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OPTIMAL ESTIMATES FOR THE GRADIENT OF HARMONIC FUNCTIONS IN THE MULTIDIMENSIONAL HALF-SPACE

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Dedicated to Louis Nirenberg on the occasion of his eighty fifth birthday

Abstract. A representation of the sharp constant in a pointwise estimate of the gradient of a harmonic function in a multidimensional half-space is obtained under the assumption that function's boundary values belong to L^p. This representation is concretized for the cases $p = 1, 2$, and ∞.

1. Introduction.

There is a series of sharp estimates for the first derivative of a function f analytic in the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im z > 0\}$ with different characteristics of the real part of f in the majorant part (see, e.g., [6]). We mention, in particular, the Lindelöf inequality in the half-plane

$$|f'(z)| \leq \frac{1}{\Im z} \sup_{\zeta \in \mathbb{C}_+} \{\Re f(\zeta) - \Re f(z)\} \tag{1.1}$$

and two equivalent inequalities

$$|f'(z)| \leq \frac{2}{\pi \Im z} \sup_{\zeta \in \mathbb{C}_+} |\Re f(\zeta)| \tag{1.2}$$

and

$$|f'(z)| \leq \frac{1}{\pi \Im z} \ \operatorname{osc}_{\mathbb{C}_+}(\Re f), \tag{1.3}$$

where $\operatorname{osc}_{\mathbb{C}_+}(\Re f)$ is the oscillation of $\Re f$ on $\mathbb{C}_+$, and z is an arbitrary point in $\mathbb{C}_+$.

Inequalities for analytic functions with certain characteristics of its real part as majorants, are called *real-part theorems* in reference to the first assertion of such a kind, the celebrated Hadamard real-part theorem

$$|f(z)| \leq \frac{C|z|}{1 - |z|} \max_{|\zeta|=1} \Re f(\zeta).$$

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Here $|z| < 1$ and f is an analytic function on the closure $\overline{\mathbb{D}}$ of the unit disk $\mathbb{D} = \{z : |z| < 1\}$ vanishing at $z = 0$. This inequality was first obtained by Hadamard with $C = 4$ in 1892 [4]. The following refinement of Hadamard real-part theorem due to Borel [1, 2], Carathéodory [8, 9] and Lindelöf [10]

$$|f(z) - f(0)| \leq \frac{2|z|}{1-|z|} \sup_{|\zeta|<1} \Re \{f(\zeta) - f(0)\}, \tag{1.4}$$

and corollaries of the last sharp estimate are often called the Borel-Carathéodory inequalities. Sometimes, (1.4) is called Hadamard-Borel-Carathéodory inequality (see, e.g. Burckel [3]). The collection of real-part theorems and related assertions is rather broad. It involves assertions of various form (see, e.g. [6] and the bibliography collected there).

Obviously, the inequalities for the first derivative of an analytic function (1.1)-(1.3) can be restated as estimates for the gradient of a harmonic function. For example, inequality (1.2) can be written in the form

$$|\nabla u(z)| \leq \frac{2}{\pi y} \sup_{\zeta \in \mathbb{R}^2_+} |u(\zeta)|, \tag{1.5}$$

where u is a harmonic function in the half-plane $\mathbb{R}^2_+ = \{z = (x, y) \in \mathbb{R}^2 : y > 0\}$.

In the present work we find a representation for the sharp coefficient $\mathcal{C}_p(x)$ in the inequality

$$|\nabla u(x)| \leq \mathcal{C}_p(x) \big\| u \big\|_p, \tag{1.6}$$

where u is harmonic function in the half-space $\mathbb{R}^n_+ = \big\{x = (x', x_n) : x' \in \mathbb{R}^{n-1}, x_n > 0\big\}$, represented by the Poisson integral with boundary values in $L_p(\mathbb{R}^{n-1})$, $|| \cdot ||_p$ is the norm in $L_p(\mathbb{R}^{n-1})$, $1 \leq p \leq \infty$, $x \in \mathbb{R}^n_+$. It is shown that

$$\mathcal{C}_p(x) = C_p \; x_n^{(1-n-p)/p}$$

and explicit formulas for C_p in (1.6) for $p = 1, 2, \infty$ are given.

Note that a direct consequence of (1.6) is the following sharp limit relation for the gradient of a harmonic function in the n-dimensional domain Ω with smooth boundary:

$$\lim_{x \to \mathcal{O}_x} \big| x - \mathcal{O}_x \big|^{(n+p-1)/p} \; \sup \big\{ |\nabla u(x)| : ||u|_{\partial\Omega}||_p \leq 1 \big\} = C_p,$$

where $\mathcal{O}_x$ is a point at $\partial\Omega$ nearest to $x \in \Omega$ (compare with Theorem 2 in [7], where a relation of the same nature for the values of solutions to elliptic systems was obtained).

In Section 2 we characterize C_p in terms of an extremal problem on the unit hemisphere in $\mathbb{R}^n$. In Section 3 we reduce this problem to that of finding of the supremum of a certain double integral, depending on a scalar parameter and show that

$$C_1 = \frac{2(n-1)}{\omega_n} = \frac{(n-1)\Gamma\left(n/2\right)}{\pi^{n/2}}, \qquad C_2 = \sqrt{\frac{n(n-1)}{2^n \omega_n}} = \sqrt{\frac{(n-1)\Gamma\left(\frac{n+2}{2}\right)}{2^n \pi^{n/2}}},$$

where ω_n is the area of the unit sphere in $\mathbb{R}^n$.

In Section 5 we treat the more difficult case of $p = \infty$. We anticipate the proof of the main result by deriving in Section 4 an algebraic inequality to be used for finding an explicit formula for C_∞. Solving the variational problem stated in Section 3, we find

$$C_\infty = \frac{4(n-1)^{(n-1)/2}\,\omega_{n-1}}{n^{n/2}\,\omega_n} = \frac{4(n-1)^{(n-1)/2}\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}n^{n/2}\Gamma\left(\frac{n-1}{2}\right)}\,.$$

In particular,

$$C_\infty = \frac{4}{3\sqrt{3}}\,, \qquad C_\infty = \frac{3\sqrt{3}}{2\pi}\,.$$

for $n = 3$ and $n = 4$, respectively.

As a trivial corollary of the inequality

$$|\nabla u(x)| \le \frac{4(n-1)^{(n-1)/2}\,\omega_{n-1}}{n^{n/2}\,\omega_n x_n} \sup_{y\in\mathbb{R}^n_+} |u(y)|\,, \tag{1.7}$$

which is equivalent to (1.6) with $p = \infty$, we find

$$|\nabla u(x)| \le \frac{2(n-1)^{(n-1)/2}\,\omega_{n-1}}{n^{n/2}\,\omega_n x_n} \operatorname{osc}_{\mathbb{R}^n_+}(u), \tag{1.8}$$

where $\operatorname{osc}_{\mathbb{R}^n_+}(u)$ is the oscillation of u on $\mathbb{R}^n_+$. The sharp inequalities (1.7) and (1.8) are multidimensional generalizations of analogues of the real-part theorems (1.2) and (1.3), respectively.

The sharp constant in the inequality

$$\left|\frac{\partial u}{\partial |x|}\Big|_{x_0}\right| \le K(x_0) \sup_{|y|<1} |u(y)|,$$

where u is a harmonic function in the three-dimensional unit ball B and $x_0 \in B$, was found by Khavinson [5], who suggested, in a private conversation, that the same constant $K(x_0)$ should appear in the stronger inequality

$$|\nabla u(x_0)| \le K(x_0) \sup_{|y|<1} |u(y)|.$$

When dealing in Section 4 with the analogue of Khavinson's problem for the multidimensional half-space, we show that in fact, the sharp constants in pointwise estimates for the absolute value of the normal derivative and of the modulus of the gradient of a harmonic function coincide. We also show that similar assertions hold for $p = 1$ and $p = 2$.

2. **Auxilliary assertion.** We introduce some notation used henceforth. Let $\mathbb{R}^n_+ = \big\{x = (x', x_n) : x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}, x_n > 0\big\}$, $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, $\mathbb{S}^{n-1}_+ = \{x \in \mathbb{R}^n : |x| = 1,\ x_n > 0\}$ and $\mathbb{S}^{n-1}_- = \{x \in \mathbb{R}^n : |x| = 1,\ x_n < 0\}$. Let $\boldsymbol{e}_\sigma$ stand for the n-dimensional unit vector joining the origin to a point σ on the sphere $\mathbb{S}^{n-1}$.

By $||\cdot||_p$ we denote the norm in the space $L^p(\mathbb{R}^{n-1})$, that is

$$||f||_p = \left\{\int_{\mathbb{R}^{n-1}} |f(x')|^p\,dx'\right\}^{1/p},$$

if $1 \le p < \infty$, and $||f||_\infty = \text{ess sup}\{|f(x')| : x' \in \mathbb{R}^{n-1}\}$.

Next, by $h^p(\mathbb{R}^n_+)$ we denote the Hardy space of harmonic functions on $\mathbb{R}^n_+$, which can be represented as the Poisson integral

$$u(x) = \frac{2}{\omega_n}\int_{\mathbb{R}^{n-1}} \frac{x_n}{|y-x|^n}\,u(y')dy' \tag{2.1}$$

with boundary values in $L^p(\mathbb{R}^{n-1})$, $1 \le p \le \infty$, where $y = (y', 0)$, $y' \in \mathbb{R}^{n-1}$.

Now, we find a representation for the best coefficient $\mathcal{C}_p(x; z)$ in the inequality for the absolute value of derivative of u at $x \in \mathbb{R}^n_+$ in the arbitrary direction $\boldsymbol{z} \in \mathbb{S}^{n-1}$ assuming that $L_p(\mathbb{R}^{n-1})$. In particular, we obtain a formula for the constant in a similar inequality for the modulus of the gradient.

Lemma 1. *Let $u \in h^p(\mathbb{R}^n_+)$, and let x be an arbitrary point in $\mathbb{R}^n_+$. The sharp coefficient $\mathcal{C}_p(x; \boldsymbol{z})$ in the inequality*

$$|\,(\nabla u(x), \boldsymbol{z})\,| \le \mathcal{C}_p(x; \boldsymbol{z}) \big\| u \big\|_p$$

is given by

$$\mathcal{C}_p(x; \boldsymbol{z}) = C_p(\boldsymbol{z}) x_n^{(1-n-p)/p}, \tag{2.2}$$

where

$$C_1(\boldsymbol{z}) = \frac{2}{\omega_n} \sup_{\sigma \in \mathbb{S}^{n-1}_+} \big|\big(\boldsymbol{e}_n - n(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma,\ \boldsymbol{z}\big)\big| \big(\boldsymbol{e}_\sigma, \boldsymbol{e}_n\big)^n, \tag{2.3}$$

$$C_p(\boldsymbol{z}) = \frac{2}{\omega_n} \left\{ \int_{\mathbb{S}^{n-1}_+} \big|\big(\boldsymbol{e}_n - n(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma, \boldsymbol{z}\big)\big|^{p/(p-1)} \big(\boldsymbol{e}_\sigma, \boldsymbol{e}_n\big)^{n/(p-1)} d\sigma \right\}^{(p-1)/p} \tag{2.4}$$

for $1 < p < \infty$, and

$$C_\infty(\boldsymbol{z}) = \frac{2}{\omega_n} \int_{\mathbb{S}^{n-1}_+} \big|\big(\boldsymbol{e}_n - n(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma,\ \boldsymbol{z}\big)\big|\, d\sigma. \tag{2.5}$$

In particular, the sharp coefficient $\mathcal{C}_p(x)$ in the inequality

$$|\nabla u(x)| \le \mathcal{C}_p(x) \big\| u \big\|_p$$

is given by

$$\mathcal{C}_p(x) = C_p\, x_n^{(1-n-p)/p}, \tag{2.6}$$

where

$$C_p = \sup_{|\boldsymbol{z}|=1} C_p(\boldsymbol{z}). \tag{2.7}$$

Proof. Let $x = (x', x_n)$ be a fixed point in $\mathbb{R}^n_+$. The representation (2.1) implies

$$\frac{\partial u}{\partial x_i} = \frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} \left[\frac{\delta_{ni}}{|y - x|^n} + \frac{n x_n (y_i - x_i)}{|y - x|^{n+2}} \right] u(y') dy',$$

that is

$$\begin{aligned} \nabla u(x) &= \frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} \left[\frac{\boldsymbol{e}_n}{|y - x|^n} + \frac{n x_n (y - x)}{|y - x|^{n+2}} \right] u(y') dy' \\ &= \frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{\boldsymbol{e}_n - n(\boldsymbol{e}_{xy}, \boldsymbol{e}_n)\boldsymbol{e}_{xy}}{|y - x|^n}\, u(y') dy', \end{aligned}$$

where $\boldsymbol{e}_{xy} = (y - x)|y - x|^{-1}$. For any $\boldsymbol{z} \in \mathbb{S}^{n-1}$,

$$(\nabla u(x), \boldsymbol{z}) = \frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{(\boldsymbol{e}_n - n(\boldsymbol{e}_{xy}, \boldsymbol{e}_n)\boldsymbol{e}_{xy},\ \boldsymbol{z})}{|y - x|^n}\, u(y') dy'. \tag{2.8}$$

Hence,

$$\mathcal{C}_1(x; \boldsymbol{z}) = \frac{2}{\omega_n} \sup_{y \in \mathbb{R}^{n-1}} \frac{|(\boldsymbol{e}_n - n(\boldsymbol{e}_{xy}, \boldsymbol{e}_n)\boldsymbol{e}_{xy},\ \boldsymbol{z})|}{|y - x|^n}, \tag{2.9}$$

and

$$\mathcal{C}_p(x;\boldsymbol{z})=\frac{2}{\omega_n}\left\{\int_{\mathbb{R}^{n-1}}\frac{\big|(\boldsymbol{e}_n-n(\boldsymbol{e}_{xy},\boldsymbol{e}_n)\boldsymbol{e}_{xy},\boldsymbol{z})\big|^q}{|y-x|^{nq}}\,dy'\right\}^{1/q} \tag{2.10}$$

for $1<p\le\infty$, where $p^{-1}+q^{-1}=1$.

Taking into account the equality

$$\frac{x_n}{|y-x|}=(\boldsymbol{e}_{xy},-\boldsymbol{e}_n), \tag{2.11}$$

by (2.9) we obtain

$$\begin{aligned}\mathcal{C}_1(x;\boldsymbol{z}) &= \frac{2}{\omega_n}\sup_{y\in\mathbb{R}^{n-1}}\frac{|(\boldsymbol{e}_n-n(\boldsymbol{e}_{xy},\boldsymbol{e}_n)\boldsymbol{e}_{xy},\ \boldsymbol{z})|}{x_n^n}\left(\frac{x_n}{|y-x|}\right)^n\\ &= \frac{2}{\omega_n x_n^n}\sup_{\sigma\in\mathbb{S}^{n-1}_-}\big|(\boldsymbol{e}_n-n(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\boldsymbol{e}_\sigma,\ \boldsymbol{z})\big|(\boldsymbol{e}_\sigma,-\boldsymbol{e}_n)^n.\end{aligned}$$

Replacing here $\boldsymbol{e}_\sigma$ by $-\boldsymbol{e}_\sigma$, we arrive at (2.2) for $p=1$ with the sharp constant (2.3).

Let $1<p\le\infty$. Using (2.11) and the equality

$$\frac{1}{|y-x|^{nq}}=\frac{1}{x_n^{nq-n+1}}\left(\frac{x_n}{|y-x|}\right)^{n(q-1)}\frac{x_n}{|y-x|^n}\,,$$

and replacing q by $p/(p-1)$ in (2.10), we conclude that (2.2) holds with the sharp constant

$$C_p(\boldsymbol{z})=\frac{2}{\omega_n}\left\{\int_{\mathbb{S}^{n-1}_-}\big|(\boldsymbol{e}_n-n(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\boldsymbol{e}_\sigma,\ \boldsymbol{z})\big|^{p/(p-1)}(\boldsymbol{e}_\sigma,-\boldsymbol{e}_n)^{n/(p-1)}\,d\sigma\right\}^{(p-1)/p},$$

where $\mathbb{S}^{n-1}_-=\{\sigma\in\mathbb{S}^{n-1}:(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)<0\}$. Replacing here $\boldsymbol{e}_\sigma$ by $-\boldsymbol{e}_\sigma$, we arrive at (2.4) for $1<p<\infty$ and at (2.5) for $p=\infty$.

By (2.8) we have

$$\big|\nabla u(x)\big|=\frac{2}{\omega_n}\sup_{|\boldsymbol{z}|=1}\int_{\mathbb{R}^{n-1}}\frac{(\boldsymbol{e}_n-n(\boldsymbol{e}_{xy},\boldsymbol{e}_n)\boldsymbol{e}_{xy},\ \boldsymbol{z})}{|y-x|^n}\,u(y')dy'.$$

Hence, by the permutation of suprema, (2.10), (2.9) and (2.2),

$$\begin{aligned}\mathcal{C}_p(x) &= \frac{2}{\omega_n}\sup_{|\boldsymbol{z}|=1}\left\{\int_{\mathbb{R}^{n-1}}\frac{\big|(\boldsymbol{e}_n-n(\boldsymbol{e}_{xy},\boldsymbol{e}_n)\boldsymbol{e}_{xy},\boldsymbol{z})\big|^q}{|y-x|^{nq}}\,dy'\right\}^{1/q}\\ &= \sup_{|\boldsymbol{z}|=1}\mathcal{C}_p(x;\boldsymbol{z})=\sup_{|\boldsymbol{z}|=1}C_p(\boldsymbol{z})x_n^{(1-n-p)/p}\end{aligned} \tag{2.12}$$

for $1<p\le\infty$, and

$$\begin{aligned}\mathcal{C}_1(x) &= \frac{2}{\omega_n}\sup_{|\boldsymbol{z}|=1}\sup_{y\in\mathbb{R}^{n-1}}\frac{|(\boldsymbol{e}_n-n(\boldsymbol{e}_{xy},\boldsymbol{e}_n)\boldsymbol{e}_{xy},\ \boldsymbol{z})|}{|y-x|^n}\\ &= \sup_{|\boldsymbol{z}|=1}\mathcal{C}_1(x;\boldsymbol{z})=\sup_{|\boldsymbol{z}|=1}C_1(\boldsymbol{z})x_n^{-n}.\end{aligned} \tag{2.13}$$

Using the notation (2.7) in (2.12) and (2.13), we arrive at (2.6). □

Remark 1. Formula (2.4) for the coefficient $C_p(\boldsymbol{z}), 1 < p < \infty$, can be written with the integral over the whole sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$,

$$C_p(\boldsymbol{z}) = \frac{2^{1/p}}{\omega_n} \left\{ \int_{\mathbb{S}^{n-1}} \left| \left(\boldsymbol{e}_n - n(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma,\ \boldsymbol{z} \right) \right|^{p/(p-1)} \left| (\boldsymbol{e}_\sigma, \boldsymbol{e}_n) \right|^{n/(p-1)} d\sigma \right\}^{(p-1)/p}.$$

A similar remark relates (2.5) as well as formula (2.3):

$$C_1(\boldsymbol{z}) = \frac{2}{\omega_n} \sup_{\sigma \in \mathbb{S}^{n-1}} \left| \left(\boldsymbol{e}_n - n(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma,\ \boldsymbol{z} \right) \left(\boldsymbol{e}_\sigma, \boldsymbol{e}_n \right)^n \right| .$$

3. **The case** $1 \le p < \infty$**.** The next assertion is based on the representation for C_p, obtained in Lemma 1.

Proposition 1. *Let $u \in h^p(\mathbb{R}^n_+)$, and let x be an arbitrary point in $\mathbb{R}^n_+$. The sharp coefficient $\mathcal{C}_p(x)$ in the inequality*

$$|\nabla u(x)| \le \mathcal{C}_p(x) \big\| u \big\|_p \tag{3.1}$$

is given by

$$\mathcal{C}_p(x) = C_p x_n^{(1-n-p)/p}\ , \tag{3.2}$$

where

$$C_1 = \frac{2(n-1)}{\omega_n}, \tag{3.3}$$

and

$$C_p = \frac{2(\omega_{n-2})^{(p-1)/p}}{\omega_n} \sup_{\gamma \ge 0} \frac{1}{\sqrt{1+\gamma^2}} \left\{ \int_0^\pi d\varphi \int_0^{\pi/2} \mathcal{F}_{n,p}(\varphi, \vartheta; \gamma)\, d\vartheta \right\}^{(p-1)/p}, \tag{3.4}$$

if $1 < p < \infty$. Here

$$\mathcal{F}_{n,p}(\varphi, \vartheta; \gamma) = \left| \mathcal{G}_n(\varphi, \vartheta; \gamma) \right|^{p/(p-1)} \cos^{n/(p-1)} \vartheta \sin^{n-2} \vartheta \sin^{n-3} \varphi \tag{3.5}$$

with

$$\mathcal{G}_n(\varphi, \vartheta; \gamma) = (n \cos^2 \vartheta - 1) + n\gamma \cos \vartheta \sin \vartheta \cos \varphi\ . \tag{3.6}$$

In particular,

$$C_2 = \sqrt{\frac{n(n-1)}{2^n \omega_n}}\ . \tag{3.7}$$

For $p = 1$ and $p = 2$ the coefficient $\mathcal{C}_p(x)$ is optimal also in the weaker inequality obtained from (3.1) by replacing ∇u by $\partial u / \partial x_n$.

Proof. The equality (3.2) was proved in Lemma 1.

(i) Let $p = 1$. Using (2.3), (2.7) and the permutability of two suprema, we find

$$\begin{aligned} C_1 &= \frac{2}{\omega_n} \sup_{|\boldsymbol{z}|=1} \sup_{\sigma \in \mathbb{S}^{n-1}_+} \left| \left(\boldsymbol{e}_n - n(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma,\ \boldsymbol{z} \right) \right| \left(\boldsymbol{e}_\sigma, \boldsymbol{e}_n \right)^n \\ &= \frac{2}{\omega_n} \sup_{\sigma \in \mathbb{S}^{n-1}_+} \left| \boldsymbol{e}_n - n(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma \right| \left(\boldsymbol{e}_\sigma, \boldsymbol{e}_n \right)^n . \end{aligned} \tag{3.8}$$

Taking into account the equality

$$\begin{aligned} |\boldsymbol{e}_n - n(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma| &= \Big(\boldsymbol{e}_n - n(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma,\ \boldsymbol{e}_n - n(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma \Big)^{1/2} \\ &= \Big(1 + (n^2 - 2n)(\boldsymbol{e}_\sigma,\ \boldsymbol{e}_n)^2 \Big)^{1/2}, \end{aligned}$$

and using (3.8), we arrive at the sharp constant (3.3).

Furthermore, by (2.3),

$$C_1(\boldsymbol{e}_n) = \frac{2}{\omega_n} \sup_{\sigma \in \mathbb{S}^{n-1}_+} \left|1 - n(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)^2\right| \big(\boldsymbol{e}_\sigma, \boldsymbol{e}_n\big)^n \geq \frac{2(n-1)}{\omega_n}.$$

Hence, by $C_1 \geq C_1(\boldsymbol{e}_n)$ and by (3.3) we obtain $C_1 = C_1(\boldsymbol{e}_n)$, which completes the proof in the case $p = 1$.

(ii) Let $1 < p < \infty$. Since the integrand in (2.4) does not change when $\boldsymbol{z} \in \mathbb{S}^{n-1}$ is replaced by $-\boldsymbol{z}$, we may assume that $z_n = (\boldsymbol{e}_n, \boldsymbol{z}) > 0$ in (2.7).

Let $\boldsymbol{z}' = \boldsymbol{z} - z_n \boldsymbol{e}_n$. Then $(\boldsymbol{z}', \boldsymbol{e}_n) = 0$ and hence $z_n^2 + |\boldsymbol{z}'|^2 = 1$. Analogously, with $\sigma = (\sigma_1, \dots, \sigma_{n-1}, \sigma_n) \in \mathbb{S}^{n-1}_+$, we associate the vector $\boldsymbol{\sigma}' = \boldsymbol{e}_\sigma - \sigma_n \boldsymbol{e}_n$.

Using the equalities $(\boldsymbol{\sigma}', \boldsymbol{e}_n) = 0$, $\sigma_n = \sqrt{1 - |\boldsymbol{\sigma}'|^2}$ and $(\boldsymbol{z}', \boldsymbol{e}_n) = 0$, we find an expression for $\big(\boldsymbol{e}_n - n(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma,\ \boldsymbol{z}\big)$ as a function of $\boldsymbol{\sigma}'$:

$$\begin{aligned}
\big(\boldsymbol{e}_n - n(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma,\ \boldsymbol{z}\big) &= z_n - n\sigma_n \big(\boldsymbol{e}_\sigma, \boldsymbol{z}\big) = z_n - n\sigma_n \big(\boldsymbol{\sigma}' + \sigma_n \boldsymbol{e}_n,\ \boldsymbol{z}' + z_n \boldsymbol{e}_n\big) \\
&= z_n - n\sigma_n \big[\big(\boldsymbol{\sigma}', \boldsymbol{z}'\big) + z_n \sigma_n\big] \\
&= -\big[n(1 - |\boldsymbol{\sigma}'|^2) - 1\big] z_n - n\sqrt{1 - |\boldsymbol{\sigma}'|^2}\, \big(\boldsymbol{\sigma}', \boldsymbol{z}'\big). \qquad (3.9)
\end{aligned}$$

Let $\mathbb{B}^{n-1} = \{x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} : |x'| < 1\}$. By (2.4) and (3.9), taking into account that $d\sigma = d\sigma' / \sqrt{1 - |\boldsymbol{\sigma}'|^2}$, we may write (2.7) as

$$\begin{aligned}
C_p &= \frac{2}{\omega_n} \sup_{\boldsymbol{z} \in \mathbb{S}^{n-1}_+} \left\{ \int_{\mathbb{B}^{n-1}} \frac{\mathcal{H}_{n,p}\big(|\boldsymbol{\sigma}'|, (\boldsymbol{\sigma}', \boldsymbol{z}')\big) \big(1 - |\boldsymbol{\sigma}'|^2\big)^{n/2(p-1)}}{\sqrt{1 - |\boldsymbol{\sigma}'|^2}}\, d\sigma' \right\}^{(p-1)/p} \\
&= \frac{2}{\omega_n} \sup_{\boldsymbol{z} \in \mathbb{S}^{n-1}_+} \left\{ \int_{\mathbb{B}^{n-1}} \mathcal{H}_{n,p}\big(|\boldsymbol{\sigma}'|, (\boldsymbol{\sigma}', \boldsymbol{z}')\big) \big(1 - |\boldsymbol{\sigma}'|^2\big)^{(n+1-p)/2(p-1)} d\sigma' \right\}^{(p-1)/p}, \qquad (3.10)
\end{aligned}$$

where

$$\mathcal{H}_{n,p}\big(|\boldsymbol{\sigma}'|, (\boldsymbol{\sigma}', \boldsymbol{z}')\big) = \Big| \big[n(1 - |\boldsymbol{\sigma}'|^2) - 1\big] z_n + n\sqrt{1 - |\boldsymbol{\sigma}'|^2}\, \big(\boldsymbol{\sigma}', \boldsymbol{z}'\big) \Big|^{p/(p-1)}. \qquad (3.11)$$

Let $B^n = \{x \in \mathbb{R}^n : |x| < 1\}$. Using the well known formula (see e.g. [11], **3.3.2(3)**),

$$\int_{B^n} g\big(|\boldsymbol{x}|, (\boldsymbol{a}, \boldsymbol{x})\big) dx = \omega_{n-1} \int_0^1 r^{n-1} dr \int_0^\pi g\big(r, |\boldsymbol{a}| r \cos\varphi\big) \sin^{n-2}\varphi \, d\varphi\, ,$$

we obtain

$$\begin{aligned}
&\int_{\mathbb{B}^{n-1}} \mathcal{H}_{n,p}\big(|\boldsymbol{\sigma}'|, (\boldsymbol{\sigma}', \boldsymbol{z}')\big) \big(1 - |\boldsymbol{\sigma}'|^2\big)^{(n+1-p)/2(p-1)}\, d\sigma' \\
&= \omega_{n-2} \int_0^1 r^{n-2} \big(1 - r^2\big)^{(n+1-p)/2(p-1)} dr \int_0^\pi \mathcal{H}_{n,p}\big(r, r|\boldsymbol{z}'| \cos\varphi, z_n\big) \sin^{n-3}\varphi d\varphi\, . \qquad (3.12)
\end{aligned}$$

Making the change of variable $r = \sin\vartheta$ in (3.12), we find

$$\begin{aligned}
&\int_{\mathbb{B}^{n-1}} \mathcal{H}_{n,p}\big(|\boldsymbol{\sigma}'|, (\boldsymbol{\sigma}', \boldsymbol{z}')\big) \big(1 - |\boldsymbol{\sigma}'|^2\big)^{(n+1-p)/2(p-1)}\, d\sigma' \qquad (3.13) \\
&= \omega_{n-2} \int_0^\pi \sin^{n-3}\varphi d\varphi \int_0^{\pi/2} \mathcal{H}_{n,p}\big(\sin\vartheta,\ |\boldsymbol{z}'| \sin\vartheta \cos\varphi\big) \sin^{n-2}\vartheta \cos^{n/(p-1)}\vartheta d\vartheta\, ,
\end{aligned}$$

where, by (3.11),

$$\mathcal{H}_{n,p}\big(\sin\vartheta,\ |\boldsymbol{z}'|\sin\vartheta\cos\varphi\big) = \Big|(n\cos^2\vartheta-1)z_n + n|\boldsymbol{z}'|\cos\vartheta\sin\vartheta\cos\varphi\Big|^{p/(p-1)}.$$

Introducing here the parameter $\gamma = |\boldsymbol{z}'|/z_n$ and using the equality $|\boldsymbol{z}'|^2 + z_n^2 = 1$, we obtain

$$\mathcal{H}_{n,p}\big(\sin\vartheta,\ |\boldsymbol{z}'|\sin\vartheta\cos\varphi\big) = (1+\gamma^2)^{-p/2(p-1)}\big|\mathcal{G}_n(\varphi,\vartheta;\gamma)\big|^{p/(p-1)}, \tag{3.14}$$

where $\mathcal{G}_n(\varphi,\vartheta;\gamma)$ is given by (3.6).

By (3.10), taking into account (3.13) and (3.14), we arrive at (3.4).

(iii) Let $p=2$. By (3.4), (3.5) and (3.6),

$$C_2 = \frac{2\sqrt{\omega_{n-2}}}{\omega_n}\sup_{\gamma\geq 0}\frac{1}{\sqrt{1+\gamma^2}}\left\{\int_0^\pi d\varphi\int_0^{\pi/2}\mathcal{F}_{n,2}(\varphi,\vartheta;\gamma)\,d\vartheta\right\}^{1/2}, \tag{3.15}$$

where

$$\mathcal{F}_{n,2}(\varphi,\vartheta;\gamma) = \big[(n\cos^2\vartheta-1)+n\gamma\cos\vartheta\sin\vartheta\cos\varphi\big]^2\cos^n\vartheta\sin^{n-2}\vartheta\sin^{n-3}\varphi. \tag{3.16}$$

The equalities (3.15) and (3.16) imply

$$C_2 = \frac{2\sqrt{\omega_{n-2}}}{\omega_n}\sup_{\gamma\geq 0}\frac{1}{\sqrt{1+\gamma^2}}\left\{\mathcal{I}_1+\gamma^2\mathcal{I}_2\right\}^{1/2}, \tag{3.17}$$

where

$$\begin{aligned}\mathcal{I}_1 &= \int_0^\pi \sin^{n-3}\varphi\,d\varphi\int_0^{\pi/2}(n\cos^2\vartheta-1)^2\sin^{n-2}\vartheta\cos^n\vartheta\,d\vartheta\\ &= \frac{\sqrt{\pi}\,n(n-1)\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{8(n-1)!},\end{aligned} \tag{3.18}$$

$$\begin{aligned}\mathcal{I}_2 &= n^2\int_0^\pi \sin^{n-3}\varphi\cos^2\varphi\,d\varphi\int_0^{\pi/2}\sin^n\vartheta\cos^{n+2}\vartheta\,d\vartheta\\ &= \frac{\sqrt{\pi}\,n\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{8(n-1)!}.\end{aligned} \tag{3.19}$$

By (3.17) we have

$$C_2 = \frac{2\sqrt{\omega_{n-2}}}{\omega_n}\max\big\{\mathcal{I}_1^{1/2},\mathcal{I}_2^{1/2}\big\},$$

which together with (3.18) and (3.19) gives

$$C_2 = \frac{2\sqrt{\omega_{n-2}}}{\omega_n}\,\mathcal{I}_1^{1/2} = \sqrt{\frac{(n-1)\Gamma\left(\frac{n+2}{2}\right)}{2^n\pi^{n/2}}}.$$

Hence (3.7) follows.

Since $\boldsymbol{z}\in\mathbb{S}^{n-1}$ and the supremum in $\gamma = |\boldsymbol{z}'|/z_n$ in (3.15) is attained for $\gamma=0$, we have $C_2 = C_2(\boldsymbol{e}_n)$. □

4. **An auxilliary algebraic inequality.** Here we prove an algebraic inequality to be used later for deriving an explicit formula for the sharp constant in the estimate for the modulus of gradient in the case $p=\infty$.

Lemma 2. *For all $x \geq 0$ and any $\mu \geq 1$ the inequality holds*

$$\left(\frac{\mu+1}{\mu+x}\right)^{\mu-1}+\left(\frac{\mu+1}{1+\mu x}\right)^{\mu-1} x^{\mu+1} \leq 2x+\frac{\mu(3\mu+1)}{(\mu+1)^2}(1-x)^2. \tag{4.1}$$

The equality sign takes place only for $\mu=1$ or $x=1$.

Proof. Clearly, the inequality (4.1) becomes equality for $\mu=1$ or $x=1$. Suppose that $\mu \in (1,\infty)$.

(i) *The case* $0 \leq x < 1$. Let us write (4.1) in the form

$$\left(\frac{\mu+1}{\mu+x}\right)^{\mu-1}+\left(\frac{\mu x+x}{1+\mu x}\right)^{\mu-1} x^{2} \leq 2x+\frac{\mu(3\mu+1)}{(\mu+1)^2}(1-x)^2.$$

Introducing the notation

$$F(x)=\left(\frac{\mu+1}{\mu+x}\right)^{\mu-1}+\left(\frac{\mu x+x}{1+\mu x}\right)^{\mu-1} x^{2}, \tag{4.2}$$

we find

$$F'(x)=-\frac{\mu-1}{\mu+1}\left(\frac{\mu+1}{\mu+x}\right)^{\mu}+x\left(2+\frac{\mu-1}{1+\mu x}\right)\left(\frac{\mu x+x}{1+\mu x}\right)^{\mu-1},$$

$$F''(x)=\frac{\mu(\mu-1)}{(\mu+1)^2}\left(\frac{\mu+1}{\mu+x}\right)^{\mu+1}+\mu\left[\frac{2(x+1)}{1+\mu x}+\frac{\mu-1}{(1+\mu x)^2}\right]\left(\frac{\mu x+x}{1+\mu x}\right)^{\mu-1}, \tag{4.3}$$

$$F'''(x)=-\frac{\mu(\mu-1)}{(\mu+1)^2}\left(\frac{\mu+1}{\mu+x}\right)^{\mu+2}+\frac{\mu(\mu^2-1)}{x(1+\mu x)^3}\left(\frac{\mu x+x}{1+\mu x}\right)^{\mu-1}. \tag{4.4}$$

By Taylor's formula with Lagrange's remainder term,

$$F(x)=F(1)+F'(1)(x-1)+\frac{1}{2}F''(t)\,(x-1)^2=2+2(x-1)+\frac{1}{2}F''(t)\,(x-1)^2, \tag{4.5}$$

where $x\in[0,1)$ and $t\in(x,1)$.

Note that $F''(0)<F''(1)$. In fact, by (4.3),

$$F''(0)=\frac{\mu-1}{\mu+1}\left(1+\frac{1}{\mu}\right)^{\mu}, \qquad F''(1)=\frac{2\mu(3\mu+1)}{(\mu+1)^2},$$

which together with the obvious inequality

$$\frac{\mu-1}{\mu+1}\,e<\frac{2\mu(3\mu+1)}{(\mu+1)^2}$$

implies $F''(0)<F''(1)$.

Next we show that

$$F''(t)<\max\left\{F''(0),\ F''(1)\right\}=\frac{2\mu(3\mu+1)}{(\mu+1)^2} \tag{4.6}$$

for any $t\in(0,1)$.

Suppose the opposite assertion holds, i.e. there exists a point $t\in(0,1)$ at which (4.6) fails. Hence $F''(t)$ attains its maximum value on $[0,1]$ at an inner point τ, i.e.,

$$F''(\tau)=\max_{t\in[0;1]}F''(t)\geq\max\left\{F''(0),\ F''(1)\right\}=\frac{2\mu(3\mu+1)}{(\mu+1)^2}. \tag{4.7}$$

Taking into account (4.4), we have

$$F'''(\tau) = -\frac{\mu(\mu-1)}{(\mu+1)^2}\left(\frac{\mu+1}{\mu+\tau}\right)^{\mu+2} + \frac{\mu(\mu^2-1)}{\tau(1+\mu\tau)^3}\left(\frac{\mu\tau+\tau}{1+\mu\tau}\right)^{\mu-1} = 0,$$

which is equivalent to

$$\left(\frac{\mu\tau+\tau}{1+\mu\tau}\right)^{\mu-1} = \frac{\tau(1+\mu\tau)^3}{(\mu+1)^3}\left(\frac{\mu+1}{\mu+\tau}\right)^{\mu+2}.$$

Combined with (4.3), this implies

$$\begin{aligned} F''(\tau) &= \frac{\mu(\mu-1)}{(\mu+1)^2}\left(\frac{\mu+1}{\mu+\tau}\right)^{\mu+1}\left\{1+\tau(1+\mu\tau)\frac{2(1+\tau)(1+\mu\tau)+\mu-1}{(\mu-1)(\mu+\tau)}\right\} \\ &< \frac{\mu(\mu-1)}{(\mu+1)^2}\left(\frac{\mu+1}{\mu+\tau}\right)^{\mu+1}\left\{2+\frac{2(1+\tau)(1+\mu\tau)}{\mu-1}\right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} F''(\tau) &< \frac{2\mu}{(\mu+1)^2}\left(\frac{\mu+1}{\mu+\tau}\right)^{\mu+1}\Big\{(\mu-1)+(1+\tau)(1+\mu\tau)\Big\} \\ &= \frac{2\mu}{(\mu+1)^2}\left(\frac{\mu+1}{\mu+\tau}\right)^{\mu+1}\Big\{\mu\tau^2+(\mu+1)\tau+\mu\Big\}. \end{aligned} \tag{4.8}$$

Setting

$$\eta(\tau) = \frac{\mu\tau^2+(\mu+1)\tau+\mu}{(\mu+\tau)^{\mu+1}}, \tag{4.9}$$

we rewrite (4.8) as

$$F''(\tau) < 2\mu(\mu+1)^{\mu-1}\eta(\tau). \tag{4.10}$$

Noting that

$$\eta'(\tau) = \frac{\mu\tau(\mu-1)(1-\tau)}{(\mu+\tau)^{\mu+2}} > 0$$

for $0<\tau<1$, by (4.9) and (4.10), we find

$$\max_{t\in[0;1]} F''(t) = F''(\tau) < 2\mu(\mu+1)^{\mu-1}\eta(1) = \frac{2\mu(3\mu+1)}{(\mu+1)^2}.$$

The latter contradicts (4.7) which proves (4.6) for all $t \in (0,1)$. Thus, it follows from (4.2), (4.5) and (4.6) that

$$\left(\frac{\mu+1}{\mu+x}\right)^{\mu-1} + \left(\frac{\mu x+x}{1+\mu x}\right)^{\mu-1} x^2 < 2x + \frac{\mu(3\mu+1)}{(\mu+1)^2}(x-1)^2$$

for all $0 \le x < 1$. This means that for $\mu > 1$ and $0 \le x < 1$ the strict inequality (4.1) holds.

(ii) *The case* $x > 1$. Since the function

$$G(x) = \left(\frac{\mu+1}{\mu+x}\right)^{\mu-1} + \left(\frac{\mu x+x}{1+\mu x}\right)^{\mu-1} x^2 - 2x - \frac{\mu(3\mu+1)}{(\mu+1)^2}(x-1)^2$$

satisfies the equality

$$G\left(\frac{1}{x}\right) = \frac{1}{x^2}\,G(x),$$

we have by part (i) that $G(x) < 0$ for $0 \le x < 1$ and hence $G(x) < 0$ for $x > 1$. Thus, the strict inequality (4.1) holds for $\mu > 1$ and $x > 1$. $\square$

Corollary 1. *For all $y \ge 0$ and any natural $n \ge 2$ the inequality holds*

$$\mathcal{P}_n^2(y) + \mathcal{P}_n^2(-y) \le \frac{2n^2 + 4(n-1)(3n-2)y^2}{n^n}, \tag{4.11}$$

where

$$\mathcal{P}_n(y) = \frac{\left(\sqrt{1+y^2}+y\right)^{n-1}}{\left(1+(n-1)\left(\sqrt{1+y^2}+y\right)^2\right)^{(n-2)/2}}. \tag{4.12}$$

Proof. Suppose that $0 < x \le 1$. We introduce the new variable

$$y = \frac{1-x}{2\sqrt{x}} \in [0, \infty). \tag{4.13}$$

Solving the equation $y = (x^{-1/2} - x^{1/2})/2$ in $\sqrt{x}$, we find $\sqrt{x} = \sqrt{1+y^2} - y$, i.e.,

$$x = \left(\sqrt{1+y^2}-y\right)^2 = \frac{1}{\left(\sqrt{1+y^2}+y\right)^2},$$

Putting $\mu = n-1$, we write (4.1) as

$$\frac{1}{\left(n-1+x\right)^{n-2}} + \frac{x^n}{\left(1+(n-1)x\right)^{n-2}} \le \frac{2n^2x + (n-1)(3n-2)(1-x)^2}{n^n}. \tag{4.14}$$

By (4.13) we have $(1-x)^2 = 4y^2x$, hence (4.14) can be rewritten in the form

$$\frac{1}{x\left(n-1+x\right)^{n-2}} + \frac{x^{n-1}}{\left(1+(n-1)x\right)^{n-2}} \le \frac{2n^2 + 4(n-1)(3n-2)y^2}{n^n}. \tag{4.15}$$

Setting $x = \left(\sqrt{1+y^2}+y\right)^{-2}$ in the first term on the left-hand side of (4.15), we obtain

$$\frac{1}{x\left(n-1+x\right)^{n-2}} = \frac{\left(\sqrt{1+y^2}+y\right)^{2n-2}}{\left(1+(n-1)\left(\sqrt{1+y^2}+y\right)^2\right)^{n-2}} = \mathcal{P}_n^2(y). \tag{4.16}$$

Similarly, putting $x = \left(\sqrt{1+y^2}-y\right)^2$ in the second term on the left-hand side of (4.15), we find

$$\frac{x^{n-1}}{\left(1+(n-1)x\right)^{n-2}} = \frac{\left(\sqrt{1+y^2}-y\right)^{2n-2}}{\left(1+(n-1)\left(\sqrt{1+y^2}-y\right)^2\right)^{n-2}} = \mathcal{P}_n^2(-y). \tag{4.17}$$

Using (4.16) and (4.17), we can rewrite (4.15) as (4.11). □

We give one more corollary of Lemma 2 containing an alternative form of (4.1) with natural $\mu \ge 2$. However, we are not going to use it henceforth.

Corollary 2. *For all $x \ge 0$ and any natural $n \ge 2$ the inequality holds*

$$\sum_{k=3}^{n+1} \binom{n+1}{k} \left\{ \frac{1}{\left(n+x\right)^{k-2}} + \frac{(-1)^k}{\left(1+nx\right)^{k-2}} \right\} (1-x)^k \le 0\,. \tag{4.18}$$

The equality sign takes place only for $x = 1$.

Proof. We set $\mu = n, n \geq 2$, in (4.1):

$$\left(\frac{n+1}{n+x}\right)^{n-1} + \left(\frac{n+1}{1+nx}\right)^{n-1} x^{n+1} \leq 2x + \frac{n(3n+1)}{(n+1)^2}(1-x)^2.$$

Multiplying the last inequality by $(n+1)^2$, we write it as

$$\frac{(n+1)^{n+1}}{\big(n+x\big)^{n-1}} + \frac{\big((n+1)x\big)^{n+1}}{\big(1+nx\big)^{n-1}} - 2(n+1)^2 x - n(3n+1)(1-x)^2 \leq 0. \tag{4.19}$$

We rewrite the first term in (4.19):

$$\frac{(n+1)^{n+1}}{\big(n+x\big)^{n-1}} = \frac{[(n+x)+(1-x)]^{n+1}}{\big(n+x\big)^{n-1}} = (n+x)^2 + (n+1)(n+x)(1-x)$$
$$+ \frac{n(n+1)}{2}(1-x)^2 + \sum_{k=3}^{n+1} \binom{n+1}{k} \frac{(1-x)^k}{(n+x)^{k-2}} . \tag{4.20}$$

Similarly, the second term in (4.19) can be written as

$$\frac{\big((n+1)x\big)^{n+1}}{\big(1+nx\big)^{n-1}} = \frac{[(1+nx)-(1-x)]^{n+1}}{\big(1+nx\big)^{n-1}} = \big(1+nx\big)^2 - (n+1)\big(1+nx\big)(1-x)$$
$$+ \frac{n(n+1)}{2}(1-x)^2 + \sum_{k=3}^{n+1}(-1)^k \binom{n+1}{k} \frac{(1-x)^k}{\big(1+nx\big)^{k-2}} . \tag{4.21}$$

Using (4.20) and (4.21) in (4.19), we arrive at (4.18) with the left-hand side as the sum of rational functions. □

5. **The case $p = \infty$.** The next assertion is the main theorem of this paper. It is based on the representation for the sharp constant C_p $(1 < p < \infty)$ obtained in Proposition 1. To find the explicit formula for C_∞ we solve an extremal problem with a scalar parameter entering the integrand in a double integral.

Theorem 1. *Let $u \in h^\infty(\mathbb{R}^n_+)$, and let x be an arbitrary point in $\mathbb{R}^n_+$. The sharp coefficient $\mathcal{C}_p(x)$ in the inequality*

$$|\nabla u(x)| \leq \mathcal{C}_\infty(x) \big\| u \big\|_\infty \tag{5.1}$$

is given by

$$\mathcal{C}_\infty(x) = \frac{4(n-1)^{(n-1)/2}\, \omega_{n-1}}{n^{n/2}\, \omega_n x_n} . \tag{5.2}$$

For $p = \infty$ the absolute value of the derivative of a harmonic function u with respect to the normal to the boundary of the half-space at any $x \in \mathbb{R}^n_+$ has the same supremum as $|\nabla u(x)|$.

Proof. We pass to the limit as $p \to \infty$ in (3.1), (3.2), (3.4) and (3.5). This results in

$$\mathcal{C}_\infty(x) = C_\infty\, x_n^{-1} , \tag{5.3}$$

where

$$C_\infty = \sup_{\gamma \geq 0} \frac{2\omega_{n-2}}{\omega_n \sqrt{1+\gamma^2}} \int_0^\pi \sin^{n-3}\varphi \, d\varphi \int_0^{\pi/2} \big|\mathcal{G}_n(\varphi, \vartheta; \gamma)\big| \sin^{n-2}\vartheta \, d\vartheta . \tag{5.4}$$

Here, by (3.6),

$$\mathcal{G}_n(\varphi, \vartheta; \gamma) = (n\cos^2\vartheta - 1) + n\gamma \cos\vartheta \sin\vartheta \cos\varphi.$$

We are looking for a solution of the equation

$$(n\cos^2\vartheta - 1) + n\gamma\cos\vartheta\sin\vartheta\cos\varphi = 0 \tag{5.5}$$

as a function ϑ of φ. We can rewrite (5.5) as the second order equation in $\tan\vartheta$:

$$\tan^2\vartheta - n\gamma\cos\varphi\tan\vartheta + 1 - n = 0.$$

Since $0 \leq \vartheta \leq \pi/2$, we find that the nonnegative root of this equation is

$$\vartheta_\gamma(\varphi) = \arctan\frac{h_\gamma(\varphi)}{2}\ , \tag{5.6}$$

where

$$h_\gamma(\varphi) = n\gamma\cos\varphi + \Big(4(n-1) + n^2\gamma^2\cos^2\varphi\Big)^{1/2}\ . \tag{5.7}$$

Taking into account that the function $\mathcal{G}_n(\varphi,\vartheta;\gamma)$ is nonnegative for $0 \leq \vartheta \leq \vartheta_\gamma(\varphi)$, $0 \leq \varphi \leq \pi$, and using the equalities

$$\int_0^\vartheta \mathcal{G}_n(\varphi,\vartheta;\gamma)\sin^{n-2}\vartheta d\vartheta = \big[\cos\vartheta + \gamma\cos\varphi\sin\vartheta\big]\sin^{n-1}\vartheta\ ,$$

$$\int_0^\pi \sin^{n-3}\varphi\, d\varphi \int_0^{\pi/2} \mathcal{G}_n(\varphi,\vartheta;\gamma)\sin^{n-2}\vartheta\, d\vartheta = \gamma\int_0^\pi \sin^{n-3}\varphi\cos\varphi\, d\varphi = 0,$$

we write (5.4) as

$$C_\infty = \sup_{\gamma\geq 0}\frac{4\omega_{n-2}}{\omega_n\sqrt{1+\gamma^2}}\int_0^\pi \sin^{n-3}\varphi\, d\varphi\int_0^{\vartheta_\gamma(\varphi)} \mathcal{G}_n(\varphi,\vartheta;\gamma)\sin^{n-2}\vartheta d\vartheta$$

$$= \sup_{\gamma\geq 0}\frac{4\omega_{n-2}}{\omega_n\sqrt{1+\gamma^2}}\int_0^\pi\big[\cos\vartheta_\gamma(\varphi) + \gamma\cos\varphi\sin\vartheta_\gamma(\varphi)\big]\sin^{n-1}\vartheta_\gamma(\varphi)\sin^{n-3}\varphi d\varphi. \tag{5.8}$$

By (5.6),

$$\sin\vartheta_\gamma(\varphi) = \frac{h_\gamma(\varphi)}{\sqrt{4+h^2_\gamma(\varphi)}}\ , \tag{5.9}$$

$$\cos\vartheta_\gamma(\varphi) = \frac{2}{\sqrt{4+h^2_\gamma(\varphi)}}\ . \tag{5.10}$$

By (5.9) and (5.10), we find

$$\cos\vartheta_\gamma(\varphi) + \gamma\cos\varphi\sin\vartheta_\gamma(\varphi) = \frac{2+\gamma h_\gamma(\varphi)\cos\varphi}{\sqrt{4+h^2_\gamma(\varphi)}}\ . \tag{5.11}$$

Using (5.9), (5.11), and the identity

$$2 + \gamma h_\gamma(\varphi)\cos\varphi = \frac{4+h^2_\gamma(\varphi)}{2n}\ ,$$

where $h_\gamma(\varphi)$ is defined by (5.7), we can write (5.8) as

$$C_\infty = \sup_{\gamma\geq 0}\ \frac{2\omega_{n-2}}{n\omega_n\sqrt{1+\gamma^2}}\int_0^\pi \frac{h^{n-1}_\gamma(\varphi)}{\big(4+h^2_\gamma(\varphi)\big)^{(n-2)/2}}\ \sin^{n-3}\varphi\, d\varphi.$$

Introducing the parameter

$$\alpha = \frac{n\gamma}{2\sqrt{n-1}},$$

and taking into account (5.7), we obtain

$$C_\infty = \sup_{\alpha \ge 0} \frac{4\omega_{n-2}(n-1)^{(n-1)/2}}{\omega_n \sqrt{n^2+4(n-1)\alpha^2}} \int_0^\pi \mathcal{P}_n\big(\alpha\cos\varphi\big)\sin^{n-3}\varphi\, d\varphi\ , \tag{5.12}$$

with $\mathcal{P}_n$ defined by (4.12). The change of variable $t=\cos\varphi$ in (5.12) implies

$$C_\infty = \sup_{\alpha \ge 0} \frac{4\omega_{n-2}(n-1)^{(n-1)/2}}{\omega_n \sqrt{n^2+4(n-1)\alpha^2}} \int_{-1}^1 \mathcal{P}_n(\alpha t)(1-t^2)^{(n-4)/2}\, dt. \tag{5.13}$$

Integrating in (5.13) over $(-1,0)$ and $(0,1)$, we have

$$C_\infty = \sup_{\alpha \ge 0} \frac{4\omega_{n-2}(n-1)^{(n-1)/2}}{\omega_n \sqrt{n^2+4(n-1)\alpha^2}} \int_0^1 \frac{\mathcal{P}_n(\alpha t)+\mathcal{P}_n(-\alpha t)}{(1-t^2)^{(4-n)/2}}\, dt.$$

Applying the Schwarz inequality, we see that

$$C_\infty \le \sup_{\alpha \ge 0} \frac{A_n}{\sqrt{n^2+4(n-1)\alpha^2}} \left\{ \int_0^1 \frac{\big(\mathcal{P}_n(\alpha t)+\mathcal{P}_n(-\alpha t)\big)^2}{(1-t^2)^{(4-n)/2}}\, dt \right\}^{1/2}, \tag{5.14}$$

where

$$A_n = \frac{4\omega_{n-2}(n-1)^{(n-1)/2}}{\omega_n} \left\{ \int_0^1 \frac{dt}{(1-t^2)^{(4-n)/2}} \right\}^{1/2}. \tag{5.15}$$

By (4.12),

$$\mathcal{P}_n(y)\mathcal{P}_n(-y) = \Big(4(n-1)y^2+n^2\Big)^{(2-n)/2},$$

which implies

$$\mathcal{P}_n(y)\mathcal{P}_n(-y) \le n^{2-n}. \tag{5.16}$$

Combining (5.16) and (4.11), we obtain

$$\big(\mathcal{P}_n(y)+\mathcal{P}_n(-y)\big)^2 \le\ \mathcal{P}_n^2(y)+\mathcal{P}_n^2(-y)+2n^{2-n} \le \frac{2n^2+4(n-1)(3n-2)y^2}{n^n} + \frac{2}{n^{n-2}}.$$

Therefore,

$$\big(\mathcal{P}_n(\alpha t)+\mathcal{P}_n(-\alpha t)\big)^2 \le \frac{4}{n^{n-2}}\left(1+\frac{(n-1)(3n-2)}{n^2}\ \alpha^2 t^2\right). \tag{5.17}$$

By (5.14), (5.15), (5.17) and by

$$\int_0^1 (1-t^2)^{(n-4)/2}\, dt = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)}{2\Gamma\left(\frac{n-1}{2}\right)},$$

$$\int_0^1 t^2(1-t^2)^{(n-4)/2}\, dt = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)}{2(n-1)\Gamma\left(\frac{n-1}{2}\right)},$$

we find

$$C_\infty \le \frac{4\omega_{n-2}\sqrt{\pi}\ (n-1)^{(n-1)/2}\Gamma\left(\frac{n-2}{2}\right)}{\omega_n n^{n/2}\Gamma\left(\frac{n-1}{2}\right)} \sup_{\alpha\ge 0}\left(\frac{n^2+(3n-2)\alpha^2}{n^2+4(n-1)\alpha^2}\right)^{1/2}. \tag{5.18}$$

Note that

$$\frac{d}{d\alpha}\left(\frac{n^2+(3n-2)\alpha^2}{n^2+4(n-1)\alpha^2}\right) = -\frac{2\alpha(n-2)n^2}{\big(n^2+4(n-1)\alpha^2\big)^2} < 0 \ \text{ for }\ \alpha>0,$$

therefore the supremum in α on the right-hand side of (5.18) is attained for $\alpha = 0$. Thus,

$$C_\infty \le \frac{4\omega_{n-2}\sqrt{\pi}\,(n-1)^{(n-1)/2}\Gamma\left(\frac{n-2}{2}\right)}{\omega_n n^{n/2}\Gamma\left(\frac{n-1}{2}\right)} = \frac{4(n-1)^{(n-1)/2}\,\omega_{n-1}}{n^{n/2}\,\omega_n}. \tag{5.19}$$

Besides, in view of (5.12) and (4.12),

$$\begin{aligned} C_\infty &\ge \frac{4\omega_{n-2}(n-1)^{(n-1)/2}}{n\omega_n}\int_0^\pi \mathcal{P}_n(0)\sin^{n-3}\varphi\, d\varphi \\ &= \frac{4\omega_{n-2}(n-1)^{(n-1)/2}}{n\omega_n}\int_0^\pi \frac{\sin^{n-3}\varphi}{n^{(n-2)/2}}\, d\varphi \\ &= \frac{4\omega_{n-2}\sqrt{\pi}\,(n-1)^{(n-1)/2}\Gamma\left(\frac{n-2}{2}\right)}{\omega_n n^{n/2}\Gamma\left(\frac{n-1}{2}\right)} = \frac{4(n-1)^{(n-1)/2}\,\omega_{n-1}}{n^{n/2}\,\omega_n}, \end{aligned}$$

which together with (5.3) and (5.19) proves the equality (5.2).

Since $\boldsymbol{z} \in \mathbb{S}^{n-1}$ and the supremum with respect to the parameter

$$\alpha = \frac{n\gamma}{2\sqrt{n-1}} = \frac{n|\boldsymbol{z}'|}{2z_n\sqrt{n-1}},$$

in (5.12) is attained for $\alpha = 0$, it follows that the absolute value of the directional derivative of a harmonic function u with respect to the normal $\boldsymbol{e}_n$ to $\partial\mathbb{R}^n_+$, taken at an arbitrary point $x \in \mathbb{R}^n_+$, has the same supremum as $|\nabla u(x)|$. □

Remark 2. Inequality (5.1) can be written in the form

$$|\nabla u(x)| \le \mathcal{C}_\infty(x) \sup_{y\in\mathbb{R}^n_+} \big|u(y)\big|. \tag{5.20}$$

Using here (5.2), we arrive at the explicit sharp inequality

$$|\nabla u(x)| \le \frac{4(n-1)^{(n-1)/2}\,\omega_{n-1}}{n^{n/2}\,\omega_n x_n} \sup_{y\in\mathbb{R}^n_+} \big|u(y)\big|\,,$$

which generalizes the real value analog (1.5) of (1.2) to harmonic functions in the n-dimensional half-space.

From (5.20) it follows that

$$|\nabla u(x)| \le \mathcal{C}_\infty(x) \sup_{y\in\mathbb{R}^n_+} \big|u(y) - \omega\big| \tag{5.21}$$

with an arbitrary constant ω. Minimizing (5.21) in ω, we obtain

$$|\nabla u(x)| \le \frac{\mathcal{C}_\infty(x)}{2}\,\mathrm{osc}_{\mathbb{R}^n_+}(u), \tag{5.22}$$

where $\mathrm{osc}_{\mathbb{R}^n_+}(u)$ is the oscillation of u on $\mathbb{R}^n_+$.

Inequalities (5.2), (5.22) imply the sharp estimate

$$|\nabla u(x)| \le \frac{2(n-1)^{(n-1)/2}\,\omega_{n-1}}{n^{n/2}\,\omega_n x_n}\,\mathrm{osc}_{\mathbb{R}^n_+}(u),$$

which is an analogue of (1.3) for harmonic functions in $\mathbb{R}^n_+$.

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