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SHARP REAL-PART THEOREMS FOR HIGH ORDER DERIVATIVES

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We obtain a representation for the sharp coefficient in an estimate of the modulus of the nth derivative of an analytic function in the upper half-plane $\mathbb{C}_+$. It is assumed that the boundary value of the real part of the function on $\partial\mathbb{C}_+$ belongs to L^p. This representation is specified for $p = 1$ and $p = 2$. For $p = \infty$ and for derivatives of odd order, an explicit formula for the sharp coefficient is found. A limit relation for the sharp coefficient in a pointwise estimate for the modulus of the n-th derivative of an analytic function in a disk is found as the point approaches the boundary circle. It is assumed that the boundary value of the real part of the function belongs to L^p. The relation in question contains the sharp constant from the estimate of the modulus of the n-th derivative of an analytic function in $\mathbb{C}_+$. As a corollary, a limit relation for the modulus of the n-th derivative of an analytic function with the bounded real part is obtained in a domain with smooth boundary. Bibliography: 8 *titles.*

1 Introduction

In this paper, we deal substantially with a class of analytic functions in the half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ represented by the Schwarz formula

$$f(z) = \frac{1}{\pi i} \int_{\infty}^{\infty} \frac{\operatorname{Re} f(\zeta)}{\zeta - z}\, d\zeta \tag{1.1}$$

and such that the boundary values on $\partial\mathbb{C}_+$ of the real part of f belong to $L^p(-\infty, \infty)$, $1 \leqslant p < \infty$.

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We consider the following inequality with sharp coefficient $\mathscr{K}_{n,p}(z;\alpha)$:

$$|\operatorname{Re}\{e^{i\alpha} f^{(n)}(z)\}| \leqslant \mathscr{K}_{n,p}(z;\alpha) || \operatorname{Re} f||_p \ , \tag{1.2}$$

where $z \in \mathbb{C}_+$ and $|| \cdot ||_p$ stands for the norm in $L^p(-\infty, \infty)$. Hereinafter, we adopt the notation $|| \operatorname{Re} f||_p$ for $|| \operatorname{Re} f|_{\partial \mathbb{C}_+}||_p$. Note that the value $K_{n,\infty}(z;\alpha)$ is obtained by passage to the limit of $K_{n,p}(z;\alpha)$ as $p \to \infty$. We find a representation for $\mathscr{K}_{n,p}(z;\alpha)$ and, as a consequence, for the sharp coefficient $\mathscr{K}_{n,p}(z)$ in the inequality

$$|f^{(n)}(z)| \leqslant \mathscr{K}_{n,p}(z) || \operatorname{Re} f||_p \ . \tag{1.3}$$

In a number of cases described below, we obtain explicit formulas for the coefficient $\mathscr{K}_{n,p}(z)$.

The concluding section of this paper concerns a limit relation for the sharp coefficient in the estimate of the modulus of the nth order derivative of an analytic function in a disk as the point approaches the boundary. It is assumed that the boundary value of the real part of the analytic function belongs to L^p. As a direct consequence, we obtain a limit relation for the modulus of the nth derivative of an analytic function with the bounded real part in a domain with smooth boundary.

Note that the inequalities (1.2) and (1.3) for analytic functions belong to the class of sharp real-part theorems (cf. [1] and the references therein) which go back to Hadamard's real-part theorem [2].

The present article extends the topic of our paper [3], where we found explicit formulas for $\mathscr{K}_{0,p}(z)$ for $p \in [1, \infty)$ and for $\mathscr{K}_{1,p}(z)$ for $p \in [1, \infty]$. Unlike [3], we consider now the case of an arbitrary $n \geqslant 1$. However, explicit formulas for $\mathscr{K}_{n,p}(z)$ are derived only for particular values of p.

Now, we describe the results of this paper in more detail. Section 2 concerns a representation for the sharp coefficient $\mathscr{K}_{n,p}(z;\alpha)$. We show that the sharp coefficient in (1.2) is given by

$$\mathscr{K}_{n,p}(z;\alpha) = \frac{K_{n,p}(\alpha)}{(\operatorname{Im} z)^{n+\frac{1}{p}}} \ ,$$

where

$$K_{n,p}(\alpha) = \frac{n!}{\pi} \left\{ \int_{-\pi/2}^{\pi/2} \left| \cos\left(\alpha - (n+1)\varphi + \frac{n\pi}{2}\right) \right|^q \cos^{(n+1)q-2} \varphi d\varphi \right\}^{1/q}$$

and $1/p + 1/q = 1$. As a consequence, we obtain a representation for the coefficient $\mathscr{K}_{n,p}(z)$ in (1.3); namely,

$$\mathscr{K}_{n,p}(z) = \frac{K_{n,p}}{(\operatorname{Im} z)^{n+\frac{1}{p}}} \ ,$$

where

$$K_{n,p} = \max_{\alpha} K_{n,p}(\alpha) \ . \tag{1.4}$$

In Section 3, we find the values of sharp constants $K_{n,1}$ and $K_{n,2}$:

$$K_{n,1} = \frac{n!}{\pi}, \qquad K_{n,2} = \sqrt{\frac{(2n)!}{2^{2n+1}\pi}} \ .$$

Note that the maximum over α in (1.4) with $p = 1$ and even n is attained at $\alpha = 0$ and with $p = 1$ and odd n at $\alpha = \pi/2$. The coefficient $K_{n,2}(\alpha)$ is independent of α.

In Sections 4 and 5, we study the case $p = \infty$ in (1.3) separately for $n = 2m + 1$ and for $n = 2m$.

Section 4 is devoted to derivatives of odd order. We show that $K_{2m+1,\infty}(\alpha)$ is independent of α and prove the following estimate with the sharp coefficient:

$$|f^{(2m+1)}(z)| \leqslant \frac{2}{\pi} \frac{[(2m+1)!!]^2}{(2m+1)(\operatorname{Im} z)^{2m+1}} \, ||\operatorname{Re} f||_\infty \, .$$

The sharp pointwise estimate for the modulus of the derivative of odd order for a bounded analytic function in a disk was established by Szász [4]. Sharp estimates for $|f^{(n)}(z)|$ with the $L^p(\partial\mathbb{D})$-norm of $|f|$ on the right-hand side for analytic functions in the unit disk $\mathbb{D}$ were obtained by Makintyre and Rogosinski [5].

Section 5 concerns sharp estimates for the modulus of derivatives of even order with $||\operatorname{Re} f||_\infty$ on the right-hand side. We rewrite the expression for the derivative in α of the integral appearing in the formula for $K_{2m,\infty}(\alpha)$ and make its further analysis. In particular, we find the sharp constants

$$K_{2,\infty} = K_{2,\infty}(0) = \frac{3\sqrt{3}}{2\pi}, \qquad K_{4,\infty} = K_{4,\infty}(\pi/2) = \frac{3}{4\pi}\big(16 + 5\sqrt{5}\big) \, .$$

Note that a sharp explicit estimate for the second order derivative of a bounded analytic function in a disk was found by Szász [4].

In Section 6, we consider an example of $K_{n,p}(\alpha)$ independent of α for $p \neq 2$ and $p \neq \infty$. Namely, we deal with the case $n = 2, p = 4$.

Section 7 is devoted to the limit relation

$$\lim_{r \to R} (R - r)^{n + \frac{1}{p}} \mathscr{H}_{n,p}(z) = K_{n,p} \, , \tag{1.5}$$

where $\mathscr{H}_{n,p}(z)$ is the sharp coefficient in the inequality

$$|f^{(n)}(z)| \leqslant \mathscr{H}_{n,p}(z) || \operatorname{Re} f|_{\partial\mathbb{D}_R}||_p \, .$$

Here, f is an analytic function in the disk $\mathbb{D}_R = \{z : |z < R|\}$ with the real part in the Hardy space $h^p(\mathbb{D}_R)$, $1 \leqslant p \leqslant \infty$, whose elements are harmonic functions in $\mathbb{D}_R$ represented by the Poisson integral with density in $L^p(\partial\mathbb{D}_R)$.

Note that the representation for $\mathscr{H}_{n,p}(z)$ is obtained in [1], where, in particular, explicit formulas for $\mathscr{H}_{n,1}(z)$ and $\mathscr{H}_{n,2}(z)$ are found. A formula for $\mathscr{H}_{1,\infty}(z)$ is due to D. Khavinson [6].

As a consequence of (1.5), we deduce the limit relation for the modulus of the nth order derivative of an analytic function f with $|| \operatorname{Re} f|_{\partial\Omega}||_\infty \leqslant 1$ in a domain $\Omega \subset \mathbb{C}_+$ with smooth boundary:

$$\lim_{d_z \to 0} d_z^n |f^{(n)}(z)| \leqslant K_{n,\infty} \, , \tag{1.6}$$

where $d_z = \operatorname{dist}(z, \partial\Omega)$. We assume that each point of $\partial\Omega$ can be touched by an interior circle of sufficiently small radius. A particular case of (1.6) is the limit relation

$$\lim_{d_z \to 0} d_z^{2m+1} \big| f^{(2m+1)}(z) \big| \leqslant \frac{2[(2m+1)!!]^2}{\pi(2m+1)}$$

containing the sharp constant on the right-hand side which was found in Section 4.

2 Representations for Sharp Coefficients in Estimates for Derivatives of Analytic Functions

In what follows, by $h^p(\mathbb{R}^2_+), 1 \leqslant p \leqslant \infty$, we mean the Hardy space of harmonic functions in the upper half-plane $\mathbb{R}^2_+$ which are represented by the Poisson integral with a density in $L^p(-\infty, \infty)$. It is well known (cf., for example, [7], Sect. 19.3) that f belongs to the Hardy space $H^p(\mathbb{C}_+)$ of analytic functions in $\mathbb{C}_+$ if $\operatorname{Re} f \in h^p(\mathbb{R}^2_+)$, $1 < p < \infty$. Moreover, any function $f \in H^p(\mathbb{C}_+)$, $1 < p < \infty$, admits the representation (1.1) since $\operatorname{Re} f \in h^p(\mathbb{R}^2_+)$.

The following assertion contains representations for the sharp coefficients in (1.2) and (1.3) for $n \geqslant 1$.

Proposition 2.1. *Let* $\operatorname{Re} f \in h^p(\mathbb{R}^2_+)$, $1 \leqslant p \leqslant \infty$, *and let* z *be an arbitrary point in* $\mathbb{C}_+$. *The sharp coefficient* $\mathscr{K}_p(z;\alpha)$ *in the inequality*

$$|\operatorname{Re}\{e^{i\alpha} f^{(n)}(z)\}| \leqslant \mathscr{K}_{n,p}(z;\alpha) ||\operatorname{Re} f||_p \tag{2.1}$$

is given by

$$\mathscr{K}_{n,p}(z;\alpha) = \frac{K_{n,p}(\alpha)}{(\operatorname{Im} z)^{n+\frac{1}{p}}} , \tag{2.2}$$

where

$$K_{n,p}(\alpha) = \frac{n!}{\pi} \left\{ \int_{-\pi/2}^{\pi/2} \left|\cos\left(\alpha - (n+1)\varphi + \frac{n\pi}{2}\right)\right|^q \cos^{(n+1)q-2} \varphi d\varphi \right\}^{1/q} \tag{2.3}$$

and $1/p + 1/q = 1$.

In particular, the best coefficient in the inequality

$$|f^{(n)}(z)| \leqslant \mathscr{K}_{n,p}(z) ||\operatorname{Re} f||_p \tag{2.4}$$

is given by

$$\mathscr{K}_{n,p}(z) = \frac{K_{n,p}}{(\operatorname{Im} z)^{n+\frac{1}{p}}} , \tag{2.5}$$

where

$$K_{n,p} = \max_{\alpha} K_{n,p}(\alpha) . \tag{2.6}$$

Proof. By (1.1), we have

$$f(z) = \frac{n!}{\pi i} \int_{\infty}^{\infty} \frac{\operatorname{Re} f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta , \tag{2.7}$$

where $z \in \mathbb{C}_+$. We have

$$\operatorname{Re}\{e^{i\alpha} f^{(n)}(z)\} = \frac{n!}{\pi} \int_{-\infty}^{\infty} \operatorname{Re}\left\{ \frac{e^{i\left(\alpha - \frac{\pi}{2}\right)}}{(\zeta - z)^{n+1}} \right\} \operatorname{Re} f(\zeta) d\zeta . \tag{2.8}$$

Let $z = x + iy$. We write (2.8) as

$$\mathrm{Re}\{e^{i\alpha} f^{(n)}(z)\} = \frac{n!}{\pi} \int\limits_{-\infty}^{\infty} \mathrm{Re}\left\{ \frac{e^{i\left(\alpha - \frac{\pi}{2}\right)} \left[(\zeta - x) + iy\right]^{n+1}}{\left[(\zeta - x)^2 + y^2\right]^{n+1}} \right\} \mathrm{Re}\, f(\zeta) d\zeta\ . \tag{2.9}$$

Hence the sharp coefficient $\mathscr{K}_{n,p}(z;\alpha)$ in (2.1) is given by

$$\mathscr{K}_{n,p}(z;\alpha) = \frac{n!}{\pi} \left\{ \int\limits_{-\infty}^{\infty} \left| \mathrm{Re}\left\{ \frac{e^{i\left(\alpha - \frac{\pi}{2}\right)} \left[(\zeta - x) + iy\right]^{n+1}}{\left[(\zeta - x)^2 + y^2\right]^{n+1}} \right\} \right|^q d\zeta \right\}^{1/q} . \tag{2.10}$$

We introduce the new integration variable $\varphi \in (-\pi/2, \pi/2)$ by the equalities

$$\sin\varphi = \frac{\zeta - x}{\sqrt{(\zeta - x)^2 + y^2}}\ , \qquad \cos\varphi = \frac{y}{\sqrt{(\zeta - x)^2 + y^2}}\ . \tag{2.11}$$

Then

$$\varphi = \arctan \frac{\zeta - x}{y} \tag{2.12}$$

and therefore,

$$d\varphi = \frac{y}{(\zeta - x)^2 + y^2}\, d\zeta\ . \tag{2.13}$$

Since

$$\begin{aligned} e^{i\left(\alpha - \frac{\pi}{2}\right)} \left\{ \frac{(\zeta - x) + iy}{\sqrt{(\zeta - x)^2 + y^2}} \right\}^{n+1} &= e^{i\left(\alpha - \frac{\pi}{2}\right)} \left(\sin\varphi + i\cos\varphi \right)^{n+1} \\ &= e^{i\left(\alpha - \frac{\pi}{2}\right)} \left[\cos\left(\frac{\pi}{2} - \varphi\right) + i \sin\left(\frac{\pi}{2} - \varphi\right) \right]^{n+1} \\ &= e^{i\left[\alpha - (n+1)\varphi + \frac{n\pi}{2}\right]} \end{aligned}$$

and

$$\frac{1}{\left[(\zeta - x)^2 + y^2\right]^{(n+1)q/2}} = \frac{1}{y^{(n+1)q-1}} \left(\frac{y}{\sqrt{(\zeta - x)^2 + y^2}} \right)^{(n+1)q-2} \frac{y}{(\zeta - x)^2 + y^2}\ ,$$

from (2.11)–(2.13) it follows that (2.10) can be written in the form (2.2) with the constant (2.3). Formulas (2.5) and (2.6) follow from (2.2) and (2.1). □

3 Cases $p = 1$ and $p = 2$

In this section, we find the values of the sharp constant in (2.4) for $p = 1$ and $p = 2$. We start with $p = 1$.

Corollary 3.1. *Let* $\mathrm{Re}\, f \in h^1(\mathbb{R}^2_+)$, *and let* z *be an arbitrary point in* $\mathbb{C}_+$. *The sharp constant* $K_{n,1}$ *in the inequality*

$$|f^{(n)}(z)| \leqslant \frac{K_{n,1}}{(\mathrm{Im}\, z)^{n+1}}\, ||\,\mathrm{Re}\, f||_1\ , \tag{3.1}$$

is given by

$$K_{n,1} = \frac{n!}{\pi}. \tag{3.2}$$

Besides,

$$K_{2m,1} = K_{2m,1}(0), \qquad K_{2m+1,1} = K_{2m+1,1}(\pi/2)\ . \tag{3.3}$$

Proof. The estimate (3.1) follows from Proposition 2.1 with

$$K_{n,1} = \max_{\alpha} K_{n,1}(\alpha) \tag{3.4}$$

and

$$K_{n,1}(\alpha) = \frac{n!}{\pi} \max_{\varphi} \left|\cos\left[\alpha - (n+1)\varphi + \frac{n\pi}{2}\right]\right| \cos^{n+1}\varphi\ . \tag{3.5}$$

Interchanging the order of maxima, we obtain

$$K_{n,1} = \frac{n!}{\pi} \max_{\varphi} \max_{\alpha} \left|\cos\left[\alpha - (n+1)\varphi + \frac{n\pi}{2}\right]\right| \cos^{n+1}\varphi = \frac{n!}{\pi}\ , \tag{3.6}$$

which proves (3.2).

If n is even, the maximum in (3.6) is attained at $\alpha = (n+1)\varphi$ and $\varphi = 0$, i.e., the maximum over α in (3.4) is attained at $\alpha = 0$. If n is odd, the maximum in (3.6) is attained at $\alpha = (n+1)\varphi + (\pi/2)$ and $\varphi = 0$, i.e., the maximum over α in (3.4) is attained at $\alpha = \pi/2$. Thus, (3.3) follows. □

The next assertion concerns the case $p = 2$.

Corollary 3.2. *Let* $\operatorname{Re} f \in h^2(\mathbb{R}^2_+)$, *and let* z *be an arbitrary point in* $\mathbb{C}_+$. *The sharp constant* $K_{n,2}$ *in the inequality*

$$|f^{(n)}(z)| \leqslant \frac{K_{n,2}}{(\operatorname{Im} z)^{n+\frac{1}{2}}}\ ||\operatorname{Re} f||_2 \tag{3.7}$$

is given by

$$K_{n,2} = \sqrt{\frac{(2n)!}{2^{2n+1}\pi}}\ . \tag{3.8}$$

The coefficient $K_{n,p}(\alpha)$ *in (2.2) is independent of* α *for* $p = 2$.

Proof. The inequality (3.7) with

$$K_{n,2} = \max_{\alpha} K_{n,2}(\alpha), \tag{3.9}$$

where

$$K_{n,2}(\alpha) = \frac{n!}{\pi} \left\{ \int_{-\pi/2}^{\pi/2} \left|\cos\left(\alpha - (n+1)\varphi + \frac{n\pi}{2}\right)\right|^2 \cos^{2n}\varphi d\varphi \right\}^{1/2}, \tag{3.10}$$

follows from (2.4)–(2.6) and (2.3). Consider the function

$$F_n(\alpha) = \int_{-\pi/2}^{\pi/2} \cos^2\left(\alpha - (n+1)\varphi + \frac{n\pi}{2}\right) \cos^{2n}\varphi d\varphi\ , \tag{3.11}$$

which appears in (3.10). Since

$$F_n(\alpha) = \frac{1}{2}\int\limits_{-\pi/2}^{\pi/2} \cos^{2n}\varphi d\varphi + \frac{1}{2}\int\limits_{-\pi/2}^{\pi/2} \cos\big(2\alpha - 2(n+1)\varphi + n\pi\big)\cos^{2n}\varphi d\varphi$$

$$= \int\limits_{0}^{\pi/2} \cos^{2n}\varphi d\varphi + (-1)^n \cos 2\alpha \int\limits_{0}^{\pi/2} \cos 2(n+1)\varphi \cos^{2n}\varphi d\varphi$$

and

$$\int\limits_{0}^{\pi/2} \cos^{\nu-2} x \cos \nu x \, dx = 0$$

(cf., for example, [8, 3.631(19)]), we find

$$F_n(\alpha) = \int\limits_{0}^{\pi/2} \cos^{2n}\varphi d\varphi = \frac{\sqrt{\pi}\Gamma\left(\frac{2n+1}{2}\right)}{2n!} = \frac{\pi(2n)!}{2^{2n+1}(n!)^2} . \tag{3.12}$$

By (3.10) and (3.11),

$$K_{n,2}(\alpha) = \frac{n!}{\pi}\sqrt{F_n(\alpha)} ,$$

which, together with (3.12), gives

$$K_{n,2}(\alpha) = \sqrt{\frac{(2n)!}{2^{2n+1}\pi}} . \tag{3.13}$$

Thus, $K_{n,2}(\alpha)$ is independent of α. Formula (3.8) follows from (3.9) and (3.13). □

4 Case $p = \infty$, $n = 2m + 1$

In this section, we prove the following assertion.

Corollary 4.1. *Let* $\mathrm{Re}\, f \in h^\infty(\mathbb{R}^2_+)$*, and let* z *be an arbitrary point in* $\mathbb{C}_+$*. The sharp constant* $K_{2m+1,\infty}$ *in the inequality*

$$|f^{(2m+1)}(z)| \leqslant \frac{K_{2m+1,\infty}}{(\mathrm{Im}\, z)^{2m+1}} \, ||\mathrm{Re}\, f||_\infty \tag{4.1}$$

is given by

$$K_{2m+1,\infty} = \frac{2}{\pi}\frac{[(2m+1)!!]^2}{2m+1} . \tag{4.2}$$

The coefficient $K_{n,p}(\alpha)$ *in (2.2) with* $n = 2m+1$ *and* $p = \infty$ *is independent of* α*.*

Proof. The inequality (4.1) follows from (2.4) and (2.5). By (2.3), we have

$$K_{2m+1,\infty}(\alpha) = \frac{(2m+1)!}{\pi} \int\limits_{-\pi/2}^{\pi/2} \big|\sin\big[\alpha - 2(m+1)\varphi\big]\big| \cos^{2m}\varphi \, d\varphi \,. \tag{4.3}$$

Setting

$$G_m(\alpha) = \int\limits_{-\pi/2}^{\pi/2} \big|\sin\big[\alpha - 2(m+1)\varphi\big]\big| \cos^{2m}\varphi \, d\varphi \tag{4.4}$$

and making the change of variable $\psi = \alpha - 2(m+1)\varphi$, we obtain

$$G_m(\alpha) = \frac{1}{2(m+1)} \int\limits_{\alpha-(m+1)\pi}^{\alpha+(m+1)\pi} \big|\sin\psi\big| \cos^{2m}\frac{\psi-\alpha}{2(m+1)} \, d\psi \,.$$

Since the integrand is $2(m+1)\pi$-periodic, it follows that

$$\begin{aligned} G_m(\alpha) &= \frac{1}{2(m+1)} \int\limits_{-(m+1)\pi}^{(m+1)\pi} \big|\sin\psi\big| \cos^{2m}\frac{\psi-\alpha}{2(m+1)} \, d\psi \\ &= \frac{1}{2(m+1)} \sum_{k=-(m+1)}^{m} \int\limits_{k\pi}^{(k+1)\pi} \big|\sin\psi\big| \cos^{2m}\frac{\psi-\alpha}{2(m+1)} \, d\psi \,. \end{aligned}$$

The change of variable $\psi - k\pi = \vartheta$ implies

$$G_m(\alpha) = \frac{1}{2(m+1)} \sum_{k=-(m+1)}^{m} \int\limits_{0}^{\pi} \big|\sin\vartheta\big| \cos^{2m}\frac{\vartheta+k\pi-\alpha}{2(m+1)} \, d\vartheta \,. \tag{4.5}$$

We introduce the notation

$$g_m(\theta) = \sum_{k=-(m+1)}^{m} \cos^{2m}\frac{\theta+k\pi}{2(m+1)} \,. \tag{4.6}$$

Since

$$\cos^{2m} x = \frac{1}{2^{2m}} \left\{ \sum_{j=0}^{m-1} 2\binom{2m}{j} \cos 2(m-j)x + \binom{2m}{m} \right\},$$

we can write (4.6) in the form

$$g_m(\theta) = \frac{2(m+1)}{2^{2m}}\binom{2m}{m} + \frac{1}{2^{2m-1}} \sum_{j=0}^{m-1}\binom{2m}{j} \sum_{k=-(m+1)}^{m} \cos\frac{(\theta+k\pi)(m-j)}{m+1} \,. \tag{4.7}$$

We set $s = m - j$ $(s = 1, 2, \dots, m)$. Consider the interior sum. We have

$$\sum_{k=-(m+1)}^{m} \cos\frac{(\theta + k\pi)s}{m+1} = \mathrm{Re}\left\{ e^{\frac{i\theta s}{m+1}} \sum_{k=-(m+1)}^{m} e^{\frac{is\pi}{m+1}k} \right\}$$

$$= \mathrm{Re}\left\{ e^{i(\frac{\theta}{m+1}-\pi)s} \sum_{\ell=0}^{2m+1} e^{\frac{is\pi}{m+1}\ell} \right\} = \mathrm{Re}\left\{ e^{i(\frac{\theta}{m+1}-\pi)s} \frac{1 - e^{2is\pi}}{1 - e^{\frac{is\pi}{m+1}}} \right\}.$$

Therefore, the second term in (4.7) vanishes and hence

$$g_m(\theta) = \frac{m+1}{2^{2m-1}} \binom{2m}{m} = \frac{(m+1)(2m)!}{2^{2m-1}(m!)^2}\ .$$

Combining this with (4.6), we write (4.5) as

$$G_m(\alpha) = \frac{1}{2(m+1)} \frac{(m+1)(2m)!}{2^{2m-1}(m!)^2} \int_0^{\pi} \sin\vartheta \, d\vartheta = \frac{(2m)!}{2^{2m-1}(m!)^2}\ .$$

Now, by (4.3) and (4.4), we find

$$K_{2m+1,\infty}(\alpha) = \frac{(2m+1)!}{\pi} G_m(\alpha) = \frac{(2m+1)!}{\pi} \cdot \frac{(2m)!}{2^{2m-1}(m!)^2} = \frac{2}{\pi} \frac{[(2m+1)!!]^2}{2m+1}\ ,$$

which proves the independence of $K_{2m+1,\infty}(\alpha)$ of α. This, together with (2.6), leads to (4.2). □

5 Cases $p = \infty$, $n = 2$ and $n = 4$

By (2.3), we have

$$K_{2m,\infty}(\alpha) = \frac{(2m)!}{\pi} \int_{-\pi/2}^{\pi/2} \left|\cos\left[\alpha - (2m+1)\varphi\right]\right| \cos^{2m-1}\varphi \, d\varphi\ . \tag{5.1}$$

Lemma 5.1. *The equality*

$$\frac{dK_{2m,\infty}}{d\alpha} = \frac{(2m)!}{\pi(2m+1)^2 2^{2(m-1)}} \int_0^{\pi/2} \big(|\cos(\alpha - \varphi)| - |\cos(\alpha + \varphi)|\big) \Lambda_m(\varphi) d\varphi \tag{5.2}$$

holds with

$$\Lambda_m(\varphi) = \sum_{\ell=1}^{m} (-1)^{\ell} (2\ell - 1) \binom{2m-1}{m-\ell} \frac{\sin\frac{(2\ell-1)\varphi}{2m+1}}{\sin\frac{(2\ell-1)\pi}{2(2m+1)}}\ . \tag{5.3}$$

Proof. Setting

$$S_m(\alpha) = \int_{-\pi/2}^{\pi/2} \left|\cos\left[\alpha - (2m+1)\varphi\right]\right| \cos^{2m-1}\varphi \, d\varphi\ , \tag{5.4}$$

we write (5.1) as

$$K_{2m,\infty}(\alpha) = \frac{(2m)!}{\pi}\, S_m(\alpha). \tag{5.5}$$

Making the change of variable $\psi = \alpha - (2m+1)\varphi$ in (5.4), we obtain

$$S_m(\alpha) = \frac{1}{2m+1} \int\limits_{\alpha-(2m+1)\frac{\pi}{2}}^{\alpha+(2m+1)\frac{\pi}{2}} \big|\cos\psi\big| \cos^{2m-1} \frac{\alpha-\psi}{2m+1}\, d\psi\ .$$

Hence

$$\frac{dS_m}{d\alpha} = -\frac{2m-1}{(2m+1)^2} \int\limits_{\alpha-(2m+1)\frac{\pi}{2}}^{\alpha+(2m+1)\frac{\pi}{2}} \big|\cos\psi\big| \cos^{2(m-1)} \frac{\alpha-\psi}{2m+1} \sin\frac{\alpha-\psi}{2m+1}\, d\psi\ .$$

Returning to the variable $\varphi = (\alpha-\psi)/(2m+1)$ and then writing the resulting integral as the sum over $(-\pi/2, 0)$ and $(0, \pi/2)$, we find

$$\frac{dS_m}{d\alpha} = -\frac{2m-1}{2m+1} \int\limits_{0}^{\pi/2} \big|\cos[\alpha-(2m+1)\varphi]\big| \cos^{2(m-1)}\varphi \sin\varphi\, d\varphi$$
$$- \frac{2m-1}{2m+1} \int\limits_{-\pi/2}^{0} \big|\cos[\alpha-(2m+1)\varphi]\big| \cos^{2(m-1)}\varphi \sin\varphi\, d\varphi\ .$$

Setting $\varphi = -\vartheta$ in the second integral, we obtain

$$\frac{dS_m}{d\alpha} = \frac{2m-1}{2m+1} \int\limits_{0}^{\pi/2} \Big\{\big|\cos[\alpha+(2m+1)\varphi]\big| - \big|\cos[\alpha-(2m+1)\varphi]\big|\Big\} \cos^{2(m-1)}\varphi \sin\varphi d\varphi\ ,$$

which after the change of variable $\theta = (2m+1)\varphi$ becomes

$$\frac{dS_m}{d\alpha} = \frac{2m-1}{(2m+1)^2} \int\limits_{0}^{(2m+1)\frac{\pi}{2}} P(\alpha,\theta) \cos^{2(m-1)} \frac{\theta}{2m+1} \sin\frac{\theta}{2m+1}\, d\theta\ , \tag{5.6}$$

where

$$P(\alpha,\theta) = \big|\cos(\alpha+\theta)\big| - \big|\cos(\alpha-\theta)\big|\ . \tag{5.7}$$

Now, (5.6) can be written as

$$\frac{dS_m}{d\alpha} = \frac{2m-1}{(2m+1)^2} \sum_{j=1}^{2m+1} \int\limits_{\frac{\pi(j-1)}{2}}^{\frac{\pi j}{2}} P(\alpha,\theta) \cos^{2(m-1)} \frac{\theta}{2m+1} \sin\frac{\theta}{2m+1}\, d\theta\ .$$

We split the last sum into two for even and odd j:

$$\frac{dS_m}{d\alpha} = \frac{2m-1}{(2m+1)^2} \sum_{s=1}^{m} \int\limits_{\frac{\pi(2s-1)}{2}}^{\pi s} P(\alpha, \theta) \cos^{2(m-1)} \frac{\theta}{2m+1} \sin \frac{\theta}{2m+1} \, d\theta$$

$$+ \frac{2m-1}{(2m+1)^2} \sum_{s=1}^{m+1} \int\limits_{\pi(s-1)}^{\frac{\pi(2s-1)}{2}} P(\alpha, \theta) \cos^{2(m-1)} \frac{\theta}{2m+1} \sin \frac{\theta}{2m+1} \, d\theta \, .$$

In the first sum, we set $\theta - s\pi = -\varphi$ under the integral sign and in the second sum we set $\theta - (s-1)\pi = \varphi$. This leads to the equality

$$\frac{dS_m}{d\alpha} = -\frac{2m-1}{(2m+1)^2} \sum_{s=1}^{m} \int\limits_{0}^{\pi/2} P(\alpha, \varphi) \cos^{2(m-1)} \frac{s\pi - \varphi}{2m+1} \sin \frac{s\pi - \varphi}{2m+1} \, d\varphi$$

$$+ \frac{2m-1}{(2m+1)^2} \sum_{s=1}^{m+1} \int\limits_{0}^{\pi/2} P(\alpha, \varphi) \cos^{2(m-1)} \frac{(s-1)\pi + \varphi}{2m+1} \sin \frac{(s-1)\pi + \varphi}{2m+1} \, d\varphi \, ,$$

which can be written as

$$\frac{dS_m}{d\alpha} = \frac{2m-1}{(2m+1)^2} \int\limits_{0}^{\pi/2} \left(\left| \cos(\alpha - \varphi) \right| - \left| \cos(\alpha + \varphi) \right| \right) \Phi_m(\varphi) d\varphi \, , \tag{5.8}$$

where

$$\Phi_m(\varphi) = \sum_{k=-m}^{m} \cos^{2(m-1)} \frac{k\pi - \varphi}{2m+1} \sin \frac{k\pi - \varphi}{2m+1} \, . \tag{5.9}$$

Next, we write (5.9) in the form

$$\Phi_m(\varphi) = \frac{2m+1}{2m-1} \frac{dQ_m}{d\varphi} \, , \tag{5.10}$$

where

$$Q_m(\varphi) = \sum_{k=-m}^{m} \cos^{2m-1} \frac{k\pi - \varphi}{2m+1} \, . \tag{5.11}$$

Since

$$\cos^{2m-1} x = \frac{1}{2^{2(m-1)}} \sum_{j=0}^{m-1} \binom{2m-1}{j} \cos(2m-2j-1)x \, ,$$

(5.11) becomes

$$Q_m(\varphi) = \frac{1}{2^{2(m-1)}} \sum_{k=-m}^{m} \sum_{j=0}^{m-1} \binom{2m-1}{j} \cos \frac{(k\pi - \varphi)(2m-2j-1)}{2m+1} \, .$$

Setting $\ell = m - j$ ($\ell = 1, 2, \ldots, m$) in the interior sum and changing the order of summation, we obtain

$$Q_m(\varphi) = \frac{1}{2^{2(m-1)}} \sum_{\ell=1}^{m} \binom{2m-1}{m-\ell} \sum_{k=-m}^{m} \cos\frac{(k\pi-\varphi)(2\ell-1)}{2m+1} . \tag{5.12}$$

We evaluate the interior sum in (5.12). Using the equalities

$$\sum_{k=-m}^{m} \cos\frac{(k\pi-\varphi)(2\ell-1)}{2m+1} = \mathrm{Re}\left\{ \sum_{k=-m}^{m} e^{i\frac{(k\pi-\varphi)(2\ell-1)}{2m+1}} \right\} = \mathrm{Re}\left\{ e^{-i\frac{(2\ell-1)\varphi}{2m+1}} \sum_{k=-m}^{m} e^{i\frac{\pi k(2\ell-1)}{2m+1}} \right\}$$

and

$$\begin{aligned} \sum_{k=-m}^{m} e^{i\frac{\pi k(2\ell-1)}{2m+1}} &= \sum_{j=0}^{2m} e^{i\frac{\pi(j-m)(2\ell-1)}{2m+1}} = e^{-i\frac{\pi m(2\ell-1)}{2m+1}} \sum_{j=0}^{2m} e^{i\frac{\pi j(2\ell-1)}{2m+1}} \\ &= \frac{1-e^{i\pi(2\ell-1)}}{1-e^{i\frac{\pi(2\ell-1)}{2m+1}}}\, e^{-i\frac{\pi m(2\ell-1)}{2m+1}} = \frac{2e^{-i\frac{\pi m(2\ell-1)}{2m+1}}}{1-e^{i\frac{\pi(2\ell-1)}{2m+1}}} , \end{aligned}$$

we find

$$\sum_{k=-m}^{m} \cos\frac{(k\pi-\varphi)(2\ell-1)}{2m+1} = 2\,\mathrm{Re}\left\{ \frac{e^{-i\frac{(2\ell-1)(\varphi+m\pi)}{2m+1}}}{1-e^{i\frac{\pi(2\ell-1)}{2m+1}}} \right\} = (-1)^{\ell+1} \frac{\cos\frac{(2\ell-1)\varphi}{2m+1}}{\sin\frac{(2\ell-1)\pi}{2(2m+1)}} .$$

Substituting the last sum into (5.12), we obtain

$$Q_m(\varphi) = \frac{1}{2^{2(m-1)}} \sum_{\ell=1}^{m} \binom{2m-1}{m-\ell} (-1)^{\ell+1} \frac{\cos\frac{(2\ell-1)\varphi}{2m+1}}{\sin\frac{(2\ell-1)\pi}{2(2m+1)}} ,$$

which, together with (5.10), implies

$$\Phi_m(\varphi) = \frac{1}{(2m-1)2^{2(m-1)}} \sum_{\ell=1}^{m} \binom{2m-1}{m-\ell} (-1)^{\ell}(2\ell-1) \frac{\sin\frac{(2\ell-1)\varphi}{2m+1}}{\sin\frac{(2\ell-1)\pi}{2(2m+1)}} .$$

Using this in (5.8) and taking into account (5.5), we arrive at (5.2) and (5.3). □

Before passing to applications of Lemma 5.1 we make two remarks.

The first one concerns the range of α in the evaluation of the maximum

$$K_{2m,\infty} = \max_{\alpha} K_{2m,\infty}(\alpha). \tag{5.13}$$

It follows from (5.1) that $K_{2m,\infty}(\alpha)$ is a π-periodic function. Hence we may assume that $\alpha \in [-\pi/2, \pi/2]$. Moreover, we can restrict our consideration to the interval $[0, \pi/2]$ since $K_{2m,\infty}(\alpha)$ is even, which is easy to check.

The second remark relates the sign of $|\cos(\varphi-\alpha)| - |\cos(\varphi+\alpha)|$. We show that

$$|\cos(\varphi-\alpha)| \geqslant |\cos(\varphi+\alpha)| \tag{5.14}$$

for $\alpha, \varphi \in [0, \pi/2]$. In fact, since

$$|\cos(\varphi-\alpha)|-|\cos(\varphi+\alpha)|=\begin{cases} \cos(\varphi-\alpha)-\cos(\varphi+\alpha) & \text{for} \quad \varphi \in \left[\,0, \frac{\pi}{2}-\alpha\right], \\ \cos(\varphi-\alpha)+\cos(\varphi+\alpha) & \text{for} \quad \varphi \in \left(\frac{\pi}{2}-\alpha, \frac{\pi}{2}\right], \end{cases}$$

it follows that

$$|\cos(\varphi-\alpha)|-|\cos(\varphi+\alpha)|=\begin{cases} 2\sin\varphi\sin\alpha & \text{for} \quad \varphi \in \left[\,0, \frac{\pi}{2}-\alpha\right] \ , \\ 2\cos\varphi\cos\alpha & \text{for} \quad \varphi \in \left(\frac{\pi}{2}-\alpha, \frac{\pi}{2}\right] \ , \end{cases}$$

and hence (5.14) holds for $\alpha, \vartheta \in [0, \pi/2]$. Moreover, the equality sign in (5.14) holds only for $\alpha = 0$ or for $\alpha = \pi/2$ provided that $\varphi \in (0, \pi/2)$.

Corollary 5.1. *Let* $\mathrm{Re}\, f \in h^\infty(\mathbb{R}^2_+)$, *and let* z *be an arbitrary point in* $\mathbb{C}_+$. *The sharp constant* $K_{2,\infty}$ *in the inequality*

$$|f''(z)| \leqslant \frac{K_{2,\infty}}{(\mathrm{Im}\, z)^2}\, ||\, \mathrm{Re}\, f||_\infty \tag{5.15}$$

is given by

$$K_{2,\infty} = K_{2,\infty}(0) = \frac{3\sqrt{3}}{2\pi} \ . \tag{5.16}$$

Proof. By Lemma 5.1,

$$\frac{dK_{2,\infty}}{d\alpha} = -\frac{4}{9\pi}\int\limits_0^{\pi/2} \big(|\cos(\alpha-\varphi)| - |\cos(\alpha+\varphi)|\big)\sin\frac{\varphi}{3}\, d\varphi \ .$$

Combined with (5.14), this implies

$$\frac{dK_{2,\infty}}{d\alpha} < 0$$

for $\alpha \in (0, \pi/2)$. Taking into account (5.1) and (5.13), we obtain

$$K_{2,\infty} = K_{2,\infty}(0) = \frac{2}{\pi}\int\limits_{-\pi/2}^{\pi/2} \big|\cos 3\varphi\big|\cos\varphi\, d\varphi$$

$$= \frac{4}{\pi}\left\{\int\limits_0^{\pi/6}\cos 3\varphi\cos\varphi\, d\varphi - \int\limits_{\pi/6}^{\pi/2}\cos 3\varphi\cos\varphi\, d\varphi\right\} = \frac{3\sqrt{3}}{2\pi} \ . \qquad \square$$

Corollary 5.2. *Let* $\mathrm{Re}\, f \in h^\infty(\mathbb{R}^2_+)$, *and let* z *be an arbitrary point in* $\mathbb{C}_+$. *The sharp constant* $K_{4,\infty}$ *in the inequality*

$$|f''''(z)| \leqslant \frac{K_{4,\infty}}{(\mathrm{Im}\, z)^4}\, ||\, \mathrm{Re}\, f||_\infty \tag{5.17}$$

is given by

$$K_{4,\infty} = K_{4,\infty}(\pi/2) = \frac{3}{4\pi}(16 + 5\sqrt{5}) \ . \tag{5.18}$$

Proof. By Lemma 5.1,

$$\frac{dK_{4,\infty}}{d\alpha} = \frac{24}{100\pi} \int_0^{\pi/2} \big(|\cos(\alpha - \varphi)| - |\cos(\alpha + \varphi)|\big) \Lambda_2(\varphi)\, d\varphi\ , \tag{5.19}$$

where

$$\Lambda_2(\varphi) = 3 \left(\frac{\sin \frac{3\varphi}{5}}{\sin \frac{3\pi}{10}} - \frac{\sin \frac{\varphi}{5}}{\sin \frac{\pi}{10}} \right). \tag{5.20}$$

Using the identity $\sin 3x = 3\sin x - 4\sin^3 x$ in (5.20), we find

$$\Lambda_2(\varphi) = 12 \frac{\sin \frac{\varphi}{5}}{\sin \frac{3\pi}{10}} \left(\sin^2 \frac{\pi}{10} - \sin^2 \frac{\varphi}{5} \right),$$

i.e., $\Lambda_2(\varphi) > 0$ for $\varphi \in (0, \pi/2)$.

Now, by (5.19) and (5.14), we conclude that

$$\frac{dK_{4,\infty}}{d\alpha} > 0$$

for $\alpha \in (0, \pi/2)$. Thus, by (5.1) and (5.13),

$$\begin{aligned} K_{4,\infty} = K_{4,\infty}(\pi/2) &= \frac{24}{\pi} \int_{-\pi/2}^{\pi/2} \big|\sin 5\varphi\big| \cos^3 \varphi\, d\varphi \\ &= \frac{48}{\pi} \left\{ \int_0^{\pi/5} \sin 5\varphi \cos^3 \varphi\, d\varphi - \int_{\pi/5}^{2\pi/5} \sin 5\varphi \cos^3 \varphi\, d\varphi + \int_{2\pi/5}^{\pi/2} \sin 5\varphi \cos^3 \varphi\, d\varphi \right\}. \end{aligned}$$

Evaluating the integrals on the right-hand side of the last equality, we arrive at (5.18). □

6 Case $p = 4$ and $n = 2$

In the previous sections, it was shown that $K_{n,p}(\alpha)$ is independent of α for $p = 2$, as well as for $p = \infty$, $n = 2m + 1$. Here, we prove that those are not all values of n and p when $K_{n,p}(\alpha)$ does not depend on α. We set

$$T_{n,p}(\alpha) = \int_{-\pi/2}^{\pi/2} \left| \cos \left(\alpha - (n+1)\varphi + \frac{n\pi}{2} \right) \right|^q \cos^{(n+1)q-2} \varphi d\varphi\ , \tag{6.1}$$

where $1/p + 1/q = 1$. Then, by (2.3) and (6.1),

$$K_{n,p}(\alpha) = \frac{n!}{\pi} \left\{ T_{n,p}(\alpha) \right\}^{1/q}.$$

Using the same argument as in the proof of (5.8) in Lemma 5.1, we can show that for any $n \geqslant 1$ and $p \in (1, \infty]$

$$\frac{dK_{n,p}}{d\alpha} = \frac{n!\big[(n-1)p+2\big]}{\pi p(n+1)^2}\,\{T_{n,p}(\alpha)\}^{-1/p}\;\frac{dT_{n,p}}{d\alpha}\,, \tag{6.2}$$

where

$$\frac{dT_{n,p}}{d\alpha} = \int\limits_0^{\pi/2}\left\{\left|\cos\left(\alpha-\varphi+\frac{n\pi}{2}\right)\right|^q - \left|\cos\left(\alpha+\varphi+\frac{n\pi}{2}\right)\right|^q\right\}\Psi_{n,p}(\varphi)d\varphi\,. \tag{6.3}$$

Here,

$$\Psi_{n,p}(\varphi) = \sum_{k=[-(n-1)/2]}^{[(n+1)/2]} \cos^{(n+1)q-3}\frac{k\pi-\varphi}{n+1}\sin\frac{k\pi-\varphi}{n+1}\,, \tag{6.4}$$

and [ ] stands for the integer part of a number.

Putting $n = 2$ and $p = 4$ (i.e., $q = 4/3$) in (6.4), we obtain

$$\Psi_{2,4}(\varphi) = \sum_{k=-1}^{1}\cos\frac{k\pi-\varphi}{3}\sin\frac{k\pi-\varphi}{3} = 0\,,$$

which, together with (6.2) and (6.3), implies

$$\frac{dK_{2,4}}{d\alpha} = 0.$$

Thus, $K_{2,4}(\alpha)$ is independent of α and hence, by Proposition 2.1, the sharp constant in

$$|f''(z)| \leqslant \frac{K_{2,4}}{(\operatorname{Im} z)^{9/4}}||\operatorname{Re} f||_4$$

can be represented, for example, as

$$K_{2,4} = K_{2,4}(0) = \frac{2}{\pi}\left\{\int\limits_{-\pi/2}^{\pi/2}|\cos 3\varphi|^{4/3}\cos^2\varphi\,d\varphi\right\}^{3/4}.$$

7 Sharp Limit Relation for Derivatives of Analytic Functions in a Disk

In what follows, by $h^p(\mathbb{D}_R)$, $1 \leqslant p \leqslant \infty$, we mean the Hardy space of harmonic functions in $\mathbb{D}_R = \{z : |z| < R\}$ which are represented by the Poisson integral with a density in $L^p(\partial\mathbb{D}_R)$.

We need the following assertion proved in Section 5.2 of the monograph [1].

Proposition 7.1. *Let f be analytic on $\mathbb{D}_R$ with $\operatorname{Re} f \in h_p(\mathbb{D}_R), 1 \leqslant p \leqslant \infty$. Further, let $n \geqslant 1$, and let $\mathscr{P}_m$ be a polynomial of degree $m \leqslant n-1$. Then for any fixed point $z, |z| = r < R$, the inequality*

$$|f^{(n)}(z)| \leqslant \mathscr{H}_{n,p}(z)||\operatorname{Re}\{f-\mathscr{P}_m\}|_{\partial\mathbb{D}_R}||_p \tag{7.1}$$

holds with the sharp factor

$$\mathscr{H}_{n,p}(z) = \frac{1}{R^{(np+1)/p}} H_{n,p}\left(\frac{r}{R}\right), \tag{7.2}$$

where

$$H_{n,p}(\gamma) = \frac{n!}{\pi} \max_{\alpha} \left\{ \int\limits_{|\zeta|=1} \left| \mathrm{Re}\left\{ \frac{\zeta e^{i\alpha}}{(\zeta-\gamma)^{n+1}} \right\} \right|^q |d\zeta| \right\}^{1/q} \tag{7.3}$$

and $1/p + 1/q = 1$. *In particular,*

$$|f^{(n)}(z)| \leqslant \mathscr{H}_{n,p}(z) \inf_{\mathscr{P} \in \{\mathfrak{P}_{n-1}\}} ||\,\mathrm{Re}\{f - \mathscr{P}\}||_p\,, \tag{7.4}$$

where $\{\mathfrak{P}_m\}$ *is the set of all polynomials of degree at most* m.

The next assertion concerns a relation between $\mathscr{H}_{n,p}(z)$ and $K_{n,p}$ which was defined by (2.6) and (2.3).

Theorem 7.1. *The following limit relation holds*:

$$\lim_{r \to R} (R-r)^{n+\frac{1}{p}} \mathscr{H}_{n,p}(z) = K_{n,p}. \tag{7.5}$$

Proof. By (7.2), we have

$$(R-r)^{n+\frac{1}{p}} \mathscr{H}_{n,p}(z) = \frac{(R-r)^{n+\frac{1}{p}}}{R^{n+\frac{1}{p}}} H_{n,p}\left(\frac{r}{R}\right) = \left(1 - \frac{r}{R}\right)^{n+\frac{1}{p}} H_{n,p}\left(\frac{r}{R}\right).$$

Hence

$$\lim_{r \to R} (R-r)^{n+\frac{1}{p}} \mathscr{H}_{n,p}(z) = \lim_{\gamma \to 1} (1-\gamma)^{n+\frac{1}{p}} H_{n,p}(\gamma)\,. \tag{7.6}$$

It follows from (7.3) that

$$(1-\gamma)^{n+\frac{1}{p}} H_{n,p}(\gamma) = \frac{n!}{\pi} \max_{\alpha} \left\{ (1-\gamma)^{n+1)q-1} E_{n,q}(\gamma;\alpha) \right\}^{1/q}\,, \tag{7.7}$$

where

$$E_{n,q}(\gamma;\alpha) = \int\limits_{|\zeta|=1} \left| \mathrm{Re}\left\{ \frac{\zeta e^{i\alpha}}{(\zeta-\gamma)^{n+1}} \right\} \right|^q |d\zeta|\,. \tag{7.8}$$

The last equality can be written as

$$E_{n,q}(\gamma;\alpha) = \int\limits_{-\pi}^{\pi} \left| \mathrm{Re}\left\{ \frac{e^{i(\varphi+\alpha)}}{(e^{i\varphi}-\gamma)^{n+1}} \right\} \right|^q d\varphi\,,$$

which after the change of variable $\varphi = 2\psi$ becomes

$$E_{n,q}(\gamma;\alpha) = 2 \int\limits_{-\pi/2}^{\pi/2} \left| \mathrm{Re}\left\{ \frac{e^{i(2\psi+\alpha)}}{(e^{2i\psi}-\gamma)^{n+1}} \right\} \right|^q d\psi\,.$$

Therefore,

$$E_{n,q}(\gamma;\alpha) = 2 \int_{-\pi/2}^{\pi/2} \left| \mathrm{Re} \left\{ \frac{e^{i(2\psi+\alpha)}(e^{-2i\psi} - \gamma)^{n+1}}{(1 - 2\gamma \cos 2\psi + \gamma^2)^{n+1}} \right\} \right|^q d\psi \ . \tag{7.9}$$

Introduce the new variable t by

$$\psi = \arctan k(\gamma)t \ , \tag{7.10}$$

with

$$k(\gamma) = \frac{1-\gamma}{1+\gamma} \ . \tag{7.11}$$

Then

$$d\psi = \frac{k(\gamma)}{1 + k^2(\gamma)t^2} \ dt \ . \tag{7.12}$$

Further, we find

$$1 - 2\gamma \cos 2\psi + \gamma^2 = \frac{(1-\gamma)^2(1+t^2)}{1 + k^2(\gamma)t^2} \ , \tag{7.13}$$

$$e^{-2i\psi} - \gamma = \frac{1 - k(\gamma)t^2 - \frac{2i}{1+\gamma}t}{1 + k^2(\gamma)t^2} \ , \tag{7.14}$$

$$e^{2i\psi} = \frac{1 - k^2(\gamma)t^2 + 2ik(\gamma)t}{1 + k^2(\gamma)t^2} \ . \tag{7.15}$$

Substituting (7.12)–(7.15) into (7.9) and making the change of limits according to (7.10), we obtain

$$E_{n,q}(\gamma;\alpha) = \frac{2}{(1+\gamma)(1-\gamma)^{(n+1)q-1}} \int_{-\infty}^{\infty} \left|\mathrm{Re}\left\{e^{i\alpha}\Psi_{n,\gamma}(t)\right\}\right|^q \frac{dt}{1 + k^2(\gamma)t^2} \ , \tag{7.16}$$

where

$$\Psi_{n,\gamma}(t) = \frac{\Big(1 - k^2(\gamma)t^2 + 2ik(\gamma)t\Big)\Big(1 - k(\gamma)t^2 - \frac{2it}{1+\gamma}\Big)^{n+1}}{\big(1+t^2\big)^{n+1}\big(1 + k^2(\gamma)t^2\big)} \ . \tag{7.17}$$

It follows from (7.7) and (7.16) that

$$(1-\gamma)^{n+\frac{1}{p}} H_{n,p}(\gamma) = \frac{n!}{\pi} \left(\frac{2}{1+\gamma}\right)^{1/q} \max_{\alpha} \left\{ \int_{-\infty}^{\infty} \left|\mathrm{Re}\left\{e^{i\alpha}\Psi_{n,\gamma}(t)\right\}\right|^q \frac{dt}{1+k^2(\gamma)t^2} \right\}^{1/q} .$$

Passing here to the limit as $\gamma \to 1$ and using (7.11) and (7.17), we get

$$\lim_{\gamma\to 1} (1-\gamma)^{n+\frac{1}{p}} H_{n,p}(\gamma) = \frac{n!}{\pi} \max_{\alpha} \left\{ \int_{-\infty}^{\infty} \left| \mathrm{Re} \left\{ \frac{e^{i\alpha}(1-it)^{n+1}}{(1+t^2)^{n+1}} \right\} \right|^q dt \right\}^{1/q} . \tag{7.18}$$

We introduce a new variable $\vartheta \in (-\pi/2, \pi/2)$ by the formula

$$\cos\vartheta = \frac{1}{\sqrt{1+t^2}}, \qquad \sin\vartheta = -\frac{t}{\sqrt{1+t^2}} \ .$$

By the equality $t = -\tan\vartheta$, we have

$$\int_{-\infty}^{\infty} \left| \mathrm{Re}\left\{ \frac{e^{i\alpha}(1-it)^{n+1}}{(1+t^2)^{n+1}} \right\} \right|^q dt = \int_{-\pi/2}^{\pi/2} \left| \mathrm{Re}\left\{ e^{i\alpha}(\cos\vartheta + i\sin\vartheta)^{n+1} \right\} \right|^q \cos^{(n+1)q-2}\vartheta \, d\vartheta$$

$$= \int_{-\pi/2}^{\pi/2} \left| \cos\big(\alpha + (n+1)\vartheta\big) \right|^q \cos^{(n+1)q-2}\vartheta \, d\vartheta \,. \tag{7.19}$$

Replacing the integral in (7.18) by (7.19) and taking into account (7.6), (2.3), and (2.6), we arrive at (7.5). □

Combining Theorem 7.1 with Propositions 2.1 and 7.1 with $p = \infty$, as well as with Corollary 4.1 and the maximum principle for harmonic functions, we obtain the following assertion.

Corollary 7.1. *Let Ω be a subdomain of $\mathbb{C}_+$ for which each point of the boundary can be touched by an interior circle of sufficiently small radius. Let f be a holomorphic function in Ω with a bounded real part and let $||\,\mathrm{Re}\, f|_{\partial\Omega}||_\infty \leqslant 1$. The limit relation*

$$\lim_{d_z \to 0} d_z^n \left| f^{(n)}(z) \right| \leqslant K_{n,\infty}$$

holds with $d_z = \mathrm{dist}\,(z, \partial\Omega)$ and

$$K_{n,\infty} = \frac{n!}{\pi} \max_\alpha \int_{-\pi/2}^{\pi/2} \left| \cos\big(\alpha + (n+1)\varphi\big) \right| \cos^{n-1}\varphi \, d\varphi \,. \tag{7.20}$$

In particular,

$$\lim_{d_z \to 0} d_z^{2m+1} \left| f^{(2m+1)}(z) \right| \leqslant \frac{2[(2m+1)!!]^2}{\pi(2m+1)} \,. \tag{7.21}$$

Remark 7.1. The pointwise estimate

$$\left| f^{(2m+1)}(z) \right| \leqslant \frac{(2m+1)!}{(1-|z|^2)^{2m+1}} \sum_{k=0}^{m} \binom{m}{k}^2 |z|^{2k} \tag{7.22}$$

for an analytic function f in the unit disk with $|f| < 1$ was derived by Szász in [4]. Using a dilation in (7.22), together with the identity

$$\sum_{k=0}^{m} \binom{m}{k}^2 = \binom{2m}{m} ,$$

and the maximum modulus principle for analytic functions, we arrive at the limit inequality (cf. (7.21))

$$\lim_{d_z \to 0} d_z^{2m+1} \left| f^{(2m+1)}(z) \right| \leqslant \frac{[(2m+1)!!]^2}{2(2m+1)}$$

for functions f with $|f(z)| < 1$ in a domain $\Omega \subset \mathbb{C}_+$ with a boundary satisfying the condition of tangency by a circle.

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