Discreteness of spectrum and positivity criteria for Schrödinger operators

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Abstract

We provide a class of necessary and sufficient conditions for the discreteness of spectrum of Schrödinger operators with scalar potentials which are semibounded below. The classical discreteness of spectrum criterion by A.M.Molchanov (1953) uses a notion of negligible set in a cube as a set whose Wiener's capacity is less than a small constant times the capacity of the cube. We prove that this constant can be taken arbitrarily between 0 and 1. This solves a problem formulated by I.M.Gelfand in 1953. Moreover, we extend the notion of negligibility by allowing the constant to depend on the size of the cube. We give a complete description of all negligibility conditions of this kind. The a priori equivalence of our conditions involving different negligibility classes is a non-trivial property of the capacity. We also establish similar strict positivity criteria for the Schrödinger operators with non-negative potentials.

1 Introduction

In 1934, K. Friedrichs [3] proved that the spectrum of the Schrödinger operator $-\Delta + V$ in $L^2(\mathbb{R}^n)$ with a locally integrable potential V is discrete provided

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$V(x) \to +\infty$ as $|x| \to \infty$ (see also [1, 11]). On the other hand, if we assume that V is semi-bounded below, then the discreteness of spectrum easily implies that for every $d > 0$

$$\int_{Q_d} V(x)dx \to +\infty \quad \text{as} \quad Q_d \to \infty, \tag{1.1}$$

where Q_d is an open cube with the edge length d and with the edges parallel to coordinate axes, $Q_d \to \infty$ means that the cube Q_d goes to infinity (with fixed d). This was first noticed by A.M.Molchanov in 1953 (see [10]) who also showed that this condition is in fact necessary and sufficient in case $n = 1$ but not sufficient for $n \geq 2$. Moreover, in the same paper Molchanov discovered a modification of condition (1.1) which is fully equivalent to the discreteness of spectrum in the case $n \geq 2$. It states that for every $d > 0$

$$\inf_F \int_{Q_d \setminus F} V(x)dx \to +\infty \quad \text{as} \quad Q_d \to \infty, \tag{1.2}$$

where infimum is taken over all compact subsets F of the closure $\bar{Q}_d$ which are called *negligible*. The negligibility of F in the sense of Molchanov means that $\operatorname{cap}(F) \leq \gamma \operatorname{cap}(Q_d)$, where cap is the Wiener capacity and $\gamma > 0$ is a sufficiently small constant. More precisely, Molchanov proved that we can take $\gamma = c_n$ where for $n \geq 3$

$$c_n = (4n)^{-4n}(\operatorname{cap}(Q_1))^{-1}.$$

Proofs of Molchanov's result can be found also in [9, 2, 6]. In particular, the books [9, 2] contain a proof which first appeared in [8] and is different from the original Molchanov proof. We will not list numerous papers related to the discreteness of spectrum conditions for one- and multidimensional Schrödinger operators. Some references can be found in [9, 6, 5].

As early as in 1953, I.M.Gelfand raised the question about the best possible constant c_n (personal communication). In this paper we answer this question by proving that c_n can be replaced by an arbitrary constant γ, $0 < \gamma < 1$.

We even establish a stronger result. We allow negligibility conditions of the form

$$\operatorname{cap}(F) \leq \gamma(d) \operatorname{cap}(Q_d) \tag{1.3}$$

and completely describe all admissible functions γ. More precisely, in the necessary condition for the discreteness of spectrum we allow arbitrary functions $\gamma : (0, +\infty) \to (0, 1)$. In the sufficient condition we can admit arbitrary functions γ with values in $(0, 1)$, defined for $d > 0$ in a neighborhood of $d = 0$ and satisfying

$$\limsup_{d \downarrow 0} d^{-2}\gamma(d) = +\infty. \tag{1.4}$$

On the other hand, if $\gamma(d) = O(d^2)$ in the negligibility condition (1.3), then the condition (1.2) is no longer sufficient, i.e. it may happen that it is satisfied but the spectrum is not discrete.

All conditions (1.2) involving functions $\gamma : (0, +\infty) \to (0, 1)$, satisfying (1.4), are necessary and sufficient for the discreteness of spectrum. Therefore two conditions with different functions γ are equivalent, which is far from being obvious a priori. This equivalence means the following striking effect: if (1.2) holds for very small sets F, then it also holds for sets F which almost fill the corresponding cubes.

Another important question is whether the operator $-\Delta + V$ with $V \geq 0$ is strictly positive, i.e. the spectrum is separated from 0. Unlike the discreteness of spectrum conditions, it is the large values of d which are relevant here. The following necessary and sufficient condition for the strict positivity was obtained in [8] (see also [9], Sect.12.5): there exist positive constants d and $\varkappa$ such that for all cubes Q_d

$$\inf_F \int_{Q_d \setminus F} V(x)dx \geq \varkappa \,, \tag{1.5}$$

where the infimum is taken over all compact sets $F \subset \bar{Q}_d$ which are negligible in the sense of Molchanov. We prove that here again an arbitrary constant $\gamma \in (0, 1)$ in the negligibility condition (1.3) is admissible.

The above mentioned results are proved in this paper in a more general context. The family of cubes Q_d is replaced by a family of arbitrary bodies homothetic to a standard bounded domain which is star-shaped with respect to a ball. Instead of locally integrable potentials $V \geq 0$ we consider positive measures. We also include operators in arbitrary open subsets of $\mathbb{R}^n$ with the Dirichlet boundary conditions.

2 Main results

Let $\mathbb{V}$ be a positive Radon measure in an open set $\Omega \subset \mathbb{R}^n$. We will consider the Schrödinger operator which is formally given by an expression $-\Delta + \mathbb{V}$. It is defined in $L^2(\Omega)$ by the quadratic form

$$h_{\mathbb{V}}(u, u) = \int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^2 \mathbb{V}(dx), \quad u \in C_0^\infty(\Omega), \tag{2.1}$$

where $C_0^\infty(\Omega)$ is the space of all C^∞-functions with compact support in Ω. For the associated operator to be well defined we need a closed form. The form above is closable in $L^2(\Omega)$ if and only if $\mathbb{V}$ is absolutely continuous with respect to the Wiener capacity, i.e. for a Borel set $B \subset \Omega$, $\operatorname{cap}(B) = 0$ implies $\mathbb{V}(B) = 0$ (see [7] and also [9], Sect. 12.4). In the present paper we will always assume that this condition is satisfied. The operator, associated with the closure of the form (2.1) will be denoted $H_{\mathbb{V}}$.

In particular, we can consider an absolutely continuous measure $\mathbb{V}$ which has a density $V \geq 0$, $V \in L^1_{loc}(\mathbb{R}^n)$, with respect to the Lebesgue measure dx. Such a measure will be absolutely continuous with respect to the capacity as well.

Instead of the cubes Q_d which we dealt with in Sect.1, a more general family of test bodies will be used. Let us start with a standard open set $\mathcal{G} \subset \mathbb{R}^n$. We assume that $\mathcal{G}$ satisfies the following conditions:

(a) $\mathcal{G}$ is bounded and star-shaped with respect to an open ball $B_\rho(0)$ of radius $\rho > 0$, with the center at $0 \in \mathbb{R}^n$;
(b) $\mathrm{diam}(\mathcal{G}) = 1$.

The first condition means that $\mathcal{G}$ is star-shaped with respect to every point of $B_\rho(0)$. It implies that $\mathcal{G}$ can be presented in the form

$$\mathcal{G} = \{x|\ x = r\omega,\ |\omega| = 1,\ 0 \leq r < r(\omega)\}, \tag{2.2}$$

where $\omega \mapsto r(\omega) \in (0, +\infty)$ is a Lipschitz function on the standard unit sphere $S^{n-1} \subset \mathbb{R}^n$ (see [9], Lemma 1.1.8).

The condition (b) is imposed for convenience of formulations.

For any positive $d > 0$ denote by $\mathcal{G}_d(0)$ the body $\{x|\ d^{-1}x \in \mathcal{G}\}$ which is homothetic to $\mathcal{G}$ with coefficient d and with the center of homothety at 0. We will denote by $\mathcal{G}_d$ a body which is obtained from $\mathcal{G}_d(0)$ by a parallel translation: $\mathcal{G}_d(y) = y + \mathcal{G}_d(0)$ where y is an arbitrary vector in $\mathbb{R}^n$.

The notation $\mathcal{G}_d \to \infty$ means that the distance from $\mathcal{G}_d$ to 0 goes to infinity.

Definition 2.1 Let $\gamma \in (0, 1)$. The *negligibility class* $\mathcal{N}_\gamma(\mathcal{G}_d; \Omega)$ consists of all compact sets $F \subset \bar{\mathcal{G}}_d$ satisfying the following conditions:

$$\bar{\mathcal{G}}_d \setminus \Omega \subset F \subset \bar{\mathcal{G}}_d\ , \tag{2.3}$$

and

$$\mathrm{cap}\,(F) \leq \gamma\, \mathrm{cap}\,(\bar{\mathcal{G}}_d). \tag{2.4}$$

Now we formulate our main result about the discreteness of spectrum.

Theorem 2.2 (i) (Necessity) *Let the spectrum of $H_\mathbb{V}$ be discrete. Then for every function $\gamma : (0, +\infty) \to (0, 1)$ and every $d > 0$*

$$\inf_{F \in \mathcal{N}_{\gamma(d)}(\mathcal{G}_d, \Omega)} \mathbb{V}(\bar{\mathcal{G}}_d \setminus F) \to +\infty \quad \text{as} \quad \mathcal{G}_d \to \infty. \tag{2.5}$$

(ii) (Sufficiency) *Let a function $d \mapsto \gamma(d) \in (0, 1)$ be defined for $d > 0$ in a neighborhood of* 0*, and satisfy* (1.4)*. Assume that there exists $d_0 > 0$ such that* (2.5) *holds for every $d \in (0, d_0)$. Then the spectrum of $H_\mathbb{V}$ in $L^2(\Omega)$ is discrete.*

Let us make some comments about this theorem.

Remark 2.3 It suffices for the discreteness of spectrum of $H_\mathbb{V}$ that the condition (2.5) holds only for a sequence of d's, i.e. $d \in \{d_1, d_2, \dots\}$, $d_k \to 0$ and $d_k^{-2}\gamma(d_k) \to +\infty$ as $k \to +\infty$.

Remark 2.4 As we will see in the proof, in the sufficiency part the condition (2.5) can be replaced by a weaker requirement: there exist $c > 0$ and $d_0 > 0$ such that for every $d \in (0, d_0)$ there exists $R > 0$ such that

$$d^{-n} \inf_{F \in \mathcal{N}_{\gamma(d)}(\mathcal{G}_d, \Omega)} \mathbb{V}(\bar{\mathcal{G}}_d \setminus F) \geq c d^{-2} \gamma(d), \tag{2.6}$$

whenever $\bar{\mathcal{G}}_d \cap (\Omega \setminus B_R(0)) \neq \emptyset$ (i.e. for distant bodies $\mathcal{G}_d$ having non-empty intersection with Ω). Moreover, it suffices that the condition (2.6) is satisfied for a sequence $d = d_k$ satisfying the condition formulated in Remark 2.3.

Note that unlike (2.5), the condition (2.6) does not require that the left hand side goes to $+\infty$ as $\mathcal{G}_d \to \infty$. What is actually required is that the left-hand side has a certain lower bound, depending on d for arbitrarily small $d > 0$ and distant test bodies $\mathcal{G}_d$. Nevertheless, the conditions (2.5) and (2.6) are equivalent because each of them is equivalent to the discreteness of spectrum.

Remark 2.5 If we take $\gamma = const \in (0, 1)$, then Theorem 2.2 gives Molchanov's result, but with the constant $\gamma = c_n$ replaced by an arbitrary constant $\gamma \in (0, 1)$. So Theorem 2.2 contains an answer to the above-mentioned Gelfand's question.

Remark 2.6 For any two functions $\gamma_1, \gamma_2 : (0, +\infty) \to (0, 1)$ satisfying the requirement (1.4), the conditions (2.5) are equivalent, and so are the conditions (2.6), because any of these conditions is equivalent to the discreteness of spectrum. In a different context an equivalence of this kind was first established in [5].

It follows that the conditions (2.5) for different constants $\gamma \in (0, 1)$ are equivalent. In the particular case, when the measure $\mathbb{V}$ is absolutely continuous with respect to the Lebesgue measure, we see that the conditions (1.2) with different constants $\gamma \in (0, 1)$ are equivalent.

Remark 2.7 The results above are new even for the operator $H_0 = -\Delta$ in $L^2(\Omega)$ (but for an arbitrary open set $\Omega \subset \mathbb{R}^n$ with the Dirichlet boundary conditions on $\partial\Omega$). In this case the discreteness of spectrum is completely determined by the geometry of Ω. More precisely, for the discreteness of spectrum of H_0 in $L^2(\Omega)$ it is necessary and sufficient that there exists $d_0 > 0$ such that for every $d \in (0, d_0)$

$$\liminf_{\mathcal{G}_d \to \infty} \operatorname{cap}(\bar{\mathcal{G}}_d \setminus \Omega) \geq \gamma(d) \operatorname{cap}(\bar{\mathcal{G}}_d), \tag{2.7}$$

where $d \mapsto \gamma(d) \in (0, 1)$ is a function, which is defined in a neighborhood of 0 and satisfies (1.4). The conditions (2.7) with different functions γ, satisfying the conditions above, are equivalent. This is a non-trivial property of capacity. It is necessary for the discreteness of spectrum that (2.7) holds for every function $\gamma : (0, +\infty) \to (0, 1)$ and every $d > 0$, but this condition may not be sufficient if γ does not satisfy (1.4) (see Theorem 2.8 below).

The following result demonstrates that the condition (1.4) is precise.

Theorem 2.8 *Assume that $\gamma(d) = O(d^2)$ as $d \to 0$. Then there exist an open set $\Omega \subset \mathbb{R}^n$ and $d_0 > 0$ such that for every $d \in (0, d_0)$ the condition (2.7) is satisfied but the spectrum of $-\Delta$ in $L^2(\Omega)$ with the Dirichlet boundary conditions is not discrete.*

Now we will state our positivity result. We will say that the operator $H_{\mathbb{V}}$ is *strictly positive* if its spectrum does not contain 0. Equivalently, we can say that the spectrum is separated from 0. Since $H_{\mathbb{V}}$ is defined by the quadratic form (2.1), the strict positivity is equivalent to the existence of $\lambda > 0$ such that

$$h_{\mathbb{V}}(u, u) \geq \lambda \|u\|^2_{L^2(\Omega)}, \quad u \in C_0^\infty(\Omega). \tag{2.8}$$

Theorem 2.9 (i) (Necessity) *Let us assume that $H_{\mathbb{V}}$ is strictly positive, so that (2.8) is satisfied with a constant $\lambda > 0$. Let us take an arbitrary $\gamma \in (0, 1)$. Then there exist $d_0 > 0$ and $\varkappa > 0$ such that*

$$d^{-n} \inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_d, \Omega)} \mathbb{V}(\bar{\mathcal{G}}_d \setminus F) \geq \varkappa \tag{2.9}$$

for every $d > d_0$ and every $\mathcal{G}_d$.

(ii) (Sufficiency) *Assume that there exist $d > 0$, $\varkappa > 0$ and $\gamma \in (0, 1)$, such that (2.9) is satisfied for every $\mathcal{G}_d$. Then the operator $H_{\mathbb{V}}$ is strictly positive.*

Instead of all bodies $\mathcal{G}_d$ it is sufficient to take only the ones from a finite multiplicity covering (or tiling) of $\mathbb{R}^n$.

Remark 2.10 Considering the Dirichlet Laplacian $H_0 = -\Delta$ in $L^2(\Omega)$ we see from Theorem 2.9 that for any choice of a constant $\gamma \in (0, 1)$ and a standard body $\mathcal{G}$, the strict positivity of H_0 is equivalent to the following condition:

$$\exists\, d > 0, \text{ such that } \operatorname{cap}(\bar{\mathcal{G}}_d \cap (\mathbb{R}^n \setminus \Omega)) \geq \gamma \operatorname{cap}(\bar{\mathcal{G}}_d) \text{ for all } \mathcal{G}_d. \tag{2.10}$$

In particular, it follows that for two different γ's these conditions are equivalent. Noting that $\mathbb{R}^n \setminus \Omega$ can be an arbitrary closed subset in $\mathbb{R}^n$, we get a property of the Wiener capacity, which is obtained as a byproduct of our spectral theory arguments.

3 Discreteness of spectrum: necessity

In this section we will prove the necessity part (i) of Theorem 2.2. We will start by recalling some definitions and introducing necessary notations.

For every subset $\mathcal{D} \subset \mathbb{R}^n$ denote by $\mathrm{Lip}(\mathcal{D})$ the space of (real-valued) functions satisfying the uniform Lipschitz condition in $\mathcal{D}$, and by $\mathrm{Lip}_c(\mathcal{D})$ the subspace in $\mathrm{Lip}(\mathcal{D})$ of all functions with compact support in $\mathcal{D}$ (this will be only used when $\mathcal{D}$ is open). By $\mathrm{Lip}_{loc}(\mathcal{D})$ we will denote the set of functions on (an open set) $\mathcal{D}$ which are Lipschitz on any compact subset $K \subset \mathcal{D}$. Note that $\mathrm{Lip}(\mathcal{D}) = \mathrm{Lip}(\bar{\mathcal{D}})$ for any bounded $\mathcal{D}$.

If F is a compact subset in an open set $\mathcal{D} \subset \mathbb{R}^n$, then the Wiener capacity of F with respect to $\mathcal{D}$ is defined as

$$\text{(3.1)} \qquad \operatorname{cap}_{\mathcal{D}}(F) = \inf\left\{ \int_{\mathbb{R}^n} |\nabla u(x)|^2 dx \,\middle|\, u \in \operatorname{Lip}_c(\mathcal{D}), u|_F = 1 \right\}.$$

By $B_d(y)$ we will denote an open ball of radius d centered at y in $\mathbb{R}^n$. We will write B_d for a ball $B_d(y)$ with unspecified center y.

We will use the notation $\operatorname{cap}(F)$ for $\operatorname{cap}_{\mathbb{R}^n}(F)$ if $F \subset \mathbb{R}^n$, $n \geq 3$, and for $\operatorname{cap}_{B_{2d}}(F)$ if $F \subset \bar{B}_d \subset \mathbb{R}^2$, where the discs B_d and B_{2d} have the same center. The choice of these discs will be usually clear from the context, otherwise we will specify them explicitly.

Note that the infimum does not change if we restrict ourselves to the Lipschitz functions u such that $0 \leq u \leq 1$ everywhere (see e.g. [9], Sect. 2.2.1).

We will also need another (equivalent) definition of the Wiener capacity $\operatorname{cap}(F)$ for a compact set $F \subset \bar{B}_d$. For $n \geq 3$ it is as follows:

$$\text{(3.2)} \qquad \operatorname{cap}(F) = \sup\{\mu(F) \,\Big|\, \int_F \mathcal{E}(x-y)d\mu(y) \leq 1 \quad \text{on } \mathbb{R}^n \setminus F\},$$

where the supremum is taken over all positive finite Radon measures μ on F and $\mathcal{E} = \mathcal{E}_n$ is the standard fundamental solution of $-\Delta$ in $\mathbb{R}^n$ i.e.

$$\text{(3.3)} \qquad \mathcal{E}(x) = \frac{1}{(n-2)\omega_n}|x|^{2-n},$$

with ω_n being the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. If $n = 2$, then

$$\text{(3.4)} \qquad \operatorname{cap}(F) = \sup\{\mu(F) \,\Big|\, \int_F G(x,y)d\mu(y) \leq 1 \quad \text{on } B_{2d} \setminus F\},$$

where G is the Green function of the Dirichlet problem for $-\Delta$ in B_{2d}, i.e.

$$-\Delta G(\cdot - y) = \delta(\cdot - y), \quad y \in B_{2d},$$

$G(\cdot, y)|_{\partial B_{2d}} = 0$ for all $y \in B_{2d}$. The maximizing measure in (3.2) or in (3.4) exists and is unique. We will denote it μ_F and call it the *equilibrium measure*. Note that

$$\operatorname{cap}(F) = \mu_F(F) = \mu_F(\mathbb{R}^n) = \langle \mu_F, 1 \rangle.$$

The corresponding potential will be denoted P_F, so

$$P_F(x) = \int_F \mathcal{E}(x-y)d\mu_F(y), \quad x \in \mathbb{R}^n \setminus F, \qquad n \geq 3,$$

$$P_F(x) = \int_F G(x,y)d\mu_F(y), \quad x \in B_{2d} \setminus F, \qquad n = 2.$$

We will call P_F the *equilibrium potential* or *capacitary potential*. We will extend it to F by setting $P_F(x) = 1$ for all $x \in F$.

It follows from the maximum principle that $0 \le P_F \le 1$ everywhere in $\mathbb{R}^n$ if $n \ge 3$ (and in B_{2d} if $n = 2$).

In case when F is a closure of an open subset with a smooth boundary, $u = P_F$ is the unique minimizer for the Dirichlet integral in (3.1) where we should take $\mathcal{D} = \mathbb{R}^n$ if $n \ge 3$ and $\mathcal{D} = B_{2d}$ if $n = 2$. In particular,

$$\int |\nabla P_F|^2 dx = \operatorname{cap}(F), \tag{3.5}$$

where the integration is taken over $\mathbb{R}^n$ (or $\mathbb{R}^n \setminus F$) if $n \ge 3$ and over B_{2d} (or $B_{2d} \setminus F$) if $n = 2$.

The following lemma provides an auxiliary estimate which is needed for the proof.

Lemma 3.1 *Assume that $\mathcal{G}$ has a C^∞ boundary, and P is the equilibrium potential of $\bar{\mathcal{G}}_d$. Then*

$$\int_{\partial \mathcal{G}_d} |\nabla P|^2 ds \le nL\rho^{-1}d^{-1} \operatorname{cap}(\bar{\mathcal{G}}_d), \tag{3.6}$$

where the gradient ∇P in the left hand side is taken along the exterior of $\bar{\mathcal{G}}_d$, ds is the $(n-1)$-dimensional volume element on $\partial\mathcal{G}_d$. The positive constants ρ, L are geometric characteristics of the standard body $\mathcal{G}$ (they depend on the choice of $\mathcal{G}$ only, but not on d): ρ was introduced at the beginning of Section 2, and

$$L = \left[\inf_{x \in \partial\mathcal{G}} \nu_r(x)\right]^{-1}, \tag{3.7}$$

where $\nu_r(x) = \frac{x}{|x|} \cdot \nu(x)$, $\nu(x)$ is the unit normal vector to $\partial\mathcal{G}$ at x which is directed to the exterior of $\bar{\mathcal{G}}$.

Proof. It suffices to consider $\mathcal{G}_d = \mathcal{G}_d(0)$. For simplicity we will write $\mathcal{G}$ instead of $\mathcal{G}_d(0)$ in this proof, until the size becomes relevant.

We will first consider the case $n \ge 3$. Note that $\Delta P = 0$ on $\complement\bar{\mathcal{G}} = \mathbb{R}^n \setminus \bar{\mathcal{G}}$. Also $P = 1$ on $\bar{\mathcal{G}}$, so in fact $|\nabla P| = |\partial P/\partial\nu|$. Using the Green formula, we obtain

$$
\begin{aligned}
0 &= \int_{\complement\bar{\mathcal{G}}} \Delta P \cdot \frac{\partial P}{\partial r} dx = \int_{\complement\bar{\mathcal{G}}} \Delta P \left(\frac{x}{|x|} \cdot \nabla P \right) dx \\
&= -\int_{\complement\bar{\mathcal{G}}} \nabla P \cdot \nabla \left(\frac{x}{|x|} \cdot \nabla P \right) dx - \int_{\partial\mathcal{G}} \frac{\partial P}{\partial \nu} \left(\frac{x}{|x|} \cdot \nabla P \right) ds \\
&= -\sum_{i,j} \int_{\complement\bar{\mathcal{G}}} \frac{\partial P}{\partial x_j} \cdot \frac{\partial}{\partial x_j} \left(\frac{x_i}{|x|} \cdot \frac{\partial P}{\partial x_i} \right) dx - \int_{\partial\mathcal{G}} \frac{\partial P}{\partial \nu} \cdot \frac{\partial P}{\partial r} ds \\
&= -\sum_{i,j} \int_{\complement\bar{\mathcal{G}}} \frac{\partial P}{\partial x_j} \cdot \frac{\delta_{ij}}{|x|} \cdot \frac{\partial P}{\partial x_i} dx + \sum_{i,j} \int_{\complement\bar{\mathcal{G}}} \frac{x_i x_j}{|x|^3} \cdot \frac{\partial P}{\partial x_i} \cdot \frac{\partial P}{\partial x_j} dx \\
&- \sum_{i,j} \int_{\complement\bar{\mathcal{G}}} \frac{x_i}{|x|} \cdot \frac{\partial P}{\partial x_j} \cdot \frac{\partial^2 P}{\partial x_i \partial x_j} dx - \int_{\partial\mathcal{G}} \frac{\partial P}{\partial \nu} \cdot \frac{\partial P}{\partial r} ds \\
&= -\int_{\complement\bar{\mathcal{G}}} \frac{1}{|x|} |\nabla P|^2 dx + \int_{\complement\bar{\mathcal{G}}} \frac{1}{|x|} \left| \frac{\partial P}{\partial r} \right|^2 dx \\
&- \frac{1}{2} \sum_i \int_{\complement\bar{\mathcal{G}}} \frac{x_i}{|x|} \cdot \frac{\partial}{\partial x_i} |\nabla P|^2 dx - \int_{\partial\mathcal{G}} |\nabla P|^2 \nu_r ds.
\end{aligned}
$$

Integrating by parts in the last integral over $\complement\bar{\mathcal{G}}$, we see that it equals

$$
\begin{aligned}
&\frac{1}{2} \sum_i \int_{\complement\bar{\mathcal{G}}} \frac{\partial}{\partial x_i} \left(\frac{x_i}{|x|} \right) \cdot |\nabla P|^2 dx + \frac{1}{2} \sum_i \int_{\partial\mathcal{G}} \frac{x_i}{|x|} |\nabla P|^2 \nu_i ds \\
&= \frac{n-1}{2} \int_{\complement\bar{\mathcal{G}}} \frac{1}{|x|} |\nabla P|^2 dx + \frac{1}{2} \int_{\partial\mathcal{G}} |\nabla P|^2 \nu_r ds,
\end{aligned}
$$

where ν_i is the ith component of ν. Returning to the calculation above, we obtain

$$
0 = \frac{n-3}{2} \int_{\complement\bar{\mathcal{G}}} \frac{1}{|x|} |\nabla P|^2 dx + \int_{\complement\bar{\mathcal{G}}} \frac{1}{|x|} \left| \frac{\partial P}{\partial r} \right|^2 dx - \frac{1}{2} \int_{\partial\mathcal{G}} |\nabla P|^2 \nu_r ds.
$$

It follows that

$$
\int_{\partial\mathcal{G}} |\nabla P|^2 \nu_r ds \leq (n-1) \int_{\complement\bar{\mathcal{G}}} \frac{1}{|x|} |\nabla P|^2 dx.
$$

Recalling that $\mathcal{G} = \mathcal{G}_d(0)$, we observe that $|x|^{-1} \leq (\rho d)^{-1}$. Now using (3.5), we obtain the desired estimate (3.6) for $n \geq 3$ (with $n-1$ instead of n).

Let us consider the case $n = 2$. Then, by definition, the equilibrium potential P for $\mathcal{G} = \mathcal{G}_d(0)$ is defined in the ball $B_{2d}(0)$. It satisfies $\Delta P = 0$ in $B_{2d}(0) \setminus \bar{\mathcal{G}}$ and the boundary conditions $P|_{\partial\mathcal{G}} = 1$, $P|_{\partial B_{2d}(0)} = 0$. Let us first modify the calculations above by taking the integrals over $B_\delta(0) \setminus \bar{\mathcal{G}}$ (instead of $\complement\bar{\mathcal{G}}$), where $d < \delta < 2d$. We will get additional boundary terms with the integration over

$\partial B_\delta(0)$. Instead of (3) we will obtain

$$0 = -\frac{1}{2}\int_{B_\delta(0)\setminus\bar{\mathcal{G}}} \frac{1}{|x|}|\nabla P|^2 dx + \int_{B_\delta(0)\setminus\bar{\mathcal{G}}} \frac{1}{|x|}\left|\frac{\partial P}{\partial r}\right|^2 dx$$
$$-\frac{1}{2}\int_{\partial\mathcal{G}} |\nabla P|^2 \nu_r ds + \frac{1}{2}\int_{\partial B_\delta(0)} \left[2\left|\frac{\partial P}{\partial r}\right|^2 - |\nabla P|^2\right] ds.$$

Therefore

$$\int_{\partial\mathcal{G}} |\nabla P|^2 \nu_r ds \le \int_{B_\delta(0)\setminus\bar{\mathcal{G}}} \frac{1}{|x|}|\nabla P|^2 dx + \int_{\partial B_\delta(0)} \left[2\left|\frac{\partial P}{\partial r}\right|^2 - |\nabla P|^2\right] ds$$
$$\le \frac{1}{\rho d}\int_{B_{2d}(0)\setminus\bar{\mathcal{G}}} |\nabla P|^2 dx + \int_{\partial B_\delta(0)} |\nabla P|^2 ds.$$

Now let us integrate both sides with respect to δ over the interval $[d, 2d]$ and divide the result by d (i.e. take average over all δ). Then the left hand side and the first term in the right hand side do not change, while the last term becomes d^{-1} times the volume integral with respect to the Lebesgue measure over $B_{2d}(0) \setminus B_d(0)$. Due to (3.5) the right hand side can be estimated by $(1+\rho)(\rho d)^{-1}\,\mathrm{cap}\,(\bar{\mathcal{G}}_d)$. Since $0 < \rho \le 1$, we get the estimate (3.6) for $n = 2$. □

Proof of Theorem 2.2, part (i). (a) We will use the same notations as above. Let us fix $d > 0$, take $\mathcal{G}_d = \mathcal{G}_d(z)$, and assume that $\mathcal{G}$ has a C^∞ boundary. Let us take a compact set $F \subset \mathbb{R}^n$ with the following properties:

(i) F is the closure of an open set with a C^∞ boundary;
(ii) $\bar{\mathcal{G}}_d \setminus \Omega \Subset F \subset B_{3d/2}(z)$;
(iii) $\mathrm{cap}\,(F) \le \gamma\,\mathrm{cap}\,(\bar{\mathcal{G}}_d)$ with $0 < \gamma < 1$.

Let us recall that the notation $\bar{\mathcal{G}}_d \setminus \Omega \Subset F$ means that $\bar{\mathcal{G}}_d \setminus \Omega$ is contained in the interior of F. This implies that $\mathbb{V}(\bar{\mathcal{G}}_d \setminus F) < +\infty$. The inclusion $F \subset B_{3d/2}(z)$ and the inequality (iii) hold, in particular, for compact sets F which are small neighborhoods (with smooth boundaries) of negligible compact subsets of $\bar{\mathcal{G}}_d$, and it is exactly such F's which we have in mind.

We will refer to the sets F satisfying (i)-(iii) above as *regular* ones.

Let P and P_F denote the equilibrium potentials of $\bar{\mathcal{G}}_d$ and F respectively. The equilibrium measure $\mu_{\bar{\mathcal{G}}_d}$ has its support in $\partial\mathcal{G}_d$ and has density $-\partial P/\partial\nu$ with respect to the $(n-1)$-dimensional Riemannian measure ds on $\partial\mathcal{G}_d$. So for $n \ge 3$ we have

$$P(y) = -\int_{\partial\mathcal{G}_d} \mathcal{E}(x-y)\frac{\partial P}{\partial\nu}(x) ds_x, \quad y \in \mathbb{R}^n;$$

$$-\int_{\partial\mathcal{G}_d} \frac{\partial P}{\partial\nu}(x) ds_x = \mathrm{cap}\,(\bar{\mathcal{G}}_d);$$

$$P(y) = 1 \text{ for all } y \in \mathcal{G}_d, \quad 0 \le P(y) \le 1 \text{ for all } y \in \mathbb{R}^n.$$

(If $n = 2$, then the same holds only with $y \in B_{2d}$ and with the fundamental solution $\mathcal{E}$ replaced by the Green function G; see notations in Section 4.) It follows that

$$-\int_{\partial \mathcal{G}_d} P_F \frac{\partial P}{\partial \nu} ds = -\int_F \int_{\partial \mathcal{G}_d} \mathcal{E}(x-y) \frac{\partial P}{\partial \nu}(x) ds_x d\mu_F(y) \le \mu_F(F) = \operatorname{cap}(F).$$

Therefore,

$$\operatorname{cap}(\bar{\mathcal{G}}_d) - \operatorname{cap}(F) \le -\int_{\partial \mathcal{G}_d} (1 - P_F) \frac{\partial P}{\partial \nu} ds,$$

and, using Lemma 3.1, we obtain

$$\begin{aligned} (\operatorname{cap}(\bar{\mathcal{G}}_d) - \operatorname{cap}(F))^2 &\le \left(\int_{\partial \mathcal{G}_d} (1 - P_F) \frac{\partial P}{\partial \nu} ds \right)^2 \\ &\le \|1 - P_F\|^2_{L^2(\partial \mathcal{G}_d)} \|\nabla P\|^2_{L^2(\partial \mathcal{G}_d)} \le nL(\rho d)^{-1} \operatorname{cap}(\mathcal{G}_d) \|1 - P_F\|^2_{L^2(\partial \mathcal{G}_d)}, \end{aligned} \tag{3.8}$$

where L is defined by (3.7).

(b) Our next goal will be to estimate the norm $\|1 - P_F\|_{L^2(\partial \mathcal{G}_d)}$ in (3.8) by the norm of the same function in $L^2(\mathcal{G}_d)$. We will use the polar coordinates (r, ω) as in (2.2), so in particular $\partial \mathcal{G}_d$ is presented as the set $\{r(\omega)\omega |\ \omega \in S^{n-1}\}$, where $r : S^{n-1} \to (0, +\infty)$ is a Lipschitz function (C^∞ as long as we assume the boundary $\partial \mathcal{G}$ to be C^∞). Assuming that $v \in \operatorname{Lip}(\bar{\mathcal{G}}_d)$, we can write

$$\begin{aligned} \int_{\partial \mathcal{G}_d} |v|^2 ds &= \int_{S^{n-1}} |v|^2 \frac{r(\omega)^{n-1}}{\nu_r} d\omega \\ &\le L \int_{S^{n-1}} |v(r(\omega), \omega)|^2 r(\omega)^{n-1} d\omega, \end{aligned} \tag{3.9}$$

where $d\omega$ is the standard $(n-1)$-dimensional volume element on S^{n-1}.

Using the inequality

$$|f(\varepsilon)|^2 \le 2\varepsilon \int_0^\varepsilon |f'(t)|^2 dt + \frac{2}{\varepsilon} \int_0^\varepsilon |f(t)|^2 dt, \quad f \in \operatorname{Lip}([0, \varepsilon]), \quad \varepsilon > 0,$$

we obtain

$$\begin{aligned} &|v(r(\omega), \omega)|^2 \\ &\le 2\varepsilon r(\omega) \int_{(1-\varepsilon)r(\omega)}^{r(\omega)} |v'_\rho(\rho, \omega)|^2 d\rho + \frac{2}{\varepsilon r(\omega)} \int_{(1-\varepsilon)r(\omega)}^{r(\omega)} |v(\rho, \omega)|^2 d\rho \\ &\le \frac{2\varepsilon r(\omega)}{[(1-\varepsilon)r(\omega)]^{n-1}} \int_{(1-\varepsilon)r(\omega)}^{r(\omega)} |v'_\rho(\rho, \omega)|^2 \rho^{n-1} d\rho \\ &+ \frac{2}{\varepsilon r(\omega)[(1-\varepsilon)r(\omega)]^{n-1}} \int_{(1-\varepsilon)r(\omega)}^{r(\omega)} |v(\rho, \omega)|^2 \rho^{n-1} d\rho. \end{aligned}$$

It follows that the integral in the right hand side of (3.9) is estimated by

$$\int_{S^{n-1}} \frac{2\varepsilon r(\omega) d\omega}{(1-\varepsilon)^{n-1}} \int_{(1-\varepsilon)r(\omega)}^{r(\omega)} |v'_\rho(\rho,\omega)|^2 \rho^{n-1} d\rho$$
$$+ \int_{S^{n-1}} \frac{2d\omega}{\varepsilon(1-\varepsilon)^{n-1} r(\omega)} |v(\rho,\omega)|^2 \rho^{n-1} d\rho.$$

Taking $\varepsilon \le 1/2$, we can majorize this by

$$2^n \varepsilon d \int_{\bar{\mathcal{G}}_d} |\nabla v|^2 dx + \frac{2^n}{\varepsilon \rho d} \int_{\bar{\mathcal{G}}_d} |v|^2 dx,$$

where $\rho \in (0,1]$ is the constant from the description of $\mathcal{G}$ in Sect. 2. Recalling (3.9), we see that the resulting estimate has the form

$$\int_{\partial \mathcal{G}_d} |v|^2 ds \le 2^n L \varepsilon d \int_{\bar{\mathcal{G}}_d} |\nabla v|^2 dx + \frac{2^n L}{\varepsilon \rho d} \int_{\bar{\mathcal{G}}_d} |v|^2 dx.$$

Now, taking $v = 1 - P_F$, we obtain

$$\int_{\partial \mathcal{G}_d} (1-P_F)^2 ds \le 2^n L \varepsilon d \operatorname{cap}(F) + \frac{2^n L}{\varepsilon \rho d} \int_{\bar{\mathcal{G}}_d} (1-P_F)^2 dx.$$

Using this estimate in (3.8), we obtain

$$\begin{aligned} &(\operatorname{cap}(\bar{\mathcal{G}}_d) - \operatorname{cap}(F))^2 \\ &\le \rho^{-1} n 2^n L^2 \operatorname{cap}(\bar{\mathcal{G}}_d) \left(\varepsilon \operatorname{cap}(F) + \frac{1}{\varepsilon \rho d^2} \int_{\mathcal{G}_d} (1-P_F)^2 dx \right). \end{aligned} \tag{3.10}$$

(c) Now let us consider $\mathcal{G}$ which is star-shaped with respect to a ball, but not necessarily has C^∞ boundary. In this case we can approximate the function $r(\omega)$ (see Section 2) from above by a decreasing sequence of C^∞ functions $r_k(\omega)$ (e.g. we can apply a standard mollifying procedure to $r(\omega) + 1/k$), so that for the the corresponding bodies $\mathcal{G}^{(k)}$ the constants L_k are uniformly bounded. It is clear that in this case we will also have $\rho_k \ge \rho$, and $\operatorname{cap}(\bar{\mathcal{G}}_d^{(k)}) \to \operatorname{cap}(\bar{\mathcal{G}}_d)$ due to the well known continuity property of the capacity (see e.g. Section 2.2.1 in [9]). So we can pass to the limit in (3.10) as $k \to +\infty$ and conclude that it holds for arbitrary $\mathcal{G}$ (which is star-shaped with respect to a ball). But for the moment we still retain the regularity condition on F.

(d) Let us define

$$\mathcal{L} = \left\{ u \,\middle|\, u \in C_0^\infty(\Omega),\ h_{\mathbb{V}}(u,u) + \|u\|^2_{L^2(\Omega)} \le 1 \right\}, \tag{3.11}$$

where $h_{\mathbb{V}}$ is defined by (2.1). By the standard functional analysis argument (see e.g. Lemma 2.3 in [6]) the spectrum of $H_{\mathbb{V}}$ is discrete if and only if $\mathcal{L}$ is precompact in $L^2(\Omega)$, which in turn holds if and only if $\mathcal{L}$ has "small tails", i.e. for every $\eta > 0$ there exists $R > 0$ such that

$$\int_{\Omega\setminus B_R(0)} |u|^2 dx \le \eta \quad \text{for every} \quad u \in \mathcal{L}, \tag{3.12}$$

Equivalently, we can write that

$$\int_{\Omega\setminus B_R(0)} |u|^2 dx \le \eta \left[\int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^2 \mathbb{V}(dx)\right], \tag{3.13}$$

for every $u \in C_0^\infty(\Omega)$.

Therefore, it follows from the discreteness of spectrum of $H_{\mathbb{V}}$ that for every $\eta > 0$ there exists $R > 0$ such that for every $\mathcal{G}_d$ with $\bar{\mathcal{G}}_d \cap (\mathbb{R}^n \setminus B_R(0)) \neq \emptyset$ and every $u \in C_0^\infty(\mathcal{G}_d \cap \Omega)$

$$\int_{\mathcal{G}_d} |u|^2 dx \le \eta \left(\int_{\mathcal{G}_d} |\nabla u|^2 dx + \int_{\bar{\mathcal{G}}_d} |u|^2 \mathbb{V}(dx)\right). \tag{3.14}$$

In other words, $\eta = \eta(\mathcal{G}_d) \to 0$ as $\mathcal{G}_d \to \infty$ for the best constant in (3.14). (Note that $\eta(\mathcal{G}_d)^{-1}$ is the bottom of the Dirichlet spectrum of $H_{\mathbb{V}}$ in $\mathcal{G}_d \cap \Omega$.)

Since $1 - P_F = 0$ on F (hence in a neighborhood of $\bar{\mathcal{G}}_d \setminus \Omega$), we can take $u = \chi_\sigma(1 - P_F)$, where $\sigma \in (0, 1)$ to be chosen later, $\chi_\sigma \in C_0^\infty(\mathcal{G}_d)$ is a cut-off function satisfying $0 \le \chi_\sigma \le 1$, $\chi_\sigma = 1$ on $\mathcal{G}_{(1-\sigma)d}$, and $|\nabla\chi_\sigma| \le Cd^{-1}$ with $C = C(\mathcal{G})$. Then, using integration by parts and the equation $\Delta P_F = 0$ on $\mathcal{G} \setminus F$, we obtain

$$\begin{aligned}\int_{\mathcal{G}_d} |\nabla u|^2 dx &= \int_{\mathcal{G}_d} \left(|\nabla\chi_\sigma|^2(1 - P_F)^2 - \nabla(\chi_\sigma^2)\cdot(1 - P_F)\nabla P_F + \chi_\sigma^2|\nabla P_F|^2\right) dx \\ &= \int_{\mathcal{G}_d} |\nabla\chi_\sigma|^2(1 - P_F)^2 dx \le C^2(\sigma d)^{-2} \int_{\mathcal{G}_d} (1 - P_F)^2 dx.\end{aligned}$$

Therefore, from (3.14)

$$\int_{\mathcal{G}_d} |u|^2 dx \le \eta \left[C^2(\sigma d)^{-2} \int_{\mathcal{G}_d} (1 - P_F)^2 dx + \mathbb{V}(\bar{\mathcal{G}}_d \setminus F)\right],$$

hence

$$\int_{\mathcal{G}_{(1-\sigma)d}} (1 - P_F)^2 dx \le \eta \left[C^2(\sigma d)^{-2} \int_{\mathcal{G}_d} (1 - P_F)^2 dx + \mathbb{V}(\bar{\mathcal{G}}_d \setminus F)\right].$$

Now, applying the obvious estimate

$$\begin{aligned}\int_{\mathcal{G}_d} (1 - P_F)^2 dx &\le \int_{\mathcal{G}_{(1-\sigma)d}} (1 - P_F)^2 dx + \operatorname{mes}(\mathcal{G}_d \setminus \mathcal{G}_{(1-\sigma)d}) \\ &\le \int_{\mathcal{G}_{(1-\sigma)d}} (1 - P_F)^2 dx + C_1\sigma d^n,\end{aligned}$$

with $C_1 = C_1(\mathcal{G})$, we see that

$$\int_{\mathcal{G}_d} (1-P_F)^2 dx \le \eta \left[C^2(\sigma d)^{-2} \int_{\bar{\mathcal{G}}_d} (1-P_F)^2 dx + \mathbb{V}(\bar{\mathcal{G}}_d \setminus F) \right] + C_1 \sigma d^n,$$

hence

$$\int_{\mathcal{G}_d} (1-P_F)^2 dx \le 2\eta \mathbb{V}(\bar{\mathcal{G}}_d \setminus F) + 2C_1 \sigma d^n, \tag{3.15}$$

provided

$$\eta C^2 (\sigma d)^{-2} \le 1/2. \tag{3.16}$$

Returning to (3.10) and using (3.15) we obtain

$$\begin{aligned} \left(1 - \frac{\operatorname{cap}(F)}{\operatorname{cap}(\bar{\mathcal{G}}_d)}\right)^2 &\le C_2 \left[\varepsilon + \varepsilon^{-1} d^{-n} \int_{\mathcal{G}_d} (1-P_F)^2 dx \right] \\ &\le C_2 [\varepsilon + 2C_1 \sigma \varepsilon^{-1} + 2\varepsilon^{-1} d^{-n} \eta \mathbb{V}(\bar{\mathcal{G}}_d \setminus F)], \end{aligned} \tag{3.17}$$

where $C_2 = C_2(\mathcal{G})$. Without loss of generality we will assume that $C_2 \ge 1/2$. Recalling that $\operatorname{cap}(F) \le \gamma \operatorname{cap}(\bar{\mathcal{G}}_d)$, we can replace the ratio $\operatorname{cap}(F)/\operatorname{cap}(\bar{\mathcal{G}}_d)$ in the left hand side by γ. Now let us choose

$$\varepsilon = \frac{(1-\gamma)^2}{4C_2}, \quad \sigma = \frac{\varepsilon(1-\gamma)^2}{8C_1} = \frac{(1-\gamma)^4}{32 C_1 C_2}. \tag{3.18}$$

Then $\varepsilon \le 1/2$ and for every fixed $\gamma \in (0,1)$ and $d > 0$ the condition (3.16) will be satisfied for distant bodies $\mathcal{G}_d$, because $\eta = \eta(\mathcal{G}_d) \to 0$ as $\mathcal{G}_d \to \infty$. (More precisely, there exists $R = R(\gamma, d) > 0$, such that (3.16) holds for every $\mathcal{G}_d$ such that $\mathcal{G}_d \cap (\mathbb{R}^n \setminus B_R(0)) \ne \emptyset$.)

If ε and σ are chosen according to (3.18), then (3.17) becomes

$$d^{-n} \mathbb{V}(\bar{\mathcal{G}}_d \setminus F) \ge (16 C_2 \eta)^{-1} (1-\gamma)^4, \tag{3.19}$$

which holds for distant bodies $\mathcal{G}_d$ if $\gamma \in (0,1)$ and $d > 0$ are arbitrarily fixed.

(e) Up to this moment we worked with "regular" sets F – see conditions (i)-(iii) in the part (a) of this proof. Now we can get rid of the regularity requirements (i) and (ii), retaining (iii). So let us assume that F is a compact set, $\bar{\mathcal{G}}_d \setminus \Omega \subset F \subset \bar{\mathcal{G}}_d$ and $\operatorname{cap}(F) \le \gamma \operatorname{cap}(\bar{\mathcal{G}}_d)$ with $\gamma \in (0,1)$. Let us construct a sequence of compact sets $F_k \Supset F$, $k = 1, 2, \dots$, such that every F_k is regular,

$$F_1 \Supset F_2 \Supset \dots, \quad \text{and} \quad \bigcap_{k=1}^{\infty} F_k = F.$$

We have then $\operatorname{cap}(F_k) \to \operatorname{cap}(F)$ as $k \to +\infty$ due to the well known continuity property of the capacity (see e.g. Section 2.2.1 in [9]). According to the previous steps of this proof, the inequality (3.19) holds for distant $\mathcal{G}_d$'s if we replace F by F_k and γ by $\gamma_k = \operatorname{cap}(F_k)/\operatorname{cap}(\bar{\mathcal{G}}_d)$. Since the measure $\mathbb{V}$ is positive, the resulting inequality will still hold if we replace $\mathbb{V}(\bar{\mathcal{G}}_d \setminus F_k)$ by $\mathbb{V}(\bar{\mathcal{G}}_d \setminus F)$. Taking limit as $k \to +\infty$, we obtain that (3.19) holds with $\gamma' = \operatorname{cap}(F)/\operatorname{cap}(\bar{\mathcal{G}}_d)$ instead of γ. Since $\gamma' \le \gamma$, (3.19) immediately follows for arbitrary compact F such that $\bar{\mathcal{G}}_d \setminus \Omega \subset F \subset \bar{\mathcal{G}}_d$ and $\operatorname{cap}(F) \le \gamma \operatorname{cap}(\bar{\mathcal{G}}_d)$ with $\gamma \in (0,1)$.

(f) Let us fix $\mathcal{G}$ and take infimum over all negligible F's (i.e. compact sets F, such that $\bar{\mathcal{G}}_d \setminus \Omega \subset F \subset \bar{\mathcal{G}}_d$ and $\operatorname{cap}(F) \le \gamma \operatorname{cap}(\bar{\mathcal{G}}_d)$) in the right hand side of (3.19). We get then for distant $\mathcal{G}_d$'s

$$d^{-n} \inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_d,\Omega)} \mathbb{V}(\bar{\mathcal{G}}_d \setminus F) \ge (16 C_2 \eta)^{-1}(1-\gamma)^4. \tag{3.20}$$

Now let us recall that the discreteness of spectrum is equivalent to the condition $\eta = \eta(\mathcal{G}_d) \to 0$ as $\mathcal{G}_d \to \infty$ (with any fixed $d > 0$). If this is the case, then it is clear from (3.20), that for every fixed $\gamma \in (0,1)$ and $d > 0$, the left hand side of (3.20) tends to $+\infty$ as $\mathcal{G}_d \to \infty$. This concludes the proof of part (i) of Theorem 2.2. $\square$

4 Discreteness of spectrum: sufficiency

In this section we will establish the sufficiency part of Theorem 2.2.

Let us recall the *Poincaré inequality* (see e.g. [4], Sect. 7.8, or [6], Lemma 5.1):

$$||u - \bar{u}||^2_{L^2(\mathcal{G}_d)} \le A(\mathcal{G}) d^2 \int_{\mathcal{G}_d} |\nabla u(x)|^2 dx, \quad u \in \operatorname{Lip}(\mathcal{G}_d),$$

where $\mathcal{G}_d \subset \mathbb{R}^n$ was described in Section 2

$$\bar{u} = \frac{1}{|\mathcal{G}_d|} \int_{\mathcal{G}_d} u(x)\, dx$$

is the mean value of u on $\mathcal{G}_d$, $|\mathcal{G}_d|$ is the Lebesgue volume of $\mathcal{G}_d$, $A(\mathcal{G}) > 0$ is independent of d. (In fact, the best $A(\mathcal{G})$ is obtained if $A(\mathcal{G})^{-1}$ is the lowest non-zero Neumann eigenvalue of $-\Delta$ in $\mathcal{G}$.)

The following Lemma generalizes (to an arbitrary body $\mathcal{G}$) a particular case of the first part of Theorem 10.1.2 in [9] (see also Lemma 2.1 in [5]).

Lemma 4.1 *There exists $C(\mathcal{G}) > 0$ such that the following inequality holds for every function $u \in \operatorname{Lip}(\bar{\mathcal{G}}_d)$ which vanishes on a compact set $F \subset \bar{\mathcal{G}}_d$ (but is not identically 0 on $\bar{\mathcal{G}}_d$):*

$$\operatorname{cap}(F) \le \frac{C(\mathcal{G}) \int_{\mathcal{G}_d} |\nabla u(x)|^2 dx}{|\mathcal{G}_d|^{-1} \int_{\mathcal{G}_d} |u(x)|^2 dx}\,. \tag{4.1}$$

Proof. Let us normalize u by

$$|\mathcal{G}_d|^{-1} \int_{\mathcal{G}_d} |u(x)|^2 dx = 1,$$

i.e. $\overline{|u|^2} = 1$. By the Cauchy-Schwarz inequality we obtain

$$\overline{|u|} \leq \left(\overline{|u|^2}\right)^{1/2} = 1 \tag{4.2}$$

Replacing u by $|u|$ does not change the denominator and may only decrease the numerator in (4.1). Therefore we can restrict ourselves to Lipschitz functions $u \geq 0$.

Let us denote $\phi = 1 - u$. Then $\phi = 1$ on F, and $\bar{\phi} = 1 - \bar{u} \geq 0$ due to (4.2). Let us estimate $\bar{\phi}$ from above. Obviously

$$\bar{\phi} = |\mathcal{G}_d|^{-1/2}(\|u\| - \|\bar{u}\|) \leq |\mathcal{G}_d|^{-1/2}\|u - \bar{u}\|,$$

where $\|\cdot\|$ means the norm in $L^2(\mathcal{G}_d)$. Hence the Poincaré inequality gives

$$\bar{\phi} \leq A^{1/2} d |\mathcal{G}_d|^{-1/2}\|\nabla u\| = A^{1/2} d |\mathcal{G}_d|^{-1/2}\|\nabla \phi\|,$$

where $A = A(\mathcal{G})$. So

$$\bar{\phi}^2 \leq A d^2 |\mathcal{G}_d|^{-1} \int_{\mathcal{G}_d} |\nabla \phi|^2 dx.$$

and

$$\|\bar{\phi}\|^2 \leq A d^2 \int_{\mathcal{G}_d} |\nabla \phi|^2 dx.$$

Using the Poincaré inequality again, we obtain

$$\|\phi\|^2 = \|(\phi - \bar{\phi}) + \bar{\phi}\|^2 \leq 2\|\phi - \bar{\phi}\|^2 + 2\|\bar{\phi}\|^2 \leq 4Ad^2 \int_{\mathcal{G}_d} |\nabla \phi|^2 dx,$$

or

$$\int_{\mathcal{G}_d} \phi^2 dx \leq 4Ad^2 \int_{\mathcal{G}_d} |\nabla \phi|^2 dx. \tag{4.3}$$

Let us extend ϕ outside $\mathcal{G}_d = \mathcal{G}_d(y)$ by inversion in each ray emanating from y. In notations introduced in (2.2) we can write that $\phi(y + r\omega) = \phi(y + r^{-1}(r(\omega))^2)\omega$ for every $r > r(\omega)$ and every $\omega \in S^{n-1}$.

It is easy to see that the extension $\tilde{\phi}$ satisfies

$$\int_{B_{3d}} |\tilde{\phi}|^2 dx \leq C_1(\mathcal{G}) \int_{\mathcal{G}_d} |\phi|^2 dx, \quad \int_{B_{3d}} |\nabla \tilde{\phi}|^2 dx \leq C_1(\mathcal{G}) \int_{\mathcal{G}_d} |\nabla \phi|^2 dx.$$

Let η be a piecewise smooth function, such that $\eta = 1$ on B_d, $\eta = 0$ outside B_{2d}, $0 \le \eta \le 1$ and $|\nabla \eta| \le d^{-1}$, i.e. $\eta(x) = 2 - d^{-1}|x|$ if $d \le |x| \le 2d$. Then

$$\mathrm{cap}\,(F) \le \int_{B_{2d}} |\nabla(\tilde{\phi}\eta)|^2 dx \le 2C_1(\mathcal{G}) \left(\int_{\mathcal{G}_d} |\nabla \phi|^2 dx + d^{-2} \int_{\mathcal{G}_d} \phi^2 dx \right).$$

Taking into account that $|\nabla \phi| = |\nabla u|$ and using (4.3), we obtain

$$\mathrm{cap}\,(F) \le 2C_1(\mathcal{G})(1 + 4A) \int_{\mathcal{G}_d} |\nabla u|^2 dx,$$

which is equivalent to (4.1) with $C(G) = 2C_1(\mathcal{G})(1 + 4A(\mathcal{G}))$. $\square$

The next lemma is an adaptation of a very general Lemma 12.1.1 from [9] (see also Lemma 2.2 in [5]) to general test bodies $\mathcal{G}_d$ (instead of cubes Q_d).

Lemma 4.2 *Let $\mathbb{V}$ be a positive Radon measure in Ω. There exists $C_2(\mathcal{G}) > 0$ such that for every $\gamma \in (0, 1)$ and $u \in \mathrm{Lip}(\bar{\mathcal{G}}_d)$ with $u = 0$ in a neighborhood of $\bar{\mathcal{G}}_d \setminus \Omega$,*

$$\int_{\mathcal{G}_d} |u|^2 dx \le \frac{C_2(\mathcal{G})d^2}{\gamma} \int_{\mathcal{G}_d} |\nabla u|^2 dx + \frac{C_2(\mathcal{G})d^n}{\mathbb{V}_\gamma(\mathcal{G}_d, \Omega)} \int_{\bar{\mathcal{G}}_d} |u|^2 \mathbb{V}(dx), \tag{4.4}$$

where

$$\mathbb{V}_\gamma(\mathcal{G}_d, \Omega) = \inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_d, \Omega)} \mathbb{V}(\mathcal{G}_d \setminus F). \tag{4.5}$$

(Here the negligibility class $\mathcal{N}_\gamma(\mathcal{G}_d, \Omega)$ was introduced in Definition 2.1.)

Proof. Let $\mathcal{M}_\tau = \{x \in \bar{\mathcal{G}}_d : |u(x)| > \tau\}$, where $\tau \ge 0$. Note that $\mathcal{M}_\tau$ is a relatively open subset of $\bar{\mathcal{G}}$, and $\mathcal{M}_\tau \subset \Omega$, hence $\bar{\mathcal{G}}_d \setminus \mathcal{M}_\tau \supset \bar{\mathcal{G}}_d \setminus \Omega$. Since

$$|u|^2 \le 2\tau^2 + 2(|u| - \tau)^2 \quad \text{on } \mathcal{M}_\tau,$$

we have for all τ

$$\int_{\mathcal{G}_d} |u|^2 dx \le 2\tau^2 |\mathcal{G}_d| + 2 \int_{\mathcal{M}_\tau} (|u| - \tau)^2 dx.$$

Let us take

$$\tau^2 = \frac{1}{4|\mathcal{G}_d|} \int_{\mathcal{G}_d} |u|^2 dx,$$

i.e. $\tau = \frac{1}{2} \left(\overline{|u|^2} \right)^{1/2}$. Then for this particular value of τ we obtain

$$\int_{\mathcal{G}_d} |u|^2 dx \le 4 \int_{\mathcal{M}_\tau} (|u| - \tau)^2 dx. \tag{4.6}$$

Assume first that $\operatorname{cap}(\bar{\mathcal{G}}_d \setminus \mathcal{M}_\tau) \geq \gamma \operatorname{cap}(\bar{\mathcal{G}}_d)$. Using (4.6) and applying Lemma 4.1 to the function $(|u|-\tau)_+$, which equals $|u|-\tau$ on $\mathcal{M}_\tau$ and 0 on $\mathcal{G}_d \setminus \mathcal{M}_\tau$, we see that

$$\operatorname{cap}(\bar{\mathcal{G}}_d \setminus \mathcal{M}_\tau) \leq \frac{C(\mathcal{G}) \int_{\mathcal{M}_\tau} |\nabla(|u|-\tau)|^2 dx}{|\mathcal{G}_d|^{-1} \int_{\mathcal{G}_d} |u|^2 dx} \leq \frac{C(\mathcal{G}) \int_{\mathcal{G}_d} |\nabla u|^2 dx}{|\mathcal{G}_d|^{-1} \int_{\mathcal{G}_d} |u|^2 dx}\ ,$$

where $C(\mathcal{G})$ is the same as in (4.1). Thus,

$$\int_{\mathcal{G}_d} |u|^2 dx \leq \frac{C(\mathcal{G})|\mathcal{G}_d| \int_{\mathcal{G}_d} |\nabla u|^2 dx}{\operatorname{cap}(\bar{\mathcal{G}}_d \setminus \mathcal{M}_\tau)} \leq \frac{C(\mathcal{G})|\mathcal{G}_d| \int_{\mathcal{G}_d} |\nabla u|^2 dx}{\gamma \operatorname{cap}(\bar{\mathcal{G}}_d)}$$

Note that $|\mathcal{G}_d| = |\mathcal{G}| d^n$ and $\operatorname{cap}(\bar{\mathcal{G}}_d) = \operatorname{cap}(\bar{\mathcal{G}}) d^{n-2}$, where for $n = 2$ the capacities of $\bar{\mathcal{G}} = \bar{\mathcal{G}}_1(0)$ and $\bar{\mathcal{G}}_d = \bar{\mathcal{G}}_d(y)$ are taken with respect to the discs $B_2(0)$ and $B_{2d}(y)$ respectively. Therefore we obtain

$$\int_{\mathcal{G}_d} |u|^2 dx \leq \frac{C(\mathcal{G})|\mathcal{G}| d^2}{\gamma \operatorname{cap}(\bar{\mathcal{G}})} \int_{\mathcal{G}_d} |\nabla u|^2 dx. \tag{4.7}$$

Now consider the opposite case $\operatorname{cap}(\bar{\mathcal{G}}_d \setminus \mathcal{M}_\tau) \leq \gamma \operatorname{cap}(\bar{\mathcal{G}}_d)$. Then we can write

$$\int_{\bar{\mathcal{G}}_d} |u|^2 \mathbb{V}(dx) \geq \int_{\mathcal{M}_\tau} |u|^2 \mathbb{V}(dx) \geq \tau^2 \mathbb{V}(\mathcal{M}_\tau) = \frac{1}{4|\mathcal{G}_d|} \int_{\mathcal{G}_d} |u|^2 dx \cdot \mathbb{V}(\mathcal{M}_\tau)$$
$$\geq \frac{1}{4|\mathcal{G}_d|} \int_{\mathcal{G}_d} |u|^2 dx \cdot \mathbb{V}_\gamma(\mathcal{G}_d, \Omega).$$

Finally we obtain in this case

$$\int_{\mathcal{G}_d} |u|^2 dx \leq \frac{4|\mathcal{G}_d|}{\mathbb{V}_\gamma(\mathcal{G}_d, \Omega))} \int_{\bar{\mathcal{G}}_d} |u|^2 \mathbb{V}(dx). \tag{4.8}$$

The desired inequality (4.4) immediately follows from (4.7) and (4.8) with $C_2(\mathcal{G}) = \max\left\{C(\mathcal{G})|\mathcal{G}|(\operatorname{cap}(\bar{\mathcal{G}}))^{-1}, 4|\mathcal{G}|\right\}$. $\square$

Now we will move to the proof of the sufficiency part in Theorem 2.2.

We will start with the following proposition which gives a general (albeit complicated) sufficient condition for the discreteness of spectrum.

Proposition 4.3 *Given an operator $H_{\mathbb{V}}$, let us assume that the following condition is satisfied: there exists $\eta_0 > 0$ such that for every $\eta \in (0, \eta_0)$ we can find $d = d(\eta) > 0$ and $R = R(\eta) > 0$, so that if $\mathcal{G}_d$ satisfies $\bar{\mathcal{G}}_d \cap (\Omega \setminus B_R(0)) \neq \emptyset$, then there exists $\gamma = \gamma(\mathcal{G}_d, \eta) \in (0,1)$ such that*

$$\gamma d^{-2} \geq \eta^{-1} \quad \text{and} \quad d^{-n} \mathbb{V}_\gamma(\mathcal{G}_d, \Omega) \geq \eta^{-1}\ . \tag{4.9}$$

Then the spectrum of $H_{\mathbb{V}}$ is discrete.

Proof. Recall that the discreteness of spectrum is equivalent to the following condition: for every $\eta > 0$ there exists $R > 0$ such that (3.13) holds for every $u \in C_0^\infty(\Omega)$. This will be true if we establish that for every $\eta > 0$ there exist $R > 0$ and $d > 0$ such that

$$\int_{\mathcal{G}_d} |u|^2 dx \le \eta \left[\int_{\mathcal{G}_d} |\nabla u|^2 dx + \int_{\bar{\mathcal{G}}_d} |u|^2 \mathbb{V}(dx) \right], \tag{4.10}$$

for all $\mathcal{G}_d$ such that $\bar{\mathcal{G}}_d \cap (\Omega \setminus B_R(0)) \neq \emptyset$ and for all $u \in C^\infty(\bar{\mathcal{G}}_d)$, such that $u = 0$ in a neighborhood of $\bar{\mathcal{G}}_d \setminus \Omega$. Indeed, assume that (4.10) is true. Let us take a covering of $\mathbb{R}^n$ by bodies $\bar{\mathcal{G}}_d$ so that it has a finite multiplicity $m = m(\mathcal{G})$ (i.e. at most m bodies $\bar{\mathcal{G}}_d$ can have non-empty intersection). Then, taking $u \in C_0^\infty(\Omega)$ and summing up the estimates (4.10) over all bodies $\mathcal{G}_d$ with $\bar{\mathcal{G}}_d \cap (\Omega \setminus B_R(0)) \neq \emptyset$, we obtain (3.13) (hence (3.12)) with $m\eta$ instead of η.

Now Lemma 4.2 and the assumptions (4.9) immediately imply (4.10) (with η replaced by $C_2(\mathcal{G})\eta$). $\square$

Instead of requiring that the conditions of Proposition 4.3 are satisfied for all $\eta \in (0, \eta_0)$, it suffices to require it for a monotone sequence $\eta_k \to +0$. We can also assume that $d(\eta_k) \to 0$ as $k \to +\infty$. Then, passing to a subsequence, we can assume that the sequence $\{d(\eta_k)\}$ is strictly decreasing. Keeping this in mind, we can replace the dependence $d = d(\eta)$ by the inverse dependence $\eta = g(d)$, so that $g(d) > 0$ and $g(d) \to 0$ as $d \to +0$ (and here we can also restrict ourselves to a sequence $d_k \to +0$). This leads to the following, essentially equivalent but more convenient reformulation of Proposition 4.3:

Proposition 4.4 *Given an operator $H_{\mathbb{V}}$, assume that the following condition is satisfied: there exists $d_0 > 0$ such that for every $d \in (0, d_0)$ we can find $R = R(d) > 0$ and $\gamma = \gamma(d) \in (0, 1)$, so that if $\bar{\mathcal{G}}_d \cap (\Omega \setminus B_R(0)) \neq \emptyset$, then*

$$d^{-2}\gamma \ge g(d)^{-1} \quad \textit{and} \quad d^{-n}\mathbb{V}_\gamma(\mathcal{G}_d, \Omega) \ge g(d)^{-1}, \tag{4.11}$$

where $g(d) > 0$ and $g(d) \to 0$ as $d \to +0$. Then the spectrum of $H_{\mathbb{V}}$ is discrete.

Proof of Theorem 2.2, part (ii). Instead of (ii) in Theorem 2.2 it suffices to prove the (stronger) statement formulated in Remark 2.4. So suppose that $\exists\, d_0 > 0$, $\exists\, c > 0$, $\forall\, d \in (0, d_0)$, $\exists\, R = R(d) > 0$, $\exists \gamma(d) \in (0, 1)$, satisfying (1.4), such that (2.6) holds for all $\mathcal{G}_d$ with $\bar{\mathcal{G}}_d \cap (\Omega \setminus B_R(0)) \neq \emptyset$.

Since the left hand side of (2.6) is exactly $d^{-n}\mathbb{V}_{\gamma(d)}(\mathcal{G}_d, \Omega)$, we see that (2.6) can be rewritten in the form

$$d^{-n}\mathbb{V}_\gamma(\mathcal{G}_d, \Omega) \ge c d^{-2}\gamma(d),$$

hence we can apply Proposition 4.4 with $g(d) = c^{-1}d^2\gamma(d)^{-1}$ to conclude that the spectrum of $H_{\mathbb{V}}$ is discrete. $\square$

5 A sufficiency precision example

In this section we will prove Theorem 2.8. We will construct a domain $\Omega \subset \mathbb{R}^n$, such that the condition (2.7) is satisfied with $\gamma(d) = Cd^2$ (with an arbitrarily large $C > 0$), and yet the spectrum of $-\Delta$ in $L^2(\Omega)$ (with the Dirichlet boundary condition) is not discrete. This will show that the condition (1.4) is precise, so Theorem 2.8 will be proved. We will assume for simplicity that $n \geq 3$.

We will use the following notations:

- $L^{(j)}$ is the spherical layer $\{x \in \mathbb{R}^n : \log j \leq |x| \leq \log(j+1)\}$. Its width is $\log(j+1) - \log j$ which is $< j^{-1}$ for all j and equivalent to j^{-1} for large j.

- $\{Q_k^{(j)}\}_{k\geq 1}$ is a collection of closed cubes which form a tiling of $\mathbb{R}^n$ and have edge length $\epsilon(n)\, j^{-1}$, where $\epsilon(n)$ is a sufficiently small constant depending on n (to be adjusted later).

- $x_k^{(j)}$ is the center of $Q_k^{(j)}$.

- $\{B_k^{(j)}\}_{k\geq 1}$ is the collection of closed balls centered at $x_k^{(j)}$ with radii ρ_j given by

  $$\omega_n (n-2)\, \rho_j^{n-2} = C(\epsilon(n)/j)^n,$$

  where ω_n is the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ and C is an arbitrary constant. The last equality can be written as

  $$\operatorname{cap}(B_k^{(j)}) = C \text{ mes } Q_k^{(j)}, \tag{5.1}$$

  where mes is the n-dimensional Lebesgue measure on $\mathbb{R}^n$. Among the balls $B_k^{(j)}$ we will select a subcollection which consists of the balls with the additional property $B_k^{(j)} \subset L^{(j)}$. We will refer to these balls as *selected* ones. We will denote selected balls by $\tilde{B}_k^{(j)}$. By an abuse of notation we will not introduce special letter for the subscripts of the selected balls. We will also denote by $\tilde{Q}_k^{(j)}$ the corresponding cubes $Q_k^{(j)}$, so that

  $$\tilde{Q}_k^{(j)} = Q_k^{(j)} \supset \tilde{B}_k^{(j)}.$$

- $\Lambda^{(j)} = \bigcup_{k\geq 1} \tilde{B}_k^{(j)} \subset L^{(j)}$.

- Ω is the complement of $\cup_{j\geq 1} \Lambda^{(j)}$.

- $B_r(P)$ is the closed ball with radius $r \leq 1$ centered at a point P. We will make a more precise choice of r later.

Proposition 5.1 *The spectrum of $-\Delta$ in Ω (with the Dirichlet boundary condition) is not discrete.*

Proof. Let $j \geq 7$ and $P \in L^{(j)}$, i.e.

$$\log j \leq |P| \leq \log(j+1).$$

Note that the ball $B_r(P)$ is a subset of the spherical layer $\cup_{l \geq s \geq m} L^{(s)}$ if and only if

$$\log m \leq |P| - r \quad \text{and} \quad |P| + r \leq \log(l+1).$$

Therefore, if

$$\log m \leq \log j - r$$

and

$$\log(j+1) + r \leq \log(l+1),$$

then $B_r(P) \subset \cup_{l \geq s \geq m} L^{(s)}$. The last two inequalities can be written as

$$(5.2) \qquad m \leq j\, e^{-r} \quad \text{and} \quad j+1 \leq (l+1)e^{-r}.$$

If we take, for example,

$$m = [j/3] \quad \text{and} \quad l = 3j,$$

then, due to the inequality $j \geq 7$, we easily deduce that

$$(5.3) \qquad B_r(P) \subset \bigcup_{[j/3] \leq s \leq 3j} L^{(s)}.$$

Using (5.2), the definition of Ω and subadditivity of capacity, we obtain:

$$\begin{aligned} \operatorname{cap}(B_r(P) \setminus \Omega) &= \operatorname{cap}(B_r(P) \cap (\cup_{s \geq 1} \Lambda^{(s)})) \\ &\leq \sum_{[j/3] \leq s \leq 3j} \sum_{k \geq 1} \operatorname{cap}(B_r(P) \cap \tilde{B}_k^{(s)}) \\ &\leq C \sum_{[j/3] \leq s \leq 3j} \sum_{\{k : B_r(P) \cap \tilde{Q}_k^{(s)} \neq \emptyset\}} \operatorname{mes} \tilde{Q}_k^{(s)}. \end{aligned}$$

It is easy to see that the multiplicity of the covering of $B_r(P)$ by the cubes $\tilde{Q}_k^{(s)}$, participating in the last sum, is at most 2, provided $\epsilon(n)$ is chosen sufficiently small. Hence,

$$(5.4) \qquad \operatorname{cap}(B_r(P) \setminus \Omega) \leq c(n)\, C\, r^n.$$

On the other hand, we know that the discreteness of spectrum guarantees that for every $r > 0$

$$\liminf_{|P| \to \infty} \operatorname{cap}(B_r(P) \setminus \Omega) \geq \gamma(n)\, r^{n-2},$$

where $\gamma(n)$ is a constant depending only on n (cf. Remark 2.7). For sufficiently small $r > 0$ this clearly contradicts (5.4). $\square$

Proposition 5.2 *The domain Ω satisfies*

$$\liminf_{|P|\to\infty} \operatorname{cap}(B_r(P)\setminus\Omega) \geq \delta(n)\, C\, r^n, \tag{5.5}$$

where $\delta(n) > 0$ depends only on n.

Proof. Let $\mu_k^{(s)}$ be the capacitary measure on $\partial\tilde{B}_k^{(s)}$ (extended by zero to $\mathbb{R}^n \setminus \partial\tilde{B}_k^{(s)}$), and let $\epsilon_1(n)$ denote a sufficiently small constant to be chosen later. We introduce the measure

$$\mu = \epsilon_1(n) \sum_{k,s} \mu_k^{(s)},$$

where the summation here and below is taken over k, s which correspond to the selected balls $\tilde{B}_k^{(s)}$. Taking $P \in L^{(j)}$, let us show that

$$\int_{B_{r/2}(P)} \mathcal{E}(x-y)d\mu(y) \leq 1 \quad \text{on} \quad \mathbb{R}^n, \tag{5.6}$$

where $\mathcal{E}(x)$ is given by (3.3). It suffices to verify (5.6) for $x \in B_r(P)$, because for $x \in \mathbb{R}^n \setminus B_r(P)$ this will follow from the maximum principle.

Obviously, the potential in (5.6) does not exceed

$$\sum_{\{s,k:\tilde{B}_k^{(s)}\cap B_{r/2}(P)\neq\emptyset\}} \epsilon_1(n) \int_{\partial\tilde{B}_k^{(s)}} \mathcal{E}(x-y)d\mu_k^{(s)}(y).$$

We divide this sum into two parts $\sum'$ and $\sum''$, the first sum being extended over all points $x_k^{(s)}$ with the distance $\leq j^{-1}$ from x. Recalling that $x \in B_r(P)$ and using (5.3), we easily see that the number of such points does not exceed a certain constant $c_1(n)$. We define the constant $\epsilon_1(n)$ by

$$\epsilon_1(n) = (2c_1(n))^{-1}.$$

Since $\mu_k^{(s)}$ is the capacitary measure, we have

$$\sum{}' \ldots \leq \epsilon_1(n)\, c_1(n) = 1/2.$$

Furthermore, by (5.1)

$$\sum{}'' \ldots \leq c_2(n) \sum{}'' \frac{\operatorname{cap}(\tilde{B}_k^{(s)})}{|x-x_k^{(s)}|^{n-2}} = c_2(n)\, C \sum{}'' \frac{\operatorname{mes}\, \tilde{Q}_k^{(s)}}{|x-x_k^{(s)}|^{n-2}}$$
$$\leq c_3(n)\; C \int_{B_r(P)} \frac{dy}{|x-y|^{n-2}} < c_4(n)\; C\; r^2.$$

We can assume that

$$r \leq (2c_4(n)C)^{-1/2}$$

which implies $\sum'' \leq 1/2$. Therefore (5.6) holds.

It follows that for large $|P|$ (i.e. for P with $|P| \geq R = R(r) > 0$), or, equivalently, for large j, we will have

$$\begin{aligned}\operatorname{cap}(B_r(P)\setminus\Omega) &\geq \sum_{\{s,k:\,\tilde{B}_k^{(s)}\subset B_{r/2}(P)\}} \epsilon_1(n)\mu_k^{(s)}(\partial\tilde{B}_k^{(s)}) \\ &= \epsilon_1(n) \sum_{\{s,k:\,\tilde{B}_k^{(s)}\subset B_{r/2}(P)\}} \operatorname{cap}(\tilde{B}_k^{(s)}) \\ &= \epsilon_1(n)\; C \sum_{\{s,k:\,\tilde{B}_k^{(s)}\subset B_{r/2}(P)\}} \operatorname{mes} Q_k^{(s)} \geq \delta(n)\; C\; r^n.\end{aligned}$$

This ends the proof of Proposition 5.2, hence of Theorem 2.8. □

Remark 5.3 Slightly modifying the construction given above, it is easy to provide an example of an operator $H = -\Delta + V(x)$ with $V \in C^\infty(\mathbb{R}^n)$, $n \geq 3$, $V \geq 0$, such that the corresponding measure $V dx$ satisfies (2.5) with $\gamma(d) = Cd^2$ and an arbitrarily large $C > 0$, but the spectrum of H in $L^2(\mathbb{R}^n)$ is not discrete. So the condition (1.4) is precise even in case of the Schrödinger operators with C^∞ potentials.

6 Positivity of $H_{\mathbb{V}}$

In this section we prove Theorem 2.9.

Proof of Theorem 2.9 (necessity). Let us assume that the operator $H_{\mathbb{V}}$ is strictly positive. This implies that the estimate (3.14) holds with some $\eta > 0$ for every $\mathcal{G}_d$ (with an arbitrary $d > 0$) and every $u \in C_0^\infty(\mathcal{G}_d \cap \Omega)$. But then we can use the arguments of Section 3 which lead to (3.20), provided (3.16) is satisfied. It will be satisfied if d is chosen sufficiently large. □

Proof of Theorem 2.9 (sufficiency). Let us assume that there exist $d > 0$, $\varkappa > 0$ and $\gamma \in (0,1)$ such that for every $\mathcal{G}_d$ the estimate (2.9) holds. Then by Lemma 4.2, for every $\mathcal{G}_d$ and every $u \in C^\infty(\bar{\mathcal{G}}_d)$, such that $u = 0$ in a neighborhood of $\bar{\mathcal{G}}_d \setminus \Omega$, we have

$$\int_{\mathcal{G}_d} |u|^2 dx \leq \frac{C_2(\mathcal{G})d^2}{\gamma} \int_{\mathcal{G}_d} |\nabla u|^2 dx + \frac{C_2(\mathcal{G})d^n}{\varkappa} \int_{\bar{\mathcal{G}}_d} |u|^2 \mathbb{V}(dx).$$

Let us take a covering of $\mathbb{R}^n$ of finite multiplicity N by bodies $\bar{\mathcal{G}}_d$. It follows that for every $u \in C_0^\infty(\Omega)$

$$\int_\Omega |u|^2 dx \leq NC_2(\mathcal{G})d^2 \max\left\{\frac{1}{\gamma}, \frac{d^{n-2}}{\varkappa}\right\} \left(\int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^2 \mathbb{V}(dx)\right),$$

which proves positivity of $H_{\mathbb{V}}$. □

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