Sharp pointwise estimates for analytic functions by the L_p-norm of the real part

Gershon Kresin * and Vladimir Maz'ya †

Abstract. We obtain sharp estimates of $|\Re\{e^{i\alpha(z)}(f(z)-f(0))\}|$ by the L_p-norm of $\Re f-\omega$ on the circle $|\zeta|=R$, where $|z|<R, 1\leq p\leq\infty$, and α is a real valued function on D_R. Here f is an analytic function in the disc $D_R=\{z:|z|<R\}$ whose real part is continuous on $\overline{D}_R$, ω is a real constant, and $\Re f-\omega$ is orthogonal to some continuous function Φ on the circle $|\zeta|=R$. We derive two types of estimates with vanishing and nonvanishing mean value of Φ. The cases $\Phi=0$ and $\Phi=1$ are discussed in more detail. In particular, we give explicit formulas for sharp constants in inequalities for $|\Re\{e^{i\alpha(z)}(f(z)-f(0))\}|$ with $p=1,2,\infty$. We also obtain estimates for $|f(z)-f(0)|$ in the class of analytic functions with two-sided bounds of $|\arg\{f(z)-f(0)\}|$. As a corollary, we find a sharp constant in the upper estimate of $|\Im f(z)-\Im f(0)|$ by $||\Re f-\Re f(0)||_p$ which generalizes the classical Carathéodory-Plemelj estimate with $p=\infty$.

Keywords: Analytic functions, real part, pointwise estimates, orthogonality condition

AMS: 30A10, 31A05, 31A20, 35J05

0 Introduction

In the present paper we consider an analytic function f on the disk $D_R=\{z:|z|<R\}$ whose real part is continuous on $\overline{D}_R$. We obtain sharp estimates for

$$\max_{|z|=r}|\Re\{e^{i\alpha(z)}(f(z)-f(0))\}|$$

by the L_p-norm of $\Re f-\omega$ on the circle ∂D_R, where $0\leq r<R,\ 1\leq p\leq\infty,\ \alpha$ is a real valued function on D_R, and ω is a real constant. We assume that

$$(\Re f-\omega,\Phi)=0, \tag{0.1}$$

where Φ is a continuous function defined on the circle $|\zeta|=R$.

*The Research Institute, The College of Judea and Samaria, Ariel, 44837, Israel

†Department of Mathematics, The Ohio State University, 231 West Eighteenth Avenue, Columbus, Ohio 43210-1174, USA

In Section 1 we prove the basic Lemma 1 which gives a general but somewhat implicit representation of the best constant $\mathcal{C}_{\Phi,\,p}(z,\alpha(z))$ in the estimate of $|\Re\{e^{i\alpha(z)}(f(z)-f(0))\}|$ by $||\Re f||_p$ with the orthogonality condition (0.1).

Section 2 concerns the inequality

$$|\Re\{e^{i\alpha(z)}(f(z)-f(0))\}| \le \mathcal{C}_{\Phi,\,p}\,(z,\alpha(z))\,||\Re f-\omega||_p \tag{0.2}$$

with $(\Re f,\Phi)=0$ and vanishing mean value of Φ for $|\zeta|=R$. By ω in (0.2) we mean an arbitrary real constant. The value $||\Re f-\omega||_p$ in the right-hand side of (0.2) can be replaced by $E_p(\Re f)$ which stands for the best approximation of $\Re f$ by a real constant in the norm of $L_p(\partial D_R)$.

The case $\Phi=0$ is treated in more details. As a corollary of Lemma 1 we find the representation for the sharp constant in (0.2)

$$\mathcal{C}_{0,\,p}(z,\alpha(z)) = R^{-1/p}C_{0,\,p}(r/R,\alpha(z)), \tag{0.3}$$

where

$$C_{0,\,p}(\gamma,\alpha)=\frac{\gamma}{\pi}\left\{\int_{-\pi}^{\pi}\left|\frac{\cos(\varphi-\alpha)-\gamma\cos\alpha}{1-2\gamma\cos\varphi+\gamma^2}\right|^{p/(p-1)}d\varphi\right\}^{(p-1)/p} \tag{0.4}$$

if $1<p<\infty$, and

$$C_{0,\,1}(\gamma,\alpha)=\frac{\gamma(1+\gamma|\cos\alpha|)}{\pi(1-\gamma^2)}, \tag{0.5}$$

$$C_{0,\,\infty}(\gamma,\alpha)=\frac{4}{\pi}\left\{\sin\alpha\log\frac{\gamma\sin\alpha+(1-\gamma^2\cos^2\alpha)^{1/2}}{(1-\gamma^2)^{1/2}}+\cos\alpha\arcsin\left(\gamma\cos\alpha\right)\right\}. \tag{0.6}$$

In particular, we obtain the equality $C_{0,\,2}(\gamma,\alpha)=\gamma[\pi(1-\gamma^2)]^{-1/2}$.

For $\Phi=0,\ p=1$, and $\omega=\mathcal{A}_f(R)=\max\{\Re f(\zeta):|\zeta|=R\}$ inequality (0.2) and formula (0.5) imply the Hadamard-Borel-Carathéodory inequality. Also note, that by (0.2) and (0.6) with $\Phi=0,\ p=\infty,\ \omega=0,\ \alpha=0$, and $\alpha=\pi/2$ one gets the estimates

$$|\Re f(z)-\Re f(0)|\le\frac{4}{\pi}\arcsin\left(\frac{r}{R}\right)||\Re f||_\infty,\quad |\Im f(z)-\Im f(0)|\le\frac{2}{\pi}\log\left(\frac{R+r}{R-r}\right)||\Re f||_\infty$$

(see, for example, [8]). The first inequality is known as the "Schwarz Arcussinus Formula". The right-hand side of the second inequality is, in fact, the sharp majorant for $|f(z)-f(0)|$.

For $\Phi=0,\ \alpha=\pi/2$ and any z with $|z|=r<R$, inequality (0.2) and formulas (0.3), (0.4) imply

$$|\Im f(z)-\Im f(0)|\le\mathcal{S}_p\,(r/R)\,||\Re f-\omega||_p \tag{0.7}$$

with the sharp constant

$$\mathcal{S}_p(\gamma) \;=\; \frac{\varkappa(\gamma)}{2\pi R^{1/p}}\left\{2\int_{-1}^{1}\frac{(1-t^2)^{(q-1)/2}}{[1-\varkappa(\gamma)t]^q}dt\right\}^{1/q}, \tag{0.8}$$

where $q = p/(p-1)$, $\varkappa(\gamma) = (2\gamma)/(1+\gamma^2)$. The integral in the right-hand side of (0.8) can be expressed in terms of hypergeometric Gauss function and is evaluated explicitly for a some values of p.

Section 3 concerns the inequality

$$|\Re\{e^{i\alpha(z)}(f(z)-f(0))\}| \le \mathcal{C}_{\Phi,\,p}\,(z,\alpha(z))\,||\Re f - (\Re f,\Phi)/(1,\Phi)||_p,$$

which is valid by Lemma 1 for Φ with a nonvanishing mean value on the circle $|\zeta| = R$.

In case $\Phi = 1$ the last inequality takes the form

$$|\Re\{e^{i\alpha(z)}(f(z)-f(0))\}| \le \mathcal{C}_{1,\,p}\,(z,\alpha(z))\,||\Re f - \Re f(0)||_p, \tag{0.9}$$

and the sharp constant is defined by

$$\mathcal{C}_{1,\,p}(z,\alpha(z)) = R^{-1/p}C_{1,\,p}(r/R,\alpha(z)),$$

where

$$C_{1,\,p}(\gamma,\alpha) = \frac{\gamma}{\pi}\min_{\lambda\in\mathbb{R}}\left\{\int_{-\pi}^{\pi}\left|\frac{\cos(\varphi-\alpha)-\gamma\cos\alpha}{1-2\gamma\cos\varphi+\gamma^2}-\lambda\right|^{p/(p-1)}d\varphi\right\}^{(p-1)/p} \tag{0.10}$$

if $1 < p < \infty$, and

$$C_{1,\,1}(\gamma,\alpha) = \frac{\gamma}{\pi(1-\gamma^2)}, \tag{0.11}$$

$$C_{1,\,\infty}(\gamma,\alpha) \;=\; \frac{2}{\pi}\left\{\sin\alpha\;\log\frac{2\gamma\sin\alpha+\sqrt{(1-\gamma^2)^2+4\gamma^2\sin^2\alpha}}{1-\gamma^2}\right.$$
$$\left.+\cos\alpha\;\arcsin\left(\frac{2\gamma\cos\alpha}{1+\gamma^2}\right)\right\}. \tag{0.12}$$

In particular, $C_{1,\,2}(\gamma,\alpha) = \gamma[\pi(1-\gamma^2)]^{-1/2}$.

For $p = \infty$, inequality (0.9) and the formula (0.12) for $C_{1,\,\infty}(\gamma,\alpha)$ imply the well-known estimates for $|\Re f(z) - \Re f(0)|, |\Im f(z) - \Im f(0)|$ and $|f(z) - f(0)|$ by $||\Re f - \Re f(0)||_\infty$ (see, for example, [3, 4, 9, 15]). In particular, we obtain the estimate

$$|\Re f(z) - \Re f(0)| \le \frac{4}{\pi}\arctan\left(\frac{r}{R}\right)||\Re f - \Re f(0)||_\infty, \tag{0.13}$$

generally known as the "Schwarz Arcustangens Formula".

In general, (0.4) i (0.10) lead to the inequality $\mathcal{C}_{1,\,p}(\gamma,\alpha) \le \mathcal{C}_{0,\,p}(\gamma,\alpha)$ which becomes equality for some values of p and α. In particular, this is the case for $p = 2$. We also show that $\mathcal{C}_{1,\,p}(\gamma,\pi/2) = \mathcal{C}_{0,\,p}(\gamma,\pi/2)$, that is the inequality

$$|\Im f(z) - \Im f(0)| \le \mathcal{S}_p\,(r/R)\,||\Re f - \Re f(0)||_p \tag{0.14}$$

holds with the sharp constant $\mathcal{S}_p(r/R)$ defined by (0.8). In conclusion we note that constant (0.8) can be written in the form

$$\mathcal{S}_p(\gamma) = \frac{\varkappa(\gamma)}{2\pi R^{1/p}} \left\{ 2\left[1-\varkappa^2(\gamma)\right]^{1/(2-2p)} \sum_{n=0}^{\infty} B\left(\frac{2p-1}{2p-2}, \frac{2n+1}{2}\right) \varkappa^{2n}(\gamma) \right\}^{(p-1)/p}, \tag{0.15}$$

where $B(u,v)$ is the Beta-function. Inequality (0.14) with the sharp constant (0.15) is a generalization of the classical estimate

$$|\Im f(z) - \Im f(0)| \le \frac{2}{\pi} \log\left(\frac{R+r}{R-r}\right) ||\Re f - \Re f(0)||_\infty,$$

due to Carathéodory and Plemelj (see [4, 3]).

The present paper extends results of our article [11] dedicated to sharp two-sided parameter dependent estimates for $\Re\{e^{i\alpha}(f(z) - f(0))\}$ by its maximal and minimal values on a circle as well as to the sharp constant for $|\Re\{e^{i\alpha}(f(z) - f(0))\}|$ involving the value $||\Re f||_\infty$.

1 Estimate of $|\Re\{e^{i\alpha(z)}(f(z) - f(0))\}|$ by $||\Re f||_p$ with associated orthogonality condition

We set for real valued functions g and h defined on the circle $|\zeta| = R$,

$$(g,h) = \int_{|\zeta|=R} g(\zeta)h(\zeta)|d\zeta|$$

and we denote the L_p-norm, $1 \le p \le \infty$, of g by $||g||_p$. We use the notations $\Delta f(z) = f(z) - f(0)$ and

$$G_{z,\alpha(z)}(\zeta) = \Re\left(\frac{e^{i\alpha(z)}z}{\zeta - z}\right). \tag{1.1}$$

Since z plays the role of a parameter in what follows, we frequently do not mark the dependence of α on z.

The following assertion is the main objective of this section.

Lemma 1. *Let f be analytic on D_R with continuous real part on $\overline{D}_R$, $1 \le p \le \infty$, and let $\alpha(z)$ be a real valued function, $|z| < R$. Further, let*

$$\int_{|\zeta|=R} \Re f(\zeta)\Phi(\zeta)|d\zeta| = 0, \tag{1.2}$$

where Φ is a real continuous function on $|\zeta| = R$. Then for any fixed point $z, |z| = r < R$, there holds

$$|\Re\{e^{i\alpha(z)}\Delta f(z)\}| \leq \mathcal{C}_{\Phi,\,p}\,(z, \alpha(z))\,||\Re f||_p \tag{1.3}$$

with the sharp constant

$$\mathcal{C}_{\Phi,\,p}(z, \alpha) = \frac{1}{\pi R}\min_{\lambda\in\mathbb{R}}||G_{z,\alpha} - \lambda\Phi||_q, \tag{1.4}$$

where $q = p/(p-1)$ for $1 < p < \infty$, $q = \infty$ for $p = 1$, and $q = 1$ for $p = \infty$.

In particular, for any fixed $z, |z| = r < R$, there holds

$$|\Delta f(z)| \leq \mathcal{C}_{\Phi,\,p}(z, -\arg\Delta f(z))||\Re f||_p. \tag{1.5}$$

Proof. Let $\Phi \in C(\partial D_R)$ and $g \in L_q(\partial D_R)$ be fixed. Consider the functional

$$F_g(h) = \int_{|\zeta|=R} g(\zeta)h(\zeta)|d\zeta|, \tag{1.6}$$

on the linear manifold $\mathbf{C}_\Phi = \{h \in C(\partial D_R) : (h, \Phi) = 0\}$ of the space $L_p(\partial D_R)$. We show that

$$\sup_{h\in\mathbf{C}_\Phi}\frac{|F_g(h)|}{||h||_p} = \min_{\lambda\in\mathbb{R}}||g - \lambda\Phi||_q. \tag{1.7}$$

It follows from the Hahn-Banach theorem that (see [10])

$$\sup_{\varphi\in\mathbf{L}_{p,\Psi}}\frac{|F_g(\varphi)|}{||\varphi||_p} = \min_{\lambda\in\mathbb{R}}||g - \lambda\Psi||_q, \tag{1.8}$$

where $\Psi \in L_q(\partial D_R)$, $\mathbf{L}_{p,\Psi} = \{\varphi \in L_p(\partial D_R) : (\varphi, \Psi) = 0\}$.

Suppose the Ψ is continuous. Let $1 \leq p < \infty$. Given any $\varepsilon > 0$, for every $\varphi \in \mathbf{L}_{p,\Psi}$ there exists $\varphi_\varepsilon \in C(\partial D_R)$ such that $||\varphi - \varphi_\varepsilon||_p \leq \varepsilon$. In the case $p = \infty$, there exists $\varphi_\varepsilon \in C(\partial D_R)$ such that the Lebesgue measure of the set $E = \{\zeta \in \partial D_R : \varphi(\zeta) \neq \varphi_\varepsilon(\zeta)\}$ does not exceed ε and $||\varphi_\varepsilon||_\infty \leq ||\varphi||_\infty$.

Assuming $||\Psi||_2 \neq 0$ we write $\varphi_\varepsilon = c_\varepsilon\Psi + (\varphi_\varepsilon - c_\varepsilon\Psi)$, where $c_\varepsilon = (\varphi_\varepsilon, \Psi)/||\Psi||_2^2$. Then $\mathring{\varphi}_\varepsilon = \varphi_\varepsilon - c_\varepsilon\Psi \in \mathbf{C}_\Psi$.

First, consider the case $1 \leq p < \infty$. If $\Psi = 0$, then (1.7) follows from (1.8) because $C(\partial D_R)$ is dense in $L_p(\partial D_R)$. For $||\Psi||_2 \neq 0$ inequality $||\varphi - \varphi_\varepsilon||_p \leq \varepsilon$ implies the estimate $||\varphi - \mathring{\varphi}_\varepsilon||_p \leq \varepsilon + |c_\varepsilon|||\Psi||_p$ with

$$|c_\varepsilon| = \frac{|(\varphi_\varepsilon, \Psi)|}{||\Psi||_2^2} = \frac{|(\varphi_\varepsilon - \varphi, \Psi)|}{||\Psi||_2^2} \leq \frac{||\varphi - \varphi_\varepsilon||_p||\Psi||_q}{||\Psi||_2^2} \leq \frac{||\Psi||_q}{||\Psi||_2^2}\,\varepsilon.$$

Hence, $||\varphi - \mathring{\varphi}_\varepsilon||_p \leq k\varepsilon$, where $k = 1 + ||\Psi||_p||\Psi||_q||\Psi||_2^{-2}$. Thus $\mathbf{C}_\Phi$ is dense in $\mathbf{L}_{p,\Phi}$, which, in view of (1.8), implies (1.7).

Now, let $p = \infty$. If $\Psi = 0$, it follows from (1.6) and $||\varphi_\varepsilon||_\infty \le ||\varphi||_\infty$ that

$$\frac{|F_g(\varphi_\varepsilon)|}{||\varphi_\varepsilon||_\infty} \ge \frac{|F_g(\varphi)|}{||\varphi||_\infty} - 2||g||_{L_1(E)}. \tag{1.9}$$

Since mes $E \le \varepsilon$, we have $||g||_{L_1(E)} \to 0$ as $\varepsilon \to 0$ which leads to (1.7) by (1.8) and (1.9). If $||\Psi||_2 \ne 0$, then taking into account the estimates

$$|F_g(\mathring{\varphi}_\varepsilon) - F_g(\varphi)| \le |F_g(\mathring{\varphi}_\varepsilon - \varphi_\varepsilon)| + |F_g(\varphi_\varepsilon - \varphi)|,$$

$$||\varphi_\varepsilon - \mathring{\varphi}_\varepsilon||_\infty \le |c_\varepsilon| ||\Psi||_\infty,$$

and

$$|c_\varepsilon| = \frac{|(\varphi_\varepsilon, \Psi)|}{||\Psi||_2^2} = \frac{|(\varphi_\varepsilon - \varphi, \Psi)|}{||\Psi||_2^2} \le \frac{2||\varphi||_\infty ||\Psi||_\infty}{||\Psi||_2^2}\, \varepsilon,$$

we obtain

$$|F_g(\mathring{\varphi}_\varepsilon)| \ge |F_g(\varphi)| - 2\left(k||g||_1\varepsilon + ||g||_{L_1(E)}\right)||\varphi||_\infty,$$

where $k = ||\Psi||_\infty^2 ||\Psi||_2^{-2}$. This, together with the estimate $||\mathring{\varphi}_\varepsilon||_\infty \le ||\varphi||_\infty + |c_\varepsilon| ||\Psi||_\infty$, implies

$$\frac{|F_g(\mathring{\varphi}_\varepsilon)|}{||\mathring{\varphi}_\varepsilon||_\infty} \ge \frac{1}{1+2k\varepsilon}\left\{\frac{|F_g(\varphi)|}{||\varphi||_\infty} - 2\left(k||g||_1\varepsilon + ||g||_{L_1(E)}\right)\right\}.$$

Combining this with(1.8) and using the arbitrariness of ε we arrive at (1.7) with $p = \infty$.

Let us apply the duality relation (1.7). The Cauchy-Schwarz formula

$$f(z) = i\, \Im f(0) + \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{\zeta + z}{(\zeta - z)\zeta}\, \Re f(\zeta) d\zeta \tag{1.10}$$

(see, for example, [3, 12]) represents an analytic function in D_R with the real part $\Re f$ continuous on $\overline{D}_R$. Clearly, (1.10) implies

$$\Delta f(z) = \frac{1}{\pi R}\int_{|\zeta|=R} \frac{z}{\zeta - z}\, \Re f(\zeta)|d\zeta|. \tag{1.11}$$

Using (1.11) and (1.1), we obtain

$$\Re\{e^{i\alpha}\Delta f(z)\} = \frac{1}{\pi R}\Re\left\{\int_{|\zeta|=R} \frac{e^{i\alpha} z}{\zeta - z}\, \Re f(\zeta)|d\zeta|\right\} = \frac{1}{\pi R}\int_{|\zeta|=R} G_{z,\alpha}(\zeta)\, \Re f(\zeta)|d\zeta|. \tag{1.12}$$

Hence and by (1.2), the sharp constant $\mathcal{C}_{\Phi,\,p}(z,\alpha)$ in

$$|\Re\{e^{i\alpha}\Delta f(z)\}| \le \mathcal{C}_{\Phi,\,p}\,(z,\alpha)\, ||\Re f||_p \tag{1.13}$$

can be written in the form

$$\mathcal{C}_{\Phi,\,p}(z,\alpha) = \frac{1}{\pi R}\sup_{h\in \mathbf{C}_\Phi} \frac{1}{||h||_p}\left|\int_{|\zeta|=R} G_{z,\alpha}(\zeta) h(\zeta)|d\zeta|\right|. \tag{1.14}$$

Therefore, applying (1.7) to the functional (1.6) with $g(\zeta) = G_{z,\alpha}(\zeta)$, we arrive at the representation (1.4) for the sharp constant in (1.3).

We have proved that inequality (1.3) with the sharp constant $\mathcal{C}_{\Phi,\,p}(z, \alpha(z))$ is valid at any point $z \in D_R$ for any fixed real valued function $\alpha(z)$ on D_R and any analytic function f on D_R with real part continuous in $\overline{D}_R$. Having f fixed in (1.3) and choosing α to satisfy $\alpha(z) = -\arg \Delta f(z)$, we arrive at (1.5). ▮

Remark 1. Obviously, (1.4) implies

$$\mathcal{C}_{0,\,2}(z, \alpha) = \frac{1}{\pi R}||G_{z,\alpha}||_2, \tag{1.15}$$

and

$$\mathcal{C}_{\Phi,\,2}(z, \alpha) = \frac{1}{\pi R}\left(||G_{z,\alpha}||_2^2 - (G_{z,\alpha}, \Phi)^2||\Phi||_2^{-2}\right)^{1/2}, \tag{1.16}$$

if $||\Phi||_2 \neq 0$.

Let us evaluate $||G_{z,\alpha}||_2$. With the notation $\zeta = Re^{it}, z = re^{i\tau}, \gamma = r/R$ one has

$$\frac{e^{i\alpha}z}{\zeta - z} = \frac{e^{i\alpha}re^{i\tau}}{Re^{it} - re^{i\tau}} = \frac{\gamma e^{i\alpha}}{e^{i(t-\tau)} - \gamma}. \tag{1.17}$$

Setting $\varphi = t - \tau$ and using (1.17) we obtain

$$\begin{aligned} ||G_{z,\alpha}||_2^2 &= \int_{|\zeta|=R}\left[\Re\left(\frac{e^{i\alpha}z}{\zeta - z}\right)\right]^2|d\zeta| = \gamma^2\int_{-\pi+\tau}^{\pi+\tau}\left[\Re\left(\frac{e^{i\alpha}}{e^{i(t-\tau)} - \gamma}\right)\right]^2 Rdt \\ &= R\gamma^2\int_{-\pi}^{\pi}\left[\Re\left(\frac{e^{i\alpha}}{e^{i\varphi} - \gamma}\right)\right]^2 d\varphi = \frac{r^2}{R}\int_{-\pi}^{\pi}\frac{(\cos(\varphi - \alpha) - \gamma\cos\alpha)^2}{(1 - 2\gamma\cos\varphi + \gamma^2)^2}d\varphi, \end{aligned} \tag{1.18}$$

and making elementary calculations, we arrive at

$$\int_{-\pi}^{\pi}\frac{(\cos(\varphi - \alpha) - \gamma\cos\alpha)^2}{(1 - 2\gamma\cos\varphi + \gamma^2)^2}d\varphi = \frac{\pi}{1 - \gamma^2},$$

which together with (1.18) gives

$$||G_{z,\alpha}||_2^2 = \frac{\pi r^2 R}{R^2 - r^2}. \tag{1.19}$$

Hence and by (1.15), (1.16) we conclude

$$\mathcal{C}_{0,\,2}(z, \alpha) = \frac{r}{\sqrt{\pi R(R^2 - r^2)}}, \tag{1.20}$$

and

$$\mathcal{C}_{\Phi,\,2}(z, \alpha) = \frac{1}{\sqrt{\pi R}}\left\{\frac{r^2}{R^2 - r^2} - \frac{(G_{z,\alpha}, \Phi)^2}{\pi R||\Phi||_2^2}\right\}^{1/2}, \tag{1.21}$$

provided $||\Phi||_2 \neq 0$. ▮

We need one more auxiliary assertion. Its proof, given in [11], is reproduced here for readers' convenience.

Lemma 2. *Let $|z| = r < R$. The relations hold*

$$\min_{|\zeta|=R} G_{z,\alpha}(\zeta) = \frac{r(r\cos\alpha - R)}{R^2 - r^2}, \quad \max_{|\zeta|=R} G_{z,\alpha}(\zeta) = \frac{r(r\cos\alpha + R)}{R^2 - r^2}. \tag{1.22}$$

Proof. Setting $\varphi = t - \tau$ in (1.17), we obtain

$$G_{z,\alpha}(\zeta) = \Re\left(\frac{e^{i\alpha} z}{\zeta - z}\right) = \Re\left(\frac{\gamma e^{i\alpha}}{e^{i\varphi} - \gamma}\right) = \frac{\gamma(\cos(\varphi - \alpha) - \gamma\cos\alpha)}{1 - 2\gamma\cos\varphi + \gamma^2}. \tag{1.23}$$

Consider the function

$$g(\varphi) = \frac{\cos(\varphi - \alpha) - \gamma\cos\alpha}{1 - 2\gamma\cos\varphi + \gamma^2}, \quad |\varphi| \le \pi. \tag{1.24}$$

We have

$$g'(\varphi) = \frac{(\gamma^2 - 1)\cos\alpha\sin\varphi + (\gamma^2 + 1)\sin\alpha\cos\varphi - 2\gamma\sin\alpha}{(1 - 2\gamma\cos\varphi + \gamma^2)^2}.$$

Solving the equation $g'(\varphi) = 0$, we find

$$\sin\varphi_+ = \frac{(1 - \gamma^2)\sin\alpha}{1 + 2\gamma\cos\alpha + \gamma^2}, \qquad \cos\varphi_+ = \frac{2\gamma + (1 + \gamma^2)\cos\alpha}{1 + 2\gamma\cos\alpha + \gamma^2}, \tag{1.25}$$

and

$$\sin\varphi_- = -\frac{(1 - \gamma^2)\sin\alpha}{1 - 2\gamma\cos\alpha + \gamma^2}, \qquad \cos\varphi_- = \frac{2\gamma - (1 + \gamma^2)\cos\alpha}{1 - 2\gamma\cos\alpha + \gamma^2}, \tag{1.26}$$

where φ_+ and φ_- are critical points of $g(\varphi)$. Setting (1.25) and (1.26) into (1.24) we arrive at

$$g(\varphi_+) = \frac{\gamma\cos\alpha + 1}{1 - \gamma^2}, \quad g(\varphi_-) = \frac{\gamma\cos\alpha - 1}{1 - \gamma^2}.$$

It follows from (1.24) that

$$g(-\pi) = g(\pi) = -\frac{\cos\alpha}{1 + \gamma} = \frac{\gamma\cos\alpha - \cos\alpha}{1 - \gamma^2}.$$

Since $g(\varphi_+) > g(\varphi_-)$ and

$$g(-\pi) = g(\pi) = \frac{\gamma\cos\alpha - \cos\alpha}{1 - \gamma^2} \le \frac{\gamma\cos\alpha + 1}{1 - \gamma^2} = g(\varphi_+),$$

$$g(-\pi) = g(\pi) = \frac{\gamma\cos\alpha - \cos\alpha}{1 - \gamma^2} \ge \frac{\gamma\cos\alpha - 1}{1 - \gamma^2} = g(\varphi_-),$$

it follows from (1.23), (1.24) that

$$\max_{|\zeta|=R} \Re\left(\frac{e^{i\alpha} z}{\zeta - z}\right) = \gamma g(\varphi_+) = \gamma\frac{\gamma\cos\alpha + 1}{1 - \gamma^2} = \frac{r(r\cos\alpha + R)}{R^2 - r^2},$$

$$\min_{|\zeta|=R} \Re\left(\frac{e^{i\alpha} z}{\zeta - z}\right) = \gamma g(\varphi_-) = \gamma\frac{\gamma\cos\alpha - 1}{1 - \gamma^2} = \frac{r(r\cos\alpha - R)}{R^2 - r^2},$$

which implies (1.22). The proof is complete. ∎

2 Estimate for $|\Re\{e^{i\alpha(z)}\Delta f(z)\}|$ by $||\Re f-\omega||_p$ if the mean value of Φ on the circle $|\zeta|=R$ is equal to zero

Let us assume that $\int_{|\zeta|=R}\Phi(\zeta)|d\zeta|=0$. Replacing f with $f-\omega$, where ω is a real constant, in Lemma 1, we obtain an estimate for $|\Re\{e^{i\alpha(z)}\Delta f(z)\}|$.

Proposition 1. *Let f be analytic on D_R with continuous real part on $\overline{D}_R$, $1\le p\le\infty$, and let $\alpha(z)$ be a real valued function, $|z|<R$. Further, let*

$$\int_{|\zeta|=R}\Re f(\zeta)\Phi(\zeta)|d\zeta|=0, \tag{2.1}$$

where Φ is a real continuous function on $|\zeta|=R$, for which

$$\int_{|\zeta|=R}\Phi(\zeta)|d\zeta|=0. \tag{2.2}$$

Then for any fixed point $z, |z|=r<R$, and arbitrary real constant ω there holds

$$|\Re\{e^{i\alpha(z)}\Delta f(z)\}|\le \mathcal{C}_{\Phi,\,p}\,(z,\alpha(z))\,||\Re f-\omega||_p \tag{2.3}$$

with the sharp constant $\mathcal{C}_{\Phi;\,p}(z,\alpha(z))$ given by (1.4).

In particular, for any fixed point $z, |z|=r<R$, and arbitrary real constant ω, the inequality holds

$$|\Delta f(z)|\le \mathcal{C}_{\Phi,\,p}(z,-\arg\Delta f(z))||\Re f-\omega||_p. \tag{2.4}$$

Remark 2. The norm $||\Re f-\omega||_p$ in (2.3) and (2.4) can be replaced with the best approximation $E_p(\Re f)$ of $\Re f$ by a real constant in the norm of the space $L_p(\partial D_R)$

$$E_p(\Re f)=\min_{\omega\in\mathbb{R}}||\Re f-\omega||_p. \tag{2.5}$$

Note that

$$E_2(\Re f)=||\Re f-\Re f(0)||_2, \tag{2.6}$$

and

$$E_\infty(\Re f)=\frac{1}{2}\Omega_f(R), \tag{2.7}$$

where $\Omega_f(R)=\mathcal{A}_f(R)-\mathcal{B}_f(R)$ is the oscillation of $\Re f$ on the circle $|\zeta|=R$. Here and in what follow, $\mathcal{A}_f(R)=\max\{\Re f(\zeta):|\zeta|=R\}$ and $\mathcal{B}_f(R)=\min\{\Re f(\zeta):|\zeta|=R\}$.

Indeed, we have

$$||\Re f-\omega||_2=\left\{\int_{|\zeta|=R}[\Re f(\zeta)-\omega]^2|d\zeta|\right\}^{1/2}=\sqrt{R}\left\{\int_{-\pi}^{\pi}[\Re f(Re^{i\varphi})-\omega]^2d\varphi\right\}^{1/2}, \tag{2.8}$$

which implies the representation

$$E_2(\Re f) = \min_{\omega\in\mathbb{R}} ||\Re f - \omega||_2 = \sqrt{R}\left\{\int_{-\pi}^{\pi}\left[\Re f(Re^{i\varphi}) - A_0\right]^2 d\varphi\right\}^{1/2},$$

where

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Re f(Re^{i\varphi})d\varphi = \Re f(0)$$

which proves (2.6).

Since the minimum value in

$$E_\infty(\Re f) = \min_{\omega\in\mathbb{R}} ||\Re f - \omega||_\infty$$

is attained at $\omega = [\mathcal{A}_f(R) + \mathcal{B}_f(R)]/2$ and hence,

$$E_\infty(\Re f) = \left|\left|\Re f - \frac{\mathcal{A}_f(R) + \mathcal{B}_f(R)}{2}\right|\right|_\infty = \mathcal{A}_f(R) - \frac{\mathcal{A}_f(R) + \mathcal{B}_f(R)}{2}$$

relation (2.7) follows. ∎

We introduce the set

$$W(\alpha_1, \alpha_2) = \{w \in \mathbb{C} : \alpha_1 \le |\arg w| \le \alpha_2 \vee \pi - \alpha_2 \le |\arg w| \le \pi - \alpha_1\}, \tag{2.9}$$

where $0 \le \alpha_1 < \alpha_2 \le \pi/2$.

The next assertion contains explicit consequences of Proposition 1 for $\Phi = 0$. It also provides an estimate of $|\Delta f(z)|$ for a class of analytic functions in D_R with continuous real part in $\overline{D}_R$ for which $\Delta f(z) \in W(\alpha_1, \alpha_2), z \in D_R$. Similarly to Remark 2, the norm $||\Re f - \omega||_p$ in the next inequalities can be replaced by $E_p(\Re f)$.

Corollary 1. *Let f be analytic on D_R with continuous real part on $\overline{D}_R$, $1 \le p \le \infty$. Further, let $\alpha(z)$ be a real valued function, $|z| < R$. Then for any fixed point $z, |z| = r < R$, and for an arbitrary real constant ω the inequality holds*

$$|\Re\{e^{i\alpha(z)}\Delta f(z)\}| \le \mathcal{C}_{0,\,p}\,(z, \alpha(z))\, ||\Re f - \omega||_p \tag{2.10}$$

with the sharp constant $\mathcal{C}_{0,\,p}(z, \alpha(z))$, where

$$\mathcal{C}_{0,\,p}(z, \alpha) = \frac{1}{R^{1/p}} C_{0,\,p}\left(\frac{r}{R}, \alpha\right), \tag{2.11}$$

and

$$C_{0,\,p}(\gamma, \alpha) = \frac{\gamma}{\pi}\left\{\int_{-\pi}^{\pi}\left|\frac{\cos(\varphi-\alpha) - \gamma\cos\alpha}{1 - 2\gamma\cos\varphi + \gamma^2}\right|^{p/(p-1)} d\varphi\right\}^{(p-1)/p} \tag{2.12}$$

if $1 < p < \infty$, *and*

$$C_{0,\,1}(\gamma,\alpha) = \frac{\gamma(1+\gamma|\cos\alpha|)}{\pi(1-\gamma^2)}, \tag{2.13}$$

$$C_{0,\,\infty}(\gamma,\alpha) = \frac{4}{\pi}\left\{\sin\alpha\log\frac{\gamma\sin\alpha + \left(1-\gamma^2\cos^2\alpha\right)^{1/2}}{\left(1-\gamma^2\right)^{1/2}} + \cos\alpha\arcsin\left(\gamma\cos\alpha\right)\right\}. \tag{2.14}$$

In particular,

$$C_{0,\,2}(\gamma,\alpha) = \frac{\gamma}{\sqrt{\pi(1-\gamma^2)}}. \tag{2.15}$$

If $\Delta f(z) \in W(\alpha_1, \alpha_2)$ *for* $z \in D_R$, *then*

$$|\Delta f(z)| \le \max_{\alpha_1\le\alpha\le\alpha_2} \mathcal{C}_{0,\,p}(z,\alpha)||\Re f - \omega||_p \tag{2.16}$$

with

$$\max_{\alpha_1\le\alpha\le\alpha_2} \mathcal{C}_{0,\,1}(z,\alpha) = \mathcal{C}_{0,\,1}(z,\alpha_1), \tag{2.17}$$

$$\max_{\alpha_1\le\alpha\le\alpha_2} \mathcal{C}_{0,\,\infty}(z,\alpha) = \mathcal{C}_{0,\,\infty}(z,\alpha_2). \tag{2.18}$$

Proof. We put $\Phi(z) = 0$ in Proposition 1. For $p = 1$, formula (2.11) with the factor (2.13) follows directly from (1.4) and (1.22). For $p = \infty$, representation (2.11) with the factor (2.14) was derived in [11].

Now, suppose $1 < p < \infty$. Combining (1.4) with (1.1) and (1.17) we have

$$\mathcal{C}_{0,\,p}(z,\alpha) = \frac{1}{\pi R}\left\{\int_{-\pi+\tau}^{\pi+\tau}\left|\Re\left(\frac{\gamma e^{i\alpha}}{e^{i(t-\tau)}-\gamma}\right)\right|^{p/(p-1)} R dt\right\}^{(p-1)/p},$$

which after the change of variable $\varphi = t - \tau$ becomes

$$\mathcal{C}_{0,\,p}(z,\alpha) = \frac{1}{\pi R^{1/p}}\left\{\int_{-\pi}^{\pi}\left|\Re\left(\frac{\gamma e^{i\alpha}}{e^{i\varphi}-\gamma}\right)\right|^{p/(p-1)} d\varphi\right\}^{(p-1)/p}. \tag{2.19}$$

Using the notation

$$C_{0,\,p}(\gamma,\alpha) = \frac{1}{\pi}\left\{\int_{-\pi}^{\pi}\left|\Re\left(\frac{\gamma e^{i\alpha}}{e^{i\varphi}-\gamma}\right)\right|^{p/(p-1)} d\varphi\right\}^{(p-1)/p}, \tag{2.20}$$

we rewrite (2.19) as

$$\mathcal{C}_{0,\,p}(z,\alpha) = \frac{1}{R^{1/p}} C_{0,\,p}\left(\frac{r}{R},\alpha\right), \tag{2.21}$$

which together with (1.23) proves (2.11) and (2.12) for $1 < p < \infty$.

Formula (2.11) with $p = 2$ and the factor (2.15) has been already derived (see (1.20)). Now, we pass to the proof of (2.16)-(2.18). First, we show the equality $\mathcal{C}_{0,\,p}(z, -\alpha) = \mathcal{C}_{0,\,p}(z, \alpha)$. For $p = 1$ and $p = \infty$ it follows directly from (2.13) and (2.14). Suppose $1 < p < \infty$. By (2.20),

$$C_{0,\,p}(\gamma, \alpha) = \frac{\gamma}{\pi} \left\{ \int_{-\pi}^{\pi} \left| \Re \left(\frac{e^{-i\alpha}}{e^{-i\varphi} - \gamma} \right) \right|^{p/(p-1)} d\varphi \right\}^{(p-1)/p}.$$

Replacing here φ by $-\psi$ we obtain

$$C_{0,\,p}(\gamma, \alpha) = \frac{\gamma}{\pi} \left\{ \int_{-\pi}^{\pi} \left| \Re \left(\frac{e^{-i\alpha}}{e^{i\psi} - \gamma} \right) \right|^{p/(p-1)} d\psi \right\}^{(p-1)/p} = C_{0,\,p}(\gamma, -\alpha).$$

This, together with (2.21), leads to $\mathcal{C}_{0,\,p}(z, -\alpha) = \mathcal{C}_{0,\,p}(z, \alpha)$. Hence, by (2.4)

$$|\Delta f(z)| \le \mathcal{C}_{0,\,p}(z, \arg \Delta f(z)) || \Re f - \omega ||_p. \tag{2.22}$$

Let $0 \le \alpha \le \pi/2$. By (2.20) and (2.21), $\mathcal{C}_{0,\,p}(z, \pi - \alpha) = \mathcal{C}_{0,\,p}(z, -\alpha)$. Combining this with $\mathcal{C}_{0,\,p}(z, -\alpha) = \mathcal{C}_{0,\,p}(z, \alpha)$ we obtain

$$\sup\{\mathcal{C}_{0,\,p}(z, \arg \Delta f(z)) : \Delta f(z) \in W(\alpha_1, \alpha_2)\} = \max\{\mathcal{C}_{0,\,p}(z, \alpha) : \alpha_1 \le \alpha \le \alpha_2\},$$

which together (2.22) implies (2.16).

Equality (2.17) follows from the last relation, (2.13) and the monotonicity of $\cos \alpha$ on $[0, \pi/2]$.

Now, we prove (2.18). In view of (2.11) and (2.14),

$$\mathcal{C}_{0,\,\infty}(z, \alpha) = \frac{4}{\pi} \left\{ \sin \alpha \log \frac{\gamma \sin \alpha + (1 - \gamma^2 \cos^2 \alpha)^{1/2}}{(1 - \gamma^2)^{1/2}} + \cos \alpha \arcsin(\gamma \cos \alpha) \right\},$$

where $\gamma = r/R$. We study the function $\mathcal{C}_{0,\,\infty}(z, \alpha)$ for $0 \le \alpha \le \pi/2$. We have

$$\frac{\partial \mathcal{C}_{0,\,\infty}(z, \alpha)}{\partial \alpha} = \frac{4}{\pi} \left(\cos \alpha \log \frac{\gamma \sin \alpha + (1 - \gamma^2 \cos^2 \alpha)^{1/2}}{(1 - \gamma^2)^{1/2}} - \sin \alpha \arcsin(\gamma \cos \alpha) \right). \tag{2.23}$$

Using the equalities

$$\cos \alpha \log \frac{\gamma \sin \alpha + (1 - \gamma^2 \cos^2 \alpha)^{1/2}}{(1 - \gamma^2)^{1/2}} = \cos \alpha \int_0^{\gamma \sin \alpha} \frac{dt}{\sqrt{1 - \gamma^2 + t^2}},$$

$$\sin \alpha \arcsin(\gamma \cos \alpha) = \sin \alpha \int_0^{\gamma \cos \alpha} \frac{dt}{\sqrt{1 - t^2}},$$

and the estimates

$$\cos \alpha \int_0^{\gamma \sin \alpha} \frac{dt}{\sqrt{1 - \gamma^2 + t^2}} > \frac{\gamma \sin \alpha \cos \alpha}{\sqrt{1 - \gamma^2 + \gamma^2 \sin^2 \alpha}},$$

$$\sin\alpha\int_0^{\gamma\cos\alpha}\frac{dt}{\sqrt{1-t^2}}<\frac{\gamma\sin\alpha\cos\alpha}{\sqrt{1-\gamma^2\cos^2\alpha}}=\frac{\gamma\sin\alpha\cos\alpha}{\sqrt{1-\gamma^2+\gamma^2\sin^2\alpha}},$$

which follow from the mean value theorem for $\alpha\in(0,\pi/2)$, we obtain from (2.23)

$$\frac{\partial\mathcal{C}_{0,\,\infty}(z,\alpha)}{\partial\alpha}>0.$$

Thus, $\mathcal{C}_{0,\,\infty}(z,\alpha)$ increases on $[0,\pi/2]$. ∎

Remark 3. Let $\alpha=\pi/2$. Since the integrand in (2.12) is an even function, we have

$$C_{0,\,p}(\gamma,\pi/2)=\frac{\gamma}{\pi}\left\{2\int_0^{\pi}\left(\frac{\sin\varphi}{1-2\gamma\cos\varphi+\gamma^2}\right)^q d\varphi\right\}^{1/q} \tag{2.24}$$

for the best constant in the inequality

$$|\Im\Delta f(z)|\le R^{-1/p}C_{0,\,p}\left(r/R,\pi/2\right)||\Re f-\omega||_p, \tag{2.25}$$

where $q=p/(p-1), 1<p<\infty$. Making the change of variable $t=\cos\varphi$ and setting $\varkappa(\gamma)=(2\gamma)/(1+\gamma^2)$, we find

$$C_{0,\,p}(\gamma,\pi/2)=\frac{\varkappa(\gamma)}{2\pi}\left\{2\int_{-1}^{1}\frac{(1-t^2)^{(q-1)/2}}{[1-\varkappa(\gamma)t]^q}dt\right\}^{1/q} \tag{2.26}$$

which together with (2.25) leads to (0.7) and (0.8). The integral

$$\mathcal{I}_q(\varkappa)=\int_{-1}^{1}\frac{(1-t^2)^{(q-1)/2}}{(1-\varkappa t)^q}dt$$

is the sum of each of two series

$$\sum_{m=0}^{\infty}(-1)^m\binom{(q-1)/2}{m}\int_{-1}^{1}\frac{t^{2m}}{(1-\varkappa t)^q}dt$$

and

$$\sum_{m=0}^{\infty}(-1)^m\binom{q/2}{m}\int_{-1}^{1}\frac{t^{2m}}{(1-\varkappa t)^q(1-t^2)^{1/2}}dt,$$

the first of which turns into a finite sum for odd q and the second one for even q.

For odd q, the recurrence relation

$$\mathcal{I}_{2n+1}(\varkappa)=\frac{2(2n-2)!!}{(2n-1)!!}\frac{1}{\varkappa^2(1-\varkappa^2)^n}-\frac{1}{\varkappa^2}\mathcal{I}_{2n-1}(\varkappa)$$

implies

$$\mathcal{I}_{2n+1}(\varkappa)=\frac{2}{\varkappa^{2n+2}}\sum_{k=1}^{n}\frac{(-1)^{n+k}(2k-2)!!}{(2k-1)!!}\left(\frac{\varkappa^2}{1-\varkappa^2}\right)^k+\frac{(-1)^n}{\varkappa^{2n+1}}\log\frac{1+\varkappa}{1-\varkappa}.$$

Hence, putting $\varkappa = (2\gamma)/(1+\gamma^2)$ and taking into account the equality

$$\mathcal{I}_1(\varkappa) = \frac{1}{\varkappa}\log\frac{1+\varkappa}{1-\varkappa}$$

and (2.26), we find

$$C_{0,\,\frac{2n+1}{2n}}(\gamma,\pi/2) = \frac{1}{2\pi}\left\{4(-1)^n\log\frac{1+\gamma}{1-\gamma}\right. \\ \left. +\frac{2(1+\gamma^2)}{\gamma}\sum_{k=1}^{n}\frac{(-1)^{n+k}(2k-2)!!}{(2k-1)!!}\left(\frac{2\gamma}{1-\gamma^2}\right)^{2k}\right\}^{\frac{1}{2n+1}}.$$

For example,

$$C_{0,\,3/2}(\gamma,\pi/2) = \frac{1}{\pi}\left\{\frac{\gamma(1+\gamma^2)}{(1-\gamma^2)^2} - \frac{1}{2}\log\frac{1+\gamma}{1-\gamma}\right\}^{1/3}.$$

For even q, by the recurrence relation

$$\mathcal{I}_{2n+2}(\varkappa) = \frac{\pi(2n-1)!!}{(2n)!!}\frac{1}{\varkappa^2(1-\varkappa^2)^{(2n+1)/2}} - \frac{1}{\varkappa^2}\mathcal{I}_{2n}(\varkappa),$$

we have

$$\mathcal{I}_{2n+2}(\varkappa) = \frac{\pi}{\varkappa^{2n+3}}\sum_{k=1}^{n}\frac{(-1)^{n+k}(2k-1)!!}{(2k)!!}\left(\frac{\varkappa^2}{1-\varkappa^2}\right)^{(2k+1)/2} + \frac{\pi(-1)^n}{\varkappa^{2n+2}}\frac{\left(1-\sqrt{1-\varkappa^2}\right)}{\sqrt{1-\varkappa^2}}.$$

Hence, using

$$\mathcal{I}_2(\varkappa) = \frac{\pi\left(1-\sqrt{1-\varkappa^2}\right)}{\varkappa^2\sqrt{1-\varkappa^2}},$$

with $\varkappa = (2\gamma)/(1+\gamma^2)$ as well as (2.26), we obtain

$$C_{0,\,\frac{2n+2}{2n+1}}(\gamma,\pi/2) = \frac{1}{2\pi}\left\{\frac{4(-1)^n\pi\gamma^2}{1-\gamma^2}\right. \\ \left. +\frac{\pi(1+\gamma^2)}{\gamma}\sum_{k=1}^{n}\frac{(-1)^{n+k}(2k-1)!!}{(2k)!!}\left(\frac{2\gamma}{1-\gamma^2}\right)^{2k+1}\right\}^{\frac{1}{2n+2}}.$$

In particular,

$$C_{0,\,4/3}(\gamma,\pi/2) = \frac{\gamma}{\pi}\left\{\frac{\pi(3-\gamma^2)}{4(1-\gamma^2)^3}\right\}^{1/4}.$$

Remark 4. Let $\alpha = 0$. Since the integrand in (2.12) is even, it follows that

$$C_{0,\,p}(\gamma,0) = \frac{\gamma}{\pi}\left\{2\int_0^{\pi}\left|\frac{\cos\varphi-\gamma}{1-2\gamma\cos\varphi+\gamma^2}\right|^q d\varphi\right\}^{1/q},$$

where $q = p/(p-1), 1 < p < \infty$. The change of variable $t = \cos\varphi$ implies the equality

$$C_{0,\,p}(\gamma, 0) = \frac{\varkappa(\gamma)}{2\pi}\left\{2\int_{-1}^{1}\frac{|t-\gamma|^q}{[1-\varkappa(\gamma)t]^q(1-t^2)^{1/2}}dt\right\}^{1/q} \tag{2.27}$$

for the sharp constant in

$$|\Re\Delta f(z)| \le C_{0,\,p}(r/R, 0)R^{-1/p}||\Re f - \omega||_p. \tag{2.28}$$

The integral in (2.27) can be evaluated for $q = 2n$, that is for $p = (2n)/(2n-1)$. We introduce the notation

$$\mathcal{J}_{2n}(\varkappa) = \int_{-1}^{1}\frac{(t-\gamma)^{2n}}{(1-\varkappa t)^{2n}(1-t^2)^{1/2}}dt.$$

By the binomial formula,

$$\mathcal{J}_{2n}(\varkappa) = \frac{1}{\varkappa^{2n}}\sum_{k=0}^{2n}\frac{(-1)^k(2n)!(1-\varkappa\gamma)^k}{k!(2n-k)!}\int_{-1}^{1}\frac{dt}{(1-\varkappa t)^k(1-t^2)^{1/2}}.$$

Evaluating the last integral and using (2.27), we conclude

$$C_{0,\,\frac{2n}{2n-1}}(\gamma, 0) = (2\pi)^{\frac{1-2n}{2n}}\left\{1 + \sum_{k=1}^{2n}\sum_{m=0}^{k-1}\frac{(-1)^{k+m}(2n)!\left(\frac{1-2m}{2}\right)_{k-1}}{k!(2n-k)!m!(k-m-1)!}\left(\frac{1+\gamma}{1-\gamma}\right)^{2m-k+1}\right\}^{\frac{1}{2n}}.$$

Remark 5. The well known inequalities (see, for instance, [8])

$$|\Re\Delta f(z)| \le \frac{4}{\pi}\arcsin\left(\frac{r}{R}\right)||\Re f||_\infty, \quad |\Im\Delta f(z)| \le \frac{2}{\pi}\log\frac{R+r}{R-r}||\Re f||_\infty$$

are particular cases of inequalities in Proposition 1. They follow from (2.10) and (2.11) with $p = \infty, \omega = 0, \alpha = 0$, and $\alpha = \pi/2$, combined with (2.14). The inequality

$$|\Delta f(z)| \le \frac{2}{\pi}\log\frac{R+r}{R-r}||\Re f||_\infty$$

(see [11]) follows from (2.18), where $p = \infty, \omega = 0$, $\alpha_2 = \pi/2$ and (2.14), (2.16).

The class of inequalities we are studying in this section embraces the following three sharp estimates

$$|\Re\Delta f(z)| \le \frac{2r}{R-r}\max_{|\zeta|=R}\Re\Delta f(\zeta), \tag{2.29}$$

$$|\Im\Delta f(z)| \le \frac{2Rr}{R^2-r^2}\max_{|\zeta|=R}\Re\Delta f(\zeta), \tag{2.30}$$

$$|\Delta f(z)| \le \frac{2r}{R-r} \max_{|\zeta|=R} \Re\Delta f(\zeta) \tag{2.31}$$

(see [3, 14, 15] and the bibliography in [3]).

Sometimes, (2.31) is called Hadamard-Borel-Carathéodory inequality (see, e.g., [3]). For the first time, the inequality

$$|f(z)| \le \frac{Cr}{R-r} \max_{|\zeta|=R} \Re f(\zeta) \tag{2.32}$$

was obtained by Hadamard (real part theorem) with $C = 4$ in 1892 [7] and used in the theory of entire functions. Here f is an analytic function on the disc D_R, continuous on $\overline{D}_R$ and vanishing at $z = 0$. Different proofs of (2.32) with $C = 2$ are given in [1, 2, 5, 13, 16, 17].

Inequalities(2.29)-(2.31) follow from Corollary 1 with $p = 1$, $\omega = \mathcal{A}_f(R)$, that is they are particular cases of the estimate

$$|\Re\{e^{i\alpha}\Delta f(z)\}| \le \frac{2r(R + r|\cos\alpha|)}{R^2 - r^2} \max_{|\zeta|=R} \Re\Delta f(\zeta)$$

for $\alpha = 0, \alpha = \pi/2$, and $\alpha = -\arg \Delta f(z)$, respectively.

Corollary 1 implies one more inequality. Putting $p = 1$ and $\omega = \mathcal{A}_f(R)$ in (2.16) and taking into account (2.11), (2.13), and (2.17), we arrive at the estimate

$$|\Delta f(z)| \le \frac{2r(R + r|\cos\alpha_1|)}{R^2 - r^2} \max_{|\zeta|=R} \Re\Delta f(\zeta), \tag{2.33}$$

valid for functions f such that $\Delta f(z) \in W(\alpha_1, \alpha_2)$ with $z \in D_R$. In particular, setting here $\alpha_1 = 0$ we arrive at (2.31). ▮

Concluding this section, we make an observation concerning Proposition 1 with $p = 2$.

Remark 6. Let m be a positive integer and let $\{\mathring{\mathcal{P}}_m\}$ be the sequence of functions on the circle $|\zeta| = R$ defined by

$$\mathring{P}_n(\zeta) = \Re \sum_{k=1}^{n} c_k \zeta^{-k}, \tag{2.34}$$

where $c_1, c_2, \dots, c_n \in \mathbb{C}$, $1 \le n \le m$. Let f be an analytic function on D_R with continuous real part on $\overline{D}_R$ and let $\alpha(z)$ be a real valued function, $|z| < R$. Suppose that

$$\int_{|\zeta|=R} \Re f(\zeta) \mathring{P}_n(\zeta) |d\zeta| = 0 \tag{2.35}$$

for all $\mathring{P}_n \in \{\mathring{\mathcal{P}}_m\}$. We show that for any fixed z with $|z| = r < R$, there holds inequality

$$|\Re\{e^{i\alpha(z)}\Delta f(z)\}| \le \frac{1}{R^{1/2}} \mathcal{K}_m\left(\frac{r}{R}\right) E_2(\Re f) \tag{2.36}$$

with the sharp constant

$$\mathcal{K}_m(\gamma) = \frac{\gamma^{m+1}}{\sqrt{\pi(1-\gamma^2)}}, \tag{2.37}$$

where $E_2(\Re f) = ||\Re \Delta f||_2$ is the best approximation of $\Re f$ by a constant in the norm of $L_2(\partial D_R)$.

Introducing the notation $\xi_k = \Re c_k, \eta_k = \Im c_k, \zeta = Re^{it}$ we write (2.34) as the trigonometric polynomial

$$\mathring{P}_n(\zeta) = \mathring{P}_n(Re^{it}) = \Re\left\{\sum_{k=1}^{n}(\xi_k + i\eta_k)R^{-k}e^{-ikt}\right\} = \sum_{k=1}^{n}\left(a_k \cos kt + b_k \sin kt\right), \tag{2.38}$$

with $a_k = \xi_k R^{-k}, b_k = \eta_k R^{-k}$.

Let $K_m(z,\alpha)$ denote the sharp constant in

$$|\Re\{e^{i\alpha(z)}\Delta f(z)\}| \le K_m(z,\alpha)||\Re f - \omega||_2, \tag{2.39}$$

where ω is an arbitrary real constant. Taking into account that the mean value of function (2.38) on the circle $|\zeta| = R$ is zero, we obtain from Proposition 1

$$\begin{aligned} K_m(z,\alpha) &= \frac{1}{\pi R}\min_{\mathring{P}_n \in \{\mathring{\mathcal{P}}_m\}}\min_{\lambda\in\mathbb{R}}||G_{z,\alpha} - \lambda\mathring{P}_n||_2 = \frac{1}{\pi R}\min_{\mathring{P}_n \in \{\mathring{\mathcal{P}}_m\}}||G_{z,\alpha} - \mathring{P}_n||_2 \\ &= \frac{1}{\pi R}\min_{\mathring{P}_n \in \{\mathring{\mathcal{P}}_m\}}\left\{R\int_{-\pi}^{\pi}\left[G_{z,\alpha}(Re^{it}) - \mathring{P}_n(Re^{it})\right]^2 dt\right\}^{1/2}. \end{aligned} \tag{2.40}$$

Let $z = re^{i\tau}$ and, as above $\zeta = Re^{it}, \gamma = r/R$. We have

$$\begin{aligned} G_{z,\alpha}(\zeta) &= \Re\left(\frac{e^{i\alpha}z\zeta^{-1}}{1 - z\zeta^{-1}}\right) = \Re\left\{e^{i\alpha}\sum_{k=1}^{\infty}\left(\frac{z}{\zeta}\right)^k\right\} = \Re\left\{\sum_{k=1}^{\infty}\gamma^k e^{i[\alpha+k(\tau-t)]}\right\} \\ &= \sum_{k=1}^{\infty}\gamma^k\cos(kt - \alpha - k\tau) = \sum_{k=1}^{\infty}\gamma^k(\cos kt\cos\beta_k + \sin kt \sin\beta_k), \end{aligned} \tag{2.41}$$

where $\beta_k = \alpha + k\tau$. Hence and by (2.40)

$$K_m(z,\alpha) = \frac{1}{\pi\sqrt{R}}\left\{\int_{-\pi}^{\pi}\left[G_{z,\alpha}(Re^{it}) - \sum_{k=1}^{m}\gamma^k(\cos kt\cos\beta_k + \sin kt\sin\beta_k)\right]^2 dt\right\}^{1/2}. \tag{2.42}$$

By (2.41), (2.42) and the Parseval equality,

$$K_m(z,\alpha) = \frac{1}{\pi\sqrt{R}}\left\{\pi\sum_{k=m+1}^{\infty}\left(\gamma^{2k}\cos^2\beta_k + \gamma^{2k}\sin^2\beta_k\right)\right\}^{1/2} = \left(\frac{1}{\pi R}\sum_{k=m+1}^{\infty}\gamma^{2k}\right)^{1/2},$$

that is

$$K_m(z,\alpha) = \frac{\gamma^{m+1}}{\sqrt{\pi R(1-\gamma^2)}}.$$

Using (2.37) we find the representation of the sharp constant $K_m(z,\alpha) = R^{-1/2}\mathcal{K}_m(r/R)$ in (2.39) which by Remark 2 implies (2.36) with the sharp constant (2.37). ∎

3 Estimate for $|\Re\{e^{i\alpha(z)}\Delta f(z)\}|$ by $||\Re f - (\Re f,\Phi)/(1,\Phi)||_p$ if the mean value of Φ is not zero

Suppose $\int_{|\zeta|=R}\Phi(\zeta)|d\zeta| \neq 0$ and replace f with $f-\omega$ in Lemma 1, where ω is a real constant. By (1.2)

$$\int_{|\zeta|=R}\{\Re f(\zeta)-\omega\}\Phi(\zeta)|d\zeta| = 0,$$

that is $\omega = (\Re f,\Phi)/(1,\Phi)$. Hence, by Lemma 1 we arrive at another type of estimates for $|\Re\{e^{i\alpha(z)}\Delta f(z)\}|$.

Proposition 2. *Let f be analytic on D_R with continuous real part on $\overline{D}_R$, $1 \leq p \leq \infty$, and let $\alpha(z)$ be a real valued function, $|z| < R$. Further, let Φ be a real continuous function on $|\zeta| = R$, for which the inequality*

$$\int_{|\zeta|=R}\Phi(\zeta)|d\zeta| \neq 0 \tag{3.1}$$

holds.

Then for any fixed point $z, |z| = r < R$, there holds

$$|\Re\{e^{i\alpha(z)}\Delta f(z)\}| \leq \mathcal{C}_{\Phi,\,p}\,(z,\alpha(z))\,||\Re f - (\Re f,\Phi)/(1,\Phi)||_p \tag{3.2}$$

with the sharp constant $\mathcal{C}_{\Phi,\,p}(z,\alpha(z))$ given by (1.4).

In particular, for any fixed point $z, |z| = r < R$, the inequality is valid

$$|\Delta f(z)| \leq \mathcal{C}_{\Phi,\,p}(z,-\arg\Delta f(z))||\Re f - (\Re f,\Phi)/(1,\Phi)||_p. \tag{3.3}$$

The following assertion is a particular case of Proposition 2 for $\Phi = 1$.

Corollary 2. *Let f be analytic on D_R with continuous real part on $\overline{D}_R$, $1 \leq p \leq \infty$. Further, let $\alpha(z)$ be a real valued function, $|z| < R$. Then for any fixed point $z, |z| = r < R$, there holds*

$$|\Re\{e^{i\alpha(z)}\Delta f(z)\}| \leq \mathcal{C}_{1,\,p}\,(z,\alpha(z))\,||\Re\Delta f||_p \tag{3.4}$$

with the sharp constant $\mathcal{C}_p(z,\alpha(z))$, where

$$\mathcal{C}_{1,\,p}(z,\alpha) = \frac{1}{R^{1/p}}C_{1,\,p}\left(\frac{r}{R},\alpha\right), \tag{3.5}$$

and

$$C_{1,\,p}(\gamma,\alpha) = \frac{\gamma}{\pi}\min_{\lambda\in\mathbb{R}}\left\{\int_{-\pi}^{\pi}\left|\frac{\cos(\varphi-\alpha)-\gamma\cos\alpha}{1-2\gamma\cos\varphi+\gamma^2}-\lambda\right|^{p/(p-1)}d\varphi\right\}^{(p-1)/p} \tag{3.6}$$

if $1 < p < \infty$, *and*

$$C_{1,\,1}(\gamma,\alpha) = \frac{\gamma}{\pi(1-\gamma^2)}, \tag{3.7}$$

$$C_{1,\,\infty}(\gamma,\alpha) = \frac{2}{\pi}\left\{\sin\alpha\log\frac{2\gamma\sin\alpha+\sqrt{(1-\gamma^2)^2+4\gamma^2\sin^2\alpha}}{1-\gamma^2}\right.$$
$$\left.+\cos\alpha\arcsin\left(\frac{2\gamma\cos\alpha}{1+\gamma^2}\right)\right\}. \tag{3.8}$$

In particular,

$$C_{1,\,2}(\gamma,\alpha) = \frac{\gamma}{\sqrt{\pi(1-\gamma^2)}}. \tag{3.9}$$

Proof. We set $\Phi(z)\equiv 1$ in Proposition 2. Then (3.2) takes the form (3.4). The equality (3.5) with

$$C_{1,\,p}(\gamma,\alpha) = \frac{1}{\pi}\min_{\lambda\in\mathbb{R}}\left\{\int_{-\pi}^{\pi}\left|\Re\left(\frac{\gamma e^{i\alpha}}{e^{i\varphi}-\gamma}\right)-\lambda\right|^{p/(p-1)}d\varphi\right\}^{(p-1)/p}, \tag{3.10}$$

where $1 < p < \infty$, can be derived from (1.4) in the same way as (2.21) with the constant (2.20) was obtained in Corollary 1. Formula (3.6) follows directly from (3.10) and (1.23).

1. The case $p = 1$. By (1.4),

$$\mathcal{C}_{1,\,1}(z,\alpha) = \frac{1}{\pi R}\min_{\lambda\in\mathbb{R}}\max_{|\zeta|=R}\left|G_{z,\alpha}(\zeta)-\lambda\right|. \tag{3.11}$$

Since λ is subject to one of the three alternatives

$$\lambda\le\min_{|\zeta|=R}G_{z,\alpha}(\zeta),\quad \min_{|\zeta|=R}G_{z,\alpha}(\zeta)<\lambda<\max_{|\zeta|=R}G_{z,\alpha}(\zeta),\quad \lambda\ge\max_{|\zeta|=R}G_{z,\alpha}(\zeta),$$

it follows that the minimum with respect to λ in (3.11) is attained at

$$\lambda = \frac{1}{2}\left\{\min_{|\zeta|=R}G_{z,\alpha}(\zeta)+\max_{|\zeta|=R}G_{z,\alpha}(\zeta)\right\},$$

which by Lemma 2 implies

$$\lambda = \frac{r^2\cos\alpha}{R^2-r^2}.$$

Putting the value of λ into (3.11) and using (1.22) we obtain

$$\mathcal{C}_{1,\,1}(z,\alpha) = \frac{1}{\pi R}\,\frac{rR}{R^2-r^2},$$

which proves (3.5) with $p=1$ and the factor (3.7).

2. The case $p=2$. From

$$\int_{|\zeta|=R} G_{z,\alpha}(z)|d\zeta| = \Re\left\{\int_{|\zeta|=R}\frac{e^{i\alpha}z}{\zeta-z}|d\zeta|\right\} = \Re\left\{\int_{|\zeta|=R}\frac{Re^{i\alpha}z}{i(\zeta-z)\zeta}d\zeta\right\} = 0,$$

and (1.21) with $\Phi(\zeta)\equiv 1$ we see that (3.5) holds with $p=2$ and the factor (3.9).

3. The case $p=\infty$. Let the function α and z with $|z|=r<R$ be fixed. It is well known (see, for example, [10]), that λ gives the minimum in (3.10) with $p=\infty$ if and only if

$$\int_{-\pi}^{\pi}\operatorname{sign}\left\{\Re\left(\frac{e^{i\alpha}}{e^{i\varphi}-\gamma}\right)-\lambda\right\}d\varphi = 0. \tag{3.12}$$

We show that this equality holds for $\lambda=-\gamma(1+\gamma^2)^{-1}\cos\alpha$ with $\gamma\in[0,1)$.

We rewrite the left-hand side of the equation

$$\Re\left(\frac{e^{i\alpha}}{e^{i\varphi}-\gamma}\right)+\frac{\gamma}{1+\gamma^2}\cos\alpha = 0 \tag{3.13}$$

as

$$\Re\left(\frac{e^{i\alpha}}{e^{i\varphi}-\gamma}\right)+\frac{\gamma}{1+\gamma^2}\cos\alpha = \Re\left(\frac{e^{i\alpha}}{e^{i\varphi}-\gamma}+\frac{\gamma e^{i\alpha}}{1+\gamma^2}\right)$$
$$= \Re\left(\frac{(1+\gamma e^{i\varphi})e^{i\alpha}}{(e^{i\varphi}-\gamma)(1+\gamma^2)}\right) = \frac{1}{1+\gamma^2}\cdot\frac{(1-\gamma^2)\cos\varphi\cos\alpha+(1+\gamma^2)\sin\varphi\sin\alpha}{1-2\gamma\cos\varphi+\gamma^2}. \tag{3.14}$$

We introduce the angle ϑ by the equalities

$$\cos\vartheta = \frac{(1-\gamma^2)}{k(\alpha,\gamma)}\cos\alpha, \qquad \sin\vartheta = \frac{(1+\gamma^2)}{k(\alpha,\gamma)}\sin\alpha, \tag{3.15}$$

where

$$k(\alpha,\gamma) = [(1-\gamma^2)^2\cos^2\alpha+(1+\gamma^2)^2\sin^2\alpha]^{1/2} = [(1+\gamma^2)^2-4\gamma^2\cos^2\alpha]^{1/2}. \tag{3.16}$$

From (3.14)-(3.15) we obtain

$$\Re\left(\frac{e^{i\alpha}}{e^{i\varphi}-\gamma}\right)+\frac{\gamma}{1+\gamma^2}\cos\alpha = \frac{k(\alpha,\gamma)}{1+\gamma^2}\cdot\frac{\cos(\varphi-\vartheta)}{1-2\gamma\cos\varphi+\gamma^2}. \tag{3.17}$$

Thus, the equation (3.13) with unknown φ is reduced to $\cos(\varphi-\vartheta)=0$. Let ϑ be the solution of system (3.15) in $(-\pi,\pi]$.

The distance between two successive roots $\varphi_n = \vartheta - \pi/2 + \pi n,\ n = 0, \pm 1, \pm 2, \dots,$ of the equation $\cos(\varphi - \vartheta) = 0$ is equal to π. We put $\zeta_0 = e^{i\varphi_0},\ \zeta_1 = e^{i\varphi_1}$ with $\varphi_0 = \vartheta - \pi/2,\ \varphi_1 = \vartheta + \pi/2$. Then

$$\Re\left(\frac{e^{i\alpha}}{\zeta_0 - \gamma}\right) + \frac{\gamma}{1+\gamma^2}\cos\alpha = \Re\left(\frac{e^{i\alpha}}{\zeta_1 - \gamma}\right) + \frac{\gamma}{1+\gamma^2}\cos\alpha = 0.$$

Thus, for fixed $\gamma \in [0, 1)$ and α, the points ζ_0 and ζ_1 divide the circle $|\zeta| = 1$ into two half-circles such that on one of them the left-hand side of (3.13) is positive and on another is negative. Hence (3.12) holds with $\lambda = -\gamma(1 + \gamma^2)^{-1}\cos\alpha$ and, therefore, by (3.5) and (3.10),

$$\mathcal{C}_{1,\,\infty}(z, \alpha) = C_{1,\,\infty}\left(\frac{r}{R}, \alpha\right), \tag{3.18}$$

where

$$C_{1,\,\infty}(\gamma, \alpha) = \frac{\gamma}{\pi}\int_{-\pi}^{\pi}\left|\Re\left(\frac{e^{i\alpha}}{e^{\varphi} - \gamma}\right) + \frac{\gamma}{1+\gamma^2}\cos\alpha\right| d\varphi.$$

This and (3.17) imply

$$C_{1,\,\infty}(\gamma, \alpha) = \frac{\gamma k(\alpha, \gamma)}{\pi(1+\gamma^2)}\int_{-\pi}^{\pi}\frac{|\cos(\varphi - \vartheta)|}{1 - 2\gamma\cos\varphi + \gamma^2}d\varphi, \tag{3.19}$$

where $k(\alpha, \gamma)$ is defined by (3.16) and ϑ is the solution of (3.15) in $(-\pi, \pi]$.

Equality (3.19) can be written as

$$C_{1,\,\infty}(\gamma, \alpha) = \frac{\gamma k(\alpha, \gamma)}{\pi(1+\gamma^2)}\left\{\int_{\vartheta-\pi/2}^{\vartheta+\pi/2}\frac{\cos(\varphi - \vartheta)}{1 - 2\gamma\cos\varphi + \gamma^2}d\varphi - \int_{\vartheta+\pi/2}^{\vartheta+3\pi/2}\frac{\cos(\varphi - \vartheta)}{1 - 2\gamma\cos\varphi + \gamma^2}d\varphi\right\}.$$

In the first integral we make the change of variable $\psi = -\varphi$ and in the second integral we put $\eta = \pi - \varphi$. Then

$$C_{1,\,\infty}(\gamma, \alpha) = \frac{\gamma k(\alpha, \gamma)}{\pi(1+\gamma^2)}\left\{\int_{-\pi/2-\vartheta}^{\pi/2-\vartheta}\frac{\cos(\psi + \vartheta)}{1 - 2\gamma\cos\psi + \gamma^2}d\psi + \int_{-\pi/2-\vartheta}^{\pi/2-\vartheta}\frac{\cos(\eta + \vartheta)}{1 + 2\gamma\cos\eta + \gamma^2}d\eta\right\},$$

which implies

$$C_{1,\,\infty}(\gamma, \alpha) = \frac{2\gamma k(\alpha, \gamma)}{\pi}\int_{-\pi/2-\vartheta}^{\pi/2-\vartheta}\frac{\cos(\psi + \vartheta)}{(1+\gamma^2)^2 - 4\gamma^2\cos^2\psi}d\psi,$$

that is

$$C_{1,\,\infty}(\gamma, \alpha) = \frac{2\gamma k(\alpha, \gamma)}{\pi}\int_{-\pi/2-\vartheta}^{\pi/2-\vartheta}\frac{\cos\psi\cos\vartheta - \sin\psi\sin\vartheta}{(1+\gamma^2)^2 - 4\gamma^2\cos^2\psi}d\psi. \tag{3.20}$$

Substituting the integrals

$$\int_{-\pi/2-\vartheta}^{\pi/2-\vartheta}\frac{\cos\psi}{(1+\gamma^2)^2 - 4\gamma^2\cos^2\psi}d\psi = \frac{1}{\gamma(1-\gamma^2)}\arctan\left(\frac{2\gamma\cos\vartheta}{1-\gamma^2}\right),$$

$$\int_{-\pi/2-\vartheta}^{\pi/2-\vartheta} \frac{\sin\psi}{(1+\gamma^2)^2 - 4\gamma^2\cos^2\psi} d\psi = -\frac{1}{2\gamma(1+\gamma^2)} \log\frac{1+\gamma^2+2\gamma\sin\vartheta}{1+\gamma^2-2\gamma\sin\vartheta}$$

into (3.20) we obtain

$$C_{1,\,\infty}(\gamma,\alpha) = \frac{2}{\pi} k(\alpha,\gamma)\left\{\frac{\cos\vartheta}{1-\gamma^2}\arctan\frac{2\gamma\cos\vartheta}{1-\gamma^2} + \frac{\sin\vartheta}{2(1+\gamma^2)}\log\frac{1+\gamma^2+2\sin\vartheta}{1+\gamma^2-2\gamma\sin\vartheta}\right\}.$$

Taking into account (3.15), (3.16), as well as the identity $\arctan[x(1-x^2)^{-1/2}] = \arcsin x$, we rewrite the last representation as

$$C_{1,\,\infty}(\gamma,\alpha) = \frac{2}{\pi}\left\{\cos\alpha \arcsin\left(\frac{2\gamma\cos\alpha}{1+\gamma^2}\right)\right.$$
$$\left. + \sin\alpha \log\frac{[(1-\gamma^2)^2+4\gamma^2\sin^2\alpha]^{1/2}+2\gamma\sin\alpha}{1-\gamma^2}\right\}. \qquad (3.21)$$

By (3.21) and (3.18) we arrive at (3.5) with $p=\infty$ with the right-hand side given by (3.8). ▮

Remark 7. Comparing the formulas (2.13), (3.7) and (2.14), (3.8) we conclude that in general $\mathcal{C}_{0,\,p}(\gamma,\alpha) \neq \mathcal{C}_{1,\,p}(\gamma,\alpha)$. However, for certain values of p and α the equality may hold. This is, clearly, the case for $p=2$ in view of (2.15) and (3.9).

Let us show now that $\mathcal{C}_{0,\,p}(\gamma,\pi/2) = \mathcal{C}_{1,\,p}(\gamma,\pi/2),\ 1\le p\le\infty$, i.e. that sharp constants in

$$|\Im\Delta f(z)| \le \mathcal{C}_{0,\,p}(\gamma,\pi/2)||\Re f - \omega||_p, \qquad (3.22)$$

$$|\Im\Delta f(z)| \le \mathcal{C}_{1,\,p}(\gamma,\pi/2)||\Re\Delta f||_p, \qquad (3.23)$$

coincide. In view of (2.11), (3.5), it suffices to prove that $C_{0,\,p}(\gamma,\pi/2) = C_{1,\,p}(\gamma,\pi/2)$. The equalities

$$\mathcal{C}_{0,\,1}(\gamma,\pi/2) = \mathcal{C}_{1,\,1}(\gamma,\pi/2) = \frac{\gamma}{\pi(1-\gamma^2)},$$
$$\mathcal{C}_{0,\,\infty}(\gamma,\pi/2) = \mathcal{C}_{1,\,\infty}(\gamma,\pi/2) = \frac{2}{\pi}\log\frac{1+\gamma}{1-\gamma}$$

follow directly from (2.13), (3.7) and (2.14), (3.8).

For $1<p<\infty$, by (3.6)

$$C_{1,\,p}(\gamma,\pi/2) = \frac{\gamma}{\pi}\min_{\lambda\in\mathbb{R}}\left\{\int_{-\pi}^{\pi}\left|\frac{\sin\varphi}{1-2\gamma\cos\varphi+\gamma^2} - \lambda\right|^q d\varphi\right\}^{1/q}. \qquad (3.24)$$

It is well-known (see, for instance, [10]), that λ gives the minimum in (3.24) if and only if

$$\int_{-\pi}^{\pi}\left|\frac{\sin\varphi}{1-2\gamma\cos\varphi+\gamma^2} - \lambda\right|^{q-1} \operatorname{sign}\left(\frac{\sin\varphi}{1-2\gamma\cos\varphi+\gamma^2} - \lambda\right) d\varphi = 0.$$

Clearly, the equality holds for $\lambda = 0$. Putting $\lambda = 0$ in (3.24) and using (2.12), we conclude that $C_{1,\,p}(\gamma, \pi/2) = C_{0,\,p}(\gamma, \pi/2)$ for $1 < p < \infty$. Thus, Remark 3 relating $C_{0,\,p}(\gamma, \pi/2)$ is also valid for $C_{1,\,p}(\gamma, \pi/2)$ and inequality (0.14) holds with the sharp constant (0.8).

We shall write the sharp constant (0.8) in (0.7) and (0.14) in a different form. Using the equality (see, for example, [6])

$$\int_0^\pi \left(\frac{\sin\varphi}{1 - 2\gamma\cos\varphi + \gamma^2} \right)^q d\varphi = B\left(\frac{q+1}{2}, \frac{1}{2} \right) F\left(q, \frac{q}{2}; \frac{q+2}{2}; \gamma^2 \right),$$

where $F(a, b; c; x)$ is the hypergeometric Gauss function, and the relation

$$F(a, b; a-b+1; x) = (1-x)^{1-2b}(1+x)^{2b-a-1} F\left(\frac{a+1}{2} - b, \frac{a}{2} + 1 - b; a - b + 1; \frac{4x}{(1+x)^2} \right),$$

we conclude by (2.24) that

$$\begin{aligned} C_{0,\,p}(\gamma, \pi/2) \;&=\; \frac{\varkappa(\gamma)}{2\pi} \left\{ 2\left[1 - \varkappa^2(\gamma)\right]^{(1-q)/2} B\left(\frac{q+1}{2}, \frac{1}{2} \right) F\left(\frac{1}{2}, 1; \frac{q+1}{2}; \varkappa^2(\gamma) \right) \right\}^{1/q} = \\ &\frac{\varkappa(\gamma)}{2\pi} \left\{ 2\left[1 - \varkappa^2(\gamma)\right]^{1/(2-2p)} \sum_{n=0}^{\infty} B\left(\frac{2p-1}{2p-2}, \frac{2n+1}{2} \right) \varkappa^{2n}(\gamma) \right\}^{(p-1)/p}, \end{aligned} \tag{3.25}$$

where $\varkappa(\gamma) = (2\gamma)/(1+\gamma^2)$. Combining (3.25) with (3.23), (3.5) and the equality $C_{1,\,p}(\gamma, \pi/2) = C_{0,\,p}(\gamma, \pi/2)$ we arrive at (0.15). ∎

The next assertion contains an estimate of $|\Delta f(z)|$ for a class of analytic functions in D_R with the real part continuous in $\overline{D}_R$ and such that $\Delta f(z) \in W(\alpha_1, \alpha_2),\ z \in D_R$, where $W(\alpha_1, \alpha_2)$ is defined by (2.9).

Corollary 3. *Let f be analytic on D_R with continuous real part on $\overline{D}_R, 1 \le p \le \infty$, and let $\Delta f(z) \in W(\alpha_1, \alpha_2)$ for $z \in D_R$. Then the inequality holds*

$$|\Delta f(z)| \le \max_{\alpha_1 \le \alpha \le \alpha_2} \mathcal{C}_{1,\,p}(z, \alpha) ||\Re \Delta f||_p, \tag{3.26}$$

where $\mathcal{C}_{1,\,p}(z, \alpha)$ is given by (3.5) – (3.8).

In particular,

$$\max_{\alpha_1 \le \alpha \le \alpha_2} \mathcal{C}_{1,\,\infty}(z, \alpha) = \mathcal{C}_{1,\,\infty}(z, \alpha_2), \tag{3.27}$$

with $\mathcal{C}_{1,\,\infty}(z, \alpha)$ defined by (3.5), (3.8).

Proof. Putting $\Phi(\zeta) \equiv 1$ in (3.3) we obtain

$$|\Delta f(z)| \le \mathcal{C}_{1,\,p}(z, -\arg \Delta f(z)) ||\Re \Delta f||_p. \tag{3.28}$$

We show that $\mathcal{C}_{1,\,p}(z, -\alpha) = \mathcal{C}_{1,\,p}(z, \alpha)$. For $p = 1$ and $p = \infty$, this follows directly from (3.5), (3.7), and (3.8).

Let $1 < p < \infty$. By (3.10),

$$C_{1,\,p}(\gamma, \alpha) = \frac{\gamma}{\pi} \min_{\lambda \in \mathbb{R}} \left\{ \int_0^{2\pi} \left| \Re\left(\frac{e^{-i\alpha}}{e^{-i\varphi} - \gamma} \right) - \lambda \right|^{p/(p-1)} d\varphi \right\}^{(p-1)/p},$$

which after the change of variable $\varphi = 2\pi - \psi$ becomes

$$C_{1,\,p}(\gamma, \alpha) = \frac{\gamma}{\pi} \min_{\lambda \in \mathbb{R}} \left\{ \int_0^{2\pi} \left| \Re\left(\frac{e^{-i\alpha}}{e^{i\psi} - \gamma} \right) - \lambda \right|^{p/(p-1)} d\psi \right\}^{(p-1)/p} = C_{1,\,p}(\gamma, -\alpha).$$

This together with (3.5) implies $\mathcal{C}_{1,\,p}(z, -\alpha) = \mathcal{C}_{1,\,p}(z, \alpha)$. Hence, (3.28) can be written as

$$|\Delta f(z)| \le \mathcal{C}_{1,\,p}(z, \arg \Delta f(z)) ||\Re \Delta f||_p. \tag{3.29}$$

Let $0 \le \alpha \le \pi/2$. By (3.5) and (3.10) we have $\mathcal{C}_{1,\,p}(z, \pi - \alpha) = \mathcal{C}_{1,\,p}(z, -\alpha)$. This and $\mathcal{C}_{1,\,p}(z, -\alpha) = \mathcal{C}_{1,\,p}(z, \alpha)$ imply

$$\sup\{\mathcal{C}_{1,\,p}(z, \arg \Delta f(z)) : \Delta f(z) \in W(\alpha_1, \alpha_2)\} = \max\{\mathcal{C}_{1,\,p}(z, \alpha) : \alpha_1 \le \alpha \le \alpha_2\},$$

which together with (3.29) leads to (3.26).

Now, we prove (3.27). Owing (3.5) and (3.8),

$$\mathcal{C}_{1,\,\infty}(z, \alpha) = \frac{2}{\pi} \left\{ \sin\alpha \log \frac{2\gamma \sin\alpha + \sqrt{(1-\gamma^2)^2 + 4\gamma^2 \sin^2\alpha}}{1-\gamma^2} + \cos\alpha \arcsin\left(\frac{2\gamma\cos\alpha}{1+\gamma^2} \right) \right\},$$

where $\gamma = r/R$.

Let us consider $\mathcal{C}_{1,\,\infty}(z, \alpha)$ for $0 \le \alpha \le \pi/2$. We have

$$\frac{\partial \mathcal{C}_{1,\,\infty}(z, \alpha)}{\partial \alpha} = \frac{2}{\pi} \left\{ \cos\alpha \log \frac{2\gamma \sin\alpha + \sqrt{(1-\gamma^2)^2 + 4\gamma^2 \sin^2\alpha}}{1-\gamma^2} - \sin\alpha \arcsin\left(\frac{2\gamma\cos\alpha}{1+\gamma^2} \right) \right\}. \tag{3.30}$$

Note that the relations

$$\cos\alpha \log \frac{2\gamma \sin\alpha + \sqrt{(1-\gamma^2)^2 + 4\gamma^2 \sin^2\alpha}}{1-\gamma^2} = \cos\alpha \int_0^{2\gamma\sin\alpha} \frac{dt}{\sqrt{(1-\gamma^2)^2 + t^2}},$$

$$\sin\alpha \arcsin\left(\frac{2\gamma\cos\alpha}{1+\gamma^2} \right) = \sin\alpha \int_0^{2\gamma\cos\alpha(1+\gamma^2)^{-1}} \frac{dt}{\sqrt{1-t^2}},$$

and the mean value theorem imply

$$\cos\alpha \int_0^{2\gamma\sin\alpha} \frac{dt}{\sqrt{(1-\gamma^2)^2 + t^2}} > \frac{2\gamma\cos\alpha\sin\alpha}{[(1-\gamma^2)^2 + 4\gamma^2\sin^2\alpha]^{1/2}},$$

$$\sin\alpha\int_0^{2\gamma\cos\alpha(1+\gamma^2)^{-1}}\frac{dt}{\sqrt{1-t^2}}<\frac{2\gamma\cos\alpha\sin\alpha}{[(1-\gamma^2)^2+4\gamma^2\sin^2\alpha]^{1/2}},$$

where $\alpha\in(0,\pi/2)$. Therefore, it follows from (3.30) that

$$\frac{\partial\mathcal{C}_{1,\,\infty}(z,\alpha)}{\partial\alpha}>0.$$

Thus, $\mathcal{C}_{1,\,\infty}(z,\alpha)$ increases on the interval $[0,\pi/2]$. ∎

Remark 8. The class of inequalities considered in this section include the following three inequalities

$$|\Re\Delta f(z)|\le\frac{4}{\pi}\arctan\left(\frac{r}{R}\right)||\Re\Delta f||_\infty, \tag{3.31}$$

$$|\Im\Delta f(z)|\le\frac{2}{\pi}\log\frac{R+r}{R-r}||\Re\Delta f||_\infty, \tag{3.32}$$

$$|\Delta f(z)|\le\frac{2}{\pi}\log\frac{R+r}{R-r}||\Re\Delta f||_\infty \tag{3.33}$$

(see [3, 4, 9, 15] and the bibliography in [3, 9]).

Inequalities (3.31), (3.32) follow from (3.4), (3.5) with $p=\infty$ combined with (3.8) with $\alpha=0$ and $\alpha=\pi/2$, respectively. Inequality (3.33) follows from Corollary 3. In fact, by (3.8),

$$C_{1,\,\infty}(\gamma,0)=\frac{2}{\pi}\arcsin\left(\frac{2\gamma}{1+\gamma^2}\right)=\frac{4}{\pi}\arctan\gamma,$$
$$C_{1,\,\infty}(\gamma,\pi/2)=\frac{2}{\pi}\log\frac{1+\gamma}{1-\gamma},$$

which together with Corollary 3 leads to

$$\max_{0\le\alpha\le\pi/2}C_{1,\,\infty}(\gamma,\alpha)=C_{1,\,\infty}(\gamma,\pi/2)=\frac{2}{\pi}\log\frac{1+\gamma}{1-\gamma}.$$

Acknowledgements. The research of the first author was supported by the KAMEA program of the Ministry of Absorption, State of Israel, and by Research Authority of the College of Judea and Samaria, Ariel.

References

[1] E. Borel, Démonstration élémentaire d'un théorème de M. Picard sur les fonctions entières, *C.R. Acad. Sci.*, **122** (1896), 1045-1048.

[2] E. Borel, *Méthodes et problèmes de Théorie des Fonctions*, Gauthier-Villars, Paris, 1922.

[3] R.B. Burckel, *An introduction to classical complex analysis, V. 1*, Academic Press, New York - San Francisco, 1979.

[4] C. Carathéodory, Elementarer Beweis für den Fundamentalsatz der konformen Abbildungen, *Mathematische Abhandlungen H.A. Schwarz Gewidmet* , 19-41, Verlag Julius Springer (1914), Berlin; Reprinted by Chelsea Publishing Co. (1974), New York.

[5] M.L. Cartwright, *Integral Functions*, Oxford Univ. Press, Oxford, 1964.

[6] I.S. Gradshtein and I.M. Ryzhik; Alan Jeffrey, editor, *Table of integrals, series and products*, Fifth edition, Academic Press, New York, 1994.

[7] J. Hadamard, Sur les fonctions entières de la forme $e^{G(X)}$, *C.R. Acad. Sci.*, **114** (1892), 1053-1055.

[8] A. Hurwitz and R. Courant, *Funktionentheorie*, Vierte Auflage, Springer-Verlag, Berlin - Göttingen - Heidelberg - New-York, 1964.

[9] P. Koebe, Über das Schwarzsche Lemma und einige damit zusammenhängende Ungleicheitsbeziehungen der Potentialtheorie und Funktionentheorie, *Math. Zeit.*, **6** (1920), 52-84.

[10] N. Korneichuk, *Exact constants in approximation theory. Encyclopedia of Mathematics and its applications, 38*, Cambridge University Press, 1991.

[11] G.I. Kresin and V.G. Maz'ya, Sharp parametric inequalities for analytic and harmonic functions related to Hadamard-Borel-Carathéodory inequalities, *Functional Differential Equations*, **9, N. 1-2** (2002), 135-163.

[12] B.Ya. Levin, *Lectures on Entire Functions*, Transl. of Math. Monogfaphs, v.150, Amer. Math. Soc., Providence, 1996.

[13] J.E. Littlewood, *Lectures on the theory of functions*, Oxford Univ. Press, Oxford, 1947.

[14] D.S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[15] G. Polya and G. Szegö, *Problems and Theorems in Analysis, v. 1*, Springer-Verlag, Berlin - Heidelberg - New-York, 1972.

[16] E.G. Titchmarsh, *The theory of functions*, Cambrige University Press, New York, 1962.

[17] L. Zalcman, Picard's theorem without tears, *Amer. Math. Monthly*, **85: 4** (1978), 265-268.