MATHEMATICAL METHODS IN THE APPLIED SCIENCES
*Math. Meth. Appl. Sci.* 2008; **31**:2055–2068
Published online 25 March 2008 in Wiley InterScience
(www.interscience.wiley.com) DOI: 10.1002/mma.1006
MOS subject classification: 35 J 25; 58 J 37

# Uniform asymptotic approximations of Green's functions in a long rod

V. Maz'ya[1,2,3] and A. Movchan[1,*,†]

[1]*Department of Mathematical Sciences, University of Liverpool, Liverpool L69 3BX, U.K.*
[2]*Department of Mathematics, Ohio State University, 231 W 18th Avenue, Columbus, OH 43210, U.S.A.*
[3]*Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden*

SUMMARY

Asymptotic approximations for Green's function for the operator $-\Delta$ in a long rod are derived. These approximations are uniformly valid over the whole domain including the end regions of the rod. Connections are established between the asymptotic approximations in a long rod and the asymptotics in thin domains. Overview of asymptotic approximations of Green's kernels in a domain with a small hole and domains with singularly perturbed smooth or conical boundaries is also given. Copyright © 2008 John Wiley & Sons, Ltd.

KEY WORDS: uniform asymptotic approximations of Green's functions; singular perturbations

*Dedicated to Professor W. L. Wendland on the occasion of his 70th birthday*

## 1. INTRODUCTION

The interest to the asymptotic analysis of Green's functions for domains with perturbed boundaries was initiated by the classical work of Hadamard [1]. The question of uniform asymptotic approximations of Green's functions for boundary value problems in singularly and regularly perturbed domains was addressed in [2], and the detailed analysis for the Dirichlet problems in domains with small holes was presented in [3]. Although some types of asymptotic approximations for Green's functions in singularly perturbed domains (for example, domains with small holes) have already been used in the existing literature (see, for example, [4, 5]), the question of uniformity of such approximations remained open until recently.

*Correspondence to: A. Movchan, Department of Mathematical Sciences, University of Liverpool, Liverpool L69 3BX, U.K.
†E-mail: abm@liv.ac.uk

 *Received 8 November 2007*

The structure of this paper can be described as follows. Section 2 gives an overview of several results for asymptotic approximations of Green's kernels in domains with small holes or small inclusions as well as in domains with singularly perturbed smooth or conical exterior boundaries. Section 3 presents a detailed analysis of Green's function of the Dirichlet–Neumann problem in a long cylindrical body. We introduce the notion of a capacitary potential and its asymptotic approximation in the elongated domain (see Section 3.1) and construct an asymptotic approximation of Green's function in the long rod in Sections 3.2 and 3.3. A version of the method of compound asymptotic expansions of solutions to boundary value problems in singularly perturbed domains, developed in [6], is used here to construct uniform asymptotic approximations of Green's kernels.

## 2. OVERVIEW OF SOME BACKGROUND WORK

The recent papers [2, 3] give uniform asymptotic approximations of Green's kernels for several types of boundary value problems in singularly perturbed domains. To illustrate a concept of uniform asymptotic approximations for Green's functions, we begin with an example of a domain with a small hole.

### *2.1. The Dirichlet problem in a domain with a small hole*

Let $\Omega$ and $\omega$ be bounded domains in $\mathbf{R}^n$, $n>2$. Assume that $\Omega$ and $\omega$ contain the origin $O$ and introduce the domain $\omega_\varepsilon=\{\mathbf{x}:\varepsilon^{-1}\mathbf{x}\in\omega\}$, as shown in Figure 1. Without loss of generality, we can also assume that the minimum distance between the origin and the points of $\partial\Omega$ as well as the maximum distance between the origin and the points of $\partial\omega$ are equal to 1. Let $G_\varepsilon$ be Green's function of the Dirichlet problem for the Laplace operator in $\Omega_\varepsilon=\Omega\setminus\overline{\omega}_\varepsilon$. The notation $|S^{n-1}|$ is used for the $(n-1)$-dimensional measure of the unit sphere. By $G$ and $\mathfrak{G}$ we denote Green's functions of the Dirichlet problems in $\Omega$ and $\mathbf{R}^n\setminus\overline{\omega}$, respectively. Also $H$ stands for the regular part of $G$, that is, $H(\mathbf{x},\mathbf{y})=((n-2)|S^{n-1}|)^{-1}|\mathbf{x}-\mathbf{y}|^{2-n}-G(\mathbf{x},\mathbf{y})$, and $P$ denotes the harmonic

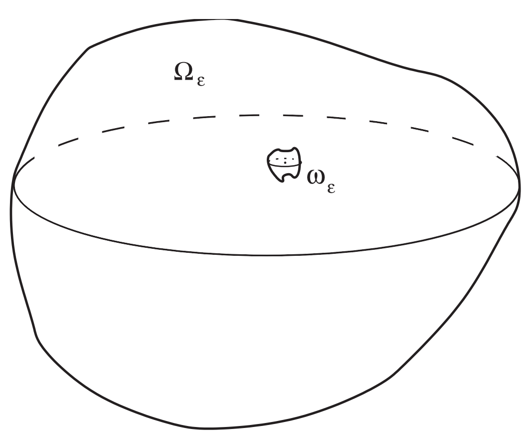


Figure 1. A domain with a small hole.



DOI: 10.1002/mma

capacitary potential of $\overline{\omega}$. The following asymptotic formula holds (see [2]):

$$\begin{aligned} G_\varepsilon(\mathbf{x},\mathbf{y}) &= G(\mathbf{x},\mathbf{y})+\varepsilon^{2-n}\mathfrak{G}(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y})-((n-2)|S^{n-1}|)^{-1}|\mathbf{x}-\mathbf{y}|^{2-n} \\ &\quad + H(0,\mathbf{y})P(\varepsilon^{-1}\mathbf{x})+H(\mathbf{x},0)P(\varepsilon^{-1}\mathbf{y})-H(0,0)P(\varepsilon^{-1}\mathbf{x})P(\varepsilon^{-1}\mathbf{y}) \\ &\quad -\varepsilon^{n-2}\operatorname{cap}\overline{\omega}H(\mathbf{x},0)H(0,\mathbf{y})+O\left(\frac{\varepsilon^{n-1}}{(\min\{|\mathbf{x}|,|\mathbf{y}|\})^{n-2}}\right) \end{aligned} \tag{1}$$

uniformly with respect to $\mathbf{x}$ and $\mathbf{y}$ in $\Omega_\varepsilon$. Note that the remainder term in (1) is $O(\varepsilon)$ on $\partial\omega_\varepsilon$ and $O(\varepsilon^{n-1})$ on $\partial\Omega$.

Although the above formula is uniformly valid in the whole domain $\Omega_\varepsilon$, it may look cumbersome. The formula can be simplified if additional constraints are imposed on the positions of the points $\mathbf{x}$ and $\mathbf{y}$. Namely, if $\min\{|\mathbf{x}|,|\mathbf{y}|\}>2\varepsilon$, then

$$G_\varepsilon(\mathbf{x},\mathbf{y})=G(\mathbf{x},\mathbf{y})-\varepsilon^{n-2}\operatorname{cap}\overline{\omega}G(\mathbf{x},0)G(0,\mathbf{y})+O\left(\frac{\varepsilon^{n-1}}{(|\mathbf{x}||\mathbf{y}|)^{n-2}\min\{|\mathbf{x}|,|\mathbf{y}|\}}\right)$$

On the other hand, if $\max\{|\mathbf{x}|,|\mathbf{y}|\}<\frac{1}{2}$, then

$$G_\varepsilon(\mathbf{x},\mathbf{y})=\varepsilon^{2-n}\mathfrak{G}\left(\frac{\mathbf{x}}{\varepsilon},\frac{\mathbf{y}}{\varepsilon}\right)-H(0,0)(P(\varepsilon^{-1}\mathbf{x})-1)(P(\varepsilon^{-1}\mathbf{y})-1)+O(\max\{|\mathbf{x}|,|\mathbf{y}|\})$$

The asymptotic approximations above employ solutions of model problems defined in $\Omega$ and $\mathbf{R}^n\setminus\overline{\omega}$, independent of the small parameter $\varepsilon$, and such solutions can be efficiently implemented into the numerical algorithms incorporating the asymptotic formulae for Green's functions.

The asymptotic approximation of Green's function for a two-dimensional domain with a small hole can also be developed (see [3]), and it has the $\log\varepsilon$ dependence, as outlined below. The capacitary potential $P_\varepsilon$ for a two-dimensional domain $\Omega_\varepsilon$ with a small hole $\omega_\varepsilon$ is defined as a solution of the boundary value problem:

$$\begin{aligned} \Delta P_\varepsilon(\mathbf{x}) &= 0, \quad \mathbf{x}\in\Omega_\varepsilon \\ P_\varepsilon(\mathbf{x}) &= 0, \quad \mathbf{x}\in\partial\Omega \\ P_\varepsilon(\mathbf{x}) &= 1, \quad \mathbf{x}\in\partial\omega_\varepsilon \end{aligned}$$

Its uniform asymptotic approximation, as $\varepsilon\to 0$, is given by the following formula:

$$P_\varepsilon(\mathbf{x})\sim\frac{-G(\mathbf{x},0)+\zeta(\mathbf{x}/\varepsilon)-(2\pi)^{-1}\log(|\mathbf{x}|/\varepsilon)-\zeta_\infty}{(2\pi)^{-1}\log\varepsilon+H(0,0)-\zeta_\infty}$$

where $H(\mathbf{x},\mathbf{y})$ is the regular part of Green's function $G(\mathbf{x},\mathbf{y})$ in the limit domain $\Omega$ without the hole, and the quantities $\zeta$ and $\zeta_\infty$ are defined as

$$\zeta(\boldsymbol{\eta})=\lim_{|\boldsymbol{\xi}|\to\infty}g(\boldsymbol{\xi},\boldsymbol{\eta})$$

and

$$\zeta_\infty=\lim_{|\boldsymbol{\eta}|\to\infty}\{\zeta(\boldsymbol{\eta})-(2\pi)^{-1}\log|\boldsymbol{\eta}|\}$$

Here, $g(\boldsymbol{\xi}, \boldsymbol{\eta})$ stands for Green's function in the unbounded model domain $\mathbf{R}^2 \setminus \overline{\omega}$. The uniform asymptotic approximation of Green's function $G_\varepsilon$ in a two-dimensional domain with a small hole has the form

$$
\begin{aligned}
G_\varepsilon(\mathbf{x}, \mathbf{y}) = {} & G(\mathbf{x}, \mathbf{y}) + g(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) + (2\pi)^{-1} \log(\varepsilon^{-1}|\mathbf{x}-\mathbf{y}|) \\
& + \frac{\left((2\pi)^{-1}\log\varepsilon + \zeta\left(\dfrac{\mathbf{x}}{\varepsilon}\right) - \zeta_\infty + H(\mathbf{x}, 0)\right)\left((2\pi)^{-1}\log\varepsilon + \zeta\left(\dfrac{\mathbf{y}}{\varepsilon}\right) - \zeta_\infty + H(0, \mathbf{y})\right)}{(2\pi)^{-1}\log\varepsilon + H(0,0) - \zeta_\infty} \\
& - \zeta(\varepsilon^{-1}\mathbf{x}) - \zeta(\varepsilon^{-1}\mathbf{y}) + \zeta_\infty + O(\varepsilon)
\end{aligned} \tag{2}
$$

We note that the structure of this asymptotic approximation resembles the one of formula (22) (and formula (23)), constructed for an elongated body in the sequel of the paper; this resemblance becomes apparent if one takes into account a logarithmic transformation of coordinates, which can be used to link the corresponding geometries.

The analysis developed for domains with small holes can be extended to singularly perturbed domains of other shapes and, in particular, to asymptotic approximations for Green's kernels in domains with singularly perturbed exterior boundaries (see Figures 2 and 4(a)).

### 2.2. *Perturbation of a smooth exterior boundary*

Consider an example of a bounded domain $\Omega_\varepsilon^-$ in $\mathbf{R}^3$, as shown in Figure 2. Let $\gamma_\varepsilon^-$ denote the perturbed small part of the boundary, and $l$ be a flat part of the boundary surrounding $\gamma_\varepsilon^-$, whereas $\Gamma^-$ is the remaining unperturbed part of the exterior surface.

Green's function for the Dirichlet–Neumann boundary value problem in $\Omega_\varepsilon^-$ is introduced as a solution of the following boundary value problem:

$$
\begin{aligned}
\Delta G_\varepsilon(\mathbf{x}, \mathbf{y}) + \delta(\mathbf{x}-\mathbf{y}) = 0, &\quad \mathbf{x}, \mathbf{y} \in \Omega_\varepsilon^- \\
G_\varepsilon(\mathbf{x}, \mathbf{y}) = 0, &\quad \mathbf{x} \in \gamma_\varepsilon^- \cup \Gamma^-, \ \ \mathbf{y} \in \Omega_\varepsilon^- \\
\frac{\partial G_\varepsilon}{\partial n_x}(\mathbf{x}, \mathbf{y}) = 0, &\quad \mathbf{x} \in l, \ \ \mathbf{y} \in \Omega_\varepsilon^-
\end{aligned}
$$

To construct an asymptotic approximation of $G_\varepsilon$, one also needs model limit domains shown in Figure 3: the unperturbed limit domain $\Omega^-$ and the unbounded domain $D^-$ corresponding

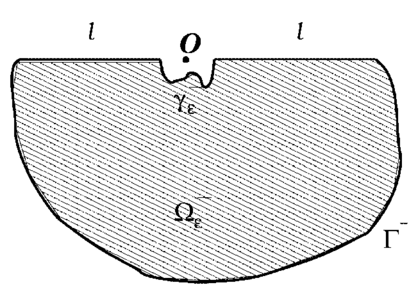


Figure 2. Domain with the perturbed boundary.



DOI: 10.1002/mma

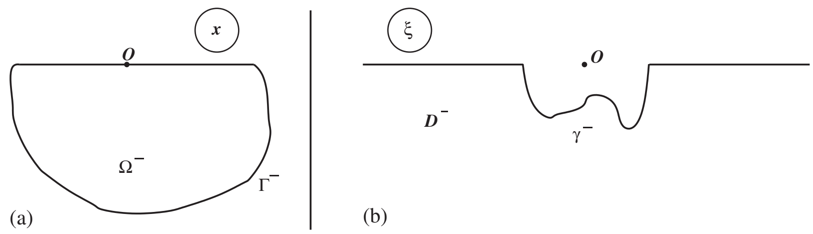


Figure 3. (a) Unperturbed domain $\Omega^-$ and (b) unbounded model domain $D^-$.

to boundary layers near the perturbed boundary. Let $G_{\Omega^-}$ and $g_{D^-}$ be Green's functions of the corresponding mixed boundary value problems in $\Omega^-$ and $D^-$. By $H_{\Omega^-}$ we denote the regular part of $G_{\Omega^-}$. The capacitary potential is introduced as a function $P_{\gamma^-}$, harmonic in $D^-$, which satisfies the homogeneous Neumann condition on $(\partial D^-)\setminus\gamma^-$, equals to 1 on $\gamma^-$, and decays at infinity. Then, the asymptotic approximation for $G_\varepsilon$ takes the following form:

$$\begin{aligned} G_\varepsilon(\mathbf{x},\mathbf{y}) = {} & G_{\Omega^-}(\mathbf{x},\mathbf{y}) + \varepsilon^{-1} g_{D^-}(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) - (4\pi|\mathbf{x}-\mathbf{y}|)^{-1} \\ & + H_{\Omega^-}(0,\mathbf{y}) P_{\gamma^-}(\varepsilon^{-1}\mathbf{x}) + H_{\Omega^-}(\mathbf{x},0) P_{\gamma^-}(\varepsilon^{-1}\mathbf{y}) \\ & - H_{\Omega^-}(0,0) P_{\gamma^-}(\varepsilon^{-1}\mathbf{x}) P_{\gamma^-}(\varepsilon^{-1}\mathbf{y}) + O(\varepsilon) \end{aligned} \tag{3}$$

For the particular example of the domain in Figure 2, one can make a mirror reflection across the flat part $l$ of the boundary so that the extended set represents a domain with a small hole. Then the method of images enables one to employ formula (1) and to deduce the asymptotic approximation (3). Indeed, other shapes of the perturbed exterior boundaries can be considered: in particular, this may include the case of a domain with a perturbed conical surface outlined below.

### 2.3. *Green's function for the Dirichlet–Neumann problem in a truncated cone*

Consider an example involving a three-dimensional domain shown in Figure 4(a). Let $K$ be an infinite cone $\{\mathbf{x}: |\mathbf{x}|>0, |\mathbf{x}|^{-1}\mathbf{x}\in\Xi\}$, where $\Xi$ is a subdomain of the unit sphere $S_1$ such that $S_1\setminus\Xi$ has a positive two-dimensional harmonic capacity. The notations $\omega$ and $\Omega$ are used for subdomains of $K$ separated from the vertex of $K$ and from infinity by surfaces $\gamma$ and $\Gamma$, respectively (see Figures 5 and 4(b)). By $\Omega_\varepsilon$ we denote a domain involving a 'small truncation' of the conical part of the boundary, i.e. $\Omega_\varepsilon = \{\mathbf{x}\in\Omega: \varepsilon^{-1}\mathbf{x}\in\omega\}$, where $\varepsilon$ stands for a small positive parameter. The conical surface is denoted by $l$, whereas $\gamma_\varepsilon = \{\mathbf{x}: \varepsilon^{-1}\mathbf{x}\in\gamma\}$ stands for the part of surface near the vertex of the truncated cone, as shown in Figure 4(a).

Let $G_\varepsilon$ and $G_{\text{cone}}$ be Green's functions for the Dirichlet–Neumann problem for $-\Delta$ in $\Omega_\varepsilon$ and the Neumann problem in $K$, respectively,

$$\Delta_x G_\varepsilon(\mathbf{x},\mathbf{y}) + \delta(\mathbf{x}-\mathbf{y}) = 0, \quad \mathbf{x},\mathbf{y}\in\Omega_\varepsilon \tag{4}$$

$$\frac{\partial G_\varepsilon}{\partial n_x}(\mathbf{x},\mathbf{y}) = 0, \quad \mathbf{x}\in l, \ \ \mathbf{y}\in\Omega_\varepsilon \tag{5}$$

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = 0, \quad \mathbf{x}\in\gamma_\varepsilon\cup\Gamma, \ \ \mathbf{y}\in\Omega_\varepsilon \tag{6}$$



DOI: 10.1002/mma

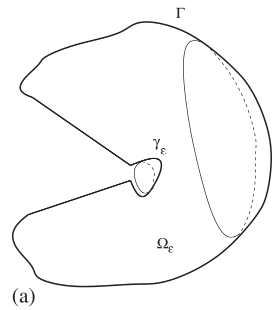


Figure 4. (a) A domain with a singularly perturbed conical boundary and (b) a limit unperturbed domain.

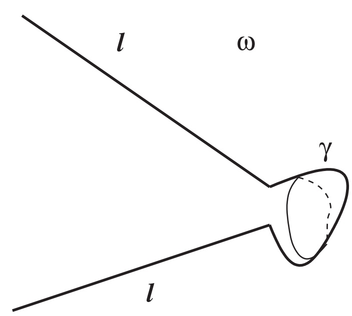


Figure 5. Scaled region in the vicinity of the perturbed boundary.

and

$$\Delta_x G_{\text{cone}}(\mathbf{x},\mathbf{y})+\delta(\mathbf{x}-\mathbf{y}) = 0, \quad \mathbf{x},\mathbf{y}\in K$$
$$\frac{\partial G_{\text{cone}}}{\partial n_x}(\mathbf{x},\mathbf{y}) = 0, \quad \mathbf{x}\in l,\ \ \mathbf{y}\in K$$
$$G_{\text{cone}}(\mathbf{x},\mathbf{y})\to 0, \quad |\mathbf{x}|\to\infty,\ \ \mathbf{y}\in K$$

Also the notation $G$ is used for Green's function of the Dirichlet–Neumann problems for $-\Delta$ in $\Omega$, that is, $G(\mathbf{x},\mathbf{y})=G_{\text{cone}}(\mathbf{x},\mathbf{y})-\Re(\mathbf{x},\mathbf{y})$, where the harmonic function $\Re(\mathbf{x},\mathbf{y})$ is a solution of the boundary value problem:

$$\Delta_x \Re(\mathbf{x},\mathbf{y})=0, \quad \mathbf{x},\mathbf{y}\in\Omega$$
$$\frac{\partial \Re}{\partial n_x}(\mathbf{x},\mathbf{y})=0, \quad \mathbf{x}\in l,\ \ \mathbf{y}\in\Omega$$
$$\Re(\mathbf{x},\mathbf{y})=G_{\text{cone}}(\mathbf{x},\mathbf{y}), \quad \mathbf{x}\in\Gamma,\ \ \mathbf{y}\in\Omega$$



DOI: 10.1002/mma

We note that

$$\Re(0,\mathbf{y}) = (s|\mathbf{y}|)^{-1} - G(0,\mathbf{y}) \quad \text{and} \quad \Re(\mathbf{x},0) = (s|\mathbf{x}|)^{-1} - G(\mathbf{x},0)$$

where $s$ is the area of $K \cap S_1$.

To describe the model fields in the unbounded domain $\omega$, we use the scaled coordinates $\boldsymbol{\xi} = \varepsilon^{-1}\mathbf{x}$, $\boldsymbol{\eta} = \varepsilon^{-1}\mathbf{y}$. Let $P(\boldsymbol{\xi})$ be a relative capacitary potential of $\gamma$, which solves the boundary value problem:

$$\begin{aligned}
\Delta P(\boldsymbol{\xi}) &= 0, \quad \boldsymbol{\xi} \in \omega \\
P(\boldsymbol{\xi}) &= 1, \quad \boldsymbol{\xi} \in \gamma \\
\frac{\partial P}{\partial n}(\boldsymbol{\xi}) &= 0, \quad \boldsymbol{\xi} \in l, \quad P(\boldsymbol{\xi}) \to 0 \quad \text{as } |\boldsymbol{\xi}| \to \infty
\end{aligned}$$

Green's function $g(\boldsymbol{\xi},\boldsymbol{\eta})$ for the unbounded domain $\omega$ is represented as $g(\boldsymbol{\xi},\boldsymbol{\eta}) = G_{\text{cone}}(\boldsymbol{\xi},\boldsymbol{\eta}) - \kappa(\boldsymbol{\xi},\boldsymbol{\eta})$, where $\kappa(\boldsymbol{\xi},\boldsymbol{\eta})$ is a solution of the model problem:

$$\begin{aligned}
\Delta_{\xi}\kappa(\boldsymbol{\xi},\boldsymbol{\eta}) &= 0, \quad \boldsymbol{\xi},\boldsymbol{\eta} \in \omega \\
\kappa(\boldsymbol{\xi},\boldsymbol{\eta}) &= G_{\text{cone}}(\boldsymbol{\xi},\boldsymbol{\eta}), \quad \boldsymbol{\xi} \in \gamma, \quad \boldsymbol{\eta} \in \omega \\
\frac{\partial \kappa}{\partial n_{\xi}}(\boldsymbol{\xi},\boldsymbol{\eta}) &= 0, \quad \boldsymbol{\xi} \in l, \quad \boldsymbol{\eta} \in \omega \\
\kappa(\boldsymbol{\xi},\boldsymbol{\eta}) &\to 0 \quad \text{as } |\boldsymbol{\xi}| \to \infty, \quad \boldsymbol{\eta} \in \omega
\end{aligned}$$

Then the required Green's function $G_{\varepsilon}(\mathbf{x},\mathbf{y})$, solving problem (4)–(6), is approximated by the following uniform asymptotic formula:

$$\begin{aligned}
G_{\varepsilon}(\mathbf{x},\mathbf{y}) = {} & G(\mathbf{x},\mathbf{y}) + \varepsilon^{-1} g(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) - G_{\text{cone}}(\mathbf{x},\mathbf{y}) + \Re(0,\mathbf{y}) P(\varepsilon^{-1}\mathbf{x}) + \Re(\mathbf{x},0) P(\varepsilon^{-1}\mathbf{y}) \\
& - \Re(0,0) P(\varepsilon^{-1}\mathbf{y}) P(\varepsilon^{-1}\mathbf{x}) + O(\varepsilon^{\lambda})
\end{aligned}$$

where $\lambda$ is a positive exponent depending on the cone opening.

In the following section, we present a new result including uniform asymptotic approximations of Green's functions for a mixed boundary value problem for the Laplacian in an elongated domain. The Dirichlet boundary conditions are set at the end regions of this domain, whereas the Neumann boundary conditions are prescribed on the lateral surface.

## 3. THE DIRICHLET–NEUMANN PROBLEM IN A LONG ROD

Let $C$ be the infinite cylinder $\{(\mathbf{x}', x_n) : \mathbf{x}' \in \omega, x_n \in \mathbf{R}\}$, where $\omega$ is a bounded domain in $\mathbf{R}^{n-1}$ with smooth boundary; here $n \geqslant 2$. Also let $C^{\pm}$ denote the Lipschitz subdomains of $C$ separated from $\pm\infty$ by surfaces $\gamma^{\pm}$, respectively.



DOI: 10.1002/mma

Let us introduce a positive number $M$ and the vector $\mathbf{M}=(\mathbf{O}', M)$, where $\mathbf{O}'$ is the origin of $\mathbf{R}^{n-1}$. It is assumed that the ratio (diam $\omega)/M$ is small.

A long rod $C_M$ is defined as follows:

$$C_M=\{\mathbf{x}:(\mathbf{x}-\mathbf{M})\in C^+, (\mathbf{x}+\mathbf{M})\in C^-\}$$

The lateral surface of the rod is denoted by $\Gamma$, as shown in Figure 6.

Let $G_M(\mathbf{x},\mathbf{y})$ denote the fundamental solution for $-\Delta$ in the domain $C_M$ subject to zero Neumann condition on the lateral surface $\Gamma$ and zero Dirichlet conditions on the end parts $\gamma^{\pm}$ of the boundary of the long rod:

$$\Delta_x G_M(\mathbf{x},\mathbf{y})+\delta(\mathbf{x}-\mathbf{y})=0, \quad \mathbf{x},\mathbf{y}\in C_M$$

$$\frac{\partial G_M}{\partial n_x}(\mathbf{x},\mathbf{y})=0, \quad \mathbf{x}\in\Gamma, \quad \mathbf{y}\in C_M$$

$$G_M(\mathbf{x},\mathbf{y})=0, \quad \mathbf{x}\in\gamma^{\pm}, \quad \mathbf{y}\in C_M$$

In order to obtain an approximation of $G_M$, we also introduce several model problems independent of the cylinder length $2M$.

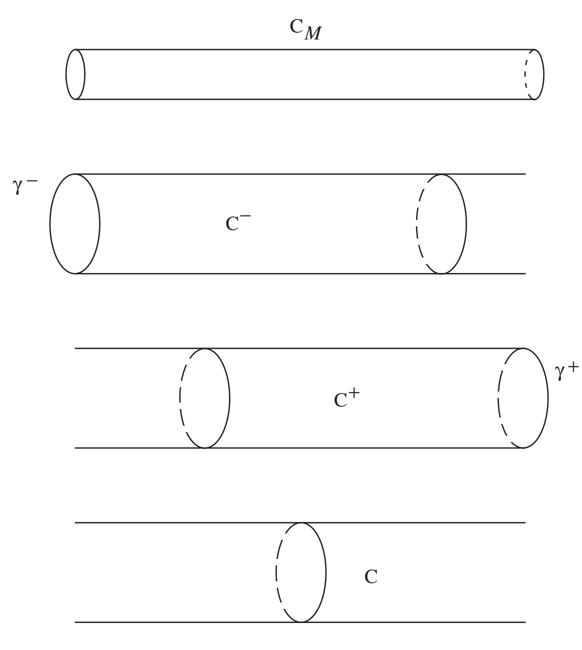


Figure 6. A long rod $C_M$ and associated unbounded model domains.



DOI: 10.1002/mma

We denote Green's function of the Neumann problem in $C$ by $G_\infty(\mathbf{x},\mathbf{y})$:

$$\Delta_x G_\infty(\mathbf{x},\mathbf{y})+\delta(\mathbf{x}-\mathbf{y})=0, \quad \mathbf{x},\mathbf{y}\in C$$

$$\frac{\partial G_\infty}{\partial n_x}(\mathbf{x},\mathbf{y})=0, \quad \mathbf{x}\in\Gamma,\ \ \mathbf{y}\in C$$

$$G_\infty(\mathbf{x},\mathbf{y})=-(2|\omega|)^{-1}|x_n-y_n|+O(\exp(-\alpha|x_n-y_n|)) \quad \text{as } |x_n|\to\infty$$

where $\alpha$ is a positive constant, and $|\omega|$ is the $(n-1)$-dimensional measure of $\omega$.

Similarly, $G^+$ and $G^-$ stand for the fundamental solutions for $-\Delta$ in the domains $C^\pm$, with the homogeneous boundary conditions defined as follows:

$$\Delta_x G^\pm(\mathbf{x}^\pm,\mathbf{y}^\pm)+\delta(\mathbf{x}^\pm-\mathbf{y}^\pm)=0, \quad \mathbf{x}^\pm,\mathbf{y}^\pm\in C^\pm$$

$$G^\pm(\mathbf{x}^\pm,\mathbf{y}^\pm)=0, \quad \mathbf{x}^\pm\in\gamma^\pm,\ \ \mathbf{y}^\pm\in C^\pm$$

$$\frac{\partial G^\pm}{\partial n_x}(\mathbf{x}^\pm,\mathbf{y}^\pm)=0, \quad \mathbf{x}^\pm\in\Gamma,\ \ \mathbf{y}^\pm\in C^\pm$$

and it is also assumed that $G^\pm(\mathbf{x}^\pm,\mathbf{y}^\pm)$ are bounded as $x_n^\pm\to\mp\infty$.

### *3.1. Capacitary potential*

The capacitary potential $P_M$ is defined as a solution of the Dirichlet–Neumann boundary value problem in $C_M$:

$$\Delta P_M(\mathbf{x})=0, \quad \mathbf{x}\in C_M \tag{7}$$

$$\frac{\partial P_M}{\partial n}(\mathbf{x})=0, \quad \mathbf{x}\in\Gamma \tag{8}$$

$$P_M(\mathbf{x})=1, \quad \mathbf{x}\in\gamma^- \ \text{ and } \ P_M(\mathbf{x})=0, \quad \mathbf{x}\in\gamma^+ \tag{9}$$

We shall also use the solutions $\zeta^\pm$ of the homogeneous Dirichlet–Neumann problems in semi-infinite domains $C^\pm$ as follows:

$$\Delta\zeta^\pm(\mathbf{x}^\pm)=0, \quad \mathbf{x}\in C^\pm \tag{10}$$

$$\frac{\partial\zeta^\pm}{\partial n}(\mathbf{x}^\pm)=0, \quad \mathbf{x}^\pm\in\Gamma \tag{11}$$

$$\zeta^\pm(\mathbf{x}^\pm)=0, \quad \mathbf{x}\in\gamma^\pm \tag{12}$$

and

$$\zeta^\pm(\mathbf{x}^\pm)=\mp x_n^\pm+\zeta_\infty^\pm+O(\exp(-\alpha|x_n^\pm|)) \quad \text{as } |x_n^\pm|\to\infty \tag{13}$$

where $\alpha$ is a positive constant, $\mathbf{x}^\pm=(\mathbf{x}',x_n\mp M)$ are local coordinates at the ends of the long rod $C_M$, and $\zeta_\infty^\pm$ are constant terms that depend on the geometry of the cross-section $\omega$ and the end parts $\gamma^\pm$ of the boundary of the long rod.



DOI: 10.1002/mma

*Theorem 1*
The following asymptotic formula, uniform with respect to $\mathbf{x}\in C_M$, for the capacitary potential $P_M(\mathbf{x})$ holds:

$$P_M(\mathbf{x})=\frac{M+x_n+\zeta_\infty^- - \zeta^-(\mathbf{x}^-)+\zeta^+(\mathbf{x}^+)}{2M+\zeta_\infty^+ + \zeta_\infty^-}+O(\exp(-\alpha M)) \tag{14}$$

Here, the functions $\zeta^\pm$, variables $\mathbf{x}^\pm$, and the constants $\zeta_\infty^\pm$ are the same as in (10)–(13), and $\alpha$ is a positive constant.

To prove this statement, we use the direct substitution of (14) into (8)–(9), which shows that the remainder term is a harmonic function satisfying homogeneous Neumann boundary conditions on the lateral surface of the rod and is exponentially small at the end parts $\gamma^\pm$ of the boundary. Then it remains to apply the estimate similar to Lemma 1.3 of Section 1.5 in [7].

### 3.2. *Asymptotic approximation of Green's function*

Let $H^\pm(\mathbf{x}^\pm,\mathbf{y}^\pm)$ be functions defined in semi-infinite domains $C^\pm$, and assume that they also satisfy the Dirichlet–Neumann boundary value problems:

$$\Delta_x H^\pm(\mathbf{x}^\pm,\mathbf{y}^\pm)=0, \quad \mathbf{x}^\pm,\mathbf{y}^\pm\in C^\pm \tag{15}$$

$$\frac{\partial H^\pm}{\partial n_x}(\mathbf{x}^\pm,\mathbf{y}^\pm)=0, \quad \mathbf{x}^\pm\in\Gamma,\ \ \mathbf{y}^\pm\in C^\pm \tag{16}$$

$$H^\pm(\mathbf{x}^\pm,\mathbf{y}^\pm)=G_\infty(\mathbf{x},\mathbf{y})+(2|\omega|)^{-1}\zeta^\pm(\mathbf{y}^\pm), \quad \mathbf{x}\in\gamma^\pm,\ \ \mathbf{y}^\pm\in C^\pm \tag{17}$$

and

$$H^\pm(\mathbf{x}^\pm,\mathbf{y}^\pm)\to 0 \quad \text{as } x_n^\pm\to\mp\infty \tag{18}$$

The asymptotic approximation is given by the following statement.

*Theorem 2*
Green's function $G_M(\mathbf{x},\mathbf{y})$ is approximated by the asymptotic formula, uniform with respect to $\mathbf{x},\mathbf{y}\in C_M$:

$$\begin{aligned}G_M(\mathbf{x},\mathbf{y})&=G_\infty(\mathbf{x},\mathbf{y})-H^+(\mathbf{x}^+,\mathbf{y}^+)-H^-(\mathbf{x}^-,\mathbf{y}^-)\\&\quad-\frac{\mathfrak{A}_M}{|\omega|}\left(\frac{1}{2}-P_M(\mathbf{x})\right)\left(\frac{1}{2}-P_M(\mathbf{y})\right)+\frac{\mathfrak{A}_M}{4|\omega|}+O(\exp(-\alpha M))\end{aligned} \tag{19}$$

where $\mathfrak{A}_M=2M+\zeta_\infty^+ + \zeta_\infty^-$, and $\alpha$ is a positive constant.

In the text below, we present a formal argument that leads to the asymptotic formula (19).



DOI: 10.1002/mma

Let

$$G_M(\mathbf{x},\mathbf{y}) = G_\infty(\mathbf{x},\mathbf{y}) - H_M^+(\mathbf{x},\mathbf{y}) - H_M^-(\mathbf{x},\mathbf{y}) \tag{20}$$

where the functions $H_M^\pm$ are defined as solutions of the boundary value problems:

$$\begin{aligned}
\Delta_x H_M^\pm(\mathbf{x},\mathbf{y}) &= 0, \quad \mathbf{x},\mathbf{y}\in C_M\\
\frac{\partial H_M^\pm}{\partial n}(\mathbf{x},\mathbf{y}) &= 0, \quad \mathbf{x}\in\Gamma, \ \ \mathbf{y}\in C_M\\
H_M^\pm(\mathbf{x},\mathbf{y}) &= G_\infty(\mathbf{x},\mathbf{y}), \quad \mathbf{x}\in\gamma^\pm, \ \ \mathbf{y}\in C_M\\
H_M^\pm(\mathbf{x},\mathbf{y}) &= 0, \quad \mathbf{x}\in\gamma^\mp, \ \ \mathbf{y}\in C_M
\end{aligned}$$

We note that the sum $\sum_\pm H_M^\pm$ is symmetric, i.e.

$$H_M^+(\mathbf{x},\mathbf{y}) + H_M^-(\mathbf{x},\mathbf{y}) = H_M^+(\mathbf{y},\mathbf{x}) + H_M^-(\mathbf{y},\mathbf{x})$$

The functions $H_M^\pm$ can be approximated by the formulae

$$\begin{aligned}
H_M^+(\mathbf{x},\mathbf{y}) &= H^+(\mathbf{x}^+,\mathbf{y}^+) - \frac{1}{2|\omega|}\zeta^+(\mathbf{y}^+)\\
&\quad - P_M(\mathbf{x})\left(H^+(\mathbf{x}^{+\prime},-\infty,\mathbf{y}^+) - \frac{1}{2|\omega|}\zeta^+(\mathbf{y}^+)\right) + h_M^+
\end{aligned}$$

and

$$\begin{aligned}
H_M^-(\mathbf{x},\mathbf{y}) &= H^-(\mathbf{x}^-,\mathbf{y}^-) - \frac{1}{2|\omega|}\zeta^-(\mathbf{y}^-)\\
&\quad - P_M(\mathbf{x})\left(H^-(\mathbf{x}^{-\prime},+\infty,\mathbf{y}^-) - \frac{1}{2|\omega|}\zeta^-(\mathbf{y}^-)\right) + h_M^-
\end{aligned}$$

with exponentially small remainder terms $h_M^\pm$. Applying Green's formula to the functions $H^\pm$ and $\zeta^\pm$ in the domains $C^\pm$, respectively, we deduce that

$$H^-(\mathbf{x}^{-\prime},+\infty,\mathbf{y}^-) = -\frac{1}{2|\omega|}\{\zeta^-(\mathbf{y}^-) - (M + \mathbf{y}_n + \zeta_-^\infty)\}$$

and

$$H^+(\mathbf{x}^{+\prime},-\infty,\mathbf{y}^+) = -\frac{1}{2|\omega|}\{\zeta^+(\mathbf{y}^+) - (M - \mathbf{y}_n + \zeta_+^\infty)\}$$



DOI: 10.1002/mma

Condition (13) yields

$$\lim_{\mathbf{y}_n^- \to +\infty} H^-(\mathbf{y}^{-\prime}, +\infty, \mathbf{y}^-) = 0$$

and

$$\lim_{\mathbf{y}_n^+ \to -\infty} H^+(\mathbf{y}^{+\prime}, -\infty, \mathbf{y}^+) = 0$$

If $\mathfrak{A} = 2M + \zeta_\infty^+ + \zeta_\infty^-$, then the following identity holds:

$$H_M^+(\mathbf{x},\mathbf{y}) + H_M^-(\mathbf{x},\mathbf{y}) = H^+(\mathbf{x}^+,\mathbf{y}^+) + H^-(\mathbf{x}^-,\mathbf{y}^-) + \frac{\mathfrak{A}}{|\omega|}\left(\frac{1}{2} - P_M(\mathbf{x})\right)\left(\frac{1}{2} - P_M(\mathbf{y})\right) - \frac{\mathfrak{A}_M}{4|\omega|} \tag{21}$$

Combining formulae (20) and (21) we deduce (19).

The direct substitution of (19) into (16) and (17) shows that the remainder term is a harmonic function satisfying homogeneous Neumann boundary conditions on the lateral surface of the rod, and it is exponentially small at the end parts $\gamma^\pm$ of the boundary. Applying the estimate similar to Lemma 1.3 of Section 1.5 in [7], we complete the proof.

*Example of Green's functions in model domains*. In some cases, Green's functions for model problems required for the above asymptotic approximation can be constructed in a simple form. As an illustration, we suggest an example involving a long rectangular strip. In this case, the function $G_\infty(\mathbf{x},\mathbf{y})$ is the Neumann function for the Laplacian in the infinite strip $\Pi = \{(x_1, x_2): -\infty < x_1 < \infty, |x_2| < 1/2\}$, given in the following form:

$$G_\infty(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(k, x_2, y_2)\exp(-\mathrm{i}k(x_1 - y_1))\,\mathrm{d}k$$

where

$$\tilde{G}(k, x_2, y_2) = \frac{\cosh(k(x_2+y_2)) + \cosh(k)\cosh(k(x_2-y_2))}{2k\sinh(k)} - \begin{cases} (2k)^{-1}\sinh(k(x_2-y_2)), & x_2 > y_2 \\ -(2k)^{-1}\sinh(k(x_2-y_2)), & x_2 < y_2 \end{cases}$$

Assuming that the end regions of the rectangular domain are 'flat', i.e. they are located on the vertical straight lines $x_1 = \pm M$, we can construct Green's functions $G_\pm$ for semi-infinite strips as follows:

$$G_\pm(\mathbf{x}^\pm, \mathbf{y}^\pm) = G_\infty(\mathbf{x}^\pm, y_1^\pm, y_2^\pm) - G_\infty(\mathbf{x}^\pm, -y_1^\pm, y_2^\pm)$$

These model fields are readily applicable in the asymptotic formula of Theorem 2.

### 3.3. *Green's function $G_M$ versus Green's functions for unbounded domains*

The result of Section 3.2 together with definitions of functions $G_\infty$ and $G^\pm$ lead to the following theorem.



DOI: 10.1002/mma

*Theorem 3*
Green's function $G_M(\mathbf{x},\mathbf{y})$ and the functions $G^{\pm}$, $G_\infty$ are related by the following asymptotic formula:

$$G_M(\mathbf{x},\mathbf{y}) = \sum_{\pm} G^{\pm}(\mathbf{x}^{\pm},\mathbf{y}^{\pm}) - G_\infty(\mathbf{x},\mathbf{y}) - \frac{1}{2|\omega|}\sum_{\pm}(\zeta^{\pm}(\mathbf{x}^{\pm})+\zeta^{\pm}(\mathbf{y}^{\pm}))$$
$$-\frac{\mathfrak{A}_M}{|\omega|}\left(\frac{1}{2}-P_M(\mathbf{x})\right)\left(\frac{1}{2}-P_M(\mathbf{y})\right)+\frac{\mathfrak{A}_M}{4|\omega|}+O(\exp(-\alpha M)) \tag{22}$$

where $\alpha$ is a positive constant independent of $M$.

*Corollary 1*
Formula (22) allows for an equivalent representation involving the model fields $\zeta^{\pm}$ defined as solutions of the boundary value problems (10)–(13):

$$G_M(\mathbf{x},\mathbf{y}) = \sum_{\pm} G^{\pm}(\mathbf{x}^{\pm},\mathbf{y}^{\pm}) - G_\infty(\mathbf{x},\mathbf{y}) + \frac{1}{4|\omega|}\left\{\mathfrak{A}_M - 2\sum_{\pm}(\zeta^{\pm}(\mathbf{x}^{\pm})+\zeta^{\pm}(\mathbf{y}^{\pm}))\right\}$$
$$-(|\omega|\mathfrak{A}_M)^{-1}\left(\mathbf{x}_n - \frac{1}{2}(\zeta^{+}_\infty-\zeta^{-}_\infty)+\zeta^{+}(\mathbf{x}^{+})-\zeta^{-}(\mathbf{x}^{-})\right)$$
$$\times\left(\mathbf{y}_n - \frac{1}{2}(\zeta^{+}_\infty-\zeta^{-}_\infty)+\zeta^{+}(\mathbf{y}^{+})-\zeta^{-}(\mathbf{y}^{-})\right)+O(\exp(-\alpha M)) \tag{23}$$

where $\alpha$ is a positive constant independent of $M$.

The above formulae can be simplified if we introduce additional constraints on the positions of the points $\mathbf{x}$ and $\mathbf{y}$ within $C_M$.

When the points $\mathbf{x}$ and $\mathbf{y}$ are 'far away' from the ends $\gamma^{\pm}$ of the long rod, the quantities $H^{\pm}$ become exponentially small; hence, we arrive to the following corollary.

*Corollary 2*
When $\min\{|\mathbf{x}\pm\mathbf{M}|/M, |\mathbf{y}\pm\mathbf{M}|/M\}\geqslant$Const, Green's function $G_M$ is approximated by the following formula:

$$G_M(\mathbf{x},\mathbf{y}) \sim G_\infty(\mathbf{x},\mathbf{y}) - (|\omega|\mathfrak{A}_M)^{-1}\left(x_n-\frac{1}{2}(\zeta^{+}_\infty-\zeta^{-}_\infty)\right)\left(y_n-\frac{1}{2}(\zeta^{+}_\infty-\zeta^{-}_\infty)\right)$$
$$+\frac{\mathfrak{A}_M}{4|\omega|} \quad \text{as } M\to\infty \tag{24}$$

Another simplified formula for Green's function can be presented for the case when the points $\mathbf{x}$ and $\mathbf{y}$ are sufficiently close to one of the ends of the rod.

*Corollary 3*
Assume that the points $\mathbf{x}$ and $\mathbf{y}$ are close to the left end $\gamma^{-}$ of the long rod $C_M$, i.e. $\max\{\mathbf{x}+M, \mathbf{y}+M\}\leqslant$Const. Then the function $G_M$ is approximated by the following formula:

$$G_M(\mathbf{x},\mathbf{y}) \sim G^{-}(\mathbf{x}^{-},\mathbf{y}^{-}) - |\omega|\frac{G^{-}(\mathbf{x}^{-\prime},+\infty,\mathbf{y}^{-})G^{-}(\mathbf{x}^{-},\mathbf{y}^{-\prime},+\infty)}{\mathfrak{A}_M} \quad \text{as } M\to\infty \tag{25}$$

Similar approximation is valid near the other end $\gamma^{+}$ of the long rod.



DOI: 10.1002/mma

### 3.4. The Dirichlet–Neumann problem in a thin rod

By rescaling, the above results can be used to find an asymptotic approximation for Green's function $G^{(\varepsilon)}$ in a thin rod rather than the long rod. Let a thin domain be defined by

$$C_\varepsilon = \{\mathbf{x} : \varepsilon^{-1}(\mathbf{x}-\mathbf{a}) \in C^+, \varepsilon^{-1}(\mathbf{x}+\mathbf{a}) \in C^-\}$$

where the notations $C^\pm$ are the same as in the beginning of Section 3 (see Figure 6), $2a$ is the length of the rod, and now $\varepsilon$ is a positive small parameter. As above, it is assumed that Green's function is subject to zero Neumann condition on the cylindrical part of $C_\varepsilon$ and zero Dirichlet condition on the remaining part of $\partial C_\varepsilon$.

*Theorem 4*

The following asymptotic formula for $G^{(\varepsilon)}(\mathbf{x},\mathbf{y})$, uniform with respect to $\mathbf{x},\mathbf{y} \in \Omega_\varepsilon$, holds:

$$\begin{aligned} G^{(\varepsilon)}(\mathbf{x},\mathbf{y}) = \varepsilon^{2-n} \Bigg\{ & G^+(\varepsilon^{-1}(\mathbf{x}-\mathbf{a}), \varepsilon^{-1}(\mathbf{y}-\mathbf{a})) + G^-(\varepsilon^{-1}(\mathbf{x}+\mathbf{a}), \varepsilon^{-1}(\mathbf{y}+\mathbf{a})) - G_\infty(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) \\ & - \varepsilon\{2|\omega|^{-1}a + \varepsilon(\zeta_\infty^+ + \zeta_\infty^-)\}^{-1} \left( \frac{x_n}{\varepsilon|\omega|} - \frac{1}{2}(\zeta_\infty^- - \zeta_\infty^+) + \zeta^+\left(\frac{\mathbf{x}-\mathbf{a}}{\varepsilon}\right) - \zeta^-\left(\frac{\mathbf{x}+\mathbf{a}}{\varepsilon}\right)\right) \\ & \times \left( \frac{y_n}{\varepsilon|\omega|} - \frac{1}{2}(\zeta_\infty^- - \zeta_\infty^+) + \zeta^+\left(\frac{\mathbf{y}-\mathbf{a}}{\varepsilon}\right) - \zeta^-\left(\frac{\mathbf{y}+\mathbf{a}}{\varepsilon}\right)\right) \\ & + \frac{1}{4}\left( (\varepsilon|\omega|)^{-1}2a + \zeta_\infty^- + \zeta_\infty^+ - 2\sum_\pm (\zeta^\pm(\varepsilon^{-1}(\mathbf{x}\mp\mathbf{a})) + \zeta^\pm(\varepsilon^{-1}(\mathbf{y}\mp\mathbf{a})))\right) \\ & + O(\exp(-\beta/\varepsilon)) \Bigg\} \end{aligned} \tag{26}$$

where $\beta$ is a positive constant independent of $\varepsilon$.

## REFERENCES


1. Hadamard J. Sur le problème d'analyse relatif à l'équilibre des plaques élastiques encastrées. *Mémoire couronné en 1907 par l'Académie des Sciences* 1907; **33**(4):515–629.
2. Maz'ya V, Movchan A. Uniform asymptotic formulae for Green's kernels in regularly and singularly perturbed domains. *Comptes Rendus de l'Académie des Sciences de Paris*, *Série I*, *Mathematique* 2006; **343**:185–190.
3. Maz'ya V, Movchan A. Uniform asymptotic formulae for Green's functions in singularly perturbed domains. *JCAM* 2007; **208**(1):194–206.
4. Ozawa S, Roppongi S. Heat kernel and singular variation of domains. *Osaka Journal of Mathematics* 1995; **32**(4):941–957.
5. Ozawa S. Approximation of Green's functions in a region with many obstacles. *Geometry and Analysis of Manifolds*, *Proceedings*, Katana-Kyoto. Springer Lecture Notes in Mathematics, vol. 1339. Springer: Berlin, 1987; 212–225.
6. Maz'ya V, Nazarov S, Plamenevskij B. *Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains*, vols 1–2. Birkhäuser: Basel, 2000.
7. Kozlov V, Maz'ya V, Movchan A. *Fields in Multi-structures*, *Asymptotic Analysis*. Oxford University Press: Oxford, 1999.




DOI: 10.1002/mma