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Sharp Pointwise Interpolation Inequalities for Derivatives*

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Abstract. We prove new pointwise inequalities involving the gradient of a function $u \in C^1(\mathbb{R}^n)$, the modulus of continuity ω of the gradient ∇u, and a certain maximal function $\mathscr{M}^\diamond u$ and show that these inequalities are sharp. A simple particular case corresponding to $n = 1$ and $\omega(r) = r$ is the Landau type inequality

$$|u'(x)|^2 \leqslant \frac{8}{3}\,\mathscr{M}^\diamond u(x)\mathscr{M}^\diamond u''(x),$$

where the constant $8/3$ is best possible and

$$\mathscr{M}^\diamond u(x) = \sup_{r>0} \frac{1}{2r}\left|\int_{x-r}^{x+r} \operatorname{sign}(y-x)u(y)\,dy\right|.$$

1. Introduction

Background. The idea of obtaining information about intermediate derivatives by using properties of a higher derivative and the function itself goes back to Hadamard [5], Kneser [10], and Hardy and Littlewood [6]. It was developed in various directions by Kolmogorov [11], Szőkefalvi-Nagy [22], Gagliardo [4], Nirenberg [20], et al. In the simplest form, this idea is expressed by the Landau inequality [12]

$$|u'(x)|^2 \leqslant 2\,\|u\|_{L_\infty}\|u''\|_{L_\infty} \tag{1}$$

on the real line. Under the additional assumption that $u \geqslant 0$ on $\mathbb{R}$, one can readily verify the estimate

$$|u'(x)|^2 \leqslant 2u(x)\|u''\|_{L_\infty}, \tag{2}$$

which has proved to be useful in various topics of the theory of differential and pseudodifferential operators [8, 9, 13, 14, 21]. Some versions and extensions of (2) were treated in [15].

Clearly, (2) fails for some smooth function u. However, one can ask whether it is possible to replace the L_∞-norms in (1) by the values at x of certain operators acting on u. There are different ways to give an affirmative answer to this question. In particular, in [16] we arrived at the pointwise inequality

$$|u'(x)|^{\alpha+1} \leqslant \frac{2^{\alpha+1}}{\alpha+2}\left(\frac{\alpha+1}{\alpha}\right)^{\alpha}(\mathscr{M}^\diamond u(x))^\alpha \sup_{y\in\mathbb{R}} \frac{|u'(y)-u'(x)|}{|y-x|^\alpha}, \tag{3}$$

where $\alpha > 0$ and $\mathscr{M}^\diamond$ is the maximal operator defined by

$$\mathscr{M}^\diamond u(x) = \sup_{r>0} \frac{1}{2r}\left|\int_{x-r}^{x+r} \operatorname{sign}(y-x)u(y)\,dy\right|. \tag{4}$$

The constant in this inequality is best possible. For $\alpha = 1$, this implies the sharp estimate

$$|u'(x)|^2 \leqslant \frac{8}{3}\,\mathscr{M}^\diamond u(x)\|u''\|_{L_\infty}. \tag{5}$$

Diverse pointwise interpolation inequalities for derivatives of integer or fractional order were obtained in [1, 7, 15–19] without best constants. We note that the operator

$$(T_\alpha u)(x) = \sup_{y\in\mathbb{R}^n} \frac{|u(y)-u(x)|}{|y-x|^\alpha},$$

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which occurs in (3), has been extensively studied (e.g., see [2]). In the above-mentioned articles, some applications of these inequalities were pointed out (e.g., the Gagliardo–Nirenberg estimates, the composition operator in fractional Sobolev spaces, pointwise multipliers, etc.).

Notation. Let $\mathscr{M}^\diamond$ be the multi-dimensional generalization of (4) defined by

$$\mathscr{M}^\diamond u(x) = \sup_{r>0} \left| \fint_{B_r(x)} \frac{y-x}{|y-x|}\, u(y)\, dy \right|, \tag{6}$$

where u is a locally integrable function on $\mathbb{R}^n$, $n \geqslant 1$, $B_r(x)$ is the ball $\{y \in \mathbb{R}^n : |x-y| < r\}$, and the bar stands for the mean value of the integral. Clearly, $\mathscr{M}^\diamond u(x)$ does not exceed the Fefferman–Stein sharp maximal function [3]

$$\mathscr{M}^\sharp u(x) = \sup_{r>0} \fint_{B_r(x)} \left| u(y) - \fint_{B_r(x)} u(z)\, dz \right| dy.$$

We write B_r instead of $B_r(0)$ and use the notation $|B_1| = \mathrm{mes}_n\, B_1$ and $|S^{n-1}| = \mathrm{mes}_{n-1}\, \partial B_1$. We introduce the mean values of a vector function $\mathbf{v}\colon \mathbb{R}^n \to \mathbb{R}^n$ over the sphere $\partial B_r(x)$ and the ball $B_r(x)$ as follows:

$$A^{(1)}\mathbf{v}(x;r) = \fint_{\partial B_r(x)} \mathbf{v}(y)\, ds_y, \qquad A^{(2)}\mathbf{v}(x;r) = \fint_{B_r(x)} \mathbf{v}(y)\, dy.$$

We also set

$$D^{(1)}_\omega(\mathbf{v};x) = \sup_{r>0} \frac{|\mathbf{v}(x) - A^{(1)}\mathbf{v}(x;r)|}{\omega(r)}, \tag{7}$$

$$D^{(2)}_\omega(\mathbf{v};x) = \sup_{r>0} \frac{|\mathbf{v}(x) - A^{(2)}\mathbf{v}(x;r)|}{\omega(r)}. \tag{8}$$

In particular, for a function v of one variable we have

$$D^{(1)}_\omega(v;x) = \sup_{r>0} \frac{|2v(x) - v(x+r) - v(x-r)|}{2\omega(r)}, \qquad D^{(2)}_\omega(v;x) = \sup_{r>0} \frac{|v(x) - \frac{1}{2r}\int_{x-r}^{x+r} v(y)\, dy|}{\omega(r)}.$$

Throughout the paper, we assume that ω is a continuous nondecreasing function on $[0,\infty)$ such that $\omega(0) = 0$ and $\omega(\infty) = \infty$.

Description of results. In this paper, we obtain new pointwise inequalities involving the gradient $\nabla u(x)$, the maximal function $\mathscr{M}^\diamond u(x)$, and one of the functions $D^{(1)}_\omega(\nabla u;x)$ and $D^{(2)}_\omega(\nabla u;x)$. For example, in Theorem 1 we prove that

$$|\nabla u(x)| \leqslant n(n+1) D^{(1)}_\omega(\nabla u;x)\, \Phi\left(\frac{\mathscr{M}^\diamond u(x)}{n D^{(1)}_\omega(\nabla u;x)} \right), \tag{9}$$

where Φ is a certain strictly increasing function generated by the modulus of continuity ω. We also show that inequality (9) and the similar inequality for $D^{(2)}_\omega(\nabla u;x)$ in Theorem 2 are sharp. Namely, we present functions for which these inequalities become equalities. These extremal functions, whose form is rather complicated, were found by guess. There is no standard approach to such a construction for the time being.

In particular, if $\omega(r) = r^\alpha$, $\alpha > 0$, then inequality (9) becomes

$$|\nabla u(x)| \leqslant C_1 (\mathscr{M}^\diamond u(x))^{\alpha/(\alpha+1)} \left(\sup_{r>0} \frac{|\nabla u(x) - A^{(1)}\nabla u(x;r)|}{r^\alpha} \right)^{1/(\alpha+1)} \tag{10}$$

with the best constant

$$C_1 = (n+1)\frac{\alpha+1}{\alpha} \left(\frac{\alpha n}{(n+\alpha)(n+\alpha+1)} \right)^{1/(\alpha+1)}. \tag{11}$$

Note that for $n = 1$ the last estimate acquires the form

$$|u'(x)| \leqslant C_1(\mathscr{M}^\diamond u(x))^{\alpha/(\alpha+1)}\left(\sup_{r>0}\frac{|2u'(x)-u'(x+r)-u'(x-r)|}{r^\alpha}\right)^{1/(\alpha+1)}, \tag{12}$$

where the best value of C_1 is given by

$$C_1 = \left(\frac{2(\alpha+1)}{\alpha(\alpha+2)^{1/\alpha}}\right)^{\alpha/(\alpha+1)}. \tag{13}$$

A similar corollary of Theorem 2 concerning $D^{(2)}_\omega(\nabla u; x)$ is the inequality

$$|u'(x)| \leqslant C_2(\mathscr{M}^\diamond u(x))^{\alpha/(\alpha+1)}\left(\sup_{r>0}\frac{|u'(x)-\frac{u(x+r)+u(x-r)}{2r}|}{r^\alpha}\right)^{1/(\alpha+1)} \tag{14}$$

with the best constant

$$C_2 = \frac{2(\alpha+1)}{\alpha}\left(\frac{\alpha}{\alpha+2}\right)^{1/(\alpha+1)}. \tag{15}$$

An interesting particular case of (12), corresponding to $\alpha = 1$, can be written as

$$|u'(x)| \leqslant \left(\frac{8}{3}\right)^{1/2}(\mathscr{M}^\diamond u(x))^{1/2}(\mathscr{M}^\diamond u''(x))^{1/2}, \tag{16}$$

which is a purely pointwise improvement of (5). The constant $(8/3)^{1/2}$ in (16) is sharp, since, as was mentioned above, it is sharp in the rougher inequality (5).

2. Pointwise Inequality in Terms of $D^{(1)}_\omega(\nabla u; x)$

The objective of this section is to obtain the following result.

Theorem 1. (i) *Let the function*

$$\Omega_1(t) := \int_0^1 (1-n+n\sigma)\sigma^{n-1}\omega(\sigma t)\,d\sigma \tag{17}$$

be strictly increasing on $[0,\infty)$, *and let* Ω_1^{-1} *be the inverse function of* Ω_1. *Further, let*

$$\Psi_1(t) = \int_0^t \Omega_1^{-1}(\tau)\,d\tau.$$

Then

$$|\nabla u(x)| \leqslant n(n+1)D^{(1)}_\omega(\nabla u; x)\Psi_1^{-1}\left(\frac{\mathscr{M}^\diamond u(x)}{nD^{(1)}_\omega(\nabla u; x)}\right) \tag{18}$$

for any $u \in C^1(\mathbb{R}^n)$,*where* Ψ_1^{-1} *is the inverse function of* Ψ_1.

(ii) *Let* $\omega \in C^1(0,\infty)$. *Suppose that the function* $t\omega'(t)$ *is nondecreasing on* $(0,\infty)$ *and, for* $n > 1$, *the function* $t\Omega_1'(t)$ *is nondecreasing on* $(0,\infty)$. *Let* R *be the unique root of the equation*

$$n(n+1)\Omega_1(t) = 1. \tag{19}$$

Inequality (18) *becomes an equality for the function*

$$u(x) = \begin{cases} x_n\left(1-n\displaystyle\int_0^1 \sigma^{n-1}\omega(\sigma|x|)\,d\sigma\right) & \text{for } 0 \leqslant |x| \leqslant R,\\ \dfrac{nx_n}{|x|}((n+1)R-|x|) & \\ \quad\times\displaystyle\int_0^1 ((n+1)\sigma-n)\sigma^{n-1}\omega\left(\sigma\frac{(n+1)R-|x|}{n}\right)d\sigma & \text{for } R < |x| < (n+1)R,\\ 0 & \text{for } |x| \geqslant (n+1)R. \end{cases} \tag{20}$$

Proof. (i) It suffices to prove (18) for $x = 0$. We have

$$\int_{B_1} (\nabla u(0) - \nabla u(y))(1 - |y|)\, dy = \frac{1}{n(n+1)}\, \nabla u(0)|S^{n-1}| - \int_{B_1} u(y)\frac{y}{|y|}\, dy. \tag{21}$$

Hence,

$$\frac{|B_1|}{n+1}\, \nabla u(0) = \int_{B_1} u(y)\frac{y}{|y|}\, dy + |S^{n-1}| \int_0^1 r^{n-1}(1 - r)(\nabla u(0) - A^{(1)}\nabla u(0; r))\, dr. \tag{22}$$

After the scaling $y \to y/t$, $r \to r/t$, Eq. (22) becomes

$$\frac{|B_1|}{n+1}\, t\nabla u(0) = \frac{1}{t^n} \int_{B_t} u(y)\frac{y}{|y|}\, dy + |S^{n-1}|\frac{1}{t^n} \int_0^t r^{n-1}(t - r)(\nabla u(0) - A^{(1)}u(0; r))\, dr. \tag{23}$$

This implies that

$$|\nabla u(0)| \leqslant \frac{n+1}{t}\, \mathscr{M}^{\diamond} u(0) + \frac{n(n+1)}{t^{n+1}}\, D_\omega^{(1)}(\nabla u; 0) \int_0^t r^{n-1}(t - r)\omega(r)\, dr,$$

which can be rewritten as

$$0 \leqslant -t\,|\nabla u(0)| + (n+1)\mathscr{M}^{\diamond} u(0) + n(n+1)D_\omega^{(1)}(\nabla u; 0) \int_0^t \Omega_1(\tau)\, d\tau. \tag{24}$$

Since Ω_1 is strictly increasing, it follows that the right-hand side of (24) attains its minimum value at

$$t_* = \Omega_1^{-1}\Big(\frac{|\nabla u(0)|}{n(n+1)D_\omega^{(1)}(\nabla u; 0)}\Big). \tag{25}$$

Thus, by (17) one has

$$\begin{aligned} 0 &\leqslant (n+1)\mathscr{M}^{\diamond} u(0) - |\nabla u(0)|\, \Omega_1^{-1}\Big(\frac{|\nabla u(0)|}{n(n+1)D_\omega^{(1)}(\nabla u; 0)}\Big) \\ &\quad + n(n+1)D_\omega^{(1)}(\nabla u; 0) \int_0^{\Omega_1^{-1}\big(\frac{|\nabla u(0)|}{n(n+1)D_\omega^{(1)}(\nabla u;0)}\big)} \Omega_1(\tau)\, d\tau \\ &= (n+1)\mathscr{M}^{\diamond} u(0) - n(n+1)D_\omega^{(1)}(\nabla u; 0) \int_0^{\Omega_1^{-1}\big(\frac{|\nabla u(0)|}{n(n+1)D_\omega^{(1)}(\nabla u;0)}\big)} x\, d\Omega_1(x). \end{aligned}$$

Therefore,

$$\mathscr{M}^{\diamond} u(0) \geqslant nD_\omega^{(1)}(\nabla u; 0) \int_0^{\frac{|\nabla u(0)|}{n(n+1)D_\omega^{(1)}(\nabla u;0)}} \Omega_1^{-1}(\tau)\, d\tau,$$

which is equivalent to (18).

(ii) First, let us prove that Eq. (19) has a unique root. Note that $\Omega_1(0) = 0$ by (17). Since $t\Omega_1'(t)$ is nondecreasing, one has $\Omega_1(\infty) = \infty$. It remains to show that $\Omega_1'(t) > 0$ for $t > 0$. To this end, we only need to check that $t\Omega_1'(t)|_{t=0} = 0$. Since the function (17) can be written as

$$\Omega_1(t) = \frac{n}{t^{n+1}} \int_0^t \tau^n \omega(\tau)\, d\tau - \frac{n-1}{t^n} \int_0^t \tau^{n-1}\omega(\tau)\, d\tau, \tag{26}$$

we see that

$$n\int_0^t \Omega_1(\tau)\, d\tau - \frac{1}{t^n} \int_0^t \tau^n \omega(\tau)\, d\tau = \frac{n}{t^{n-1}} \int_0^t \tau^{n-1}\omega(\tau)\, d\tau - \frac{n+1}{t^n} \int_0^t \tau^n \omega(\tau)\, d\tau. \tag{27}$$

Hence,

$$\frac{1}{t^n} \int_0^t \tau^n \omega(\tau)\, d\tau - (n-1) \int_0^t \Omega_1(\tau)\, d\tau = t\Omega_1(t), \tag{28}$$

and thus

$$t\Omega_1'(t) = \omega(t) - \frac{n}{t^{n+1}} \int_0^t \tau^n \omega(\tau)\, d\tau - n\Omega_1(t). \tag{29}$$

The last relation, combined with $\omega(0) = \Omega_1(0) = 0$, shows that $t\Omega_1'(t)$ vanishes at $t = 0$.

Now let us prove that u defined by (20) is in $C^1(\mathbb{R}^n)$. We claim that u is continuous on the sphere $|x| = R$ together with its first partial derivatives. We use spherical coordinates to write $x_n = r\cos\theta$. Denote the function $u(x)/\cos\theta$ by $u_1(r)$ for $0 \leqslant |x| \leqslant R$ and by $u_2(r)$ for $R < |x| < (n+1)R$; i.e.,

$$u_1(r) = r - \frac{n}{r^{n-1}} \int_0^r t^{n-1}\omega(t)\,dt, \tag{30}$$

$$u_2(r) = n^2\left(\frac{n+1}{\kappa_r^n}\int_0^{\kappa_r} t^n\omega(t)\,dt - \frac{n}{\kappa_r^{n-1}}\int_0^{\kappa_r} t^{n-1}\omega(t)\,dt\right), \tag{31}$$

where

$$\kappa_r = n^{-1}((n+1)R - r). \tag{32}$$

Clearly,

$$u_2(R) = n^2\left(\frac{n+1}{R^n}\int_0^R t^n\omega(t)\,dt - \frac{n}{R^{n-1}}\int_0^R t^{n-1}\omega(t)\,dt\right). \tag{33}$$

By (26), we can write

$$u_2(R) = n\left((n+1)R\,\Omega_1(R) - \frac{1}{R^{n-1}}\int_0^R t^{n-1}\omega(t)\,dt\right).$$

It follows from (19) and (30) that the right-hand side is equal to $u_1(R)$. Let us show that

$$u_2'(R) = u_1'(R). \tag{34}$$

From (30) we obtain

$$u_1'(r) = 1 + \frac{n(n-1)}{r^n}\int_0^r t^{n-1}\omega(t)\,dt - n\omega(r). \tag{35}$$

By (31),

$$u_2'(r) = n^2\left(\frac{n+1}{\kappa_r^{n+1}}\int_0^{\kappa_r} t^n\omega(t)\,dt - \frac{n-1}{\kappa_r^n}\int_0^{\kappa_r} t^{n-1}\omega(t)\,dt - \frac{\omega(\kappa_r)}{n}\right). \tag{36}$$

Therefore,

$$u_2'(R) = n^2\left(\frac{n+1}{R^{n+1}}\int_0^R t^n\omega(t)\,dt - \frac{n-1}{R^n}\int_0^R t^{n-1}\omega(t)\,dt - \frac{\omega(R)}{n}\right),$$

which by (26) can be rewritten as

$$u_2'(R) = n(n+1)\Omega_1(R) + \frac{n(n-1)}{R^n}\int_0^R t^{n-1}\omega(t)\,dt - n\omega(R).$$

Using (19) and (35), we arrive at (34). It remains to note that

$$u_2((n+1)R) = 0, \qquad u_2'((n+1)R) = 0$$

by (31) and (36). Hence, $u \in C^1(\mathbb{R}^n)$.

Our next goal is to show that

$$\mathscr{M}^\diamond u(0) = \frac{R}{n+1} - n\int_0^R \Omega_1(t)\,dt. \tag{37}$$

Let us find the maxima of the function

$$r \to M_r u := \frac{1}{|B_r|}\left|\int_{B_r} \frac{y}{|y|}\,u(y)\,dy\right| \tag{38}$$

on $[0, R]$ and $[R, (n+1)R]$ separately. Recall that for $0 \leqslant |x| \leqslant R$ the function u can be represented as $\cos\theta\, u_1(r)$, where u_1 is defined by (30). It is clear that the function (38) is equal to

$$\frac{2|S^{n-2}|}{|B_1|r^n}\left(\int_0^r \rho^n d\rho\int_0^{\pi/2}(\cos\theta)^2(\sin\theta)^{n-2}d\theta - n\int_0^r\int_0^\rho t^{n-1}\omega(t)\,dt\,d\rho\int_0^{\pi/2}(\cos\theta)^2(\sin\theta)^{n-2}d\theta\right).$$

Since

$$2\,|S^{n-2}|\int_0^{\pi/2}(\cos\theta)^2(\sin\theta)^{n-2}d\theta = n^{-1}|S^{n-1}|, \tag{39}$$

it follows that

$$M_r u = \frac{r}{n+1} - \frac{n}{r^n}\int_0^r\int_0^\rho t^{n-1}\omega(t)\,dt\,d\rho = \frac{r}{n+1} - n\bigg(\frac{1}{r^{n-1}}\int_0^r \tau^{n-1}\omega(\tau)\,d\tau - \frac{1}{r^n}\int_0^r \tau^n\omega(\tau)\,d\tau\bigg),$$

and by (28) we arrive at

$$M_r u = \frac{r}{n+1} - n\int_0^r \Omega_1(t)\,dt. \tag{40}$$

As was proved above, $\Omega_1'(t) > 0$ for $t > 0$. Therefore,

$$\max_{0\leqslant r\leqslant R} M_r u = M_R u. \tag{41}$$

Now let us prove that

$$\max_{R\leqslant r\leqslant (n+1)R} M_r u = M_R u. \tag{42}$$

By (38), one has

$$M_r u = \frac{1}{|B_1|r^n}\bigg(|B_1|R^n\bigg(\frac{R}{n+1} - n\int_0^R \Omega_1(t)\,dt\bigg) + 2|S^{n-2}|\int_0^{\pi/2}(\cos\theta)^2(\sin\theta)^{n-2}\,d\theta\int_R^r u_2(\rho)\rho^{n-1}\,d\rho\bigg).$$

In view of (39), we obtain

$$M_r u = r^{-n}\bigg(R^n\bigg(\frac{R}{n+1} - n\int_0^R \Omega_1(t)\,dt\bigg) + \int_R^r u_2(\rho)\rho^{n-1}\,d\rho\bigg).$$

Combining this with (40), we see that to prove (42) we need to show that the function

$$A(r) := (r^n - R^n)\bigg(\frac{R}{n+1} - n\int_0^R \Omega_1(t)\,dt\bigg) - \int_R^r u_2(\rho)\rho^{n-1}\,d\rho \tag{43}$$

is nonnegative on the interval $[R, (n+1)R]$. Clearly,

$$A'(r) = nr^{n-1}\bigg(\frac{R}{n+1} - n\int_0^R \Omega_1(t)\,dt - \frac{1}{n}\,u_2(r)\bigg).$$

Note that by (31) and (26)

$$-\frac{1}{n}\,u_2(r) = n\bigg(\int_0^{\kappa_r}\Omega_1(t)\,dt - \kappa_r\Omega_1(\kappa_r)\bigg) = -n\int_0^{\kappa_r} t\Omega_1'(t)\,dt. \tag{44}$$

Integrating by parts and using (19), we find that

$$A'(r) = nr^{n-1}\bigg(n\int_0^R t\Omega_1'(t)\,dt - n\int_0^{\kappa_r} t\Omega_1'(t)\,dt\bigg) \geqslant 0.$$

Since $A(R) = 0$, it follows from the last inequality that $A(r) \geqslant 0$ for $R \leqslant r \leqslant (n+1)R$. Thus, (37) holds.

Let us now justify the relation

$$\sup_{r>0}\frac{|\nabla u(0) - A^{(1)}\nabla u(0;r)|}{\omega(r)} = 1. \tag{45}$$

Let $0 \leqslant r \leqslant R$. It follows from (20) that

$$\frac{\partial u}{\partial x_n} = 1 - \frac{n}{r^n}\int_0^r t^{n-1}\omega(t)\,dt - (\cos\theta)^2\bigg(n\omega(r) - \frac{n^2}{r^n}\int_0^r t^{n-1}\omega(t)\,dt\bigg), \tag{46}$$

which, together with (39), implies that

$$A^{(1)}\frac{\partial u}{\partial x_n}(0;r)=1-\omega(r).$$

Hence,

$$\frac{\partial u}{\partial x_n}(0)-A^{(1)}\frac{\partial u}{\partial x_n}(0;r)=\omega(r).$$

Combining this fact with the formulas

$$\frac{\partial u}{\partial x_k}(0)=0,\quad A^{(1)}\frac{\partial u}{\partial x_k}(0;r)=0,\qquad k=1,\dots,n-1,$$

we obtain

$$\frac{|\nabla u(0)-A^{(1)}\nabla u(0;r)|}{\omega(r)}=1\quad\text{for } 0\leqslant r\leqslant R. \tag{47}$$

We now claim that

$$\omega(\kappa_r)\leqslant\frac{\partial u}{\partial x_n}(0)-A^{(1)}\frac{\partial u}{\partial x_n}(0;r)\leqslant\omega(r)\quad\text{for } R\leqslant r\leqslant(n+1)R. \tag{48}$$

Note that

$$\frac{\partial u}{\partial x_n}=\frac{1}{r}\,u_2(r)-(\cos\theta)^2\left(\frac{u_2(r)}{r}-u_2'(r)\right),$$

where u_2 is given by (31). In view of (39), one has

$$A^{(1)}\frac{\partial u}{\partial x_n}(0;r)=\frac{n-1}{n}\,\frac{u_2(r)}{r}+\frac{1}{n}\,u_2'(r). \tag{49}$$

By (31) and (36),

$$\begin{aligned}A^{(1)}\frac{\partial u}{\partial x_n}(0;r)=\frac{n(n-1)}{r}&\left(\frac{n+1}{\kappa_r^n}\int_0^{\kappa_r}t^n\omega(t)\,dt-\frac{n}{\kappa_r^{n-1}}\int_0^{\kappa_r}t^{n-1}\omega(t)\,dt\right)\\&+n\left(\frac{n+1}{\kappa_r^{n+1}}\int_0^{\kappa_r}t^n\omega(t)\,dt-\frac{n-1}{\kappa_r^n}\int_0^{\kappa_r}t^{n-1}\omega(t)\,dt\right)-\omega(\kappa_r).\end{aligned} \tag{50}$$

Next, observe that

$$\frac{n+1}{\kappa_r^n}\int_0^{\kappa_r}t^n\omega(t)\,dt-\frac{n}{\kappa_r^{n-1}}\int_0^{\kappa_r}t^{n-1}\omega(t)\,dt=\int_0^{\kappa_r}\left(1-\frac{t}{\kappa_r}\right)\left(\frac{t}{\kappa_r}\right)^{n-1}t\omega'(t)\,dt\geqslant 0,$$

since ω is nondecreasing. This, together with (50) and the inequality $r\geqslant\kappa_r$, yields

$$A^{(1)}\frac{\partial u}{\partial x_n}(0;r)\leqslant n(n+1)\left(\frac{n}{\kappa_r^{n+1}}\int_0^{\kappa_r}t^n\omega(t)\,dt-\frac{n-1}{\kappa_r^n}\int_0^{\kappa_r}t^{n-1}\omega(t)\,dt\right)-\omega(\kappa_r).$$

By (26), this inequality can be rewritten as

$$A^{(1)}\frac{\partial u}{\partial x_n}(0;r)\leqslant n(n+1)\Omega_1(\kappa_r)-\omega(\kappa_r). \tag{51}$$

Since Ω_1 is strictly increasing and $\kappa_r<R$, it follows that

$$A^{(1)}\frac{\partial u}{\partial x_n}(0;r)\leqslant n(n+1)\Omega_1(R)-\omega(\kappa_r).$$

We now use the identity

$$n(n+1)\Omega_1(R)=1=\frac{\partial u}{\partial x_n}(0)$$

to obtain

$$A^{(1)}\frac{\partial u}{\partial x_n}(0;r)\leqslant\frac{\partial u}{\partial x_n}(0)-\omega(\kappa_r),$$

which implies the left inequality in (48).

To prove the right inequality in (48), we note that relation (50) can be rewritten as

$$A^{(1)}\frac{\partial u}{\partial x_n}(0;r) = \frac{n(n-1)}{r}\left(\frac{1}{\kappa_r^n}\int_0^{\kappa_r} t^n\omega(t)\,dt - n\int_0^{\kappa_r}\Omega_1(t)\,dt\right) + \frac{n}{\kappa_r^{n+1}}\int_0^{\kappa_r} t^n\omega(t)\,dt + n\Omega_1(\kappa_r) - \omega(\kappa_r) \tag{52}$$

in view of (27).

Let

$$B(r) := r\omega(r) - r + rA^{(1)}\frac{\partial u}{\partial x_n}(0;r). \tag{53}$$

The right inequality in (48) is equivalent to the inequality $B(r) \geqslant 0$, $r \in [R,(n+1)R]$. By (52), we have

$$B(r) := r\omega(r) - r + n(n-1)\left(\frac{1}{\kappa_r^n}\int_0^{\kappa_r} t^n\omega(t)\,dt - n\int_0^{\kappa_r}\Omega_1(t)\,dt\right) + r\left(\frac{n}{\kappa_r^{n+1}}\int_0^{\kappa_r} t^n\omega(t)\,dt + n\Omega_1(\kappa_r) - \omega(\kappa_r)\right). \tag{54}$$

Using the relations $\kappa_R = R$ and $n(n+1)\Omega_1(R) = 1$, we obtain

$$B(R) = \frac{n^2}{R^n}\int_0^R t^n\omega(t)\,dt - n^2(n-1)\int_0^R \Omega_1(t)\,dt - \frac{n}{n+1}R. \tag{55}$$

We note that relation (55), together with (28) for $t = R$, gives $B(R) = 0$.

The next step is to show that $B'(r) \geqslant 0$ for $r \in [R,(n+1)R]$. Combining (49) with (44), we see that

$$A^{(1)}\frac{\partial u}{\partial x_n}(0;r) = \frac{n(n-1)}{r}\int_0^{\kappa_r} t\Omega_1'(t)\,dt - \kappa_r\Omega_1'(\kappa_r)$$

which, together with (53), gives

$$B(r) = r\omega(r) - r + n(n-1)\int_0^{\kappa_r} t\Omega_1'(t)\,dt - r\kappa_r\Omega_1'(\kappa_r).$$

Clearly,

$$B'(r) = (r\omega(r))' - 1 - n\kappa_r\Omega_1'(\kappa_r) + \frac{r}{n}(t\Omega_1'(t))'|_{t=\kappa_r}. \tag{56}$$

For $n = 1$, by (29) and (26), one has

$$t\Omega_1'(t) = \omega(t) - \frac{2}{t^2}\int_0^t \tau\omega(\tau)\,d\tau = \frac{1}{t^2}\int_0^t \tau^2\omega'(\tau)\,d\tau.$$

Since $t\omega'(t)$ is nondecreasing, it follows that

$$\left(t^{-2}\int_0^t \tau^2\omega'(\tau)\,d\tau\right)' \geqslant 0.$$

Thus, $t\Omega_1'(t)$ is also nondecreasing for $n = 1$. Hence, it follows from the assumption of the theorem that both functions $t\Omega_1'(t)$ and $t\omega'(t)$ are nondecreasing for $n \geqslant 1$. Therefore, the last term on the right-hand side in (56) is nonnegative, and $\kappa_r\Omega_1'(\kappa_r) \leqslant R\Omega_1'(R)$ for $r \geqslant R$. Thus,

$$B'(r) \geqslant \omega(R) + R\omega'(R) - 1 - nR\Omega_1'(R) \tag{57}$$

for $r \in [R,(n+1)R]$. Owing to relation (29) for $t = R$, the last inequality can be rewritten as

$$B'(r) \geqslant R\omega'(R) - (n-1)\omega(R) + \frac{n^2}{R^{n+1}}\int_0^R t^n\omega(t)\,dt - \frac{1}{n+1}\,. \tag{58}$$

By (26) for $t = R$, relation (58) gives

$$B'(r) \geqslant R\omega'(R) - (n-1)\omega(R) + \frac{n(n-1)}{R^n}\int_0^R t^{n-1}\omega(t)\,dt.$$

Integrating by parts on the right-hand side, we obtain

$$B'(r) \geqslant R\omega'(R) - \frac{n-1}{R^n}\int_0^R t^n\omega'(t)\,dt \geqslant \frac{1}{R^n}\int_0^R t^n(t\omega'(t))'\,dt.$$

Since the function $t\omega'(t)$ is nondecreasing, it follows that the right-hand side is nonnegative. This implies the right inequality in (48) and, together with (47), leads to (45).

Finally, we must show that inequality (18) becomes an equality for u given by (24). It follows from (19) and (37) that

$$n\int_0^{1/(n(n+1))}\Omega_1^{-1}(\tau)\,d\tau = n\int_0^R t\,d\Omega_1(t) = n\biggl(R\,\Omega_1(R) - \int_0^R \Omega_1(t)\,dt\biggr)$$
$$= \frac{R}{n+1} - n\int_0^R \Omega_1(t)\,dt = \mathscr{M}^\diamond u(0).$$

By (45), the right-hand side of (18) is equal to $n(n+1)\Psi_1^{-1}(\Psi_1(1/(n(n+1)))) = 1$. The proof is complete.

Remark 1. In general, for $n > 1$ the assumption that $t\omega'(t)$ is nondecreasing does not imply that $t\Omega_1'(t)$ is nondecreasing. Let us prove this. Note that

$$(t\Omega_1'(t))' = \omega'(t) + \frac{n(n+1)}{t^{n+2}}\int_0^t \tau^n\omega(\tau)\,d\tau - \frac{n}{t}\,\omega(t) - n\Omega_1'(t)$$
$$= \omega'(t) + \frac{n(n+1)^2}{t^{n+2}}\int_0^t \tau^n\omega(\tau)\,d\tau - \frac{n^2(n-1)}{t^{n+1}}\int_0^t \tau^{n-1}\omega(\tau)\,d\tau - \frac{2n}{t}\,\omega(t)$$

by (29) and (26). Integrating by parts, we obtain

$$(t\Omega_1'(t))' = \omega'(t) - \frac{n}{t}\biggl(\frac{n+1}{t^{n+1}}\int_0^t \tau^{n+1}\omega'(\tau)\,d\tau - \frac{n-1}{t^n}\int_0^t \tau^n\omega'(\tau)\,d\tau\biggr),$$

and therefore,

$$(t\Omega_1'(t))' = -\frac{n}{t}\biggl(\frac{n-1}{nt^n}\int_0^t \tau^n(\tau\omega'(\tau))'\,d\tau - \frac{1}{t^{n+1}}\int_0^t \tau^{n+1}(\tau\omega'(\tau))'\,d\tau\biggr). \tag{59}$$

We set

$$\omega(t) = \begin{cases} t/\varepsilon & \text{for } 0 < t < \varepsilon, \\ \log(te/\varepsilon) & \text{for } t \geqslant \varepsilon. \end{cases}$$

Then

$$t\omega'(t) = \begin{cases} t/\varepsilon & \text{for } 0 < t < \varepsilon, \\ 1 & \text{for } t \geqslant \varepsilon. \end{cases}$$

By (59), one has

$$(t\Omega_1'(t))'\big|_{t=1} = -n\biggl(\frac{n-1}{n}\int_0^\varepsilon \frac{\tau^n}{\varepsilon}\,d\tau - \int_0^\varepsilon \frac{\tau^{n+1}}{\varepsilon}\,d\tau\biggr) = \varepsilon^n\biggl(\frac{n}{n+2}\,\varepsilon - \frac{n-1}{n+1}\biggr),$$

which is negative if $\varepsilon < (n-1)(n+2)/(n(n+1))$ and $n > 1$. Hence, $t\Omega_1'(t)$ is not nondecreasing, while $t\omega'(t)$ is.

3. Pointwise Inequality in Terms of $D_\omega^{(2)}(\nabla u; x)$

In this section, we prove the following analog of Theorem 1.

Theorem 2. (i) *Let the function*

$$\Omega_2(t) := \omega(t) - n\int_0^1 \sigma^n\omega(\sigma t)\,d\sigma \tag{60}$$

be strictly increasing on $[0,\infty)$*, and let* Ω_2^{-1} *be the inverse function of* Ω_2*. Further, let*

$$\Psi_2(t) = \int_0^t \Omega_2^{-1}(\tau)\, d\tau.$$

Then

$$|\nabla u(x)| \leqslant (n+1) D_\omega^{(2)}(\nabla u; x) \Psi_2^{-1}\bigg(\frac{\mathscr{M}^\diamond u(x)}{D_\omega^{(2)}(\nabla u; x)} \bigg), \tag{61}$$

for any $u \in C^1(\mathbb{R}^n)$*, where* Ψ_2^{-1} *is the inverse function of* Ψ_2 *and* $D_\omega^{(2)}$ *is defined by* (8)*.*

(ii) *Let* $\omega \in C^1(0,\infty)$*. Suppose that the function* $x\Omega_2'(x)$ *is nondecreasing on* $(0,\infty)$*. Let* $\mathscr{R}$ *be the unique root of the equation*

$$(n+1)\Omega_2(x) = 1. \tag{62}$$

Inequality (61) *becomes an equality for the function*

$$u(x) = \begin{cases} x_n(1-\omega(|x|)) & \text{for } 0 \leqslant |x| \leqslant \mathscr{R}, \\ \dfrac{x_n}{|x|}\bigg(((n+1)\mathscr{R} - |x|)\omega(n^{-1}((n+1)\mathscr{R}-|x|)) & \\ \qquad - \dfrac{(n+1)n^{n+1}}{((n+1)\mathscr{R}-|x|)^n} \displaystyle\int_0^{\frac{(n+1)\mathscr{R}-|x|}{n}} t^n \omega(t)\, dt \bigg) & \text{for } \mathscr{R} < |x| < (n+1)\mathscr{R}, \\ 0 & \text{for } |x| \geqslant (n+1)\mathscr{R}. \end{cases} \tag{63}$$

Proof. (i) It suffices to prove inequality (61) for $x = 0$. One clearly has

$$\begin{aligned} \int_{B_1} (\nabla u(0) - A^{(2)}\nabla u(0; |y|))|y|\, dy &= \frac{|S^{n-1}|}{|B_1|} \int_0^1 \int_0^t \int_{S^{n-1}} (\nabla u(0) - \nabla u(\theta\rho))\, d\theta\, \rho^{n-1}\, d\rho\, dt \\ &= n|S^{n-1}| \int_0^1 \int_0^t (\nabla u(0) - A^{(1)}\nabla u(0;\rho))\rho^{n-1}\, d\rho\, dt. \end{aligned}$$

Changing the order of integration in the last two integrals and using (22), we see that

$$\begin{aligned} \int_{B_1} (\nabla u(0) - A^{(2)}\nabla u(0; |y|))|y|\, dy &= n|S^{n-1}| \int_0^1 (\nabla u(0) - A^{(1)}\nabla u(0;t))(1-t)t^{n-1} dt \\ &= \frac{n|B_1|}{n+1}\nabla u(0) - n\int_{B_1} u(y)\, \frac{y}{|y|}\, dy. \end{aligned}$$

After rescaling $y \to y/t$, $r \to r/t$, we obtain

$$\frac{1}{t^n}\int_{B_t} (\nabla u(0) - A^{(2)}\nabla u(0; |y|))\, dy = \frac{n|B_1|}{n+1}\, t\nabla u(0) - \frac{n}{t^n}\int_{B_t} u(y)\, \frac{y}{|y|}\, dy.$$

This implies that

$$|\nabla u(0)| \leqslant \frac{n+1}{t}\mathscr{M}^\diamond u(0) + \frac{n+1}{t^{n+1}}\, D_\omega^{(2)}(\nabla u; 0)\int_0^t \rho^n\omega(\rho)\, d\rho,$$

which can be rewritten as

$$0 \leqslant -t|\nabla u(0)| + (n+1)\mathscr{M}^\diamond u(0) + (n+1)D_\omega^{(2)}(\nabla u; 0)\int_0^t \Omega_2(\tau)\, d\tau. \tag{64}$$

Since Ω_2 is strictly inreasing, it follows that the right-hand side attains its minimum value at

$$t_* = \Omega_2^{-1}\bigg(\frac{|\nabla u(0)|}{(n+1)D_\omega^{(2)}(\nabla u; 0)} \bigg). \tag{65}$$

Thus,

$$
\begin{aligned}
0 &\leqslant (n+1)\mathscr{M}^{\diamond}u(0) - |\nabla u(0)|\Omega_2^{-1}\left(\frac{|\nabla u(0)|}{(n+1)D_\omega^{(2)}(\nabla u;0)}\right) \\
&\quad + (n+1)D_\omega^{(2)}(\nabla u;0)\int_0^{\Omega_2^{-1}\left(\frac{|\nabla u(0)|}{(n+1)D_\omega^{(2)}(\nabla u;0)}\right)} \Omega_2(\tau)\,d\tau \\
&= (n+1)\mathscr{M}^{\diamond}u(0) - (n+1)D_\omega^{(2)}(\nabla u;0)\int_0^{\Omega_2^{-1}\left(\frac{|\nabla u(0)|}{(n+1)D_\omega^{(2)}(\nabla u;0)}\right)} x\,d\Omega_2(x) \\
&= (n+1)\mathscr{M}^{\diamond}u(0) - (n+1)D_\omega^{(2)}(\nabla u;0)\int_0^{\frac{|\nabla u(0)|}{(n+1)D_\omega^{(2)}(\nabla u;0)}} \Omega_2^{-1}(\tau)\,d\tau
\end{aligned}
$$

by (64). Therefore,

$$
\mathscr{M}^{\diamond}u(0) \geqslant D_\omega^{(2)}(\nabla u;0)\int_0^{\frac{|\nabla u(0)|}{(n+1)D_\omega^{(2)}(\nabla u;0)}} \Omega_2^{-1}(\tau)\,d\tau,
$$

which is equivalent to (61).

(ii) We have

$$
\Omega_2'(t) = \omega'(t) - \frac{n}{t}\int_0^1 \sigma^n \omega'(\sigma t)\sigma t\,d\sigma, \tag{66}
$$

which implies that

$$
\omega'(t) > \Omega_2'(t) > \frac{\omega'(t)}{n+1}\,. \tag{67}
$$

Since $\omega' \geqslant 0$, it follows that the function Ω_2 is strictly increasing, and therefore Eq. (62) has a unique root.

Let us verify that the function (63) belongs to $C^1(\mathbb{R}^n)$. We denote the ratio $u(x)/\cos\theta$ by $u_1(r)$ for $0 \leqslant |x| \leqslant \mathscr{R}$ and by $u_2(r)$ for $\mathscr{R} < |x| < (n+1)\mathscr{R}$, where u is defined by (63). One has

$$
u_1(r) = r - r\omega(r), \tag{68}
$$

$$
u_2(r) = n\left(\kappa_r\omega(\kappa_r) - \frac{n+1}{\kappa_r^n}\int_0^{\kappa_r} t^n\omega(t)\,dt\right), \tag{69}
$$

where

$$
\kappa_r = n^{-1}((n+1)\mathscr{R} - r). \tag{70}
$$

Obviously,

$$
u_2(\mathscr{R}) = n\left(\mathscr{R}\,\omega(\mathscr{R}) - \frac{n+1}{\mathscr{R}^n}\int_0^{\mathscr{R}} t^n\omega(t)\,dt\right),
$$

which by (60) can be rewritten as

$$
u_2(\mathscr{R}) = n\mathscr{R}\left(\frac{n+1}{n}\,\Omega_2(\mathscr{R}) - \frac{\omega(\mathscr{R})}{n}\right). \tag{71}
$$

Using (62), we see that $u_2(\mathscr{R}) = u_1(\mathscr{R})$.

Let us now show that

$$
u_2'(\mathscr{R}) = u_1'(\mathscr{R}). \tag{72}
$$

By (69), one has

$$
u_2'(r) = n\omega(\kappa_r) - \kappa_r\omega'(\kappa_r) - \frac{n(n+1)}{\kappa_r^{n+1}}\int_0^{\kappa_r} t^n\omega(t)\,dt. \tag{73}
$$

Therefore, it follows from (60) that

$$
u_2'(\mathscr{R}) = n\omega(\mathscr{R}) - \mathscr{R}\,\omega'(\mathscr{R}) - \frac{n(n+1)}{\mathscr{R}^{n+1}}\int_0^{\mathscr{R}} t^n\omega(t)\,dt = (n+1)\Omega_2(\mathscr{R}) - \mathscr{R}\,\omega'(\mathscr{R}) - \omega(\mathscr{R}),
$$

which, together with (62), implies (72). It remains to note that formulas (69) and (73) and the relation $\kappa_{(n+1)\mathscr{R}} = 0$ imply that

$$u_2((n+1)\mathscr{R}) = 0, \qquad u_2'((n+1)\mathscr{R}) = 0.$$

Our next objective is to show that the function u defined by (63) satisfies

$$\mathscr{M}^\diamond u(0) = \frac{\mathscr{R}}{n+1} - \int_0^{\mathscr{R}} \Omega_2(t)\,dt. \tag{74}$$

For $0 \leqslant |x| \leqslant \mathscr{R}$, the function u is equal to $\cos\theta u_1(\rho)$, where u_1 is given by (68). Clearly, M_r, which is defined by (38), can be rewritten as

$$\frac{2n|S^{n-2}|}{|S^{n-1}|r^n}\int_0^r \rho^n(1-\omega(\rho))\,d\rho \int_0^{\pi/2}(\cos\theta)^2(\sin\theta)^{n-2}\,d\theta;$$

by (39) and the definition of Ω_2, this can be represented as

$$\frac{r}{n+1} - \frac{1}{r^n}\int_0^r \rho^n\omega(\rho)\,d\rho = \frac{r}{n+1} - \int_0^r \Omega_2(t)\,dt.$$

It follows from (67) that $\Omega_2'(t) > 0$ for $t > 0$. Therefore,

$$\max_{0\leqslant r\leqslant\mathscr{R}} M_r u = M_{\mathscr{R}} u = \frac{\mathscr{R}}{n+1} - \int_0^{\mathscr{R}} \Omega_2(t)\,dt. \tag{75}$$

We now claim that

$$\max_{\mathscr{R}\leqslant r\leqslant(n+1)\mathscr{R}} M_r u = M_{\mathscr{R}} u. \tag{76}$$

By (69),

$$M_r u = \frac{n}{|S^{n-1}|r^n}\Bigg(n|S^{n-1}|\mathscr{R}^n\bigg(\frac{\mathscr{R}}{n+1} - \int_0^{\mathscr{R}}\Omega_2(t)\,dt\bigg) + 2n|S^{n-2}|\int_0^{\pi/2}(\cos\theta)^2(\sin\theta)^{n-2}\,d\theta\int_{\mathscr{R}}^r u_2(\rho)\rho^{n-1}\,d\rho\Bigg).$$

Applying (39), we obtain

$$M_r u = r^{-n}\bigg(\mathscr{R}^n\bigg(\frac{\mathscr{R}}{n+1} - \int_0^{\mathscr{R}}\Omega_2(t)\,dt\bigg) + \int_{\mathscr{R}}^r u_2(\rho)\rho^{n-1}\,d\rho\bigg).$$

To justify (76), we must show that the function

$$\mathscr{A}(r) := (r^n - \mathscr{R}^n)\bigg(\frac{\mathscr{R}}{n+1} - \int_0^{\mathscr{R}}\Omega_2(t)\,dt\bigg) - \int_{\mathscr{R}}^r u_2(\rho)\rho^{n-1}\,d\rho \tag{77}$$

is nonnegative on the interval $[\mathscr{R}, (n+1)\mathscr{R}]$. Obviously, $\mathscr{A}(\mathscr{R}) = 0$ and

$$\mathscr{A}'(r) = nr^{n-1}\bigg(\frac{\mathscr{R}}{n+1} - \int_0^{\mathscr{R}}\Omega_2(t)\,dt - \frac{1}{n}\,u_2(r)\bigg).$$

Observing that

$$-\frac{1}{n}\,u_2(r) = \int_0^{\kappa_r}\Omega_2(t)\,dt - \kappa_r\Omega_2(\kappa_r)$$

by (69) and (60) and integrating by parts, we conclude that

$$\mathscr{A}'(r) = nr^{n-1}\bigg(\int_0^{\mathscr{R}} t\Omega_2'(t)\,dt - \int_0^{\kappa_r} t\Omega_2'(t)\,dt\bigg).$$

It follows from the inequality $t\Omega_2'(t) > 0$ that $\mathscr{A}'(r) > 0$. This proves (76) and, together with (75), implies relation (74).

Let us now prove that

$$\sup_{r>0} \frac{|\nabla u(0) - A^{(2)} \nabla u(0;r)|}{\omega(r)} = 1. \tag{78}$$

We first assume that $0 \leqslant r \leqslant \mathscr{R}$. By (68),

$$\frac{\partial u}{\partial x_n} = 1 - \omega(r) - \frac{x_n^2}{r}\,\omega'(r). \tag{79}$$

Therefore,

$$A^{(2)} \frac{\partial u}{\partial x_n}(0;r) = \frac{n}{r^n} \int_0^r (1-\omega(\rho))\rho^{n-1}\,d\rho - \frac{2n|S^{n-2}|}{|S^{n-1}|r^n} \int_0^r \omega'(\rho)\rho^n\,d\rho \int_0^{\pi/2} (\cos\theta)^2 (\sin\theta)^{n-2}\,d\theta.$$

By (39), this can be rewritten as

$$A^{(2)} \frac{\partial u}{\partial x_n}(0;r) = \frac{n}{r^n}\left(\frac{r^n}{n} - \int_0^r \rho^{n-1}\omega(\rho)\,d\rho\right) - \frac{1}{r^n}\int_0^r \omega'(\rho)\rho^n\,d\rho = 1-\omega(r). \tag{80}$$

Let us show that the function $r\omega'(r)$ is strictly increasing. In fact, by (60),

$$\Omega_2(t) = \frac{d}{dt}\left(t^{-n}\int_0^t \tau^n \omega(\tau)\,d\tau\right)$$

and hence

$$t^n\omega(t) = \frac{d}{dt}\left(t^n \int_0^t \Omega_2(\tau)\,d\tau\right).$$

Therefore,

$$t\omega(t) = n\int_0^t \Omega_2(\tau)\,d\tau + t\Omega_2(t). \tag{81}$$

This implies the relation

$$t\omega'(t) = t\Omega_2'(t) + \frac{n}{t}\int_0^t \tau\Omega_2'(\tau)\,d\tau.$$

The derivative of the second term on the right-hand side is equal to

$$\frac{n}{t^2}\left(t^2\Omega_2'(t) - \int_0^t \tau\Omega_2'(\tau)\,d\tau\right) \tag{82}$$

and is positive, since $t\Omega_2'(t)$ is nondecreasing. Combining this with (82), we see that $t\omega'(t)$ is a strictly increasing function. Since $\omega(0)=0$, it follows that $r\omega'(r) \to 0$ as $r \to 0$. In conjunction with (79) and (80), this implies that

$$\frac{\partial u}{\partial x_n}(0) - A^{(2)}\frac{\partial u}{\partial x_n}(0;r) = \omega(r).$$

Combining this with the relations

$$\frac{\partial u}{\partial x_k}(0) = 0, \quad A^{(2)}\frac{\partial u}{\partial x_k}(0;r) = 0, \qquad k = 1,\dots,n-1,$$

we find that

$$\frac{|\nabla u(0) - A^{(2)}\nabla u(0;r)|}{\omega(r)} = 1 \quad \text{for } 0 \leqslant r \leqslant \mathscr{R}.$$

Let us now show that

$$0 \leqslant \frac{\partial u}{\partial x_n}(0) - A^{(2)}\frac{\partial u}{\partial x_n}(0;r) \leqslant \omega(r) \quad \text{for } \mathscr{R} \leqslant r \leqslant (n+1)\mathscr{R}. \tag{83}$$

Applying Green's formula, we obtain

$$A^{(2)}\frac{\partial u}{\partial x_n}(0;r) = \frac{1}{|B_1|r^n}\int_{\partial B_r} u\cos\theta\,ds = u_2(r)\frac{2|S^{n-2}|}{|B_1|r}\int_0^{\pi/2}(\cos\theta)^2(\sin\theta)^{n-2}\,d\theta.$$

This, together with (39), gives

$$A^{(2)}\frac{\partial u}{\partial x_n}(0;r)=\frac{u_2(r)}{r}\,. \tag{84}$$

Since $(\partial u/\partial x_n)(0)=1$, we see that the left inequality in (83) can be rewritten as $u_2(r)\leqslant r$. The last inequality is valid since

$$u_2(r)=n\kappa_r\left(\frac{n+1}{n}\,\Omega_2(\kappa_r)-\frac{\omega(\kappa_r)}{n}\right)\leqslant\kappa_r\Omega_2(\kappa_r)(n+1)\leqslant\mathscr{R}\leqslant r.$$

Let us prove the right inequality in (83), which is equivalent to the inequality

$$\omega(r)+\frac{u_2(r)}{r}-1\geqslant 0,\qquad r\in[\mathscr{R},(n+1)\mathscr{R}]. \tag{85}$$

The function $g(r):=r\omega(r)+u_2(r)-r$ vanishes for $r=\mathscr{R}$, since $u_2(\mathscr{R})=u_1(\mathscr{R})$ and (68) holds. Further, (73) yields

$$\begin{aligned}g'(r)&=(r\omega(r))'-\left(\kappa_r\omega'(\kappa_r)+\frac{n(n+1)}{\kappa_r^{n+1}}\int_0^{\kappa_r}t^n\omega(t)\,dt-n\omega(\kappa_r)\right)-1\\&=(r\omega(r))'-\kappa_r\omega'(\kappa_r)+\frac{n}{\kappa_r^{n+1}}\int_0^{\kappa_r}t^{n+1}\omega'(t)\,dt-1.\end{aligned}$$

Using (66), we obtain

$$g'(r)=(r\omega(r))'-\kappa_r\Omega_2'(\kappa_r)-1. \tag{86}$$

Since $(t\omega(t))'$ is strictly increasing, Eq. (86) implies that

$$g'(r)>(\mathscr{R}\omega(\mathscr{R}))'-\mathscr{R}\Omega_2'(\mathscr{R})-1,\qquad r\in(\mathscr{R},(n+1)\mathscr{R}].$$

In view of (66), the right-hand side is equal to

$$\omega(\mathscr{R})+\frac{n}{\mathscr{R}^{n+1}}\int_0^{\mathscr{R}}\tau^{n+1}\omega'(\tau)\,d\tau-1=\omega(\mathscr{R})+(n+1)\Omega_2(\mathscr{R})-\omega(\mathscr{R})-1=0.$$

Hence, g increases on $[\mathscr{R},(n+1)\mathscr{R}]$.

It remains to prove that inequality (61) becomes an equality at $x=0$ for the function u given by (63). By (62), one has

$$\int_0^{1/(n+1)}\Omega_2^{-1}(\tau)\,d\tau=\int_0^{\mathscr{R}}t\,d\Omega_2(t)=\mathscr{R}\Omega_2(\mathscr{R})-\int_0^{\mathscr{R}}\Omega_2(t)\,dt=\frac{\mathscr{R}}{n+1}-\int_0^{\mathscr{R}}\Omega_2(t)\,dt.$$

This, together with relations (74) and (78), shows that the right-hand side of (61) is equal to $(n+1)\Psi_2^{-1}(\Psi_2(1/(n+1)))=1$. This completes the proof.

Remark 2. In the proof of Theorem 2, we showed the assumption that $t\Omega_2'(t)$ is nondecreasing to imply that $t\omega'(t)$ is strictly increasing. Let us prove that the converse is not true, that is, the function $t\Omega_2'(t)$ need not be monotone if $t\omega'(t)$ increases. For any small $t>0$, we set

$$\omega(t)=\int_0^t\int_0^\tau\xi(\sin\xi^{-1})^2d\xi\,\frac{d\tau}{\tau}\,.$$

The function $t\omega'(t)$ increases, because

$$(t\omega'(t))'=t(\sin t^{-1})^2.$$

Hence, it follows from (66) that the derivative

$$(t\Omega_2'(t))'=(t\omega'(t))'-\frac{n}{t^{n+2}}\int_0^t\tau^{n+1}(\tau\omega'(\tau))'\,d\tau \tag{87}$$

is negative for $t_k=(\pi k)^{-1}$, $k=1,2,\dots$. Now by (87) one has

$$(t\Omega_2'(t))'=t(\sin t^{-1})^2-\frac{n}{2(n+3)}\,t(1+O(t)).$$

Here the right-hand side is positive for $t_k = (\pi(k+1/2))^{-1}$ if k is sufficiently large. Hence, $t\Omega_2'(t)$ is not monotone.

4. The Special Case $\omega(r) = r^\alpha$, $\alpha > 0$

Setting $\omega(r) = r^\alpha$ with $\alpha > 0$ in Theorem 1 and using the notation

$$D_\alpha^{(1)}(\nabla u; x) = \sup_{r>0} \frac{|\nabla u(x) - A^{(1)}\nabla u(x;r)|}{r^\alpha},$$

we obtain the following corollary to Theorem 1.

Corollary 1. *Let $u \in C^1(\mathbb{R}^n)$, and let $\alpha > 0$. Then inequality* (10) *holds with the best constant* (11).

Inequality (10) *becomes an equality for the function*

$$u(x) = \begin{cases} x_n\left(1 - \dfrac{n}{n+\alpha}|x|^\alpha\right) & \text{for } 0 \leqslant |x| \leqslant R, \\ \dfrac{\alpha n^{1-\alpha}((n+1)R - |x|)^{\alpha+1}}{(n+\alpha)(n+\alpha+1)}\,\dfrac{x_n}{|x|} & \text{for } R < |x| < (n+1)R, \\ 0 & \text{for } |x| \geqslant (n+1)R, \end{cases}$$

where

$$R = \left(\frac{(n+\alpha)(n+\alpha+1)}{(\alpha+1)n(n+1)}\right)^{1/\alpha}.$$

Corollary 2 (a local version of Corollary 1). *Let $\mathscr{M}_1^\diamond$ denote the modified maximal operator given by*

$$\mathscr{M}_1^\diamond u(x) = \sup_{0<r<1}\left|\fint_{B_r(x)} \frac{y-x}{|y-x|}\,u(y)\,dy\right|,$$

and let

$$D_{1,\alpha}^{(1)}(\nabla u; x) = \sup_{0<r<1} \frac{|\nabla u(x) - A^{(1)}\nabla u(x;r)|}{r^\alpha}.$$

Then for any $\alpha > 0$ the inequality

$$|\nabla u(x)| \leqslant \big(C_1(D_{1,\alpha}^{(1)}(\nabla u;x))^{1/(\alpha+1)} + C_2(\mathscr{M}_1^\diamond u(x))^{1/(\alpha+1)}\big)(\mathscr{M}_1^\diamond u(x))^{\alpha/(\alpha+1)} \tag{88}$$

holds with the best constants C_1 defined by (11) *and $C_2 = n+1$.*

Proof. It suffices to set $x = 0$. It follows from (23) that

$$|\nabla u(0)| \leqslant (n+1)\left(\mathscr{M}_1^\diamond u(0)t^{-1} + \frac{n}{(n+\alpha)(n+\alpha+1)}\,D_{1,\alpha}^{(1)}(\nabla u;0)t^\alpha\right).$$

The right-hand side attains its minimum value either at

$$t = \left(\frac{(n+\alpha)(n+\alpha+1)}{\alpha n}\,\frac{\mathscr{M}_1^\diamond u(0)}{D_{1,\alpha}^{(1)}(\nabla u;0)}\right)^{1/(\alpha+1)} < 1$$

or at $t = 1$. Thus we arrive at the following alternatives: either

$$\mathscr{M}_1^\diamond u(0) \leqslant \frac{\alpha n}{(n+\alpha)(n+\alpha+1)}\,D_{1,\alpha}^{(1)}(\nabla u;0)$$

and

$$|\nabla u(0)| \leqslant C_1(\mathscr{M}_1^\diamond u(0))^{\alpha/(\alpha+1)}(D_{1,\alpha}^{(1)}(\nabla u;0))^{1/(\alpha+1)} \tag{89}$$

with C_1 defined by (11), or

$$\mathscr{M}_1^\diamond u(0) \geqslant \frac{\alpha n}{(n+\alpha)(n+\alpha+1)}\,D_{1,\alpha}^{(1)}(\nabla u;0)$$

and

$$|\nabla u(0)| \leqslant \left(\frac{C_1}{\alpha+1}(D^{(1)}_{1,\alpha}(\nabla u;0))^{1/(\alpha+1)} + C_2(\mathscr{M}_1^\diamond u(0))^{1/(\alpha+1)}\right)(\mathscr{M}_1^\diamond u(0))^{\alpha/(\alpha+1)}. \qquad (90)$$

Inequalities (89) and (90) imply (88).

To show that the constant C_1 is sharp, we make the dilation $x \to \delta x$, $0 < \delta < 1$, in (88). Then

$$|\nabla u(x)| \leqslant (C_1(D^{(1)}_\alpha(\nabla u;x))^{1/(\alpha+1)} + C_2\delta(\mathscr{M}^\diamond u(x))^{1/(\alpha+1)})(\mathscr{M}^\diamond u(x))^{\alpha/(\alpha+1)}.$$

Passing to the limit as $\delta \to 0$, we arrive at (88) with the best constant C_1.

To prove that the constant C_2 is sharp, we set $u(x) = x_n$. Then (88) becomes

$$|\nabla u(x)| \leqslant (n+1)\mathscr{M}_1^\diamond u(x). \qquad (91)$$

Clearly,

$$\begin{aligned}\mathscr{M}_1^\diamond u(0) &= \sup_{0<r<1}\left|\fint_{B_r}\frac{x}{|x|}\,x_n\,dx\right| = \sup_{0<r<1}\fint_{B_r}\frac{x_n^2}{|x|}\,dx \\ &= \sup_{0<r<1}\frac{2n|S^{n-2}|}{|S^{n-1}|r^n}\int_0^{\pi/2}(\cos\theta)^2(\sin\theta)^{n-2}\,d\theta\int_0^r\rho^n\,d\rho,\end{aligned}$$

which, together with (39), implies

$$\mathscr{M}_1^\diamond u(0) = (n+1)^{-1}.$$

Thus, inequality (91) becomes an equality. This completes the proof of the corollary.

Setting $\omega(r) = r^\alpha$, $\alpha > 0$, and using the notation

$$D^{(2)}_\alpha(\nabla u;x) = \sup_{r>0}\frac{|\nabla u(x) - A^{(2)}\nabla u(x;r)|}{r^\alpha},$$

we obtain the following corollary to Theorem 2.

Corollary 3. *Let $u \in C^1(\mathbb{R}^n)$, and let $\alpha > 0$. Then the inequality*

$$|\nabla u(x)| \leqslant C_2(\mathscr{M}^\diamond u(x))^{\alpha/(\alpha+1)}(D^{(2)}_\alpha(\nabla u;x))^{1/(\alpha+1)} \qquad (92)$$

holds with the best constant

$$C_2 = (n+1)\frac{\alpha+1}{\alpha}\left(\frac{\alpha}{n+\alpha+1}\right)^{1/(\alpha+1)}. \qquad (93)$$

Inequality (92) becomes an equality for the odd function given for $x \geqslant 0$ by the formula

$$u(x) = \begin{cases} x_n(1-|x|^\alpha) & \text{for } 0 \leqslant |x| \leqslant \mathscr{R}, \\ \dfrac{\alpha n^{-\alpha}}{n+\alpha+1}((n+1)\mathscr{R} - |x|)^{\alpha+1}\dfrac{x_n}{|x|} & \text{for } \mathscr{R} < |x| < (n+1)\mathscr{R}, \\ 0 & \text{for } |x| \geqslant (n+1)\mathscr{R}, \end{cases}$$

where

$$\mathscr{R} = \left(\frac{n+\alpha+1}{(\alpha+1)(n+1)}\right)^{1/\alpha}.$$

5. The One-Dimensional Case

For $n = 1$,

$$D^{(1)}_\omega(u';x) = \sup_{r>0}\frac{|2u'(x) - u'(x+r) - u'(x-r)|}{2\omega(r)}, \qquad D^{(2)}_\omega(u';x) = \sup_{r>0}\frac{|u'(x) - \frac{u(x+r)+u(x-r)}{2r}|}{\omega(r)},$$

and $\mathscr{M}^\diamond$ is defined by (4).

The next two corollaries readily follow from Theorems 1 and 2.

Corollary 4. *Let $u \in C^1(\mathbb{R})$. Then the inequality*

$$|u'(x)| \leqslant 2D_\omega^{(1)}(u';x)\Psi_1^{-1}\left(\frac{\mathscr{M}^\diamond u}{D_\omega^{(1)}(u';x)}\right) \tag{94}$$

holds, where Ψ_1^{-1} is the inverse function of

$$\Psi_1(t) = \int_0^t \Omega^{-1}(\tau)\,d\tau$$

with Ω^{-1} being the inverse function of

$$\Omega(t) = \int_0^1 \sigma\omega(\sigma t)\,d\sigma.$$

Suppose that $t\omega'(t)$ is nondecreasing on $(0,\infty)$. Then inequality (94) becomes an equality for the odd function u given on the half-line $x \geqslant 0$ by the formula

$$u(x) = \begin{cases} x\left(1 - \displaystyle\int_0^1 \omega(\sigma x)\,d\sigma\right) & \text{for } 0 \leqslant x \leqslant R, \\ (2R - x)\displaystyle\int_0^1 (2\sigma - 1)\omega(\sigma(2R - x))\,d\sigma & \text{for } R < x < 2R, \\ 0 & \text{for } x \geqslant 2R, \end{cases}$$

where R is the unique root of the equation $2\Omega_1(t) = 1$.

Corollary 5. *Let $u \in C^1(\mathbb{R})$. Then the inequality*

$$|u'(x)| \leqslant 2\,D_\omega^{(2)}(u';x)\Psi_2^{-1}\left(\frac{\mathscr{M}^\diamond u}{D_\omega^{(2)}(u';x)}\right) \tag{95}$$

holds, where Ψ_2^{-1} is the inverse function of

$$\Psi_2(t) = \int_0^t (\omega - \Omega)^{-1}(\tau)\,d\tau$$

with $(\omega - \Omega)^{-1}$ being the inverse function of $\omega - \Omega$.

Suppose that the function $t(\omega(t) - \Omega(t))'$ is nondecreasing on $(0,\infty)$. Then inequality (95) becomes an equality for the odd function u given on the half-line $x \geqslant 0$ by

$$u(x) = \begin{cases} x(1 - \omega(x)) & \text{for } 0 \leqslant x \leqslant \mathscr{R}, \\ (2\mathscr{R} - x)\omega(2\mathscr{R} - x) - \dfrac{2}{2\mathscr{R} - x}\displaystyle\int_0^{2\mathscr{R}-x} t\omega(t)\,dt & \text{for } \mathscr{R} < x < 2\mathscr{R}. \\ 0 & \text{for } x \geqslant 2\mathscr{R}, \end{cases}$$

where $\mathscr{R}$ is the unique root of the equation $2\Omega_2(t) = 1$.

Set

$$D_\omega(u';x) = \sup_{y\in\mathbb{R}} \frac{|u'(x) - u'(y)|}{\omega(|x - y|)}$$

and note that $D_\omega^{(1)}(u';x) \leqslant D_\omega(u';x)$. Moreover, if u is odd, then $D_\omega^{(1)}(u';0) = D_\omega(u';0)$. Therefore, Corollary 4 implies the following assertion.

Corollary 6. *Let $u \in C^1(\mathbb{R})$. Then*

$$|u'(x)| \leqslant 2\,D_\omega(u';x)\Psi_1^{-1}\left(\frac{\mathscr{M}^\diamond u}{D_\omega(u';x)}\right). \tag{96}$$

Inequality (96) becomes an equality for the same function as in Corollary 4. As in Corollary 4, here we assume that $r\omega'(r)$ is nondecreasing on $(0,\infty)$.

In the special case $\omega(t) = t^\alpha$, $\alpha > 0$, Corollaries 4 and 6 can be stated as follows.

Corollary 7. *Let $u \in C^1(\mathbb{R})$, and let $\alpha > 0$. Inequality* (12) *holds with the best constant* (13). *The rougher inequality* (3) *follows from* (12). *Inequality* (12) (*and even* (3)) *becomes an equality for the odd function u whose values for $x \geqslant 0$ are given by*

$$u(x) = \begin{cases} (\alpha+1)x - x^{\alpha+1} & \text{for } 0 \leqslant x \leqslant \left(\dfrac{\alpha+2}{2}\right)^{1/\alpha}, \\ \dfrac{\alpha}{\alpha+2}\left(2\left(\dfrac{\alpha+2}{2}\right)^{1/\alpha} - x\right)^{\alpha+1} & \text{for } \left(\dfrac{\alpha+2}{2}\right)^{1/\alpha} < x < 2\left(\dfrac{\alpha+2}{2}\right)^{1/\alpha}, \\ 0 & \text{for } x \geqslant 2\left(\dfrac{\alpha+2}{2}\right)^{1/\alpha}. \end{cases} \tag{97}$$

Note that (3) is a special case of (96) for $\omega(t) = t^\alpha$.

In conclusion, we present an assertion that readily follows from Corollary 5 for $\omega(t) = t^\alpha$, $\alpha > 0$.

Corollary 8. *Let $u \in C^1(\mathbb{R})$, and let $\alpha > 0$. Inequality* (14) *holds with the best constant* (15). *Inequality* (14) *becomes an equality for the odd function u whose values on the half-line $x \geqslant 0$ are given by*

$$u(x) = \begin{cases} x - x^{\alpha+1} & \text{for } 0 \leqslant x \leqslant \left(\dfrac{\alpha+2}{2(\alpha+1)}\right)^{1/\alpha}, \\ \dfrac{\alpha}{\alpha+2}\left(2\left(\dfrac{\alpha+2}{2}\right)^{1/\alpha} - x\right)^{\alpha+1} & \text{for } \left(\dfrac{\alpha+2}{2(\alpha+1)}\right)^{1/\alpha} < x < 2\left(\dfrac{\alpha+2}{2(\alpha+1)}\right)^{1/\alpha}, \\ 0 & \text{for } x \geqslant 2\left(\dfrac{\alpha+2}{2(\alpha+1)}\right)^{1/\alpha}. \end{cases}$$

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