\mathcal{K}\\ |x|\le 2}} \rho^{-3+q\varepsilon} \prod \Big(\frac{r_j}{\rho}\Big)^{-2+q\varepsilon} dx\Big)^{2/q},$$

where $\frac{2}{p} + \frac{2}{q} = 1$. The second integral on the right is finite if $\varepsilon > 0$. Consequently, $\phi u \in W^{2,2}_{\beta'+\varepsilon,\vec{\delta}+\varepsilon\vec{1}}(\mathcal{K})^\ell$. Analogously, we obtain $(1-\phi)u \in W^{2,2}_{\beta'-\varepsilon,\vec{\delta}-\varepsilon\vec{1}}(\mathcal{K})^\ell$. This implies

$$L(\phi u) = -L\big((1-\phi)u\big) \in W^{0,2}_{\beta'-\varepsilon,\vec{\delta}-\varepsilon\vec{1}}(\mathcal{K})^\ell \quad \text{and} \quad B(\phi u)|_{\Gamma_j} \in W^{1/2,2}_{\beta'-\varepsilon,\vec{\delta}-\varepsilon\vec{1}}(\Gamma_j)^\ell.$$

From this and from [16, Th. 4.2] it follows that ϕu and, therefore, also u belong to $W^{2,2}_{\beta'-\varepsilon,\vec{\delta}-\varepsilon\vec{1}}(\mathcal{K})^\ell$. Hence, by Theorem [16, Th. 4.1], $u = 0$. The proof of the theorem is complete. □

The solution in Theorem 4.1 was constructed by means of the Green function introduced in Sect. 4.4 with $\kappa = \beta + \frac{3}{p} - \frac{5}{2}$. In fact, we can suppose that κ is an arbitrary real number such that $\Lambda_- < -\kappa - 1/2 < \Lambda_+$, where $\Lambda_- < \operatorname{Re}\lambda < \Lambda_+$ is the widest strip in the complex plane which contains the line $\operatorname{Re}\lambda = 2 - \beta - 3/p$ and is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$. This implies the following regularity assertion for the solution u.

Theorem 4.3. *Suppose that there are no eigenvalues of the pencil $\mathfrak{A}$ in the closed strip between the lines $\operatorname{Re}\lambda = 2-\beta-3/p$ and $\operatorname{Re}\lambda = 2 - \beta' - 3/p'$, and that the components of $\vec{\delta}$, $\vec{\delta}'$ satisfy the inequalities $\max(2-\mu_j, 0) < \delta_j + 2/p < 2$, $\max(2-\mu_j, 0) < \delta'_j + 2/p' < 2$ for $j = 1, \dots, n$. If*

$$f \in W^{0,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell \cap W^{0,p'}_{\beta',\vec{\delta}'}(\mathcal{K})^\ell \ \ \textit{and} \ \ g_j \in W^{1-1/p,p}_{\beta,\vec{\delta}}(\Gamma_j)^\ell \cap W^{1-1/p',p'}_{\beta',\vec{\delta}'}(\Gamma_j)^\ell$$

for $j = 1, \dots, n$, then the solution $u \in W^{2,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ of problem (1.2)–(1.4) belongs to the space $W^{2,p'}_{\beta',\vec{\delta}'}(\mathcal{K})^\ell$.

Remark 4.3. *If the assumptions (i)–(iii) of Remark 4.2 are satisfied for some j, then $\mu_j = 1$, and in the conditions $\max(2-\mu_j, 0) < \delta_j + 2/p < 2$, $\max(2-\mu_j, 0) < \delta'_j + 2/p' < 2$ of Theorems 4.2 and 4.3 the number μ_j can be replaced by the real part $\mu_j^{(2)}$ of the first eigenvalue of the pencil $A_j(\lambda)$ on the right of the line $\operatorname{Re}\lambda = 1$. However, then the boundary data must satisfy a compatibility condition on the edge M_j (see Sect. 4.4) if $\delta_j + 2/p \le 1$ and $\delta'_j + 2/p' \le 1$, respectively.*

4.7 Solvability of the Dirichlet and mixed problems

We consider now problem (1.2)–(1.4), where $f \in W^{0,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$, $g_j \in \mathcal{W}^{1+d_j-1/p,p}_{\beta,\vec{\delta}}(\Gamma_j; \tilde{J})^\ell$ for $j = 1, \dots, n$. Here

$$d_j = 1 \ \text{ for } j \in J_0, \quad d_j = 0 \text{ for } j \in J_1.$$

Due to Lemma 4.2, we can restrict ourselves to the case $g_j = 0$. Then the solution of problem (1.2)–(1.4) is given by

$$u(x) = \int_{\mathcal{K}} G(x,\xi)\, f(\xi)\, d\xi,$$

where $G(x,\xi)$ is Green 's matrix introduced in Sect. 4.4 with $\kappa = \beta + \frac{3}{p} - \frac{5}{2}$. The following theorem can be proved in the same way as for the Neumann problem by means of the estimates (4.6)–(4.9).

Theorem 4.4. *Let $f \in W^{0,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$, $g_j \in \mathcal{W}^{1+d_j-1/p,p}_{\beta,\vec{\delta}}(\Gamma_j; \tilde{J})^\ell$ for $j = 1, \dots, n$. Suppose that the line $\operatorname{Re}\lambda = 2 - \beta - 3/p$ does not contain eigenvalues of the pencil $\mathfrak{A}$ and that the components of $\vec{\delta}$ satisfy the inequalities*

$$2 - \mu_j < \delta_j + 2/p < 2 \ \textit{ for } j \in \tilde{J}, \quad \max(2-\mu_j, 0) < \delta_j + 2/p < 2 \ \textit{ for } j \in J\backslash\tilde{J}. \tag{4.20}$$

Then problem (1.2)–(1.4) has a unique solution $u \in \mathcal{W}^{2,p}_{\beta,\vec{\delta}}(\mathcal{K}; \tilde{J})^\ell$.

Furthermore, a regularity assertion analogous to Theorem 4.3 holds.

5 Weak solutions of the boundary value problem

5.1 Existence of weak solutions to the Neumann problem

Let $V^{l,2}_\beta(\mathcal{K}) = W^{l,2}_{\beta,\vec{0}}(\mathcal{K})$ be the closure of the set $C_0^\infty(\overline{\mathcal{K}}\backslash\{0\})$ with respect to the norm

$$\|u\|_{V^{l,2}_\beta(\mathcal{K})} = \Big(\int_{\mathcal{K}} \sum_{|\alpha|\le l} \rho^{2(\beta-l+|\alpha|)} \big|\partial_x^\alpha u(x)\big|^2 dx\Big)^{1/2}$$

and let $V_\beta^{-1,2}(\mathcal{K})$ be the dual space of $V_{-\beta}^{1,2}(\mathcal{K})$. It can be shown (cf. [1, Th. 3.8]) that every functional $F \in V_\beta^{-1,2}(\mathcal{K})$ has the form

$$F(v) = \int_{\mathcal{K}} \bar{f}_0\, v\, dx + \sum_{j=1}^{3} \int_{\mathcal{K}} \bar{f}_j\, \partial_{x_j} v\, dx \tag{5.1}$$

for $v \in V_{-\beta}^{1,2}(\mathcal{K})$, where $f_0 \in V_{\beta+1}^{0,2}(\mathcal{K})$ and $f_j \in V_\beta^{0,2}(\mathcal{K})$ for $j = 1,2,3$. In [16, Th. 4.3] it was proved that the problem

$$b_{\mathcal{K}}(u,v) = \overline{F(v)} \qquad \text{for all } v \in V_{-\beta}^{1,2}(\mathcal{K}) \tag{5.2}$$

has a unique solution $u \in V_\beta^{1,2}(\mathcal{K})^\ell$ for arbitrary $F \in V_\beta^{-1,2}(\mathcal{K})^\ell$ if the line $\operatorname{Re}\lambda = -\beta - 1/2$ does not contain eigenvalues of the pencil $\mathfrak{A}(\lambda)$.

Now we are interested in weak solutions of the Neumann problem in the weighted Sobolev space $W_{\beta,\vec{\delta}}^{1,p}(\mathcal{K})^\ell$. We denote by $V_{\beta,\vec{\delta}}^{-1,p}(\mathcal{K})$ the dual space of $V_{-\beta,-\vec{\delta}}^{1,q}(\mathcal{K})$, $q = p/(p-1)$, or, what is the same, the set of all functionals on $V_{-\beta,-\vec{\delta}}^{1,q}(\mathcal{K})$ which have the form (5.1) with $f_0 \in V_{\beta+1,\vec{\delta}+\vec{1}}^{0,p}(\mathcal{K})$ and $f_j \in V_{\beta,\vec{\delta}}^{0,p}(\mathcal{K})$ for $j = 1,2,3$. Note that $V_{-\beta,-\vec{\delta}}^{1,q}(\mathcal{K}) = W_{-\beta,-\vec{\delta}}^{1,q}(\mathcal{K})$ if the components δ_j of $\vec{\delta}$ are less than $1 - 2/p$. Hence $V_{\beta,\vec{\delta}}^{-1,p}(\mathcal{K})$ is also the dual space of $W_{-\beta,-\vec{\delta}}^{1,q}(\mathcal{K})$ if $\delta_j < 1 - 2/p$ for $j = 1,\dots,n$.

Our goal is to prove that for arbitrary $F \in V_{\beta,\vec{\delta}}^{-1,p}(\mathcal{K})^\ell$ there exists a unique $u \in V_{\beta,\vec{\delta}}^{1,p}(\mathcal{K})^\ell$ satisfying

$$b_{\mathcal{K}}(u,v) = \overline{F(v)} \qquad \text{for all } v \in V_{-\beta,-\vec{\delta}}^{1,q}(\mathcal{K}). \tag{5.3}$$

To this end, we consider the vector function

$$u(x) = \int_{\mathcal{K}} G(x,\xi)\, f_0(\xi)\, d\xi + \sum_{j=1}^{3} \int_{\mathcal{K}} \partial_{\xi_j} G(x,\xi)\, f_j(\xi)\, d\xi, \tag{5.4}$$

where $f_0 \in V_{\beta+1,\vec{\delta}+\vec{1}}^{0,p}(\mathcal{K})^\ell$, $f_j \in V_{\beta,\vec{\delta}}^{0,p}(\mathcal{K})^\ell$ for $j = 1,2,3$, and $G(x,\xi)$ is the Green matrix introduced in Sect. 4.4 with $\kappa = \beta + \frac{3}{p} - \frac{3}{2}$. Our goal is to show that the vector function (5.4) belongs to $V_{\beta,\vec{\delta}}^{1,p}(\mathcal{K})^\ell$ if there are no eigenvalues of the pencil $\mathfrak{A}$ on the line $\operatorname{Re}\lambda = -\beta + 1 - 3/p$ and the components of $\vec{\delta}$ satisfy the inequalities

$$\max(1-\mu_j, 0) < \delta_j + 2/p < 1 \quad \text{for } j = 1,\dots,n. \tag{5.5}$$

Let the operator S_j be defined on $V_{\beta,\vec{\delta}}^{0,p}(\mathcal{K})^\ell$ by

$$(S_j f)(x) = \int_{\mathcal{K}} \partial_{\xi_j} G(x,\xi)\, f(\xi)\, d\xi. \tag{5.6}$$

Lemma 5.1. *Let ζ_k, $k = 0, \pm 1, \dots$, be smooth functions on $\mathcal{K}$ depending only on $\rho = |x|$ such that* (2.3) *is satisfied. Suppose that the line* $\operatorname{Re}\lambda = -\beta + 1 - 3/p$ *does not contain eigenvalues of the pencil $\mathfrak{A}$ and that for the components of $\vec{\delta}$ the inequalities* (5.5) *are valid. Then for arbitrary $f \in V_{\beta,\vec{\delta}}^{0,p}(\mathcal{K})^\ell$, $|\alpha| \le 1$, and $|l-k| \ge 3$ we have*

$$\|\zeta_k \partial_x^\alpha S_j \zeta_l f\|_{V_{\beta-1+|\alpha|,\vec{\delta}}^{0,p}(\mathcal{K})^\ell} \le c\, 2^{-|k-l|\sigma}\, \|\zeta_l f\|_{V_{\beta,\vec{\delta}}^{0,p}(\mathcal{K})^\ell}$$

with positive constants c and σ independent of k, l, and f.

P r o o f. By means of Hölder's inequality, we get

$$\begin{aligned}
&\|\zeta_k \partial_x^\alpha S_j \zeta_l f\|^p_{V_{\beta-1+|\alpha|,\vec{\delta}}^{0,p}(\mathcal{K})^\ell} \\
&= \int_{\mathcal{K}} |x|^{p(\beta-1+|\alpha|)} \prod \left(\frac{r_j(x)}{|x|}\right)^{p\delta_j} |\zeta_k(x)|^p \left| \int_{\mathcal{K}} \partial_x^\alpha \partial_{\xi_j} G(x,\xi)\, \zeta_l(\xi)\, f(\xi)\, d\xi \right|^p dx \\
&\le \|\zeta_l f\|^p_{V_{\beta,\vec{\delta}}^{0,p}(\mathcal{K})^\ell} \int_{\mathcal{K}} |x|^{p(\beta-1+|\alpha|)} \prod \left(\frac{r_j(x)}{|x|}\right)^{p\delta_j} |\zeta_k(x)|^p \\
&\qquad \times \Bigg(\int\limits_{\substack{\mathcal{K} \\ 2^{l-1}<|\xi|<2^{l+1}}} |\xi|^{-q\beta} \prod \left(\frac{r_j(\xi)}{|\xi|}\right)^{-q\delta_j} \left|\partial_x^\alpha \partial_{\xi_j} G(x,\xi)\right|^q d\xi \Bigg)^{p/q} dx,
\end{aligned}$$

where $q = p/(p-1)$. Let $k \ge l+3$. Then $|\xi| < |x|/2$ for $x \in \operatorname{supp}\zeta_k$, $\xi \in \operatorname{supp}\zeta_l$ and (4.9) implies

$$\|\zeta_k \partial_x^\alpha S\zeta_l f\|^p_{V^{0,p}_{\beta-1+|\alpha|,\vec\delta}(\mathcal{K})^\ell} \le \|\zeta_l f\|^p_{V^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell} \int\limits_{\substack{\mathcal{K}\\ 2^{k-1}<|x|<2^{k+1}}} |x|^{p(\beta-1+\Lambda_-+\varepsilon)} \prod \Big(\frac{r_j(x)}{|x|}\Big)^{p(\delta_j+\delta_{j,\alpha})}\, dx$$

$$\times \Bigg(\int\limits_{\substack{\mathcal{K}\\ 2^{l-1}<|\xi|<2^{l+1}}} |\xi|^{-q(\beta+2+\Lambda_-+\varepsilon)} \prod \Big(\frac{r_j(\xi)}{|\xi|}\Big)^{q(\delta_{j,1}-\delta_j)}\, d\xi\Bigg)^{p/q},$$

where $\delta_{j,\alpha} = \min(0,\mu_j - |\alpha| - \varepsilon)$, $\delta_{j,1} = \min(0,\mu_j - 1 - \varepsilon)$. Under our assumptions on $\vec\delta$, we have $p(\delta_j + \delta_{j,\alpha}) > -2$ for $|\alpha| \le 1$ and $q(\delta_{j,1} - \delta_j) > -2$. Consequently,

$$\|\zeta_k \partial_x^\alpha S_j\zeta_l f\|^p_{V^{0,p}_{\beta-1+|\alpha|,\vec\delta}(\mathcal{K})^\ell} \le 2^{kp(\beta-1+3/p+\Lambda_-+\varepsilon)}\, 2^{lp(-\beta+1-3/p-\Lambda_--\varepsilon)}\, \|\zeta_l f\|^p_{V^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell}.$$

This proves the lemma for $k \ge l+3$. Analogously, it can be proved for $k \le l-3$. □

Lemma 5.2. *Let ζ_k, β, and $\vec\delta$ be as in Lemma 5.1. Then for arbitrary $f \in V^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell$ and $|l-k| \le 2$ we have*

$$\|\zeta_k S_j \zeta_l f\|_{W^{1,p}_{\beta,\vec\delta}(\mathcal{K})^\ell} \le c\, \|\zeta_l f\|_{V^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell} \tag{5.7}$$

with a constant c independent of k, l, and f.

Proof. Let $\chi^\pm$ be the same functions as in the proof of Lemma 4.7, and let

$$u_l^\pm(x) = \int_{\mathcal{K}} \chi^\pm(x,\xi)\, \partial_{\xi_j} G(x,\xi)\, \zeta_l(\xi)\, f(\xi)\, d\xi.$$

Then $S_j\zeta_l f = u_l^+ + u_l^-$. We show first that

$$\|\zeta_k u_l^+\|_{V^{0,p}_{\beta-1,\vec\delta-\vec 1}(\mathcal{K})^\ell} \le c\, \|\zeta_l f\|_{V^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell}. \tag{5.8}$$

From Hölder's inequality and (4.6) it follows that

$$\Big|\int_{\mathcal{K}} \chi^+(x,\xi)\, \partial_{\xi_j} G(x,\xi)\, \zeta_l(\xi)\, f(\xi)\, d\xi\Big|^p \le \int_{\mathcal{K}} |x-\xi|^{-2}\, \big|\chi^+ \zeta_l(\xi)\, f(\xi)\big|^p\, d\xi\, \Bigg(\int\limits_{|x-\xi|<r(x)} |x-\xi|^{-2}\, d\xi\Bigg)^{p-1}$$

$$\le c\, r(x)^{p-1} \int_{\mathcal{K}} |x-\xi|^{-2}\, \big|\chi^+(x,\xi)\, \zeta_l(\xi)\, f(\xi)\big|^p\, d\xi$$

for $x \in \operatorname{supp}\zeta_k$. Since $2^{-4}|x| \le |\xi| \le 2^4|x|$ and $c_1\, r_i(x) \le r_i(\xi) \le c_2 r_i(x)$ for $x \in \operatorname{supp}\zeta_k$, $\xi \in \operatorname{supp}\zeta_l \cap \operatorname{supp}\chi^+(x,\cdot)$, the last inequality implies

$$\|\zeta_k u_l^+\|^p_{V^{0,p}_{\beta-1,\vec\delta-\vec 1}(\mathcal{K})^\ell}$$

$$\le c \int_{\mathcal{K}} |x|^{p(\beta-1)} \prod_{i=1}^n \Big(\frac{r_i(x)}{|x|}\Big)^{p(\delta_i-1)} |\zeta_k(x)|^p\, r(x)^{p-1} \Big(\int_{\mathcal{K}} |x-\xi|^{-2}\, \big|\chi^+(x,\xi)\, \zeta_l(\xi)\, f(\xi)\big|^p\, d\xi\Big)\, dx$$

$$\le c \int_{\mathcal{K}} |\xi|^{p(\beta-1)} \prod_{i=1}^n \Big(\frac{r_i(\xi)}{|\xi|}\Big)^{p(\delta_i-1)} r(\xi)^{p-1}\, \big|\zeta_l(\xi)\, f(\xi)\big|^p \Bigg(\int\limits_{|x-\xi|<3r(\xi)} |x-\xi|^{-2}\, dx\Bigg)\, d\xi$$

$$\le c \int_{\mathcal{K}} |\xi|^{p\beta} \prod_{i=1}^n \Big(\frac{r_i(\xi)}{|\xi|}\Big)^{p\delta_i} \big|\zeta_l(\xi)\, f(\xi)\big|^p\, d\xi = c\, \|\zeta_l f\|^p_{W^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell}.$$

Here we used the inequality $r(x) \le c\,|x| \prod_i r_i(x/|x|)$. Next we show that

$$\|\zeta_k u_l^-\|_{W^{1,p}_{\beta,\vec\delta}(\mathcal{K})^\ell} \le c\, \|\zeta_l f\|_{V^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell}. \tag{5.9}$$

By our assumptions on $\vec\delta$, there exist real numbers s_i such that

$$\max\Big(-\frac{\delta_{i,1}}{q}\,,\ \delta_i - \frac{\delta_{i,1}}{p}\Big) < s_i < \min\Big(\frac{2+\delta_{i,1}}{q}\,,\ \delta_i + \frac{2+\delta_{i,1}}{p}\Big),$$

where $\delta_{i,1} = \min(0, \mu_i - 1 - \varepsilon)$, $q = p/(p-1)$. Using (4.7), Hölder's inequality and (4.12), we obtain

$$\big|\zeta_k(x)\partial_{x_\nu} u_l^-(x)\big|^p \le c\,|\zeta_k(x)|^p \Bigg(\int\limits_{\substack{\mathcal{K}\\ |x-\xi|>r(x)/2}} |x-\xi|^{-3} \prod_i \left(\frac{r_i(x)}{|x-\xi|}\right)^{\delta_{i,1}} \prod_i \left(\frac{r_i(\xi)}{|x-\xi|}\right)^{\delta_{i,1}} |\zeta_l f|^p \, d\xi \Bigg)^p$$

$$\le c\,|\zeta_k(x)|^p \int\limits_{\substack{\mathcal{K}\\ |x-\xi|>r(x)/2}} |x-\xi|^{-3} \prod_i \left(\frac{r_i(x)}{|x-\xi|}\right)^{\delta_{i,1}} \prod_i \left(\frac{r_i(\xi)}{|x-\xi|}\right)^{\delta_{i,1}} \prod_i \left(\frac{r_i(\xi)}{|\xi|}\right)^{ps_i} |\zeta_l f|^p \, d\xi$$

$$\times \Bigg(\int\limits_{\substack{|x|/16<|\xi|<16|x|\\ |x-\xi|>r(x)/2}} |x-\xi|^{-3} \prod_i \left(\frac{r_i(x)}{|x-\xi|}\right)^{\delta_{i,1}} \prod_i \left(\frac{r_i(\xi)}{|x-\xi|}\right)^{\delta_{i,1}} \prod_i \left(\frac{r_i(\xi)}{|\xi|}\right)^{-qs_i} d\xi \Bigg)^{p-1}$$

$$\le c\,|\zeta_k(x)|^p \prod_{i=1}^n \left(\frac{r_i(x)}{|x|}\right)^{-ps_i} \int\limits_{\substack{\mathcal{K}\\ |x-\xi|>r(x)/2}} \prod_i \left(\frac{r_i(x)}{|x-\xi|}\right)^{\delta_{i,1}} \prod_i \left(\frac{r_i(\xi)}{|x-\xi|}\right)^{\delta_{i,1}} \prod_i \left(\frac{r_i(\xi)}{|\xi|}\right)^{ps_i} \frac{|\zeta_l f|^p}{|x-\xi|^3} \, d\xi.$$

Consequently,

$$\|\zeta_k \partial_{x_\nu} u_l^-\|^p_{W^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell} \le c \int_{\mathcal{K}} |x|^{p\beta} \prod_{i=1}^n \left(\frac{r_i(x)}{|x|}\right)^{p(\delta_i - s_i)} |\zeta_k(x)|^p$$

$$\times \Bigg(\int\limits_{\substack{\mathcal{K}\\ |x-\xi|>r(x)/2}} |x-\xi|^{-3} \prod_i \left(\frac{r_i(x)}{|x-\xi|}\right)^{\delta_{i,1}} \prod_i \left(\frac{r_i(\xi)}{|x-\xi|}\right)^{\delta_{i,1}} \prod_i \left(\frac{r_i(\xi)}{|\xi|}\right)^{ps_i} \big|\zeta_l(\xi)\, f(\xi)\big|^p \, d\xi \Bigg) dx$$

$$\le c \int_{\mathcal{K}} |\xi|^{p\beta} \prod_{i=1}^n \left(\frac{r_i(\xi)}{|\xi|}\right)^{ps_i} \big|\zeta_l(\xi)\, f(\xi)\big|^p$$

$$\times \Bigg(\int\limits_{\substack{\mathcal{K}\\ |x-\xi|>r(\xi)/3}} |\zeta_k(x)|^p\, |x-\xi|^{-3} \prod_i \left(\frac{r_i(x)}{|x-\xi|}\right)^{\delta_{i,1}} \prod_i \left(\frac{r_i(\xi)}{|x-\xi|}\right)^{\delta_{i,1}} \prod_i \left(\frac{r_i(x)}{|x|}\right)^{p(\delta_i - s_i)} dx \Bigg) d\xi.$$

Estimating the inner integral on the right by means of (4.12), we obtain

$$\|\zeta_k \partial_{x_\nu} u_l^-\|_{W^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell} \le c\,\|\zeta_l f\|_{V^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell}.$$

Analogously, the inequalities

$$\|u_l^-\, \partial_{x_\nu} \zeta_k\|_{W^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell} \le c\,\|\zeta_l f\|_{V^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell} \quad \text{and} \quad \|\zeta_k u_l^-\|_{W^{0,p}_{\beta-1,\vec\delta}(\mathcal{K})^\ell} \le c\,\|\zeta_l f\|_{V^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell}$$

can be shown. This proves (5.9). By the definition of u_l^+ and u_l^-, we have

$$b_{\mathcal{K}}(u_l^+ + u_l^-\,,\, v) = \int_{\mathcal{K}} \zeta_l\, f \cdot \partial_{x_j} \overline{v}\, dx$$

for all $v \in C_0^\infty(\overline{\mathcal{K}}\backslash\mathcal{S})^\ell$ and, therefore,

$$b_{\mathcal{K}}(u_l^+, v) = F(v) \ \text{ for all } v \in C_0^\infty(\overline{\mathcal{K}}\backslash\mathcal{S})^\ell, \ \text{ where } F(v) = \int_{\mathcal{K}} \zeta_l\, f \cdot \partial_{x_j} \overline{v}\, dx - b_{\mathcal{K}}(u_l^-, v).$$

Let $\eta_k = \zeta_{k-1} + \zeta_k + \zeta_{k+1}$. Then $\eta_k u_l^+ \in V^{0,p}_{\beta-1,\vec\delta-1}(\mathcal{K})^\ell$ and $\eta_k F \in V^{-1,p}_{\beta,\vec\delta}(\mathcal{K})^\ell$ for $|l-k| \le 2$. Analogously to Lemma 4.3, it can be proved that $\zeta_k u_l^+ \in V^{1,p}_{\beta,\vec\delta}(\mathcal{K})^\ell$ and

$$\|\zeta_k u_l^+\|_{V^{1,p}_{\beta,\vec\delta}(\mathcal{K})^\ell} \le c \left(\|\eta_k u_l^+\|_{V^{0,p}_{\beta-1,\vec\delta-1}(\mathcal{K})^\ell} + \|\eta_k F\|_{V^{-1,p}_{\beta,\vec\delta}(\mathcal{K})^\ell} \right).$$

Here the constant c is independent of k. From this and from (5.8) and (5.9) we conclude that

$$\|\zeta_k u_l^+\|_{V^{1,p}_{\beta,\vec\delta}(\mathcal{K})^\ell} \le c\,\|\zeta_l f\|_{V^{0,p}_{\beta,\vec\delta}(\mathcal{K})^\ell}.$$

The last inequality together with (5.9) implies (5.7). □

Now we can prove the existence of weak solutions to the Neumann problem.

Theorem 5.1. *Suppose that there are no eigenvalues of the pencil* $\mathfrak{A}$ *on the line* $\mathrm{Re}\,\lambda = 1-\beta-3/p$ *and the components of* $\vec{\delta}$ *satisfy* (5.5)*. Then problem* (5.3) *has a unique solution* $u \in W^{1,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ *for arbitrary* $F \in V^{-1,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$.

P r o o f. Let F be a functional of the form (5.1), where $f_0 \in V^{0,p}_{\beta+1,\vec{\delta}+\vec{1}}(\mathcal{K})^\ell$ and $f_j \in V^{0,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ for $j=1,2,3$. By Lemmas 4.5–4.8, the mapping

$$V^{0,p}_{\beta+1,\vec{\delta}+\vec{1}}(\mathcal{K})^\ell \ni f_0 \to u(\cdot) = \int_{\mathcal{K}} G(\cdot,\xi)\, f_0(\xi)\, d\xi \in W^{1,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$$

is continuous. Furthermore, it follows from Lemmas 4.8, 5.1, and 5.2 that the mappings

$$V^{0,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell \ni f_j \to u(\cdot) = \int_{\mathcal{K}} \partial_{\xi_j} G(\cdot,\xi)\, f_j(\xi)\, d\xi \in W^{1,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell, \quad j=1,2,3,$$

are continuous. Hence, the vector-function (5.4) is a solution of problem (5.3) in the space $V^{1,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$. The uniqueness of the solution follows from Lemma 4.4 and Theorem 4.2. □

Remark 5.1. It can be proved analogously to Theorem 4.1 that (5.4) defines also a continuous mapping

$$(f_0,f_1,f_2,f_3) \ni V^{0,2}_{\beta+1}(\mathcal{K})^\ell \times V^{0,2}_{\beta}(\mathcal{K})^{3\ell} \to u \in V^{1,2}_{\beta}(\mathcal{K})^\ell$$

if the line $\mathrm{Re}\,\lambda = -\beta-1/2$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$ and $G(x,\xi)$ is the Green matrix introduced in Sect. 4.4 with $\kappa=\beta$. Hence the unique solution $u \in V^{1,2}_\beta(\mathcal{K})^\ell$ of problem (5.2) (see [16, Th. 4.3]) has the representation (5.4) if the functional $F \in V^{-1,2}_\beta(\mathcal{K})^\ell$ is given by (5.1) with $f_0 \in V^{0,2}_{\beta+1}(\mathcal{K})$ and $f_j \in V^{0,2}_\beta(\mathcal{K})$, $j=1,2,3$.

5.2 Regularity results for weak solutions to the Neumann problem

Theorem 5.2. *Suppose that there are no eigenvalues of the pencil* $\mathfrak{A}$ *in the closed strip between the lines* $\mathrm{Re}\,\lambda = 1-\beta-3/p$ *and* $\mathrm{Re}\,\lambda = 1-\beta'-3/p'$ *and that the components of* $\vec{\delta}$ *and* $\vec{\delta}'$ *satisfy the inequalities* $\max(1-\mu_j,0) < \delta_j + 2/p < 1$, $\max(1-\mu_j,0) < \delta'_j + 2/p' < 1$, $j=1,\dots,n$. *If* $u \in W^{1,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ *is a solution of problem* (5.3)*, where* $F \in V^{-1,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell \cap V^{-1,p'}_{\beta',\vec{\delta}'}(\mathcal{K})^\ell$*, then* $u \in W^{1,p'}_{\beta',\vec{\delta}'}(\mathcal{K})^\ell$.

P r o o f. By Theorem 5.1, problem (5.3) is solvable in $W^{1,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ and $W^{1,p'}_{\beta',\vec{\delta}'}(\mathcal{K})^\ell$. Both solutions coincide, since they are given by (5.4), where $f \in V^{0,p}_{\beta+1,\vec{\delta}+\vec{1}}(\mathcal{K})^\ell \cap V^{0,p'}_{\beta'+1,\vec{\delta}'+\vec{1}}(\mathcal{K})^\ell$, $f_j \in V^{0,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell \cap V^{0,p'}_{\beta',\vec{\delta}'}(\mathcal{K})^\ell$ and $G(x,\xi)$ is the Green function introduced in Sect. 4.4 with arbitrary κ between $\beta+\frac{3}{p}-\frac{3}{2}$ and $\beta'+\frac{3}{p'}-\frac{3}{2}$. □

Analogously, the following statement holds (cf. Remark 5.1).

Lemma 5.3. *Let* $u \in V^{1,2}_\beta(\mathcal{K})^\ell$ *be a solution of problem* (5.2)*, where* $F \in V^{-1,2}_\beta(\mathcal{K})^\ell \cap V^{-1,p}_{\beta',\vec{\delta}}(\mathcal{K})^\ell$*. If the closed strip between the lines* $\mathrm{Re}\,\lambda = -\beta-1/2$ *and* $\mathrm{Re}\,\lambda = 1-\beta'-3/p$ *is free of eigenvalues of the pencil* $\mathfrak{A}(\lambda)$ *and the components of* $\vec{\delta}$ *satisfy* (5.5)*, then* $u \in W^{1,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$.

In the following lemma we consider the case when the strip between the lines $\mathrm{Re}\,\lambda = 1-\beta-3/p$ and $\mathrm{Re}\,\lambda = 1-\beta'-3/p'$ contains eigenvalues of the pencil $\mathfrak{A}(\lambda)$.

Lemma 5.4. *Let* $u \in W^{1,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ *be a solution of problem* (5.3)*, where* $F \in V^{-1,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell \cap V^{-1,p'}_{\beta',\vec{\delta}'}(\mathcal{K})^\ell$*. Suppose that the lines* $\mathrm{Re}\,\lambda = 1-\beta-3/p$ *and* $\mathrm{Re}\,\lambda = 1-\beta'-3/p'$ *do not contain eigenvalues of the pencil* $\mathfrak{A}$ *and that the components of* $\vec{\delta}$ *and* $\vec{\delta}'$ *satisfy the inequalities* $\max(1-\mu_j,0) < \delta_j+2/p < 1$, $\max(1-\mu_j,0) < \delta'_j + 2/p' < 1$, $j=1,\dots,n$*. Then* u *admits the decomposition*

$$u = \sum_{\nu=1}^{N} \sum_{j=1}^{I_\nu} \sum_{s=0}^{\kappa_{\nu,j}-1} c_{\nu,j,s}\, \rho^{\lambda_\nu} \sum_{\sigma=0}^{s} \frac{1}{\sigma!}\, (\log\rho)^\sigma\, u^{(\nu,j,s-\sigma)}(\omega) + w, \tag{5.10}$$

where $w \in W^{1,p'}_{\beta',\vec{\delta}'}(\mathcal{K})^\ell$, λ_ν *are the eigenvalues of the pencil* $\mathfrak{A}(\lambda)$ *between the lines* $\mathrm{Re}\,\lambda = 1-\beta-3/p$ *and* $\mathrm{Re}\,\lambda = 1-\beta'-3/p'$*, and* $u^{(\nu,j,s)}$ *are the eigenvectors (*$s=0$*) and generalized eigenvectors of the pencil* $\mathfrak{A}(\lambda)$ *corresponding to the eigenvalue* λ_ν.

P r o o f. Let $\{F_k\} \subset C_0^\infty(\overline{\mathcal{K}}\backslash\{0\})^\ell$ be a sequence converging to F in $V^{-1,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell \cap V^{-1,p'}_{\beta',\vec{\delta}'}(\mathcal{K})^\ell$. Then, by [16, Th. 4.3] and Lemma 5.3, there exist solutions $u_k \in V^{1,2}_{\beta-3/2+3/p}(\mathcal{K})^\ell \cap W^{1,p}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ and $v_k \in V^{1,2}_{\beta'-3/2+3/p'}(\mathcal{K})^\ell \cap W^{1,p'}_{\beta',\vec{\delta}'}(\mathcal{K})^\ell$ of the problem

$$b_{\mathcal{K}}(u,v) = \int_{\mathcal{K}} \overline{F}_k \cdot \overline{v}\, dx \quad \text{for all } v \in C_0^\infty(\overline{\mathcal{K}}\backslash\{0\})^\ell.$$

By Theorem 5.1, the sequence $\{u_k\}$ converges to u in $W^{1,p}_{\beta,\vec\delta}(\mathcal{K})^\ell$, while $\{v_k\}$ converges to the unique solution $v \in W^{1,p'}_{\beta',\vec\delta'}(\mathcal{K})^\ell$ of problem (5.3). Let $\mathcal{X}$ be the linear span of the functions

$$\rho^{\lambda_\nu} \sum_{\sigma=0}^{s} \frac{1}{\sigma!}\,(\log\rho)^\sigma\, u^{(\nu,j,s-\sigma)}(\omega).$$

According to [16, Cor. 4.3], we have $u_k - v_k \in \mathcal{X}$ for all k and, consequently, $u - v \in \mathcal{X}$. This proves the lemma. □

Furthermore, using Theorems 4.2 and 5.1, we obtain the following result.

Lemma 5.5. *Let the functional* $F \in V^{-1,p}_{\beta,\delta}(\mathcal{K})^\ell$ *have the representation*

$$F(v) = \int_{\mathcal{K}} \bar f \cdot v\, dx + \sum_{j=1}^{n} \int_{\Gamma_j} \bar g_j \cdot v\, dx \quad \textit{for all } v \in C_0^\infty(\overline{\mathcal{K}}\backslash\{0\})^\ell, \tag{5.11}$$

where $f \in W^{0,p'}_{\beta',\vec\delta'}(\mathcal{K})^\ell$, $g_j \in W^{1-1/p',p'}_{\beta',\vec\delta'}(\Gamma_j)^\ell$. *Suppose that the closed strip between the lines* $\operatorname{Re}\lambda = 1-\beta-3/p$ *and* $\operatorname{Re}\lambda = 2-\beta'-3/p'$ *does not contain eigenvalues of the pencil* $\mathfrak{A}$ *and the components of* $\vec\delta$, $\vec\delta'$ *satisfy the inequalities*

$$\max(1-\mu_j,0) < \delta_j + 2/p < 1, \qquad \max(2-\mu_j,0) < \delta'_j + 2/p' < 2.$$

Then the solution $u \in W^{1,p}_{\beta,\vec\delta}(\mathcal{K})^\ell$ *of problem* (5.3) *belongs to* $W^{2,p'}_{\beta',\vec\delta'}(\mathcal{K})^\ell$.

P r o o f. According to Theorems 4.2 and 5.1, there exist a unique solution $u \in W^{1,p}_{\beta,\vec\delta}(\mathcal{K})^\ell$ of problem (5.3) and a unique solution $u \in W^{2,p'}_{\beta',\vec\delta'}(\mathcal{K})^\ell$ of problem (1.2), (1.4). Both solutions coincide, since they are represented by the same Green matrix $G(x,\xi)$. □

Next we will show that the solution in the last lemma belongs to $W^{l,p'}_{\beta',\vec\delta'}(\mathcal{K})^\ell$ if $f \in W^{l-2,p'}_{\beta',\vec\delta'}(\mathcal{K})^\ell$, $g_j \in W^{l-1-1/p',p'}_{\beta',\vec\delta'}(\Gamma_j)^\ell$ and β', $\vec\delta'$ satisfy analogous conditions to (5.5). For this end, we prove the following lemma.

Lemma 5.6. *Let* $u \in W^{l,p}_{\beta,\vec\delta}(\mathcal{K})^\ell$ *be a solution of the Neumann problem* (1.2), (1.4) *such that* $\rho\partial_\rho u \in W^{l,p}_{\beta,\vec\delta}(\mathcal{K})^\ell$. *If* $f \in W^{l-1,p}_{\beta+1,\vec\delta}(\mathcal{K})^\ell$, $g_j \in W^{l-1/p,p}_{\beta+1,\vec\delta}(\Gamma_j)^\ell$, *and the strip* $l-\delta-2/p \le \operatorname{Re}\lambda \le l+1-\delta-2/p$ *is free of eigenvalues of the pencil* $A_j(\lambda)$, $j=1,\dots,n$, *then* $u \in W^{l+1,p}_{\beta+1,\vec\delta}(\mathcal{K})^\ell$ *and*

$$\|u\|_{W^{l+1,p}_{\beta+1,\vec\delta}(\mathcal{K})^\ell} \le c\left(\sum_{j=0}^{1} \|(\rho\partial_\rho)^j u\|_{W^{l,p}_{\beta,\vec\delta}(\mathcal{K})^\ell} + \|f\|_{W^{l-1,p}_{\beta+1,\vec\delta}(\mathcal{K})^\ell} + \sum_{j=1}^{n} \|g_j\|_{W^{l-1/p,p}_{\beta+1,\vec\delta}(\Gamma_j)^\ell}\right).$$

P r o o f. First note that, according to Lemma 4.4, we have $u \in W^{l+1,p}_{\beta+1,\vec\delta+\vec 1}(\mathcal{K})^\ell$. We denote by ζ_k, η_k the same functions as in the proof of Lemma 4.3. Furthermore, we set $\tilde\zeta_k(x) = \zeta_k(2^k x)$, $\tilde\eta_k(x) = \eta_k(2^k x)$ and $v(x) = u(2^k x)$. Since $\operatorname{supp}\zeta_k \subset \{x:\ 1/2 < |x| < 2\}$, we conclude from Lemma 3.4 and from the assumptions on u, f and g_j that $\tilde\zeta_k u \in W^{l+1,p}_{\beta+1,\vec\delta}(\mathcal{K})^\ell$ and

$$\|\tilde\zeta_k v\|^p_{W^{l+1,p}_{\beta+1,\vec\delta}(\mathcal{K})^\ell} \le c\left(\sum_{j=0}^{1} \|\tilde\eta_k(\rho\partial_\rho)^j v\|^p_{W^{l,p}_{\beta,\vec\delta}(\mathcal{K})^\ell} + \|\tilde\eta_k L v\|^p_{W^{l-1,p}_{\beta+1,\vec\delta}(\mathcal{K})^\ell} + \sum_{j=1}^{n} \|\tilde\eta_k B v\|^p_{W^{l-1/p,p}_{\beta+1,\vec\delta}(\Gamma_j)^\ell}\right),$$

where c is independent of k. Multiplying the last inequality by $2^{kp(\beta-l)+3k}$ and substituting $2^k = y$, we obtain the same inequality with ζ_k, η_k instead of $\tilde\zeta_k, \tilde\eta_k$ for the vector function u. Now the lemma follows from the equivalence of the norm in $\mathcal{W}^{l,p}_{\beta,\vec\delta}(\mathcal{K};\tilde J)$ with the norm (2.4) and from the analogous result for the trace spaces. □

Theorem 5.3. *Let the functional in* $F \in V^{-1,p}_{\beta,\delta}(\mathcal{K})^\ell$ *have the representation* (5.11) *with* $f \in W^{l-2,p'}_{\beta',\vec\delta'}(\mathcal{K})^\ell$, $g_j \in W^{l-1-1/p',p'}_{\beta',\vec\delta'}(\Gamma_j)^\ell$. *Suppose that the closed strip between the lines* $\operatorname{Re}\lambda = 1-\beta-3/p$ *and* $\operatorname{Re}\lambda = l-\beta'-3/p'$ *does not contain eigenvalues of the pencil* $\mathfrak{A}$ *and the components of* $\vec\delta$, $\vec\delta'$ *satisfy the inequalities*

$$\max(1-\mu_j,0) < \delta_j + 2/p < 1, \qquad \max(l-\mu_j,0) < \delta'_j + 2/p' < l.$$

Then the solution $u \in W^{1,p}_{\beta,\vec\delta}(\mathcal{K})^\ell$ *of problem* (5.3) *belongs to* $W^{l,p'}_{\beta',\vec\delta'}(\mathcal{K})^\ell$.

P r o o f. 1) If $l-\delta'_j-2/p'<2$ for $j=1,\dots,n$, then $f\in W^{0,p'}_{\beta'-l+2,\vec\delta'-(l-2)\vec 1}(\mathcal K)^\ell$, $g_j\in W^{1-1/p',p'}_{\beta'-l+2,\vec\delta'-(l-2)\vec 1}(\Gamma_j)^\ell$, and Lemma 5.5 implies $u\in W^{2,p'}_{\beta'-l+2,\vec\delta'-(l-2)\vec 1}(\mathcal K)^\ell$. Applying Lemma 4.4, we obtain $u\in W^{l,p'}_{\beta',\vec\delta'}(\mathcal K)^\ell$.

2) Suppose that $l-\delta'_j-2/p'\in[k-1,k)$ for $j=1,\dots,n$, where k is an integer, $3\le k\le l$. This is only possible if $\mu_j>k-1$. We prove by induction in k that $u\in W^{l,p'}_{\beta',\vec\delta'}(\mathcal K)^\ell$. Let first $k=3$, i.e., $l-3<\delta'_j+2/p'\le l-2$. Then $f\in W^{1,p'}_{\beta'-l+3,\vec\delta'-(l-3)\vec 1}(\mathcal K)^\ell\subset W^{0,p'}_{\beta'-l+2,\vec\delta'-(l-3+\varepsilon)\vec 1}(\mathcal K)^\ell$, $g_j\in W^{1-1/p',p'}_{\beta'-l+2,\vec\delta'-(l-3+\varepsilon)\vec 1}(\Gamma_j)^\ell$, where $\varepsilon\ge 0$ is such that $l-3<\delta'_j-\varepsilon+2/p'<l-2$. Since $0<\delta'_j-l+3-\varepsilon+2/p'<2$, Lemma 5.5 implies $u\in W^{2,p'}_{\beta'-l+2,\vec\delta'-(l-3+\varepsilon)\vec 1}(\mathcal K)^\ell$. From this and from Lemma 4.4 we conclude that $u\in W^{3,p'}_{\beta'-l+3,\vec\delta'-(l-4+\varepsilon)\vec 1}(\mathcal K)^\ell$ and, therefore, $\rho\partial_\rho u\in W^{2,p'}_{\beta'-l+2,\vec\delta'-(l-4+\varepsilon)\vec 1}(\mathcal K)^\ell$. Furthermore,

$$L\rho\partial_\rho u=\rho\partial_\rho f+2f\in W^{0,p'}_{\beta'-l+2,\vec\delta'-(l-3)\vec 1}(\mathcal K)^\ell,\quad B\rho\partial_\rho u\big|_{\Gamma_j}=\rho\partial_\rho g_j+g_j\in W^{1-1/p',p'}_{\beta'-l+2,\vec\delta'-(l-3)\vec 1}(\Gamma_j)^\ell,$$

where $0<\delta'_j-l+3+2/p'\le 1$ for $j=1,\dots,n$. Consequently, by Theorem 4.3, we have $\rho\partial_\rho u\in W^{2,p'}_{\beta'-l+2,\vec\delta'-(l-3)\vec 1}(\mathcal K)^\ell$. Since u belongs to the same space, we obtain from Lemma 5.6 that $u\in W^{3,p'}_{\beta'-l+3,\vec\delta'-(l-3)\vec 1}(\mathcal K)^\ell$. Using again Lemma 4.4, we get $u\in W^{l,p'}_{\beta',\vec\delta'}(\mathcal K)^\ell$. Thus, the theorem is proved for $l-\delta'_j-2/p'\in[2,3)$.

Suppose that $l-\delta'_j-2/p'\in[k-1,k)$, $k\ge 4$, and that the theorem is proved for $l-\delta'_j-2/p'\in[k-2,k-1)$. Since $f\in W^{l-2,p'}_{\beta',\vec\delta'}(\mathcal K)^\ell\subset W^{l-3,p'}_{\beta'-1,\vec\delta'}(\mathcal K)^\ell$, the induction hypothesis implies $u\in W^{l-1,p'}_{\beta'-1,\vec\delta'}(\mathcal K)^\ell$ and, consequently, $\rho\partial_\rho u\in W^{l-2,p'}_{\beta'-2,\vec\delta'}(\mathcal K)^\ell$. On the other hand,

$$L\rho\partial_\rho u=\rho\partial_\rho f+2f\in W^{l-3,p'}_{\beta'-1,\vec\delta'}(\mathcal K)^\ell,\quad B\rho\partial_\rho u\big|_{\Gamma_j}=\rho\partial_\rho g_j+g_j\in W^{l-2-1/p',p'}_{\beta'-1,\vec\delta'}(\Gamma_j)^\ell.$$

Hence, by the induction hypothesis, we have $\rho\partial_\rho u\in W^{l-1,p'}_{\beta'-1,\vec\delta'}(\mathcal K)^\ell$. From this and from Lemma 5.6 we conclude that $u\in W^{l,p'}_{\beta',\vec\delta'}(\mathcal K)^\ell$.

3) Finally, we assume that $l-\delta'_j-2/p'\in[k_j-1,k_j)$ for $j=1,\dots,n$ with different $k_j\in\{1,\dots,l\}$. Then let $\psi_1,\dots,\psi_n$ be smooth functions on $\overline\Omega$ such that $\psi_j\ge 0$, $\psi_j=1$ near $M_j\cap S^2$, and $\sum\psi_j=1$. We extend ψ_j to $\mathcal K$ by the equality $\psi_j(x)=\psi_j(x/|x|)$. Then $\partial_x^\alpha\psi_j(x)\le c\,|x|^{-|\alpha|}$. Using the first and second parts of the proof, we can show, by induction in l that $\psi_j u\in W^{l,p'}_{\beta',\vec\delta'}(\mathcal K)^\ell$ for $j=1,\dots,n$. This completes the proof. □

Remark 5.2. Suppose that the conditions (i)–(iii) of Remark 4.2 are satisfied for some j. Then the result of Theorem 5.3 remains true if δ'_j satisfies the inequalities $\max(l-\mu_j^{(2)},0)<\delta'_j+2/p'<l$, where $\mu_j^{(2)}$ is the first eigenvalue of the pencil $A_j(\lambda)$ on the right of the line $\mathrm{Re}\,\lambda=1$.

5.3 Weak solutions of the Dirichlet and mixed problems

Let $F\in V^{-1,p}_{\beta,\vec\delta}(\mathcal K)$ and $g_j\in V^{1-1/p,p}_{\beta,\vec\delta}(\Gamma_j)^\ell$, $j\in J_0$, be given. By a weak solution of the boundary value problem (1.2)–(1.4) we mean a vector function $u\in\mathcal W^{1,p}_{\beta,\vec\delta}(\mathcal K;\tilde J)^\ell$ satisfying

$$b_{\mathcal K}(u,v)=\overline{F(v)}\quad\text{for all } v\in V^{1,q}_{-\beta,-\vec\delta}(\mathcal K)^\ell,\ v=0 \text{ on } \Gamma_j \text{ for } j\in J_0, \tag{5.12}$$

$$u=g_j \text{ on } \Gamma_j \text{ for } j\in J_0, \tag{5.13}$$

where $q=p/(p-1)$. According to Lemma 4.1, we can restrict ourselves to the case of homogeneous Dirichlet condition (5.13). Then the vector function (5.4) is a solution of problem (5.12), (5.13), where F is given by (5.1). Analogously to Theorem 5.1, the following statement holds.

Theorem 5.4. *Suppose that there are no eigenvalues of the pencil $\mathfrak A$ on the line* $\mathrm{Re}\,\lambda=1-\beta-3/p$ *and the components of $\vec\delta$ satisfy the inequalities*

$$1-\mu_j<\delta_j+2/p<1 \text{ for } j\in\tilde J,\qquad \max(1-\mu_j,0)<\delta_j+2/p<1 \text{ for } j\in J\backslash\tilde J. \tag{5.14}$$

Then problem (5.12), (5.13) *has a unique solution $u\in\mathcal W^{1,p}_{\beta,\vec\delta}(\mathcal K;\tilde J)^\ell$ for arbitrary $F\in V^{-1,p}_{\beta,\vec\delta}(\mathcal K)^\ell$, $g_j\in V^{1-1/p,p}_{\beta,\vec\delta}(\Gamma_j)^\ell$, $j\in J_0$.*

Furthermore, the following regularity assertion holds analogously to Theorem 5.2.

Theorem 5.5. *Let the functional* $F \in V_{\beta,\vec\delta}^{-1,p}(\mathcal{K})^\ell$ *have the representation*

$$F(v) = \int_{\mathcal{K}} \bar f \cdot v\, dx + \sum_{j=\in J_1} \int_{\Gamma_j} \bar g_j \cdot v\, dx \quad \text{for all } v \in C_0^\infty(\overline{\mathcal{K}})\backslash\{0\})^\ell$$

with $f \in \mathcal{W}_{\beta',\vec\delta'}^{l-2,p'}(\mathcal{K};\tilde J)^\ell$, $g_j \in \mathcal{W}_{\beta',\vec\delta'}^{l-1-1/p',p'}(\Gamma_j;\tilde J)^\ell$, $j \in J_1$. *Suppose further that* $g_j \in V_{\beta',\vec\delta'}^{l-1/p',p'}(\Gamma_j)^\ell$ *for* $j \in J_0$, *that the closed strip between the lines* $\mathrm{Re}\,\lambda = 1-\beta-3/p$ *and* $\mathrm{Re}\,\lambda = l-\beta'-3/p'$ *does not contain eigenvalues of the pencil* $\mathfrak{A}$, *that the components of* $\vec\delta$ *satisfy* (5.14) *and the components of* $\vec\delta'$ *satisfy the inequalities*

$$l-\mu_j < \delta_j' + 2/p' < l \text{ for } j \in \tilde J, \quad \max(l-\mu_j, 0) < \delta_j' + 2/p' < l \text{ for } j \in J\backslash\tilde J.$$

Then the solution $u \in \mathcal{W}_{\beta,\vec\delta}^{1,p}(\mathcal{K};\tilde J)^\ell$ *of problem* (5.12), (5.13) *belongs to* $\mathcal{W}_{\beta',\vec\delta'}^{l,p'}(\mathcal{K};\tilde J)^\ell$.

6 Examples

6.1 The Neumann problem for the Laplace equation

We consider the problem

$$-\Delta u = f \text{ in } \mathcal{K}, \qquad \frac{\partial u}{\partial n} = g_j \text{ on } \Gamma_j. \tag{6.1}$$

As is known (see, e.g. [9, Ch.2]), the eigenvalues of the operator pencils $A_j(\lambda)$ are $k\pi/\theta_j$, where θ_j is the angle at the edge M_j and k is an arbitrary integer. Thus, we have $\mu_j = \pi/\theta_j$ for $j = 1,\dots,n$. The eigenvalues of the pencil $\mathfrak{A}(\lambda)$ are real. Consequently, the following results are valid (see Theorems 4.2, 5.3):

1) *Problem* (6.1) *is uniquely solvable in* $W_{\beta,\vec\delta}^{l,p}(\mathcal{K})$ *for arbitrary* $f \in W_{\beta,\vec\delta}^{l-2,p}(\mathcal{K})$ *and* $g_j \in W_{\beta,\vec\delta}^{l-1-1/p,p}(\Gamma_j)$, $l \ge 2$, *if* $l-\beta-3/p$ *is not an eigenvalue of the pencil* $\mathfrak{A}(\lambda)$ *and the components of* $\vec\delta$ *satisfy the inequalities*

$$\max(l-\frac{\pi}{\theta_j}, 0) < \delta_j + \frac{2}{p} < l \tag{6.2}$$

for $j = 1,\dots,n$.

2) *If* $1-\beta-3/p$ *is not an eigenvalue of the pencil* $\mathfrak{A}(\lambda)$ *and the components of* $\vec\delta$ *satisfy the inequalities* $\max(1-\pi/\theta_j, 0) < \delta_j + \frac{2}{p} < 1$, *then for every* $F \in V_{\beta,\vec\delta}^{-1,p}(\mathcal{K})$ *there exists a unique weak solution of the Neumann problem, i.e. a unique function* $u \in W_{\beta,\vec\delta}^{1,p}(\mathcal{K})$ *satisfying*

$$\int_{\mathcal{K}} \nabla u \cdot \nabla \bar v\, dx = \overline{F(v)} \text{ for all } v \in V_{-\beta,-\vec\delta}^{1,q}(\mathcal{K}),\ q = p/(p-1). \tag{6.3}$$

This solution belongs to $W_{\beta',\vec\delta'}^{l,p'}(\mathcal{K})^\ell$ *if* F *has the form* (5.11) *with* $f \in W_{\beta'\vec\delta'}^{l-2,p'}(\mathcal{K})$, $g_j \in W_{\beta',\vec\delta'}^{l-1-1/p',p'}(\mathcal{K})$, *there are no eigenvalues of the pencil* $\mathfrak{A}(\lambda)$ *in the closed interval between* $1-\beta-3/p$ *and* $l-\beta'-3/p'$ *and the components of* $\vec\delta'$ *satisfy the inequalities* $\max(l-\pi/\theta_j, 0) < \delta_j' + 2/p' < l$.

Let us consider e.g. the solution $u \in \mathcal{H} = V_0^{1,2}(\mathcal{K})$ of the problem

$$\int_{\mathcal{K}} \nabla u \cdot \nabla \bar v\, dx = \overline{F(v)} \text{ for all } v \in \mathcal{H},$$

where $F \in \mathcal{H}^*$ has the representation (5.11) with $f \in W_{\beta,\vec\delta}^{l-2,p}(\mathcal{K})$, $g_j \in W_{\beta,\vec\delta}^{l-1-1/p,p}(\Gamma_j)$. If $-1/2 \le l-\beta-3/p < 0$ and the components of $\vec\delta$ satisfy the inequalities (6.2), then $u \in W_{\beta,\vec\delta}^{l,p}(\mathcal{K})$. If $l-\beta-3/p > 0$, the interval $[0, l-\beta-3/p]$ contains only the eigenvalue $\lambda = 0$ of the pencil $\mathfrak{A}(\lambda)$ and the components of $\vec\delta$ satisfy (6.2), then u admits the representation $u = c + w$, where c is a constant and $w \in W_{\beta,\vec\delta}^{l,p}(\mathcal{K})$. In particular, there is the representation $u = c + w$ with $w \in V_0^{2,2}(\mathcal{K})$ if $f \in L_2(\mathcal{K})$, $g_j \in V_0^{1/2,2}(\Gamma_j)$ and the cone $\mathcal{K}$ is convex. The last assertion follows from the estimate $\lambda_1 \ge (\sqrt5 - 1)/2$ for the first positive eigenvalue of the pencil $\mathfrak{A}(\lambda)$ obtained by Dauge [4].

6.2 The Neumann problem for the Lamé system

We consider the problem

$$\Delta u + \frac{1}{1-2\nu}\, \nabla\,\nabla\cdot u = f \ \text{ in } \mathcal{K}, \quad \sigma(u)\, n = g_j \ \text{ on } \Gamma_j\,, \tag{6.4}$$

where $\sigma(u) = \{\sigma_{i,j}(u)\}$ is the stress tensor connected with the strain tensor

$$\Big\{\varepsilon_{i,j}(u)\Big\} = \Big\{\frac{1}{2}\,(\partial_{x_j} u_i + \partial_{x_i} u_j)\Big\}$$

by the Hooke law

$$\sigma_{i,j}(u) = 2\mu\, \left(\frac{\nu}{1-2\nu}(\varepsilon_{1,1}+\varepsilon_{2,2}+\varepsilon_{3,3})\,\delta_{i,j} + \varepsilon_{i,j}\right)$$

(μ is the shear modulus, ν is the Poisson ratio, $\nu < 1/2$, and $\delta_{i,j}$ denotes the Kronecker symbol).

If the angle θ_j at the edge M_j is greater than π, then the eigenvalue of the pencil $A_j(\lambda)$ with smallest positive real part is $\lambda_1^{(j)} = \xi_+(\theta_j)/\theta_j$, where $\xi_+(\theta)$ is the smallest positive solution of the equation

$$\frac{\sin\xi}{\xi} + \frac{\sin\theta}{\theta} = 0 \tag{6.5}$$

(see, e.g., [9, Sect. 4.2]). Note that $\xi_+(\theta_j) < \pi$ and, therefore, $\lambda_1^{(j)} < 1$ for $\theta_j > \pi$. If $\theta_j < \pi$, then the eigenvalues with smallest positive real parts are $\lambda_1^{(j)} = 1$ and $\lambda_2^{(j)} = \pi/\theta_j$.

For the existence of a vector function $u \in W^{l,p}_{\beta,\delta}(\mathcal{K})^\ell$ satisfying the boundary conditions $\sigma(u)\, n = g_j$ on Γ_j, $j = 1,\dots,n$, it is necessary that the boundary data g_j belong to $W^{l-1-1/p,p}_{\beta,\delta}(\Gamma_j)^\ell$ and satisfy certain compatibility conditions on the edge M_j if $\delta_j + 2/p \le l-1$. Let Γ_{j_+} and Γ_{j_-} be the faces of $\mathcal{K}$ adjacent to the edge M_j and let n^{j+}, n^{j-} be the exterior normals to Γ_{j_+} and Γ_{j_-}, respectively. Then one has to suppose that

$$n^{j-}\cdot g_{j_+} = n^{j+}\cdot g_{j_-} \ \text{ on } M_j \ \text{ if } \delta_j + 2/p < l-1, \tag{6.6}$$

$$\int_0^\infty \int_0^{\varepsilon\rho} \rho^{p(\beta-l+1)+2}\, r_j^{-1} \big|n^{j-}\cdot g_{j_+}(r_j,\rho) - n^{j+}\cdot g_{j_-}(r_j,\rho)\big|^p\, dr_j\, d\rho < \infty \ \text{ if } \delta_j + 2/p = l-1, \tag{6.7}$$

where ε is a sufficiently small positive number (cf. Remark 3.1). If $\delta_j + 2/p > l-1$ for $j = 1,\dots,n$, then $W^{l-1-1/p,p}_{\beta,\delta}(\Gamma_j) = V^{l-1-1/p,p}_{\beta,\delta}(\Gamma_j)$ and, according to Lemma 4.1, for arbitrary $g_j \in W^{l-1-1/p,p}_{\beta,\delta}(\Gamma_j)^\ell$ there exists a vector function $u \in V^{l,p}_{\beta,\delta}(\mathcal{K})^\ell \subset W^{l,p}_{\beta,\delta}(\mathcal{K})^\ell$ satisfying $\sigma(u)\, n = g_j$ on Γ_j, $j = 1,\dots,n$.

Using Theorem 4.2, 5.3 and Remark 5.2, we obtain the following results.

1) *Let* $f \in W^{l-2,p}_{\beta,\vec\delta}(\mathcal{K})^3$ *and* $g_j \in W^{l-1-1/p,p}_{\beta,\vec\delta}(\Gamma_j)^3$, $l \ge 2$, *where* β *and* $\vec\delta$ *are such that the line* $\operatorname{Re}\lambda = l - \beta - 3/p$ *is free of eigenvalues of the pencil* $\mathfrak{A}(\lambda)$ *and*

$$l - \frac{\xi_+(\theta_j)}{\theta_j} < \delta_j + \frac{2}{p} < l \ \text{ if } \theta_j > \pi, \quad \max(l - \frac{\pi}{\theta_j}, 0) < \delta_j + \frac{2}{p} < l \ \text{ if } \theta_j < \pi. \tag{6.8}$$

Furthermore, we assume that the boundary data g_j *satisfy the compatibility conditions* (6.6), (6.7). *Then problem* (6.4) *is uniquely solvable in* $W^{l,p}_{\beta,\vec\delta}(\mathcal{K})^3$.

2) *If there are no eigenvalues of the pencil* $\mathfrak{A}(\lambda)$ *on the line* $\operatorname{Re}\lambda = 1 - \beta - 3/p$ *and the components of* $\vec\delta$ *satisfy the inequalities*

$$1 - \frac{\xi_+(\theta_j)}{\theta_j} < \delta_j + \frac{2}{p} < 1 \ \text{ if } \theta_j > \pi, \quad 0 < \delta_j + \frac{2}{p} < 1 \ \text{ if } \theta_j < \pi, \tag{6.9}$$

then for every $F \in V^{-1,p}_{\beta,\vec\delta}(\mathcal{K})$ *there exists a unique weak solution of the Neumann problem, i.e. a unique function* $u \in W^{1,p}_{\beta,\vec\delta}(\mathcal{K})^3$ *satisfying*

$$b_{\mathcal{K}}(u,v) \stackrel{\text{def}}{=} \int_{\mathcal{K}} \sum_{i,j=1}^3 \sigma_{i,j}(u)\, \varepsilon_{i,j}(\overline{v})\, dx = \overline{F(v)} \ \text{ for all } v \in V^{1,q}_{-\beta,-\vec\delta}(\mathcal{K})^3, \ q = p/(p-1). \tag{6.10}$$

This solution belongs to $W^{l,p'}_{\beta',\vec{\delta}'}(\mathcal{K})^\ell$ *if* F *has the form* (5.11) *with* $f \in W^{l-2,p'}_{\beta'\vec{\delta}'}(\mathcal{K})$, $g_j \in W^{l-1-1/p',p'}_{\beta',\vec{\delta}'}(\mathcal{K})$, $l \geq 2$, *there are no eigenvalues of the pencil* $\mathfrak{A}(\lambda)$ *in the closed interval between* $1-\beta-3/p$ *and* $l-\beta'-3/p'$, *and the components of* $\vec{\delta}'$ *satisfy the inequalities* $l-\xi_+(\theta_j)/\theta_j < \delta'_j + 2/p' < l$ *for* $\theta_j > \pi$, $\max(l-\pi/\theta_j, 0) < \delta'_j + 2/p' < l$ *for* $\theta_j < \pi$.

We consider the solution $u \in \mathcal{H} = V_0^{1,2}(\mathcal{K})^3$ of the problem

$$b_{\mathcal{K}}(u,v) = \overline{F(v)} \quad \text{for all } v \in \mathcal{H},$$

where $F \in \mathcal{H}^*$ has the representation (5.11) with $f \in W^{l-2,p}_{\beta,\vec{\delta}}(\mathcal{K})^3$, $g_j \in W^{l-1-1/p,p}_{\beta,\vec{\delta}}(\Gamma_j)^3$. By [9, Th. 4.3.1]), the strip $-1/2 \leq \operatorname{Re}\lambda \leq 0$ contains only the eigenvalue $\lambda = 0$ if the cone $\mathcal{K}$ is Lipschitz. The eigenvectors to this eigenvalue are constant vectors, while generalized eigenvectors corresponding to $\lambda = 0$ do not exist. Consequently, we obtain $u \in W^{l,p}_{\beta,\vec{\delta}}(\mathcal{K})^3$ if $-1/2 \leq l-\beta-3/p < 0$, the components of $\vec{\delta}$ satisfy the inequalities (6.8), and the boundary data satisfy the compatibility conditions (6.6), (6.7). If $l-\beta-3/p > 0$, the strip $0 \leq \operatorname{Re}\lambda \leq l-\beta-3/p$ contains only the eigenvalue $\lambda = 0$ of the pencil $\mathfrak{A}(\lambda)$, the components of $\vec{\delta}$ satisfy (6.8) and the boundary data g_j satisfy (6.6), (6.7), then u admits the representation $u = c + w$, where c is a constant vector and $w \in W^{l,p}_{\beta,\vec{\delta}}(\mathcal{K})^3$. In particular, we have $u = c + w$, where $c \in \mathbb{C}^3$, $w \in V^{2,2}_\beta(\mathcal{K})^3$ if $f \in V^{0,2}_\beta(\mathcal{K})^3$, $g_j \in V^{1/2,2}_\beta(\Gamma_j)^3$, the strip $-1/2 \leq \operatorname{Re}\lambda \leq -\beta + 1/2$ contains at most the eigenvalue $\lambda = 0$, and condition (6.7) is satisfied.

7 The problem in a bounded domain

7.1 Formulation of the problem

Let $\mathcal{G}$ be a bounded domain of polyhedral type in $\mathbb{R}^3$. This means that

(i) the boundary $\partial\mathcal{G}$ consists of smooth (of class C^∞) open two-dimensional manifolds Γ_j (the faces of $\mathcal{G}$), $j = 1,\dots,n$, smooth curves M_k (the edges), $k = 1,\dots,m$, and corners $x^{(1)},\dots,x^{(d)}$,

(ii) for every $\xi \in M_k$ there exist a neighborhood $\mathcal{U}_\xi$ and a diffeomorphism (a C^∞ mapping) κ_ξ which maps $\mathcal{G} \cap \mathcal{U}_\xi$ onto $\mathcal{D}_\xi \cap B_1$, where $\mathcal{D}_\xi$ is a dihedron of the form (2.1) and B_1 is the unit ball,

(iii) for every corner $x^{(j)}$ there exist a neighborhood $\mathcal{U}_j$ and a diffeomorphism κ_j mapping $\mathcal{G} \cap \mathcal{U}_j$ onto $\mathcal{K}_j \cap B_1$, where $\mathcal{K}_j$ is a cone with vertex at the origin.

Let $\mathcal{S}$ denote the set of all edge points and corners. We consider the problem

$$Lu = f \ \text{ in } \mathcal{G}, \quad u = g_j \ \text{ on } \Gamma_j \text{ for } j \in J_0, \quad Bu = g_j \ \text{ on } \Gamma_j \text{ for } j \in J_1, \tag{7.1}$$

where

$$Lu = -\sum_{i,j=1}^{3} \partial_{x_j}\big(A_{i,j}(x)\partial_{x_i}u\big) + \sum_{i=1}^{3} A_i(x)\,\partial_{x_i}u + A_0(x)\,u, \quad Bu = \sum_{i,j=1}^{3} A_{i,j}(x)\,n_j\,\partial_{x_i}u,$$

$J_0 \cup J_1 = \{1,2,\dots,n\}$, $J_0 \cap J_1 = \emptyset$. The corresponding sesquilinear form is

$$b(u,v) = \int_{\mathcal{G}} \Big(\sum_{i,j=1}^{3} A_{i,j}\partial_{x_i}u \cdot \partial x_j\overline{v} + \sum_{i=1}^{3} A_i\partial_{x_i}u \cdot \overline{v} + A_0 u \cdot \overline{v} \Big)\, dx.$$

Let $\mathcal{H} = \{u \in W^{1,2}(\mathcal{G})^\ell : u = 0 \text{ on } \Gamma_j \text{ for } j \in J_0\}$, where $W^{1,2}(\mathcal{G})$ denotes the Sobolev space of all functions quadratically summable on $\mathcal{G}$ together with their derivatives of first order. As in the previous sections, we assume that $A_{i,j} = A^*_{j,i}$ for $i,j = 1,2,3$. Furthermore, we suppose that

$$|b(u,u)| \geq c_1\,\|u\|^2_{W^{1,2}(\mathcal{G})^\ell} - c_2\,\|u\|^2_{L^2(\mathcal{G})^\ell} \quad \text{for all } u \in \mathcal{H} \tag{7.2}$$

with certain positive constants c_1 and c_2.

7.2 Model problems and corresponding operator pencils

We introduce the operator pencils generated by problem (7.1) for the points of $\mathcal{S}$.

1) Let ξ be an edge point, and let Γ_{j+}, Γ_{j-} be the faces of $\mathcal{G}$ adjacent to ξ. Then by $\mathcal{D}_\xi$ we denote the dihedron which is bounded by the half-planes $\Gamma^\circ_{j\pm}$ tangential to $\Gamma_{j\pm}$ at ξ and consider the model problem

$$L_0(\xi,\partial_x)\, u = f \ \text{ in } \mathcal{D}_\xi, \quad u = g_{j_\pm} \ \text{ on } \Gamma^\circ_{j\pm} \text{ for } j_\pm \in J_0, \quad B(\xi,\partial_x)\, u = g_{j_\pm} \ \text{ on } \Gamma^\circ_{j\pm} \text{ for } j_\pm \in J_1,$$

where

$$L_0(\xi,\partial_x) = -\sum_{i,j=1}^{3} A_{i,j}(\xi)\,\partial_{x_i}\partial_{x_j}\,, B(\xi,\partial_x) = \sum_{i,j=1}^{3} A_{i,j}(\xi)\, n_j\,\partial_{x_i}$$

are the principal parts of L and B with coefficients frozen at ξ. The operator pencil corresponding to this model problem (see Sect. 4.1) is denoted by $A_\xi(\lambda)$. Furthermore, we denote by $\lambda_1(\xi)$ the eigenvalue with smallest positive real part of $A_\xi(\lambda)$ and set

$$\mu_j = \inf_{\xi\in M_j} \operatorname{Re}\lambda_1(\xi) \ \text{ for } j = 1,\dots,m.$$

2) Let $x^{(k)}$ be a corner of $\mathcal{G}$ and let $J^{(k)}$ be the set of all indices j such that $x^{(k)} \in \overline{\Gamma}_j$. By our assumptions, there exist a neighborhood $\mathcal{U}$ of $x^{(k)}$ and a diffeomorphism κ mapping $\mathcal{G}\cap\mathcal{U}$ onto $\mathcal{K}\cap B_1$ and $\Gamma_j\cap\mathcal{U}$ onto $\Gamma^\circ_j\cap B_1$ for $j\in J^{(k)}$, where $\mathcal{K}$ is a polyhedral cone with vertex 0 and Γ°_j are the faces of this cone. Without loss of generality, we may assume that the Jacobian matrix $\kappa'(x)$ is equal to the identity matrix I at the point $x^{(k)}$. We consider the model problem

$$L_0(x^{(k)},\partial_x)\, u = f \ \text{ in } \mathcal{K}, \quad u = g_j \ \text{ on } \Gamma^\circ_j \text{ for } j\in J_0^{(k)}, \quad B(x^{(k)},\partial_x)\, u = g_j \ \text{ on } \Gamma^\circ_j \text{ for } j \in J_1^{(k)},$$

where $J_0^{(k)} = J_0 \cap J^{(k)},\ J_1^{(k)} = J_1^{(k)}$. The operator pencil generated by this model problem (see Sect. 4.1) is denoted by $\mathfrak{A}_k(\lambda)$. Note that (due to the condition $\kappa'(x^{(k)}) = I$)

$$L_0(x^{(k)},\partial_x) = \tilde{L}_0(0,\partial_x) \ \text{ and } \ B(x^{(k)},\partial_x) = \tilde{B}(0,\partial_x),$$

where $\tilde{L},\ \tilde{B}$ are given by $\tilde{L}(y,\partial_y) = L(x,\partial_x)$ and $\tilde{B}(y,\partial_y) = B(x,\partial_x)$ with $y = \kappa(x)$.

7.3 Sobolev spaces in $\mathcal{G}$

We denote by $r_j(x)$ the distance of x to the edge M_j, by $\rho_k(x)$ the distance to the corner $x^{(k)}$ and by $\rho(x)$ the distance to the set $X = \{x^{(1)},\dots,x^{(d)}\}$. Furthermore, let $\tilde{J}$ be an arbitrary subset of $\{1,\dots,m\}$. Then $\mathcal{W}^{l,p}_{\vec\beta,\vec\delta}(\mathcal{G};\tilde{J})$ is defined as the weighted Sobolev space with the norm

$$\|u\|_{\mathcal{W}^{l,p}_{\vec\beta,\vec\delta}(\mathcal{G};\tilde{J})} = \Big(\int_{\mathcal{G}} \sum_{|\alpha|\le l} |\partial_x^\alpha u|^p \prod_{k=1}^{d} \rho_k^{p(\beta_k-l+|\alpha|)} \prod_{j\in\tilde{J}} \big(\frac{r_j}{\rho}\big)^{p(\delta_j-l+|\alpha|)} \prod_{j\notin\tilde{J}} \big(\frac{r_j}{\rho}\big)^{p\delta_j}\, dx\Big)^{1/p}.$$

Here $1<p<\infty$, $\vec\beta = (\beta_1,\dots,\beta_d)\in\mathbb{R}^d$, $\vec\delta = (\delta_1,\dots,\delta_m)\in\mathbb{R}^m$, and l is a nonnegative integer. For $\tilde{J} = \{1,\dots,m\}$ and $\tilde{J} = \emptyset$ we will use the notation $\mathcal{W}^{l,p}_{\vec\beta,\vec\delta}(\mathcal{G};\{1,\dots,m\}) = V^{l,p}_{\vec\beta,\vec\delta}(\mathcal{G})$ and $\mathcal{W}^{l,p}_{\vec\beta,\vec\delta}(\mathcal{G};\emptyset) = W^{l,p}_{\vec\beta,\vec\delta}(\mathcal{G})$.

The trace spaces on Γ_j for $V^{l,p}_{\vec\beta,\vec\delta}(\mathcal{G})$, $W^{l,p}_{\vec\beta,\vec\delta}(\mathcal{G})$, and $\mathcal{W}^{l,p}_{\vec\beta,\vec\delta}(\mathcal{G};\tilde{J})$ are denoted by $V^{l-1/p,p}_{\vec\beta,\vec\delta}(\Gamma_j)$, $W^{l,p}_{\vec\beta,\vec\delta}(\Gamma_j)$, and $\mathcal{W}^{l,p}_{\vec\beta,\vec\delta}(\Gamma_j;\tilde{J})$, respectively.

7.4 Regularity results for weak solutions

In the sequel, let $\tilde{J}$ denote the set of all $j = 1,\dots,m$ such that $M_j \subset \overline{\Gamma}_k$ for at least one $k\in J_0$ (i.e., the Dirichlet condition is given on at least one face Γ_k adjacent to the the edge M_j). Note that $\mathcal{H}$ is continuously imbedded into $\mathcal{W}^{1,2}_{0,0}(\mathcal{G};\tilde{J})$ and that on $\mathcal{H}$ the norms in $W^{1,2}(\mathcal{G})$, $W^{1,2}_{0,0}(\mathcal{G})$, and $\mathcal{W}^{1,2}_{0,0}(\mathcal{G};\tilde{J})$ are equivalent. Furthermore, we have $\mathcal{W}^{l-1/p,p}_{\beta,\vec\delta}(\Gamma_j,\tilde{J}) = V^{l-1/p,p}_{\beta,\vec\delta}(\Gamma_j)$ for $j\in J_0$.

We consider the solution $u\in\mathcal{W}^{1,2}_{0,0}(\mathcal{G};\tilde{J})^\ell$ of the problem

$$b(u,v) = \overline{F(v)} \ \text{ for all } v\in\mathcal{H}, \quad u = g_j \ \text{ on } \Gamma_j \ \text{ for } j\in J_0, \tag{7.3}$$

where F is a given linear and continuous functional on $\mathcal{H}$ and $g_j \in \mathcal{W}^{1/2,2}_{0,0}(\Gamma_j;\tilde{J})^\ell$.

Let $\mathcal{U}$ be a neighborhood of the corner $x^{(k)}$, and let κ be a diffeomorphism mapping $\mathcal{G}\cap\mathcal{U}$ onto $\mathcal{K}\cap B_1$ such that $\kappa'(x^{(k)}) = I$. Here $\mathcal{K}$ is a cone with vertex 0 and B_1 denotes the unit ball. For functions with support in $\mathcal{K}\cap B_1$ we introduce the sesquilinear form

$$\tilde{b}(u,v) = \int_{\mathcal{K}} \Big(\sum_{i,j=1}^{3} \tilde{A}_{i,j}(x)\,\partial_{x_i}u\cdot\partial_{x_j}\overline{v} + \sum_{i=1}^{3} \tilde{A}_i(x)\,\partial_{x_i}u\cdot\overline{v} + \tilde{A}_0(x)u\cdot v\Big)\,dx$$

which is defined by

$$\tilde{b}(u,v) = b(\tilde{u},\tilde{v}), \quad \text{where } \tilde{u}(x) = u\big(\kappa(x)\big),\ \tilde{v}(x) = v\big(\kappa(x)\big) \ \text{ for } x\in\mathcal{G}\cap\mathcal{U}.$$

Furthermore, we set

$$\tilde{b}_0(u,v) = \int_{\mathcal{K}} \sum_{i,j=1}^{3} \tilde{A}_{i,j}(0)\,\partial_{x_i}u\cdot\partial_{x_j}\overline{v}\,dx.$$

Note that $\tilde{A}_{i,j}(0) = A_{i,j}(x^{(k)})$ since $\kappa'(x^{(k)}) = I$.

We denote by $\mathcal{H}_0$ the closure of the set $\{u\in C_0^\infty(\overline{\mathcal{K}})^\ell:\ u=0 \text{ on } \Gamma_j^\circ \text{ for } j\in J_0^{(k)}\}$ with respect to the norm

$$\|u\|_{\mathcal{H}_0} = \Big(\int_{\mathcal{K}} \sum_{j=1}^{3} |\partial_{x_j}u|^2\,dx\Big)^{1/2}.$$

Here we used the same notation as in the preceding subsections.

Lemma 7.1. *Suppose that inequality* (7.2) *is satisfied for all* $u\in\mathcal{H}$. *Then there exists a positive constant* c *such that*

$$|\tilde{b}_0(u,u)| \ge c\,\|u\|^2_{\mathcal{H}_0} \quad \textit{for all } u\in\mathcal{H}_0\,.$$

P r o o f. Let $u\in C_0^\infty(\overline{\mathcal{K}})^\ell$, $u=0$ on Γ_j° for $j\in J_0^{(k)}$, and $\operatorname{supp} u\subset B_\varepsilon$, where B_ε is the ball with radius ε and center 0, and ε is sufficiently small. Using the inequality (7.2) for the function $\tilde{u}(x) = u(\kappa(x))$, we obtain

$$|\tilde{b}(u,u)| \ge c_0'\,\|u\|^2_{W^{1,2}(\mathcal{K})^\ell} - c_1'\,\|u\|^2_{L^2(\mathcal{K})^\ell}$$

with certain positive constants c_0', c_1'. By Hardy's inequality, we have

$$\|u\|^2_{L^2(\mathcal{K})^\ell} \le \varepsilon^2 \int_{\mathcal{K}} \rho^{-2}\,|u|^2\,dx \le c\,\varepsilon^2 \int_{\mathcal{K}} |\nabla u|^2\,dx.$$

Furthermore,

$$\big|\tilde{b}(u,u) - \tilde{b}_0(u,u)\big| \le c\,\varepsilon\,\|\nabla u\|^2_{L^2(\mathcal{K})^\ell} + c\,\|u\|_{L^2(\mathcal{K})^\ell}\,\big(\|u\|_{L^2(\mathcal{K})^\ell} + \|\nabla u\|_{L^2(\mathcal{K})^\ell}\big) \le c'\,\varepsilon\,\|\nabla u\|^2_{L^2(\mathcal{K})^\ell}\,.$$

Thus, for small ε, we have

$$\big|\tilde{b}_0(u,u)\big| \ge \tfrac{c_0'}{2}\,\|\nabla u\|^2_{L^2(\mathcal{K})^\ell}.$$

Applying the similarity mapping $x = y/N$, we obtain this estimate (with the same constant $c_0'/2$) for the function $v(x) = u(x/N)$. Consequently, the assertions holds for all $u\in C_0^\infty(\overline{\mathcal{K}})^\ell$ satisfying $u=0$ on Γ_j° for $j\in J_0^{(k)}$. This proves the lemma. □

Let $\tilde{J}_k = \{j\in\tilde{J}:\ \overline{M}_j \ni x^{(k)}\}$. We define the operator

$$\mathcal{W}^{1,2}_{0,0}(\mathcal{K};\tilde{J}_k)^\ell \ni u \to \tilde{\mathcal{A}}_0 u = \big(F,\,\{g_j\}_{j\in J_0^{(k)}}\big) \in \mathcal{H}_0^* \times \prod_{j\in J_0^{(k)}} V^{1/2,2}_{0,0}(\Gamma_j^\circ)^\ell$$

by

$$\overline{F(v)} = \tilde{b}_0(u,v) \ \text{ for all } v\in\mathcal{H}_0, \quad g_j = u\big|_{\Gamma_j^\circ} \ \text{ for } j\in J_0^{(k)}.$$

From Lemmas 4.1 and 7.1 it follows that $\tilde{\mathcal{A}}_0$ is an isomorphism. Furthermore, let $\mathcal{H}^{l,p}_{\beta,\vec{\delta}}(\mathcal{K})$ be the space of all $F\in\mathcal{H}_0^*$ having the representation

$$F(v) = \int_{\mathcal{K}} \bar{f}\cdot v\,dx + \sum_{j\in J_1^{(k)}} \int_{\Gamma_j^\circ} \bar{g}_j\cdot v\,dx$$

for all $v \in C_0^\infty(\overline{\mathcal{K}}\backslash\{0\})^\ell$, $v = 0$ on Γ_j° for $j \in J_0^{(k)}$, where $f \in \mathcal{W}^{l-2,p}_{\beta,\vec{\delta}}(\mathcal{K};\tilde{J}_k)^\ell$, $g_j \in \mathcal{W}^{l-1-1/p,p}_{\beta,\vec{\delta}}(\Gamma_j^\circ;\tilde{J}_k)^\ell$. The norm in $\mathcal{H}^{l,p}_{\beta,\vec{\delta}}(\mathcal{K})$ is given by

$$\|F\|_{\mathcal{H}^{l,p}_{\beta,\vec{\delta}}(\mathcal{K})} = \|F\|_{\mathcal{H}_0^*} + \|f\|_{\mathcal{W}^{l-2,p}_{\beta,\vec{\delta}}(\mathcal{K};\tilde{J}_k)^\ell} + \sum_{j\in J_1^{(k)}} \|g_j\|_{\mathcal{W}^{l-1-1/p,p}_{\beta,\vec{\delta}}(\Gamma_j^\circ;\tilde{J}_k)^\ell}.$$

By Lemma 5.3, Theorem 5.3, and the analogous results for the Dirichlet and mixed problems, the operator $\tilde{\mathcal{A}}_0$ realizes also an isomorphism

$$\mathcal{W}^{1,2}_{0,0}(\mathcal{K};\tilde{J}_k)^\ell \cap \mathcal{W}^{l,p}_{\beta,\vec{\delta}}(\mathcal{K};\tilde{J}_k)^\ell \to \mathcal{H}^{l,p}_{\beta,\vec{\delta}}(\mathcal{K}) \times \prod_{j\in J_0^{(k)}} \Big(V^{1/2,2}_{0,0}(\Gamma_j^\circ)^\ell \cap V^{l-1/p,p}_{\beta,\vec{\delta}}(\Gamma_j^\circ)^\ell \Big) \tag{7.4}$$

if there are no eigenvalues of the pencil $\mathfrak{A}_k(\lambda)$ in the strip $-1/2 < \operatorname{Re}\lambda \le l-\beta-3/p$ and the components of $\vec{\delta}$ satisfy the inequalities

$$l-\mu_j < \delta_j + 2/p < l \ \text{ for } j \in \tilde{J}, \quad \max(l-\mu_j, 0) < \delta_j + 2/p < l \ \text{ for } j \notin \tilde{J}. \tag{7.5}$$

Theorem 7.1. *Let $u \in \mathcal{W}^{1,2}_{0,0}(\mathcal{G};\tilde{J})^\ell$ be a solution of problem* (7.3). *Suppose that $F \in \mathcal{H}^*$ has the form*

$$F(v) = \int_{\mathcal{G}} \bar{f}\cdot v\,dx + \sum_{j\in J_1} \int_{\Gamma_j} \bar{g}_j \cdot v\,dx$$

for all $v \in C_0^\infty(\overline{\mathcal{G}}\backslash X)^\ell$, $v = 0$ on Γ_j for $j \in J_0$, where $f \in \mathcal{W}^{l-2,p}_{\vec{\beta},\vec{\delta}}(\mathcal{G};\tilde{J})^\ell$, $g_j \in \mathcal{W}^{l-1-1/p,p}_{\vec{\beta},\vec{\delta}}(\Gamma_j;\tilde{J})^\ell$, and that $g_j \in V^{1/2,2}_{0,0}(\Gamma_j)^\ell \cap V^{l-1/p,p}_{\vec{\beta},\vec{\delta}}(\Gamma_j)^\ell$ for $j \in J_0$. If the strip $-1/2 < \operatorname{Re}\lambda \le l-\beta_k-3/p$ is free of eigenvalues of the pencil $\mathfrak{A}_k(\lambda)$, $k = 1,\dots,d$, and the components of $\vec{\delta}$ satisfy (7.5), *then $u \in \mathcal{W}^{l,p}_{\vec{\beta},\vec{\delta}}(\mathcal{G};\tilde{J})^\ell$.*

P r o o f. Suppose first that the support of u is contained in a neighborhood of the vertex $x^{(k)}$. We denote by χ a smooth function on $[0,\infty)$ such $\chi = 1$ in $[0,1]$, $\chi = 0$ in $[2,\infty)$. Furthermore, let $\chi_\varepsilon(x) = \chi(|x|/\varepsilon)$. We introduce the sesquilinear form

$$\tilde{b}_\varepsilon(u,v) = \int_{\mathcal{K}} \bigg(\sum_{i,j=1}^3 \tilde{A}^{(\varepsilon)}_{i,j}\partial_{x_i}u\cdot\partial_{x_j}\overline{v} + \sum_{i=1}^3 \tilde{A}^{(\varepsilon)}_i \partial_{x_i}u\cdot\overline{v} + \tilde{A}^{(\varepsilon)}_0 u\cdot\overline{v}\bigg)\,dx,$$

where

$$\tilde{A}^{(\varepsilon)}_{i,j}(x) = \chi_\varepsilon\tilde{A}_{i,j}(x) + (1-\chi_\varepsilon)\,\tilde{A}_{i,j}(0) \ \text{ for } i,j = 1,2,3, \quad \tilde{A}^{(\varepsilon)}_i(x) = \chi_\varepsilon\,\tilde{A}_i(x) \text{ for } i = 0,1,2,3.$$

Obviously, $\tilde{b}_\varepsilon(u,v) = \tilde{b}(u,v)$ if $u(x) = 0$ for $|x| > \varepsilon$. The form $\tilde{b}_\varepsilon$ generates a linear and continuous operator

$$\mathcal{W}^{1,2}_{0,0}(\mathcal{K};\tilde{J}_k)^\ell \ni u \to \tilde{\mathcal{A}}_\varepsilon u = \big(F,\,\{g_j\}_{j\in J_0^{(k)}}\big) \in \mathcal{H}_0^* \times \prod_{j\in J_0^{(k)}} V^{1/2,2}_{0,0}(\Gamma_j^\circ)^\ell \tag{7.6}$$

by

$$\overline{F(v)} = \tilde{b}_\varepsilon(u,v) \ \text{ for all } v \in \mathcal{H}_0, \quad g_j = u\big|_{\Gamma_j^\circ} \ \text{ for } j \in J_0^{(k)}.$$

The restriction of $\tilde{\mathcal{A}}_\varepsilon$ to $\mathcal{W}^{1,2}_{0,0}(\mathcal{K};\tilde{J}_k)^\ell \cap \mathcal{W}^{l,p}_{\beta_k,\vec{\delta}}(\mathcal{K};\tilde{J}_k)^\ell$ represents a continuous mapping (7.4). Since the norm of the operator $\tilde{\mathcal{A}}_\varepsilon - \tilde{\mathcal{A}}_0$ is small for small ε, we conclude that the operator (7.6) and its restriction to $\mathcal{W}^{1,2}_{0,0}(\mathcal{K};\tilde{J}_k)^\ell \cap \mathcal{W}^{l,p}_{\beta_k,\vec{\delta}}(\mathcal{K};\tilde{J}_k)^\ell$ are isomorphisms if ε is sufficiently small. Hence, under the conditions of the theorem, we obtain $u \in \mathcal{W}^{l,p}_{\vec{\beta},\vec{\delta}}(\mathcal{G};\tilde{J})^\ell$. Analogously, this results holds if the support of u is contained in a sufficiently small neighborhood of an edge point. Using a partition of unity on $\overline{\mathcal{G}}$, we obtain the assertion of the theorem for arbitrary $u \in \mathcal{W}^{1,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G};\tilde{J})^\ell$. □

7.5 Solvability of the boundary value problem

We denote the operator

$$\mathcal{W}^{l,p}_{\vec{\beta},\vec{\delta}}(\mathcal{G};\tilde{J})^\ell \ni u \ \to \big(Lu, \{u|_{\Gamma_j}\}_{j\in J_0}, \{Bu|_{\Gamma_j}\}_{j\in J_1}\big)$$
$$\in \mathcal{W}^{l-2,p}_{\vec{\beta},\vec{\delta}}(\mathcal{G};\tilde{J})^\ell \times \prod_{j\in J_0} V^{l-1/p,p}_{\vec{\beta},\vec{\delta}}(\Gamma_j)^\ell \times \prod_{j\in J_1} \mathcal{W}^{l-1-1/p,p}_{\vec{\beta},\vec{\delta}}(\Gamma_j;\tilde{J})^\ell \tag{7.7}$$

of the boundary value problem (7.1) by $\mathcal{A}$. Our goal is to prove that this operator is Fredholm if the line $\mathrm{Re}\,\lambda = l - \beta_k - 3/p$ is free eigenvalues of the pencil $\mathfrak{A}_k(\lambda)$ for $k = 1,\dots,d$ and the components of $\vec{\delta}$ satisfy (7.5). To this end, we construct a left and right regularizer for $\mathcal{A}$.

Lemma 7.2. *Let $\mathcal{U}$ be a sufficiently small subset of $\mathcal{G}$ and let ϕ be a smooth function with support in $\mathcal{U}$. Suppose that there are no eigenvalues of the pencil $\mathfrak{A}_k(\lambda)$ on the line* $\mathrm{Re}\,\lambda = l - \beta_k - 3/p$ *for $k = 1,\dots,d$ and that the components of $\vec{\delta}$ satisfy* (7.5)*. Then there exists an operator $\mathcal{R}$ continuously mapping the space*

$$\{(f,g) \in \mathcal{W}^{l-2,p}_{\vec{\beta},\vec{\delta}}(\mathcal{G};\tilde{J})^\ell \times \prod_{j\in J_0} V^{l-1/p,p}_{\vec{\beta},\vec{\delta}}(\Gamma_j)^\ell \times \prod_{j\in J_1} \mathcal{W}^{l-1-1/p,p}_{\vec{\beta},\vec{\delta}}(\Gamma_j;\tilde{J})^\ell :\ \mathrm{supp}\,(f,g) \subset \overline{\mathcal{U}}\}$$

into $\mathcal{W}^{l,p}_{\vec{\beta},\vec{\delta}}(\mathcal{G};\tilde{J})^\ell$ such that $\phi\mathcal{A}\mathcal{R}(f,g) = \phi(f,g)$ for all (f,g) with support in $\overline{\mathcal{U}}$ and $\phi\mathcal{R}\mathcal{A}u = \phi u$ for all $u \in \mathcal{W}^{l,p}_{\vec{\beta},\vec{\delta}}(\mathcal{G};\tilde{J})^\ell$, $\mathrm{supp}\, u \subset \overline{\mathcal{U}}$.

P r o o f. In order to simplify the notation, we restrict ourselves here to the Neumann problem. The proof for the Dirichlet and mixed problems proceeds analogously. There exists a diffeomorphism κ mapping $\mathcal{U}$ onto a subset $\mathcal{V}$ of a cone $\mathcal{K}$. The coordinate transformation $y = \kappa(x)$ applied to problem (7.1) yields the equations

$$\tilde{L}\tilde{u} = \tilde{f} \ \text{ in } \mathcal{V}, \quad \tilde{B}\tilde{u} = \tilde{g}_j \ \text{ on } \mathcal{V}\cap\Gamma_j\,,$$

where $\tilde{u} = u\circ\kappa^{-1}$.

Suppose first that $\overline{\mathcal{U}}$ contains the corner $x^{(1)}$ and $\overline{\mathcal{V}}$ contains the vertex $\kappa(x^{(1)}) = 0$ of $\mathcal{K}$. Then we denote by $\tilde{L}_0$ and $\tilde{B}_0$ the principal parts of $\tilde{L}$ and $\tilde{B}$, respectively, with coefficients frozen at $x = 0$. Due to the assumptions on $\vec{\beta}$ and $\vec{\delta}$, the operator $\tilde{\mathcal{A}}_0 = (\tilde{L}_0, \tilde{B}_0)$ realizes an isomophism

$$W^{l,p}_{\beta_1,\vec{\delta}}(\mathcal{K})^\ell \to W^{l-2,p}_{\beta_1,\vec{\delta}}(\mathcal{K})^\ell \times \prod_j W^{l-1-1/p,p}_{\beta_1,\vec{\delta}}(\Gamma_j)^\ell. \tag{7.8}$$

We introduce the differential operators $\tilde{L}_\varepsilon = \chi_\varepsilon\tilde{L} + (1-\chi_\varepsilon)\tilde{L}_0$ and $\tilde{B}_\varepsilon = \chi_\varepsilon\tilde{B} + (1-\chi_\varepsilon)\tilde{B}_0$, where χ_ε is the same cut-off function as in the proof of Theorem 7.1. Since the difference of the operators $\tilde{\mathcal{A}}_0 = (\tilde{L}_0, \tilde{B}_0)$ and $\tilde{\mathcal{A}}_\varepsilon = (\tilde{L}_\varepsilon, \tilde{B}_\varepsilon)$ in the operator norm (7.8) is small for small ε, the operator $\tilde{\mathcal{A}}_\varepsilon$ realizes also an isomorphism (7.8) if ε is sufficiently small. We assume that $\mathcal{V}$ is contained in the ball $|y| < \varepsilon$. Then the coefficients of $\tilde{L}_\varepsilon$, $\tilde{B}_\varepsilon$ coincide with that of $\tilde{L}$ and $\tilde{B}$, respectively, on the support of $(\tilde{f}, \tilde{g})$. Let

$$u(x) = \tilde{u}(\kappa(x)) \ \text{ for } x \in \mathcal{U}, \quad \text{where } \tilde{u} = \tilde{\mathcal{A}}_\varepsilon^{-1}\,(\tilde{f}, \tilde{g}). \tag{7.9}$$

Outside $\mathcal{U}$ let u be continuously extended to a vector function from $W^{l,p}_{\vec{\beta},\vec{\delta}}(\mathcal{G})^\ell$. The so defined mapping $(f,g) \to u$ is denoted by $\mathcal{R}$. It can be easily verified that $\mathcal{R}$ has the desired properties.

Suppose now that $\overline{\mathcal{U}}$ contains an edge point $\xi \in M_1$ but no points of other edges and no corners of $\mathcal{G}$. Then we denote by $\tilde{L}_0$, $\tilde{B}_0$ the principal parts of $\tilde{L}$ and $\tilde{B}$, respectively, with coefficients frozen at $\eta = \kappa(\xi)$. The operators $\tilde{L}_\varepsilon$ and $\tilde{B}_\varepsilon$ are defined as above, where $\chi_\varepsilon(y) = \chi(|y-\eta|/\varepsilon)$ and χ is a smooth function on $[0,\infty)$, $\chi = 1$ in $[0,1]$, $\chi = 0$ in $[2,\infty)$. There exist a number β_0 and a tuple $\vec{\delta}'$, $\delta'_1 = \delta_1$, such that the operator $\tilde{\mathcal{A}}_0 = (\tilde{L}_0, \tilde{B}_0)$ and for sufficiently small ε also the operator $\tilde{\mathcal{A}}_\varepsilon = (\tilde{L}_\varepsilon, \tilde{B}_\varepsilon)$ realize isomorphisms

$$W^{l,p}_{\beta_0,\vec{\delta}'}(\mathcal{K})^\ell \to W^{l-2,p}_{\beta_0,\vec{\delta}'}(\mathcal{K})^\ell \times \prod_j W^{l-1-1/p,p}_{\beta_0,\vec{\delta}'}(\Gamma_j)^\ell.$$

Hence, if $\mathcal{V}$ is contained in the ball $|y-\eta| < \varepsilon$, the conditions of the lemma are satisfied for the operator $(f,g) \to u$ defined by (7.9). Analogously, the lemma can be proved for the case $\mathcal{U}\cap\mathcal{S} = \emptyset$. □

Theorem 7.2. *Suppose that the line* $\mathrm{Re}\,\lambda = l - \beta_k - 3/p$ *does not contain eigenvalues of the pencil $\mathfrak{A}_k(\lambda)$ for $k = 1,\dots,d$ and that the components of $\vec{\delta}$ satisfy* (7.5)*. Then the operator* (7.7) *is Fredholm.*

Proof. Let $\{\mathcal{U}_\nu\}$ be a sufficiently fine open covering of $\mathcal{G}$ and let ϕ_ν, ψ_ν be infinitely differentiable functions such that $\operatorname{supp}\phi_\nu \subset \operatorname{supp}\psi_\nu \subset \mathcal{U}_\nu$, $\phi_\nu\psi_\nu = \phi_\nu$, and $\sum \phi_\nu = 1$. For every ν there exists an operator $\mathcal{R}_\nu$ having the properties of the operator $\mathcal{R}$ in Lemma 7.2 for $\mathcal{U} = \mathcal{U}_\nu \cap \mathcal{G}$. We prove that the operator $\mathcal{R}$ defined by

$$\mathcal{R}(f,g) = \sum_\nu \phi_\nu \mathcal{R}\, \psi_\nu(f,g).$$

is a left and right regularizer for the operator (7.7). We show first that $\mathcal{RA} - I$ is a compact operator on $\mathcal{W}^{l,p}_{\vec{\beta},\vec{\delta}}(\mathcal{G};\tilde{J})^\ell$. Here I denotes the identity operator. Obviously,

$$\mathcal{RA}u = \sum_\nu \phi_\nu \mathcal{R}_\nu \left(\mathcal{A}\psi_\nu u - [\mathcal{A},\psi_\nu]u\right) = u - \sum_\nu \phi_\nu \mathcal{R}_\nu [\mathcal{A},\psi_\nu]u,$$

where $[\mathcal{A},\psi_\nu] = \mathcal{A}\psi_\nu - \psi_\nu\mathcal{A}$ is the commutator of $\mathcal{A}$ and ψ_ν. Here the mapping $u \to [\mathcal{A},\psi_\nu]u$ is continuous from $\mathcal{W}^{l,p}_{\vec{\beta},\vec{\delta}}(\mathcal{G};\tilde{J})^\ell$ into

$$\mathcal{W}^{l-1,p}_{\vec{\beta}',\vec{\delta}'}(\mathcal{G};\tilde{J})^\ell \times \prod_{j\in J_0} V^{l+1-1/p,p}_{\vec{\beta}',\vec{\delta}'}(\Gamma_j)^\ell \times \prod_{j\in J_1} \mathcal{W}^{l-1/p,p}_{\vec{\beta}',\vec{\delta}'}(\Gamma_j;\tilde{J})^\ell, \tag{7.10}$$

where the components of $\vec{\beta}'$, $\vec{\delta}'$ satisfy the inequalities $\beta_k \le \beta_k' < \beta_k + 1$ for $k = 1,\dots,d$ and $\delta_j \le \delta_j' < \delta_j + 1$ for $j = 1,\dots,m$. We can choose $\vec{\beta}'$ and $\vec{\delta}'$ such that the strip $l - \beta_k - 3/p \le \operatorname{Re}\lambda \le l + 1 - \beta_k' - 3/p$ is free of eigenvalues of the pencil $\mathfrak{A}_k(\lambda)$ for $k = 1,\dots,d$, $l + 1 - \mu_j < \delta_j' + 2/p < l + 1$ for $j \in \tilde{J}$, and $\max(l + 1 - \mu_j, 0) < \delta_j' + 2/p < l + 1$ for $j = 1,\dots,m$, $j \notin \tilde{J}$. Then, by Theorem 7.1, the operator $\mathcal{R}_\nu$ maps the space (7.10) into $\mathcal{W}^{l+1,p}_{\vec{\beta}',\vec{\delta}'}(\mathcal{G};\tilde{J})^\ell$. Since the last space is compactly imbedded into $\mathcal{W}^{l,p}_{\vec{\beta},\vec{\delta}}(\mathcal{G};\tilde{J})^\ell$, it follows that the operator $\mathcal{RA} - I$ is compact. Analogously the compactness of $\mathcal{AR} - I$ can be proved. This means that $\mathcal{R}$ is a left and right regularizer. Consequently, the operator $\mathcal{A}$ is Fredholm. □

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