Schauder estimates for solutions to boundary value problems for second order elliptic systems in polyhedral domains

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Abstract

Boundary value problems for second order elliptic differential equations and systems in a polyhedral domain are considered. The authors prove Schauder estimates and obtain regularity assertions for the solutions.

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1 Introduction

Schauder estimates, i.e. coercive estimates of Hölder norms, of solutions to linear elliptic equations and systems in domains with smooth boundaries have important applications to linear and especially nonlinear boundary value problems (see, e.g., Agmon, Douglis, Nirenberg [1] and Gilbarg, Trudinger [4]). In the present paper, these estimates are obtained for solutions of elliptic systems of second order in polyhedral domains. The Dirichlet and Neumann conditions are given on the boundary or its parts. We prove the solvability in weighted Hölder spaces, where the weights are powers of the distances to the edges and vertices of the domain, and obtain regularity assertions for the solutions. As an example, we consider boundary value problems in linear elasticity. We only mention that the results obtained can be of use for various nonlinear equations in polyhedral domains.

There is an extensive bibliography concerning elliptic boundary value problems in domains with edges (see e.g. the references in the books of Dauge [2], Nazarov and Plamenevskiĭ [15]). However, most of the works in this field deal with solutions in Sobolev spaces with or without weight. Moreover, often the Neumann problem is not included. Whereas for the Dirichlet and mixed problems there is a solvability theory in weighted spaces with so-called homogeneous norms, the study of the Neumann problem requires the use of other classes of weighted spaces. For the Neumann problem to the Laplace equation we refer here to the papers of Zajaczkowski and Solonnikov [17] (solutions in weighted L_2 Sobolev spaces), Dauge [3] (L_p Sobolev spaces without weight) and the preprints of Solonnikov [16], Grachev and Maz'ya [5] (weighted Sobolev and Hölder spaces), for more general problems to the book of Nazarov and Plamenevskiĭ [15] (weighted L_2 Sobolev spaces) and to our previous paper [13] (weighted L_p Sobolev spaces). In the sequel, we apply the estimates of Green's matrices given in our article [12] in order to prove the solvability in weighted Hölder spaces. Note that pointwise estimates of Green's functions of boundary value problems in domains with edges were first obtained by Maz'ya and Plamenevskiĭ [9, 10] and used for the proof of solvability theorems in weighted Hölder spaces. Their results are applicable e.g. to the Dirichlet problem but not to the Neumann problem.

We outline the main results of the paper. Let

$$\mathcal{K} = \{x \in \mathbb{R}^3 : \omega = x/|x| \in \Omega\} \tag{1.1}$$

be a polyhedral cone with faces $\Gamma_j = \{x : x/|x| \in \gamma_j\}$ and edges M_j, $j = 1, \dots, n$. Here Ω is a curvilinear polygon on the unit sphere bounded by the arcs $\gamma_1, \dots, \gamma_n$. Suppose that $\mathcal{K}$ coincides with a dihedral angle $\mathcal{D}_j$ in a neighborhood of an arbitrary edge point $x \in M_j$. We consider the boundary value problem

$$L(\partial_x)\, u = -\sum_{i,j=1}^{3} A_{i,j}\, \partial_{x_i}\partial_{x_j} u = f \quad \text{in } \mathcal{K}, \tag{1.2}$$

$$u = g_j \ \text{on } \Gamma_j \text{ for } j \in J_0, \tag{1.3}$$

$$B(\partial_x)\, u = \sum_{i,j=1}^{3} A_{i,j}\, n_j\, \partial_{x_i} u = g_k \quad \text{on } \Gamma_k \text{ for } k \in J_1. \tag{1.4}$$

where $A_{i,j}$ are constant $\ell \times \ell$ matrices such that $A_{i,j} = A^*_{j,i}$, $J_0 \cup J_1 = J = \{1, \dots, n\}$, $J_0 \cap J_1 = \emptyset$, u, f, g are vector-valued functions, and (n_1, n_2, n_3) denotes the exterior normal to Γ_k. We denote by $\mathcal{H}$ the closure of the set $\{u \in C_0^\infty(\overline{\mathcal{K}})^\ell : \, u = 0 \text{ on } \Gamma_j \text{ for } j \in J_0\}$ with respect to the norm

$$\|u\|_{\mathcal{H}} = \Big(\int_{\mathcal{K}} \sum_{j=1}^{3} |\partial_{x_j} u(x)|^2\, dx \Big)^{1/2}. \tag{1.5}$$

($C_0^\infty(G)$ is the set of all infinitely differentiable functions u such that $\operatorname{supp} u$ is compact and contained in G.) Throughout the paper, it is assumed that the sesquilinear form

$$b_{\mathcal{K}}(u, v) = \int_{\mathcal{K}} \sum_{i,j=1}^{3} A_{i,j} \partial_{x_i} u \cdot \partial_{x_j} \overline{v}\, dx \tag{1.6}$$

is $\mathcal{H}$-coercive, i.e.,

$$b_{\mathcal{K}}(u, u) \ge c\, \|u\|^2_{\mathcal{H}} \text{ for all } u \in \mathcal{H}. \tag{1.7}$$

Due to Lax-Milgram's lemma, this guarantees the existence and uniqueness of a weak solution of problem (1.2)–(1.4).

In Section 2 we introduce weighted Hölder spaces and in the cone $\mathcal{K}$. Let $r_j(x)$ denote the distance of x to the edge M_j. For nonnegative integer l, $\sigma \in (0,1)$, $\beta \in \mathbb{R}$ and $\vec{\delta} = (\delta_1, \dots, \delta_n) \in \mathbb{R}^n$ the space $C^{l,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})$ contains all l times differentiable functions u satisfying the condition

$$\sup_{x \in \mathcal{K}} |x|^{\beta - l - \sigma + |\alpha|} \prod_{j=1}^{n} \Big(\frac{r_j(x)}{|x|} \Big)^{\max(0, \delta_j - l - \sigma + |\alpha|)} |\partial_x^\alpha u(x)| < \infty \quad \text{for } |\alpha| \le l$$

and some weighted Hölder conditions, while the space $\Lambda^{l,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})$ consists of l times differentiable functions satisfying

$$\sup_{x \in \mathcal{K}} |x|^{\beta - l - \sigma + |\alpha|} \prod_{j=1}^{n} \Big(\frac{r_j(x)}{|x|} \Big)^{\delta_j - l - \sigma + |\alpha|} |\partial_x^\alpha u(x)| < \infty \quad \text{for } |\alpha| \le l$$

and a weighted Hölder condition. We study, in particular, the relations between these spaces.

Section 3 contains some auxiliary results for the problem in a dihedron. We give here a regularity assertion for the solution and study inhomogeneous boundary conditions.

In Section 4 we prove the existence and the uniqueness of solutions of problem (1.2)–(1.4) in weighted Hölder spaces. The solvability holds under certain assumptions on the spectrum of operator pencils $\mathfrak{A}(\lambda)$ and $A_j(\lambda)$ generated by the boundary value problem at the vertex of the cone and at the edges, respectively. Here $\mathfrak{A}(\lambda)$ is the operator of the parameter-dependent boundary value problem

$$\mathcal{L}(\lambda) u = f \ \text{ in } \Omega, \quad u = g_j \ \text{ on } \gamma_j \text{ for } j \in J_0, \quad \mathcal{B}(\lambda) u = g_k \ \text{ on } \gamma_k \text{ for } k \in J_1,$$

where the differential operators $\mathcal{L}(\lambda)$ and $\mathcal{B}(\lambda)$ are defined by

$$\mathcal{L}(\lambda) u = \rho^{2-\lambda}\, L(\partial_x)\, \big(\rho^\lambda u(\omega)\big), \quad \mathcal{B}(\lambda) u = \rho^{1-\lambda}\, B(\partial_x)\, \big(\rho^\lambda u(\omega)\big),$$

$\rho = |x|$, $\omega = x/|x|$. Furthermore, $A_j(\lambda)$ are operators of boundary value problems for a system of parameter-dependent ordinary differential equations. The eigenvalues of the pencils $A_j(\lambda)$ are the roots of certain transcendental equations. For example, the Dirichlet problem for the Lamé system in a cone is uniquely solvable in $\Lambda^{2,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^3$ for arbitrary $f \in \Lambda^{0,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^3$, $g_j \in \Lambda^{1,\sigma}_{\beta,\vec{\delta}}(\Gamma_j)^3$ if the line $\operatorname{Re}\lambda = 2+\sigma-\beta$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$ and the components of $\vec{\delta}$ satisfy the inequalities

$$2 - \lambda_1^{(j)} < \delta_j - \sigma < 2,$$

where $\lambda_1^{(j)}$ is the smallest positive solution of the equation $(3-4\nu)\sin(\lambda\theta_j) = \pm\lambda\sin\theta_j$. Here ν is the Poisson ratio, $\nu < 1/2$, and θ_j denotes the angle at the edge M_j. The Neumann problem for the Lamé system in a cone $\mathcal{K}$ with edge angles $\theta_j < \pi$ is uniquely solvable in $C^{2,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^3$ for arbitrary $f \in C^{0,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^3$, $g_j \in C^{1,\sigma}_{\beta,\vec{\delta}}(\Gamma_j)$ if the line $\operatorname{Re}\lambda = 2+\sigma-\beta$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$, the components of $\vec{\delta}$ satisfy the inequalities

$$\max\big(2 - \frac{\pi}{\theta_j}, 0\big) < \delta_j - \sigma < 2, \quad \delta_j - \sigma \neq 1, \tag{1.8}$$

and (in the case $\delta_j - \sigma < 1$) the boundary data satisfy certain compatibility conditions on M_j. In general, $2+\sigma-\delta_j$ must be less than the first positive eigenvalue of the pencil $A_j(\lambda)$. A feature of the Neumann problem for the Lamé system is that $\lambda = 1$ is always an eigenvalue of this pencil. For $\theta_j < \pi$ this is the smallest positive eigenvalue, while the second one is π/θ_j. Condition (1.8) allows that the number $2+\sigma-\delta_j$ exceeds the eigenvalue $\lambda = 1$. However, then the boundary data must satisfy the compatibility condition

$$\vec{n}_{j_+} \cdot g_{j_+} = \vec{n}_{j_-} \cdot g_{j_-} \quad \text{on } M_j. \tag{1.9}$$

Here $\vec{n}_{j_+}$, $\vec{n}_{j_-}$ are the exterior normals to the sides Γ_{j_+} and Γ_{j_-}, respectively, adjacent to M_j.

Furthermore, we study the smoothness of solutions. For example, we obtain the following result for the weak solution $u \in \mathcal{H}$ of the Dirichlet problem for the Lamé system. If $f \in \Lambda^{l-2,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^3$, $g_j \in \lambda^{l-1,\sigma}_{\beta,\vec{\delta}}(\Gamma_j)^3$, the strip $-1/2 \le \operatorname{Re}\lambda \le l+\sigma-\beta$ contains no eigenvalues of the pencil $\mathfrak{A}(\lambda)$, and the components of $\vec{\delta}$ satisfy the inequalities $l - \lambda_1^{(j)} < \delta_j - \sigma < l$, then $u \in \Lambda^{l,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^3$. In particular, for a convex cone we obtain $u \in \Lambda^{2,\sigma}_{1,\vec{1}}(\mathcal{K})^3$ with $\vec{1} = (1,\dots,1)$ and certain positive σ if $f \in \Lambda^{0,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^3$, $g_j \in \Lambda^{2,\sigma}_{\beta,\vec{\delta}}(\Gamma_j)^3$. This follows from a result of Kozlov, Mazya and Schwab [7] (see also [6, Th.3.5.3]) who proved that the strip $-2 \le \operatorname{Re}\lambda \le 1$ does not contain eigenvalues of the pencil $\mathfrak{A}(\lambda)$ and from the obvious fact that $\lambda_1^{(j)} > 1$ for $\theta_j < \pi$. The same is true for the Dirichlet problem to system (1.2) if

$$\sum_{i,j=1}^{3} \big(A_{i,j} f^{(j)}, f^{(i)}\big)_{\mathbb{C}^\ell} \ge c \sum_{j=1}^{3} |f^{(j)}|_{\mathbb{C}^\ell} \quad \text{for all } f^{(j)} \in \mathbb{C}^\ell$$

(see [6, Th.8.6.2,Th.11.4.1]). The inclusion $u \in \Lambda^{2,\sigma}_{1,\vec{1}}(\mathcal{K})^3$ implies, in particular, that the first derivatives of u are Hölder continuous. For a nonconvex cone we obtain $u \in \Lambda^{2,\sigma}_{2,\vec{2}}(\mathcal{K})^3$ with certain $\sigma > 0$ if $f \in \Lambda^{0,\sigma}_{2,\vec{2}}(\mathcal{K})^3$, $g_j \in \Lambda^{2,\sigma}_{2,\vec{2}}(\Gamma_j)^3$. This means, in particular, that u satisfies a Hölder condition.

Analogous results hold for the weak solution $u \in \mathcal{H}$ of the Neumann problem for the Lamé system. If $f \in C^{l-2,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^3$, $g_j \in C^{l-1,\sigma}_{\beta,\vec{\delta}}(\Gamma_j)$, the strip $-1/2 \le \operatorname{Re}\lambda \le l+\sigma-\beta$ contains at most the eigenvalue $\lambda = 0$ of the pencil $\mathfrak{A}(\lambda)$, the edge angles θ_j are less than π, the components of $\vec{\delta}$ satisfy the inequalities

$$\max\big(l - \frac{\pi}{\theta_j}, 0\big) < \delta_j - \sigma < l, \quad \delta_j - \sigma \neq 1, 2, \dots, l-1,$$

and (in the case $\delta_j - \sigma < l-1$) the compatibility condition (1.9), then there exists a constant vector c such that $u - c \in C^{l,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^3$.

In the last section we consider the problem with variable coefficients in a bounded domain of polyhedral type. By means of the results of Section 5, we prove a regularity assertion for weak solutions.

2 Weighted Hölder spaces

We introduce here weighted Hölder spaces with homogeneous and inhomogeneous norms in angles, dihedrons and polyhedral cones. For the case of an angle the imbeddings and relations between these spaces are essentially proved in [8].

2.1 Weighted Hölder spaces in an angle

The space $\Lambda_\delta^{l,\sigma}(K)$. Let K be the angle $\{x=(x_1,x_2) : \ 0<r<\infty, \ 0<\varphi<\theta\}$, where r,φ are the polar coordinates of the point x. For arbitrary integer $l \ge 0$ and real $\delta \ge 0$, $\sigma \in (0,1]$ we define $\Lambda_\delta^{l,\sigma}(K)$ as the set of all l times continuously differentiable functions on $\overline{K}\backslash\{0\}$ with finite norm

$$\|u\|_{\Lambda_\delta^{l,\sigma}(K)} = \sum_{|\alpha|\le l} \sup_{x\in K} |x|^{\delta-l-\sigma+|\alpha|} |\partial_x^\alpha u(x)| + \langle u\rangle_{l,\sigma,\delta;K} \, ,$$

where

$$\langle u\rangle_{l,\sigma,\delta;K} = \sum_{|\alpha|=l} \sup_{\substack{x,y\in K\\ |x-y|\le |x|/2}} |x|^\delta \, \frac{|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)|}{|x-y|^\sigma} \, .$$

It can be easily shown that the norm in $\Lambda_\delta^{l,\sigma}(K)$ is equivalent to

$$\|u\| = \sum_{|\alpha|\le l} \sup_{x\in K} |x|^{\delta-l-\sigma+|\alpha|} |\partial_x^\alpha u(x)| + \sum_{|\alpha|=l} \sup_{x,y\in K} \frac{\big||x|^\delta \partial_x^\alpha u(x) - |y|^\delta \partial_y^\alpha u(y)\big|}{|x-y|^\sigma} \, .$$

Lemma 2.1 *The space $\Lambda_\delta^{l,\sigma}(K)$ is continuously imbedded into $\Lambda_{\delta'}^{l',\sigma'}(K)$ if $l+\sigma > l'+\sigma'$, $l+\sigma-\delta = l'+\sigma'-\delta'$.*

Proof: For $l=l'$ the assertion of the lemma is obvious. Let $l>l'$ and $|\alpha|=l'$. By the mean value theorem, there exists a real number $t\in(0,1)$ such that

$$\partial_x^\alpha u(x) - \partial_y^\alpha u(y) = (\nabla \partial^\alpha u)\big(x+t(y-x)\big)\cdot(x-y).$$

Furthermore, for $|x-y|<|x|/2$ we have $|x-y|<|x|/2<\big|x+t(y-x)\big|$ and, therefore,

$$|x|^{\delta'} \frac{|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)|}{|x-y|^{\sigma'}} \le c\, \big|x+t(y-x)\big|^{\delta'-\sigma'+1} \, \big|(\nabla\partial^\alpha u)\big(x+t(y-x)\big)\big|. \tag{2.1}$$

Consequently, $\langle u\rangle_{l',\sigma',\delta';K}$ can be estimated by the norm of u in $\Lambda_\delta^{l,\sigma}(K)$. This proves the lemma for the case $l>l'$. ■

The space $C_\delta^{l,\sigma}(K)$. We introduce another class of weighted spaces. For integer $l\ge 0$ and real δ, σ, $0\le\delta<l+\sigma$, $0<\sigma\le 1$, let $C_\delta^{l,\sigma}(K)$ be the space of all l times continuously differentiable functions on $\overline{K}\backslash\{0\}$ with finite norm

$$\|u\|_{C_\delta^{l,\sigma}(K)} = \sum_{|\alpha|\le l} \sup_{x\in K} |x|^{\max(\delta-l-\sigma+|\alpha|,0)} |\partial_x^\alpha u(x)| + \langle u\rangle_{l,\sigma,\delta;K} \, .$$

For $\delta \ge l+\sigma$ we set $C_\delta^{l,\sigma}(K) = \Lambda_\delta^{l,\sigma}(K)$.

Analogously to Lemma 2.1, the following result holds.

Lemma 2.2 *The space $C_\delta^{l,\sigma}(K)$ is continuously imbedded into $C_{\delta'}^{l',\sigma'}(K)$ if $l+\sigma>l'+\sigma'$, $l+\sigma-\delta = l'+\sigma'-\delta'$, $\delta'\ge 0$.*

We describe a relation between the weighted Hölder space $C_\delta^{l,\sigma}(K)$ and the usual Hölder space $C^{l,\sigma}(K)$.

Lemma 2.3 *The space $C_0^{l,\sigma}(K)$ coincides with $C^{l,\sigma}(K)$.*

Proof: Obviously $C^{l,\sigma}(K)$ is continuously imbedded into $C_0^{l,\sigma}(K)$. We prove the imbedding $C_0^{l,\sigma}(K) \subset C^{l,\sigma}(K)$ for $l = 0$. First we show that any function $u \in C_0^{0,\sigma}(K)$ is continuous at $x = 0$. We consider the sequence of the points $x_n = 2^{-n}x_0$, where x_0 is an arbitrary point in K. Since $|x_n - x_{n+1}| = |x_n|/2$, we have

$$|u(x_n) - u(x_{n+1})| \le c_0\, |x_n - x_{n+1}|^\sigma = c_0\, |x_0|^\sigma\, 2^{-(n+1)\sigma}, \quad \text{where } c_0 = \langle u\rangle_{0,\sigma,0;K}.$$

Consequently,

$$|u(x_n) - u(x_m)| \le c_0\, |x_0|^\sigma \left(2^{-(n+1)\sigma} + 2^{-(n+2)\sigma} + \cdots + 2^{-m\sigma}\right) < c_0\, |x_0|^\sigma\, \frac{1}{2^\sigma - 1}\, 2^{-n\sigma}$$

for $m > n$. Thus, $\{u(x_n)\}$ is a Cauchy sequence. The limit of this sequence is denoted by a. It can be easily shown that the sequence $\{u(x_n)\}$ has the same limit if $\{x_n\}$ is an arbitrary sequence on the ray from 0 to x_0 such that $x_n \to 0$ as $n \to \infty$. Now let $\{x_n\}$ be an arbitrary sequence in the angle $\{x \in K : |\varphi - \varphi_0| < \arctan 1/2\}$, where r, φ are the polar coordinates of x and r_0, φ_0 that of x_0. We denote by y_n the orthogonal projection of x_n onto the line through 0 and x_0. Then $|x_n - y_n| \le |y_n|/2$ and, therefore, $|u(x_n) - u(y_n)| \le c_0\, |x_n - y_n|^\sigma < c_0\, |y_n|^\sigma$. If x_n tends to 0 as $n \to \infty$, then $\lim |y_n| = 0$ and $\lim u(y_n) = a$. This implies $\lim u(x_n) = a$. Repeating this argument, we conclude that $u(x_n) \to a$ for an arbitrary sequence $\{x_n\}$ in K converging to 0. Thus, the function u is continuous at the vertex of K. Next we show that

$$\frac{|u(x) - u(y)|}{|x - y|^\sigma} \le c\, \langle u\rangle_{0,\sigma,0;K} \quad \text{for } u \in C_0^{0,\sigma}(K),\ |x - y| > |x|/2. \tag{2.2}$$

Indeed, for $|x - y| > |x|/2$ we have $|x - y| > |y|/3$ and, therefore,

$$\frac{|u(x) - u(y)|}{|x - y|^\sigma} \le 2^\sigma\, \frac{|u(x) - u(0)|}{|x|^\sigma} + 3^\sigma\, \frac{|u(y) - u(0)|}{|y|^\sigma}\,.$$

Using the inequalities

$$\big|u(x) - u(0)\big| \le \sum_{n=0}^\infty \big|u(2^{-n}x) - u(2^{-n-1}x)\big| \le c_0 \sum_{n=0}^\infty |2^{-n-1}x|^\sigma = \frac{c_0}{2^\sigma - 1}\, |x|^\sigma,$$

we obtain (2.2). Thus, we have proved that $C_0^{0,\sigma}(K) = C^{0,\sigma}(K)$. ¿From this we can easily deduce the equality $C_0^{l,\sigma}(K) = C^{l,\sigma}(K)$ for $l \ge 1$. ■

Corollary 2.1 *If $k - 1 \le \delta - \sigma < k$, $k \in \{0, 1, \ldots, l\}$, then $C_\delta^{l,\sigma}(K)$ is continuously imbedded into $C^{l-k,\sigma-\delta+k}(K)$.*

A relation between the spaces $\Lambda_\delta^{l,\sigma}(K)$ and $C_\delta^{l,\sigma}(K)$. A relation between the above introduced weighted spaces holds by means of the following version of Hardy's inequality.

Lemma 2.4 *Let u be a differentiable function on $\overline{K}\backslash\{0\}$ and let $r^\delta|\nabla u(x)| < c < \infty$ for $x \in K$. Furthermore, we assume that*

$$\text{(i) } \delta > 1 \text{ and } u(x) \to 0 \text{ as } |x| \to \infty \qquad \text{or} \qquad \text{(ii) } \delta < 1 \text{ and } u(x) \to 0 \text{ as } |x| \to 0.$$

Then $\displaystyle \sup_{x\in K} r^{\delta-1}\, |u(x)| \le \frac{1}{|\delta - 1|} \sup_{x\in K} r^\delta\, |\partial_r u(x)|.$

Proof: If condition (i) is satisfied, then the assertion follows from the inequality

$$|u(r,\varphi)| \le \int_r^\infty |\partial_t u(t,\varphi)|\, dt \le \sup_{0\le t\le r} \big|t^\delta \partial_t u(t,\varphi)\big| \cdot \int_r^\infty t^{-\delta}\, dt.$$

Replacing the integration interval (r,∞) in the above inequality by $(0,r)$, we can estimate $|u(r,\varphi)|$ in the same way for case (ii). ■

Let $u \in C_\delta^{l,\sigma}(K)$, $k-1 \le \delta-\sigma < k$, where k is an integer, $0 \le k \le l$. Then, by Corollary 2.1, the derivatives of u up to order $l-k$ are continuous at $x=0$. We denote by

$$p_{l-k}(u) = \sum_{|\alpha|\le l-k} \frac{1}{\alpha!} (\partial^\alpha u)(0)\, x^\alpha$$

the Taylor polynomial of degree $l-k$ of u.

Lemma 2.5 *Let $u \in C_\delta^{l,\sigma}(K)$, where $k-1 \le \delta-\sigma < k \le l$. For the inclusion $u \in \Lambda_\delta^{l,\sigma}(K)$ it is necessary and sufficient that $p_{l-k} = 0$.*

Proof: If $u \in \Lambda_\delta^{l,\sigma}(K)$, then $\partial^\alpha u(x) = O(|x|^{l+\sigma-\delta-|\alpha|})$ and, consequently, $(\partial^\alpha u)(0) = 0$ for $|\alpha| \le l-k$. Suppose now that $u \in C_\delta^{l,\sigma}(K)$ and $(\partial^\alpha u)(0) = 0$ for $|\alpha| \le l-k$. We show that

$$\sup |x|^{\delta-l-\sigma+|\alpha|}\, \big|\partial^\alpha u(x)\big| \le c\, \|u\|_{C_\delta^{l,\sigma}(K)} \tag{2.3}$$

for $|\alpha| \le l$. If $k>0$, then (2.3) is obviously satisfied for $|\alpha| \ge l-k+1$. Furthermore, we conclude from Lemma 2.4 that

$$\sup |x|^{\delta-\sigma-l} |u(x)| \le c_1 \sup |x|^{\delta-\sigma-l+1} |\nabla u(x)| \le \cdots \le c_{l-k+1} \sum_{|\alpha|=l-k+1} \sup |x|^{\delta-\sigma-k+1} |\partial^\alpha u(x)| \tag{2.4}$$

what proves (2.3) for $|\alpha| \le l$. If $k=0$, then the space $C_\delta^{l,\sigma}(K)$ is continuously imbedded into $C^{l,\sigma-\delta}(K)$. Consequently,

$$\sup |x|^{\delta-\sigma} |\partial^\alpha u(x)| = \sup |x|^{\delta-\sigma} \big|(\partial^\alpha u)(x) - (\partial^\alpha u)(0)\big| \le c\, \|u\|_{C_\delta^{l,\sigma}(K)} \quad \text{for } |\alpha| = l.$$

Using Lemma 2.4, we obtain (2.3) for $|\alpha| \le l$. Hence, $u \in \Lambda_\delta^{l,\sigma}(K)$. ■

Corollary 2.2 *If $u \in C_\delta^{l,\sigma}(K)$, $k-1 \le \delta-\sigma < k \le l$, then $u - p_{l-k}(u) \in \Lambda_\delta^{l,\sigma}(K)$ and*

$$\|u - p_{l-k}(u)\|_{\Lambda_\delta^{l,\sigma}(K)} \le c\, \|u\|_{C_\delta^{l,\sigma}(K)}$$

with a constant c independent of u.

2.2 Weighted Hölder spaces in a dihedron

Definition of the spaces $\Lambda_\delta^{l,\sigma}(\mathcal{D})$ and $C_\delta^{l,\sigma}(\mathcal{D})$. Let $\mathcal{D}$ be the dihedron $\{x = (x', x_3) :\, x' = (x_1, x_2) \in K,\ x_3 \in \mathbb{R}\}$, where K is an angle in the (x_1, x_2)-plane with vertex at the origin. The boundary of $\mathcal{D}$ consists of two half-planes $\Gamma^\pm$ and the edge M. For arbitrary integer $l \ge 0$ and real δ, σ, $0 < \sigma \le 1$, we define $\Lambda_\delta^{l,\sigma}(\mathcal{D})$ as the space of all functions with continuous derivatives up to order l on $\overline{\mathcal{D}}\backslash M$ such that

$$\|u\|_{\Lambda_\delta^{l,\sigma}(\mathcal{D})} = \sum_{|\alpha|\le l} \sup_{x\in\mathcal{D}} |x'|^{\delta-l-\sigma+|\alpha|} |\partial_x^\alpha u(x)| + \langle u\rangle_{l,\sigma,\delta;\mathcal{D}} < \infty, \tag{2.5}$$

where

$$\langle u\rangle_{l,\sigma,\delta;\mathcal{D}} = \sum_{|\alpha|=l} \sup_{\substack{x,y\in\mathcal{D}\\ |x-y|<|x'|/2}} |x'|^{\delta} \frac{|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)|}{|x-y|^\sigma}\,.$$

It can be easily shown that $\Lambda_\delta^{l,\sigma}(\mathcal{D})$ is continuously imbedded into $C^{l-k,k-\delta+\sigma}(\mathcal{D})$ if $k-1 \le \delta-\sigma < k$, where k is nonnegative integer, $k \le l$. Indeed for $|\alpha| = l-k$ and $|x-y| > |x'|/2$ we have $|x-y| > |y'|/3$ and, consequently,

$$\frac{|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)|}{|x-y|^{k-\delta+\sigma}} \le (|x'|/2)^{\delta-\sigma-k} |\partial_x^\alpha u(x)| + (|y'|/3)^{\delta-\sigma-k} |\partial_y^\alpha u(y)| \le c\, \|u\|_{\Lambda_\delta^{l,\sigma}(\mathcal{D})},$$

while for $|x-y| < |x'|/2$ the expression on the left hand side can be estimated by $\langle u\rangle_{l,\sigma,\delta;\mathcal{D}}$ if $k=0$ and by the supremum of $|x'|^{\delta-\sigma-k+1}|\nabla\partial^\alpha u(x)|$ if $k\ge 1$.

Let $0\le\delta<l+\sigma$ and $0<\sigma\le 1$. Then by $C^{l,\sigma}_\delta(\mathcal{D})$ we denote the weighted Hölder space with the norm

$$\|u\|_{C^{l,\sigma}_\delta(\mathcal{D})} = \|u\|_{C^{l-k,k-\delta+\sigma}(\mathcal{D})} + \sum_{|\alpha|=l-k+1}^{l} \sup_{x\in\mathcal{D}} |x'|^{\delta-l-\sigma+|\alpha|}|\partial_x^\alpha u(x)| + \langle u\rangle_{l,\sigma,\delta;\mathcal{D}}\,,$$

where $k=[\delta-\sigma]+1$, $[s]$ denotes the greatest integer less or equal to s. In the case $\delta\ge l+\sigma$ we set $C^{l,\sigma}_\delta(\mathcal{D}) = \Lambda^{l,\sigma}_\delta(\mathcal{D})$.

The following lemma can be proved analogously to Lemma 2.1 (see also [11, Prop.1.4]).

Lemma 2.6 *The space $\Lambda^{l,\sigma}_\delta(\mathcal{D})$ is continuously imbedded into $\Lambda^{l',\sigma'}_{\delta'}(\mathcal{D})$ if $l+\sigma>l'+\sigma'$ and $l+\sigma-\delta = l'+\sigma'-\delta'$. If additionally $\delta>\delta'\ge 0$, then there is the continuous imbedding $C^{l,\sigma}_\delta(\mathcal{D})\subset C^{l',\sigma'}_{\delta'}(\mathcal{D})$.*

Traces on the edge. Obviously, the trace of an arbitrary function $u\in C^{l,\sigma}_\delta(\mathcal{D})$ on the edge M lies in the space $C^{l-k,k-\delta+\sigma}(M)$ if $k-1\le\delta-\sigma<k$, $k\in\{0,1,\dots,l\}$. The following lemma shows that every function $f\in C^{l-k,k-\delta+\sigma}(M)$ can be extended to a function $u\in C^{l,\sigma}_\delta(\mathcal{D})$.

Lemma 2.7 *Let $f\in C^{l-k,k-\delta+\sigma}(M)$, where k,l are integers, $0\le k\le l$, and σ,δ are real numbers, $0<\sigma\le 1$, $k-1\le\delta-\sigma<k$. Then there exists a function $u\in C^{l,\sigma}_\delta(\mathcal{D})$ such that $\partial^j_{x_3}u|_M = \partial^j_{x_3}f$ for $j=0,1,\dots,l-k$ and the following inequalities are satisfied:*

(i) $|x'|^{\delta-l-\sigma+|\alpha|}\,\big|\partial_x^\alpha u(x)\big| \le c_\alpha\,\|f\|_{C^{l-k,k-\delta+\sigma}(M)}$ *for $\alpha=(\alpha_1,\alpha_2,\alpha_3)$, $\alpha_1+\alpha_2>0$, and for $|\alpha|>l-k$, $x\in\mathcal{D}$.*

(ii) $|x'|^{\delta-l-\sigma+j}\,\big|\partial^j_{x_3}\big(u(x)-f(x_3)\big)\big| \le c_j\,\|f\|_{C^{l-k,k-\delta+\sigma}(M)}$ *for $j=0,1,\dots,l-k$, $x\in\mathcal{D}$,*

Here the constants c_j and c_α are independent of f and x.

Proof: We set

$$u(x',x_3) = (Ef)(x',x_3) \stackrel{def}{=} \int_0^1 f(x_3+tr)\,\psi(t)\,dt, \tag{2.6}$$

where $r=|x'|$ and $\psi\in C_0^\infty(\mathbb{R})$ is a function with support in $(0,1)$ satisfying the condition

$$\int_0^1 \psi(t)\,dt = 1, \qquad \int_0^1 t^j\,\psi(t)\,dt = 0 \quad \text{for } j=1,\dots,l-k.$$

Since $\partial^j_{x_3}f$ is continuous for $j=0,1\dots,l-k$, we have $\partial^j_{x_3}u(0,x_3)=f^{(j)}(x_3)$. Furthermore,

$$\big|\partial_x^\alpha u(x)| \le c\,\sup_{x_3}|f^{(j)}(x_3)| \quad \text{for } x\in\mathcal{D},\ |\alpha|=j\le l-k. \tag{2.7}$$

We prove that

$$\big|\partial^j_{x_3}\partial^\nu_r u(x)\big| \le c\,r^{l+\sigma-\delta-j-\nu}\,\|f\|_{C^{l-k,k-\delta+\sigma}(M)} \tag{2.8}$$

if $\nu\ge 1$ or $j+\nu>l-k$. First let $j+\nu\le l-k$, $\nu\ge 1$. We set $\psi_{j,\nu}(t)=t^\nu\,\psi(t)$ if $j+\nu=l-k$ and

$$\psi_{j,\nu}(t) = \int_0^t \frac{(t-\tau)^{l-k-j-\nu-1}}{(l-k-j-\nu-1)!}\,\tau^\nu\,\psi(\tau)\,d\tau \quad \text{if } j+\nu<l-k.$$

Obviously, $\operatorname{supp}\psi_{j,\nu}\subset(0,1)$, $\int_0^1\psi_{j,\nu}(t)\,dt=0$ and $\psi_{j,\nu}^{(l-k-j-\nu)}(t)=t^\nu\,\psi(t)$. Hence, partial integration yields

$$\begin{aligned}\partial^j_{x_3}\partial^\nu_r u &= \int_0^1 t^\nu\,f^{(j+\nu)}(x_3+tr)\,\psi(t)\,dt = (-1)^{l-k-j-\nu}\int_0^1 r^{l-k-j-\nu}f^{(l-k)}(x_3+tr)\,\psi_{j,\nu}(t)\,dt\\ &= (-1)^{l-k-j-\nu}\int_0^1 r^{l-k-j-\nu}\big(f^{(l-k)}(x_3+tr)-f^{(l-k)}(x_3)\big)\,\psi_{j,\nu}(t)\,dt\end{aligned}$$

what implies (2.8). Now let $j+\nu > l-k$. For $j = j'+j''$, $\nu = \nu'+\nu''$, $j'+\nu' = l-k$, we have

$$\begin{aligned}
\partial_{x_3}^j \partial_r^\nu u &= \partial_{x_3}^{j''} \partial_r^{\nu''} \int_0^1 t^{\nu'}\, f^{(l-k)}(x_3+tr)\,\psi(t)\,dt \\
&= \int_{\mathbb{R}} f^{(l-k)}(\tau)\, \partial_{x_3}^{j''} \partial_r^{\nu''} \frac{1}{r} \Big(\frac{\tau-x_3}{r}\Big)^{\nu'} \psi\Big(\frac{\tau-x_3}{r}\Big)\,d\tau \\
&= \int_{\mathbb{R}} \big(f^{(l-k)}(\tau) - f^{(l-k)}(x_3)\big)\, \partial_{x_3}^{j''} \partial_r^{\nu''} \frac{1}{r} \Big(\frac{\tau-x_3}{r}\Big)^{\nu'} \psi\Big(\frac{\tau-x_3}{r}\Big)\,d\tau.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\big|\partial_{x_3}^j \partial_r^\nu v\big| &\le c\,r^{-1-j''-\nu''} \int_{x_3}^{x_3+r} \big|f^{(l-k)}(\tau) - f^{(l-k)}(x_3)\big|\,d\tau \\
&\le c\,r^{-1-j''-\nu''} \sup_{x_3\le\tau\le x_3+r} \frac{\big|f^{(l-k)}(\tau) - f^{(l-k)}(x_3)\big|}{|\tau-x_3|^{k+\sigma-\delta}} \int_{x_3}^{x_3+r} (\tau-x_3)^{k+\sigma-\delta}\,d\tau.
\end{aligned}$$

This implies (2.8) for $j+\nu > l-k$. Since

$$\big|\partial_x^\alpha u(x)\big| \le c \sum_{\nu=1}^{\alpha_1+\alpha_2} r^{\nu-\alpha_1-\alpha_2}\,\big|\partial_{x_3}^{\alpha_3}\partial_r^\nu u(x)\big|$$

for $\alpha = (\alpha_1,\alpha_2,\alpha_3)$, $\alpha_1+\alpha_2 > 0$, we obtain assertion (i). Analogously to (2.1), we get

$$|x'|^\delta \frac{\big|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)\big|}{|x-y|^\sigma} \le c\,\big|x'+t(y'-x')\big|^{\delta-\sigma+1}\,\big|(\nabla\partial^\alpha u)\big(x+t(y-x)\big)\big|$$

with a certain $t\in(0,1)$ for $|x-y| < |x'|/2$. According to (i), the right-hand side of the last inequality can be estimated by the norm of f in $C^{l-k,k-\delta+\sigma}(M)$ for $|\alpha| = l$. Hence,

$$\langle u\rangle_{l,\sigma,\delta;\mathcal{D}} \le c\,\|f\|_{C^{l-k,k-\delta+\sigma}(M)}\,. \tag{2.9}$$

Together with (i) this implies that $\partial_x^\alpha u \in \Lambda_\delta^{k,\sigma}(\mathcal{D}) \subset \Lambda^{0,k-\delta+\sigma}(\mathcal{D})$ for $|\alpha| = l-k$, $\alpha_1+\alpha_2 \ne 0$. ¿From this and from (2.7) we conclude that $\partial_x^\alpha u \in C^{0,k-\delta+\sigma}(\mathcal{D})$ for $|\alpha| = l-k$, $\alpha_1+\alpha_2\ne 0$. Furthermore, from the definition of u it follows that

$$\frac{\big|\partial_{x_3}^{l-k}u(x) - \partial_{y_3}^{l-k}u(y)\big|}{|x-y|^{k-\delta+\sigma}} \le c\,\|f^{(l-k)}\|_{C^{0,k-\delta+\sigma}(M)} \quad \text{for } x,y\in\mathcal{D}.$$

Consequently, $\partial_x^\alpha u \in C^{0,k-\delta+\sigma}(\mathcal{D})$ for $|\alpha| = l-k$. This together with (2.7), (2.9) and assertion (i) implies $u \in C_\delta^{l,\sigma}(\mathcal{D})$. It remains to prove (ii). Since $\partial_{x_3}^j u(0,x_3) = f^{(j)}(x_3)$ and $\delta-\sigma < k$, we have (see Lemma 2.4)

$$r^{\delta-l-\sigma+j}\,\big|\partial_{x_3}^j\big(u(x) - f(x_3)\big)\big| \le \sup_{x\in\mathcal{D}} r^{\delta-l-\sigma+j+1}\,\big|\partial_{x_3}^j\partial_r u\big| \le c\,\|f\|_{C^{l-k,k+\sigma-\delta}(M)}$$

for $j = 0,1,\dots,l-k$. The proof is complete. ■

A relation between the spaces $\Lambda_\delta^{l,\sigma}(\mathcal{D})$ and $C_\delta^{l,\sigma}(\mathcal{D})$. The following lemma can be proved analogously to Lemma 2.5.

Lemma 2.8 *Let $u\in C_\delta^{l,\sigma}(\mathcal{D})$, where $k-1\le\delta-\sigma<k$, $k\in\{0,1,\dots,l\}$. For the inclusion $u\in\Lambda_\delta^{l,\sigma}(K)$ it is necessary and sufficient that $\partial^\alpha u = 0$ on M for $|\alpha|\le l-k$.*

Now we are able to prove a relation between the spaces $C_\delta^{l,\sigma}(\mathcal{D})$ and $\Lambda_\delta^{l,\sigma}(\mathcal{D})$ analogous to that given in Corollary 2.2.

Lemma 2.9 *Let $u \in C_\delta^{l,\sigma}(\mathcal{D})$, where $\delta \ge 0$, $0 < \sigma \le 1$, $k-1 \le \delta - \sigma < k$, and k is an integer, $0 \le k \le l$. Furthermore, let $f_{i,j} = \partial_{x_1}^i \partial_{x_2}^j u|_M$ for $i+j \le l-k$ and $u_{i,j}(x) = \chi(|x'|)\,(Ef_{i,j})(x)$ where E is the operator (2.6) and χ is a smooth cut-off function on $[0,\infty)$, supp $\chi \subset [0,2)$, $\chi = 1$ on $[0,1]$. Then there is the following decomposition for u:*

$$u = v + w, \quad \textit{where } v \in \Lambda_\delta^{l,\sigma}(\mathcal{D}),\ w = \sum_{i+j \le l-k} \frac{1}{i!\,j!}\, u_{i,j}\, x_1^i x_2^j \in C_{\delta+m}^{l+m,\sigma}(\mathcal{D}),\ m = 0,1,2,\ldots.$$

Proof: We show first that $w \in C_\delta^{l,\sigma}(\mathcal{D})$. Since $f_{i,j} \in C^{l-k-i-j,k+\sigma-\delta}(M)$, it follows from Lemma 2.7 that

$$|x'|^{\max(0,\delta-l-\sigma+|\alpha|)}\, \big|\partial_x^\alpha w(x)\big| \le c \sum_{i+j\le l-k} \|f_{i,j}\|_{C^{l-k-i-j,k+\sigma-\delta}(M)} \le c\, \|u\|_{C_\delta^{l,\sigma}(\mathcal{D})}\,.$$

Furthermore,

$$\langle w\rangle_{l+m,\sigma,\delta+m;\mathcal{D}} \le c \sum_{|\alpha|=l+m+1} \sup_{x\in\mathcal{D}} |x'|^{\delta-\sigma+j+1}\, \big|\partial_x^\alpha w(x)\big| \le c\, \|u\|_{C_\delta^{l,\sigma}(\mathcal{D})}\,.$$

In order to show that $w \in C^{l-k,k-\delta+\sigma}(\mathcal{D})$, we consider the function $\partial_x^\alpha\,(u_{i,j}\, x_1^i x_2^j)$ for $|\alpha| = l-k$. Obviously, this is a linear combination of functions

$$(\partial_x^\gamma u_{i,j})\, x_1^{i-\mu}\, x_2^{j-\nu} \quad \text{where } \mu \le i,\ \nu \le j,\ |\gamma| = l-k-\mu-\nu. \tag{2.10}$$

Since $u_{i,j} \in C^{l-k-i-j,k-\delta+\sigma}(\mathcal{D})$, the term (2.10) with $\mu = i$, $\nu = j$ belongs to $C^{0,k-\delta+\sigma}(\mathcal{D})$. If $\mu+\nu < i+j$, then it follows from Lemma 2.7 that $\partial_x^\gamma u_{i,j} \in \Lambda_{\delta+i+j+|\gamma|}^{l,\sigma}(\mathcal{D})$. Consequently, $(\partial_x^\gamma u_{i,j})\, x_1^{i-\mu} x_2^{j-\nu} \in \Lambda_{\delta-k+l}^{l,\sigma}(\mathcal{D}) \subset \Lambda_0^{0,k-\delta+\sigma}(\mathcal{D})$ if $\mu+\nu < i+j$. This implies that $\partial_x^\alpha w \in C^{0,k-\delta+\sigma}(\mathcal{D})$ for $|\alpha| = l-k$. Thus, it is shown that $w \in C_{\delta+j}^{l+j,\sigma}(\mathcal{D})$. Furthermore, the norm of w can be estimated by the norm of u in $C_\delta^{l,\sigma}(\mathcal{D})$.

By means of Theorem 2.7, it can be easily shown that $\partial^\alpha w = f_{i,j}$ on M for $\alpha = (i,j,0)$, $i+j \le l-k$. From this we conclude that $\partial^\alpha(u-w) = 0$ on M for $|\alpha| \le l-k$. Applying Lemma 2.8, we obtain $u - w \in \Lambda_\delta^{l,\sigma}(\mathcal{D})$. ■

2.3 Weighted Hölder spaces in a polyhedral cone

Let $\mathcal{K}$ be the cone (1.1). We denote by $r_j(x)$ the distance to the edge M_j and by $r(x)$ the distance to the set $\mathcal{S} = M_1 \cup \cdots \cup M_n \cup \{0\}$. The subset $\{x \in \mathcal{K} :\ r_j(x) < 3r(x)/2\}$ is denoted by $\mathcal{K}_j$. Furthermore, let $\tilde{J}$ be an arbitrary subset of $J = \{1,2,\ldots,n\}$, l a nonnegative integer, $0 < \sigma < 1$, $\beta \in \mathbb{R}$, $\vec{\delta} = (\delta_1,\ldots,\delta_n) \in \mathbb{R}^n$, $\delta_j \ge 0$ if $j \in J\backslash\tilde{J}$, and $k_j = [\delta_j - \sigma] + 1$. We introduce the functions $h_j(t) = t$ for $j \in \tilde{J}$, $h_j(t) = \max(t,0)$ for $j \in J\backslash\tilde{J}$ and define $C_{\beta,\vec{\delta}}^{l,\sigma}(\mathcal{K};\tilde{J})$ as the set of all l times continuously differentiable functions on $\overline{\mathcal{K}}\backslash\mathcal{S}$ with finite norm

$$\begin{aligned}
\|u\|_{C_{\beta,\vec{\delta}}^{l,\sigma}(\mathcal{K};\tilde{J})} &= \sum_{|\alpha|\le l} \sup_{x\in\mathcal{K}} |x|^{\beta-l-\sigma+|\alpha|} \prod_{j=1}^n \Big(\frac{r_j(x)}{|x|}\Big)^{h_j(\delta_j-l-\sigma+|\alpha|)}\, \big|\partial_x^\alpha u(x)\big| \\
&\quad + \sum_{j\in J\backslash\tilde{J}}\ \sum_{|\alpha|=l-k_j}\ \sup_{\substack{x,y\in\mathcal{K}_j\\ |x-y|<|x|/2}} |x|^{\beta-\delta_j}\, \frac{\big|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)\big|}{|x-y|^{k_j+\sigma-\delta_j}} \\
&\quad + \sum_{|\alpha|=l}\ \sup_{|x-y|<r(x)/2} |x|^\beta \prod \Big(\frac{r_j(x)}{|x|}\Big)^{\delta_j} \frac{\big|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)\big|}{|x-y|^\sigma}\,.
\end{aligned} \tag{2.11}$$

Furthermore, we define $C_{\beta,\vec{\delta}}^{l,\sigma}(\mathcal{K}) = C_{\beta,\vec{\delta}}^{l,\sigma}(\mathcal{K};\emptyset)$ and $\Lambda_{\beta,\vec{\delta}}^{l,\sigma}(\mathcal{K}) = C_{\beta,\vec{\delta}}^{l,\sigma}(\mathcal{K};J)$. The trace spaces for $C_{\beta,\vec{\delta}}^{l,\sigma}(\mathcal{K};\tilde{J})$, $C_{\beta,\vec{\delta}}^{l,\sigma}(\mathcal{K})$ and $\Lambda_{\beta,\vec{\delta}}^{l,\sigma}(\mathcal{K})$ on Γ_j are denoted by $C_{\beta,\vec{\delta}}^{l,\sigma}(\Gamma_j,\tilde{J})$, $C_{\beta,\vec{\delta}}^{l,\sigma}(\Gamma_j)$ and $\Lambda_{\beta,\vec{\delta}}^{l,\sigma}(\Gamma_j)$, respectively.

It can be easily shown (cf. Lemma 2.6) that $C_{\beta,\vec{\delta}}^{l,\sigma}(\mathcal{K};\tilde{J}) \subset C_{\beta',\vec{\delta}}^{l',\sigma'}(\mathcal{K};\tilde{J})$ if $l+\sigma > l'+\sigma'$, $l+\sigma-\beta = l'+\sigma'-\beta'$, $l+\sigma-\delta_j \ge l'+\sigma'-\delta'_j$ for $j = 1,\ldots,n$, $\delta_j,\delta'_j \ge 0$ for $j \in J\backslash\tilde{J}$.

3 The model problem in a dihedron

3.1 The operator pencil generated by the boundary value problem

Let $d^\pm \in \{0,1\}$. We consider the problem

$$Lu = f \ \text{ in } \mathcal{D}, \quad d^\pm u + (1-d^\pm)Bu = g^\pm \ \text{ on } \Gamma^\pm, \tag{3.1}$$

where L and B are the same differential operators as in (1.2) and (1.4), respectively. It is assumed in this section that the sesquilinear form corresponding to this problem (i.e. the form (1.6) with $\mathcal{D}$ instead of $\mathcal{K}$) satisfies (1.7), where $\mathcal{K}$ has to be replaced by $\mathcal{D}$.

We introduce the following operator pencil $A(\lambda)$. Let

$$L(\partial_{x'},0) = -\sum_{i,j=1}^{2} A_{i,j}\, \partial_{x_i}\partial_{x_j}\,, \qquad B^\pm(\partial_{x'},0) = \sum_{i,j=1}^{2} A_{i,j}\, n_j^\pm \partial_{x_i}.$$

Here $n_j^\pm$ are the components of the exterior normal to $\Gamma^\pm$. We define the differential operators $\mathcal{L}(\lambda)$ and $\mathcal{B}^\pm(\lambda)$ depending on the complex parameter λ by

$$\mathcal{L}(\lambda)\, u(\varphi) = r^{2-\lambda}\, L(\partial_{x'},0)\, \big(r^\lambda u(\varphi)\big), \quad \mathcal{B}^\pm(\lambda)\, u(\varphi) = d^\pm u(\varphi) + (1-d^\pm)\, r^{1-\lambda}\, B^\pm(\partial_{x'},0)\, \big(r^\lambda u(\varphi)\big),$$

where again r,φ are the polar coordinates in the (x_1,x_2)-plane. Then $A(\lambda)$ denotes the operator

$$W^2(0,\theta)^\ell \ni u \to \Big(\mathcal{L}(\lambda)u\,,\, \mathcal{B}^+(\lambda)\, u(\varphi)\big|_{\varphi=0}\,, \mathcal{B}^-(\lambda)\, u(\varphi)\big|_{\varphi=\theta}\Big) \in L_2(0,\theta)^\ell \times \mathbb{C}^\ell \times \mathbb{C}^\ell.$$

As is known, $A(\lambda)$ is an isomorphism for all $\lambda \in \mathbb{C}$ except a countable set of isolated points, the eigenvalues of the pencil $A(\lambda)$. All eigenvalues have finite geometric and algebraic multiplicities.

3.2 Boundary conditions on the sides of the dihedron

Lemma 3.1 *Let $g^\pm \in \Lambda_\delta^{l+d^\pm-1,\sigma}(\Gamma^\pm)^\ell$, $l \ge 1-\min d^\pm$. There exists a vector function $u \in \Lambda_\delta^{l,\sigma}(\mathcal{D})^\ell$ such that $d^\pm u + (1-d^\pm)\, Bu = g^\pm$ on $\Gamma^\pm$ and*

$$\|u\|_{\Lambda_\delta^{l,\sigma}(\mathcal{D})^\ell} \le c \sum_\pm \|g^\pm\|_{\Lambda_\delta^{l+d^\pm-1,\sigma}(\Gamma^\pm)^\ell}, \tag{3.2}$$

where c is independent of g^+ and g^-.

Proof: Let ζ_k be infinitely differentiable functions on $(0,\infty)$ such that

$$supp\, \zeta_k \subset \big(2^{k-1}, 2^{k+1}\big), \quad |\partial_r^j \zeta_k(r)| \le c\, 2^{-kj}, \quad \text{and} \quad \sum_{k=-\infty}^{+\infty} \zeta_k = 1. \tag{3.3}$$

Setting $\zeta_k(x) = \zeta_k(|x'|)$ we can consider ζ_k as a function on $\mathcal{D}$. Furthermore, let $\tilde\zeta_k(x) = \zeta_k(2^k x)$ and $\tilde g_k(x) = g(2^k x)$. Then the support of $\tilde\zeta_k$ is contained in $\{x :\ 1/2 < |x'| < 2\}$. Consequently, there exists a function $\tilde u_k \in C^{l,\sigma}(\mathcal{D})^\ell$ with support in $\{x :\ 1/4 < |x'| < 4\}$ such that $d^\pm \tilde u_k + (1-d^\pm) B\tilde u_k = 2^{k(1-d^\pm)}\tilde\zeta_k \tilde g_k$ on $\Gamma^\pm$ and

$$\|\tilde u_k\|_{C^{l,\sigma}(\mathcal{D})^\ell} \le c_1 \sum_\pm 2^{k(1-d^\pm)}\, \|\tilde\zeta_k \tilde g_k\|_{C^{l+d^\pm-1,\sigma}(\mathcal{D})^\ell} \le c_2\, 2^{k(l+\sigma-\delta)}\, \|\zeta_k g\|_{\Lambda_\delta^{l+d^\pm-1,\sigma}(\Gamma^\pm)^\ell}$$

with constants c_1, c_2 independent of g and k. We set $u = \sum_{k=-\infty}^{+\infty} u_k$, where $u_k(x) = \tilde u_k(2^{-k}x)$. Then $d^\pm u + (1-d^\pm)Bu = g$ on $\Gamma^\pm$. Since the support of u_k is contained in $\{x :\ 2^{k-2} < |x'| < 2^{k+2}\}$, we get

$$\begin{aligned} \|u\|_{\Lambda_\delta^{l,\sigma}(\mathcal{D})^\ell} &\le c \sup_k \|u_k\|_{\Lambda_\delta^{l,\sigma}(\mathcal{D})^\ell} \le c \sup_k 2^{k(\delta-l-\sigma)} \|\tilde u_k\|_{C^{l,\sigma}(\mathcal{D})^\ell} \le c \sup_k \|\zeta_k g\|_{\Lambda_\delta^{l+d^\pm-1,\sigma}(\Gamma^\pm)^\ell} \\ &\le c\, \|g\|_{\Lambda_\delta^{l+d^\pm-1,\sigma}(\Gamma^\pm)^\ell}. \end{aligned}$$

This proves the lemma. ■

We need an analogous assertion for the Neumann problem in the class of the spaces $C_\delta^{l,\sigma}$.

Lemma 3.2 *Let $d^+ = d^- = 0$, $g^\pm \in C_\delta^{1,\sigma}(\Gamma^\pm)^\ell$, $\delta \ge \sigma$, $g^\pm(x) = 0$ for $|x'| > 1$. In the case $\delta < \sigma + 1$ we suppose further that for every $x_3 \in \mathbb{R}$ there exist vectors $c(x_3), d(x_3) \in \mathbb{C}^\ell$ such that*

$$\left(A_{1,1}n_1^\pm + A_{1,2}n_2^\pm\right) c(x_3) + \left(A_{2,1}n_1^\pm + A_{2,2}n_2^\pm\right) d(x_3) = g_0^\pm(x_3), \tag{3.4}$$

where $g_0^\pm = g^\pm|_M$ and $n^\pm = (n_1^\pm, n_2^\pm, 0)$ is the exterior normal to $\Gamma^\pm$. Then there exists a vector function $u \in C_\delta^{2,\sigma}(\mathcal{D})^\ell$ satisfying $Bu = g^\pm$ on $\Gamma^\pm$ and an estimate analogous to (3.2).

Proof: If $\delta \ge \sigma + 1$, then $C_\delta^{1,\sigma}(\Gamma^\pm) = \Lambda_\delta^{1,\sigma}(\Gamma^\pm)$ and the assertion follows from Lemma 3.1.
Suppose that $\sigma \le \delta < \sigma + 1$. Then $g_0^\pm \in C^{0,\sigma+1-\delta}(M)^\ell$, and there is the representation

$$g^\pm = Eg_0^\pm + G^\pm, \quad \text{where } (E\phi)(x) = \int_0^1 \phi(x_3 + t|x'|)\, \psi(t)\, dt, \quad G^\pm \in \Lambda_\delta^{1,\sigma}(\Gamma^\pm)^\ell,$$

and ψ is the same function as in the proof of Lemma 2.7. By the assumptions of the lemma, there exist vector functions $c, d \in C^{0,\sigma+1-\delta}(M)^\ell$ satisfying (3.4). We set $v(x) = x_1\,(Ec)(x) + x_2\,(Ed)(x)$. Then

$$\begin{aligned} B(\partial_x)v\big|_{\Gamma^\pm} &= \left(A_{1,1}n_1^\pm + A_{1,2}n_2^\pm\right) Ec + \left(A_{2,1}n_1^\pm + A_{2,2}n_2^\pm\right) Ed \\ &\quad + \sum_{i=1}^{3} \left(A_{i,1}n_1^\pm + A_{i,2}n_2^\pm\right) \left(x_1\partial_{x_i}Ec + x_2\partial_{x_i}Ed\right) \end{aligned}$$

¿From Lemma 2.7 it follows that $x_1\partial_{x_i}Ec + x_2\partial_{x_i}Ed \in \Lambda_\delta^{1,\sigma}(\mathcal{D})^\ell$ for $i = 1, 2, 3$. Furthermore,

$$\left(A_{1,1}n_1^\pm + A_{1,2}n_2^\pm\right) Ec + \left(A_{2,1}n_1^\pm + A_{2,2}n_2^\pm\right) Ed - Eg_0^\pm \in \Lambda_\delta^{1,\sigma}(\mathcal{D})^\ell.$$

Consequently, $B(\partial_x)v|_{\Gamma^\pm} - g^\pm \in \Lambda_\delta^{1,\sigma}(\Gamma^\pm)^\ell$. Applying Lemma 3.1, we obtain the assertion of the lemma in the case $\sigma \le \delta < \sigma + 1$. ■

Remark 3.1 The equations (3.4) are solvable for all $g_0^\pm(x_3)$ if and only if the homogeneous system

$$\left(A_{1,1}n_1^\pm + A_{1,2}n_2^\pm\right) c + \left(A_{2,1}n_1^\pm + A_{2,2}n_2^\pm\right) d = 0 \tag{3.5}$$

has only the trivial solution $(c, d) = 0$ or, what is the same, if there does not exist a linear vector-function $p(x') = cx_1 + dx_2$ satisfying $B(\partial_x)p = 0$ on $\Gamma^\pm$. This is the case, for example, if $\lambda = 1$ is not an eigenvalue of the pencil $A(\lambda)$.

If the system (3.5) has nontrivial solution, then $\lambda = 1$ is an eigenvalue of the pencil $A(\lambda)$, and there is an eigenfunction of the form $(c_1x_1 + c_2x_2)/|x'|$ corresponding to this eigenvalue. In this case for the existence of a vector function $u \in C_\delta^{1,\sigma}(\mathcal{D})^\ell$, $\sigma \le \delta < \sigma + 1$, satisfying $B(\partial_x)\,u = g^\pm$ on $\Gamma^\pm$ it is sufficient that g^+ and g^- satisfy $2\ell - r'$ compatibility conditions $l_k(g^+, g^-) = 0$ on M, $k = 1, \ldots, 2\ell - r'$, where r' is the rank of the matrix $\begin{pmatrix} A_{1,1} & A_{2,1} \\ A_{1,2} & A_{2,2} \end{pmatrix}$ and l_k are linear functionals on $\mathbb{C}^{2\ell}$. For example, in the case of the Neumann problem for the Lamé system g^+ and g^- must satisfy the condition $n^- \cdot g^+ = n^+ \cdot g^-$ on M (see [12, Sect.2.5]).

3.3 Regularity assertions for solutions in the spaces $\Lambda_\delta^{l,\sigma}$

The next lemma follows from [14, Th.6.3.7] and [1, Th.9.3].

Lemma 3.3 *Let G_1, G_2 be bounded subdomains of $\mathbb{R}^3$ such that $\overline{G}_1 \subset G_2$, $G_1 \cap \mathcal{D} \ne \emptyset$ and $\overline{G}_1 \cap M = \emptyset$. If u is a solution of (3.1), $u \in W^{2,p}(\mathcal{D} \cap G_2)^\ell$, $f \in C^{l-2,\sigma}(\mathcal{D} \cap G_2)^\ell$, $g^\pm \in C^{l+d^\pm-1,\sigma}(\Gamma^\pm \cap G_2)^\ell$, $l \ge 2$, $0 < \sigma < 1$, then $u \in C^{l,\sigma}(\mathcal{D} \cap G_1)^\ell$. Furthermore,*

$$\|u\|_{C^{l,\sigma}(\mathcal{D}\cap G_1)^\ell} \le c\left(\|f\|_{C^{l-2,\sigma}(\mathcal{D}\cap G_2)^\ell} + \sum_\pm \|g^\pm\|_{C^{l+d^\pm-1,\sigma}(\Gamma^\pm\cap G_2)^\ell} + \|u\|_{C(\mathcal{D}\cap G_2)^\ell}\right)$$

with a constant c independent of u.

Let $W^{l,p}_{loc}(\overline{\mathcal{D}}\backslash M)$ be the set of all functions u such that $\phi u \in W^{l,p}(\mathcal{D})$ for all $\phi \in C_0^\infty(\overline{\mathcal{D}}\backslash M)$.

Lemma 3.4 *Let $u \in W^{2,p}_{loc}(\overline{\mathcal{D}}\backslash M)^\ell$ be a solution of problem* (3.1). *If* $\sup |x'|^{\delta-l-\sigma}|u(x)| < \infty$, $f \in \Lambda_\delta^{l-2,\sigma}(\mathcal{D})^\ell$, $l \ge 2$, $0<\sigma<1$, *and* $g^\pm \in \Lambda_\delta^{l+d^\pm-1,\sigma}(\Gamma^\pm)^\ell$, *then* $u \in \Lambda_\delta^{l,\sigma}(\mathcal{D})^\ell$ *and*

$$\|u\|_{\Lambda_\delta^{l,\sigma}(\mathcal{D})^\ell} \le c\Big(\|f\|_{\Lambda_\delta^{l-2,\sigma}(\mathcal{D})^\ell} + \sum_\pm \|g^\pm\|_{\Lambda_\delta^{l+d^\pm-1,\sigma}(\Gamma^\pm)^\ell} + \sup |x'|^{\delta-l-\sigma}|u(x)|\Big). \tag{3.6}$$

Proof: Due to Lemma 3.1, we may restrict ourselves to the case $g^\pm = 0$. Let x,y be points in $\mathcal{D}$ such that $|x-y| < |x'|/2$. We introduce the functions $\tilde{u}(\xi) = u(|x'|\xi)$ and $\tilde{f}(\xi) = f(|x'|\xi)$. Since $L(\partial_\xi)\,\tilde{u}(\xi) = |x'|^2\,\tilde{f}(\xi)$ in $\mathcal{D}$ and $d^\pm\tilde{u}(\xi) + (1-d^\pm)\,B(\partial_\xi)\,\tilde{u}(\xi) = 0$ on $\Gamma^\pm$, Lemma 3.3 implies

$$\sum_{|\alpha|\le l} \big|\partial_\xi^\alpha \tilde{u}(\xi)\big| + \sum_{|\alpha|=l} \frac{\big|\partial_\xi^\alpha\tilde{u}(\xi) - \partial_\eta^\alpha\tilde{u}(\eta)\big|}{|\xi-\eta|^\sigma} \le c\,\Big(|x'|^2\,\|\tilde{f}\|_{C^{l-2,\sigma}(\mathcal{U}_\xi)^\ell} + \|\tilde{u}\|_{C(\bar{\mathcal{U}}_\xi)^\ell}\Big)$$

for $|\xi'|=1$, $|\xi-\eta|<1/2$, where $\mathcal{U}_\xi = \{\zeta\in\mathcal{D}:\ |\zeta-\xi|<3/4\}$. Here the constant c is independent of ξ_3. Setting $\xi = x/|x'|$ and $\eta = y/|x'|$, we conclude that

$$\begin{aligned}\sum_{|\alpha|=l} |x'|^{|\alpha|}|\partial_x^\alpha u(x)| + \sum_{|\alpha|=l} |x'|^{l+\sigma}\,\frac{\big|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)\big|}{|x-y|^\sigma} &\le c\,\Big(\sum_{|\alpha|\le l-2} |x'|^{|\alpha|+2} \sup_{X\in\mathcal{V}_x} \big|(\partial^\alpha f)(X)\big| \\ &+ \sum_{|\alpha|=l-2} |x'|^{l+\sigma} \sup_{\substack{X,Y\in\mathcal{V}_x\\ |X-Y|<|x'|/2}} \frac{\big|(\partial^\alpha f)(X) - (\partial^\alpha f)(Y)\big|}{|x-y|^\sigma} + \sup_{X\in\mathcal{V}_x}\big|u(X)\big|\Big),\end{aligned} \tag{3.7}$$

where $\mathcal{V}_x = \{X\in\mathcal{D}:\ |X-x| < 3|x'|/4\}$. Multiplying this inequality by $|x'|^{\delta-l-\sigma}$ and using the fact that $|x'|/4 < r(X) < 7|x'|/4$ for $X\in\mathcal{V}_x$, we obtain (3.6). ∎

We further need the following modification of Lemma 3.4.

Lemma 3.5 *Let ϕ, ψ be smooth functions with compact supports, $\psi = 1$ in a neighborhood of* $\operatorname{supp}\phi$. *Suppose that $u \in W^{2,p}_{loc}(\overline{\mathcal{D}}\backslash M)^\ell$ is a solution of problem* (3.1), $\sup|x'|^{\delta-l-\sigma}\big|\psi(x)\,u(x)\big| < \infty$, $\psi f \in \Lambda_\delta^{l-2,\sigma}(\mathcal{D})^\ell$ *and* $\psi g^\pm \in \Lambda_\delta^{l+d^\pm-1,\sigma}(\Gamma^\pm)^\ell$, $l\ge 2$, $0<\sigma<1$. *Then* $\phi u \in \Lambda_\delta^{l,\sigma}(\mathcal{D})^\ell$ *and*

$$\|\phi u\|_{\Lambda_\delta^{l,\sigma}(\mathcal{D})^\ell} \le c\,\Big(\|\psi f\|_{\Lambda_\delta^{l-2,\sigma}(\mathcal{D})^\ell} + \sum_\pm \|\psi g^\pm\|_{\Lambda_\delta^{l+d^\pm-1,\sigma}(\Gamma^\pm)^\ell} + \sup_{x\in\mathcal{D}} |x'|^{\delta-l-\sigma}\big|\psi(x)\,u(x)\big|\Big).$$

Proof: Let $\mathcal{U}$ be the set $\{x\in\mathcal{D}:\ \psi(x)=1\}$. Then

$$\begin{aligned}&\sum_{|\alpha|=l} |x'|^{\delta-l-\sigma+|\alpha|}|\partial_x^\alpha u(x)| + \sum_{|\alpha|=l} |x'|^\delta\,\frac{\big|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)\big|}{|x-y|^\sigma} \\ &\le c\,\Big(\|f\|_{\Lambda_\delta^{l-2,\sigma}(\mathcal{U})^\ell} + \sum_\pm \|g^\pm\|_{\Lambda_\delta^{l+d^\pm-1,\sigma}(\Gamma^\pm\cap\bar{\mathcal{U}})^\ell} + \sup_{X\in\mathcal{U}} r(X)^{\delta-l-\sigma}\big|u(X)\big|\Big)\end{aligned}$$

for $x\in\operatorname{supp}\phi$, where the norm in $\Lambda_\delta^{l,\sigma}(\mathcal{U})$ is defined by (2.5) with $\mathcal{U}$ instead of $\mathcal{D}$. For $|x'|>\varepsilon>0$ this estimate follows from Lemma 3.3, while for sufficiently small $|x'|$ it is a consequence of (3.7). This implies the assertion of the lemma. ∎

In the next lemma the eigenvalues of the pencil $A(\lambda)$ play an important role.

Lemma 3.6 *Let ϕ, ψ be as in Lemma* 3.5, *and let u be a solution of problem* (3.1) *such that* $\psi u \in \Lambda_\delta^{l,\sigma}(\mathcal{D})^\ell$, $\psi\partial_{x_3}u \in \Lambda_\delta^{l,\sigma}(\mathcal{D})^\ell$, $\psi f \in \Lambda_\delta^{l-1,\sigma}(\mathcal{D})^\ell$, $\psi g^\pm \in \Lambda_\delta^{l+d^\pm,\sigma}(\Gamma^\pm)^\ell$. *If there are no eigenvalues of the pencil $A(\lambda)$ in the strip* $l+\sigma-\delta \le \operatorname{Re}\lambda \le l+1+\sigma-\delta$, *then* $\phi u \in \Lambda_\delta^{l+1,\sigma}(\mathcal{D})^\ell$ *and*

$$\|\phi u\|_{\Lambda_\delta^{l+1,\sigma}(\mathcal{D})^\ell} \le c\,\Big(\sum_{j=0}^1 \|\psi\partial_{x_3}^j u\|_{\Lambda_\delta^{l,\sigma}(\mathcal{D})^\ell} + \|\psi f\|_{\Lambda_\delta^{l-1,\sigma}(\mathcal{D})^\ell} + \sum_\pm \|\psi g^\pm\|_{\Lambda_\delta^{l+d^\pm,\sigma}(\Gamma^\pm)^\ell}\Big).$$

Proof: Let χ be an infinitely differentiable function on $\overline{\mathcal{D}}$ such that $\chi = 1$ in a neighborhood of $\operatorname{supp}\phi$, $\psi = 1$ in a neighborhood of $\operatorname{supp}\chi$. Obviously,

$$L(\partial_{x'},0)\,(\chi u) = F \stackrel{def}{=} \chi f + \chi L_1 \partial_{x_3} u + [L(\partial_{x'},0),\chi]\,u,$$

where L_1 is a differential operator of first order and $[L(\partial_{x'},0),\chi] = L(\partial_{x'},0)\chi - \chi L(\partial_{x'},0)$ denotes the commutator of $L(\partial_{x'},0)$ and χ. By our assumptions on u and f, the function $F(\cdot,x_3)$ belongs to $\Lambda_\delta^{l-1,\sigma}(K)^\ell$ for arbitrary fixed x_3. Analogously, we have $d^\pm u + (1-d^\pm)\,B^\pm(\partial_{x'},0)\,(\chi u) = G^\pm$, where $G^\pm(\cdot,x_3)|_{\gamma^\pm} \in \Lambda_\delta^{l+d^\pm,\sigma}(\gamma^\pm)^\ell$ for fixed x_3. Consequently, by [9, Th.8.4], we obtain $(\chi u)(\cdot,x_3) \in \Lambda_\delta^{l+1,\sigma}(K)^\ell$ and

$$\|(\chi u)(\cdot,x_3)\|_{\Lambda_\delta^{l+1,\sigma}(K)^\ell} \le c\Big(\|F(\cdot,x_3)\|_{\Lambda_\delta^{l-1,\sigma}(K)^\ell} + \sum_\pm \|G^\pm(\cdot,x_3)\|_{\Lambda_\delta^{l+d^\pm,\sigma}(\Gamma^\pm)^\ell}\Big)$$

with a constant c independent of x_3. From this and from the inclusion $\psi\partial_{x_3}u \in \Lambda_\delta^{l,\sigma}(\mathcal{D})^\ell$ we conclude that

$$\sup_{x\in\mathcal{D}} |x'|^{\delta-l-1-\sigma+|\alpha|}\,\big|\partial_x^\alpha(\chi u)(x)\big| \le c\Big(\sum_{j=0}^{1}\|\psi\partial_{x_3}^j u\|_{\Lambda_\delta^{l,\sigma}(\mathcal{D})^\ell} + \|\psi f\|_{\Lambda_\delta^{l-1,\sigma}}(\mathcal{D})^\ell + \sum_\pm \|\psi g^\pm\|_{\Lambda_\delta^{l+d^\pm,\sigma}(\Gamma^\pm)^\ell}\Big)$$

for $|\alpha| \le l+1$. This implies, in particular, that $\chi u \in \Lambda_{\delta-1}^{l,\sigma}(\mathcal{D})^\ell$ (cf. Lemma 2.6). Applying Lemma 3.5, we obtain the assertion of the lemma. ∎

3.4 Regularity assertions for solutions of the Neumann problem in the spaces $C_\delta^{l,\sigma}$

For the case $d^+ = d^- = 0$ (the case of the Neumann problem) we need annalogous assertions in the spaces $C_\delta^{l,\sigma}$.

Lemma 3.7 *Let ϕ, ψ be the same functions as in Lemma* 3.5, *and let $u \in W_{loc}^{2,p}(\overline{\mathcal{D}}\backslash M)^\ell$ be a solution of problem* (3.1) *with $d^+ = d^- = 0$. If $\psi u \in C_{\delta-l-\sigma+k+\tau}^{k,\tau}(\mathcal{D})^\ell$, $\psi f \in C_\delta^{l-2,\sigma}(\mathcal{D})^\ell$, $\psi g^\pm \in C_\delta^{l-1,\sigma}(\Gamma^\pm)^\ell$, where $l \ge 2$, $k \ge 0$, $\delta \ge l+\sigma-k-\tau > 0$, $0<\sigma<1$, $0<\tau<1$, then $\phi u \in C_\delta^{l,\sigma}(\mathcal{D})^\ell$ and*

$$\|\phi u\|_{C_\delta^{l,\sigma}(\mathcal{D})^\ell} \le c\Big(\|\psi f\|_{C_\delta^{l-2,\sigma}(\mathcal{D})^\ell} + \sum_\pm \|\psi g^\pm\|_{C_\delta^{l-1,\sigma}(\Gamma^\pm)^\ell} + \|\psi u\|_{C_{\delta-l-\sigma+k+\tau}^{k,\tau}(\mathcal{D})^\ell}\Big).$$

Proof: By Lemma 2.9, ψu admits the decomposition $\psi u = v + w$, where $v \in \Lambda_{\delta-l-\sigma+k+\tau}^{k,\tau}(\mathcal{D})^\ell$, $w \in C_\delta^{l,\sigma}(\mathcal{D})^\ell$ and

$$\|v\|_{\Lambda_{\delta-l-\sigma+k+\tau}^{k,\tau}(\mathcal{D})^\ell} + \|w\|_{C_\delta^{l,\sigma}(\mathcal{D})^\ell} \le c\,\|\psi u\|_{C_{\delta-l-\sigma+k+\tau}^{k,\tau}(\mathcal{D})^\ell}\,.$$

Let χ be a smooth function such that $\psi = 1$ in a neighborhood of $\operatorname{supp}\phi$ and $\psi = 1$ in a neighborhood of $\operatorname{supp}\chi$. Then $\chi Lv = \chi f - \chi Lw \in C_\delta^{l-2,\sigma}(\mathcal{D})^\ell$. In the cases $k=0$ and $k=1$ it follows from the condition on δ that $\delta > l-2+\sigma$ and, consequently, $C_\delta^{l-2,\sigma}(\mathcal{D}) = \Lambda_\delta^{l-2,\sigma}(\mathcal{D})$. In the case $k \ge 2$, the inclusions $v \in \Lambda_{\delta-l-\sigma+k+\tau}^{k,\tau}(\mathcal{D})^\ell$ and $\chi Lv \in C_\delta^{l-2,\sigma}(\mathcal{D})^\ell$ imply $\chi Lv \in \Lambda_{\delta-l-\sigma+k+\tau}^{k-2,\tau}(\mathcal{D})^\ell \cap C_\delta^{l-2,\sigma}(\mathcal{D})^\ell \subset \Lambda_\delta^{l-2,\sigma}(\mathcal{D})^\ell$. Analogously, $\chi Bv \in \Lambda_\delta^{l-1,\sigma}(\Gamma^\pm)^\ell$. Using Lemma 3.5, we obtain $\phi v \in \Lambda_\delta^{l,\sigma}(\mathcal{D})^\ell$ and

$$\begin{aligned}
\|\phi v\|_{\Lambda_\delta^{l,\sigma}(\mathcal{D})^\ell} &\le c\Big(\|\chi Lv\|_{\Lambda_\delta^{l-2,\sigma}(\mathcal{D})^\ell} + \sum_\pm \|\chi Bv\|_{\Lambda_\delta^{l-1,\sigma}(\Gamma^\pm)^\ell} + \|\chi v\|_{\Lambda_{\delta-l-\sigma+k+\tau}^{k,\tau}(\mathcal{D})^\ell}\Big) \\
&\le c\Big(\|\psi f\|_{C_\delta^{l-2,\sigma}(\mathcal{D})^\ell} + \sum_\pm \|\psi g^\pm\|_{C_\delta^{l-1,\sigma}(\Gamma^\pm)^\ell} + \|\psi w\|_{C_\delta^{l,\sigma}(\mathcal{D})^\ell} + \|\psi v\|_{\Lambda_{\delta-l-\sigma+k+\tau}^{k,\tau}(\mathcal{D})^\ell}\Big).
\end{aligned}$$

The result follows. ∎

Lemma 3.8 *Let ϕ, ψ be as in Lemma 3.5, and let u be a solution of problem (3.1) with $d^+ = d^- = 0$ such that $\psi u \in C_\delta^{l,\sigma}(\mathcal{D})^\ell$, $\psi\partial_{x_3} u \in C_\delta^{l,\sigma}(\mathcal{D})^\ell$, $\psi f \in C_\delta^{l-1,\sigma}(\mathcal{D})^\ell$, $\psi g^\pm \in C_\delta^{l,\sigma}(\Gamma^\pm)^\ell$, $l \ge 2$, $\delta > \sigma$. If there are no eigenvalues of the pencil $A(\lambda)$ in the strip $l+\sigma-\delta \le \operatorname{Re}\lambda \le l+\sigma-\delta+1$, then $\phi u \in C_\delta^{l+1,\sigma}(\mathcal{D})^\ell$ and*

$$\|\phi u\|_{C_\delta^{l+1,\sigma}(\mathcal{D})^\ell} \le c\Big(\sum_{j=0}^{1} \|\psi\partial_{x_3}^j u\|_{C_\delta^{l,\sigma}(\mathcal{D})^\ell} + \|\psi f\|_{C_\delta^{l-1,\sigma}(\mathcal{D})^\ell} + \sum_\pm \|\psi g^\pm\|_{C_\delta^{l,\sigma}(\Gamma^\pm)^\ell}\Big).$$

Proof: Suppose that $k-1 < l+\sigma-\delta \le k$, where k is an integer, $k \le l$. Then both u and $\partial_{x_3} u$ belong to $C^{l-k,k+\sigma-\delta}(\mathcal{D})^\ell$. Consequently, the traces $u_{i,j}$ of $\partial_{x_1}^i \partial_{x_2}^j u$ on M are from $C^{l-k-i-j+1,k-\sigma-\delta}(M)^\ell$ for $i+j \le l-k$. By Lemma 2.9, there is the representation

$$\psi u = \psi \sum_{i+j\le l-k} \frac{Eu_{i,j}}{i!j!}\, x_1^i\, x_2^j + v,$$

where E is the extension operator (2.6) in the proof of Lemma 2.7, $v \in \Lambda_\delta^{l,\sigma}(\mathcal{D})^\ell$, $\partial_{x_3} v \in \Lambda_\delta^{l,\sigma}(\mathcal{D})^\ell$. By means of Lemma 2.7, it can be shown that $L(\psi u - v) \in C_\delta^{l-1,\sigma}(\mathcal{D})^\ell$ and $B(\psi u - v) \in C_\delta^{l,\sigma}(\Gamma^\pm)^\ell$. Therefore, also $Lv \in C_\delta^{l-1,\sigma}(\mathcal{D})^\ell$ and $Bv|_{\Gamma^\pm} \in C_\delta^{l,\sigma}(\Gamma^\pm)^\ell$. From the inclusion $v \in \Lambda_\delta^{l,\sigma}(\mathcal{D})^\ell$ it follows that $\partial_x^\alpha Lv = 0$ on M for $|\alpha| \le l-k-2$ and $\partial_r^j Bv = 0$ on M gor $j \le l-k-1$. We denote by $F_{i,j}$ the traces of $\partial_{x_1}^i \partial_{x_2}^j Lv$ on M, $i+j = l-k-1$, and by $G_{l-k}^\pm$ the trace of $\partial_r^{l-k} Bv|_{\Gamma^\pm}$ on M. Obviously, $F_{i,j} \in C^{0,k+\sigma-\delta}(M)^\ell$ and $G_{l-k}^\pm \in C^{0,k+\sigma-\delta}(M)^\ell$. Furthermore, by Lemma 2.9,

$$\psi\, Lv - \psi \sum_{i+j=l-k-1} \frac{EF_{i,j}}{i!\,j!}\, x_1^i\, x_2^j \in \Lambda_\delta^{l-1,\sigma}(\mathcal{D})^\ell, \quad \psi\, Bv - \psi\, \frac{EG_{l-k}^\pm}{(l-k)!}\, r^{l-k} \in \Lambda_\delta^{l,\sigma}(\Gamma^\pm)^\ell.$$

Since $\lambda = l-k+1$ is not an eigenvalue of the pencil $A(\lambda)$, there exist homogeneous matrix-valued polynomials $p_{i,j}(x_1,x_2)$ and $q^\pm(x_1,x_2)$ of degree $l-k+1$ such that

$$L(\partial_x)\, p_{i,j} = L(\partial_{x'},0)\, p_{i,j} = \frac{x_1^i\, x_2^j}{i!\,j!}\, I_\ell, \quad B(\partial_x)\, p_{i,j}\big|_{\Gamma^\pm} = 0, \tag{3.8}$$

$$L(\partial_x)\, q^\pm = 0, \quad B(\partial_x)\, q^\pm\big|_{\Gamma^\pm} = \frac{r^{l-k}}{(l-k)!}\, I_\ell, \quad B(\partial_x)\, q^\pm\big|_{\Gamma^\mp} = 0, \tag{3.9}$$

where I_ℓ denotes the $\ell \times \ell$ identity matrix (see [12, Le.2.4]). We set

$$w(x) = \sum_{i+j=l-k-1} p_{i,j}(x')\,(EF_{i,j})(x) + \sum_\pm q^\pm(x')\,(EG_{l-k}^\pm)(x).$$

¿From Lemma 2.7 it follows that $\partial_{x_k} EF_{i,j} \in \Lambda_{\delta+l-k}^{l-1,\sigma}(\mathcal{D})^\ell$ and $\partial_{x_k} EG_{l-k} \in \Lambda_{\delta+l-k}^{l-1,\sigma}(\mathcal{D})^\ell$ for $k = 1,2,3$. This together with (3.8) and (3.9) implies that $\psi L(v-w) \in \Lambda_\delta^{l-1,\sigma}(\mathcal{D})^\ell$ and $\psi B(v-w) \in \Lambda_\delta^{l,\sigma}(\Gamma^\pm)^\ell$. Furthermore, $\psi\,(v-w) \in \Lambda_\delta^{l,\sigma}(\mathcal{D})^\ell$ and $\psi\partial_{x_3}(v-w) \in \Lambda_\delta^{l,\sigma}(\mathcal{D})^\ell$. Applying Lemma 3.6, we conclude that $\phi(v-w) \in \Lambda_\delta^{l+1,\sigma}(\mathcal{D})^\ell$ and, consequently, $\phi u \in C_\delta^{l+1,\sigma}(\mathcal{D})^\ell$. ∎

4 The boundary value problem in a cone

We consider problem (1.2)–(1.4) in the cone (1.1).

Henceforth, $\tilde{J}$ denotes the set of all $j \in J = \{1,\dots,n\}$ such that $M_j \subset \overline{\Gamma}_k$ for at least one $k \in J_0$. Furthermore we set $d_j = 1$ for $j \in J_0$ and $d_j = 0$ for $j \in J_1$.

4.1 A regularity result for the solution

Using Lemmas 3.5 and 3.7, we can prove the following assertion.

Lemma 4.1 *Let* $u \in W^{2,p}_{loc}(\overline{\mathcal{K}}\backslash\mathcal{S})^\ell$ *be a solution of problem* (1.2)–(1.4). *Suppose that* $u \in C^{k,\tau}_{\beta',\vec{\delta}'}(\mathcal{K};\tilde{J})^\ell$, $f \in C^{l-2,\sigma}_{\beta,\vec{\delta}}(\mathcal{K};\tilde{J})^\ell$ *and* $g_j \in C^{l+d_j-1,\sigma}_{\beta,\vec{\delta}}(\Gamma_j;\tilde{J})^\ell$, *where* $\beta' = \beta - l - \sigma + k + \tau$, $\delta'_j = \delta_j - l - \sigma + k + \tau \ge 0$, $l + \sigma > k + \tau$, $0 < \sigma < 1$, $0 < \tau < 1$. *Then* $u \in C^{l,\sigma}_{\beta,\vec{\delta}}(\mathcal{K};\tilde{J})^\ell$.

Proof: We restrict ourselves in the proof to the case $\tilde{J} = \emptyset$. The proof for $\tilde{J} \neq \emptyset$ proceeds analogously.

Let x be an arbitrary point in $\mathcal{K}$, and let M_1 be the nearest edge to x. We introduce the functions $\tilde{u}(\xi) = u(|x|\xi)$, $\tilde{f}(\xi) = f(|x|\xi)$ and $\tilde{g}_j(\xi) = g_j(|x|\xi)$. Since $\tilde{u}$ is a solution of the problem $L(\partial_\xi)\,\tilde{u}(\xi) = |x|^2 \tilde{f}(\xi)$ in $\mathcal{K}$, $B(\partial_\xi)\,\tilde{u}(\xi) = |x|\,\tilde{g}_j(\xi)$ on Γ_j, Lemma 3.7 implies

$$\prod_{j=1}^n r_j(\xi)^{\max(0,\delta_j - l - \sigma + |\alpha|)}\, \big|\partial_\xi^\alpha \tilde{u}(\xi)\big|$$
$$\le c\,\Big(|x|^2\, \|\tilde{f}\|_{C^{l-2,\sigma}_{\beta,\vec{\delta}}(\mathcal{U}_\xi)^\ell} + \sum_{j=1}^n |x|\, \|\tilde{g}_j\|_{C^{l-1,\sigma}_{\beta,\vec{\delta}}(\Gamma_j\cap\overline{\mathcal{U}}_\xi)^\ell} + \|\tilde{u}\|_{C^{k,\tau}_{\beta',\vec{\delta}'}(\mathcal{U}_\xi)^\ell}\Big)$$

for $|\alpha| \le l$ and $|\xi| = 1$, where $\mathcal{U}_\xi = \{\eta \in \mathcal{K}:\ |\eta - \xi| < 3/4\}$ and the norm in $C^{l,\sigma}_{\beta,\vec{\delta}}(\mathcal{U}_\xi)$ is defined by (2.11) with $\mathcal{U}_\xi$ instead of $\mathcal{K}$. Setting $\xi = x/|x|$ and multiplying by $|x|^{\beta-l-\sigma}$, we obtain

$$|x|^{\beta-l-\sigma+|\alpha|} \prod_{j=1}^n r_j(\xi)^{\max(0,\delta_j - l - \sigma + |\alpha|)}\, \big|\partial_x^\alpha u(x)\big|$$
$$\le c\,\Big(\|f\|_{C^{l-2,\sigma}_{\beta,\vec{\delta}}(\mathcal{V}_x)^\ell} + \sum_{j=1}^n \|\tilde{g}_j\|_{C^{l-1,\sigma}_{\beta,\vec{\delta}}(\Gamma_j\cap\overline{\mathcal{V}}_x)^\ell} + \|u\|_{C^{k,\sigma}_{\beta-l-k,\vec{\delta}-(l-k)\vec{1}}(\mathcal{V}_x)^\ell}\Big), \tag{4.1}$$

where $\mathcal{V}_x = \{X \in \mathcal{K}:\ |X - x| < 3|x|/4\}$. Analogously, using Lemma 3.7, we can estimate

$$|x|^{\beta-\delta_1} \frac{\big|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)\big|}{|x-y|^{1+[\delta_1-\sigma]-\delta_1+\sigma}} \quad \text{for } |\alpha| = l - 1 - [\delta_1 - \sigma],\ r_1(y) = r(y),\ |x-y| < |x|/2 \ \text{ and}$$
$$|x|^\beta \prod \Big(\frac{r_j(x)}{|x|}\Big)^{\delta_j} \frac{\big|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)\big|}{|x-y|^\sigma} \quad \text{for } |\alpha| = l,\ |x-y| < r(x)/2$$

by the right-hand side of (4.1). This proves the lemma. ■

Remark 4.1 Lemma 3.7 allows also to prove the following generalization of Lemma 4.1: If $u \in W^{2,p}_{loc}(\overline{\mathcal{K}}\backslash\mathcal{S})^\ell$ is a solution of problem (1.2)–(1.4) such that $\psi u \in C^{k,\tau}_{\beta',\vec{\delta}'}(\mathcal{K};\tilde{J})^\ell$, $\psi f \in C^{l-2,\sigma}_{\beta,\vec{\delta}}(\mathcal{K};\tilde{J})^\ell$ and $\psi g_j \in C^{l+d_j-1,\sigma}_{\beta,\vec{\delta}}(\Gamma_j;\tilde{J})^\ell$, where ϕ, ψ are smooth functions on $\overline{\Omega}$, $\psi = 1$ in a neighborhood of $\operatorname{supp}\phi$, and $\beta, \beta', \vec{\delta}, \vec{\delta'}$ are as in Lemma 4.1, then $\phi u \in C^{l,\sigma}_{\beta,\vec{\delta}}(\mathcal{K};\tilde{J})^\ell$.

4.2 Operator pencils generated by the boundary value problem

We introduce the following operator pencils $\mathfrak{A}$ and A_j.

1. Let $\mathcal{H}_\Omega = \{u \in W^{1,2}(\Omega)^\ell:\ u = 0 \text{ on } \gamma_j \text{ for } j \in J_0\}$ and

$$a(u,v;\lambda) = \frac{1}{\log 2} \int\limits_{\substack{\mathcal{K}\\ 1<|x|<2}} \sum_{i,j=1}^3 A_{i,j}\partial_{x_i} U \cdot \partial_{x_j} \overline{V}\, dx,$$

where $U(x) = \rho^\lambda(\omega)$, $V(x) = \rho^{-1-\overline{\lambda}} v(\omega)$, $u, v \in \mathcal{H}_\Omega$, and $\lambda \in \mathbb{C}$. Then the operator $\mathfrak{A}(\lambda):\ \mathcal{H}_\Omega \to \mathcal{H}^*_\Omega$ is defined by

$$\big(\mathfrak{A}(\lambda)u, v\big)_\Omega = a(u,v;\lambda), \quad u, v \in \mathcal{H}_\Omega\,.$$

Here $(\cdot,\cdot)_\Omega$ denotes the extension of the L_2 scalar product to $\mathcal{H}^*_\Omega \times \mathcal{H}_\Omega$.

2. Let Γ_{j_+}, Γ_{j_-} be the faces of $\mathcal{K}$ adjacent to the edge M_j. We introduce new Cartesian coordinates $y = (y_1, y_2, y_3)$ such that M_j coincides with the positive y_3-axis and Γ_{j_+}, Γ_{j_-} are contained in the half-planes $\{y \in \mathbb{R}^3 : \varphi = 0\}$ and $\{y \in \mathbb{R}^3 : \varphi = \theta_j\}$, respectively, where r, φ are the polar coordinates in the (y_1, y_2)-plane. Furthermore, we define the operators $\mathcal{L}_j(\lambda)$ and $\mathcal{B}_{j_\pm}(\lambda)$ on the Sobolev space $W^{2,2}(0,\theta_j)^\ell$ by

$$\mathcal{L}_j(\lambda)\, u(\varphi) = r^{2-\lambda}\, L\big(r^\lambda u(\varphi)\big), \quad \mathcal{B}_{j_\pm}(\lambda)\, u(\varphi) = \left\{ \begin{array}{cl} u(\varphi) & \text{if } j_\pm \in J_0, \\ r^{1-\lambda}\, B\big(r^\lambda u(\varphi)\big) & \text{if } j_\pm \in J_1. \end{array} \right.$$

By $A_j(\lambda)$ we denote the operator

$$W^2(0,\theta_j)^\ell \ni u \to \Big(\mathcal{L}_j(\lambda)u\, ,\, \mathcal{B}_{j_+}(\lambda)\, u(\varphi)\,\big|_{\varphi=0}\, ,\, \mathcal{B}_{j_-}(\lambda)\, u(\varphi)\,\big|_{\varphi=\theta_j}\Big) \in L_2(0,\theta_j)^\ell \times \mathbb{C}^\ell \times \mathbb{C}^\ell.$$

As is known, the spectra of the pencils $\mathfrak{A}$ and A_j consist of isolated points, the eigenvalues. We denote by $\lambda_1^{(j)}$ the eigenvalue of the pencil A_j with smallest positive real part and set $\mu_j = \operatorname{Re} \lambda_1^{(j)}$.

4.3 Boundary data on the sides of the cone

We prove analogous assertions to Lemmas 3.1 and 3.2 for boundary conditions on the sides of the cone $\mathcal{K}$.

Lemma 4.2 *Let $g_j \in \Lambda^{l+d_j-1,\sigma}_{\beta,\vec\delta}(\Gamma_j)^\ell$ for $j = 1, \dots, n$, where $0 < \sigma < 1$, $l \ge 2$ if $J_1 \neq \emptyset$ and $l \ge 1$ else. Then there exists a vector function $u \in \Lambda^{l,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell$ such that $u = g_j$ on Γ_j for $j \in J_0$, $Bu = g_j$ on Γ_j for $j \in J_1$, and*

$$\|u\|_{\Lambda^{l,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell} \le c \sum_{j=1}^{n} \|g_j\|_{\Lambda^{l+d_j-1,\sigma}_{\beta,\vec\delta}(\Gamma_j)^\ell} \tag{4.2}$$

with a constant c independent of g_j, $j = 1, \dots, n$.

Proof: Let ζ_k be smooth functions on $(0,\infty)$ satisfying (3.3). Setting $\zeta_k(x) = \zeta_k(|x|)$ we can consider ζ_k as a function on $\mathcal{K}$. We set $h_{k,j}(x) = \zeta_k(2^k x)\, g_j(2^k x)$ for $j \in J_0$ and $h_{k,j}(x) = 2^k\, \zeta_k(2^k x)\, g_j(2^k x)$ for $j \in J_1$. The support of $h_{k,j}$ is contained in $\{x : \frac12 < |x| < 2\}$. Consequently, by Lemma 3.1, there exists a vector function $w_k \in \Lambda^{l,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell$ such that $w_k(x) = 0$ for $|x| < 1/4$ and $|x| > 4$, $w_k = h_{k,j}$ on Γ_j for $j \in J_0$, $Bw_k = h_{k,j}$ on Γ_j for $j \in J_1$,

$$\|w_k\|_{\Lambda^{l,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell} \le c \sum_{j\in J_0} \|h_{k,j}\|_{\Lambda^{l+d_j-1,\sigma}_{\beta,\vec\delta}(\Gamma_j)^\ell} \tag{4.3}$$

where c is independent of k. From this we conclude that the function $u_k(x) = w_k(2^{-k}x)$ satisfies $u_k = \zeta_k g_j$ on Γ_j for $j \in J_0$, $Bu_k = \zeta_k g_j$ for $j \in J_1$ and the estimate (4.3) with $\zeta_k g_j$ instead of $h_{k,j}$. Thus, $u = \sum u_k$ has the desired properties. ■

Analogously, using Lemma 3.2, one can prove the following result

Lemma 4.3 *Let $g_j \in \Lambda^{2,\sigma}_{\beta,\vec\delta}(\Gamma_j)^\ell$ for $j \in J_0$, $g_j \in C^{1,\sigma}_{\beta,\vec\delta}(\Gamma_j; \tilde J)^\ell$ for $j \in J_1$. For $j \in J\backslash\tilde J$ we assume that $\delta_j \ge \sigma$ and that $\lambda = 1$ is not an eigenvalue of the pencil $A_j(\lambda)$ if $\delta_j < 1 + \sigma$. Then there exists a vector function $u \in C^{2,\sigma}_{\beta,\vec\delta}(\mathcal{K}; \tilde J)^\ell$ such that $u = g_j$ on Γ_j for $j \in J_0$, $Bu = g_j$ on Γ_j for $j \in J_1$, and*

$$\|u\|_{C^{2,\sigma}_{\beta,\vec\delta}(\mathcal{K};\tilde J)^\ell} \le c \Big(\sum_{j\in J_0} \|g_j\|_{C^{2,\sigma}_{\beta,\vec\delta}(\Gamma_j)^\ell} + \sum_{j\in J_1} \|g_j\|_{C^{1,\sigma}_{\beta,\vec\delta}(\Gamma_j;\tilde J)^\ell} \Big).$$

4.4 Estimates of Green's matrix

Let κ be a fixed real number such that the line $\mathrm{Re}\,\lambda = -\kappa - 1/2$ is free of eigenvalues of the pencil $\mathfrak{A}$. We denote by $V_\kappa^{1,2}(\mathcal{K})$ the space of all functions u in $\mathcal{K}$ such that $|x|^{\kappa-1}u \in L_2(\mathcal{K})$ and $|x|^\kappa \nabla u \in L_2(\mathcal{K})^3$ According to [12], there exists a unique solution $G(x,\xi)$ of the problem

$$L(\partial_x)\, G(x,\xi) = \delta(x-\xi)\, I_\ell, \quad x,\xi \in \mathcal{K}, \tag{4.4}$$

$$G(x,\xi) = 0, \quad x \in \Gamma_j,\ \xi \in \mathcal{K},\ j \in J_0, \tag{4.5}$$

$$B(\partial_x)\, G(x,\xi) = 0, \quad x \in \Gamma_j,\ \xi \in \mathcal{K},\ j \in J_1 \tag{4.6}$$

(I_ℓ denotes the $\ell \times \ell$ identity matrix) such that the function $x \to \zeta\Big(\frac{|x-\xi|}{r(\xi)}\Big)\, G(x,\xi)$ belongs to the space $V_\kappa^{1,2}(\mathcal{K})^{\ell\times\ell}$ for every fixed $\xi \in \mathcal{K}$ and for every smooth function ζ on $(0,\infty)$, $\zeta(t) = 0$ for $t < \frac12$, $\zeta(t) = 1$ for $t > 1$. We denote by $\Lambda_- < \mathrm{Re}\,\lambda < \Lambda_+$ the widest strip in the complex plane which contains the line $\mathrm{Re}\,\lambda = -\kappa - 1/2$ and is free of eigenvalues of the pencil $\mathfrak{A}$. By [12], Green's function $G(x,\xi)$ satisfies the following estimates:

$$\big|\partial_x^\alpha \partial_\xi^\gamma G(x,\xi)\big| \le c\,|x-\xi|^{-1-|\alpha|-|\gamma|} \quad \text{if } |\xi|/2 < |x| < 2|\xi|,\ |x-\xi| < \min(r(x), r(\xi)), \tag{4.7}$$

$$\big|\partial_x^\alpha \partial_\xi^\gamma G(x,\xi)\big| \le c\,|x-\xi|^{-1-|\alpha|-|\gamma|} \prod_{j=1}^n \Big(\frac{r_j(x)}{|x-\xi|}\Big)^{\delta_{j,\alpha}} \prod_{j=1}^n \Big(\frac{r_j(\xi)}{|x-\xi|}\Big)^{\delta_{j,\gamma}} \tag{4.8}$$
$$\text{if } |\xi|/2 < |x| < 2|\xi|,\ |x-\xi| > \min(r(x), r(\xi)),$$

$$\big|\partial_x^\alpha \partial_\xi^\gamma G(x,\xi)\big| \le c\,|x|^{\Lambda_+-|\alpha|-\varepsilon}\ |\xi|^{-1-\Lambda_+-|\gamma|+\varepsilon} \prod_{j=1}^n \Big(\frac{r_j(x)}{|x|}\Big)^{\delta_{j,\alpha}} \prod_{j=1}^n \Big(\frac{r_j(\xi)}{|\xi|}\Big)^{\delta_{j,\gamma}} \quad \text{if } |x| < |\xi|/2, \tag{4.9}$$

$$\big|\partial_x^\alpha \partial_\xi^\gamma G(x,\xi)\big| \le c\,|x|^{\Lambda_--|\alpha|+\varepsilon}\ |\xi|^{-1-\Lambda_--|\gamma|-\varepsilon} \prod_{j=1}^n \Big(\frac{r_j(x)}{|x|}\Big)^{\delta_{j,\alpha}} \prod_{j=1}^n \Big(\frac{r_j(\xi)}{|\xi|}\Big)^{\delta_{j,\gamma}} \quad \text{if } |x| > 2|\xi|. \tag{4.10}$$

Here $\delta_{j,\alpha} = \mu_j - |\alpha| - \varepsilon$ for $j \in \tilde{J}$ and $\delta_{j,\alpha} = \min(0, \mu_j - |\alpha| - \varepsilon)$ for $j \notin \tilde{J}$ (ε is an arbitrarily small positive number).

Note that there are sharper estimates for the derivatives of $G(x,\xi)$ with respect to $\rho = |x|$ (see [12, Rem.4.2]). In particular,

$$\big|\partial_x^\alpha \partial_\rho^k G(x,\xi)\big| \le c\,|x-\xi|^{-1-|\alpha|-k} \prod_{j=1}^n \Big(\frac{r_j(x)}{|x-\xi|}\Big)^{\delta_{j,\alpha}} \tag{4.11}$$
$$\text{if } |\xi|/2 < |x| < 2|\xi|,\ |x-\xi| > \min(r(x), r(\xi)),$$

$$\big|\partial_x^\alpha \partial_\rho^k G(x,\xi)\big| \le c\,|x|^{\Lambda_+-|\alpha|-k-\varepsilon}\ |\xi|^{-1-\Lambda_++\varepsilon} \prod_{j=1}^n \Big(\frac{r_j(x)}{|x|}\Big)^{\delta_{j,\alpha}} \quad \text{if } |x| < |\xi|/2, \tag{4.12}$$

$$\big|\partial_x^\alpha \partial_\rho^k G(x,\xi)\big| \le c\,|x|^{\Lambda_--|\alpha|-k+\varepsilon}\ |\xi|^{-1-\Lambda_--\varepsilon} \prod_{j=1}^n \Big(\frac{r_j(x)}{|x|}\Big)^{\delta_{j,\alpha}} \quad \text{if } |x| > 2|\xi|. \tag{4.13}$$

Remark 4.2 In some cases, when $\mu_j = 1$, estimates (4.8)–(4.10) can be improved (see [12, Rem.4.3]). Let the following conditions be satisfied for a certain index j:

(i) The strip $0 < \mathrm{Re}\,\lambda < 1$ does not contain eigenvalues of the pencil $A_j(\lambda)$ and $\lambda = 1$ is the only eigenvalue on the line $\mathrm{Re}\,\lambda = 1$.

(ii) The eigenvectors of $A_j(\lambda)$ corresponding to the eigenvalue $\lambda = 1$ are restrictions of linear vector functions to the unit circle, while generalized eigenvectors corresponding to this eigenvalue do not exist.

(iii) The ranks of the matrices $\mathcal{N}\mathcal{A}$ and $\mathcal{N}\mathcal{A}\mathcal{N}^T$, where

$$\mathcal{A} = \begin{pmatrix} A_{1,1} & A_{2,1} & A_{3,1} \\ A_{1,2} & A_{2,2} & A_{3,2} \\ A_{1,3} & A_{2,3} & A_{3,3} \end{pmatrix} \quad \text{and} \quad \mathcal{N} = \begin{pmatrix} n_1^+ I_\ell & n_2^+ I_\ell & n_3^+ I_\ell \\ n_1^- I_\ell & n_2^- I_\ell & n_3^- I_\ell \end{pmatrix}$$

(n^+, n^- are the normal vectors to the faces Γ_{j_+} and Γ_{j_-} adjacent to the edge M_j, I_ℓ denotes the $\ell\times\ell$ identity matrix, and $\mathcal{N}^T$ denotes the transposed matrix of $\mathcal{N}$), coincide.

Then the number $\mu_j = 1$ can be replaced by the real part $\mu_j^{(2)}$ of the first eigenvalue of the pencil $A_j(\lambda)$ on the right of the line $\operatorname{Re}\lambda = 1$.

Note that conditions (i)–(iii) are satisfied, e.g., for the Neumann problem to the Lamé system and in anisotropic elasticity if the angle θ_j at the edge M_j is less than π. Here the matrices $\mathcal{N}\mathcal{A}$ and $\mathcal{N}\mathcal{A}\mathcal{N}^T$ have rank 5.

4.5 Solvability of the Neumann problem

In this subsection we restrict ourselves to the Neumann problem (1.2), (1.4) with $J_1 = \{1,\dots,n\}$. We prove the existence of solutions in the space $C^{2,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell$, where $0<\sigma<1$, β is a real number such that the line $\operatorname{Re}\lambda = 2+\sigma-\beta$ is free of eigenvalues of the pencil $\mathfrak{A}$, and the components δ_j of $\vec\delta$ satisfy the conditions

$$\max(2-\mu_j,0) < \delta_j - \sigma < 2 \quad\text{and}\quad \delta_j-\sigma\neq 1 \quad\text{for } j=1,\dots,n. \tag{4.14}$$

Lemma 4.3 allows us to restrict ourselves to homogeneous boundary conditions.

Representation of the solution by Green's matrix. We introduce the operator S which is defined on $C^{0,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell$ by

$$(Sf)(x) = \int_{\mathcal{K}} G(x,\xi)\cdot f(\xi)\,d\xi, \tag{4.15}$$

where $G(x,\xi)$ is the Green matrix introduced in the foregoing subsection with $\kappa = \beta-\sigma-5/2$. Our goal is to show that S realizes a continuous mapping from $C^{0,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell$ into $C^{2,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell$.

Note that, under the above conditions on $\vec\delta$, we have $C^{0,\sigma}_{\beta,\vec\delta}(\mathcal{K}) = \Lambda^{0,\sigma}_{\beta,\vec\delta}(\mathcal{K})$. Let χ be an arbitrary cut-off function, $\chi(x)=1$ for $|x|<1$, $\chi(x)=0$ for $|x|>2$. Then $\chi f\in W^{0,p}_{\beta'+\varepsilon,\vec\delta'}(\mathcal{K})^\ell$ and $(1-\chi)f\in W^{0,p}_{\beta'-\varepsilon,\vec\delta'}(\mathcal{K})^\ell$ for arbitrary $f\in C^{0,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell$, where $\beta'=\beta-\sigma-3/p$, $\delta'_j=\delta_j-\sigma-2/p+\varepsilon$, $\varepsilon>0$, $p>1$. Here $W^{l,p}_{\beta,\vec\delta}(\mathcal{K})$ is the Sobolev space with the norm

$$\|u\|_{W^{l,p}_{\beta,\vec\delta}(\mathcal{K})} = \Big(\int_{\mathcal{K}}\sum_{|\alpha|\le l}|x|^{p(\beta-l+|\alpha|)}\prod_{j=1}^n\Big(\frac{r_j(x)}{|x|}\Big)^{p\delta_j}\big|\partial_x^\alpha u(x)\big|^p\,dx\Big)^{1/p}.$$

¿From [13, Th.4.1] it follows that $S\chi f\in W^{2,p}_{\beta'+\varepsilon,\vec\delta'}(\mathcal{K})^\ell$ and $S(1-\chi)f\in W^{2,p}_{\beta'-\varepsilon,\vec\delta'}(\mathcal{K})^\ell$ if ε is a sufficiently small positive number. In particular, $Sf\in W^{2,p}_{loc}(\overline{\mathcal{K}}\backslash\mathcal{S})^\ell$.

A weighted L_∞ estimate for the solution.

Lemma 4.4 *Let $f\in C^{0,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell$, where β is such that the line $\operatorname{Re}\lambda=2+\sigma-\beta$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$ and the components of δ satisfy (4.14). Then $u=Sf$ satisfies the estimate*

$$\sup_{x\in\mathcal{K}}|x|^{\beta-\sigma-2}\big|u(x)\big| \le c\,\|f\|_{C^{0,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell}$$

Moreover, if $\delta_j<1+\sigma$, then $\quad\sup\limits_{x\in\mathcal{K}_j}|x|^{\beta-\sigma-1}\big|\nabla u(x)\big|\le c\,\|f\|_{C^{0,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell}.$

Proof: Using (4.10), we obtain

$$\Big|\int\limits_{\substack{\mathcal{K}\\|\xi|<|x|/2}} G(x,\xi)\,f(\xi)\,d\xi\Big| \le c\,\|f\|_{C^{0,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell}\int\limits_{\substack{\mathcal{K}\\|\xi|<|x|/2}}|G(x,\xi)|\,|\xi|^{\sigma-\beta}\prod r_j(\xi/|\xi|)^{\sigma-\delta_j}\,d\xi$$

$$\le c\,|x|^{\Lambda_-+\varepsilon}\,\|f\|_{C^{0,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell}\int\limits_{\substack{\mathcal{K}\\|\xi|<|x|/2}}|\xi|^{\sigma-\beta-1-\Lambda_--\varepsilon}\prod r_j(\xi/|\xi|)^{\sigma-\delta_j}\,d\xi \le c\,|x|^{2+\sigma-\beta}\,\|f\|_{C^{0,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell}.$$

Analogously, this estimate holds for the integration domain $\{\xi \in \mathcal{K} : |\xi| > 2|x|\}$. For the integration domain $\{\xi \in \mathcal{K} : |x|/2 < |\xi| < 2|x|\}$, we obtain, by means of (4.7), (4.8),

$$\Big| \int G(x,\xi)\, f(\xi)\, d\xi \Big| \le c\, \|f\|_{C^{0,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^\ell} \int |x-\xi|^{-1} |\xi|^{\sigma-\beta} \prod r_j(\xi/|\xi|)^{\sigma-\delta_j}\, d\xi \le c\, |x|^{\sigma-\beta+2}\, \|f\|_{C^{0,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^\ell}.$$

This proves the first part of the lemma. The proof of the second part proceeds analogously. Here one can make use of the fact that $\mu_j > 1$ if $\delta_j < 1+\sigma$. Consequently, it follows from (4.7), (4.8) that $|\partial_x^\alpha G(x,\xi)| \le c\, |x-\xi|^{-2}$ for $|x|/2 < |\xi| < 2|x|$, $|\alpha| = 1$. ■

Two auxiliary inequalities. For the proof of Hölder estimates we need the following lemma.

Lemma 4.5 1) *Let $\alpha + \beta > 3$ and $0 \le \beta < 2$. Then*

$$\int\limits_{\substack{\mathcal{D} \\ |x-\xi|>R}} |x-\xi|^{-\alpha}\, r(\xi)^{-\beta}\, d\xi \le c_{\alpha,\beta}\, R^{3-\alpha-\beta}$$

with a constant $c_{\alpha,\beta}$ independent of x and R.
2) *If $\alpha+\beta < 3$, $\alpha \ge 0$ and $\beta < 2$, then*

$$\int\limits_{\substack{\mathcal{D} \\ |x-\xi|<R}} |x-\xi|^{-\alpha}\, r(\xi)^{-\beta}\, d\xi \le c_{\alpha,\beta}\, R^{3-\alpha-\beta}$$

Proof: 1) Let $\alpha+\beta > 3$, $0 \le \beta \le 2$. Then

$$\int\limits_{\substack{\mathcal{D} \\ r(\xi)>|x-\xi|>R}} |x-\xi|^{-\alpha}\, r(\xi)^{-\beta}\, d\xi \le c \int\limits_{\substack{\mathcal{D} \\ |x-\xi|>R}} |x-\xi|^{-\alpha-\beta}\, d\xi \le c\, R^{3-\alpha-\beta}$$

We denote by x^* and ξ^* the nearest points to x and ξ on the edge M. If $|x-\xi| > r(\xi) > R$, then $|\xi - x^*| \ge r(\xi) > R$ and $|\xi - x^*| \le |\xi - \xi^*| + |\xi^* - x^*| \le r(\xi) + |\xi - x| < 2|\xi - x|$ and, therefore,

$$\int\limits_{\substack{\mathcal{D} \\ |x-\xi|>r(\xi)>R}} |x-\xi|^{-\alpha}\, r(\xi)^{-\beta}\, d\xi \le c \int\limits_{\substack{\mathcal{D} \\ |\xi-x^*|>R}} |\xi-x^*|^{-\alpha}\, r(\xi)^{-\beta}\, d\xi = c\, R^{3-\alpha-\beta}.$$

Finally, since $2|x-\xi| > R + |x_3 - \xi_3|$ for $|x-\xi| > R$, we obtain

$$\int\limits_{\substack{\mathcal{D} \\ |x-\xi|>R>r(\xi)}} |x-\xi|^{-\alpha}\, r(\xi)^{-\beta}\, d\xi \le c \int_{-\infty}^{+\infty} (R + |x_3-\xi_3|)^{-\alpha}\, d\xi_3 \int_0^R r^{1-\beta}\, dr = c' R^{3-\alpha-\beta}.$$

This proves the first part of the lemma.
2) Let $\alpha+\beta < 3$, $\alpha \ge 0$ and $\beta < 2$. Obviously,

$$\int\limits_{\substack{\mathcal{D} \\ |x-\xi|<\min(R,r(\xi))}} |x-\xi|^{-\alpha}\, r(\xi)^{-\beta}\, d\xi \le \int\limits_{\substack{\mathcal{D} \\ |x-\xi|<R}} |x-\xi|^{-\alpha-\beta}\, d\xi = c\, R^{3-\alpha-\beta}\,.$$

We denote again by x^* the nearest point to x on M. Since $|\xi - x^*| < 2|\xi - x|$ for $r(\xi) < |x-\xi|$, we obtain

$$\int\limits_{\substack{\mathcal{D} \\ r(\xi)<|\xi-x|<R}} |x-\xi|^{-\alpha}\, r(\xi)^{-\beta}\, d\xi \le \int\limits_{\substack{\mathcal{D} \\ |\xi-x^*|<2R}} |\xi-x^*|^{-\alpha}\, r(\xi)^{-\beta}\, d\xi = c\, R^{3-\alpha-\beta}.$$

The proof of the lemma is complete. ■

Corollary 4.1 *Let x an arbitrary point of $\mathcal{K}$ such that $r_k(x) < 2\,r(x)$ and let $\mathcal{K}_x = \{\xi \in \mathcal{K} :\ N^{-1}\,|x| < |\xi| < N\,|x|\}$, where N is an arbitrary real number, $N > 1$. If $\alpha+\beta_k > 3$ and $0 \le \beta_j < 2$ for $j = 1,\dots,n$, then*

$$\int\limits_{\substack{\mathcal{K}_x\\ |x-\xi|>R}} |x-\xi|^{-\alpha} \prod_{j=1}^{n} \Big(\frac{r_j(\xi)}{|\xi|}\Big)^{-\beta_j} d\xi \le c\,|x|^{\beta_k} R^{3-\alpha-\beta_k}.$$

If $\alpha + \beta_k < 3$, $\alpha \ge 0$ and $\beta_j \le 2$ for $j = 1,\dots,n$, then

$$\int\limits_{\substack{\mathcal{K}_x\\ |x-\xi|<R}} |x-\xi|^{-\alpha} \prod_{j=1}^{n} \Big(\frac{r_j(\xi)}{|\xi|}\Big)^{-\beta_j} d\xi \le c\,|x|^{\beta_k} R^{3-\alpha-\beta_k}.$$

Here the constant c is independent of x and R.

Proof: Let $\alpha+\beta_k > 3$ and $0 \le \beta_j < 2$, $j = 1,\dots,n$. We set $S_1 = \{\xi \in \mathcal{K}_x :\ |x-\xi| > R,\ r_k(\xi) < 3r(\xi)\}$ and $S_2 = \{\xi \in \mathcal{K}_x :\ |x-\xi| > R,\ r_k(\xi) > 3r(\xi)\}$. If $\xi \in S_1$, then $r_j(\xi) \ge c\,|\xi|$ for $j \ne k$ with a certain $c > 0$. Hence, by means of the substitution $\xi = |x|\,\eta$, we obtain

$$\int\limits_{S_1} |x-\xi|^{-\alpha} \prod_{j=1}^{n} \Big(\frac{r_j(\xi)}{|\xi|}\Big)^{-\beta_j} \le c\,|x|^{3-\alpha} \int\limits_{|x|^{-1}S_1} \Big|\frac{x}{|x|} - \eta\Big|^{-\alpha} \big(r_k(\eta)\big)^{-\beta_k} d\eta.$$

Here $|x|^{-1}S_1 = \{\eta \in \mathcal{K} :\ N^{-1} < |\eta| < N,\ \big|\eta - |x|^{-1}x\big| > R/|x|,\ r_k(\eta) < 2\,r(\eta)\}$. Due to Lemma 4.5, the right-hand side of the last inequality is majorized by $c\,|x|^{\beta_k} R^{3-\alpha-\beta_k}$.

The set S_2 is nonempty only if $R < (N+1)|x|$. Since $|x-\xi| > c\,|x|$ for $r_k(x) < 2\,r(x)$, $r_k(\xi) > 3\,r(\xi)$, we obtain

$$\int\limits_{S_2} |x-\xi|^{-\alpha} \prod_{j=1}^{n} \Big(\frac{r_j(\xi)}{|\xi|}\Big)^{-\beta_j} d\xi \le c\,|x|^{-\alpha} \int\limits_{\substack{\mathcal{K}\\ |x|/N<|\xi|<N\,|x|}} \prod_{j=1}^{n} \Big(\frac{r_j(\xi)}{|\xi|}\Big)^{-\beta_j} d\xi$$
$$= c'\,|x|^{3-\alpha} \le c''\,|x|^{\beta_k} R^{3-\alpha-\beta_k}.$$

This proves the first part of the lemma. The proof of the second part proceeds analogously. ∎

Weighted Hölder estimates. Let $\mathcal{K}_j = \{x \in \mathcal{K} :\ r_j(x) < 3r(x)/2\}$, $j = 1,\dots,n$. We denote by ϕ_j a smooth function on $\overline{\Omega}$ with support in $\overline{\mathcal{K}_j} \cap \overline{\Omega}$ which is extended to $\mathcal{K}$ by $\phi_j(x) = \phi_j(x/|x|)$. Our goal is to show that $\phi_j u \in C^{0,2+\sigma-\delta_j}_{\beta-\delta_j,0}(\mathcal{K})^\ell$ for $\delta_j > \sigma+1$ and $\phi_j u \in C^{1,1+\sigma-\delta_j}_{\beta-\delta_j,0}(\mathcal{K})^\ell$ for $\delta_j < \sigma+1$.

Lemma 4.6 *Let f be as in Lemma 4.4 and let $s = 0$ if $\delta_j > \sigma+1$, $s = 1$ if $\delta_j < \sigma+1$. Then*

$$|x|^{\beta-\delta_j} \frac{|\partial^s_{x_i} u(x) - \partial^s_{y_i} u(y)|}{|x-y|^{2+\sigma-s-\delta_j}} \le c\,\|f\|_{C^{0,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^\ell} \tag{4.16}$$

for $x \in \mathcal{K}_j$, $|x-y| < r_j(x)/2$, where c is independent of x and y.

Proof: First note that $\mu_j > 1$ if $s = 1$. Obviously

$$\big|\partial^s_{x_i} u(x) - \partial^s_{y_i} u(y)\big| \le \|f\|_{C^{0,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^\ell} \int_{\mathcal{K}} \big|\partial^s_{x_i} G(x,\xi) - \partial^s_{y_i} G(y,\xi)\big|\,|\xi|^{\sigma-\beta} \prod_k r_k(\xi/|\xi|)^{\sigma-\delta_k} d\xi \tag{4.17}$$

We consider the integral on the right over the subdomains $\mathcal{K}_x^{(1)} = \{x \in \mathcal{K} :\ |\xi| < |x|/3\}$, $\mathcal{K}_x^{(2)} = \{x \in \mathcal{K} :\ |\xi| > 3|x|\}$, $\mathcal{K}_x^{(3)} = \{x \in \mathcal{K} :\ |x|/3 < |\xi| < 3|x|,\ |x-\xi| < \min(r(x), r(\xi)\}$, and $\mathcal{K}_x^{(4)} = \{x \in \mathcal{K} :\ |x|/3 < |\xi| < 3|x|,\ |x-\xi| > \min(r(x), r(\xi)\}$. By (4.10), we have

$$\big|\partial^s_{x_i} G(x,\xi) - \partial^s_{y_i} G(y,\xi)\big| \le |x-y| \sum_{|\alpha|=s+1} \big|(\partial^\alpha_x G)(\tilde{x},\xi)\big| \le c\,|x-y|\,|\tilde{x}|^{\Lambda_- - s - 1 + \varepsilon}\,|\xi|^{-1-\Lambda_- - \varepsilon} \Big(\frac{r(\tilde{x})}{|\tilde{x}|}\Big)^{\delta_{j,s+1}}$$

for $\xi \in \mathcal{K}_x^{(1)}$, where $\tilde{x}$ is a point on the line between x and y and $\delta_{j,s+1} = \min(0, \mu_j - s - 1 - \varepsilon)$. Using the inequalities $|x|/2 < |\tilde{x}| < 3|x|/2$ and $r(x)/2 < r(\tilde{x}) < 3r(x)/2$, we obtain

$$
\begin{aligned}
&\int_{\mathcal{K}_x^{(1)}} \big|\partial_{x_i}^s G(x,\xi) - \partial_{y_i}^s G(y,\xi)\big|\, |\xi|^{\sigma-\beta} \prod_k r_k(\xi/|\xi|)^{\sigma-\delta_k}\, d\xi \\
&\le c\, |x-y|\, |x|^{\Lambda_- - s - 1 + \varepsilon}\, r(x/|x|)^{\delta_{j,s+1}} \int_{\mathcal{K}_x^{(1)}} |\xi|^{\sigma-\beta-1-\Lambda_- - \varepsilon} \prod_k r_k(\xi/|\xi|)^{\sigma-\delta_k}\, d\xi \\
&\le c\, |x-y|\, |x|^{\sigma-\beta-s+1} r(x/|x|)^{\delta_{j,s+1}} \le c\, |x|^{\delta_j-\beta}\, |x-y|^{2+\sigma-s-\delta_j}. \qquad (4.18)
\end{aligned}
$$

Analogously, this inequality can be proved for $\mathcal{K}_x^{(2)}$ by means of (4.9).

Using (4.7), (4.8) and Corollary 4.1, we obtain

$$
\begin{aligned}
&\int\limits_{\substack{\mathcal{K}_x^{(3)} \cup \mathcal{K}_x^{(4)} \\ |x-\xi|<2|x-y|}} \big|\partial_{x_i}^s G(x,\xi) - \partial_{y_i}^s G(y,\xi)\big|\, |\xi|^{\sigma-\beta} \prod_k r_k(\xi/|\xi|)^{\sigma-\delta_k}\, d\xi \\
&\le c \int\limits_{\substack{\mathcal{K}_x^{(3)} \cup \mathcal{K}_x^{(4)} \\ |x-\xi|<2|x-y|}} \big(|x-\xi|^{-1-s} + |y-\xi|^{-1-s}\big) |\xi|^{\sigma-\beta} \prod_k r_k(\xi/|\xi|)^{\sigma-\delta_k}\, d\xi \\
&\le c\, |x|^{\sigma-\beta} \big(|x|^{\delta_j-\sigma} + |y|^{\delta_j-\sigma}\big)\, |x-y|^{2+\sigma-s-\delta_j} \le c'\, |x|^{\delta_j-\beta}\, |x-y|^{2+\sigma-s-\delta_j}. \qquad (4.19)
\end{aligned}
$$

If $\xi \in \mathcal{K}_x^{(3)}$ and $|x-\xi| > 2|x-y|$, then every point $\tilde{x}$ on the line between x an y satisfies the inequalities $|\tilde{x}-\xi| < 3|x-\xi|/2 < 3r(x)/2 < 3r(\tilde{x})$, and $|\tilde{x}-\xi| > |x-y|$. Consequently, by (4.7) and (4.8,

$$
\big|\partial_{x_i}^s G(x,\xi) - \partial_{y_i}^s G(y,\xi)\big| \le c\, |x-y|\, |\tilde{x}-\xi|^{-2-s}.
$$

Thus, using Corollary 4.1, we obtain

$$
\begin{aligned}
&\int\limits_{\substack{\mathcal{K}_x^{(3)} \\ |x-\xi|>2|x-y|}} \big|\partial_{x_i}^s G(x,\xi) - \partial_{y_i}^s G(y,\xi)\big|\, |\xi|^{\sigma-\beta} \prod_k r_k(\xi/|\xi|)^{\sigma-\delta_k}\, d\xi \\
&\le c\, |x|^{\sigma-\beta}\, |x-y| \int\limits_{\substack{\mathcal{K}_x^{(3)} \\ |\tilde{x}-\xi|>|x-y|}} |\tilde{x}-\xi|^{-2-s} \prod_k r_k(\xi/|\xi|)^{\sigma-\delta_k}\, d\xi \le c\, |x|^{\delta_j-\beta}\, |x-y|^{2+\sigma-s-\delta_j}. \qquad (4.20)
\end{aligned}
$$

Finally, we consider the integral over the set $\{\xi \in \mathcal{K}_x^{(4)} : |x-\xi| > 2|x-y|\}$. By (4.8), we have

$$
\big|\partial_{x_i}^s G(x,\xi) - \partial_{y_i}^s G(y,\xi)\big| \le c\, |x-y|\, |\tilde{x}-\xi|^{-2-s} \Big(\frac{r_j(\tilde{x})}{|\tilde{x}-\xi|}\Big)^{\delta_{j,s+1}},
$$

where again $\tilde{x}$ is a point on the line between x and y. Since $|x-\xi| < |\tilde{x}-\xi|$, we obtain, by means of Corollary 4.1,

$$
\begin{aligned}
&\int\limits_{\substack{\mathcal{K}_x^{(4)} \\ |x-\xi|>2|x-y|}} \big|\partial_{x_i}^s G(x,\xi) - \partial_{y_i}^s G(y,\xi)\big|\, |\xi|^{\sigma-\beta} \prod_k r_k(\xi/|\xi|)^{\sigma-\delta_k}\, d\xi \\
&\le c\, |x|^{\sigma-\beta}\, |x-y|\, r_j(\tilde{x})^{\delta_{j,s+1}} \int\limits_{\substack{\mathcal{K}_x^{(4)} \\ |\tilde{x}-\xi|>|x-y|}} |\tilde{x}-\xi|^{-2-s-\delta_{j,s+1}} \prod_k r_k(\xi/|\xi|)^{\sigma-\delta_k}\, d\xi \\
&\le c\, |x|^{\delta_j-\beta}\, |x-y|^{2+\sigma-s-\delta_j-\delta_{j,s+1}}\, r_j(\tilde{x})^{\delta_{j,s+1}} \le c'\, |x|^{\delta_j-\beta}\, |x-y|^{2+\sigma-s-\delta_j}. \qquad (4.21)
\end{aligned}
$$

Inequality (4.17) together with (4.18)–(4.21) imply the assertion of the lemma. ∎

Lemma 4.7 *Estimate* (4.16) *is valid for* $x \in \mathcal{K}_j$, $y = tx$, $1/2 < t < 3/2$.

Proof: As in the proof of Lemma 4.6, we show that the inequalities (4.18)–(4.21) are satisfied. By (4.13), we have

$$\big|\partial_{x_i}^s G(x,\xi) - \partial_{y_i}^s G(y,\xi)\big| = |x-y|\,\big|(\partial_x^\alpha \partial_\rho G)(\tilde{x},\xi)\big| \le c\,|x-y|\,|\tilde{x}|^{\Lambda_- - s - 1 + \varepsilon}\,|\xi|^{-1-\Lambda_- - \varepsilon}$$

for $\xi \in \mathcal{K}_x^{(1)}$, where $\tilde{x}$ is a point on the line between x and y. This implies (4.18). The proof of (4.19) and (4.20) is the same as in the case $|x-y| < r_j(x)/2$. Finally, we can prove (4.21) by means of the estimate

$$\big|\partial_{x_i}^s G(x,\xi) - \partial_{y_i}^s G(y,\xi)\big| = |x-y|\,\big|(\partial_x^\alpha \partial_\rho G)(\tilde{x},\xi)\big| \le c\,|x-y|\,|\tilde{x}-\xi|^{-2-s}$$

which follows from (4.11). ■

Corollary 4.2 *Estimate* (4.16) *is valid for* $x \in \mathcal{K}_j$, $|x-y| < |x|/2$.

Proof: By Lemma 4.6, inequality (4.16) is valid for $x \in \mathcal{K}_j$, $|x-y| < r_j(x)$. Analogously to the proof of Lemma 2.3, one can conclude from this inequality that the restriction of u to the set $\{x \in \mathcal{K}_j : |x| = \rho\}$ is continuous at $x^* = M_j \cap \{x : |x| = \rho\}$ for any ρ and that (4.16) is valid for $x \in \mathcal{K}_j$, $|x| = |y| = \rho$, $|x-y| < |x|/2$. ¿From this and from Lemma 4.7 it follows that

$$\begin{aligned}\big|\partial_{x_i}^s u(x) - \partial_{y_i}^s u(y)\big| &\le \big|(\partial_{x_i}^s u)(x) - (\partial_{x_i}^s u)(y|x|/|y|)\big| + \big|(\partial_{x_i}^s u)(y|x|/|y|) - (\partial_{x_i}^s u)(y)\big| \\ &\le c\,|x|^{\delta_j - \sigma}|x-y|^{2+\sigma-s-\delta-j}\end{aligned}$$

for $x \in \mathcal{K}_j$, $|x-y| < |x|/2$. The lemma is proved. ■

Existence and uniqueness of solutions in $C^{2,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$.

Theorem 4.1 *Let* $f \in C^{0,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ *and* $g_j \in C^{1,\sigma}_{\beta,\vec{\delta}}(\Gamma_j)^\ell$ *for* $j = 1,\dots,n$, *where* $0 < \sigma < 1$, β *is such that the line* $\operatorname{Re}\lambda = 2 + \sigma - \beta$ *is free of eigenvalues of the pencil* $\mathfrak{A}$, *and the components* δ_j *of* $\vec{\delta}$ *satisfy* (4.14). *Then there exists a unique solution* $u \in C^{2,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ *of problem* (1.2), (1.4).

Proof: Due to Lemma 4.3, we may assume, without loss of generality, that $g_j = 0$ for $j = 1,\dots,n$. We consider the vector function $u = Sf$, where the operator S is defined by (4.15). Let ϕ_j, ψ_j, $j = 1,\dots,n$ be smooth functions on $\overline{\Omega}$, $\psi_j = 1$ in a neighborhood of $\operatorname{supp}\phi_j$, $\psi_j = 0$ outside $\overline{\mathcal{K}}_j \cap \overline{\Omega}$, $\phi_1 + \cdots \phi_n = 1$. We extend ϕ_j and ψ_j to $\overline{\mathcal{K}}\backslash\{0\}$ by $\phi_j(x) = \phi_j(x/|x|)$, $\psi_j(x) = \psi_j(x/|x|)$. It follows from Lemma 4.4 and Corollary 4.2 that $\psi_j u \in C^{0,2-\sigma-\delta_j}_{\beta-\delta_j,0}(\mathcal{K})^\ell$ if $\delta_j > 1-\sigma$ and $\psi_j u \in C^{1,1-\sigma-\delta_j}_{\beta-\delta_j,0}(\mathcal{K})^\ell$ if $\delta_j < 1-\sigma$. Since $\psi_j L u \in C^{0,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ we conclude from Lemma 4.1 and Remark 4.1 that $\phi_j u \in C^{2,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ for $j = 1,\dots,n$. This implies $u \in C^{2,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$.

We prove the uniqueness of the solution. Suppose that $u \in C^{2,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$, $Lu = 0$ in $\mathcal{K}$, and $Bu = 0$ on Γ_j, $j = 1,\dots,n$. Let χ be a smooth cut-off function on $\overline{\mathcal{K}}$ equal to one for $|x| < 1$ and to zero for $|x| > 2$. Furthermore, let $\beta' = \beta - \sigma - 3/2$ and δ'_j be real numbers such that $\max(0, \delta_j - \sigma - 1) < \delta'_j < 1$. From the inclusion $u \in C^{2,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ it follows that $\chi u \in W^{2,2}_{\beta'+\varepsilon,\vec{\delta}'}(\mathcal{K})^\ell$ and $(1-\chi)u \in W^{2,2}_{\beta'-\varepsilon,\vec{\delta}'}(\mathcal{K})^\ell$, where ε is an arbitrary positive number. Consequently, $L(\chi u) = -L\big((1-\chi)u\big) \in W^{0,2}_{\beta'-\varepsilon,\vec{\delta}'}(\mathcal{K})^\ell$ and $B(\chi u) = -B\big((1-\chi)u\big) \in W^{1/2,2}_{\beta'-\varepsilon,\vec{\delta}'}(\Gamma_j)^\ell$. Applying [12, Th.4.2], we obtain $\chi u \in W^{0,2}_{\beta'-\varepsilon,\vec{\delta}'}(\mathcal{K})^\ell$ if ε is sufficiently small. Hence, $u \in W^{0,2}_{\beta'-\varepsilon,\vec{\delta}'}(\mathcal{K})^\ell$ and [12, Th.4.1] implies $u = 0$. The proof of the theorem is complete. ■

Remark 4.3 Suppose that conditions (i)–(iii) of Remark 4.2 are satisfied for some j. Then in Theorem 4.1 the condition (4.14) for δ_j can be replaced by $\max(2-\mu_j^{(2)}, 0) < \delta_j - \sigma < 2$, $\delta_j - \sigma \ne 1$, where $\mu_j^{(2)}$ is the real part of the first eigenvalue on the right of the line $\operatorname{Re}\lambda = 1$. However, in this case, g^+ and g^- have to satisfy some compatibility conditions on the edge M_j if $\delta_j < 1 + \sigma$ (see Remark 3.1).

4.6 Regularity assertions for the solution of the Neumann problem

Next we will show that the solution in Theorem 4.1 belongs to $C^{l,\sigma'}_{\beta',\vec\delta'}(\mathcal{K})^\ell$ if, additionally to the conditions of Theorem 4.1, $f \in C^{l-2,\sigma'}_{\beta',\vec\delta'}(\mathcal{K})^\ell$ and $g_j \in C^{l-1,\sigma'}_{\beta',\vec\delta'}(\Gamma_j)^\ell$, where l, σ', β' and the components of $\vec\delta'$ satisfy the following conditions:

(i) The closed strip between the lines $\operatorname{Re}\lambda = 2+\sigma-\beta$ and $\operatorname{Re}\lambda = l'+\sigma'-\beta'$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$.

(ii) $\max(l-\mu_j,0) < \delta'_j - \sigma' < l, \quad \delta'_j - \sigma' \neq 1,2,\dots,l-1$ for $j=1,\dots,n$.

If, additionally to the conditions of Theorem 4.1, $f \in C^{0,\sigma'}_{\beta',\vec\delta'}(\mathcal{K})^\ell$ and $g_j \in C^{1,\sigma'}_{\beta',\vec\delta'}(\Gamma_j)^\ell$, where σ', β' and $\vec\delta'$ satisfy (i) and (ii) with $l=2$, then in all estimates in the proof of Theorem 4.1 one can replace σ, β and $\vec\delta$ by σ', β' and $\vec\delta'$. Consequently, the following assertion holds.

Lemma 4.8 *Let $u \in C^{2,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell$ be a solution of problem* (1.2), (1.4), *where $f \in C^{0,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell \cap C^{0,\sigma'}_{\beta',\vec\delta'}(\mathcal{K})^\ell$ and $g_j \in C^{1,\sigma}_{\beta,\vec\delta}(\Gamma_j)^\ell \cap C^{1,\sigma'}_{\beta',\vec\delta'}(\Gamma_j)^\ell$. Suppose that for $\sigma,\beta,\vec\delta$ the conditions of Theorem* 4.1 *are fulfilled, while $\sigma',\beta',\vec\delta'$ satisfy* (i) *and* (ii) *with $l=2$. Then $u \in C^{2,\sigma'}_{\beta',\vec\delta'}(\mathcal{K})^\ell$.*

Analogously, using the estimates in the proof [12, Th.4.1], we obtain the following result.

Lemma 4.9 *Let $u \in C^{2,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell$ be a solution of problem* (1.2), (1.4), *where $f \in C^{0,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell \cap W^{0,p}_{\beta',\vec\delta'}(\mathcal{K})^\ell$ and $g_j \in C^{1,\sigma}_{\beta,\vec\delta}(\Gamma_j)^\ell \cap W^{1-1/p,p}_{\beta',\vec\delta'}(\Gamma_j)^\ell$. Suppose that for $\sigma,\beta,\vec\delta$ the conditions of Theorem* 4.1 *are fulfilled, that the closed strip between the lines $\operatorname{Re}\lambda = 2+\sigma-\beta$ and $\operatorname{Re}\lambda = 2-\beta-3/p$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$, and that $\max(2-\mu_j,0) < \delta'_j + 2/p < 2$ for $j=1,\dots,n$. Then $u \in W^{2,p}_{\beta',\vec\delta'}(\mathcal{K})^\ell$.*

Conversely, under the above assumptions on f and g_j, every solution $u \in W^{2,p}_{\beta',\vec\delta'}(\mathcal{K})^\ell$ belongs to $C^{2,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell$.

The following lemma can be proved analogously to Lemma 4.1 using Lemma 3.8.

Lemma 4.10 *Let $u \in C^{l,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell$ be a solution of the Neumann problem* (1.2), (1.4) *such that $\rho\partial_\rho u \in C^{l,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell$. If $f \in C^{l-1,p}_{\beta+1,\vec\delta}(\mathcal{K})^\ell$, $g_j \in C^{l-1/p,p}_{\beta+1,\vec\delta}(\Gamma_j)^\ell$, and the strip $l+\sigma-\delta \le \operatorname{Re}\lambda \le l+\sigma-\delta+1$ is free of eigenvalues of the pencil $A_j(\lambda)$, $j=1,\dots,n$, then $u \in C^{l+1,p}_{\beta+1,\vec\delta}(\mathcal{K})^\ell$ and*

$$\|u\|_{C^{l+1,\sigma}_{\beta+1,\vec\delta}(\mathcal{K})^\ell} \le c\Big(\sum_{j=0}^{1} \|(\rho\partial_\rho)^j u\|_{C^{l,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell} + \|f\|_{C^{l-1,\sigma}_{\beta+1,\vec\delta}(\mathcal{K})^\ell} + \sum_{j=1}^{n} \|g_j\|_{C^{l,\sigma}_{\beta+1,\vec\delta}(\Gamma_j)^\ell}\Big).$$

Theorem 4.2 *Let $u \in C^{2,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell$ be a solution of problem* (1.2), (1.4), *where $f \in C^{0,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell \cap C^{l-2,\sigma'}_{\beta',\vec\delta'}(\mathcal{K})^\ell$ and $g_j \in C^{1,\sigma}_{\beta,\vec\delta}(\Gamma_j)^\ell \cap C^{l-1,\sigma'}_{\beta',\vec\delta'}(\Gamma_j)^\ell$, $l \ge 2$. We suppose that for $\sigma,\beta,\vec\delta$ the conditions of Theorem* 4.1 *are fulfilled, while $l,\sigma',\beta',\vec\delta'$ satisfy* (i) *and* (ii). *Then $u \in C^{l,\sigma'}_{\beta',\vec\delta'}(\mathcal{K})^\ell$.*

Proof: If $l+\sigma'-\delta'_j < 2$ for $j=1,\dots,n$, then $f \in C^{0,\sigma'}_{\beta'-l+2,\vec\delta'-(l-2)\vec 1}(\mathcal{K})^\ell$, $g_j \in C^{1,\sigma'}_{\beta'-l+2,\vec\delta'-(l-2)\vec 1}(\Gamma_j)^\ell$, where $\max(2-\mu_j,0) < \delta'_j - l + 2 - \sigma' < 2$. Consequently, Lemma 4.8 implies $u \in C^{2,\sigma'}_{\beta'-l+2,\vec\delta'-(l-2)\vec 1}(\mathcal{K})^\ell$. Applying Lemma 4.1, we obtain $u \in C^{l,\sigma'}_{\beta',\vec\delta'}(\mathcal{K})^\ell$.

Suppose that $2 < l+\sigma'-\delta'_j < 3$ for $j=1,\dots,n$. This is only possible if $\mu_j > 2$. Then $f \in C^{0,\sigma'}_{\beta'-l+2,\vec\delta'-(l-2-\varepsilon)\vec 1}(\mathcal{K})^\ell$, $g_j \in C^{1,\sigma'}_{\beta'-l+2,\vec\delta'-(l-2-\varepsilon)\vec 1}(\Gamma_j)^\ell$, where ε is such that $0 < \delta'_j - l + 2 + \varepsilon - \sigma' < 2$.

Consequently, Lemma 4.8 implies $u \in C^{2,\sigma'}_{\beta'-l+2,\vec\delta'-(l-2-\varepsilon)\vec 1}(\mathcal{K})^\ell$. From this and from Lemma 4.1 we conclude that $u \in C^{3,\sigma'}_{\beta'-l+3,\vec\delta'-(l-3-\varepsilon)\vec 1}(\mathcal{K})^\ell$ and, therefore, $\rho\partial_\rho u \in C^{2,\sigma'}_{\beta'-l+2,\vec\delta'-(l-3-\varepsilon)\vec 1}(\mathcal{K})^\ell$. Furthermore,

$$L\rho\partial_\rho u = \rho\partial_\rho f + 2f \in C^{0,\sigma'}_{\beta'-l+2,\vec\delta'-(l-3)\vec 1}(\mathcal{K})^\ell, \quad B\rho\partial_\rho u\big|_{\Gamma_j} = \rho\partial_\rho g_j + g_j \in C^{1,\sigma'}_{\beta'-l+2,\vec\delta'-(l-3)\vec 1}(\Gamma_j)^\ell,$$

where $0 < \delta'_j - l + 3 - \sigma' < 1$ for $j = 1,\dots,n$. Consequently, by Theorem 4.8, we have $\rho\partial_\rho u \in C^{2,\sigma'}_{\beta'-l+2,\vec\delta'-(l-3)\vec 1}(\mathcal{K})^\ell$. Since u belongs to the same space, Lemma 4.10 implies $u \in C^{3,\sigma'}_{\beta'-l+3,\vec\delta'-(l-3)\vec 1}(\mathcal{K})^\ell$. Using again Lemma 4.1, we get $u \in C^{l,\sigma'}_{\beta',\vec\delta'}(\mathcal{K})^\ell$. Thus, the theorem is proved for $2 < l + \sigma' - \delta'_j < 3$.

We prove by induction in k that the assertion of the lemma is true if $k - 1 < l + \sigma - \delta'_j < k$, where k is an integer, $k \ge 3$. For $k = 3$ this is shown above. Suppose that $k \ge 4$ and that the theorem is proved for $k-2 < l + \sigma' - \delta'_j < k-1$. Since $f \in C^{l-2,\sigma'}_{\beta',\vec\delta'}(\mathcal{K})^\ell \subset C^{l-3,\sigma'}_{\beta'-1,\vec\delta'}(\mathcal{K})^\ell$ and $g_j \in C^{l-2,\sigma'}_{\beta'-1,\vec\delta'}(\Gamma_j)^\ell$, the induction hypothesis implies $u \in C^{l-1,\sigma'}_{\beta'-1,\vec\delta'}(\mathcal{K})^\ell$ and, consequently, $\rho\partial_\rho u \in C^{l-2,\sigma'}_{\beta'-2,\vec\delta'}(\mathcal{K})^\ell$. On the other hand, $L\rho\partial_\rho u = \rho\partial_\rho f + 2f \in C^{l-3,\sigma'}_{\beta'-1,\vec\delta'}(\mathcal{K})^\ell$ and $B\rho\partial_\rho u\big|_{\Gamma_j} = \rho\partial_\rho g_j + g_j \in C^{l-2,\sigma'}_{\beta'-1,\vec\delta'}(\Gamma_j)^\ell$. Hence, by the induction hypothesis, we have $\rho\partial_\rho u \in C^{l-1,\sigma'}_{\beta'-1,\vec\delta'}(\mathcal{K})^\ell$. From this and from Lemma 4.10 we conclude that $u \in C^{l,\sigma'}_{\beta',\vec\delta'}(\mathcal{K})^\ell$.

Finally, we assume that $l + \sigma' - \delta'_j \in [k_j - 1, k_j)$ for $j = 1,\dots,n$ with different $k_j \in \{1,\dots,l\}$. Then let $\psi_1,\dots,\psi_n$ be smooth functions on $\overline{\Omega}$ such that $\psi_j \ge 0$, $\psi_j = 1$ near $M_j \cap S^2$, and $\sum \psi_j = 1$. We extend ψ_j to $\mathcal{K}$ by the equality $\psi_j(x) = \psi_j(x/|x|)$. Then $\partial_x^\alpha \psi_j(x) \le c\,|x|^{-|\alpha|}$. Using the first part of the proof, one can show by induction in l that $\psi_j u \in C^{l,\sigma'}_{\beta',\vec\delta'}(\mathcal{K})^\ell$ for $j = 1,\dots,n$. The proof of the theorem is complete. ■

Remark 4.4 If conditions (i)–(iii) of Remark 4.2 are satisfied for some j, then in the conditions of Theorem 4.2 on $\sigma, \delta_j, \sigma', \delta_j$ and l the number μ_j can be replaced by the real part $\mu_j^{(2)}$ of the first eigenvalue of the pencil $A_j(\lambda)$ on the right of the line $\operatorname{Re}\lambda = 1$. However, if $\delta'_j - \sigma' < l - 1$, then g^+ and g^- must satisfy certain compatibility conditions on the edge M_j.

4.7 Solvability of the Dirichlet and mixed problems in weighted Hölder spaces

We consider problem (1.2)–(1.4) and denote by $\tilde J$ the set of all $j \in J = \{1,\dots,n\}$ such that $M_j \subset \overline{\Gamma}_k$ for at least one $k \in J_0$. Furthermore, we set $d_j = 1$ for $j \in J_0$ and $d_j = 0$ for $j \in J_1$. The following theorem can be proved analogously to Theorem 4.1. In the case $\tilde J = \{1, 2, \dots, n\}$ (when $C^{l,\sigma}_{\beta,\vec\delta}(\mathcal{K};\tilde J) = \Lambda^{l,\sigma}_{\beta,\vec\delta}(\mathcal{K})$), the proof is even easier. Then it suffices to show that the solution of (1.2) with homogeneous boundary conditions (1.3), (1.4) satisfies the estimate

$$\sup_{x\in\mathcal{K}} |x|^{\beta-2-\sigma} \prod_{j=1}^n \Big(\frac{r_j(x)}{|x|}\Big)^{\delta_j-2-\sigma} \big|u(x)\big| \le c\, \|f\|_{\Lambda^{0,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell}$$

and to apply Lemmas 4.1 and 4.2.

Theorem 4.3 *Suppose that there are no eigenvalues of the pencil $\mathfrak{A}(\lambda)$ on the line $\operatorname{Re}\lambda = 2 + \sigma - \beta$ and that the components of $\vec\delta$ satisfy the inequalities*

$$2 - \mu_j < \delta_j - \sigma < 2 \ \text{ for } j \in \tilde J, \qquad \max(2-\mu_j, 0) < \delta_j - \sigma < 2,\ \ \delta_j - \sigma \ne 1 \ \text{ for } j \in J\backslash\tilde J.$$

Then for all $f \in C^{0,\sigma}_{\beta,\vec\delta}(\mathcal{K};\tilde J)^\ell$, $g_j \in C^{1+d_j,\sigma}_{\beta,\vec\delta}(\Gamma_j,\tilde J)^\ell$ there exists a unique solution $u \in C^{2,\sigma}_{\beta,\vec\delta}(\mathcal{K},\tilde J)^\ell$ of problem (1.2)–(1.4).

Furthermore, the following regularity assertions are valid.

Theorem 4.4 *Let $u \in C^{2,\sigma}_{\beta,\vec\delta}(\mathcal{K},\tilde J)^\ell$ be a solution of problem* (1.2)–(1.4)*, where $\beta,\vec\delta$ satisfy the conditions of Theorem* 4.3*. If $f \in C^{l-2,\sigma'}_{\beta',\vec\delta'}(\mathcal{K};\tilde J)^\ell$, $g_j \in C^{l-1+d_j,\sigma'}_{\beta',\vec\delta'}(\Gamma_j,\tilde J)^\ell$, the closed strip between the lines $\operatorname{Re}\lambda = 2+\sigma-\beta$ and $\operatorname{Re}\lambda = l+\sigma'-\beta'$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$, and the components of $\vec\delta'$ satisfy the inequalities*

$$l-\mu_j < \delta'_j - \sigma' < l \ \text{ for } j\in\tilde J, \qquad \max(l-\mu_j,0) < \delta'_j-\sigma' < l,\ \delta'_j-\sigma' \neq 1,2,\dots,l-1 \ \text{ for } j\in J\backslash\tilde J,$$

then $u \in C^{l,\sigma'}_{\beta',\vec\delta'}(\mathcal{K};\tilde J)^\ell$.

4.8 Boundary value problems for the Lamé system

The Dirichlet for the Lamé system. We consider the problem

$$-\mu\Big(\Delta u + (1-2\nu)^{-1}\nabla\nabla\cdot u\Big) = f \ \text{ in } \mathcal{K}, \quad u = g_j \ \text{ on } \Gamma_j,\ j=1,2,\dots,J, \tag{4.22}$$

where μ is the shear modulus and ν is the Poisson ratio. In order to apply Theorems 4.3 and 4.4, we need some information on the eigenvalues of the pencils $A_j(\lambda)$ and $\mathfrak{A}(\lambda)$. Let θ_j be the angle at the edge M_j. Then the eigenvalue $\lambda_1^{(j)}$ with smallest positive real part of $A_j(\lambda)$ is the the smallest positive solution of the equation $(3-4\nu)\sin(\theta_j\lambda) = \pm\lambda\sin\theta_j$ (see, e.g., [6, Sec.3.1]). Note that $1 < \lambda_1^{(j)}(\theta_j) < \pi/\theta_j$ for $\theta_j < \pi$ and $1/2 < \lambda_1^{(j)}(\theta_j) < \pi/\theta_j$ for $\pi < \theta_j < 2\pi$. Furthermore, we mention that the eigenvalues of the pencil $\mathfrak{A}(\lambda)$ lie outside the strip $-1 \le \operatorname{Re}\lambda \le 0$. In the case when Ω is a subset of a half-sphere S^2_+ and $S^2_+\backslash\Omega$ contains a nonempty open set even the strip $-2 \le \operatorname{Re}\lambda \le 1$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$ (see [6, Th.3.5.3, 3.6.1]).

We consider the weak solution of problem (4.22), i.e., a vector function $u \in V_0^{1,2}(\mathcal{K})^3 = W^{1,2}_{0,\vec 0}(\mathcal{K})^3$ satisfying

$$\int_{\mathcal{K}} \sum_{i,j=1}^{3} \sigma_{i,j}(u)\,\varepsilon_{i,j}(v)\,dx = \int_{\mathcal{K}} f\cdot v\,dx + \sum_{j=1}^{n}\int_{\Gamma_j} g_j\cdot v\,dx \ \text{ for all } v\in\mathcal{H}, \quad u = g_j \ \text{ on } \Gamma_j,\ j=1,\dots,n. \tag{4.23}$$

Here $\sigma(u) = \{\sigma_{i,j}(u)\}$ is the stress tensor connected with the strain tensor $\{\varepsilon_{i,j}(u)\} = \{\frac12(\partial_{x_i}u_j + \partial_{x_j}u_i)\}$ by the Hooke law

$$\sigma_{i,j} = 2\mu\left(\frac{\nu}{1-2\nu}\,(\varepsilon_{1,1}+\varepsilon_{2,2}+\varepsilon_{3,3})\,\delta_{i,j} + \varepsilon_{i,j}\right)$$

Theorem 4.5 1) *Suppose that $l-\lambda_1^{(j)} < \delta_j - \sigma < l$ for $j=1,\dots,n$ and that the line $\operatorname{Re}\lambda = l+\sigma-\beta$ does not contain eigenvalues of the pencil $\mathfrak{A}(\lambda)$. Then Problem* (4.22) *has a unique solution $u\in\Lambda^{l,\sigma}_{\beta,\vec\delta}(\mathcal{K})^3$ for arbitrary $f\in\Lambda^{l-2,\sigma}_{\beta,\vec\delta}(\mathcal{K})^3$ and $g_j\in\Lambda^{l-1,\sigma}_{\beta,\vec\delta}(\Gamma_j)^3$.*

2) *Let $u\in V_0^{1,2}(\mathcal{K})^3$ be a solution of problem* (4.23)*, where $f\in\mathcal{H}^*\cap\Lambda^{l-2,\sigma}_{\beta,\vec\delta}(\mathcal{K})^3$ and $g_j\in\Lambda^{l-1,\sigma}_{\beta,\vec\delta}(\Gamma_j)^3$. Suppose that $l-\lambda_1^{(j)} < \delta_j-\sigma < l$ for $j=1,\dots,n$ and that there are no eigenvalues of the pencil $\mathfrak{A}(\lambda)$ in the strip $-1/2 \le \operatorname{Re}\lambda \le l+\sigma-\beta$. Then $u\in\Lambda^{l,\sigma}_{\beta,\vec\delta}(\mathcal{K})^3$.*

Proof: The first part of the theorem follows immediately from Theorems 4.3 and 4.4. We prove the second part for $l=2$. Let $V^{2,2}_{\beta,\vec\delta}(\mathcal{K}) = W^{2,2}_{\beta,\vec\delta}(\mathcal{K})\cap W^{1,2}_{\beta-1,\vec\delta-\vec 1}(\mathcal{K})\cap W^{2,2}_{\beta-2,\vec\delta-2\cdot\vec 1}(\mathcal{K})$ and let $V^{3/2,2}_{\beta,\vec\delta}(\mathcal{K})$ be the corresponding trace space. Furthermore, let χ_+ be a smooth cut-off function on $\overline{\mathcal{K}}$ equal to zero near the vertex of the cone and $\chi_- = 1-\chi_+$. Then $\chi_\pm f \in V^{0,2}_{\beta-\sigma\pm\varepsilon-3/2,\vec\delta'}(\mathcal{K})^3$ and $\chi g_\pm \in V^{3/2,2}_{\beta-\sigma\pm\varepsilon-3/2,\vec\delta'}(\Gamma_j)^3$, where $\delta'_j = \delta_j-\sigma-1+\varepsilon$ for $j=1,\dots,n$ and ε is an arbitrarily small positive number. Using [13, Th.5.5], we conclude that $\chi_\pm u\in V^{2,2}_{\beta-\sigma\pm\varepsilon-3/2,\vec\delta'}(\mathcal{K})^\ell$ Now it suffices to note that a result analogous to Lemma 4.9 is valid for the Dirichlet problem in the spaces $\Lambda^{l,\sigma}_{\beta,\vec\delta}$. This implies $\chi_\pm u\in\Lambda^{2,\sigma}_{\beta,\vec\delta}(\mathcal{K})^\ell$. Thus, the theorem is proved for $l=2$. Applying Theorem 4.4, we obtain the result for $l>2$. ■

For example, for an arbitrary polyhedral cone one can choose $l=2$, $\beta=2$ and $\delta_j=2$ for $j=1,\dots,n$. We conclude from the second part of the last theorem that the weak solution $u\in V_0^{1,2}(\mathcal{K})^3$ belongs to $\Lambda^{2,\sigma}_{2,\vec{2}}(\mathcal{K})^3$ if $f\in\mathcal{H}^*\cap\Lambda^{0,\sigma}_{2,\vec{2}}(\mathcal{K})^3$, $g_j\in\Lambda^{2,\sigma}_{2,\vec{2}}(\Gamma_j)^3$ and σ is sufficiently small. Since $\Lambda^{2,\sigma}_{2,\vec{2}}(\mathcal{K})^3\subset\Lambda^{0,\sigma}_{0,\vec{0}}(\mathcal{K})^3$ this means that u is Hölder continuous on $\overline{\mathcal{K}}$.

If $\mathcal{K}$ is convex, then one can choose, e.g., $l=2$, $\beta=1$ and $\delta_j=1$ for $j=1,\dots,n$. Provided σ is sufficiently small, we obtain $u\in\Lambda^{2,\sigma}_{1,\vec{1}}(\mathcal{K})^3$ if $f\in\mathcal{H}^*\cap\Lambda^{0,\sigma}_{1,\vec{1}}(\mathcal{K})^3$ and $g_j\in\Lambda^{2,\sigma}_{1,\vec{1}}(\Gamma_j)^3$. This means that even the first derivatives of u are Hölder continuous on $\overline{\mathcal{K}}$.

The Neumann problem for the Lamé system. We consider the problem

$$-\mu\Big(\Delta u+\frac{1}{1-2\nu}\nabla\nabla\cdot u\Big)=f\ \text{ in }\mathcal{K},\quad \sigma(u)\,\vec{n}=g_j\ \text{ on }\Gamma_j. \tag{4.24}$$

Again we give some results concerning the spectra of the pencils $A_j(\lambda)$ and $\mathfrak{A}(\lambda)$. If the angle θ_j at the edge M_j is greater than π, then the eigenvalue of the pencil $A_j(\lambda)$ with smallest positive real part is $\lambda_1^{(j)}=\xi_+(\theta_j)/\theta_j$, where $\xi_+(\theta)$ is the smallest positive solution of the equation $\xi^{-1}\sin\xi+\theta^{-1}\sin\theta=0$ (see, e.g. [6, Sect.4.2]). Note that $1/2\le\lambda_1^{(j)}<1$ for $\pi<\theta_j\le 2\pi$. If $\theta_j<\pi$, then the eigenvalues with smallest positive real parts are $\lambda_1^{(j)}=1$ and $\lambda_2^{(j)}=\pi/\theta_j$.

Furthermore, we mention that the eigenvalues of the operator pencil $\mathfrak{A}(\lambda)$ lie outside the strip $-1<\operatorname{Re}\lambda<0$ if the cone $\mathcal{K}$ is Lipschitz. The number $\lambda=0$ is the only eigenvalue on the line $\operatorname{Re}\lambda=0$. The eigenvectors corresponding to the eigenvalue $\lambda=0$ are the constant vectors. Generalized eigenvectors corresponding to this eigenvalue do not exist (see [6, Th.4.3.1]).

By a weak solution of problem (4.24) we mean a vector function $u\in\mathcal{H}=W^{1,2}_{0,\vec{0}}(\mathcal{K})^3$ satisfying

$$\int_{\mathcal{K}}\sum_{i,j=1}^{3}\sigma_{i,j}(u)\,\varepsilon_{i,j}(v)\,dx=\int_{\mathcal{K}}f\cdot v\,dx+\sum_{j=1}^{n}\int_{\Gamma_j}g_j\cdot v\,dx \tag{4.25}$$

for all $v\in\mathcal{H}$. We assume that the right-hand side of (4.25) defines a continuous functional on $\mathcal{H}$. This assumption is satisfied, e.g., for vector functions $f\in C^{0,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^3$ and $g_j\in C^{1,\sigma}_{\beta,\vec{\delta}}(\Gamma_j)^3$ with compact supports if $\beta-\sigma<5/2$ and $\delta_j-\sigma<2$, $j=1,\dots,n$.

Theorem 4.6 1) *Let $f\in C^{l-2,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^3$, $g_j\in C^{l-1,\sigma}_{\beta,\vec{\delta}}(\Gamma_j)^3$, $l\ge 2$. Suppose that the line $\operatorname{Re}\lambda=l+\sigma-\beta$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$ and the components of $\vec{\delta}$ satisfy the inequalities*

$$l-\frac{\xi_+(\theta_j)}{\theta_j}<\delta_j-\sigma<l\ \text{ if }\theta_j>\pi,\quad \max\big(l-\frac{\pi}{\theta_j},0\big)<\delta_j-\sigma<l,\ \delta_j-\sigma\ne 1,2,\dots,l-1\ \text{ for }\theta_j<\pi. \tag{4.26}$$

If $\delta_j-1<l-1$ we assume additionally that the boundary data satisfy the compatibility condition (1.9). *Then there exists a unique solution $u\in C^{l,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^3$ of problem* (4.24).

2) *Suppose that $f\in C^{l-2,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^3$ and $g_j\in C^{l-1,\sigma}_{\beta,\vec{\delta}}(\Gamma_j)^3$, where β is such that the strip $-1/2<\operatorname{Re}\lambda\le 2+\sigma-\beta$ contains at most one eigenvalue, the eigenvalue $\lambda=0$, of the pencil $\mathfrak{A}(\lambda)$ and the components of $\vec{\delta}$ satisfy* (4.26). *Furthermore, we assume that the boundary data satisfy the compatibility condition* (1.9) *if $\delta_j-\sigma<l-1$. Then the weak solution u has the representation $u=c+v$, where c is a constant vector and $v\in C^{l,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^3$.*

Proof: The first part follows from Theorems 4.1, 4.2 and Remark 4.3. The second part can be proved analogously to Theorem 4.5. Let first $l=2$ and let $\chi_\pm$ be the same cut-off functions as in the proof of Theorem 4.5. Then $\chi_\pm f\in W^{0,2}_{\beta-\sigma\pm\varepsilon-3/2,\vec{\delta}'}(\mathcal{K})^3$ and $\chi g_\pm\in W^{1/2,2}_{\beta-\sigma\pm\varepsilon-3/2,\vec{\delta}'}(\Gamma_j)^3$, where $\delta_j'=\delta_j-\sigma-1+\varepsilon$ for $j=1,\dots,n$ and ε is an arbitrarily small positive number. Using [13, Le.5.4,Th.5.3] and the above given property of the pencil $\mathfrak{A}(\lambda)$, we conclude that $\chi_\pm u=c+v_\pm$, where $v_\pm\in W^{2,2}_{\beta-\sigma\pm\varepsilon-3/2,\vec{\delta}'}(\mathcal{K})^\ell$. Lemma 4.9 yields $v_\pm\in C^{2,\sigma}_{\beta,\vec{\delta}}(\mathcal{K})^3$. This implies the desired representation for u in the case $l=2$. Using

Theorem 4.2, we obtain the result for $l > 2$. ■

Let, for example, $\mathcal{K}$ be a Lipschitz cone, $f \in C^{0,\sigma}_{2,\vec{2}}(\mathcal{K})^3$, $g_j \in C^{1,\sigma}_{2,\vec{2}}(\Gamma_j)^3$, where $\sigma \in (0,1)$ is such that $\sigma < \min(\xi_+(\theta_j)/\theta_j)$ (here the minimum is taken over all j with $\theta_j > \pi$) and the strip $0 < \mathrm{Re}\,\lambda \le \sigma$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$. Then there exists a constant vector c such that $u - c \in C^{2,\sigma}_{2,\vec{2}}(\mathcal{K})^3 \subset C^{0,\sigma}_{0,0}(\mathcal{K})^3$. This means, in particular, that u is Hölder continuous in $\overline{\mathcal{K}}$.

5 The problem in a bounded domain

5.1 Formulation of the problem

Let $\mathcal{G}$ be a bounded domain of polyhedral type in $\mathbb{R}^3$. This means that

(i) the boundary $\partial\mathcal{G}$ consists of smooth (of class C^∞) open two-dimensional manifolds Γ_j (the faces of $\mathcal{G}$), $j = 1, \dots, n$, smooth curves M_k (the edges), $k = 1, \dots, m$, and corners $x^{(1)}, \dots, x^{(d)}$,

(ii) for every $\xi \in M_k$ there exist a neighborhood $\mathcal{U}_\xi$ and a diffeomorphism (a C^∞ mapping) κ_ξ which maps $\mathcal{G} \cap \mathcal{U}_\xi$ onto $\mathcal{D}_\xi \cap B_1$, where $\mathcal{D}_\xi$ is a dihedron of the form $K_\xi \times \mathbb{R}$ with a plane wedge K_ξ and B_1 is the unit ball,

(iii) for every corner $x^{(j)}$ there exist a neighborhood $\mathcal{U}_j$ and a diffeomorphism κ_j mapping $\mathcal{G} \cap \mathcal{U}_j$ onto $\mathcal{K}_j \cap B_1$, where $\mathcal{K}_j$ is a cone with vertex at the origin.

We consider the problem

$$Lu = f \ \text{ in } \mathcal{G}, \quad u = g_j \ \text{ on } \Gamma_j \text{ for } j \in J_0, \quad Bu = g_j \ \text{ on } \Gamma_j \text{ for } j \in J_1, \tag{5.27}$$

where

$$Lu = -\sum_{i,j=1}^{3} \partial_{x_j}\big(A_{i,j}(x)\partial_{x_i} u\big) + \sum_{i=1}^{3} A_i(x)\,\partial_{x_i} u + A_0(x)\,u, \quad Bu = \sum_{i,j=1}^{3} A_{i,j}(x)\, n_j\, \partial_{x_i} u,$$

$J_0 \cup J_1 = \{1, 2, \dots, n\}$, $J_0 \cap J_1 = \emptyset$. The corresponding sesquilinear form is

$$b(u,v) = \int_{\mathcal{G}} \Big(\sum_{i,j=1}^{3} A_{i,j}\partial_{x_i} u \cdot \partial x_j \overline{v} + \sum_{i=1}^{3} A_i \partial_{x_i} u \cdot \overline{v} + A_0 u \cdot \overline{v} \Big)\, dx.$$

Let $\mathcal{H} = \{u \in W^{1,2}(\mathcal{G})^\ell \, : \, u = 0 \text{ on } \Gamma_j \text{ for } j \in J_0\}$, where $W^{1,2}(\mathcal{G})$ denotes the Sobolev space of all functions quadratically summable on $\mathcal{G}$ together with their derivatives of first order. As in the previous sections, we assume that $A_{i,j} = A^*_{j,i}$ for $i, j = 1, 2, 3$. Furthermore, we suppose that

$$|b(u,u)| \ge c_1\, \|u\|^2_{W^{1,2}(\mathcal{G})^\ell} - c_2\, \|u\|^2_{L^2(\mathcal{G})^\ell} \quad \text{for all } u \in \mathcal{H} \tag{5.28}$$

with certain positive constants c_1 and c_2.

5.2 Model problems and corresponding operator pencils

We introduce the operator pencils generated by problem (5.27) for the singular boundary points.

1) Let ξ be an edge point, and let $\Gamma_{j_+}, \Gamma_{j_-}$ be the faces of $\mathcal{G}$ adjacent to ξ. Then by $\mathcal{D}_\xi$ we denote the dihedron which is bounded by the half-planes $\Gamma^\circ_{j_\pm}$ tangential to $\Gamma_{j_\pm}$ at ξ and consider the model problem

$$L^\circ(\xi, \partial_x)\, u = f \ \text{ in } \mathcal{D}_\xi, \quad u = g_{j_\pm} \ \text{ on } \Gamma^\circ_{j_\pm} \text{ for } j_\pm \in J_0, \quad B(\xi, \partial_x)\, u = g_{j_\pm} \ \text{ on } \Gamma^\circ_{j_\pm} \text{ for } j_\pm \in J_1,$$

where

$$L^\circ(\xi, \partial_x) = -\sum_{i,j=1}^{3} A_{i,j}(\xi)\, \partial_{x_i} \partial_{x_j}\,, \quad B(\xi, \partial_x) = \sum_{i,j=1}^{3} A_{i,j}(\xi)\, n_j\, \partial_{x_i}.$$

The operator pencil corresponding to this model problem (see Section 4.2) is denoted by $A_\xi(\lambda)$. Furthermore, we denote by $\lambda_1(\xi)$ the eigenvalue with smallest positive real part of $A_\xi(\lambda)$ and set

$$\mu_j = \inf_{\xi \in M_j} \operatorname{Re} \lambda_1(\xi) \quad \text{for } j = 1, \dots, m.$$

2) Let $x^{(k)}$ be a corner of $\mathcal{G}$ and let $J^{(k)}$ be the set of all indices j such that $x^{(k)} \in \overline{\Gamma}_j$. By our assumptions, there exist a neighborhood $\mathcal{U}$ of $x^{(k)}$ and a diffeomorphism κ mapping $\mathcal{G} \cap \mathcal{U}$ onto $\mathcal{K} \cap B_1$ and $\Gamma_j \cap \mathcal{U}$ onto $\Gamma_j^\circ \cap B_1$ for $j \in J^{(k)}$, where $\mathcal{K}$ is a polyhedral cone with vertex 0 and Γ_j° are the faces of this cone. Without loss of generality, we may assume that the Jacobian matrix $\kappa'(x)$ coincides with the identity matrix I at $x^{(k)}$. We consider the model problem

$$L^\circ(x^{(k)}, \partial_x)\, u = f \quad \text{in } \mathcal{K}, \qquad u = g_j \quad \text{on } \Gamma_j^\circ \text{ for } j \in J_0^{(k)}, \qquad B(x^{(k)}, \partial_x)\, u = g_j \quad \text{on } \Gamma_j^\circ \text{ for } j \in J_1^{(k)},$$

where $J_0^{(k)} = J_0 \cap J^{(k)}$, $J_1^{(k)} = J_1^{(k)}$. The operator pencil generated by this model problem (see Section 4.2) is denoted by $\mathfrak{A}_k(\lambda)$.

5.3 Smoothness of weak solutions

We introduce the weighted Hölder space $C^{l,\sigma}_{\vec\beta,\vec\delta}(\mathcal{G};\tilde J)$, where $\tilde J$ denotes the set of all $j = 1, \dots, m$ such that $M_j \subset \overline{\Gamma}_k$ for at least one $k \in J_0$ (i.e. the Dirichlet condition is given on at least one face Γ_k adjacent to the the edge M_j), l is a nonnegative integer, $0 < \sigma < 1$, $\vec\beta = (\beta_1, \dots, \beta_d) \in \mathbb{R}^d$, $\vec\delta = (\delta_1, \dots, \delta_m) \in \mathbb{R}^m$, $\delta_j \ge 0$ for $j \notin \tilde J$. We denote by $r_j(x)$ the distance of x to the edge M_j, by $\rho_k(x)$ the distance to the corner $x^{(k)}$, by $r(x)$ the distance to $\mathcal{S}$ (the set of all edge points and corners), and by $\rho(x)$ the distance to the set $X = \{x^{(1)}, \dots, x^{(d)}\}$. Furthermore, let $\mathcal{G}_{j,k} = \{x \in \mathcal{G} : \, r_j(x) < 3r(x)/2, \ \rho_k(x) < 3\rho(x)/2\}$ and $k_j = [\delta_j - \sigma] + 1$. Then $C^{l,\sigma}_{\vec\beta,\vec\delta}(\mathcal{G};\tilde J)$ is defined as the set of all l times continuously differentiable functions on $\overline{\mathcal{G}}\backslash\mathcal{S}$ with finite norm

$$\begin{aligned}
\|u\|_{C^{l,\sigma}_{\vec\beta,\vec\delta}(\mathcal{G};\tilde J)} \;=\; & \sum_{|\alpha|\le l} \sup_{x\in\mathcal{G}} \prod_{k=1}^d \rho_k(x)^{\beta_k - l - \sigma + |\alpha|} \prod_{j=1}^m \Big(\frac{r_j(x)}{\rho(x)}\Big)^{h_j(\delta_j - l - \sigma + |\alpha|)} \big|\partial_x^\alpha u(x)\big| \\
& + \sum_{1\le j\le m,\, j\notin\tilde J} \ \sum_{k=1}^d \ \sum_{|\alpha| = l - k_j} \ \sup_{\substack{x,y\in\mathcal{G}_{j,k} \\ |x-y|<\rho_k(x)/2}} \rho_k(x)^{\beta_k - \delta_j} \frac{\big|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)\big|}{|x-y|^{k_j+\sigma-\delta_j}} \\
& + \sum_{|\alpha|=l} \ \sup_{|x-y|<r(x)/2} \prod_{k=1}^d \rho_k(x)^{\beta_k} \prod_{j=1}^m \Big(\frac{r_j(x)}{\rho(x)}\Big)^{\delta_j} \frac{\big|\partial_x^\alpha u(x) - \partial_y^\alpha u(y)\big|}{|x-y|^\sigma}\,.
\end{aligned}$$

Here, the functions h_j are defined as $h_j(t) = t$ for $j \in \tilde J$, $h_j(t) = \max(t,0)$ for $j \notin \tilde J$. For $\tilde J = \{1,\dots,m\}$ and $\tilde J = \emptyset$ we will use the notation $C^{l,\sigma}_{\vec\beta,\vec\delta}(\mathcal{G};\{1,\dots,m\}) = \Lambda^{l,\sigma}_{\vec\beta,\vec\delta}(\mathcal{G})$ and $C^{l,\sigma}_{\vec\beta,\vec\delta}(\mathcal{G};\emptyset) = C^{l,\sigma}_{\vec\beta,\vec\delta}(\mathcal{G})$. The trace spaces on Γ_j for $\Lambda^{l,\sigma}_{\vec\beta,\vec\delta}(\mathcal{G})$, $C^{l,\sigma}_{\vec\beta,\vec\delta}(\mathcal{G})$ and $C^{l,\sigma}_{\vec\beta,\vec\delta}(\mathcal{G};\tilde J)$ are denoted by $\Lambda^{l,\sigma}_{\vec\beta,\vec\delta}(\Gamma_j)$, $C^{l,\sigma}_{\vec\beta,\vec\delta}(\Gamma_j)$ and $C^{l,\sigma}_{\vec\beta,\vec\delta}(\Gamma_j;\tilde J)$, respectively.

We consider the solution $u \in \mathcal{H}$ of problem (5.27) with homogeneous Dirichlet conditions (i.e., $g_j = 0$ for $j \in J_0$). This means that u satisfies the equation

$$b(u,v) = \int_{\mathcal{G}} f \cdot \overline{v}\, dx + \sum_{j\in J_1} g_j \cdot \overline{v}\, dx \quad \text{for all } v \in \mathcal{H}. \tag{5.29}$$

Suppose that $f \in C^{l-2,\sigma}_{\vec\beta,\vec\delta}(\mathcal{G};\tilde J)^\ell$, $g_j \in C^{l-1,\sigma}_{\vec\beta,\vec\delta}(\Gamma_j;\tilde J)^\ell$ for $j \in J_1$, where $l \ge 2$, $0 < \sigma < 1$, the strip $-1/2 < \operatorname{Re}\lambda < l + \sigma - \beta_k$ contains no eigenvalues of the pencil $\mathfrak{A}(\lambda)$, $k = 1, \dots, d$, and the components of $\vec\delta$ satisfy the inequalities

$$l - \mu_j < \delta_j - \sigma < l \quad \text{for } j \in \tilde J, \qquad \max(l - \mu_j, 0) < \delta_j - \sigma < l, \ \ \delta_j - \sigma \ne 1, 2, \dots, l-1 \quad \text{for } j \notin \tilde J.$$

Under these conditions, the right-hand side of (5.29) defines a continuous functional on $\mathcal{H}$.

Theorem 5.1 *Under the above assumptions on f and g, the solution $u \in \mathcal{H}$ of problem* (5.29) *belongs to $C^{l-2,\sigma}_{\vec{\beta},\vec{\delta}}(\mathcal{G})^\ell$.*

Proof: We restrict ourselves in the proof to the Neumann problem. The proof for the Dirichlet and mixed problems proceeds analogously. Suppose first that the support of u is contained in a sufficiently small neighborhood $\mathcal{U}$ of the corner $x^{(k)}$. By κ we denote a diffeomorphism mapping $\mathcal{G} \cap \mathcal{U}$ onto $\mathcal{K} \cap \mathcal{V}$, where $\mathcal{K}$ is a cone with sides Γ_j° and vertex at the the origin, and $\mathcal{V}$ is a neighborhood of the origin. We assume that $\kappa'(x^{(k)}) = I$. Then the vector function $\tilde{u}(x) = u(\kappa^{-1}(x))$ is a solution of the problem

$$\tilde{L}(x, \partial_x)\, \tilde{u} = \tilde{f} \ \text{ in } \mathcal{K}, \quad \tilde{B}(x, \partial_x)\, \tilde{u} = \tilde{g}_j \ \text{ on } \Gamma_j^\circ,\ j \in J^{(k)},$$

where $\tilde{f}(x) = f(\kappa^{-1}(x))$, $\tilde{g}_j(x) = g_j(\kappa^{-1}(x))$, and $\tilde{L}, \tilde{B}$ are differential operators of second and first order, respectively. Here the principal part $\tilde{L}^\circ(0, \partial_x)$ of $\tilde{L}$ with coefficients frozen at the origin coincides with $L^\circ(x^{(k)}, \partial_x)$. Analogously, $\tilde{B}(0, \partial_x) = B(x^{(k)}, \partial_x)$. ¿From the inclusions $\tilde{f} \in C^{l-2,\sigma}_{\beta_k,\vec{\delta}}(\mathcal{K})^\ell$, $\tilde{g}_j \in C^{l-1,\sigma}_{\beta_k,\vec{\delta}}(\Gamma_j^\circ)^\ell$ it follows that $\tilde{f} \in W^{l-2,2}_{\beta'_k,\vec{\delta}'}(\mathcal{K})^\ell$, $\tilde{g}_j \in W^{l-3/2,3}_{\beta'_k,\vec{\delta}'}(\Gamma_j^\circ)^\ell$, where $\beta'_k = \beta - \sigma + \varepsilon - 3/2$, $\delta'_j = \delta_j - \sigma - 1 + \varepsilon$, ε is an arbitrarily small positive number. Hence, by [13, Th.7.1], we have $u \in W^{l,2}_{\beta'_k,\vec{\delta}'}(\mathcal{K})^\ell$. Let χ be a smooth cut-off function equal to one near the origin and to zero outside the unit ball, and let $\chi_\varepsilon(x) = \chi(x/\varepsilon)$. We introduce the operators

$$\tilde{L}_\varepsilon = \chi_\varepsilon \tilde{L}(x, \partial_x) + (1 - \chi_\varepsilon)\tilde{L}^\circ(0, \partial_x), \qquad \tilde{B}_\varepsilon = \chi_\varepsilon \tilde{B}(x, \partial_x) + (1 - \chi_\varepsilon)\tilde{B}(0, \partial_x)$$

For a vector function $\tilde{u}$ with sufficiently small support in a neighborhood of the origin we have $\tilde{L}\tilde{u} = \tilde{L}_\varepsilon \tilde{u}$ and $\tilde{B}\tilde{u} = \tilde{B}_\varepsilon \tilde{u}$. Obviously, the operators $\tilde{\mathcal{A}}_\varepsilon = (\tilde{L}_\varepsilon, \tilde{B}_\varepsilon)$ and $\tilde{\mathcal{A}}_0 = \big(\tilde{L}^\circ(0, \partial_x), \tilde{B}(0, \partial_x)\big)$ are continuous operators

$$C^{l,\sigma}_{\beta_k,\vec{\delta}}(\mathcal{K})^\ell \cap W^{l,2}_{\beta'_k,\vec{\delta}'}(\mathcal{K})^\ell \to C^{l-2,\sigma}_{\beta_k,\vec{\delta}}(\mathcal{K})^\ell \cap W^{l-2,2}_{\beta'_k,\vec{\delta}'}(\mathcal{K})^\ell \times \prod \big(C^{l-1,\sigma}_{\beta_k,\vec{\delta}}(\Gamma_j^\circ)^\ell \cap W^{l-3/2,2}_{\beta'_k,\vec{\delta}'}(\Gamma_j^\circ)^\ell\big). \tag{5.30}$$

¿From Theorems 4.1 and 4.2, Lemma 4.9 and [13, Th.4.2,Th.5.3] it follows that $\tilde{\mathcal{A}}_0$ is an isomorphism. Since, the difference $\tilde{\mathcal{A}}_\varepsilon - \tilde{\mathcal{A}}_0$ is small in the operator norm (5.30) for small ε, it follows that $\tilde{\mathcal{A}}_\varepsilon$ is an isomorphism (5.30) if ε is sufficiently small. From this we conclude that $\tilde{u} \in C^{l,\sigma}_{\beta_k,\vec{\delta}}(\mathcal{K})^\ell$. This proves the theorem for solutions u with support in a small neighborhood of the angle $x^{(k)}$. Analogously, this assertion can be proved for the case when the support of u is contained in a small neighborhood of an arbitrary edge point. Using a partition of unity on $\overline{\mathcal{G}}$, we obtain the result for arbitrary $u \in \mathcal{H}$. ∎

Remark 5.1 In the case $J^{(k)} \subset J_1$ the number $\lambda = 0$ is always an eigenvalue of the pencil $\mathfrak{A}_k(\lambda)$. If this is the only eigenvalue in the strip $-1/2 < \mathrm{Re}\,\lambda < l + \sigma - \beta_k$, the eigenvectors are constant and generalized eigenvectors corresponding to $\lambda = 0$ do not exist, then the solution $u \in \mathcal{H}$ has the representation $u = c + v$ in a neighborhood of $x^{(k)}$, where c is a constant vector and $v \in C^{l-2,\sigma}_{\vec{\beta},\vec{\delta}}(\mathcal{G})^\ell$.

Using a result analogous to Lemma 4.2, one can prove the same regularity assertion for the weak solution of problem (5.27) with inhomogeneous Dirichlet conditions and boundary data $g_j \in \Lambda^{l,\sigma}_{\vec{\beta},\vec{\delta}}(\Gamma_j)^\ell$, $j \in J_0$.

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