Contemporary Mathematics

Criteria for Invariance of Convex Bodies for Linear Parabolic Systems

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Dedicated to David Shoikhet on his 60th birthday

Abstract. We consider systems of linear partial differential equations, which contain only second and first derivatives in the x variables and which are uniformly parabolic in the sense of Petrovskiĭ in the layer $\mathbb{R}^n \times [0, T]$. For such systems we obtain necessary and, separately, sufficient conditions for invariance of a convex body. These necessary and sufficient conditions coincide if the coefficients of the system do not depend on t. The above mentioned criterion is formulated as an algebraic condition describing a relation between the geometry of the invariant convex body and coefficients of the system. The criterion is concretized for certain classes of invariant convex sets: polyhedral angles, cylindrical and conical bodies.

1. Main results and background

We consider the Cauchy problem for parabolic systems of the form

$$\frac{\partial \boldsymbol{u}}{\partial t} - \sum_{j,k=1}^{n} \mathcal{A}_{jk}(x,t)\frac{\partial^2 \boldsymbol{u}}{\partial x_j \partial x_k} - \sum_{j=1}^{n} \mathcal{A}_j(x,t)\frac{\partial \boldsymbol{u}}{\partial x_j} = \boldsymbol{0}, \tag{1.1}$$

where $\boldsymbol{u} = (u_1, \dots, u_m)$ and $(x,t) \in \mathbb{R}^{n+1}_T = \mathbb{R}^n \times (0, T]$.

By $\mathfrak{S}$ we denote the closure of an arbitrary convex proper subdomain of $\mathbb{R}^m$. We say that $\mathfrak{S}$ is invariant for system (1.1) in $\mathbb{R}^{n+1}_T$ if any solution $\boldsymbol{u}$ of (1.1), which is continuous and bounded in $\overline{\mathbb{R}^{n+1}_T}$, belongs to $\mathfrak{S}$ under the assumption that $\boldsymbol{u}(\cdot, 0) \in \mathfrak{S}$. Note that the classical maximum modulus principle and the componentwise maximum principle for parabolic and elliptic systems can be obviously interpreted as statements on the invariance of a ball and an orthant, respectively.

In the present paper we are interested in algebraic conditions on the coefficients $\mathcal{A}_{jk}, \mathcal{A}_j$ ensuring the invariance of an arbitrary convex $\mathfrak{S}$.

The notion of invariant set for parabolic and elliptic systems and the first results concerning these sets appeared in the paper by Weinberger [**26**]. Nowadays, there

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exists a large literature on invariant sets for nonlinear parabolic and elliptic systems with principal part subjected to various structural conditions such as scalarity, diagonality and others (see, for example, Alikakos [**1, 2**], Amann [**3**], Bates [**4**], Bebernes and Schmitt [**6**], Bebernes, Chueh and Fulks [**5**], Chueh, Conley and Smoller [**7**], Conway, Hoff and Smoller [**8**], Cosner and Schaefer [**9**], Kuiper [**15**], Lemmert [**16**], Redheffer and Walter [**20, 21**], Schaefer [**23**], Smoller [**24**], Weinberger [**27**] and references there).

We note that maximum principles for weakly coupled parabolic systems are discussed in the books by Protter and Weinberger [**19**], and Walter [**25**] which also contain rich bibliographies on this subject. The criteria on validity of the componentwise maximum principle for linear parabolic system of the general form in $\mathbb{R}^{n+1}_T$ were obtained in the paper by Otsuka [**18**]. In our papers [**11**]-[**13**] and [**17**] (see also monograph [**14**] and references therein) the criteria for validity of other type of maximum principles for parabolic systems were established, which are interpreted as conditions for the invariance of compact convex bodies.

Henceforth we assume:

(i) real $(m\times m)$-matrix-valued functions $\mathcal{A}_{jk}=\mathcal{A}_{kj}$ and $\mathcal{A}_j$ are defined in $\overline{\mathbb{R}^{n+1}_T}$ and have continuous and bounded derivatives in x up to the second and first order, respectively, which satisfy the uniform Hölder condition on $\overline{\mathbb{R}^{n+1}_T}$ with exponent $\alpha\in(0,1]$ with respect to the parabolic distance $\big(|x-x'|^2+|t-t'|\big)^{1/2}$;

(ii) system (1.1) is uniformly parabolic in the sense of Petrovskiĭ in $\overline{\mathbb{R}^{n+1}_T}$, i.e., for any point $(x,t)\in\overline{\mathbb{R}^{n+1}_T}$, the real parts of the λ-roots of the equation $\det\Big(\sum_{j,k=1}^n \mathcal{A}_{jk}(x,t)\sigma_j\sigma_k+\lambda I\Big)=0$ satisfy the inequality $\mathrm{Re}\ \lambda(x,t,\boldsymbol{\sigma})\le -\delta|\boldsymbol{\sigma}|^2$, where δ=const >0, for any $\boldsymbol{\sigma}=(\sigma_1,\dots,\sigma_n)\in\mathbb{R}^n$, I is the identity matrix of order m, and $|\cdot|$ is the Euclidean length of a vector.

The main result of the paper is the following assertion.

Theorem. (i) *Let the unit outward normal $\boldsymbol{\nu}(a)$ to $\partial\mathfrak{S}$ at any point $a\in\partial\mathfrak{S}$ for which it exists, is an eigenvector of all matrices $\mathcal{A}^*_{jk}(x,t)$, $\mathcal{A}^*_j(x,t)$, $1\le j,k\le n$, $(x,t)\in\mathbb{R}^{n+1}_T$. Then $\mathfrak{S}$ is invariant for system* (1.1) *in $\mathbb{R}^{n+1}_T$. Here and henceforth * means passage to the transposed matrix.*

(ii) *Let $\mathfrak{S}$ be invariant for system* (1.1) *in $\mathbb{R}^{n+1}_T$. Then the unit outward normal $\boldsymbol{\nu}(a)$ to $\partial\mathfrak{S}$ at any point $a\in\partial\mathfrak{S}$ for which it exists, is an eigenvector of all matrices $\mathcal{A}^*_{jk}(x,0)$, $\mathcal{A}^*_j(x,0)$, $1\le j,k\le n$, $x\in\mathbb{R}^n$.*

We note that this result was obtained in our paper [**13**] for the case of a compact $\mathfrak{S}$ and $\mathcal{A}_j=0, 1\le j\le n$.

If the coefficients of the system do not depend on t, the theorem just formulated contains the following exhaustive criterion of the invariance of $\mathfrak{S}$.

Corollary. *A convex body $\mathfrak{S}$ is invariant for parabolic system*

$$\frac{\partial\boldsymbol{u}}{\partial t}-\sum_{j,k=1}^n \mathcal{A}_{jk}(x)\frac{\partial^2\boldsymbol{u}}{\partial x_j\partial x_k}-\sum_{j=1}^n \mathcal{A}_j(x)\frac{\partial\boldsymbol{u}}{\partial x_j}=\boldsymbol{0} \tag{1.2}$$

in $\mathbb{R}^{n+1}_T$ if and only if the unit outward normal $\boldsymbol{\nu}(a)$ to $\partial\mathfrak{S}$ at any point $a\in\partial\mathfrak{S}$ for which it exists, is an eigenvector of all matrices $\mathcal{A}^*_{jk}(x)$, $\mathcal{A}^*_j(x)$, $1\le j,k\le n$, $x\in\mathbb{R}^n$.

We give four examples of invariant convex bodies $\mathfrak{S}$ to show usefulness of this corollary. Here conditions on the coefficients are quite explicit.

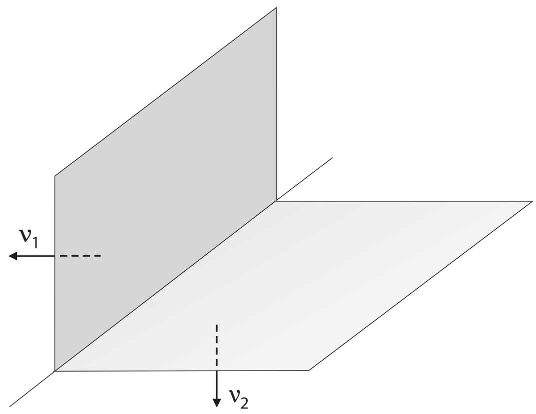

FIGURE 1. Invariant dihedral angle in $\mathbf{R}^3$. Nondiagonal elements of the second and third rows of all (3×3)-matrix-valued functions $\mathcal{A}_{jk}$ and $\mathcal{A}_j$ are equal to zero.

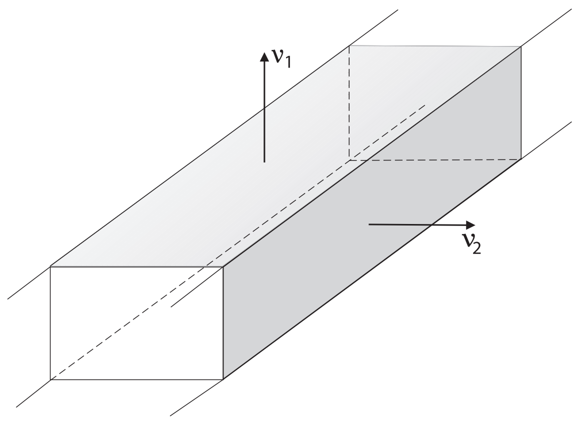

FIGURE 2. Invariant rectangular cylinder in $\mathbf{R}^3$. Nondiagonal elements of the second and third rows of all (3×3)-matrix-valued functions $\mathcal{A}_{jk}$ and $\mathcal{A}_j$ are equal to zero.

The next two examples concern the case of invariant cones. Here conditions on the coefficients are different for polyhedral and smooth cones.

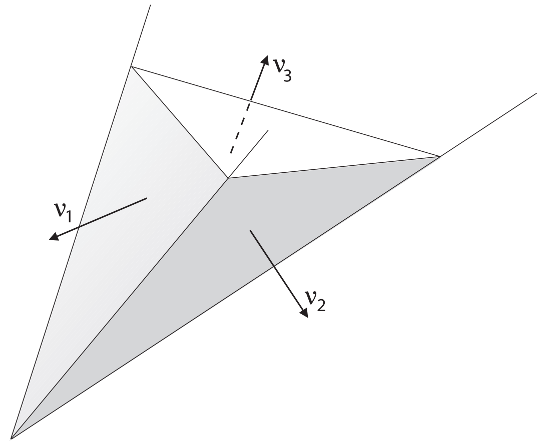

FIGURE 3. Invariant polyhedral cone in $\mathbf{R}^3$ with three facets. All (3×3)-matrix-valued functions $\mathcal{A}_{jk}$ and $\mathcal{A}_j$ are similar to diagonal with the transforming matrix $[\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \boldsymbol{\nu}_3]^*$.

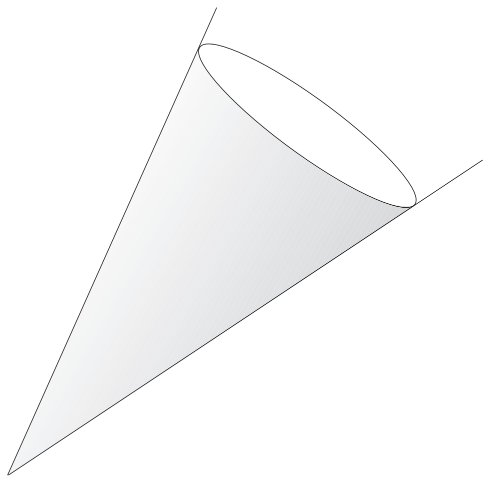

FIGURE 4. Invariant cone in $\mathbf{R}^3$ with a smooth guide. All (3×3)-matrix-valued functions $\mathcal{A}_{jk}$ and $\mathcal{A}_j$ are scalar.

We note that the conditions of smoothness of the coefficients of system (1.1) in Theorem can be relaxed but we leave this extension outside the scope of the present paper.

2. Necessary conditions for invariance of a convex body

By $[\mathrm{C_b}(\overline{\mathbb{R}^{n+1}_T})]^m$ we denote the space of continuous and bounded m-component vector-valued functions defined on $\overline{\mathbb{R}^{n+1}_T}$. By $[\mathrm{C}^{(2,1)}(\mathbb{R}^{n+1}_T)]^m$ we mean the space of m-component vector-valued functions on $\mathbb{R}^{n+1}_T$ whose derivatives with respect to x up to the second order and first derivative with respect to t are continuous.

Let $\boldsymbol{\nu}$ be a fixed m-dimensional unit vector, let $\boldsymbol{a}$ be a fixed m-dimensional vector, and let $\mathbb{R}^m_{\boldsymbol{\nu}}(\boldsymbol{a}) = \{\boldsymbol{u} \in \mathbb{R}^m : (\boldsymbol{u} - \boldsymbol{a}, \boldsymbol{\nu}) \leq 0\}$.

For the convex body $\mathfrak{S}$ by $\partial^*\mathfrak{S}$ we mean the set of points $a \in \partial\mathfrak{S}$ for which there exists the unit outward normal $\boldsymbol{\nu}(a)$ to $\partial\mathfrak{S}$. We denote $\mathfrak{N}_{\mathfrak{S}} = \{\boldsymbol{\nu}(a) : a \in \partial^*\mathfrak{S}\}$.

The next assertion contains a necessary condition for the invariance of a convex body for parabolic system (1.1) in $\mathbb{R}^{n+1}_T$.

Proposition 2.1. *Let a convex body $\mathfrak{S}$ be invariant for the system* (1.1) *in $\mathbb{R}^{n+1}_T$. Then there exists a function $g : \mathbb{R}^{n+1}_T \times \mathbb{R}^n \times \mathfrak{N}_{\mathfrak{S}} \to \mathbb{R}$ such that*

$$G^*(t, 0, x, \eta)\boldsymbol{\nu} = g(t, x; \eta; \boldsymbol{\nu})\boldsymbol{\nu} \ , \tag{2.1}$$

where $G(t, \tau, x, \eta)$ is the fundamental matrix of solutions for system (1.1).

Proof. Suppose that $\mathfrak{S}$ is invariant for system (1.1) in $\mathbb{R}^{n+1}_T$. According to Eidel'man [**10**] (Theorem 1.3), there exists a unique vector-valued function in $[\mathrm{C}^{(2,1)}(\mathbb{R}^{n+1}_T)]^m \cap [\mathrm{C_b}(\overline{\mathbb{R}^{n+1}_T})]^m$, which satisfies the Cauchy problem

$$\begin{aligned} &\frac{\partial \boldsymbol{u}}{\partial t} - \sum_{j,k=1}^{n} \mathcal{A}_{jk}(x,t)\frac{\partial^2 \boldsymbol{u}}{\partial x_j \partial x_k} - \sum_{j=1}^{n} \mathcal{A}_j(x,t)\frac{\partial \boldsymbol{u}}{\partial x_j} = \mathbf{0} \quad \text{in } \mathbb{R}^{n+1}_T, \\ &\boldsymbol{u}\big|_{t=0} = \boldsymbol{\psi}, \end{aligned} \tag{2.2}$$

where $\boldsymbol{\psi}$ is a bounded and continuous vector-valued function on $\mathbb{R}^n$. This solution can be represented in the form

$$\boldsymbol{u}(x,t) = \int_{\mathbb{R}^n} G(t, 0, x, \eta)\boldsymbol{\psi}(\eta)d\eta.$$

We fix a point $a \in \partial^*\mathfrak{S}$ and denote $\boldsymbol{\nu}(a)$ by $\boldsymbol{\nu}$. Since

$$\int_{\mathbb{R}^n} G(t, 0, x, \eta)d\eta = I, \tag{2.3}$$

the vector-valued function

$$\boldsymbol{u}_a(x,t) = \boldsymbol{u}(x,t) - \boldsymbol{a} = \int_{\mathbb{R}^n} G(t, 0, x, \eta)\big(\boldsymbol{\psi}(\eta) - \boldsymbol{a}\big)d\eta \tag{2.4}$$

satisfies the Cauchy problem

$$\begin{aligned} &\frac{\partial \boldsymbol{u}_a}{\partial t} - \sum_{j,k=1}^{n} \mathcal{A}_{jk}(x,t)\frac{\partial^2 \boldsymbol{u}_a}{\partial x_j \partial x_k} - \sum_{j=1}^{n} \mathcal{A}_j(x,t)\frac{\partial \boldsymbol{u}_a}{\partial x_j} = \mathbf{0} \quad \text{in } \mathbb{R}^{n+1}_T, \\ &\boldsymbol{u}_a\big|_{t=0} = \boldsymbol{\psi} - \boldsymbol{a}. \end{aligned} \tag{2.5}$$

We fix a point $(x,t) \in \mathbb{R}^{n+1}_T$ and represent $G^*(t, 0, x, \eta)\boldsymbol{\nu}$ as

$$G^*(t, 0, x, \eta)\boldsymbol{\nu} = g(t, x; \eta; \boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{f}(t, x; \eta; \boldsymbol{\nu}) \ , \tag{2.6}$$

where

$$g(t,x;\eta;\boldsymbol{\nu}) = \big(G^*(t,0,x,\eta)\boldsymbol{\nu},\boldsymbol{\nu}\big) \tag{2.7}$$

and

$$\boldsymbol{f}(t,x;\eta;\boldsymbol{\nu}) = G^*(t,0,x,\eta)\boldsymbol{\nu} - \big(G^*(t,0,x,\eta)\boldsymbol{\nu},\boldsymbol{\nu}\big)\boldsymbol{\nu}\ . \tag{2.8}$$

Let us fix a point $(x,t), t>0$. By the boundedness and continuity in η of $G(t,0,x,\eta)$ (see, e.g., Eidel'man [**10**], pp. 72, 93), $\boldsymbol{f}(t,x;\eta;\boldsymbol{\nu})$ is also bounded and continuous in η.

Suppose there exists a set $\mathcal{M}\subset\mathbb{R}^n$, $\text{meas}_n\mathcal{M}>0$, such that for all $\eta\in\mathcal{M}$, the inequality

$$\boldsymbol{f}(t,x;\eta;\boldsymbol{\nu}) \neq \boldsymbol{0} \tag{2.9}$$

holds, and for all $\eta\in\mathbb{R}^n\backslash\mathcal{M}$ the equality $\boldsymbol{f}(t,x;\eta;\boldsymbol{\nu})=\boldsymbol{0}$ is valid.

Further, we set

$$\boldsymbol{\psi}(\eta)-\boldsymbol{a} = \alpha\boldsymbol{f}(t,x;\eta;\boldsymbol{\nu}) - \beta\boldsymbol{\nu}\ , \tag{2.10}$$

where $\alpha,\beta>0$. It follows from (2.8) and (2.10) that

$$(\boldsymbol{\psi}(\eta)-\boldsymbol{a},\boldsymbol{\nu}) = -\beta<0, \qquad |\boldsymbol{\psi}(\eta)-\boldsymbol{a}| = \big(\alpha^2|\boldsymbol{f}(t,x;\eta;\boldsymbol{\nu})|^2+\beta^2\big)^{1/2} \tag{2.11}$$

and

$$(\boldsymbol{\psi}(\eta)-\boldsymbol{a}, G^*(t,0,x,\eta)\boldsymbol{\nu}) = \alpha|\boldsymbol{f}(t,x;\eta;\boldsymbol{\nu})|^2 - \beta\big(G^*(t,0,x,\eta)\boldsymbol{\nu},\boldsymbol{\nu}\big). \tag{2.12}$$

We introduce a Cartesian coordinate system $\mathcal{O}\xi_1\ldots\xi_{m-1}$ in the plane, tangent to $\partial\mathfrak{S}$ with the origin at the point $\mathcal{O}=a$. We direct the axis $\mathcal{O}\xi_m$ along the interior normal to $\partial\mathfrak{S}$. Let $\boldsymbol{e}_1,\ldots,\boldsymbol{e}_m$ denote the coordinate orthonormal basis of this system and let $\xi'=(\xi_1,\ldots,\xi_{m-1})$.

We use the notation

$$\mu = \sup\{|\boldsymbol{f}(t,x;\eta;\boldsymbol{\nu})| : \eta\in\mathbb{R}^n\}.$$

Let $\partial\mathfrak{S}$ be described by the equation $\xi_m=F(\xi')$ in a neighbourhood of $\mathcal{O}$, where F is convex and differentiable at $\mathcal{O}$.

We put $\beta=\max\{F(\xi'):|\xi'|=\alpha\mu\}$. By (2.11),

$$(\boldsymbol{\psi}(\eta)-\boldsymbol{a},\boldsymbol{e}_m)=\beta>0, \quad |\boldsymbol{\psi}(\eta)-\boldsymbol{a}|\leq(\alpha^2\mu^2+\beta^2)^{1/2},$$

which implies $\boldsymbol{\psi}(\eta)\in\mathfrak{S}$ for all $\eta\in\mathbb{R}^n$.

By invariance of $\mathfrak{S}$, this gives

$$\begin{aligned}\big(\boldsymbol{u}_a(x,t),\boldsymbol{\nu}\big) &= \int_{\mathbb{R}^n}\big(G(t,0,x,\eta)\big(\boldsymbol{\psi}(\eta)-\boldsymbol{a}\big),\boldsymbol{\nu}\big)\,d\eta\\ &= \int_{\mathbb{R}^n}(\boldsymbol{\psi}(\eta)-\boldsymbol{a},G^*(t,0,x,\eta)\boldsymbol{\nu})\,d\eta\leq 0\ .\end{aligned} \tag{2.13}$$

Now, by (2.13) and (2.12),

$$0\geq\big(\boldsymbol{u}_a(x,t),\boldsymbol{\nu}\big)=\int_{\mathbb{R}^n}\big[\alpha|\boldsymbol{f}(t,x;\eta;\boldsymbol{\nu})|^2-\beta\big(G^*(t,0,x,\eta)\boldsymbol{\nu},\boldsymbol{\nu}\big)\big]\,d\eta,$$

which along with (2.3) leads to

$$0\geq\big(\boldsymbol{u}_a(x,t),\boldsymbol{\nu}\big)=\alpha\left(\int_{\mathcal{M}}|\boldsymbol{f}(t,x;\eta;\boldsymbol{\nu})|^2d\eta-\frac{\beta}{\alpha}\right). \tag{2.14}$$

By the differentiability of F at $\mathcal{O}$, we have $\beta/\alpha \to 0$ as $\alpha \to 0$. Consequently, one can choose α so small that the second factor on the right-hand side of (2.14) becomes positive, which contradicts the condition $\text{meas}_n \mathcal{M} > 0$. Therefore, $\boldsymbol{f}(t, x; \eta; \boldsymbol{\nu}) = \mathbf{0}$ for almost all $\eta \in \mathbb{R}^n$. This together with (2.8) and the continuity of $G(t, 0, x, \eta)$ in η shows that $\boldsymbol{f}(t, x; \eta; \boldsymbol{\nu}) = \mathbf{0}$ for all $\eta \in \mathbb{R}^n$.

Since $(x, t) \in \mathbb{R}^{n+1}_T$ and $a \in \partial^* \mathfrak{S}$ are arbitrary, we arrive at (2.1) by (2.6). □

We introduce the space $[\mathrm{C}^{k,\alpha}_{\mathrm{b}}(\mathbb{R}^n)]^m$ of m-component vector-valued functions defined in $\mathbb{R}^n$ and having continuous and bounded derivatives up to order k, which satisfy the uniform Hölder condition with exponent $\alpha, 0 < \alpha \le 1$.

By $[\mathrm{C}^{k,\alpha}_{\mathrm{b}}(\overline{\mathbb{R}^{n+1}_T})]^m$ we denote the space of m-component vector-valued functions defined in $\overline{\mathbb{R}^{n+1}_T}$, having continuous and bounded x-derivatives up to order k, which satisfy the uniform Hölder condition with exponent α with respect to the parabolic distance $\big(|x - x'|^2 + |t - t'|\big)^{1/2}$ between the points (x, t) and (x', t') in $\mathbb{R}^{n+1}_T$. For the space of $(m \times m)$-matrix-valued functions, defined on $\overline{\mathbb{R}^{n+1}_T}$ and having similar properties, we use the notation $[\mathrm{C}^{k,\alpha}_{\mathrm{b}}(\overline{\mathbb{R}^{n+1}_T})]^{m\times m}$.

Let

$$\mathfrak{A}(x, t, D_x) = \sum_{j,k=1}^{n} \mathcal{A}_{jk}(x, t) \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^{n} \mathcal{A}_j(x, t) \frac{\partial}{\partial x_j} + \mathcal{A}_0(x, t).$$

We quote the following known assertion (see Eidel'man [**10**], Theorem 5.3), which will be used in the sequel.

Theorem 2.1. *Let $(m \times m)$-matrix valued coefficients $\mathcal{A}_{jk}, \mathcal{A}_j, \mathcal{A}_0$ of the operator $\mathfrak{A}(x, t, D_x)$ belong to $[\mathrm{C}^{0,\alpha}_{\mathrm{b}}(\overline{\mathbb{R}^{n+1}_T})]^{m\times m}$ and let $\boldsymbol{u}_0 \in [\mathrm{C}^{2,\alpha}_{\mathrm{b}}(\mathbb{R}^n)]^m$. Let, further, the system*

$$\frac{\partial \boldsymbol{u}}{\partial t} - \mathfrak{A}(x, t, D_x)\boldsymbol{u} = \mathbf{0},$$

$\boldsymbol{u} = (u_1, \dots, u_m)$, be uniformly parabolic in the sense of Petrovskiĭ in the layer $\overline{\mathbb{R}^{n+1}_T}$ and let $G(t, \tau, x, \eta)$ be its fundamental matrix.

Then the vector-valued function

$$\boldsymbol{u}(x, t) = \int_{\mathbb{R}^n} G(t, 0, x, \eta)\boldsymbol{u}_0(\eta) d\eta$$

belongs to $[\mathrm{C}^{2,\alpha}_{\mathrm{b}}(\overline{\mathbb{R}^{n+1}_T})]^m$ and it is a unique solution of the Cauchy problem

$$\frac{\partial \boldsymbol{u}}{\partial t} - \mathfrak{A}(x, t, D_x)\boldsymbol{u} = \mathbf{0} \ \text{ in } \mathbb{R}^{n+1}_T, \quad \boldsymbol{u}\big|_{t=0} = \boldsymbol{u}_0 \, .$$

The following assertion gives a necessary condition for the invariance of $\mathfrak{S}$ which is formulated in terms of the coefficients of system (1.1). It settles the necessity part of Theorem from Sect. 1.

Proposition 2.2. *Let a convex body $\mathfrak{S}$ be invariant for system* (1.1) *in $\mathbb{R}^{n+1}_T$. Then there exist functions $a_{jk}, \ a_j : \mathbb{R}^n \times \mathfrak{N}_{\mathfrak{S}} \to \mathbb{R}, \ 1 \le j, k \le n$, such that*

$$\mathcal{A}^*_{jk}(x, 0)\boldsymbol{\nu} = a_{jk}(x; \boldsymbol{\nu})\boldsymbol{\nu}, \quad \mathcal{A}^*_j(x, 0)\boldsymbol{\nu} = a_j(x; \boldsymbol{\nu})\boldsymbol{\nu}.$$

PROOF. Suppose that $\mathfrak{S}$ is invariant for system (1.1) in $\mathbb{R}^{n+1}_T$. We fix a point $a \in \partial^* \mathfrak{S}$ and denote $\boldsymbol{\nu}(a)$ by $\boldsymbol{\nu}$. Let the function $\boldsymbol{\psi}$ in (2.5) is defined by

$$\boldsymbol{\psi}(x) = \boldsymbol{a} + \left(\sum_{j,k=1}^{n} \alpha_{jk}(x_j - y_j)(x_k - y_k) + \sum_{j=1}^{n} \beta_j (x_j - y_j) \right) \zeta_r(x-y)\boldsymbol{\tau}, \tag{2.15}$$

where α_{jk}, β_j are constants, y is a fixed point in $\mathbb{R}^n$, $\zeta_r \in \mathrm{C}_0^\infty(\mathbb{R}^n)$, $0 \le \zeta_r(x) \le 1$, $\zeta_r(x) = 1$ for $|x| \le r/2$ and $\zeta_r(x) = 0$ for $|x| \ge r$, $\boldsymbol{\tau}$ is a unit m-dimensional vector which is orthogonal to $\boldsymbol{\nu}$.

It follows from (2.4) and Proposition 2.1 that

$$\begin{aligned} \big(\boldsymbol{u}_a(x,t), \boldsymbol{\nu}\big) &= \int_{\mathbb{R}^n} (\boldsymbol{\psi}(\eta) - \boldsymbol{a}, G^*(t,0,x,\eta)\boldsymbol{\nu})\, d\eta \\ &= \int_{\mathbb{R}^n} g(t,x;\eta;\boldsymbol{\nu})\, (\boldsymbol{\psi}(\eta) - \boldsymbol{a},\ \boldsymbol{\nu})\, d\eta \ , \end{aligned}$$

which, by (2.15), gives $\big(\boldsymbol{u}_a(x,t), \boldsymbol{\nu}\big) = 0$. This and (2.5) imply

$$\sum_{j,k=1}^{n} \left(\frac{\partial^2 \boldsymbol{u}_a}{\partial x_j \partial x_k},\ \mathcal{A}^*_{jk}(x,t)\boldsymbol{\nu} \right) + \sum_{j=1}^{n} \left(\frac{\partial \boldsymbol{u}_a}{\partial x_j},\ \mathcal{A}_j(x,t)^*\boldsymbol{\nu} \right) = \boldsymbol{0} \ .$$

By Theorem 2.1, we pass to the limit as $t \to 0$ to obtain

$$\sum_{j,k=1}^{n} \left(\frac{\partial^2 \boldsymbol{\psi}_a}{\partial x_j \partial x_k},\ \mathcal{A}^*_{jk}(x,0)\boldsymbol{\nu} \right) + \sum_{j=1}^{n} \left(\frac{\partial \boldsymbol{\psi}_a}{\partial x_j},\ \mathcal{A}^*_j(x,0)\boldsymbol{\nu} \right) = \boldsymbol{0} \ , \tag{2.16}$$

where $\boldsymbol{\psi}_a(x) = \boldsymbol{\psi}(x) - \boldsymbol{a}$. Now, (2.15) leads to

$$\frac{\partial^2 \boldsymbol{\psi}_a}{\partial x_j \partial x_k}\Big|_{x=y} = \alpha_{jk}\boldsymbol{\tau} \ , \qquad \frac{\partial \boldsymbol{\psi}_a}{\partial x_j}\Big|_{x=y} = \beta_j \boldsymbol{\tau} \ .$$

Then, by (2.16),

$$\sum_{j,k=1}^{n} \alpha_{jk} \Big(\boldsymbol{\tau},\ \mathcal{A}^*_{jk}(y,0)\boldsymbol{\nu} \Big) + \sum_{j=1}^{n} \beta_j \Big(\boldsymbol{\tau},\ \mathcal{A}^*_j(y,0)\boldsymbol{\nu} \Big) = \boldsymbol{0} \ .$$

Hence, by arbitrariness of α_{jk}, β_j and $\boldsymbol{\tau}$, we arrive at the equalities

$$\mathcal{A}^*_{jk}(y,0)\boldsymbol{\nu} = a_{jk}(y)\boldsymbol{\nu}, \quad \mathcal{A}^*_j(y,0)\boldsymbol{\nu} = a_j(y)\boldsymbol{\nu}, \quad 1 \le j,k \le n$$

with $\boldsymbol{\nu} = \boldsymbol{\nu}(a)$, where $y \in \mathbb{R}^n$ and $a \in \partial^* \mathfrak{S}$ are arbitrary fixed points. The proof is complete. □

3. Sufficient condition for invariance of a convex body

Let $\boldsymbol{\nu}$ be a fixed m-dimensional unit vector and let a stand for a fixed point in $\mathbb{R}^m$.

PROPOSITION 3.1. *Let the equalities*

$$\mathcal{A}^*_{jk}(x,t)\boldsymbol{\nu} = a_{jk}(x,t)\boldsymbol{\nu}, \quad \mathcal{A}^*_j(x,t)\boldsymbol{\nu} = a_j(x,t)\boldsymbol{\nu}, \quad 1 \le j,k \le n, \tag{3.1}$$

hold for all $(x,t) \in \mathbb{R}^{n+1}_T$ *with* $a_{jk},\ a_j : \mathbb{R}^{n+1}_T \to \mathbb{R}$. *Then the half-space* $\mathbb{R}^m_{\boldsymbol{\nu}}(\boldsymbol{a})$ *is an invariant set for system* (1.1) *in* $\mathbb{R}^{n+1}_T$.

Proof. Let $\boldsymbol{u} \in [\mathrm{C_b}(\overline{\mathbb{R}^{n+1}_T})]^m \cap [\mathrm{C}^{(2,1)}(\mathbb{R}^{n+1}_T)]^m$ be a solution of the Cauchy problem (2.2). Then the vector-valued function $\boldsymbol{u}_a = \boldsymbol{u} - \boldsymbol{a}$ is solution of the Cauchy problem (2.5).

Hence,

$$\frac{\partial}{\partial t}(\boldsymbol{u}_a, \boldsymbol{\nu}) - \sum_{j,k=1}^{n}\left(\mathcal{A}_{jk}(x,t)\frac{\partial^2 \boldsymbol{u}_a}{\partial x_j \partial x_k},\ \boldsymbol{\nu}\right) - \sum_{j=1}^{n}\left(\mathcal{A}_{j}(x,t)\frac{\partial \boldsymbol{u}_a}{\partial x_j},\ \boldsymbol{\nu}\right)$$
$$= \frac{\partial}{\partial t}(\boldsymbol{u}_a, \boldsymbol{\nu}) - \sum_{j,k=1}^{n}\left(\frac{\partial^2 \boldsymbol{u}_a}{\partial x_j \partial x_k},\ \mathcal{A}^*_{jk}(x,t)\boldsymbol{\nu}\right) - \sum_{j=1}^{n}\left(\frac{\partial \boldsymbol{u}_a}{\partial x_j \partial x_k},\ \mathcal{A}^*_{j}(x,t)\boldsymbol{\nu}\right) = 0\,.$$

By (3.1) we arrive at

$$\frac{\partial}{\partial t}(\boldsymbol{u}_a, \boldsymbol{\nu}) - \sum_{j,k=1}^{n}\left(\frac{\partial^2 \boldsymbol{u}_a}{\partial x_j \partial x_k},\ a_{jk}(x,t)\boldsymbol{\nu}\right) - \sum_{j=1}^{n}\left(\frac{\partial \boldsymbol{u}_a}{\partial x_j},\ a_{j}(x,t)\boldsymbol{\nu}\right)$$
$$= \frac{\partial}{\partial t}(\boldsymbol{u}_a, \boldsymbol{\nu}) - \sum_{j,k=1}^{n} a_{jk}(x,t)\frac{\partial^2}{\partial x_j \partial x_k}(\boldsymbol{u}_a, \boldsymbol{\nu}) - \sum_{j=1}^{n} a_{j}(x,t)\frac{\partial}{\partial x_j}(\boldsymbol{u}_a, \boldsymbol{\nu}) = 0\,.$$

Thus the function $u_a = (\boldsymbol{u}_a, \boldsymbol{\nu})$ satisfies

$$\frac{\partial u_a}{\partial t} - \sum_{j,k=1}^{n} a_{jk}(x,t)\frac{\partial^2 u_a}{\partial x_j \partial x_k} - \sum_{j=1}^{n} a_{j}(x,t)\frac{\partial u_a}{\partial x_j} = 0 \ \text{ in } \ \mathbb{R}^{n+1}_T,\ u_a\Big|_{t=0} = (\boldsymbol{\psi} - \boldsymbol{a}, \boldsymbol{\nu}).$$

Therefore, by the maximum principle for solutions to the scalar parabolic equation in $\mathbb{R}^{n+1}_T$ with the unknown function u_a, we conclude

$$\inf_{y\in\mathbb{R}^n}\big(\boldsymbol{u}(y,0) - \boldsymbol{a}, \boldsymbol{\nu}\big) \le \big(\boldsymbol{u}(x,t) - \boldsymbol{a}, \boldsymbol{\nu}\big) \le \sup_{y\in\mathbb{R}^n}\big(\boldsymbol{u}(y,0) - \boldsymbol{a}, \boldsymbol{\nu}\big),$$

i.e., the half-space $\mathbb{R}^m_{\boldsymbol{\nu}}(\boldsymbol{a})$ is invariant for system (1.1) in $\mathbb{R}^{n+1}_T$. □

The next assertion results directly from Proposition 3.1 and the known assertion (Rockafellar [**22**], Theorem 18.8):

$$\mathfrak{S} = \bigcap_{a\in\partial^*\mathfrak{S}} \mathbb{R}^m_{\boldsymbol{\nu}(a)}(\boldsymbol{a}). \tag{3.2}$$

Proposition 3.2. *Let $\mathfrak{S}$ be a convex body and let the equalities*

$$\mathcal{A}^*_{jk}(x,t)\boldsymbol{\nu} = a_{jk}(x,t;\boldsymbol{\nu})\boldsymbol{\nu},\quad \mathcal{A}^*_{j}(x,t)\boldsymbol{\nu} = a_{j}(x,t;\boldsymbol{\nu})\boldsymbol{\nu},\quad 1 \le j,k \le n,$$

hold for all $(x,t) \in \mathbb{R}^{n+1}_T$ and $\boldsymbol{\nu} \in \mathfrak{N}_{\mathfrak{S}}$ with $a_{jk},\ a_j : \mathbb{R}^{n+1}_T \times \mathfrak{N}_{\mathfrak{S}} \to \mathbb{R}$.

Then $\mathfrak{S}$ is an invariant for system (1.1) *in $\mathbb{R}^{n+1}_T$.*

Hence, the proof of sufficiency in Theorem from Sect.1 is obtained.

4. Corollaries

Let us introduce a layer $\mathbb{R}^{n+1}_{\tau,T} = \mathbb{R}^n \times (\tau, T]$, where $\tau \in [0, T)$. We say that a convex body $\mathfrak{S}$ is invariant for system (1.1) in $\mathbb{R}^{n+1}_{\tau,T}$, if any solution $\boldsymbol{u}$ of (1.1), which is continuous and bounded in $\overline{\mathbb{R}^{n+1}_{\tau,T}}$, belongs to $\mathfrak{S}$ under the assumption that its initial values $\boldsymbol{u}(\cdot, \tau)$ lie in $\mathfrak{S}$.

Let $\tau \in [0, T)$. Repeating almost word for word all previous proofs replacing $\boldsymbol{u}|_{t=0}$ by $\boldsymbol{u}|_{t=\tau}$, $\mathbb{R}^{n+1}_{0,T}$ by $\mathbb{R}^{n+1}_{\tau,T}$, $G(t, 0, x, \eta)$ by $G(t, \tau, x, \eta)$ and making obvious similar changes, we arrive at the following criterion for the invariance of $\mathfrak{S}$ for the parabolic system (1.1) in any layer $\mathbb{R}^{n+1}_{\tau,T}$ with $\tau \in [0,T)$.

PROPOSITION 4.1. *A convex body $\mathfrak{S}$ is invariant for system* (1.1) *in the layer $\mathbb{R}^{n+1}_{\tau,T}$ for all $\tau \in [0,T)$ simultaneously, if and only if the unit outward normal $\boldsymbol{\nu}(a)$ to $\partial\mathfrak{S}$ at any point $a \in \partial\mathfrak{S}$ for which it exists, is an eigenvector of all matrices $\mathcal{A}^*_{jk}(x,t)$, $\mathcal{A}^*_j(x,t)$, $1 \le j,k \le n$, $(x,t) \in \mathbb{R}^{n+1}_T$.*

All criteria, formulated below, concern invariant convex bodies for system (1.2) in $\mathbb{R}^{n+1}_T$. We note that similar assertions are valid also for system (1.1) in any layer $\mathbb{R}^{n+1}_{\tau,T}$ with $\tau \in [0,T)$.

Polyhedral angles. We introduce a polyhedral angle

$$\mathbf{R}^m_+(\alpha_{m-k+1},\dots,\alpha_m) = \{u = (u_1,\dots,u_m) : u_{m-k+1} \ge \alpha_{m-k+1},\dots,u_m \ge \alpha_m\},$$

where $k = 1,\dots,m$. In particular, $\mathbf{R}^m_+(\alpha_m)$ is a half-space, $\mathbf{R}^m_+(\alpha_{m-1},\alpha_m)$ is a dihedral angle, and $\mathbf{R}^m_+(\alpha_1,\dots\alpha_m)$ is an orthant in $\mathbb{R}^m$.

Using Corollary stated in Sect. 1, we derive

COROLLARY 4.1. *The polyhedral angle $\mathbf{R}^m_+(\alpha_{m-k+1},\dots,\alpha_m)$ is invariant for system* (1.2) *in $\mathbb{R}^{n+1}_T$ if and only if all nondiagonal elements of $m-k+1$-th,…, m-th rows of the matrix-valued functions $\mathcal{A}_{jk}$ and $\mathcal{A}_j$, $1 \le j,k \le n$, are equal to zero.*

In particular, a half-plane $\mathbf{R}^2_+(\alpha_2)$ is invariant for system (1.2) *in $\mathbb{R}^{n+1}_T$ if and only if all* (2×2)*-matrix-valued functions $\mathcal{A}_{jk}$ and $\mathcal{A}_j$, $1 \le j,k \le n$, are upper triangular.*

Cylinders. Let

$$\mathbf{R}^m_-(\beta_{m-k+1},\dots,\beta_m) = \{u = (u_1,\dots,u_m) : u_{m-k+1} \le \beta_{m-k+1},\dots,u_m \le \beta_m\}$$

be a polyhedral angle and $\alpha_{m-k+1} < \beta_{m-k+1},\dots,\alpha_m < \beta_m$.

Let us introduce a polyhedral cylinder $\mathbf{C}^m(\alpha_{m-k+1},\dots,\alpha_m;\beta_{m-k+1},\dots,\beta_m) = \mathbf{R}^m_+(\alpha_{m-k+1},\dots,\alpha_m) \cap \mathbf{R}^m_-(\beta_{m-k+1},\dots,\beta_m)$, $k < m$.

In particular, $\mathbf{C}^m(\alpha_m;\beta_m)$ is a layer and $\mathbf{C}^m(\alpha_{m-1},\alpha_m;\beta_{m-1},\beta_m)$ is a rectangular cylinder.

The following criterion stems from Corollary stated in Sect. 1.

COROLLARY 4.2. *The polyhedral cylinder $\mathbf{C}^m(\alpha_{m-k+1},\dots,\alpha_m;\beta_{m-k+1},\dots,\beta_m)$ is invariant for system* (1.2) *in $\mathbb{R}^{n+1}_T$ if and only if all nondiagonal elements of $m-k+1$-th, $m-k+2$-th,…, m-th rows of matrix-valued functions $\mathcal{A}_{jk}$ and $\mathcal{A}_j$, $1 \le j,k \le n$, are equal to zero.*

In particular, a strip $\mathbf{C}^2(\alpha_2;\beta_2)$ is invariant for system (1.2) *in $\mathbb{R}^{n+1}_T$ if and only if all* (2×2)*-matrix-valued functions $\mathcal{A}_{jk}$ and $\mathcal{A}_j$, $1 \le j,k \le n$, are upper triangular.*

Let us introduce the body

$$\mathbf{S}^m_k(R) = \{u = (u_1,\dots,u_m) : u^2_{m-k+1} + \dots + u^2_m \le R^2\},$$

which is a spherical cylinder for $k < m$.

Using Corollary stated in Sect. 1, we arrive at the following criterion.

COROLLARY 4.3. *The body $\mathbf{S}^m_k(R)$ is invariant for system* (1.2) *in $\mathbb{R}^{n+1}_T$ if and only if:*

(i) *all nondiagonal elements of* $m-k+1$*-th,* $m-k+2$*-th,...,* m*-th rows of matrix-valued functions* $\mathcal{A}_{jk}$ *and* $\mathcal{A}_j$, $1 \le j,k \le n$, *are equal to zero;*

(ii) *all* $m-k+1$*-th,* $m-k+2$*-th,...,* m*-th diagonal elements of matrix* $\mathcal{A}_{jk}(x)$ $\big(\mathcal{A}_j(x)\big)$ *are equal for any fixed point* $x \in \mathbb{R}^n$ *and indices* $j,k = 1,\dots,n$.

Cones. By $\mathbf{K}_p^m$ we denote a convex polyhedral cone in $\mathbb{R}^m$ with p facets. Let, further, $\{\boldsymbol{\nu}_1,\dots,\boldsymbol{\nu}_p\}$ be the set of unit outward normals to the facets of this cone. By $[\boldsymbol{v}_1,\dots,\boldsymbol{v}_m]$ we mean the $(m\times m)$-matrix whose columns are m-component vectors $\boldsymbol{v}_1,\dots,\boldsymbol{v}_m$.

We give an auxiliary assertion of geometric character.

LEMMA 4.1. *Let* $\mathbf{K}_p^m$ *be a convex polyhedral cone in* $\mathbb{R}^m$ *with* p *facets,* $p \ge m$. *Then any system* $\boldsymbol{\nu}_1,\dots,\boldsymbol{\nu}_m$ *of unit outward normals to* m *different facets of* $\mathbf{K}_p^m$ *is linear independent.*

PROOF. By F_i we denote the facet of $\mathbf{K}_p^m$ for which the vector $\boldsymbol{\nu}_i$ is normal, $1 \le i \le m$. Let T_i be the supporting plane of this facet. We place the origin of the coordinate system with the orthonormal basis $\boldsymbol{e}_1,\dots,\boldsymbol{e}_m$ at an interior point $\mathcal{O}$ of $\mathbf{K}_p^m$ and use the notation $x = \mathcal{O}q$, where q is the vertex of the cone. Further, let $d_i = \operatorname{dist}\,(\mathcal{O},F_i),\ i=1,\dots,m$. Since

$$q = \bigcap_{i=1}^m T_i\ ,$$

it follows that $x=(x_1,\dots,x_m)$ is the only solution of the system

$$(\boldsymbol{\nu}_i,x) = d_i,\ \ i=1,2,\dots,m,$$

or, which is the same,

$$\sum_{j=1}^m (\boldsymbol{\nu}_i,\boldsymbol{e}_j)x_j = d_i,\ \ i=1,2,\dots,m.$$

The matrix of this system is $[\boldsymbol{\nu}_1,\dots,\boldsymbol{\nu}_m]^*$. Consequently,

$$\det[\boldsymbol{\nu}_1,\dots,\boldsymbol{\nu}_m]^* \ne 0.$$

This implies the linear independence of the system $\boldsymbol{\nu}_1,\dots,\boldsymbol{\nu}_m$. □

COROLLARY 4.4. *The convex polyhedral cone* $\mathbf{K}_m^m$ *is invariant for system* (1.2) *in* $\mathbb{R}_T^{n+1}$ *if and only if*

$$\mathcal{A}_{jk}(x) = \big([\boldsymbol{\nu}_1,\dots,\boldsymbol{\nu}_m]^*\big)^{-1}\,\mathcal{D}_{jk}(x)\,[\boldsymbol{\nu}_1,\dots,\boldsymbol{\nu}_m]^* \tag{4.1}$$

and

$$\mathcal{A}_{j}(x) = \big([\boldsymbol{\nu}_1,\dots,\boldsymbol{\nu}_m]^*\big)^{-1}\,\mathcal{D}_{j}(x)\,[\boldsymbol{\nu}_1,\dots,\boldsymbol{\nu}_m]^* \tag{4.2}$$

for all $x \in \mathbb{R}^n$, $1 \le j,k \le n$, *where* $\mathcal{D}_{jk}$ *and* $\mathcal{D}_j$ *are diagonal* $(m\times m)$*-matrix-valued functions.*

The convex polyhedral cone $\mathbf{K}_p^m$ *with* $p > m$ *and convex cone with a smooth guide are invariant for system* (1.2) *in* $\mathbb{R}_T^{n+1}$ *if and only if all matrix-valued functions* $\mathcal{A}_{jk}$ *and* $\mathcal{A}_j$, $1 \le j,k \le n$, *are scalar.*

PROOF. We fix a point $x \in \mathbb{R}^n$. By $\mathcal{A}$ we denote any of the $(m \times m)$-matrices $\mathcal{A}_{jk}(x)$ and $\mathcal{A}_j(x)$, $1 \leq j, k \leq n$.

By Corollary stated in Sect. 1, a necessary and sufficient condition for invariance of $\mathfrak{S}$ is equation

$$\mathcal{A}^* \boldsymbol{\nu} = \mu \boldsymbol{\nu} \ \text{ for any } \ \boldsymbol{\nu} \in \mathfrak{N}_{\mathfrak{S}}, \tag{4.3}$$

where $\mu = \mu(\boldsymbol{\nu})$ is a real number.

(i) If $\mathfrak{S} = \mathbf{K}^m_m$, we write (4.3) as

$$\mathcal{A}^* \boldsymbol{\nu}_1 = \mu_1 \boldsymbol{\nu}_1, \ldots, \mathcal{A}^* \boldsymbol{\nu}_m = \mu_m \boldsymbol{\nu}_m \ , \tag{4.4}$$

where $\{\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_m\}$ is the set of unit outward normals to the facets of the $\mathbf{K}^m_m$. These normals are linear independent by Lemma 4.1. Let $\mathcal{D} = \text{diag}\ \{\mu_1, \ldots, \mu_m\}$. Equations (4.4) can be written as

$$\mathcal{A}^*[\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_m] = [\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_m]\ \mathcal{D},$$

which leads to the representation

$$\mathcal{A} = \left([\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_m]^*\right)^{-1} \mathcal{D}\ [\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_m]^* \ . \tag{4.5}$$

Now, (4.5) is equivalent to (4.1) and (4.2).

(ii) Let us consider the cone $\mathbf{K}^m_p$ with $p > m$. By $\{\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_m\}$ we denote a system of unit outward normals to m facets of $\mathbf{K}^m_p$. Let also $\boldsymbol{\nu}$ be a normal to a certain $m+1$-th facet. By Lemma 4.1, arbitrary m vectors in the collection $\{\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_m, \boldsymbol{\nu}\}$ are linear independent. Hence there are no zero coefficients α_i in the representation $\boldsymbol{\nu} = \alpha_1 \boldsymbol{\nu}_1 + \cdots + \alpha_m \boldsymbol{\nu}_m$.

Let (4.3) hold. Then

$$\mathcal{A}^* \boldsymbol{\nu} = \lambda \boldsymbol{\nu},\ \mathcal{A}^* \boldsymbol{\nu}_1 = \mu_1 \boldsymbol{\nu}_1, \ldots, \mathcal{A}^* \boldsymbol{\nu}_m = \mu_m \boldsymbol{\nu}_m \ . \tag{4.6}$$

Therefore,

$$\lambda \sum_{i=1}^m \alpha_i \boldsymbol{\nu}_i = \lambda \boldsymbol{\nu} = \mathcal{A}^* \boldsymbol{\nu} = \mathcal{A}^* \sum_{i=1}^m \alpha_i \boldsymbol{\nu}_i = \sum_{i=1}^m \alpha_i \mu_i \boldsymbol{\nu}_i.$$

Thus,

$$\sum_{i=1}^m (\lambda - \mu_i) \alpha_i \boldsymbol{\nu}_i = \mathbf{0}.$$

Hence, $\mu_i = \lambda$ for $i = 1, \ldots, m$ and consequently $\mathcal{A}$ is a scalar matrix.

Conversely, if $\mathcal{A} = \lambda\ \text{diag}\ \{1, \ldots, 1\}$, then (4.3) with $\mu = \lambda$ holds for $\mathfrak{S} = \mathbf{K}^m_p$ with $p > m$.

The proof is complete for $p > m$.

(iii) Let (4.3) hold for the cone $\mathbf{K}$ with a smooth guide. This cone $\mathbf{K}$ can be inscribed into a polyhedral cone $\mathbf{K}^m_{m+1}$. Let $\{\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_m, \boldsymbol{\nu}\}$ be a system of unit outward normals to the facets of $\mathbf{K}^m_{m+1}$. This system is a subset of the collection of normals to the boundary of $\mathbf{K}$. By Lemma 4.1, arbitrary m vectors in the set $\{\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_m, \boldsymbol{\nu}\}$ are linear independent. Repeating word by word the argument used in (ii) we arrive at the scalarity of $\mathcal{A}$.

Conversely, (4.3) is an obvious consequence of the scalarity of $\mathcal{A}$ for $\mathfrak{S} = \mathbf{K}$.

The proof is complete. $\square$

References

1. N. Alikakos, *Remarks on invariance in reaction-diffusion equations*, Nonlinear Analysis. Theory, Methods & Applications, **5**:6 (1981), 593–614.
2. N. Alikakos, *Quantitative maximum principle and strongly coupled gradient-like reaction-diffusion systems*, Proc. Royal Soc. Edinburg, Sect. A, **94**:3-4 (1983), 265–286.
3. H. Amann, *Invariant sets and existence theorems for semilinear parabolic and elliptic systems*, J. Math. Anal. Appl., **65** (1978), 432–467.
4. P.W. Bates, *Containment for weakly coupled parabolic systems*, Houston J. Math., **11**:2 (1985), 151–158.
5. J.W. Bebernes, K.N. Chueh, and W. Fulks, *Some applications of invariance for parabolic systems*, Indiana Univ. Math. J., **28**:2 (1979), 269–277.
6. J.W. Bebernes and K. Schmitt, *Invariant sets and the Hukuhara-Kneser property for systems of parabolic partial differential equations*, Rocky Mountain J. Math., **7** (1977), 557–567.
7. K.N. Chueh, C.C. Conley, and J.A. Smoller, *Positively invariant regions for systems of non-linear diffusion equations*, Indiana Univ. Math. J., **26** (1977), 373–391.
8. E. Conway, D. Hoff, and J. Smoller, *Large time behavior of solutions of systems of nonlinear reaction-diffusion equations*, SIAM J. Appl. Math., **35** (1) (1978), 1–16.
9. C. Cosner and P.W. Schaefer, *On the development of functionals which satisfy a maximum principle*, Appl. Analysis, **26** (1987), 45–60.
10. S.D. Eidel'man, *Parabolic Systems*, North-Holland and Noordhoff, Amsterdam, 1969.
11. G.I. Kresin and V.G. Maz'ya, *Criteria for validity of the maximum modulus principle for solutions of linear parabolic systems*, Ark. Math., **32** (1994), 121–155.
12. G.I. Kresin and V.G. Maz'ya, *On the maximum principle with respect to smooth norms for linear strongly coupled parabolic systems*, Functional Differential Equations, **5**:3-4 (1998), 349–376.
13. G.I. Kresin and V.G. Maz'ya, *Criteria for validity of the maximum norm principle for parabolic systems*, Poten. Anal., **10** (1999), 243–272.
14. G. Kresin and V. Maz'ya, *Maximum Principles and Sharp Constants for Solutions of Elliptic and Parabolic Systems*, Math. Surveys and Monographs, **183**, Amer. Math. Soc., Providence, Rhode Island, 2012.
15. H.J. Kuiper, *Invariant sets for nonlinear elliptic and parabolic systems*, SIAM J. Math. Anal., **11**:6 (1980), 1075–1103.
16. R. Lemmert, *Über die Invarianz konvexer Teilmengen eines normierten Raumes in bezug auf elliptische Differentialgleichungen*, Comm. Partial Diff. Eq., **3**:4 (1978), 297–318.
17. V.G. Maz'ya and G.I. Kresin, *On the maximum principle for strongly elliptic and parabolic second order systems with constant coefficients*, Mat. Sb., **125(167)** (1984), 458–480 (Russian); English transl.: Math. USSR Sb., **53** (1986) ,457–479.
18. K. Otsuka, *On the positivity of the fundamental solutions for parabolic systems*, J. Math. Kioto Univ., **28** (1988), 119–132.
19. M.H. Protter and H.F. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1967; Springer-Verlag, New York Inc., 1984.
20. R. Redheffer and W. Walter, *Invariant sets for systems of partial differential equations. I. Parabolic equations*, Arch. Rat. Mech. Anal., **67** (1978), 41–52.
21. R. Redheffer and W. Walter, *Invariant sets for systems of partial differential equations. II. First-order and elliptic equations*, Arch. Rat. Mech. Anal., **73** (1980), 19–29.
22. R.T. Rockafellar, *Convex Analysis*, Princeton Univ. Press, Princton, N.J., 1970.
23. C. Schaefer, *Invariant sets and contractions for weakly coupled systems of parabolic differential equations*, Rend. Mat., **13** (1980), 337–357.
24. J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, Berlin-Heidelberg-New York, 1983.
25. W. Walter, *Differential and Integral Inequalities*, Springer, Berlin-Heidelberg-New York, 1970.
26. H.F. Weinberger, *Invariant sets for weakly coupled parabolic and elliptic systems*, Rend. Mat., **8** (1975), 295–310.
27. H.F. Weinberger, *Some remarks on invariant sets for systems*, in “Maximum principles and eigenvalue problems in partial differential equations”, P.W. Schaefer ed., Pittman Research Notes, Math. ser., **175** (1988), pp. 189–207.

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