Conductor and capacitary inequalities for functions on topological spaces and their applications to Sobolev type imbeddings

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Abstract. In 1972 the author proved the so called *conductor* and *capacitary inequalities* for the Dirichlet type integrals of a function on a Euclidean domain. Both were used to derive necessary and sufficient conditions for Sobolev type inequalities involving arbitrary domains and measures.

The present article contains new conductor inequalities for nonnegative functionals acting on functions defined on topological spaces. Sharp capacitary inequalities, stronger than the classical Sobolev inequality, with the best constant and the sharp form of the Yudovich inequality (1961) due to Moser (1971) are found.

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1 Introduction

Let Ω be an open set in $\mathbb{R}^n$ and let f be an arbitrary function in $C_0^\infty(\Omega)$, i.e. an infinitely differentiable function with compact support in Ω.

In this paper we discuss generalizations and applications of the inequality

$$\int_0^\infty \mathrm{cap}_p(\overline{M_{at}}, M_t)\, d(t^p) \le c(a,p) \int_\Omega |\mathrm{grad} f|^p dx, \tag{1}$$

where $a = const > 1$, $1 \le p < \infty$, $M_t = \{x \in \Omega : |f(x)| > t\}$, and cap_p is the so called conductor p-capacitance (see (22) below). A discrete version of (1) and its analogue involving second order derivatives of a nonnegative f were obtained by the author in 1972 [M4] (see also [M6] and [M7]).

By monotonicity of cap_p the *conductor inequality* (1) implies

$$\int_0^\infty \mathrm{cap}_p(\overline{M_t}, \Omega)\, d(t^p) \le C(p) \int_\Omega |\mathrm{grad} f|^p dx, \tag{2}$$

which was also proved in [M4] with the best constant

$$C(p) = p^p (p-1)^{1-p}. \tag{3}$$

(For $p = 2$ inequality (2) with $C(2) = 4$ was used without explicit formulation already in [M2], [M3].)

Inequality (2) and its various extensions are of independent interest and have numerous applications to the theory of Sobolev spaces on Euclidean domains, Riemannian manifolds, and metric spaces, to linear and nonlinear partial differential equations, calculus of variations, theories of Dirichlet forms and Markov processes, etc. ([M4], [Ad], [M5], [Dah], [Han], [Ko1], [Ko2], [Ra], [Ne], [AP], [Ka], [MN], [Vo], [AH], [MP], [HMV], [Ai], [V1], [V2], [Gr], [Haj], [Ta], [Fi], [FU1], [FU2], [CS], [AX1], [AX2], *et al*).

Note that the left-hand side in (2) can be zero for all $f \in C_0^\infty(\Omega)$. (This happens if and only if either $p > n$ and $\Omega = \mathbb{R}^n$, or $p = n$ and the complement of Ω has zero n-capacity.) At the same time, the left-hand side in (1) is always positive if $f \neq 0$. The layer cake texture of the left-hand side in the conductor inequality (1) allows for significant corollaries which cannot be directly deduced from inequality (2). For instance, as a straightforward consequence of (1) and the classical isocapacitance property of a conductor (see (44) below), one deduces

$$\int_0^\infty \frac{d(t^n)}{\left(\log \dfrac{m_n(M_t)}{m_n(M_{at})}\right)^{n-1}} \leq c(a) \int_\Omega |\text{grad} f|^n dx, \tag{4}$$

where $n > 1$, m_n is the n-dimensional Lebesgue measure, and $a > 1$. Note that (4) is stronger than the well-known inequality

$$\int_0^\infty \frac{d(t^n)}{\left(\log \dfrac{m_n(\Omega)}{m_n(M_t)}\right)^{n-1}} \leq c \int_\Omega |\text{grad} f|^n dx, \tag{5}$$

(see [M4], [Han], [BW]) which is informative only if the volume of Ω is finite.

In the case $p \neq n$ and $p > 1$, another straightforward consequence of (1) in a similar flavor is the following improvement of the classical Sobolev inequality

$$\int_0^\infty |m_n(M_t)^{\frac{p-n}{n(p-1)}} - m_n(M_{at})^{\frac{p-n}{n(p-1)}}|^{1-p} d(t^p) \leq c(p, a) \int_\Omega |\text{grad} f|^p dx. \tag{6}$$

Among other applications of conductor inequalities which seem to be unattainable with the help of capacitary inequalities is a necessary and sufficient condition for the two measure Sobolev type inequality ([M4], [M6], [M7]):

$$\left(\int_\Omega |f|^q d\mu\right)^{1/q} \leq C \left(\int_\Omega |\phi(x, \text{grad} f)|^p dx + \int_\Omega |f|^p d\nu\right)^{1/p}, \tag{7}$$

where $q \geq p$, μ and ν are locally finite Radon measures on Ω, and the function: $\Omega \times \mathbb{R}^n \ni (x, y) \to \phi(x, y)$ is continuous and positively homogeneous in y of degree 1. The characterization just mentioned is formulated in terms of the conductor capacitance generated by the integral

$$\int_\Omega |\phi(x, \text{grad} f)|^p dx. \tag{8}$$

In the one-dimensional case, when this capacitance is calculated explicitly (see either Lemma 4 in [M4] or Lemma 2.2.2/2 in [M6]), this characterization takes the following simple form:

$$\mu(\sigma_d(x))^{p/q} \le \text{const}(\tau^{1-p} + \nu(\sigma_{d+\tau}(x))), \tag{9}$$

where x, d and τ are such that $\overline{\sigma_{d+\tau}(x)} \subset \Omega$.

In Sections 2 and 4 of the present article we derive some new conductor inequalities for functions defined on a locally compact Hausdorff space $\mathcal{X}$. It is worth mentioning that, unlike the Sobolev inequalities, the conductor inequalities do not depend on the dimension of $\mathcal{X}$. Furthermore, with a lower estimate for the p-conductance by a certain measure on $\mathcal{X}$, one can readily deduce the Sobolev- Lorentz type inequalities involving this measure.

In Section 2 we are interested in conductor inequalities for the Dirichlet type integral

$$\int_{\mathcal{X}} \mathcal{F}_p[f], \tag{10}$$

where $\mathcal{F}_p$ is a measure valued operator acting on a function f and satisfying locality and contractivity conditions. A prototype of (10) is the functional

$$\int_\Omega |\phi(x, \text{grad}\, f(x))|^p d\mu + \int_\Omega |f(x)|^p d\nu, \tag{11}$$

where ϕ is the same as in (7).

In Theorem 1 proved is Section 2 we obtain the conductor inequality

$$\Phi^{-1}\Big(\int_0^\infty \Phi(t^p \text{cap}_p(\overline{M_{at}}, M_t))\frac{dt}{t}\Big) \le c(a,p) \int_{\mathcal{X}} \mathcal{F}_p[f], \tag{12}$$

where and elsewhere $M_t = \{x \in \mathcal{X} : |f(x)| > t\}$, Φ is a positive convex function on $(0,\infty)$, $\Phi(+0) = 0$, and Φ^{-1} stands for the inverse of Φ. By cap_p the p-conductance generated by the operator $\mathcal{F}_p$ is meant.

The short Section 3 is dedicated to a discussion of inequalities (4)-(6).

In Section 4 we derive the conductor inequality

$$\Big(\int_0^\infty \text{cap}_{p,\Gamma}(\overline{M_{at}}, M_t)^{q/p} d(t^q)\Big)^{p/q} \le c(a,p,q)\, \langle f\rangle^p_{p,\Gamma}, \tag{13}$$

where $q \ge p \ge 1$,

$$\langle f\rangle_{p,\Gamma} := \Big(\int_{\mathcal{X}}\int_{\mathcal{X}} |f(x) - f(y)|^p\, \Gamma(dx \times dy)\Big)^{1/p} \tag{14}$$

and $\text{cap}_{p,\Gamma}$ is the p-capacitance corresponding to the seminorm (14). We apply (13) to obtain a necessary and sufficient condition for a two measure Sobolev inequality involving $\langle f\rangle_{p,\Gamma}$.

In the last Section 5 we handle variants of the sharp capacitary inequality (2). We show in Theorem 3 that a fairly general capacitary inequality is a direct consequence of a one-dimensional inequality for functions with the first derivative in $L_p(0,\infty)$. A corollary of this result is the following inequality with the best constant, complementing (2)

$$\Big(\int_\Omega \text{cap}_p(\overline{M_t}, \Omega)^{q/p} d(t^q)\Big)^{1/q} \le \Big(\frac{\Gamma\big(\frac{pq}{q-p}\big)}{\Gamma\big(\frac{q}{q-p}\big)\Gamma\big(p\frac{q-1}{q-p}\big)}\Big)^{1/p-1/q}\Big(\int_\Omega |\text{grad}\, f|^p dx\Big)^{1/p}, \tag{15}$$

where $q > p \geq 1$. Combined with an isocapacitary inequality, estimate (15) with $q = pn/(n-p)$, $n > p$, immediately gives the classical Sobolev estimate

$$\Big(\int_\Omega |f|^{\frac{pn}{n-p}} dx\Big)^{1-p/n} \leq c \int_\Omega |\mathrm{grad}\, f|^p dx \tag{16}$$

with the best constant (see [FF], [M1], [Ro], [Au], [Tal]). Another example of application of Theorem 3 is the inequality

$$\sup \int_0^\infty \exp\big(-c\,\mathrm{cap}_p(\overline{M_t}, \Omega)^{1/(1-p)}\big) d \exp(c\,t^{p/(p-1)}) < \infty, \tag{17}$$

where $c = const$, the supremum is taken over all $f \in C_0^\infty(\Omega)$ subject to $\|\mathrm{grad}\, f\|_{L_p(\Omega)} \leq 1$, and cap_p is the capacity generated by the norm $\|\mathrm{grad}\, f\|_{L_p(\Omega)}$. Inequality (17) with $p = n$ is stronger than the sharp form of the Yudovich inequality [Yu] due to Moser [Mo], which immediately follows from (17) and an isocapacitary inequality.

2 Conductor inequalities for a Dirichlet type integral with a locality property

Let $\mathcal{X}$ denote a locally compact Hausdorff space and let $C(\mathcal{X})$ stand for the space of continuous real valued functions given on $\mathcal{X}$. By $C_0(\mathcal{X})$ we denote the set of the functions $f \in C(\mathcal{X})$ with compact supports in $\mathcal{X}$.

We introduce an operator $\mathcal{F}_p$ defined on a subset $\mathrm{dom}(\mathcal{F}_p)$ of $C(\mathcal{X})$ and taking values in the cone of nonnegative locally finite Borel measures on $\mathcal{X}$. We suppose that $1 \in \mathrm{dom}(\mathcal{F}_p)$ and $\mathcal{F}_p$ is positively homogeneous of order $p \geq 1$, i.e. for every real α, $f \in \mathrm{dom}(\mathcal{F}_p)$ implies $\alpha f \in \mathrm{dom}(\mathcal{F}_p)$ and

$$\mathcal{F}_p[\alpha f] = |\alpha|^p \mathcal{F}_p[f] \tag{18}$$

It is also assumed that $\mathcal{F}_p$ is contractive, that is $\lambda(f) \in \mathrm{dom}(\mathcal{F}_p)$ and

$$\mathcal{F}_p[\lambda(f)] \leq \mathcal{F}_p[f] \tag{19}$$

for all $f \in \mathrm{dom}(\mathcal{F}_p)$, where λ is an arbitrary real valued Lipschitz function on the line $\mathbb{R}$ such that $|\lambda'| \leq 1$ and $\lambda(0) = 0$. We suppose that the following locality condition holds:

$$f(x) = c \in \mathbb{R} \ \textit{ on a compact set } \mathcal{C} \implies \int_{\mathcal{C}} \mathcal{F}_p[f] = \int_{\mathcal{C}} \mathcal{F}_p[c]. \tag{20}$$

An example of the measure satisfying conditions (18)-(20) is given by (11), where

$$\Omega \times \mathbb{R}^n \ni (x, z) \to \phi(x, z) \in \mathbb{R} \tag{21}$$

is a continuous function, positive homogeneous of degree 1 with respect to z. One can take the space of locally Lipschitz functions on Ω as $\mathrm{dom}(\mathcal{F}_p)$.

Let g and G denote open sets in $\mathcal{X}$ such that that the closure $\bar{g}$ is a compact subset of G. We introduce the p-capacitance of the conductor $G \backslash \bar{g}$ (in other terms, the relative p-capacity of the set $\bar{g}$ with respect to G) as

$$\mathrm{cap}_p(\bar{g}, G) = \inf\Big\{ \int_{\mathcal{X}} \mathcal{F}_p[\varphi] \colon \ \varphi \in \mathrm{dom}(\mathcal{F}_p), \ 0 \leq \varphi \leq 1 \ \text{ on } G,$$

$$\left.\begin{array}{l}\varphi = 0 \ \text{ outside a compact subset of } G\\ \text{and } \ \varphi = 1 \ \text{ on a neighborhood of } \overline{g}\end{array}\right\}. \tag{22}$$

Using the truncation

$$\lambda(\xi) = \min\Big\{\frac{(\xi-\varepsilon)_+}{1-\varepsilon}, 1\Big\}$$

with $\varepsilon \in (0,1)$ and $\xi \in \mathbb{R}$, we see that the infimum in (22) does not change if the class of admissible functions φ is enlarged to

$$\{\, \varphi \in \operatorname{dom}(\mathcal{F}_p) \cap C_0(\mathcal{X}) : \varphi \geq 1 \text{ on } g, \quad \varphi \leq 0 \text{ on } \mathcal{X}\backslash G\,\} \tag{23}$$

(compare with Sect. 2.2 in [M6]).

Lemma 1. *Let $f \in \operatorname{dom}(\mathcal{F}_p) \cap C_0(\mathcal{X})$ and let $a = const > 1$ and $M_t = \{x \in \mathcal{X} : |f(x)| > t\}$. Then the function $t \to \operatorname{cap}_p(\overline{M_{at}}, M_t)$ is upper semicontinuous.*

Proof. It follows from (20) that

$$\int_{\mathcal{X}} \mathcal{F}_p[f] = \int_{\operatorname{supp} f} \mathcal{F}_p[f] < \infty. \tag{24}$$

Let $t_0 > 0$ and $\varepsilon > 0$. There exist open sets g and G such that

$$\overline{M_{at}} \subset g, \quad \overline{g} \subset G, \quad \overline{G} \subset M_t. \tag{25}$$

It follows from the definition of cap_p that for all compact sets $C \subset g$

$$\operatorname{cap}_p(C, G) \leq \operatorname{cap}_p(\overline{M_{at_0}}, M_{t_0}) + \varepsilon \tag{26}$$

(compare with Sect. 2.2.1 in [M6]). By (25),

$$\max\{f(x) : x \in \overline{g}\} < a\, t_0, \quad \text{and} \quad \min\{f(x) : x \in \overline{G}\} > t_0.$$

We denote

$$\delta_1 = t_0 - a^{-1}\max\{f(x) : x \in \overline{g}\}$$

and

$$\delta_2 = \min\{f(x) : x \in \overline{G}\} - t_0.$$

Then

$$\overline{M_{a(t_0-\delta)}} \subset g \quad \text{and} \quad \overline{G} \subset M_{t_0+\delta}$$

for every $\delta \in (0, \min\{\delta_1, \delta_2\})$. Putting $C = \overline{M_{a(t_0-\delta)}}$ in (26) and recalling that cap_p decreases with enlargement of the conductor, we obtain

$$\operatorname{cap}_p(\overline{M_{a(t_0-\delta)}}, M_{t_0+\delta}) \leq \operatorname{cap}_p(\overline{M_{at_0}}, M_{t_0}) + \varepsilon. \tag{27}$$

Using the monotonicity of cap_p again, we deduce from (27) that

$$\operatorname{cap}_p(\overline{M_{at}}, M_t) \leq \operatorname{cap}_p(\overline{M_{at_0}}, M_{t_0}) + \varepsilon$$

for every t sufficiently close to t_0. In other words, the function $t \to \operatorname{cap}_p(\overline{M_{at}}, M_t)$ is upper semicontinuous. The result follows.

We prove a general conductor inequality in the integral form for the functional (10).

Theorem 1. *Let Φ denote an increasing convex (not necessarily strictly convex) function given on $[0,\infty)$, $\Phi(0)=0$. Then inequality* (12) *holds for all $f\in \mathrm{dom}(\mathcal{F}_p)\cap C_0(\mathcal{X})$ and for an arbitrary $a>1$.*

Proof. We have

$$\mathrm{cap}_p(\overline{M_{at}}, M_t) \le \int_{\mathcal{X}} \mathcal{F}_p[\varphi]$$

for every $\varphi\in \mathrm{dom}(\mathcal{F}_p)\cap C_0(\mathcal{X})$ satisfying

$$\varphi = 1 \ \text{ on } \ \overline{M_{at}},\ \ \varphi = 0 \ \text{ on } \ \mathcal{X}\backslash M_t, \ \text{ and } \ 0\le\varphi\le 1 \ \text{ on } \ \mathcal{X}.$$

By the homogeneity of $\mathcal{F}_p$ and by (20),

$$t^p\mathrm{cap}_p(\overline{M_{at}}, M_t) \le \int_{M_t} \mathcal{F}_p[t\varphi].$$

We set here

$$\varphi_(x) = \frac{\Lambda_t(f(x))}{(a-1)t},$$

where

$$\Lambda_t(\xi) = \min\big\{(|\xi|-t)_+,\ (a-1)t\big\},\ \ \xi\in\mathbb{R}, \tag{28}$$

with $\xi_+ = (|\xi|+\xi)/2$. By $\Lambda_t = const$ on $\overline{M_{at}}$ and by (20) we have

$$t^p\mathrm{cap}_p(\overline{M_{at}}, M_t) \le \frac{1}{(a-1)^p}\int_{M_t\backslash\overline{M_{at}}} \mathcal{F}_p[\Lambda_t(f)].$$

Since the mapping $\xi\to\Lambda_t(\xi)$ is contractive and since the function $t\to\int_{M_t}\mathcal{F}_p[f]$ has at most a countable set of discontinuities, it follows that

$$t^p\mathrm{cap}_p(\overline{M_{at}}, M_t) \le \frac{1}{(a-1)^p}\int_{M_t\backslash M_{at}} \mathcal{F}_p[f] + t^p\int_{M_{at}}\mathcal{F}_p[1] \tag{29}$$

for almost every $t>0$. Hence,

$$\int_0^\infty \Phi(t^p\mathrm{cap}_p(\overline{M_{at}}, M_t))\frac{dt}{t} \le \int_0^\infty \Phi\Big(\frac{1}{(a-1)^p}\int_{M_t\backslash M_{at}} \mathcal{F}_p[f] + t^p\int_{M_{at}}\mathcal{F}_p[1]\Big)\frac{dt}{t}$$

$$\le \frac{1}{2}\int_0^\infty \Phi\Big(\frac{2}{(a-1)^p}\int_{M_t\backslash M_{at}} \mathcal{F}_p[f]\Big)\frac{dt}{t} + \frac{1}{2}\int_0^\infty \Phi\Big(2t^p\int_{M_{at}}\mathcal{F}_p[1]\Big)\frac{dt}{t}. \tag{30}$$

Let γ denote a locally integrable function on $(0,\infty)$ such that there exist the limits $\gamma(0)$ and $\gamma(\infty)$. Then the identity

$$\int_0^\infty (\gamma(t)-\gamma(at))\frac{dt}{t} = (\gamma(0)-\gamma(\infty))\log a \tag{31}$$

holds. Setting here

$$\gamma(t) := \Phi\Big(\frac{1}{(a-1)^p}\int_{M_t}\mathcal{F}_p[f]\Big)$$

and using the convexity of Φ we obtain

$$\int_0^\infty \Phi\Big(\frac{2}{(a-1)^p}\int_{M_t\setminus M_{at}} \mathcal{F}_p[f]\Big)\frac{dt}{t}$$

$$\le \int_0^\infty \Big\{\Phi\Big(\frac{2}{(a-1)^p}\int_{M_t} \mathcal{F}_p[f]\Big) - \Phi\Big(\frac{2}{(a-1)^p}\int_{M_{at}} \mathcal{F}_p[f]\Big)\Big\}\frac{dt}{t}$$

$$= \log a\, \Phi\Big(\frac{2}{(a-1)^p}\int_{\mathcal{X}} \mathcal{F}_p[f]\Big). \tag{32}$$

By convexity of Φ,

$$\int_0^\infty \Phi\Big(2t^p \int_{M_{at}} \mathcal{F}_p[1]\Big)\frac{dt}{t} \le 2\int_0^\infty \Phi'\Big(2t^p\int_{M_{at}}\mathcal{F}_p[1]\Big)t^{p-1}\int_{M_{at}}\mathcal{F}_p[1]\,dt$$

$$\le 2\int_0^\infty \Phi'\Big(2p\int_0^t \tau^{p-1}\int_{M_{a\tau}}\mathcal{F}_p[1]\,d\tau\Big)t^{p-1}\int_{M_{at}}\mathcal{F}_p[1]\ dt$$

$$= \frac{1}{p}\Phi\Big(2p\int_0^\infty \tau^{p-1}\int_{M_{a\tau}}\mathcal{F}_p[1]\ d\tau\Big). \tag{33}$$

Clearly,

$$\int_0^\infty \tau^{p-1}\int_{M_{a\tau}}\mathcal{F}_p[1]\,d\tau = (a^p-1)\int_0^\infty \tau^{p-1}\int_{M_\tau\setminus M_{a\tau}}\mathcal{F}_p[1]\ d\tau. \tag{34}$$

Using the truncation

$$\lambda(\xi) = \begin{cases} |\xi| & \text{for } |\xi| > a\tau \\ a\tau & \text{for } |\xi| \le a\tau \end{cases}$$

together with (19), we deduce from (34) and (31) that

$$\int_0^\infty \tau^{p-1}\int_{M_{a\tau}}\mathcal{F}_p[1]\,d\tau \le \frac{a^p-1}{a^p}\int_0^\infty\int_{M_\tau\setminus M_{a\tau}}\mathcal{F}_p[f]\,\frac{d\tau}{\tau}$$

$$= \log a\,\frac{a^p-1}{a^p}\int_{\mathcal{X}}\mathcal{F}_p[f].$$

Combining this with (34), we arrive at

$$\int_0^\infty \Phi\Big(2t^p\int_{M_{at}}\mathcal{F}_p[1]\Big)\frac{dt}{t} \le \frac{1}{p}\Phi\Big(2p\log a\,\frac{a^p-1}{a^p}\int_{\mathcal{X}}\mathcal{F}_p[f]\Big).$$

Summing up (32) and the last inequality, we conclude by (30) that

$$\int_0^\infty \Phi\big(t^p \mathrm{cap}_p(\overline{M_{at}}, M_t)\big)\frac{dt}{t}$$

$$\le \frac{1}{2}\log a\ \Phi\Big(\frac{2}{(a-1)^p}\int_{\mathcal{X}}\mathcal{F}_p[f]\Big) + \frac{1}{2p}\Phi\Big(2p\log a\,\frac{a^p-1}{a^p}\int_{\mathcal{X}}\mathcal{F}_p[f]\Big),$$

and (12) follows. $\square$

Remark 1. Suppose that (20) is replaced with the following more restrictive locality condition:

$$f(x) = const \quad on \ a \ compact \ set \ \mathcal{C} \quad \Longrightarrow \int_{\mathcal{C}} \mathcal{F}_p[f] = 0, \tag{35}$$

which holds, for example, if the measure ν in (11) is zero.

Then the above proof becomes simpler. In fact, we can replace (30) with

$$\int_0^\infty \Phi(t^p \mathrm{cap}_p(\overline{M_{at}}, M_t)) \frac{dt}{t} \le \int_0^\infty \Phi\Big(\frac{1}{(a-1)^p} \int_{M_t \setminus M_{at}} \mathcal{F}_p[f]\Big).$$

Estimating the right-hand side by (32) we obtain the inequality

$$\int_0^\infty \Phi(t^p \mathrm{cap}_p(\overline{M_{at}}, M_t)) \frac{dt}{t} \le \log a \ \Phi\Big(\frac{1}{(a-1)^p} \int_{\mathcal{X}} \mathcal{F}_p[f]\Big). \ \square \tag{36}$$

The next statement follows directly from (12) and (36) by setting $\Phi(\xi) = \xi^{q/p}$ for $\xi \ge 0$.

Corollary 1. *Let $q \ge p$ and let $\mathcal{F}_p$ satisfy the locality condition* (20)*. Then for all $f \in \mathrm{dom}(\mathcal{F}_p) \cap C_0(\mathcal{X})$ and for an arbitrary $a > 1$*

$$\Big(\int_0^\infty \big(\mathrm{cap}_p(\overline{M_{at}}, M_t)\big)^{q/p} d(t^q)\Big)^{1/q} \le C\Big(\int_{\mathcal{X}} \mathcal{F}_p[f]\Big)^{1/p}. \tag{37}$$

If additionally $\mathcal{F}_p$ is subject to (35)*, then one can choose*

$$C = \frac{(q \ \log a)^{1/q}}{a-1}.$$

Remark 2. Let $\mathcal{F}_p$ satisfy (35). Then one can easily see that for every sequence $\{t_k\}_{k=-\infty}^{\infty}$, such that $0 < t_k < t_{k+1}$,

$$t_k \to 0 \ \text{ as } \ k \to -\infty \quad \text{and} \ \ t_k \to \infty \ \text{ as } \ k \to \infty,$$

the following discrete conductor inequality holds:

$$\sum_{k=-\infty}^{\infty} (t_{k+1} - t_k)^p \mathrm{cap}_p(\overline{M_{t_{k+1}}}, M_{t_k}) \le \int_{\mathcal{X}} \mathcal{F}_p[f]. \tag{38}$$

Putting $t_k = a^k$, where $a > 1$, we see that

$$\sum_{k=-\infty}^{\infty} a^{pk} \mathrm{cap}_p(\overline{M_{a^{k+1}}}, M_{a^k}) \le (a-1)^{-p} \int_{\mathcal{X}} \mathcal{F}_p[f]. \tag{39}$$

Using Lemma 1 and monotonicity properties of the capacitance, we check that inequality (39) is equivalent to (37) with $q = p$ modulo the value of the coefficient c. $\square$

The capacitary inequality

$$\Big(\int_0^\infty \big(\mathrm{cap}_p(\overline{M_t}, \mathcal{X})\big)^{q/p} d(t^q)\Big)^{1/q} \le C\Big(\int_{\mathcal{X}} \mathcal{F}_p[f]\Big)^{1/p} \tag{40}$$

results directly from (37).

An immediate consequence of (40) is the following criterion for the Sobolev type inequality

$$\|f\|_{L_q(\mu)} \le C\Big(\int_{\mathcal{X}} \mathcal{F}_p[f]\Big)^{1/p}, \tag{41}$$

where μ is a locally finite Radon measure on $\mathcal{X}$, $q \ge p$, and f is an arbitrary function in $\mathrm{dom}(\mathcal{F}_p) \cap C_0(\mathcal{X})$.

Corollary 2. *Inequality* (41) *holds if and only if*

$$\sup \frac{\mu(g)^{p/q}}{\mathrm{cap}_p(\overline{g}, \mathcal{X})} < \infty. \tag{42}$$

Criteria of such a kind were first obtained in [M1] - [M4].

Proof. The necessity of (42) is obvious and its sufficiency follows from the well-known and easily checked inequality

$$\Big(\int_0^\infty \mu(M_t)d(t^q)\Big)^{1/q} \le \Big(\int_0^\infty \mu(M_t)^{p/q}d(t^p)\Big)^{1/p},$$

where $q \ge p \ge 1$ (see [HLP]).

3 Inequalities (4) and (6)

Let $\mathcal{X} = \Omega$, where Ω is an open set in $\mathbb{R}^n$, and let $\mathcal{F}_p$ be defined by

$$\mathcal{F}_p[f] = |\mathrm{grad} f(x)|^p dx. \tag{43}$$

Inequalities (4) and (6) follow directly from (1) combined with the isocapacitance inequalities

$$\mathrm{cap}_p(\bar{g}, G) \ge \begin{cases} c_n\Big(\log\dfrac{m_n(G)}{m_n(\overline{g})}\Big)^{1-n} & \text{for } \ p = n \\[2ex] c_p\, |m_n(G)^{\frac{p-n}{n(p-1)}} - m_n(\overline{g})^{\frac{p-n}{n(p-1)}}|^{1-p} & \text{for } \ p \ne n, \end{cases} \tag{44}$$

with

$$c_n = n^{n-1}\omega_n \ \text{ and } \ c_p = |n-p|^{p-1}(p-1)^{1-p}\omega_n \ \text{for } \ p \ne n, \tag{45}$$

where ω_n is the $(n-1)$-dimensional area of the unit ball in $\mathbb{R}^n$ (see either [M4] or Sect. 2.2.3 in [M6]).

Remark 3. Let us compare the integrals in the left-hand sides of (4) and (5):

$$\int_0^\infty \frac{d(t^n)}{\Big(\log\dfrac{m_n(M_t)}{m_n(M_{at})}\Big)^{n-1}} \tag{46}$$

and

$$\int_0^\infty \frac{d(t^n)}{\Big(\log\dfrac{m_n(\Omega)}{m_n(M_t)}\Big)^{n-1}}\,, \tag{47}$$

where $m_n(\Omega) < \infty$. Clearly, the first of them exceeds the second. However, the convergence of the second integral does not imply the convergence of the first. In fact, let $B_r = \{x \in \mathbb{R}^n : |x| < r\}$, $\Omega = B_2$, and

$$f(x) = \begin{cases} 5 - |x| & \text{for } 0 \le |x| < 1 \\ 2 - |x| & \text{for } 1 \le |x| < 2. \end{cases} \tag{48}$$

We have

$$M_t = \begin{cases} B_{2-t} & \text{for } 0 \le t < 1, \\ B_1 & \text{for } 1 \le t \le 4, \\ B_{5-t} & \text{for } 4 < t \le 5. \end{cases}$$

Let $1 < a < 4$. Then both sets M_t and M_{at} for $t \in (1, 4a^{-1})$ coincide with the ball B_1, which makes (46) divergent whereas integral (47) is finite. Furthermore, integral (46) is convergent for $a \ge 4$.

Therefore, inequality (4) is strictly better than (5), even for domains Ω of finite volume. We see also that the convergence of integral (46) for a bounded function f may depend on the value of a.

The same argument shows that inequality (6) for all $f \in C_0^\infty(\Omega)$ with $1 < p < n$, i.e.

$$\int_0^\infty \frac{d(t^p)}{\left(\dfrac{1}{m_n(M_{at})^{\frac{n-p}{n(p-1)}}} - \dfrac{1}{m_n(M_t)^{\frac{n-p}{n(p-1)}}}\right)^{p-1}} \le c \int_\Omega |\operatorname{grad} f|^p dx, \tag{49}$$

improves the Lorentz space $L_{\frac{np}{n-p},p}(\Omega)$ inequality

$$\int_0^\infty \big(m_n(M_t)\big)^{\frac{n-p}{n}} d(t^p) \le c \int_\Omega |\operatorname{grad} f|^p dx$$

which results from (2) and is stronger, in its turn, than the Sobolev inequality (16).

In conclusion we add that the convergence of integral in the left-hand side of (49) may depend on the choice of a, as shown by function (48). $\square$

4 Conductor inequality for a Dirichlet type integral without locality conditions

Here the notation $\mathcal{X}$ has the same meaning as in Section 2. Let $\times$ stand for the Cartesian product of sets and let Γ denote a nonnegative symmetric Radon measure on $\mathcal{X}^2 := \mathcal{X} \times \mathcal{X}$, locally finite outside the diagonal $\{(x, y) \in \mathcal{X}^2 : x = y\}$. We shall derive a conductor inequality for the seminorm (14), where f is an arbitrary function in $C_0(\mathcal{X})$ such that

$$\langle f \rangle_{p,\Gamma} < \infty. \tag{50}$$

Clearly, the seminorm $\langle f \rangle_{p,\Gamma}$ is contractive, that is

$$\langle \lambda(f) \rangle_{p,\Gamma} \le \langle f \rangle_{p,\Gamma}$$

with the same λ as in (19).

Let as before g and G denote open sets in $\mathcal{X}$ such that $\overline{g}$ is a compact subset of G. Similarly to Section 2, we introduce the capacitance of the conductor $G\backslash\overline{g}$

$$\begin{aligned}\operatorname{cap}_{p,\Gamma}(\bar{g}, G) = \inf\{\, \langle f\rangle_{p,\Gamma}^p : \varphi \in C_0(\mathcal{X}),\ 0 \le \varphi \le 1 \text{ on } G,\\ \varphi = 0 \text{ outside a compact subset of } G\\ \text{and } \varphi = 1 \text{ on a neighborhood of } \overline{g}\ \}.\end{aligned}$$

It is straightforward that this infimum does not change if the class of admissible functions φ is replaced with (23) (compare with the definition of $\operatorname{cap}_p(\bar{g}, G)$ in Sect. 2).

Theorem 2. *For all $f \in C_0(\mathcal{X})$ subject to* (50), *for all $q \ge p \ge 1$, and for an arbitrary $a > 1$ the conductor inequality* (13) *holds.*

Proof. The measurability of the function $t \to \operatorname{cap}_{p,\Gamma}(\overline{M_{at}}, M_t)$ is proved word for word as in Lemma 1.

Clearly,

$$(a-1)^p t^p \operatorname{cap}_{p,\Gamma}(\overline{M_{at}}, M_t) \le \langle\, \Lambda_t(f)\,\rangle_{p,\Gamma}^p \tag{51}$$

with Λ_t defined by (28). Let K_t denote the conductor $M_t\backslash\overline{M_{at}}$. Since

$$S^2 \subset (S \times T) \cup (T \times S) \cup (S\backslash T)^2$$

for all sets S and T and since Γ is symmetric, it follows that

$$(a-1)^p t^p \operatorname{cap}_{p,\Gamma}(\overline{M_{at}}, M_t) \le \Big(2\int_{K_t\times\mathcal{X}} + \int_{(\mathcal{X}\backslash K_t)^2}\Big)|\Lambda_t(f(x)) - \Lambda_t(f(y))|^p\, \Gamma(dx \times dy)$$

$$\le 2\int_{K_t\times\mathcal{X}} |f(x) - f(y)|^p\, \Gamma(dx \times dy) + 2(a-1)^p t^p \Gamma(\overline{M_{at}} \times (\mathcal{X}\backslash M_t)). \tag{52}$$

By Minkowski's inequality,

$$(a-1)^p\Big(\int_0^\infty \big(\operatorname{cap}_{p,\Gamma}(\overline{M_{at}}, M_t)\big)^{q/p} t^{q-1} dt\Big)^{p/q} \le A + B,$$

where

$$A = 2\Big(\int_0^\infty \Big(\int_{K_t\times\mathcal{X}} |f(x) - f(y)|^p \Gamma(dx \times dy)\Big)^{q/p} \frac{dt}{t}\Big)^{p/q}$$

and

$$B = 2(a-1)^p\Big(\int_0^\infty \Gamma(\overline{M_{at}} \times (\mathcal{X}\backslash M_t)^{q/p} t^{q-1} dt\Big)^{p/q}.$$

Since $q \ge p$ we have

$$A \le 2\Big(\int_0^\infty \frac{\gamma(t) - \gamma(at)}{t} dt\Big)^{p/q},$$

where

$$\gamma(t) = \Big(\int_{M_t\times\mathcal{X}} |f(x) - f(y)|^p \Gamma(dx \times dy)\Big)^{q/p}.$$

Using (31), we obtain

$$A \le 2(\log a)^{p/q}\, \langle f\rangle_p^p. \tag{53}$$

Let us estimate B. Clearly,

$$B = 2(a-1)^p\Big(\int_0^\infty \Big(\int_{\mathcal{X}^2} \varkappa(x,\overline{M_{at}})\,\varkappa(y,\mathcal{X}\backslash M_t)\,\Gamma(dx\times dy)\Big)^{q/p} t^{q-1}dt\Big)^{p/q},$$

where $\varkappa(\cdot, S)$ is the characteristic function of a set S. By Minkowski's inequality,

$$B \le 2(a-1)^p \int_{\mathcal{X}^2}\Big(\int_0^\infty \varkappa(x,\overline{M_{at}})\,\varkappa(y,\mathcal{X}\backslash M_t)\,t^{q-1}dt\Big)^{p/q}\Gamma(dx\times dy)$$
$$= \frac{2(a-1)^p}{q^{p/q}a^p}\int_{\mathcal{X}^2}\big(|f(x)|^q - a^q|f(y)|^q\big)_+^{p/q}\Gamma(dx\times dy).$$

Obviously, the inequality $|f(x)| \ge a|f(y)|$ implies

$$|f(x)|^q - a^q|f(y)|^q \le |f(x)|^q \le \frac{a^q}{(a-1)^q}\big(|f(x)| - |f(y)|\big)_+^q.$$

Hence

$$B \le q^{-p/q}\langle f\rangle_p^p.$$

Summing up this estimate and (53), we arrive at (13) with

$$c(a,p,q) = \frac{1+2(q\log a)^{p/q}}{(a-1)^p q^{p/q}}.$$

The proof is complete. $\Box$

To show the usefulness of inequality (13) we characterize a two-weight Sobolev type inequality involving the seminorm $\langle f\rangle_{p,\Gamma}$ (see [M7] for other applications of (13)).

Corollary 3. *Let $q \ge p \ge 1$, $q \ge r > 0$, and let μ and ν be locally finite nonnegative Radon measures on $\mathcal{X}$. Inequality*

$$\int_{\mathcal{X}} |f|^q d\mu \le C\Big(\langle f\rangle_{p,\Gamma}^q + \Big(\int_{\mathcal{X}}|f|^r d\nu\Big)^{q/r}\Big) \tag{54}$$

holds for every $f \in \mathrm{dom}(\mathcal{F}_p)\cap C_0(\mathcal{X})$ if and only if all bounded open sets g and G in $\mathcal{X}$ such that $\overline{g}\subset G$, satisfy the inequality

$$\mu(g) \le Q\big(\mathrm{cap}_{p,\Gamma}(\overline{g}, G)^{q/p} + \nu(G)^{q/r}\big). \tag{55}$$

The best constants C and Q in (54) *and* (55) *are related by* $Q \le C \le c(p,q)\,Q$.

Proof. The necessity of (55) and the estimate $Q \le C$ are obtained by putting an arbitrary function $f \in \mathrm{dom}(\mathcal{F}_p)\cap C_0(\mathcal{X})$ subject to $f = 1$ on g, $f = 0$ on $\mathcal{X}\backslash G$, $0 \le f \le 1$, into (54).

The sufficiency of (55) results by the following argument:

$$\int_{\mathcal{X}} |f|^q d\mu = \int_0^\infty \mu(M_t)d(t^q)$$
$$\le Q\Big(\int_0^\infty \mathrm{cap}_{p,\Gamma}(\overline{M_{at}}, M_t)^{q/p}d(t^q) + \int_0^\infty \nu(M_t)^{q/r}d(t^q)\Big)$$
$$\le Q\Big(c(a,p,q)^{q/p}\langle f\rangle_{p,\Gamma}^q + \Big(\int_{\mathcal{X}}|f|^r d\nu\Big)^{q/r}\Big),$$

where $c(a,p,q)$ is the same constant as in (13). The proof is complete. $\Box$

Remark 4. Using the obvious identity

$$\langle |f|\rangle_{1,\Gamma} = \int_0^\infty \Gamma(M_t \times (\mathcal{X}\backslash M_t))dt$$

instead of the conductor inequality (13), we deduce with the same argument that the inequality

$$\Big(\int_{\mathcal{X}} |f|^q d\mu\Big)^{1/q} \le C\Big(\langle f\rangle_{1,\Gamma} + \int_{\mathcal{X}} |f| d\nu\Big) \tag{56}$$

with $q \ge 1$ holds if and only if for all bounded open sets $g \subset \mathcal{X}$

$$\mu(g)^{1/q} \le C(\ \Gamma(g \times (\mathcal{X}\backslash g)) + \nu(g)\ )$$

with the same value of C as in (56).

5 Sharp capacitary inequalities and their applications

Let Ω denote an open set in $\mathbb{R}^n$ and let the function

$$\Omega \times \mathbb{R}^n \ni (x,z) \to \phi(x,z) \in \mathbb{R}$$

be continuous and positively homogeneous of degree 1 with respect to y. Clearly, the measure

$$\mathcal{F}_p[f] := |\phi(x, \operatorname{grad} f(x))|^p dx$$

satisfies (18), (19), and (35). Hence, (40) implies the inequality

$$\Big(\int_0^\infty \operatorname{cap}_p(\overline{M_t}, \Omega)^{q/p} d(t^q)\Big)^{1/q} \le C\Big(\int_\Omega |\phi(x, \operatorname{grad} f(x))|^p dx\Big)^{1/p}, \tag{57}$$

where cap_p is the p-capacitance corresponding to the integral (8), $C = const > 0$, and f is an arbitrary function in $C_0^\infty(\Omega)$. The next assertion gives the sharp value of C for $q > p$. In case $q = p$ the sharp value of C is given by (3) and is obtained by the same argument.

Proposition 1. *Inequality* (57) *with* $q > p \ge 1$ *holds with*

$$C = \Big(\frac{\Gamma\big(\frac{pq}{q-p}\big)}{\Gamma\big(\frac{q}{q-p}\big)\Gamma\big(p\frac{q-1}{q-p}\big)}\Big)^{1/p-1/q}. \tag{58}$$

This value of C *is sharp if either* Ω *is a ball or* $\Omega = \mathbb{R}^n$.

Proof. Let

$$\psi(t) = \int_t^\infty \Big(\int_{|f(x)|=\tau} |\phi(x, N(x))|^p |\operatorname{grad} f(x)|^{p-1} ds(x)\Big)^{1/(1-p)} d\tau$$

with ds standing for the surface element and $N(x)$ denoting the normal vector at x directed inward M_τ. Further, let $t(\psi)$ denote the inverse function of $\psi(t)$. Then

$$\int_\Omega |\phi(x, \operatorname{grad} f(x))|^p dx = \int_0^\infty |t'(\psi)|^p d\psi \tag{59}$$

(see either [M4] or Sect. 22.2 and 2.3 of [M6] for more details). By Bliss' inequality [Bl]

$$\Big(\int_0^\infty t(\psi)^q \frac{d\psi}{\psi^{1+q(p-1)/p}}\Big)^{1/q} \le \Big(\frac{p}{q(p-1)}\Big)^{1/q} C\, \Big(\int_0^\infty |t'(\psi)|^p d\psi\Big)^{1/p}, \tag{60}$$

with C as in (58), and by (59) this is equivalent to

$$\Big(\int_0^\infty \frac{d(t(\psi)^q)}{\psi^{q(p-1)/p}}\Big)^{1/q} \le C\Big(\int_\Omega |\phi(x, \operatorname{grad} f(x))|^p dx\Big)^{1/p}.$$

In order to obtain (57) with C given by (58) it remains to recall that

$$\operatorname{cap}_p(\overline{M_t}) \le \frac{1}{\psi(t)^{p-1}} \tag{61}$$

(see either [M4] or Lemma 2.2.2/1 in [M6]). The constant (58) is best possible since (57) becomes equality for radial functions. $\square$

Following [M4] (see also [M6], Sect. 2.2.1), we introduce the weighted perimeter minimizing function σ on $(0, \infty)$ by

$$\sigma(m) := \inf \int_{\partial g} |\phi(x, N(x))|^p ds(x), \tag{62}$$

where the infimum is extended over all bounded open sets g with smooth boundaries subject to

$$m_n(g) \ge m.$$

According to [M4] (see also Corollary 2.2.3/2 in [M6]), the following isocapacitance inequality holds

$$\operatorname{cap}_p(\overline{g}, G) \ge \Big(\int_{m_n(g)}^{m_n(G)} \frac{dm}{\sigma(m)^{p'}}\Big)^{1-p}. \tag{63}$$

Therefore, (57) leads to

Corollary 4. *For all $f \in C_0^\infty(\Omega)$*

$$\Big(\int_0^\infty \Big(\int_{m_n(M_t)}^{m_n(\Omega)} \frac{dm}{\sigma(m)^{p'}}\Big)^{-q/p'} d(t^q)\Big)^{1/q} \le C\Big(\int_\Omega |\phi(x, \operatorname{grad} f(x))|^p dx\Big)^{1/p} \tag{64}$$

with $q > p$ and C defined by (58). *For $p = 1$ the last inequality should be replaced by*

$$\Big(\int_0^\infty \sigma(m_n(M_t))^q d(t^q)\Big)^{1/q} \le \int_\Omega |\phi(x, \operatorname{grad} f(x))| dx \tag{65}$$

with $q \ge 1$. $\square$

By the way, this corollary, combined with the classical isoperimetric inequality

$$s(\partial g) \ge n^{1/n'} \omega_n^{1/n} m_n(g)^{1/n'},$$

immediately gives the following well-known sharp result.

Corollary 5. ([FF] and [M1] for $p = 1$, [Ro], [Au], [Tal] for $p > 1$) *Let $n > p \ge 1$ and $q = pn(n-p)^{-1}$. Then every $f \in C_0^\infty(\mathbb{R}^n)$ satisfies the Sobolev inequality* (16) *with*

$$c = \pi^{-1/2} n^{-1/2} \Big(\frac{p-1}{n-p}\Big)^{1/p'} \Big(\frac{\Gamma(n)\Gamma(1+n/2)}{\Gamma(n/p)\Gamma(1+n-n/p)}\Big)^{1/n}.$$

The next assertion resulting from (59) and (61) shows that a quite general capacitary inequality is a consequence of a certain inequality for functions of one variable.

Theorem 3. *Let α and β be positive nondecreasing functions on $(0,\infty)$ such that*

$$\sup \int_0^\infty \beta(\psi^{1-p}) d(\alpha(t(\psi))) < \infty, \tag{66}$$

with the supremum taken over all absolutely continuous functions $[0,\infty) \ni \psi \to t(\psi) \geq 0$ subject to $t(0) = 0$ and

$$\int_0^\infty |t'(\psi)|^p d\psi \leq 1. \tag{67}$$

Then

$$\sup \int_0^\infty \beta(\mathrm{cap}_p(\overline{M_t}, \Omega)) d\alpha(t) < \infty \tag{68}$$

with the supremum extended over all f subject to

$$\int_\Omega |\phi(x, \mathrm{grad}\, f(x))|^p dx \leq 1. \tag{69}$$

The least upper bounds (66) *and* (68) *coincide.* $\Box$

In fact, the above Proposition 1 is a particular case of Theorem 3 corresponding to the choice

$$\alpha(t) = t^q \text{ and } \beta(\xi) = \xi^{q/p}.$$

The next result is another consequence of Theorem 3.

Proposition 2. *For every $c > 0$*

$$\sup \int_0^\infty \exp\Big(\frac{-c}{\mathrm{cap}_p(\overline{M_t}, \Omega)^{1/(p-1)}}\Big) d(\exp(c\, t^{p'})) < \infty, \tag{70}$$

where the supremum is taken over all $f \in C_0^\infty(\Omega)$ subject to (69) *and $p' = p/(p-1)$, $p > 1$.*

Proof. It follows from a theorem by Jodeit [Jo] that

$$\sup \int_0^\infty \exp(t(\psi)^{p'} - \psi) d\psi < \infty,$$

with the supremum taken over all absolutely continuous functions $[0,\infty) \ni \psi \to t(\psi) \geq 0$ subject to $t(0) = 0$ and (67). Hence, for every $c > 0$,

$$\sup \int_0^\infty \exp(c\, t(\psi)^{p'} - c\, \psi) d\psi < \infty.$$

It remains to refer to Theorem 3 with

$$\alpha(t) = \exp(c\, t^{p'}) \text{ and } \beta(\xi) = \exp(-c\, \xi^{1/(1-p)}). \ \Box$$

A direct consequence of Proposition 2 and the isocapacitance inequality (44) is the following celebrated Moser's result.

Corollary 6. [Mo] *Let* $m_n(\Omega) < \infty$ *and let*

$$\{f\} := \{f \in C_0^\infty(\Omega) : \ \|\mathrm{grad}\, f\|_{L_n(\Omega)} \leq 1\}.$$

Then

$$\sup_{\{f\}} \int_\Omega \exp(n\omega_n^{1/(n-1)}|f(x)|^{n'})\, dx < \infty.$$

Proof. The first inequality (44) can be written as

$$m_n(\overline{g}) \leq m_n(G) \exp\Big(-n\omega_n^{1/(n-1)} \mathrm{cap}_n(\overline{g}, G)^{1/(1-n)}\Big).$$

Hence, putting $c = n\omega_n^{1/(n-1)}$ and $p = n$ in (70) we obtain

$$\int_0^\infty m_n(M_t) d \exp\big(n\omega_n^{1/(n-1)} t^{n'}\big) < \infty.$$

The result follows.

Remark 5. One needs no changes in proofs to see that the main results of this section, Propositions 1 and 2 as well as Theorem 3, hold true if Ω is an open subset of a Riemannian manifold, $\mathrm{grad}\, f$ is the Riemannian gradient, and m_n is the Riemannian measure.

We can go even further extending the results just mentioned to the measure valued operator $\mathcal{F}_p[f]$ in Section 2 subject to the condition

$$\mathcal{F}_p[\lambda(f)] = |\lambda'(f)|^p \mathcal{F}_p[f] \tag{71}$$

with the same λ as in (19). In fact, (71) implies

$$\int_{\mathcal{X}} \mathcal{F}_p[f] = \int_0^\infty |t'(\psi)|^p d\psi, \tag{72}$$

where $t(\psi)$ is the inverse of the function

$$\psi(t) = \int_t^\infty \Big| \frac{d}{d\tau} \mathcal{F}_p[f](M_\tau) \Big|^{1/(1-p)} d\tau.$$

Identity (72) is the core of the proof of Theorem 3.

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