THE FORM BOUNDEDNESS CRITERION FOR THE RELATIVISTIC SCHRÖDINGER OPERATOR

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ABSTRACT. We establish necessary and sufficient conditions for the boundedness of the relativistic Schrödinger operator $\mathcal{H} = \sqrt{-\Delta} + Q$ from the Sobolev space $W_2^{1/2}(\mathbb{R}^n)$ to its dual $W_2^{-1/2}(\mathbb{R}^n)$, for an arbitrary real- or complex-valued potential Q on $\mathbb{R}^n$. In other words, we give a complete solution to the problem of the domination of the potential energy by the kinetic energy in the relativistic case characterized by the inequality

$$\left| \int_{\mathbb{R}^n} |u(x)|^2 \, Q(x) \, dx \right| \le \text{const} \, ||u||^2_{W_2^{1/2}}, \qquad \forall u \in C_0^\infty(\mathbb{R}^n),$$

where the "indefinite weight" Q is a locally integrable function (or, more generally, a distribution) on $\mathbb{R}^n$. Along with necessary and sufficient results, we also present new broad classes of admissible potentials Q in the scale of Morrey spaces of negative order, and discuss their relationship to well-known L_p and Fefferman-Phong conditions.

1. Introduction

In the present paper we establish necessary and sufficient conditions for the relative form boundedness of the potential energy operator Q with respect to the *relativistic* kinetic energy operator $\mathcal{H}_0 = \sqrt{-\Delta}$, which is fundamental to relativistic quantum systems. Here Q is an arbitrary real- or complex-valued potential (possibly a distribution), and $\mathcal{H}_0$ is a nonlocal operator which replaces the standard Laplacian $H_0 = -\Delta$ used in the nonrelativistic theory.

More precisely, we characterize all potentials $Q \in \mathcal{D}'(\mathbb{R}^n)$ such that

$$|\langle Q \, u, \, u\rangle| \le a \, \langle \sqrt{-\Delta} \, u, \, u\rangle + b \, \langle u, \, u\rangle, \qquad \forall u \in \mathcal{D}(\mathbb{R}^n),$$

for some $a > 0$, $b \in \mathbb{R}$, where $\mathcal{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$.

In particular, if Q is real-valued, and the form bound $a < 1$, then this inequality makes it possible to define, via the classical KLMN Theorem (see, e.g., [RS], Theorem X.17), the relativistic Schrödinger operator $\mathcal{H} = \sqrt{-\Delta}+Q$,

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where the sum $\sqrt{-\Delta}+Q$ is a uniquely defined self-adjoint operator associated with the sum of the corresponding quadratic forms whose form domain $\mathcal{Q}(\mathcal{H})$ coincides with the Sobolev space $W_2^{1/2}(\mathbb{R}^n)$. (For complex-valued Q, this sum defines an m-sectorial operator provided $a < 1/2$; see [EE], Theorem IV.4.2.)

Equivalently, we give a complete characterization of the class of admissible potentials Q such that the relativistic Schrödinger operator $\mathcal{H} = \sqrt{-\Delta} + Q$ is bounded from $W_2^{1/2}(\mathbb{R}^n)$ to the dual space $W_2^{-1/2}(\mathbb{R}^n)$.

A nice introduction to the theory of the relativistic Schrödinger operator is given in [LL]. We observe that it is customary to develop the relativistic theory in parallel to its nonrelativistic counterpart, without making a connection between them. One of the advantages of our general approach where distributional potentials Q are admissible is that it provides a direct link between the two theories.

In Sec. 2, we develop an extension principle which establishes a connection between the relativistic Schrödinger operator $\mathcal{H} = \sqrt{-\Delta} + Q$ and the nonrelativistic one, $H = -\Delta + \widetilde{Q}$, where $\widetilde{Q}$ is a distribution defined on a higher dimensional Euclidean space. Note that the *nonrelativistic* form boundedness problem was settled in full generality only recently by the authors in [MV2]. (The one-dimensional case of the Sturm-Liouville operator $H = -\frac{d^2}{dx^2} + Q$ on the real axis and half-axis is treated in [MV3].)

It is worth noting that in the above discussion of the relative form boundedness $\mathcal{H}_0 = \sqrt{-\Delta}$ can be replaced by $\mathcal{H}_{\mathfrak{m}} = \sqrt{-\Delta + \mathfrak{m}^2} - \mathfrak{m}$, where $\mathfrak{m}$ represents the mass of the particle under consideration. This operator appears in the relativistic Schrödinger equation:

$$\mathcal{H}_{\mathfrak{m}}\psi + Q\,\psi = E\,\psi \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n). \tag{1.1}$$

One of the central questions of the relativistic theory is the domination of the potential energy $\int_{\mathbb{R}^n} |u|^2\, Q(x)\, dx$ by the kinetic energy associated with $||u||^2_{W_2^{1/2}}$, which explains a special role of the Sobolev space $W_2^{1/2}$ in this context (see [LL], Sec. 7.11 and 11.3). We address this problem by characterizing the weighted norm inequality with "indefinite weights":

$$\left| \int_{\mathbb{R}^n} |u(x)|^2\, Q(x)\, dx \right| \le \text{const}\, ||u||^2_{W_2^{1/2}}, \qquad \forall u \in \mathcal{D}(\mathbb{R}^n). \tag{1.2}$$

Here Q is a locally integrable real- or complex-valued function, or more generally, a distribution. In the latter case, the left-hand side of (1.2) is understood as $|\langle Q\, u,\, u\rangle|$, where $\langle Q\cdot,\, \cdot\rangle$ is the quadratic form associated with the corresponding multiplication operator.

An analogous inequality characterized in [MV2],

$$(1.3)\qquad \left|\int_{\mathbb{R}^n} |u(x)|^2\, Q(x)\, dx\right| \le \text{const}\, ||u||^2_{W^1_2}, \qquad \forall u \in \mathcal{D}(\mathbb{R}^n),$$

with the Sobolev norm of order 1 in place of 1/2, is used extensively in spectral theory of the nonrelativistic Schrödinger operator $H = -\Delta + Q$. (See [AiS], [Fef], [M1], [M2], [MV2], [Nel], [RS], [Sch], [Sim].) In particular, (1.3) is equivalent to the relative form boundedness of the potential energy operator Q with respect to the traditional kinetic energy operator $H_0 = -\Delta$.

We remark that, for *nonnegative* (or nonpositive) potentials Q (possibly measures on $\mathbb{R}^n$ which may be singular with respect to n-dimensional Lebesgue measure), the inequalities (1.2) and (1.3) have been thoroughly studied, and are well understood by now. (See [ChWW], [Fef], [KeS], [M1], [MV1], [Ver].) On the other hand, for real-valued Q which may change sign, or complex-valued Q, only sufficient conditions, as well as examples of potentials with strong cancellation properties have been known, mostly in the framework of the nonrelativistic Schrödinger operator theory and Sobolev multipliers ([AiS], [CoG], [MSh], [Sim]).

We now state our main results on the relativistic Schrödinger operator with "indefinite" potentials Q in the form of the following two theorems. Simpler sufficient and necessary conditions in the scales of Sobolev, Lorentz-Sobolev, and Morrey spaces of negative order are obtained as corollaries. Their relationship to more conventional L_p and Fefferman-Phong classes is discussed at the end of the Introduction, and in Sec. 3 in more detail.

Note that rigorous definitions of the expressions like $\langle Q\cdot,\, \cdot\rangle$ or $(-\Delta+1)^{-1/4}Q$ are given in the main body of the paper.

Theorem I. *Let $Q \in \mathcal{D}'(\mathbb{R}^n)$, $n \ge 1$. The following statements are equivalent:*

(i) *The relativistic Schrödinger operator $\mathcal{H} = \sqrt{-\Delta} + Q$ is bounded from $W^{1/2}_2(\mathbb{R}^n)$ to $W^{-1/2}_2(\mathbb{R}^n)$.*

(ii) *The inequality*

$$(1.4)\qquad |\langle Qu,\, u\rangle| \le \text{const}\, ||u||^2_{W^{1/2}_2}, \qquad \forall u \in \mathcal{D}(\mathbb{R}^n),$$

holds, where the constant does not depend on u.

(iii) $\Phi = (-\Delta + 1)^{-1/4}Q \in L_{2,\,loc}(\mathbb{R}^n)$, *and the inequality*

$$(1.5)\qquad \int_{\mathbb{R}^n} |u(x)|^2\, |\Phi(x)|^2\, dx \le \text{const}\, ||u||^2_{W^{1/2}_2}, \qquad \forall u \in \mathcal{D}(\mathbb{R}^n),$$

holds, where the constant does not depend on u.

Theorem II. *Let* $Q \in \mathcal{D}'(\mathbb{R}^n)$, $n \geq 1$, *and let* $\mathcal{H} = \sqrt{-\Delta} + Q$. *Then* $\mathcal{H} : W_2^{1/2}(\mathbb{R}^n) \to W_2^{-1/2}(\mathbb{R}^n)$ *is bounded if and only if* $\Phi = (-\Delta + 1)^{-1/4} Q \in L_{2,\,loc}(\mathbb{R}^n)$, *and any one of the following equivalent conditions holds:*

(i) *For every compact set* $e \subset \mathbb{R}^n$,

$$\int_e |\Phi(x)|^2 \, dx \leq \text{const } \text{cap}\,(e,\, W_2^{1/2}), \tag{1.6}$$

where the constant does not depend on e. *Here* $\text{cap}\,(\cdot,\, W_2^m)$ *is the capacity associated with the Sobolev space* $W_2^m(\mathbb{R}^n)$ *defined by:*

$$\text{cap}\,(e,\, W_2^m) = \inf\,\{||u||^2_{W_2^m} : \quad u \in \mathcal{D}(\mathbb{R}^n), \qquad u \geq 1 \quad \textit{on} \quad e\}.$$

(ii) *The function* $J_{1/2}\,|\Phi|^2$ *is finite* a.e., *and*

$$J_{1/2}\,\left(J_{1/2}\,|\Phi|^2\right)^2(x) \leq \text{const}\, J_{1/2}\,|\Phi|^2(x) \qquad \text{a.e.} \tag{1.7}$$

Here $J_{1/2} = (-\Delta + 1)^{-1/4}$ *is the Bessel potential of order* $1/2$.

(iii) *For every dyadic cube* P_0 *in* $\mathbb{R}^n$ *of sidelength* $\ell(P_0) \leq 1$,

$$\sum_{P \subseteq P_0} \left[\frac{\int_P |\Phi(x)|^2 \, dx}{|P|^{1-1/(2n)}} \right]^2 |P| \leq \text{const} \int_{P_0} |\Phi(x)|^2 \, dx, \tag{1.8}$$

where the sum is taken over all dyadic cubes P *contained in* P_0, *and the constant does not depend on* P_0.

We observe that statement (iii) of Theorem I reduces the problem of characterizing general weights Q such that either (i) or equivalently (ii) holds, to a similar problem for the *nonnegative* weight $|\Phi|^2$.

The proof of Theorem I makes use of the connection mentioned above between the boundedness problem for the relativistic operator

$$\mathcal{H} = \sqrt{-\Delta} + Q : \, W_2^{1/2}(\mathbb{R}^n) \to W_2^{-1/2}(\mathbb{R}^n),$$

and its nonrelativistic counterpart,

$$H = -\Delta + \widetilde{Q} : \, W_2^1(\mathbb{R}^{n+1}) \to W_2^{-1}(\mathbb{R}^{n+1}).$$

The latter is acting on a pair of Sobolev spaces of integer order in the higher dimensional Euclidean space, and the corresponding potential $\widetilde{Q} \in \mathcal{D}'(\mathbb{R}^{n+1})$. We also employ extensively a calculus of maximal and Fourier multiplier operators on the space of functions $f \in L_{2,\,loc}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} |f(x)|^2 \, |u(x)|^2 \, dx \leq \text{const}\, ||u||^2_{W_2^m}, \qquad \forall u \in \mathcal{D}(\mathbb{R}^n),$$

developed in [MV1], [MV2], and based on the theory of Muckenhoupt weights and use of equilibrium measures associated with arbitrary compact sets of positive capacity.

Combining Theorem I with the characterizations of the inequality (1.4) for nonnegative weights established earlier (see, e.g., [ChWW], [Fef], [KeS], [M1], [M2], [MV1], [MV2], [Ver]) we obtain more explicit characterizations of admissible weights Q stated in Theorem II.

We now recall the well-known isoperimetric inequalities (see, e.g., [MSh], Sec. 2.1.2):

$$\operatorname{cap}(e,\, W_2^{1/2}(\mathbb{R}^n)) \geq c\,|e|^{(n-1)/n}, \qquad \operatorname{diam}(e) \leq 1, \qquad n \geq 2,$$

$$\operatorname{cap}(e,\, W_2^{1/2}(\mathbb{R}^1)) \geq \frac{c}{\log \frac{2}{|e|}}, \qquad \operatorname{diam}(e) \leq 1, \qquad n = 1,$$

where $|e|$ is Lebesgue measure of a compact set $e \subset \mathbb{R}^n$. Note that the one-dimensional case is special in this setting, since $m = 1/2$ is the critical Sobolev exponent for $W_2^m(\mathbb{R}^n)$ if $n = 1$. Thus, it requires certain modifications in comparison to the general case $n \geq 2$.

These estimates together with statement (i) of Theorem II (note that it is enough to verify (1.6) only for compact sets e such that $\operatorname{diam}(e) \leq 1$), yield sharp sufficient conditions for (1.4) to hold.

Corollary 1. *Suppose $Q \in \mathcal{D}'(\mathbb{R}^n)$, $n \geq 1$. Then $\mathcal{H} = \sqrt{-\Delta} + Q$ is a bounded operator from $W_2^{1/2}(\mathbb{R}^n)$ to $W_2^{-1/2}(\mathbb{R}^n)$ if one of the following conditions holds:*

$$\int_e |\Phi(x)|^2\, dx \leq c\,|e|^{(n-1)/n}, \quad \operatorname{diam}(e) \leq 1, \qquad n \geq 2, \tag{1.9}$$

or

$$\int_e |\Phi(x)|^2\, dx \leq \frac{c}{\log \frac{2}{|e|}}, \quad \operatorname{diam}(e) \leq 1, \qquad n = 1, \tag{1.9$'$}$$

where the constant c does not depend on $e \subset \mathbb{R}^n$.

Remark 1. We observe that (1.9) holds if $\Phi \in L_{2n,\infty}(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$, $n \geq 2$, where $L_{p,\infty}$ denotes the weak L_p (Lorentz) space. Similarly, in the one-dimensional case, (1.9′) holds if $\Phi \in L_{1+\epsilon}(\mathbb{R}^1) + L_\infty(\mathbb{R}^1)$, $\epsilon > 0$.

Remark 2. The class of admissible potentials Q satisfying (1.9) is substantially broader than the standard (in the relativistic case) class $Q \in L_n(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$, $n \geq 2$. In particular, it contains highly oscillating functions with significant growth of $|Q|$ at infinity, along with singular measures and distributions. Similarly, in the one-dimensional case, the class of potentials defined by (1.9′) is much wider than the standard class $Q \in L_{1+\epsilon}(\mathbb{R}^1) + L_\infty(\mathbb{R}^1)$, $\epsilon > 0$. (See [LL], Sec. 11.3.)

These relations, along with sharper estimates in terms of Morrey spaces of negative order which follow from Theorems I and II, are discussed in Sec. 3. They extend significantly relativistic analogues of the Fefferman-Phong class introduced in [Fef], as well as other known classes of admissible potentials.

2. The form boundedness criterion

For positive integers m, the Sobolev space $W_2^m(\mathbb{R}^n)$ is defined as the space of weakly differentiable functions such that

$$(2.1) \qquad ||f||_{W_2^m} = \left[\int_{\mathbb{R}} (|f(x)|^2 + |\nabla^m f(x)|^2)\, dx\right]^{\frac{1}{2}} < \infty.$$

More generally, for real $m > 0$, $W_2^m(\mathbb{R}^n)$ is the space of all $f \in L_2(\mathbb{R}^n)$ which can be represented in the form $f = (-\Delta + 1)^{-m/2} g$, where $g \in L_2(\mathbb{R}^n)$. Here $(-\Delta + 1)^{-m/2} g = J_m \star g$ is the convolution of g with the Bessel kernel J_m of order m, and $||f||_{W_2^m} = ||g||_{L_2^m}$ (see [M2], [St1]). This definition is consistent with the previous one for integer m, and defines an equivalent norm on $W_2^m(\mathbb{R}^n)$. Note that another equivalent norm on $W_2^m(\mathbb{R}^n)$ is given by

$$||f||_{W_2^m} = ||f||_{L_2} + \|\, |D|^m f\, \|_{L_2}, \quad f \in W_2^m(\mathbb{R}^n),$$

where $|D| = (-\Delta)^{1/2}$.

The dual space $W_2^{-m}(\mathbb{R}^n) = W_2^m(\mathbb{R}^n)^*$ can be identified with the space of distributions f of the form $f = (-\Delta + 1)^{m/2} g$, where $g \in L_2(\mathbb{R}^n)$.

Let $\gamma \in \mathcal{D}'(\mathbb{R}^n)$ be a (complex-valued) distribution on $\mathbb{R}^n$. We will use the same notation for the corresponding multiplication operator $\gamma : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ defined by:

$$\langle \gamma u, v\rangle = \langle \gamma, \bar{u}\, v\rangle \qquad u, v \in \mathcal{D}(\mathbb{R}^n).$$

For $m, l \in \mathbb{R}$, we denote by Mult $(W_2^m \to W_2^l)$ the class of bounded multiplication operators (multipliers) from W_2^m to W_2^l generated by $\gamma \in \mathcal{D}'(\mathbb{R}^n)$ so that the corresponding sesquilinear form $\langle \gamma\, \cdot, \, \cdot\rangle$ is bounded:

$$(2.2) \qquad |\langle \gamma\, u, \, v\rangle| = |\langle \gamma, \, \bar{u}\, v\rangle| \le C\, ||u||_{W_2^m}\, ||v||_{W_2^{-l}}, \qquad \forall u, v \in \mathcal{D}(\mathbb{R}^n),$$

where C does not depend on u, v. The *multiplier norm* denoted by $||\gamma||_{W_2^m \to W_2^l}$ is equal to the least bound C in the preceding inequality.

It is easy to see that, in the case $l = -m$, (2.2) is equivalent to the quadratic form inequality:

$$(2.2') \qquad |\langle \gamma\, u, \, u\rangle| = |\langle \gamma, \, |u|^2\rangle| \le C'\, ||u||^2_{W_2^m}, \qquad \forall u \in \mathcal{D}(\mathbb{R}^n).$$

To verify this, suppose that $||u||_{W_2^m} \le 1$, $||v||_{W_2^m} \le 1$, where $u, v \in \mathcal{D}(\mathbb{R}^n)$. Applying (2.2′) together with the polarization identity:

$$\bar{u}\, v = \frac{1}{4} \left(|u+v|^2 - |u-v|^2 - i|u-iv|^2 + i|u+iv|^2 \right),$$

and the parallelogram identity, we get:

$$\begin{aligned} |\langle \gamma, \, \bar{u}\, v\rangle| &\le \frac{C'}{4} \left(||u+v||^2_{W_2^m} + ||u-v||^2_{W_2^m} + ||u+iv||^2_{W_2^m} + ||u-iv||^2_{W_2^m} \right) \\ &\le 2C'. \end{aligned}$$

Hence, (2.2) holds for $l = -m$ with $C = 2C'$. Moreover, the least bound C' in (2.2′) satisfies the inequality:

$$C' \le ||\gamma||_{W_2^m \to W_2^{-m}} \le 2C'.$$

Let $|D| = (-\Delta)^{1/2}$. We define the relativistic Schrödinger operator as

$$\mathcal{H} = |D| + Q : \, \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n),$$

(see [LL]), where $Q : \, \mathcal{D}(\mathbb{R}^n) \to \in \mathcal{D}'(\mathbb{R}^n)$ is a multiplication operator defined by $Q \in \mathcal{D}'(\mathbb{R}^n)$. It is well-known that actually $|D|$ is a bounded operator from $W_2^{1/2}(\mathbb{R}^n)$ to $W_2^{-1/2}(\mathbb{R}^n)$. Thus, $\mathcal{H}$ can be extended to a bounded operator:

$$\mathcal{H} : \, W_2^{1/2}(\mathbb{R}^n) \to W_2^{-1/2}(\mathbb{R}^n),$$

if and only if $Q \in \text{Mult}\,(W_2^{1/2}(\mathbb{R}^n) \to W_2^{-1/2}(\mathbb{R}^n))$, or, equivalently, if the quadratic form inequality (2.2′) holds for $\gamma = Q$ and $m = 1/2$.

From the preceding discussion it follows that $\mathcal{H} : \, W_2^{1/2}(\mathbb{R}^n) \to W_2^{-1/2}(\mathbb{R}^n)$ is bounded if and only if

$$|\langle Q\, u, \, u\rangle| \le a \, \langle |D| \, u, \, u\rangle + b \, \langle u, \, u\rangle, \qquad \forall u \in \mathcal{D}(\mathbb{R}^n), \tag{2.3}$$

for some $a, b > 0$. By definition this means that Q is *relatively form bounded* with respect to $|D|$.

In particular, if Q is real-valued, and $0 < a < 1$ in the preceding inequality, then by the so-called KLMN Theorem ([RS], Theorem X.17), $\mathcal{H} = |D| + Q$ is defined as a unique self-adjoint operator such that

$$\langle \mathcal{H} u, \, v\rangle = \langle |D| \, u, \, v\rangle + \langle Q\, u, v\rangle, \qquad \forall u \in \mathcal{D}(\mathbb{R}^n).$$

For complex-valued Q such that (2.3) holds with $0 < a < 1/2$, it follows that $\mathcal{H} = |D| + Q$, understood in a similar sense, is an m-sectorial operator ([EE], Theorem IV.4.2).

In the case where $Q \in L_{1,\, loc}(\mathbb{R}^n)$, (2.3) is equivalent to the inequality:

$$\left| \int_{\mathbb{R}^n} |u(x)|^2 \, Q(x) \, dx \right| \le \text{const} \, ||u||^2_{W_2^{1/2}}, \qquad \forall u \in \mathcal{D}(\mathbb{R}^n), \tag{2.4}$$

and hence to the boundedness of the corresponding sesquilinear form:

$$\left| \int_{\mathbb{R}^n} u(x) \, \overline{v(x)} \; Q(x) \, dx \right| \le \text{const} \, ||u||_{W_2^{1/2}(\mathbb{R}^n)} \, ||v||_{W_2^{1/2}(\mathbb{R}^n)},$$

where the constant is independent of $u, v \in \mathcal{D}(\mathbb{R}^n)$.

Our characterization of potentials Q such that $\mathcal{H}: \, W_2^{1/2}(\mathbb{R}^n) \to W_2^{-1/2}(\mathbb{R}^n)$ is based on a series of lemmas and propositions presented below, and the results of [MV2] for the nonrelativistic Schrödinger operator.

By $L_{2,unif}(\mathbb{R}^n)$, we denote the class of $f \in L_{2,loc}(\mathbb{R}^n)$ such that

$$||f||_{L_{2,unif}} = \sup_{x \in \mathbb{R}^n} \, ||\chi_{B_1(x)} \, f||_{L_2(\mathbb{R}^n)} < \infty, \tag{2.5}$$

where $B_r(x)$ denotes a Euclidean ball of radius r centered at x.

Lemma 2.1. *Let* $0 < l < 1$, *and* $m > l$. *Then* $\gamma \in \text{Mult}\,(W_2^m \to W_2^l)$ *if and only if* $\gamma \in W_2^{m-l} \to L_2$, *and* $|D|^l \gamma \in \text{Mult}\,(W_2^m \to L_2)$. *Moreover,*

$$\|\gamma\|_{W_2^m \to W_2^l} \sim \big\| |D|^l \gamma \big\|_{W_2^m \to L_2} + \|\gamma\|_{W_2^{m-l} \to L_2} \, . \tag{2.6}$$

Proof. We first prove the lower estimate for $\|\gamma\|_{W_2^m \to W_2^l}$:

$$\big\| |D|^l \gamma \big\|_{W_2^m \to L_2} + \|\gamma\|_{W_2^{m-l} \to L_2} \le c \, \|\gamma\|_{W_2^m \to W_2^l} \, . \tag{2.7}$$

Here and below c denotes a constant which depends only on l, m, and n.

Let $u \in C_0^\infty(\mathbb{R}^n)$. Using the integral representation (which follows by inspecting the Fourier transforms of both sides),

$$|D|^l u(x) = c(n, l) \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+l}} dy, \tag{2.8}$$

we obtain:

$$\begin{aligned} &|D|^l \, (\gamma \, u)(x) - \gamma(x) \, |D|^l \, u(x) - u(x) \, |D|^l \, \gamma(x) \\ &= -c(n, l) \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(\gamma(x) - \gamma(y))}{|x - y|^{n+l}} dy. \end{aligned}$$

Hence,

$$\big| \, |D|^l \, (\gamma \, u) - \gamma \, |D|^l \, u - u \, |D|^l \gamma \big| \le c \, \mathcal{D}_{l/2} \, u \cdot \mathcal{D}_{l/2} \, \gamma, \tag{2.9}$$

where

$$\mathcal{D}_s \, u(x) = \left(\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dy \right)^{\frac{1}{2}}, \quad s > 0.$$

Next, we estimate:

$$\begin{aligned} ||u \cdot |D|^l \gamma||_{L_2} &\le \big\| \, |D|^l (\gamma \, u) \big\|_{L_2} + \big\| \gamma \, |D|^l u \big\|_{L_2} + c \, ||\mathcal{D}_{l/2} u \cdot \mathcal{D}_{l/2} \gamma||_{L_2} \\ &\le \; ||\gamma u||_{W_2^l} + ||\gamma||_{W_2^{m-l} \to L_2} \, \big\| \, |D|^l u \big\|_{W_2^{m-l}} + c \, ||\mathcal{D}_{l/2} u \cdot \mathcal{D}_{l/2} \gamma||_{L_2} \end{aligned}$$

$$\leq \, ||\gamma||_{W_2^m\to W_2^l}\,||u||_{W_2^m} + ||\gamma||_{W_2^{m-l}\to L_2}\,||u||_{W_2^m} + c\,||\mathcal{D}_{l/2}u\cdot\mathcal{D}_{l/2}\gamma||_{L_2}$$
$$\leq c\,||\gamma||_{W_2^m\to W_2^l}\,||u||_{W_2^m} + c\,||\mathcal{D}_{l/2}u\cdot\mathcal{D}_{l/2}\gamma||_{L_2}.$$

In the last line we have used the known inequality ([MSh], Sec. 2.2.2):

$$||\gamma||_{W_2^{m-l}\to L_2} \leq c\,||\gamma||_{W_2^m\to W_2^l}.$$

To estimate the term $||\mathcal{D}_{l/2}u\cdot\mathcal{D}_{l/2}\gamma||_{L_2}$, we apply the pointwise estimate (Lemma 1 in [MSh], Sec. 3.1.1):

$$\mathcal{D}_{l/2}u \leq \, J_s\mathcal{D}_{l/2}((-\Delta+1)^{s/2}\,u),$$

with $s = m - l/2$, where $J_s = (-\Delta+1)^{-s/2}$ is the Bessel potential of order s. Hence

$$||\mathcal{D}_{l/2}u\cdot\mathcal{D}_{l/2}\gamma||_{L_2} \leq \, ||J_{m-l/2}\mathcal{D}_{l/2}((-\Delta+1)^{m/2-l/4}\,u))\cdot\mathcal{D}_{l/2}\gamma||_{L_2}$$
$$\leq \, c\,||\mathcal{D}_{l/2}\gamma||_{W_2^{m-l/2}\to L_2}\,||J_{m-l/2}\mathcal{D}_{l/2}((-\Delta+1)^{m/2-l/4}\,u))||_{W_2^{m-l/2}}$$
$$\leq \, c\,||\mathcal{D}_{l/2}\gamma||_{W_2^{m-l/2}\to L_2}\,||\mathcal{D}_{l/2}(-\Delta+1)^{m/2-l/4}u||_{L_2}$$
$$\leq \, c\,||\mathcal{D}_{l/2}\gamma||_{W_2^{m-l/2}\to L_2}\,||u||_{W_2^m}.$$

We next show that

$$||\mathcal{D}_{l/2}\gamma||_{W_2^{m-l/2}\to L_2} \leq c\,||\gamma||_{W_2^m\to W_2^l}.$$

By the Lemma in [MSh], Sec. 3.2.5 in the case $p = 2$, we have:

$$||\mathcal{D}_l\gamma||_{W_2^m\to L_2} + ||\gamma||_{W_2^{m-l}\to L_2} \leq c\,||\gamma||_{W_2^m\to W_2^l},$$

where $m \geq l > 0$. Applying the preceding estimate with $m - l/2$ in place of m and $l/2$ in place of l respectively, we get:

$$||\mathcal{D}_{l/2}\gamma||_{W_2^{m-l/2}\to L_2} + ||\gamma||_{W_2^{m-l}\to L_2} \leq c\,||\gamma||_{W_2^{m-l/2}\to W_2^{l/2}}.$$

Now by interpolation,

$$||\gamma||_{W_2^{m-l/2}\to W_2^{l/2}} \leq ||\gamma||^{1/2}_{W_2^{m-l}\to L_2}\,||\gamma||^{1/2}_{W_2^m\to W_2^l}.$$

Since $||\gamma||_{W_2^{m-l}\to L_2} \leq c\,||\gamma||_{W_2^m\to W_2^l}$, it follows that

$$||\gamma||_{W_2^{m-l/2}\to W_2^{l/2}} \leq c\,||\gamma||_{W_2^m\to W_2^l}.$$

Hence,

$$||\mathcal{D}_{l/2}\gamma||_{W_2^{m-l/2}\to L_2} \leq c\,||\gamma||_{W_2^{m-l/2}\to W_2^{l/2}} \leq c\,||\gamma||_{W_2^m\to W_2^l}.$$

Combining these estimates, we obtain:

$$\left\|u\cdot|D^l|\gamma\right\|_{L_2} \leq c\,||\gamma||_{W_2^m\to W_2^l}\,||u||_{W_2^m},$$

which is equivalent to the inequality

$$\left\||D^l|\gamma\right\|_{W_2^m\to L_2} \leq c\,||\gamma||_{W_2^m\to W_2^l}.$$

This, together with the inequality $||\gamma||_{W_2^{m-l}\to L_2} \le c\,||\gamma||_{W_2^m\to W_2^l}$ used above, completes the proof of (2.7).

We now prove the upper estimate

$$\|\gamma\|_{W_2^m\to W_2^l} \le c\left(\left\||D|^l\gamma\right\|_{W_2^m\to L_2} + ||\gamma||_{W_2^{m-l}\to L_2}\right). \tag{2.10}$$

By (2.9),

$$\left\||D|^l(\gamma u)\right\|_{L_2} \le \left\|\gamma|D|^l u\right\|_{L_2} + \left\||D|^l\gamma\cdot u\right\|_{L_2} + c\,||\mathcal{D}_{l/2}u\cdot\mathcal{D}_{l/2}\gamma||_{L_2}.$$

Using an elementary estimate $||u||_{W_2^{m-l}} \le c\,||u||_{W_2^m}$, we have:

$$\|\gamma u\|_{L_2} \le ||\gamma||_{W_2^{m-l}\to L_2}\,||u||_{W_2^{m-l}} \le c\,||\gamma||_{W_2^{m-l}\to L_2}\,||u||_{W_2^m}.$$

From these inequalities, combined with the estimate

$$||\mathcal{D}_{l/2}u\cdot\mathcal{D}_{l/2}\gamma||_{L_2} \le c\,||\gamma||_{W_2^{m-l/2}\to W_2^{l/2}}\,||u||_{W_2^m}$$

established above, it follows:

$$\begin{aligned}||\gamma u||_{W_2^l} \le\, & c\,(||\gamma||_{W_2^{m-l}\to L_2}\,||u||_{W_2^m} + \left\||D|^l\gamma\right\|_{W_2^m\to L_2}\,||u||_{W_2^m})\\ & +c\,||\gamma||_{W_2^{m-l/2}\to W_2^{l/2}}\,||u||_{W_2^m}.\end{aligned}$$

As above, by an interpolation argument,

$$||\gamma||_{W_2^{m-l/2}\to W_2^{l/2}} \le ||\gamma||^{1/2}_{W_2^{m-l}\to L_2}\,||\gamma||^{1/2}_{W_2^m\to W_2^l}.$$

Thus,

$$\|\gamma\|_{W_2^m\to W_2^l} \le c\left(\left\||D|^l\gamma\right\|_{W_2^m\to L_2} + ||\gamma||_{W_2^{m-l}\to L_2} + ||\gamma||^{1/2}_{W_2^{m-l}\to L_2}\,||\gamma||^{1/2}_{W_2^m\to W_2^l}\right).$$

Clearly, the preceding estimate yields:

$$\|\gamma\|_{W_2^m\to W_2^l} \le c\left(\left\||D|^l\gamma\right\|_{W_2^m\to L_2} + ||\gamma||_{W_2^{m-l}\to L_2}\right).$$

This completes the proof of Lemma 2.1. □

Lemma 2.2. *Let $0 < l < 1$, and $\frac{n}{2} \ge m > l$. Then $\gamma \in \mathrm{Mult}\,(W_2^m \to W_2^l)$ if and only if $(-\Delta+1)^{l/2}\gamma \in \mathrm{Mult}\,(W_2^m \to L_2)$, and*

$$||\gamma||_{W_2^m\to W_2^l} \sim ||(-\Delta+1)^{l/2}\gamma||_{W_2^m\to L_2}. \tag{2.11}$$

Proof. We denote by M the Hardy-Littlewood maximal operator:

$$Mf(x) = \sup_{r>0}\frac{1}{|B_r(x)|}\int_{B_r(x)}|f(y)|\,dy, \qquad x\in\mathbb{R}^n.$$

Recall that a nonnegative weight $w \in L_{1,\,loc}(\mathbb{R}^n)$ is said to be in the Muckenhoupt class $A_1(\mathbb{R}^n)$ if

$$Mw(x) \le \mathrm{const}\, w(x) \qquad \text{a.e.}$$

The least constant on the right-hand side of the preceding inequality is called the A_1-bound of w.

We will need the following statement established earlier in [MV1], Lemma 3.1 (see also [MSh], Sec. 2.6.5) for the homogeneous Sobolev spaces $\dot{W}_p^m(\mathbb{R}^n)$ defined as the completion of $\mathcal{D}(\mathbb{R}^n)$ with respect to the norm

$$||u||_{\dot{W}_p^m} = ||(-\Delta)^{m/2} u||_{L_p}, \qquad u \in \mathcal{D}(\mathbb{R}^n).$$

Lemma 2.3. *Let $\gamma \in \mathrm{Mult}\,(\dot{W}_p^m \to L_p)$, where $1 < p < \infty$, and $0 < m < \frac{n}{p}$. Suppose that T is a bounded operator on the weighted space $L_p(w)$ for every $w \in A_1(\mathbb{R}^n)$. Suppose additionally that, for all $f \in L_p(w)$, the inequality*

$$||Tf||_{L_p(w)} \le C\, ||f||_{L_p(w)}$$

holds with a constant C which depends only on the A_1-bound of the weight w. Then $T\gamma \in \mathrm{Mult}\,(\dot{W}_p^m \to L_p)$, and

$$||T\gamma||_{\dot{W}_p^m \to L_p} \le C_1 ||\gamma||_{\dot{W}_p^m \to L_p},$$

where the constant C_1 does not depend on γ.

We will also need a Fourier multiplier theorem of Mikhlin type for L_p spaces with weights. Let $m \in L_\infty(\mathbb{R}^n)$. Then the Fourier multiplier operator with symbol m is defined on $L_2(\mathbb{R}^n)$ by $T_m = \mathcal{F}^{-1}\, m\, \mathcal{F}$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are respectively the direct and inverse Fourier transforms.

The following lemma follows from the results of Kurtz and Wheeden [KWh], Theorem 1.

Lemma 2.4. *Suppose $1 < p < \infty$ and $w \in A_1(\mathbb{R}^n)$. Suppose that $m \in C^\infty(\mathbb{R}^n \setminus \{0\})$ satisfies the Mikhlin multiplier condition:*

$$|D^\alpha\, m(x)| \le C_\alpha\, |x|^{-|\alpha|}, \qquad x \in \mathbb{R}^n \setminus \{0\}, \tag{2.12}$$

for every multi-index α such that $0 \le |\alpha| \le n$. Then the inequality

$$||T_m\, f||_{L_p(w)} \le C\, ||f||_{L_p(w)}, \qquad f \in L_p(w) \cap L_2(\mathbb{R}^n),$$

holds with the constant that depends only on p, n, the A_1-bound of w, and the constant C_α in (2.12).

Corollary 2.5. *Suppose $1 < p < \infty$ and $w \in A_1(\mathbb{R}^n)$. Suppose $0 < l \le 2$. Define*

$$m_l(x) = (1 + |x|^2)^{l/2} - |x|^l. \tag{2.13}$$

Then

(2.14) $$||T_{m_l} f||_{L_p(w)} \le C \, ||f||_{L_p(w)}, \qquad f \in L_p(w) \cap L_2(\mathbb{R}^n),$$

where the constant C depends only on l, p, n, and the A_1-constant of w.

Remark. It is well known that in the unweighted case the operator $T_{n_l} = (1-\Delta)^{-l/2} \, T_{m_l}$, is bounded on $L_p(\mathbb{R}^n)$ for all $l > 0$ and $1 \le p \le \infty$, including the endpoints ([St1], Sec. 5.3.2, Lemma 2).

Proof of Corollary 2.5. Clearly,

$$0 \le m_l(x) \le C \, (1 + |x|)^{l-2}, \qquad x \in \mathbb{R}^n.$$

Furthermore, it is easy to see by induction that, for any multi-index α, $|\alpha| \ge 1$, we have the following estimates:

$$|D^\alpha \, m_l(x)| \le C_{\alpha,l} \, |x|^{l-2-|\alpha|}, \qquad |x| \to \infty,$$

and

$$|D^\alpha \, m_l(x)| \le C_{\alpha,l} \, |x|^{l-|\alpha|}, \qquad |x| \to 0.$$

Since $0 < l \le 2$, from this it follows that m_l satisfies (2.12), and hence by Lemma 2.4 the inequality

$$||T_{m_l} f||_{L_p(w)} \le C \, ||f||_{L_p(w)}$$

holds with a constant that depends only on l, p, and the A_1-bound of w. □

We are now in a position to complete the proof of Lemma 2.2. Suppose that $\gamma \in \text{Mult}\,(W_2^m \to W_2^l)$, where $\frac{n}{2} \ge m > l$ and $0 < l < 1$. By Corollary 2.5, the operator $T_{m_l} = (1-\Delta)^{l/2} - |D|^l$ is bounded on $L_2(w)$ for every $w \in A_1$, and its norm is bounded by a constant which depends only on l, n, and the A_1-bound of w. Hence by Lemma 2.3 it follows that $\gamma \in \text{Mult}\,(\dot{W}_2^m \to L_2)$ yields $T_{m_l}\gamma = \left((1-\Delta)^{l/2} - |D|^l\right) \gamma \in \text{Mult}\,(\dot{W}_2^m \to L_2)$, and

$$||T_{m_l}\gamma||_{\dot{W}_2^m \to L_2} \le c \, ||\gamma||_{\dot{W}_2^m \to L_2},$$

where c depends only on l, m, and n.

We need to replace $\dot{W}_2^m$ in the preceding inequality by W_2^m. To this end, let $B = B_1(x_0)$ denote a ball of radius 1 in $\mathbb{R}^n$, and $2B = B_2(x_0)$. Suppose that $m < \frac{n}{2}$ (the case $m = \frac{n}{2}$ requires usual modifications). Then $\gamma \in \text{Mult}\,(W_2^m \to L_2)$ if and only if $\sup_B \, ||\chi_B \, \gamma||_{\dot{W}_2^m \to L_2} < +\infty$, and (see [MSh], Sec. 1.1.4): $||\gamma||_{W_2^m \to L_2}$ is equivalent to $\sup_B \, ||\chi_B \, \gamma||_{\dot{W}_2^m \to L_2}$.

Hence,

$$\|T_{m_l}\gamma\|_{W_2^m \to L_2} \le c \, \sup_B \, \|\chi_B \, T_{m_l}\gamma\|_{\dot{W}_2^m \to L_2}.$$

We set $\gamma = \chi_{2B}\,\gamma + \chi_{(2B)^c}\,\gamma$, and estimate each term separately. By Lemma 2.3,

$$\|\chi_B\, T_{m_l}(\chi_{2B}\,\gamma)\|_{\dot W_2^m \to L_2} \le c \sup_B \|\chi_{2B}\,\gamma\|_{\dot W_2^m \to L_2} \le c\,\|\gamma\|_{W_2^m \to L_2}.$$

To estimate the second term, notice that $T_{m_l}(\chi_{(2B)^c}\gamma) \in L_\infty(B)$, and hence

$$\|\chi_B\, T_{m_l}(\chi_{(2B)^c}\gamma)\|_{\dot W_2^m \to L_2} \le c\,||T_{m_l}(\chi_{(2B)^c}\gamma)(x)||_{L_\infty(B)} \le c\,\|\gamma\|_{W_2^m \to L_2}.$$

Indeed, for $x \in B$,

$$|T_{m_l}(\chi_{(2B)^c}\gamma)(x)| \le c \int_{|x-y|\ge 1} \frac{|\gamma(y)|}{|x-y|^{n+l}}\, dy \le c \int_1^{+\infty} \frac{\int_{B_r(x)} |\gamma(y)| dy}{r^{n+l+1}} dr.$$

Since $\gamma \in \text{Mult}\,(W_2^m \to L_2)$, it follows that $\gamma \in L_{2,unif}$, and hence

$$\int_{B_r(x)} |\gamma(y)|^2\, dy \le c\, r^n\, ||\gamma||^2_{W_2^m \to L_2}, \qquad r \ge 1.$$

Consequently,

$$\int_{B_r(x)} |\gamma(y)|\, dy \le c\, r^{n/2} ||\gamma||_{L_2(B_r(x))} \le c\; r^n\, ||\gamma||_{W_2^m \to L_2}, \qquad r \ge 1.$$

Hence,

$$||T_{m_l}(\chi_{(2B)^c}\gamma)(x)||_{L_\infty(B)} \le c\|\gamma\|_{W_2^m \to L_2}.$$

Thus, we have proved the inequality:

$$\|\left((1-\Delta)^{l/2} - |D|^l\right)\gamma\|_{W_2^m \to L_2} \le c\,\|\gamma\|_{W_2^m \to L_2}.$$

Clearly, $||\gamma||_{W_2^m \to L_2} \le ||\gamma||_{W_2^m \to W_2^l}$. Using these estimates and Lemma 2.1, we obtain:

$$||(1-\Delta)^{l/2}\,\gamma||_{W_2^m \to L_2} \le c\,\left(\left\||D|^l\gamma\right\|_{W_2^m \to L_2} + ||\gamma||_{W_2^m \to L_2}\right) \le c\,||\gamma||_{W_2^m \to W_2^l}.$$

Conversely, suppose that $(1-\Delta)^{l/2}\,\gamma \in \text{Mult}\,(W_2^m \to L_2)$. It follows from the above estimate of $\|\left((1-\Delta)^{l/2} - |D|^l\right)\gamma\|_{W_2^m \to L_2}$ that

$$\left\||D|^l\gamma\right\|_{W_2^m \to L_2} \le c\,\left(||(1-\Delta)^{l/2}\,\gamma||_{W_2^m \to L_2} + ||\gamma||_{W_2^m \to L_2}\right).$$

Obviously, $||\gamma||_{W_2^m \to L_2} \le c\,||\gamma||_{W_2^{m-l} \to L_2}$. Applying again Lemma 2.1 together with the preceding estimates, we have:

$$\begin{aligned} ||\gamma||_{W_2^m \to W_2^l} &\le c\,\left(\left\|\,|D|^l\gamma\right\|_{W_2^m \to L_2} + ||\gamma||_{W_2^{m-l} \to L_2}\right) \\ &\le c\,\left(||(1-\Delta)^{l/2}\,\gamma||_{W_2^m \to L_2} + ||\gamma||_{W_2^{m-l} \to L_2}\right). \end{aligned}$$

It remains to obtain the estimate

$$||\gamma||_{W_2^{m-l} \to L_2} \le c\,||(1-\Delta)^{l/2}\,\gamma||_{W_2^m \to L_2},$$

whose proof is similar to the argument used in [MSh], Sec. 2.6, and is outlined below.

Since $(1-\Delta)^{l/2}\,\gamma \in \text{Mult}\,(W_2^m \to L_2)$, it follows that

$$\int_e |(1-\Delta)^{l/2}\,\gamma|^2\,dx \le \;||(1-\Delta)^{l/2}\,\gamma||^2_{W_2^m\to L_2}\,\text{cap}\,(e,\,W_2^m),$$

for every compact set $e \subset \mathbb{R}^n$. Hence, for every ball $B_r(a)$,

$$\int_{B_r(a)} |(1-\Delta)^{l/2}\,\gamma|^2\,dx \le c\,||(1-\Delta)^{l/2}\,\gamma||^2_{W_2^m\to L_2}\,r^{n-2m}, \quad 0<r\le 1,$$

and in particular

$$||(1-\Delta)^{l/2}\,\gamma||_{L_2,unif} \le c\,||(1-\Delta)^{l/2}\,\gamma||_{W_2^m\to L_2}.$$

Notice that $\gamma = J_l\,(1-\Delta)^{l/2}\,\gamma$, where the Bessel potential $J_l = (1-\Delta)^{-l/2}$ can be represented as a convolution operator, $J_l f = G_l\star f$. Here G_l is a positive radially decreasing function whose behavior at 0 and infinity respectively is given by

$$G_l(x) \asymp |x|^{l-n} \quad \text{as} \quad x\to 0, \qquad \text{if} \quad 0<l<n,$$
$$G_l(x) \asymp |x|^{(l-n-1)/2}\,e^{-|x|} \quad \text{as} \quad |x|\to+\infty.$$

From this, it is easy to derive the pointwise estimate

$$|\gamma(x)| \le \int_{\mathbb{R}^n} G_l(x-t)\,|(1-\Delta)^{l/2}\,\gamma(t)|\,dt$$
$$\le c\left(\int_{|z|\le 1}\frac{|(1-\Delta)^{l/2}\,\gamma(x+z)|}{|z|^{n-l}}\,dz + ||(1-\Delta)^{l/2}\,\gamma||_{L_2,unif}\right).$$

Using Hedberg's inequality together with the preceding pointwise estimate, as in the proof of Lemma 2.6.2 in [MSh], we deduce:

$$|\gamma(x)| \le c\,(M\,(1-\Delta)^{l/2}\,\gamma(x))^{1-\frac{l}{m}}\left(\sup_{0<r\le 1,\,a\in\mathbb{R}^n}\frac{\int_{B_r(a)}|(1-\Delta)^{l/2}\,\gamma|^2\,dy}{r^{n-2m}}\right)^{\frac{l}{2m}}$$
$$+\,c\,||(1-\Delta)^{l/2}\gamma||_{L_2,unif} \le c\,(M\,(1-\Delta)^{l/2}\,\gamma(x))^{1-\frac{l}{m}}||(1-\Delta)^{l/2}\gamma||^{\frac{l}{m}}_{W_2^m\to L_2}$$
$$+\,c\,||(1-\Delta)^{l/2}\gamma||_{W_2^m\to L_2},$$

where M is the Hardy-Littlewood maximal operator. Using the preceding estimates, together with the boundedness of M on the space $\text{Mult}\,(W_2^m\to L_2)$ (see [MSh], Sec. 2.6) we obtain:

$$\left\|\,|\gamma|^{\frac{m}{m-l}}\right\|^{1-\frac{l}{m}}_{W_2^m\to L_2} \le c\,||(1-\Delta)^{l/2}\,\gamma||_{W_2^m\to L_2}.$$

By Lemma 2 in [MSh], Sec. 2.2.1, it follows:

$$\|\gamma\|_{W_2^{m-l}\to L_2} \le c\, \left\| |\gamma|^{\frac{m}{m-l}} \right\|^{1-\frac{l}{m}}_{W_2^m \to L_2} \le c\, ||(1-\Delta)^{l/2}\, \gamma||_{W_2^m\to L_2}.$$

The proof of Lemma 2.2 is complete. □

Theorem 2.6. *Let* $\gamma \in \mathcal{D}'(\mathbb{R}^n)$*. Then* $\gamma \in \text{Mult}\,(W_2^{1/2}(\mathbb{R}^n) \to W_2^{-1/2}(\mathbb{R}^n))$ *if and only if* $\Phi = (-\Delta+1)^{-1/4}\gamma \in \text{Mult}\,(W_2^{1/2}(\mathbb{R}^n) \to L_2(\mathbb{R}^n))$*. Furthermore,*

$$||\gamma||_{W_2^{1/2}\to W_2^{-1/2}} \sim ||\Phi||_{W_2^{1/2}\to L_2}.$$

Proof. To prove the "if" part, it suffices to verify that, for every $u \in C_0^\infty(\mathbb{R}^n)$ and $\Phi = (-\Delta+1)^{-1/4}\gamma \in \text{Mult}\,(W_2^{1/2} \to L_2)$, the inequality

$$\left| \int_{\mathbb{R}^n} |u|^2 \gamma \right| \le C\, ||\Phi||_{W_2^{1/2}\to L_2}\, ||u||^2_{W_2^{1/2}} \tag{2.15}$$

holds. Here the integral on the left-hand side is understood in the sense of quadratic forms:

$$\int_{\mathbb{R}^n} |u|^2 \gamma = \langle\, \gamma u,\, u\rangle,$$

where $\langle\, \gamma\cdot,\, \cdot\rangle$ is the quadratic form associated with the multiplier operator γ, as explained in detail in [MV2].

Since $\gamma = (-\Delta+1)^{1/4}\Phi$, we have:

$$\begin{aligned}\left| \int_{\mathbb{R}^n} |u|^2\, \gamma \right| &= \left| \int_{\mathbb{R}^n} (-\Delta+1)^{1/4}\Phi \cdot\, |u|^2 \right| \\ &\le \left| \int_{\mathbb{R}^n} \left((-\Delta+1)^{1/4} - |D|^{1/2}\right) \Phi \cdot\, |u|^2 \right| + \left| \int_{\mathbb{R}^n} |D|^{1/2}\, \Phi \cdot\, |u|^2 \right|.\end{aligned}$$

Note that $(-\Delta+1)^{1/4} - |D|^{1/2} = T_{m_{1/2}}$, where T_{m_l} is the Fourier multiplier operator defined by (2.13). By Corollary 2.5, $T_{m_{1/2}}$ is a bounded operator on $L_2(w)$ for any A_1-weight w, and its norm depends only on the A_1-bound of w. Hence by Lemma 2.3 it follows that $\left((-\Delta+1)^{1/4} - |D|^{1/2}\right) \Phi \in \text{Mult}\,(W_2^{1/2} \to L_2)$, and

$$||\left((-\Delta+1)^{1/4} - |D|^{1/2}\right) \Phi||_{W_2^{1/2}\to L_2} \le C\, ||\Phi||_{W_2^{1/2}\to L_2}.$$

Using this estimate and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\left| \int_{\mathbb{R}^n} \left((-\Delta+1)^{1/4} - |D|^{1/2}\right) \Phi \cdot\, |u|^2 \right| \\ &\le C\, ||((-\Delta+1)^{1/4} - |D|^{1/2})\Phi \cdot u||_{L_2}\, ||u||_{L_2}\end{aligned}$$

$$\leq C\,||\Phi||_{W_2^{1/2}\to L_2}||u||^2_{W_2^{1/2}}.$$

Hence, in order to prove (2.15) it suffices to establish the inequality:

$$\left|\int_{\mathbb{R}^n}|D|^{1/2}\,\Phi\cdot\,|u|^2\right|\leq C\,||\Phi||_{W_2^{1/2}\to L_2}\,||u||^2_{W_2^{1/2}}. \tag{2.16}$$

By duality,

$$\left|\int_{\mathbb{R}^n}|D|^{1/2}\,\Phi\cdot\,|u|^2\right| = \left|\int_{\mathbb{R}^n}\Phi(x)\,(|D|^{1/2}\,|u|^2)(x)\,dx\right|,$$

where $\Phi\in L_{2,loc}$, and the integral on the right-hand side is well-defined (see details in [MV2]).

Notice that, for $u\in C_0^\infty(\mathbb{R}^n)$,

$$|D|^{1/2}\,|u|^2(x) = c\int_{\mathbb{R}^n}\frac{|u(x)|^2-|u(y)|^2}{|x-y|^{n+1/2}}\,dy.$$

Using the identity $|a|^2-|b|^2 = |a-b|^2 - 2\mathrm{Re}\,[\bar{b}\,(b-a)]$ with $b=u(x)$ and $a=u(y)$, and integrating against $\frac{dy}{|x-y|^{n+1/2}}$, we get:

$$\int_{\mathbb{R}^n}\frac{|u(x)|^2-|u(y)|^2}{|x-y|^{n+1/2}}\,dy = \int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+1/2}}\,dy - 2\,\mathrm{Re}\left[\overline{u(x)}\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+1/2}}\,dy\right].$$

Hence,

$$\left|\,|D|^{1/2}\,|u|^2(x)\right|\leq c\left(2\,|u(x)|\left|\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+1/2}}\,dy\right|+\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+1/2}}\,dy\right) = 2c\,|u(x)|\,\left|\,|D|^{1/2}\,u(x)\right| + c\,|\mathcal{D}_{1/4}u(x)|^2.$$

Using the preceding inequality, we estimate:

$$\left|\int_{\mathbb{R}^n}\Phi\,|D|^{1/2}\,|u|^2\,dx\right| \leq c\,||\Phi\,u||_{L_2}\,\left\||D|^{1/2}\,u\right\|_{L_2} + c\int_{\mathbb{R}^n}|\Phi|\,|\mathcal{D}_{1/4}u|^2\,dx \leq c\,||\Phi||_{W_2^{1/2}\to L_2}\,||u||^2_{W_2^{1/2}} + c\int_{\mathbb{R}^n}|\Phi|\,|\mathcal{D}_{1/4}J_{1/2}f|^2\,dx,$$

where $f=(-1+\Delta)^{1/4}\,u$. The last integral is bounded by:

$$\int_{\mathbb{R}^n}|\Phi|\,|J_{1/4}\,\mathcal{D}_{1/4}\,J_{1/4}\,f|^2\,dx \leq c\int_{\mathbb{R}^n}|\Phi|\,M(\mathcal{D}_{1/4}\,J_{1/4}\,f)\,|J_{1/2}\,\mathcal{D}_{1/4}\,J_{1/4}\,f|\,dx$$

$$\leq c\,||M(\mathcal{D}_{1/4}\, J_{1/4}\, f)||_{L_2}\, ||\Phi\, J_{1/2}\, \mathcal{D}_{1/4}\, J_{1/4}\, f||_{L_2}$$
$$\leq c\,||\mathcal{D}_{1/4}\, J_{1/4}\, f||_{L_2}\, ||\Phi||_{W_2^{1/2}\to L_2}\, ||J_{1/2}\, \mathcal{D}_{1/4}\, J_{1/4}\, f||_{W_2^{1/2}}$$
$$\leq c\,||\Phi||_{W_2^{1/2}\to L_2}\, ||f||^2_{L_2} = c\,||\Phi||_{W_2^{1/2}\to L_2}\, ||u||^2_{W_2^{1/2}}.$$

In the preceding chain of inequalities we first applied Hedberg's inequality (see, e.g., [MSh], Sec. 1.1.3 and Sec. 3.1.2):

$$J_{1/4}\, g \leq c\,(Mg)^{1/2}\,(J_{1/2}\, g)^{1/2},$$

with $g = |\mathcal{D}_{1/4}\, J_{1/4}\, f|$, and then the Hardy-Littlewood maximal inequality for the operator M. This completes the proof of (2.15).

To prove the "only if" part of the Theorem, we will show that

$$||\Phi||_{W_2^{1/2}(\mathbb{R}^n)\to L_2(\mathbb{R}^n)} \leq c\,||\gamma||_{W_2^{1/2}(\mathbb{R}^n)\to W_2^{-1/2}(\mathbb{R}^n)}.$$

The proof of this estimate is based on the extension of the distribution $\gamma \in \text{Mult}\,(W_2^{1/2}(\mathbb{R}^n) \to W_2^{-1/2}(\mathbb{R}^n))$ to the higher dimensional Euclidean space, and subsequent application of the characterization of the class of multipliers $\text{Mult}\,(W_2^{1}(\mathbb{R}^{n+1}) \to W_2^{-1}(\mathbb{R}^{n+1}))$ obtained by the authors in [MV2].

We denote by $\gamma\otimes\delta$ the distribution on $\mathbb{R}^{n+1}$ defined by

$$\langle \gamma\otimes\delta,\, u(x, x_{n+1})\rangle = \langle \gamma,\, u(x,0)\rangle,$$

where $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, and $\delta = \delta(x_{n+1})$ is the delta-function supported on $x_{n+1} = 0$. It is not difficult to see that

$$||\gamma\otimes\delta||_{W_2^1(\mathbb{R}^{n+1})\to W_2^{-1}(\mathbb{R}^{n+1})} \sim ||\gamma||_{W_2^{1/2}(\mathbb{R}^n)\to W_2^{-1/2}(\mathbb{R}^n)}.$$

This follows from the well-known fact that the space of traces on $\mathbb{R}^n$ of functions in $W_2^1(\mathbb{R}^{n+1})$ coincides with $W_2^{1/2}(\mathbb{R}^n)$, with the equivalence of norms (see, e.g., [MSh], Sec. 5.1). Indeed, for any $U, V \in C_0^\infty(\mathbb{R}^{n+1})$ let $u(x) = U(x,0)$ and $v(x) = V(x,0)$. Then by the trace estimate mentioned above $||u||_{W_2^{1/2}(\mathbb{R}^n)} \leq c\,||U||_{W_2^1(\mathbb{R}^{n+1})}$, and hence

$$|\langle \gamma\otimes\delta,\, \overline{U}\, V\rangle| = |\langle \gamma,\, \overline{u}\, v\rangle| \leq ||\gamma||_{W_2^{1/2}(\mathbb{R}^n)\to W_2^{-1/2}(\mathbb{R}^n)} ||u||_{W_2^{1/2}(\mathbb{R}^n)}\, ||v||_{W_2^{1/2}(\mathbb{R}^n)}$$
$$\leq c^2\, ||\gamma||_{W_2^{1/2}(\mathbb{R}^n)\to W_2^{-1/2}(\mathbb{R}^n)}\, ||U||_{W_2^1(\mathbb{R}^{n+1})}\, ||V||_{W_2^1(\mathbb{R}^{n+1})}.$$

This gives the estimate:

$$||\gamma\otimes\delta||_{W_2^1(\mathbb{R}^{n+1})\to W_2^{-1}(\mathbb{R}^{n+1})} \leq c^2\, ||\gamma||_{W_2^{1/2}(\mathbb{R}^n)\to W_2^{-1/2}(\mathbb{R}^n)}.$$

The converse inequality (which is not used below) follows similarly by extending $u, v \in C_0^\infty(\mathbb{R}^n)$ to $U, V \in W_2^1(\mathbb{R}^{n+1})$ with the corresponding estimates of norms.

For the rest of the proof, it will be convenient to introduce the notation $J_s^{(n+1)} = (-\Delta_{n+1} + 1)^{-s/2}$, $s > 0$, for the Bessel potential of order s on $\mathbb{R}^{n+1}$; here Δ_{n+1} denotes the Laplacian on $\mathbb{R}^{n+1}$.

Now by Theorem 4.2, [MV2] we obtain that $\gamma \otimes \delta \in \text{Mult}\,(W_2^1(\mathbb{R}^{n+1}) \to W_2^{-1}(\mathbb{R}^{n+1}))$ if and only if $J_1^{(n+1)}(\gamma \otimes \delta) \in \text{Mult}\,(W_2^1(\mathbb{R}^{n+1}) \to L_2(\mathbb{R}^{n+1}))$, and

$$\begin{aligned} ||J_1^{(n+1)}(\gamma \otimes \delta)||_{W_2^1(\mathbb{R}^{n+1}) \to L_2(\mathbb{R}^{n+1})} &\le c\,||\gamma \otimes \delta||_{W_2^1(\mathbb{R}^{n+1}) \to W_2^{-1}(\mathbb{R}^{n+1})} \\ &\le c_1\,||\gamma||_{W_2^{1/2}(\mathbb{R}^n) \to W_2^{-1/2}(\mathbb{R}^n)}. \end{aligned}$$

Next, pick $0 < \epsilon < 1/2$ and observe that $J_1^{(n+1)} = (-1 + \Delta_{n+1})^{1/4+\epsilon/2}\, J_{\epsilon+3/2}^{(n+1)}$. Using Lemma 2.2 with $l = 1/2 + \epsilon$, $m = 1$, and $J_{\epsilon+3/2}^{(n+1)}(\gamma \otimes \delta)$ in place of γ, we deduce:

$$||J_1^{(n+1)}(\gamma \otimes \delta)||_{W_2^1(\mathbb{R}^{n+1}) \to L_2(\mathbb{R}^{n+1})} \sim ||J_{\epsilon+3/2}^{(n+1)}(\gamma \otimes \delta)||_{W_2^1(\mathbb{R}^{n+1}) \to W_2^{1/2+\epsilon}(\mathbb{R}^{n+1})}.$$

As was proved above, the left-hand side of the preceding relation is bounded by a constant multiple of $||\gamma||_{W_2^{1/2}(\mathbb{R}^n) \to W_2^{-1/2}(\mathbb{R}^n)}$.

Thus,

$$||J_{\epsilon+3/2}^{(n+1)}(\gamma \otimes \delta)||_{W_2^1(\mathbb{R}^{n+1}) \to W_2^{1/2+\epsilon}(\mathbb{R}^{n+1})} \le c\,||\gamma||_{W_2^{1/2}(\mathbb{R}^n) \to W_2^{-1/2}(\mathbb{R}^n)}.$$

Passing to the trace on $\mathbb{R}^n = \{x_{n+1} = 0\}$ in the multiplier norm on the left-hand side (see [MSh], Sec. 5.2), we obtain:

$$||\text{Trace}\, J_{\epsilon+3/2}^{(n+1)}(\gamma \otimes \delta)||_{W_2^{1/2}(\mathbb{R}^n) \to W_2^{\epsilon}(\mathbb{R}^n)} \le c\,||\gamma||_{W_2^{1/2}(\mathbb{R}^n) \to W_2^{-1/2}(\mathbb{R}^n)}.$$

We now observe that

$$\text{Trace}\, J_{\epsilon+3/2}^{(n+1)}(\gamma \otimes \delta) = \text{const}\, J_{\epsilon+1/2}^{(n)}(\gamma),$$

which follows immediately by inspecting the corresponding Fourier transforms.

In other words,

$$||J_{\epsilon+1/2}^{(n)}\,\gamma||_{W_2^{1/2}(\mathbb{R}^n) \to W_2^{\epsilon}(\mathbb{R}^n)} \le c\,||\gamma||_{W_2^{1/2}(\mathbb{R}^n) \to W_2^{-1/2}(\mathbb{R}^n)}. \tag{2.17}$$

From this estimate and Lemma 2.2 with $l = \epsilon$, $m = 1/2$, and with γ replaced by $J_{\epsilon+1/2}^{(n)}\gamma$, it follows:

$$\begin{aligned} ||J_{1/2}^{(n)}\,\gamma||_{W_2^{1/2}(\mathbb{R}^n) \to L_2(\mathbb{R}^n)} &= ||(-\Delta + 1)^{\epsilon/2} J_{\epsilon+1/2}^{(n)}\gamma||_{W_2^{1/2}(\mathbb{R}^n) \to L_2(\mathbb{R}^n)} \\ \le c\,||J_{\epsilon+1/2}^{(n)}\gamma||_{W_2^{1/2}(\mathbb{R}^n) \to W_2^{\epsilon}(\mathbb{R}^n)} &\le C\,||\gamma||_{W_2^{1/2}(\mathbb{R}^n) \to W_2^{-1/2}(\mathbb{R}^n)}. \end{aligned}$$

Thus, $\Phi = J_{1/2}^{(n)}\,\gamma \in \text{Mult}\,(W_2^{1/2}(\mathbb{R}^n) \to L_2(\mathbb{R}^n))$, and

$$||\Phi||_{W_2^{1/2}(\mathbb{R}^n) \to L_2(\mathbb{R}^n)} \le C\,||\gamma||_{W_2^{1/2}(\mathbb{R}^n) \to W_2^{-1/2}(\mathbb{R}^n)}.$$

The proof of Theorem 2.6 is complete. □

3. Some corollaries of the form boundedness criterion

Theorem 2.6 proved in Sec. 2, combined with the known criteria for nonnegative potentials, yields Theorem II stated in the Introduction. In particular, it follows that, if $Q \in \mathcal{D}'(\mathbb{R}^n)$, and $\Phi = (-\Delta + 1)^{-1/4} Q$, then the multiplier defined by Q, and hence $\mathcal{H} = \sqrt{-\Delta} + Q$, is a bounded operator from $W_2^{1/2}(\mathbb{R}^n)$ to $W_2^{-1/2}(\mathbb{R}^n)$ if and only if

$$\int_e |\Phi(x)|^2 \, dx \le c \, \text{cap}\,(e, W_2^{1/2}(\mathbb{R}^n)), \tag{3.1}$$

for every compact set $e \subset \mathbb{R}^n$ such that $\text{diam}\,(e) \le 1$.

Some simpler conditions which do not involve capacities are discussed in this section.

The following *necessary* condition is immediate from (3.1) and the known estimates of the capacity of the ball in $\mathbb{R}^n$ ([MSh], Sec. 2.1.2).

Corollary 3.1. *Suppose $Q \in \mathcal{D}'(\mathbb{R}^n)$, $n \ge 1$. Suppose $\mathcal{H} = \sqrt{-\Delta} + Q : W_2^{1/2}(\mathbb{R}^n) \to W_2^{-1/2}(\mathbb{R}^n)$ is a bounded operator. Then, for every ball $B_r(a)$ in $\mathbb{R}^n$,*

$$\int_{B_r(a)} |\Phi(x)|^2 \, dx \le c \, r^{n-1}, \quad 0 < r \le 1, \qquad n \ge 2, \tag{3.2}$$

and

$$\int_{B_r(a)} |\Phi(x)|^2 \, dx \le \frac{c}{\log \frac{2}{r}}, \quad 0 < r \le 1, \qquad n = 1, \tag{3.3}$$

where the constant does not depend on $a \in \mathbb{R}^n$ and r.

We notice that the class of distributions Q such that $\Phi = (-\Delta + 1)^{-1/4} Q$ satisfies (3.2) can be regarded as a Morrey space of order $-1/2$.

Combining Theorem II with the Fefferman-Phong condition ([Fef]) applied to $|\Phi|^2$, we arrive at *sufficient* conditions in terms of Morrey spaces of negative order. (Strictly speaking, the Fefferman-Phong condition [Fef] was originally established for estimates in the homogeneous Sobolev space $\dot{W}_2^1$ of order $m = 1$. However, it can be carried over to Sobolev spaces W_2^m for all $0 < m \le n/2$. See, e.g., [KeS] or [MV1], p. 98.)

Corollary 3.2. *Suppose $Q \in \mathcal{D}'(\mathbb{R}^n)$, $n \ge 2$. Suppose $\Phi = (-\Delta + 1)^{-1/4} Q$, and $s > 1$. Then $\mathcal{H}$ is a bounded operator from $W_2^{1/2}(\mathbb{R}^n)$ to $W_2^{-1/2}(\mathbb{R}^n)$ if*

$$\int_{B_r(a)} |\Phi(x)|^{2s} \, dx \le \text{const} \, r^{n-s}, \quad 0 < r \le 1, \tag{3.4}$$

where the constant does not depend on $a \in \mathbb{R}^n$ and r.

Remark. It is worth mentioning that condition (3.4) defines a class of potentials which is strictly broader than the (relativistic) Fefferman-Phong class of Q such that

$$\int_{B_r(a)} |Q(x)|^s \, dx \le \text{const}\, r^{n-s}, \quad 0 < r \le 1, \qquad n \ge 2, \tag{3.5}$$

for some $s > 1$.

This follows from the observation that if one replaces Q by $|Q|$ in (3.4), then obviously the resulting class defined by:

$$\int_{B_r(a)} (J_{1/2}|Q|)^{2s} \, dx \le \text{const}\, r^{n-s}, \quad 0 < r \le 1, \qquad n \ge 2, \tag{3.6}$$

becomes smaller, but still contains some singular measures, together with all functions in the Fefferman-Phong class (3.5). (The latter was noticed earlier in [MV1], Proposition 3.5.)

A smaller but more conventional class of admissible potentials appears when one replaces $\text{cap}\,(e, W_2^{1/2}(\mathbb{R}^n))$ on the right-hand side of (3.1) by its lower estimate in terms of Lebesgue measure of $e \subset \mathbb{R}^n$. This yields the following result (stated as Corollary 1 in the Introduction).

Corollary 3.3. *Suppose $Q \in \mathcal{D}'(\mathbb{R}^n)$, $n \ge 1$. Suppose $\Phi = (-\Delta + 1)^{-1/4} Q$. Then $\mathcal{H} = \sqrt{-\Delta} + Q$ is a bounded operator from $W_2^{1/2}(\mathbb{R}^n)$ to $W_2^{-1/2}(\mathbb{R}^n)$ if, for every measurable set $e \subset \mathbb{R}^n$,*

$$\int_e |\Phi(x)|^2 \, dx \le c\, |e|^{(n-1)/n}, \quad \text{diam}\,(e) \le 1, \qquad n \ge 2, \tag{3.7}$$

or

$$\int_e |\Phi(x)|^2 \, dx \le \frac{c}{\log \frac{2}{|e|}}, \quad \text{diam}\,(e) \le 1, \qquad n = 1, \tag{3.8}$$

where the constant c does not depend on e.

We remark that (3.7), without the extra assumption $\text{diam}\,(e) \le 1$, is equivalent to $\Phi \in L_{2n,\infty}(\mathbb{R}^n)$, where $L_{p,\infty}(\mathbb{R}^n)$ is the Lorentz (weak L_p) space of functions f such that

$$|\{x \in \mathbb{R}^n : \, |f(x)| > t\}| \le \frac{C}{t^p}, \qquad t > 0.$$

In particular, (3.7) holds if $\Phi \in L_{2n}(\mathbb{R}^n)$, or equivalently, $Q \in W_{2n}^{-1/2}(\mathbb{R}^n)$.

Furthermore, if $\Phi \in L_\infty(\mathbb{R}^n)$, then obviously (3.7) holds as well, since

$$\mathrm{cap}\,(e, W_2^{1/2}(\mathbb{R}^n)) \geq C\,|e|,$$

if $\mathrm{diam}\,(e) \leq 1$. This leads to the sufficient condition $\Phi \in L_{2n}(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$, $n \geq 2$.

It is worth noting that (3.7) defines a substantially broader class of admissible potentials than the standard (in the relativistic case) class $Q \in L_n(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$, $n \geq 2$ ([LL], Sec. 11.3). This is a consequence of the imbedding:

$$L_n(\mathbb{R}^n) \subset W_{2n}^{-1/2}(\mathbb{R}^n), \qquad n \geq 2,$$

which follows from the classical Sobolev imbedding $W_p^{1/2}(\mathbb{R}^n) \subset L_r(\mathbb{R}^n)$, for $p = 2n/(2n-1)$ and $r = n/(n-1)$, $n \geq 2$. Indeed, by duality, the latter is equivalent to:

$$L_n(\mathbb{R}^n) = L_r(\mathbb{R}^n)^* \subset W_p^{1/2}(\mathbb{R}^n)^* = W_{2n}^{-1/2}(\mathbb{R}^n).$$

Similarly, in the one-dimensional case, the class of potentials defined by (3.8) is wider than the standard class $L_{1+\epsilon}(\mathbb{R}^1) + L_\infty(\mathbb{R}^1)$, $\epsilon > 0$.

It is easy to see that actually $Q \in L_n(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$ if $n \geq 2$, or $Q \in L_{1+\epsilon}(\mathbb{R}^1) + L_\infty(\mathbb{R}^1)$ if $n = 1$, is sufficient for the inequality

$$\int_{\mathbb{R}^n} |u(x)|^2\,|Q(x)|\,dx \leq \mathrm{const}\,||u||^2_{W_2^{1/2}}, \quad u \in C_0^\infty(\mathbb{R}^n),$$

which is a "naïve" version of (1.2) where Q is replaced by $|Q|$.

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