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Approximate Approximations with Data on a Perturbed Uniform Grid

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Abstract. The aim of this paper is to extend the approximate quasi-interpolation on a uniform grid by dilated shifts of a smooth and rapidly decaying function to the case that the data are given on a perturbed uniform grid. It is shown that high order approximation of smooth functions up to some prescribed accuracy is possible.

Keywords. scattered data quasi-interpolation, multivariate approximation, error estimates

Mathematics Subject Classification (2000). Primary 41A30, secondary 65D15, 41A63, 41A25

1. Introduction

The approximation of multivariate functions from scattered data is an important theme in numerical mathematics. One of the methods to attack this problem is quasi-interpolation. One takes values $u(\mathbf{x}_j)$ of a function u on a set of nodes $\{\mathbf{x}_j\}_{j\in J}$ and constructs an approximant of u by linear combinations

$$\sum_{j\in J} u(\mathbf{x}_j)\eta_j(\mathbf{x})\,,$$

where $\eta_j(\mathbf{x})$ is a set of basis functions. Using quasi-interpolation there is no need to solve large algebraic systems. The approximation properties of quasi-interpolants in the case that $\mathbf{x}_j$ are the nodes of a uniform grid are well-understood. For example, the quasi-interpolant

$$\sum_{\mathbf{j}\in\mathbb{Z}^n} u(h\mathbf{j})\varphi\Big(\frac{\mathbf{x}-h\mathbf{j}}{h}\Big) \tag{1.1}$$

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can be studied via the theory of principal shift-invariant spaces, which has been developed in several articles by de Boor, DeVore and Ron (see *e.g.* [2, 3]). Here φ is supposed to be a compactly supported or rapidly decaying function. Based on the Strang-Fix condition for φ, which is equivalent to polynomial reproduction, convergence and approximation orders for several classes of basis functions were obtained (see also Schaback/Wu [17], Jetter/Zhou [6]). Scattered data quasi-interpolation by functions, which reproduce polynomials, has been studied by Buhmann, Dyn, Levin in [1] and Dyn, Ron in [4] (see also [19] for further references).

In order to extend the quasi-interpolation (1.1) to general classes of approximating functions, another concept of approximation procedures, called *Approximate Approximations*, was proposed in [8] and [9]. These procedures have the common feature, that they are accurate without being convergent in a rigorous sense. Consider, for example, the quasi-interpolant on the uniform grid

$$\mathcal{M}_{h,\mathcal{D}}\, u(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{\mathbf{j}\in\mathbb{Z}^n} u(h\mathbf{j})\, \eta\Big(\frac{\mathbf{x}-h\mathbf{j}}{h\sqrt{\mathcal{D}}}\Big)\,, \tag{1.2}$$

where η is sufficiently smooth and of rapid decay, h and $\mathcal{D}$ are two positive parameters. It was shown that if $\mathcal{F}\eta - 1$ has a zero of order N at the origin ($\mathcal{F}\eta$ denotes the Fourier transform of η), then $\mathcal{M}_{h,\mathcal{D}}u$ approximates u pointwise

$$|\mathcal{M}_{h,\mathcal{D}}u(\mathbf{x}) - u(\mathbf{x})| \le c_{N,\eta}\, (h\sqrt{\mathcal{D}})^N \|\nabla_N u\|_{L_\infty(\mathbb{R}^n)} + \varepsilon \sum_{k=0}^{N-1} (h\sqrt{\mathcal{D}})^k |\nabla_k u(\mathbf{x})| \tag{1.3}$$

with a constant $c_{N,\eta}$ not depending on u, h, and $\mathcal{D}$, and ε can be made arbitrarily small if $\mathcal{D}$ is sufficiently large (see [12, 13]). Here $\nabla_k u$ denotes the vector of all partial derivatives $\partial^{\boldsymbol{\alpha}} u$ of order $|\boldsymbol{\alpha}| = k$.

In general, there is no convergence of the *approximate quasi-interpolant* $\mathcal{M}_{h,\mathcal{D}}u(\mathbf{x})$ to $u(\mathbf{x})$ as $h \to 0$. However, one can fix $\mathcal{D}$ such that up to any prescribed accuracy $\mathcal{M}_{h,\mathcal{D}}u$ approximates u with order $O(h^N)$. The lack of convergence as $h \to 0$, which is not perceptible in numerical computations for appropriately chosen $\mathcal{D}$, is compensated by a greater flexibility in the choice of approximating functions η. In applications, this flexibility enables one to obtain simple and accurate formulae for values of various integral and pseudo-differential operators of mathematical physics (see [11, 14, 16] and the review paper [18]) and to develop explicit semi-analytic time-marching algorithms for initial boundary value problems for linear and non-linear evolution equations ([10, 7]).

Up to now the approximate quasi-interpolation approach was extended to nonuniform grids in two directions. The case that the set of nodes is a smooth image of a uniform grid was studied in [15]. It was shown that formulae similar to (1.2) preserve the basic properties of approximate quasi-interpolation. A similar result for quasi-interpolation on piecewise uniform grids was obtained in [5].

It is the purpose of the present paper to generalize the method of approximate quasi-interpolation to functions given on a set of nodes $\{\mathbf{x}_j\}$ close to a uniform, not necessarily rectangular, grid Λ_h of size h. More precisely, we suppose that for some positive constant κ the κh-neighborhood of any grid point $\mathbf{y}_j$ of Λ_h contains at least one node $\mathbf{x}_j$.

Then under some additional assumption on the nodes we construct a quasi-interpolant with centers at the grid point of Λ_h

$$\mathbb{M}_{h,\mathcal{D}}u(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{\mathbf{y}_j \in \Lambda_h} F_{j,h}(u)\eta\Big(\frac{\mathbf{x}-\mathbf{y}_j}{h\sqrt{\mathcal{D}}}\Big) , \tag{1.4}$$

and show that the estimate (1.3) remains true for $\mathbb{M}_{h,\mathcal{D}}u$ under the same assumptions on the function η. Here $F_{j,h}$ are linear functionals of the data at a finite number of nodes around $\mathbf{x}_j$.

By a suitable choice of η it is possible to obtain explicit semi-analytic or other efficient approximation formulae for multi-dimensional integral and pseudo-differential operators which are based on the quasi-interpolant (1.4). So the cubature of those integrals, which is one of the applications of the approximate quasi-interpolation on uniform grids, can be carried over to the case when the integral operators are applied to functions given on a perturbed uniform grid.

We give a simple example of formula (1.4). Let $\{x_j\}$ be a sequence of points on $\mathbb{R}$ close to the uniform grid $\{hj\}_{j\in\mathbb{Z}}$ such that $x_{j+1} - x_j \geq c\,h > 0$. Consider a rapidly decaying function η satisfying the conditions

$$\Big|1 - \sum_{j\in\mathbb{Z}} \eta(x-j)\Big| < \varepsilon\,, \quad \Big|\sum_{j\in\mathbb{Z}} (x-j)\eta(x-j)\Big| < \varepsilon.$$

One can easily see that the quasi-interpolant

$$M_h u(x) = \sum_{j\in\mathbb{Z}} \left(\frac{x_{j+1}-h\,j}{x_{j+1}-x_j}u(x_j) + \frac{hj - x_j}{x_{j+1}-x_j}u(x_{j+1})\right)\eta\left(\frac{x}{h} - j\right)$$

satisfies the estimate

$$|M_h u(x) - u(x)| \leq C\,h^2\,\|u''\|_{L_\infty(\mathbb{R})} + \varepsilon(|u(x)| + h|u'(x)|)\,,$$

where the constant C depends on the function η.

The outline of the paper is as follows. In Section 2 we consider some examples of uniform non-cubic grids and establish error estimates for approximate quasi-interpolation on these grids. As an interesting example we consider quasi-interpolants on a regular hexagonal grid. In Section 3 we consider an extension of the approximate quasi-interpolation to a perturbed uniform grid. We construct the quasi-interpolant $\mathbb{M}_{h,\mathcal{D}}u$ with gridded centers and coefficients depending on scattered data and obtain approximation estimates. The results of some numerical experiments are presented in Section 4, which confirm the predicted approximation orders.

2. Quasi-interpolants on uniform non-cubic grids

In this section we study quasi-interpolants on uniform grids of the form $\{hA\mathbf{j}\}$, $\mathbf{j} \in \mathbb{Z}^n$, where A is a nonsingular matrix. As special examples we consider two-dimensional tridiagonal and hexagonal grids.

2.1. Approximation results. Suppose that for some $K > N + n$ and the smallest integer $n_0 > n/2$ the function $\eta(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, satisfies the conditions

$$(1+|\mathbf{x}|)^K |\partial^{\boldsymbol{\beta}} \eta(\mathbf{x})| \le C_{\boldsymbol{\beta}}\,,\ \mathbf{x} \in \mathbb{R}^n\,, \tag{2.1}$$

for all $0 \le |\boldsymbol{\beta}| \le n_0$, and

$$\partial^{\boldsymbol{\alpha}}(\mathcal{F}\eta - 1)(\mathbf{0}) = 0\,,\ 0 \le |\boldsymbol{\alpha}| < N. \tag{2.2}$$

It was shown in [15] that the quasi-interpolant $\mathcal{M}_{h,\mathcal{D}}u$ defined by (1.2) on the cubic grid $\{h\mathbf{j}\}$, $\mathbf{j} \in \mathbb{Z}^n$, approximates a sufficiently smooth function $u \in W_\infty^N(\mathbb{R}^n)$ with

$$\begin{aligned} |\mathcal{M}_{h,\mathcal{D}}u(\mathbf{x}) \ - u(\mathbf{x})| &\le c_{N,\eta}\,(h\sqrt{\mathcal{D}})^N \|\nabla_N u\|_{L_\infty(\mathbb{R}^n)} \\ &+ \sum_{k=0}^{N-1} \Big(\frac{\sqrt{\mathcal{D}}h}{2\pi}\Big)^k \sum_{|\boldsymbol{\alpha}|=k} \frac{|\partial^{\boldsymbol{\alpha}} u(\mathbf{x})|}{\boldsymbol{\alpha}!} \sum_{\boldsymbol{\nu}\in\mathbb{Z}^n\setminus\mathbf{0}} |\partial^{\boldsymbol{\alpha}}\mathcal{F}\eta(\sqrt{\mathcal{D}}\boldsymbol{\nu})|\,, \end{aligned} \tag{2.3}$$

where the constant $c_{N,\eta}$ is independent of u, h, and D. Moreover, under the above assumptions on η

$$\sum_{\boldsymbol{\nu}\in\mathbb{Z}^n\setminus\mathbf{0}} |\partial^{\boldsymbol{\alpha}}\mathcal{F}\eta(\sqrt{\mathcal{D}}\boldsymbol{\nu})| \to 0 \quad \text{as} \quad \mathcal{D}\to\infty\,,$$

hence for any $\varepsilon > 0$ there exist $\mathcal{D}$ such that the estimate (1.3) is satisfied. Another consequence of the inequality (2.3) is the local approximation result that for any $\varepsilon > 0$ there exist sufficiently large $\mathcal{D}$ and $\kappa > 0$ such that

$$\begin{aligned} &|\mathcal{M}_{h,\mathcal{D}}u(\mathbf{x}) - u(\mathbf{x})| \\ &\le c_{N,\eta}\,(h\sqrt{\mathcal{D}})^N \sup_{B(\mathbf{x},\kappa h)} |\nabla_N u| + \varepsilon\Big(\|u\|_{L_\infty(\mathbb{R}^n)} + \sum_{k=1}^{N-1} (h\sqrt{\mathcal{D}})^k |\nabla_k u(\mathbf{x})|\Big), \end{aligned} \tag{2.4}$$

where $B(\mathbf{x},\kappa h)$ is the ball of radius κh with center in $\mathbf{x}$.

The quasi–interpolation formula (1.2) and corresponding approximation results can be easily generalized to the case when the values of u are given on a lattice

$$\Lambda_h := \{hA\mathbf{j}\,,\ \mathbf{j}\in\mathbb{Z}^n\}$$

with a real nonsingular $n\times n$-matrix A. We define the quasi–interpolant

$$\mathcal{M}_{\Lambda_h}u(\mathbf{x}) := \frac{\det A}{\mathcal{D}^{n/2}}\sum_{\mathbf{j}\in\mathbb{Z}^n} u(hA\mathbf{j})\,\eta\Big(\frac{\mathbf{x}-hA\mathbf{j}}{\sqrt{\mathcal{D}}h}\Big)\,. \tag{2.5}$$

Using the notation $u_A = u(A\,\cdot\,)$, $\eta_A = \det A\,\eta(A\,\cdot\,)$, $\mathbf{t} = A^{-1}\mathbf{x}$ the sum (2.5) transforms to

$$\mathcal{M}_{\Lambda_h}u(\mathbf{x}) = \mathcal{D}^{-n/2}\sum_{\mathbf{j}\in\mathbb{Z}^n} u_A(h\mathbf{j})\eta_A\Big(\frac{\mathbf{t}-h\mathbf{j}}{\sqrt{\mathcal{D}}h}\Big) = \mathcal{M}_{h,\mathcal{D}}u_A(\mathbf{t})\ ,$$

i.e. coincides with quasi-interpolation formula (1.2) with the transformed generating function η_A applied to the function u_A. Since

$$\int_{\mathbb{R}^n}\mathbf{x}^{\boldsymbol{\alpha}}\eta_A(\mathbf{x})\,d\mathbf{x} = \det A\int_{\mathbb{R}^n}\mathbf{x}^{\boldsymbol{\alpha}}\eta(A\mathbf{x})\,d\mathbf{x} = \int_{\mathbb{R}^n}(A^{-1}\mathbf{x})^{\boldsymbol{\alpha}}\eta(\mathbf{x})\,d\mathbf{x}\,,$$

the generating function η_A satisfies the decay and the moment conditions (2.1) and (2.2) together with η. Denoting by $(A\nabla)_j$ for the j-th component of the vector $A\nabla$ and using the notation

$$(A\nabla)^{\boldsymbol{\alpha}} = (A\nabla)_1^{\alpha_1}\ldots(A\nabla)_n^{\alpha_n}$$

we have that

$$\partial^{\boldsymbol{\alpha}}u_A(\mathbf{t}) = (A^t\nabla)^{\boldsymbol{\alpha}}u(A\mathbf{t})\ ,\quad \partial^{\boldsymbol{\alpha}}\mathcal{F}\eta_A(\boldsymbol{\lambda})) = ((A^t)^{-1}\nabla)^{\boldsymbol{\alpha}}\mathcal{F}\eta((A^t)^{-1}\boldsymbol{\lambda})\,,$$

where A^t denotes the transpose to the matrix A. Then estimate (2.3) takes the form

$$\begin{aligned}|\mathcal{M}_{\Lambda_h}u(\mathbf{x}) - u(\mathbf{x})| \le\ & c_{A,\eta}\,(h\sqrt{\mathcal{D}})^N\|\nabla_N u\|_{L_\infty(\mathbb{R}^n)}\\ &+\sum_{k=0}^{N-1}\Big(\frac{h\sqrt{\mathcal{D}}}{2\pi}\Big)^k\sum_{|\boldsymbol{\alpha}|=k}\frac{|(A^t\nabla)^{\boldsymbol{\alpha}}u(\mathbf{x})|}{\boldsymbol{\alpha}!}\sum_{\boldsymbol{\nu}\in\mathbb{Z}^n\setminus\mathbf{0}}\big|((A^t)^{-1}\nabla)^{\boldsymbol{\alpha}}\mathcal{F}\eta(\sqrt{\mathcal{D}}(A^t)^{-1}\boldsymbol{\nu})\big|\,,\end{aligned} \tag{2.6}$$

where the constant $c_{A,\eta}$ is independent of u, h and $\mathcal{D}$. We see that it is always possible to choose $\mathcal{D}$ such that the quasi-interpolant $\mathcal{M}_{\Lambda_h}u$ satisfies an estimate of the form (1.3) or (2.4) for any $\varepsilon > 0$.

Note that Poisson's summation formula on the affine lattice $\Lambda = \{A\mathbf{j}\}_{\mathbf{j}\in\mathbb{Z}^n}$ has the form

$$\frac{\det A}{\mathcal{D}^{n/2}}\sum_{\mathbf{j}\in\mathbb{Z}^n}\eta\Big(\frac{\mathbf{x}-A\mathbf{j}}{\sqrt{\mathcal{D}}}\Big) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^n}\mathcal{F}\eta(\sqrt{\mathcal{D}}(A^t)^{-1}\boldsymbol{\nu})\,\mathrm{e}^{2\pi i(\mathbf{x},(A^t)^{-1}\boldsymbol{\nu})}\,. \tag{2.7}$$

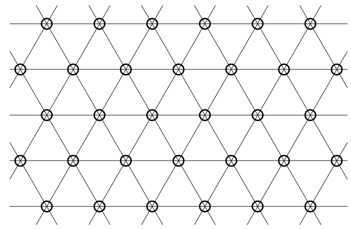

Figure 1: Tridiagonal grid

2.2. Examples. In the following we consider some 2d-examples:
1. First we consider quasi-interpolants on a regular triangular grid. It is easy to check, that the matrix

$$A = \begin{pmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix}$$

maps the integer vectors $\mathbf{j} \in \mathbb{Z}^2$ onto the vertices $\mathbf{y}_{\mathbf{j}}^{\triangle} = A\mathbf{j}$ of a partition of the plane into equilateral triangles of side length 1 indicated in Fig. 1. From (2.5) we see that a quasi-interpolant on the nodes $\{h\mathbf{y}_{\mathbf{j}}^{\triangle} = hA\mathbf{j}\}_{\mathbf{j}\in\mathbb{Z}^2}$ of a regular tridiagonal partition of $\mathbb{R}^2$ can be given as

$$\mathcal{M}_h^{\triangle} u(\mathbf{x}) := \frac{\sqrt{3}}{2\mathcal{D}} \sum_{\mathbf{j}\in\mathbb{Z}^2} u(h\mathbf{y}_{\mathbf{j}}^{\triangle})\, \eta\Big(\frac{\mathbf{x} - h\mathbf{y}_{\mathbf{j}}^{\triangle}}{\sqrt{\mathcal{D}}h}\Big)\,.$$

In particular, the function system $\dfrac{\sqrt{3}}{2\mathcal{D}}\eta\Big(\dfrac{\mathbf{x} - \mathbf{y}_{\mathbf{j}}^{\triangle}}{\sqrt{\mathcal{D}}}\Big)$ forms a approximate partition of unity and

$$\Big|1 - \frac{\sqrt{3}}{2\mathcal{D}} \sum_{\mathbf{j}\in\mathbb{Z}^2} \eta\Big(\frac{\mathbf{x} - \mathbf{y}_{\mathbf{j}}^{\triangle}}{\sqrt{\mathcal{D}}}\Big)\Big| \le \sum_{\boldsymbol{\nu}\in\mathbb{Z}^2\setminus\mathbf{0}} \Big|\int_{\mathbb{R}^2} \eta(\mathbf{y})\, \mathrm{e}^{-2\pi i\sqrt{\mathcal{D}}(A^{-1}\mathbf{y},\boldsymbol{\nu})} d\mathbf{y}\Big|\,.$$

From the relation

$$A^{-1} = \begin{pmatrix} 1 & -1/\sqrt{3} \\ 0 & 2/\sqrt{3} \end{pmatrix}$$

we obtain from (2.7) Poisson's summation formula for Gaussians $\eta(\mathbf{x}) = \pi^{-1}\mathrm{e}^{-|\mathbf{x}|^2}$ on the triangular grid

$$\begin{aligned}
&\frac{\sqrt{3}}{2\pi\mathcal{D}} \sum_{\mathbf{j}\in\mathbb{Z}^2} \mathrm{e}^{-|\mathbf{x}-\mathbf{y}_{\mathbf{j}}^{\triangle}|^2/\mathcal{D}} \\
&= \frac{1}{\pi} \sum_{\boldsymbol{\nu}\in\mathbb{Z}^2} \mathrm{e}^{2\pi i(x_1\nu_1+x_2(2\nu_2-\nu_1)/\sqrt{3})} \int_{\mathbb{R}^2} \mathrm{e}^{-|\mathbf{y}|^2}\, \mathrm{e}^{-2\pi i\sqrt{\mathcal{D}}(y_1\nu_1+y_2(2\nu_2-\nu_1)/\sqrt{3})} d\mathbf{y} \qquad (2.8) \\
&= \sum_{\boldsymbol{\nu}\in\mathbb{Z}^2} \mathrm{e}^{-4\pi^2\mathcal{D}(\nu_1^2-\nu_1\nu_2+\nu_2^2)/3}\, \mathrm{e}^{2\pi i(x_1\nu_1+x_2(2\nu_2-\nu_1)/\sqrt{3})} .
\end{aligned}$$

Hence the factor of the main term of the saturation error in (2.6), which corresponds to $\boldsymbol{\alpha} = (0,0)$, is bounded by

$$\begin{aligned}
&\Big| 1 - \frac{\sqrt{3}}{2\pi\mathcal{D}} \sum_{\mathbf{j}\in\mathbb{Z}^2} \mathrm{e}^{-|\mathbf{x}-\mathbf{y}_{\mathbf{j}}^{\triangle}|^2/\mathcal{D}} \Big| \\
&\le \sum_{(\nu_1,\nu_2)\neq(0,0)} \mathrm{e}^{-4\pi^2\mathcal{D}(\nu_1^2-\nu_1\nu_2+\nu_2^2)/3} = 6\,\mathrm{e}^{-4\pi^2\mathcal{D}/3} + O(\mathrm{e}^{-4\pi^2\mathcal{D}}) .
\end{aligned}$$

Note that this difference is less than single and double precision of floating point arithmetics of modern computers if the parameter $\mathcal{D} \ge 1.5$ and $\mathcal{D} \ge 3.0$, respectively.

2. Next we consider a hexagonal grid. To construct a quasi-interpolant with

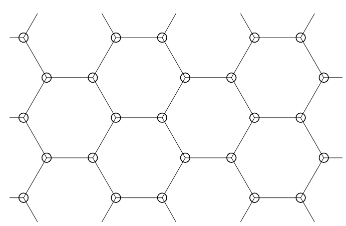

Figure 2: Hexagonal grid

functions centered at the nodes of the regular grid depicted in Fig. 2 we note that this grid can be obtained if from the nodes of the regular triangular lattice of side length 1 the nodes of another triangular grid with side length $\sqrt{3}$ are removed. This is indicate in Fig. 3, where the eliminated triangular grid is depicted with dashed lines.

The removed nodes can be written in the form $B\mathbf{j}$, $\mathbf{j} \in \mathbb{Z}^2$, with the matrix

$$B = \begin{pmatrix} 3/2 & 0 \\ \sqrt{3}/2 & \sqrt{3} \end{pmatrix} .$$

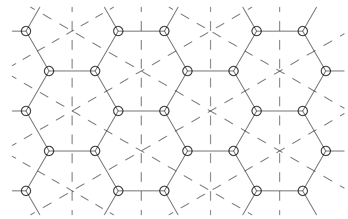

Figure 3: Nodes of a hexagonal grid

Hence, the set of nodes $\mathbf{X}^\diamond$ of the regular hexagonal grid are given by

$$\mathbf{X}^\diamond = \{A\mathbf{j}\}_{\mathbf{j}\in\mathbb{Z}^2} \setminus \{B\mathbf{j}\}_{\mathbf{j}\in\mathbb{Z}^2}\,,$$

and the sum of the shifted basis functions $\eta(\cdot/\sqrt{\mathcal{D}})$ centered at the nodes of $\mathbf{X}^\diamond$ can be written as

$$\sum_{\mathbf{y}^\diamond\in\mathbf{X}^\diamond} \eta\Big(\frac{\mathbf{x}-\mathbf{y}^\diamond}{\sqrt{\mathcal{D}}}\Big) = \sum_{\mathbf{j}\in\mathbb{Z}^2} \eta\Big(\frac{\mathbf{x}-A\mathbf{j}}{\sqrt{\mathcal{D}}}\Big) - \sum_{\mathbf{j}\in\mathbb{Z}^2} \eta\Big(\frac{\mathbf{x}-B\mathbf{j}}{\sqrt{\mathcal{D}}}\Big).$$

Under the condition $\mathcal{F}\eta(0) = 1$ we have from (2.7)

$$\sum_{\mathbf{j}\in\mathbb{Z}^2} \eta\Big(\frac{\mathbf{x}-A\mathbf{j}}{\sqrt{\mathcal{D}}}\Big) = \frac{\mathcal{D}}{\det A}\Big(1 + \sum_{\boldsymbol{\nu}\in\mathbb{Z}^2\setminus\mathbf{0}} \mathcal{F}\eta(\sqrt{\mathcal{D}}(A^t)^{-1}\boldsymbol{\nu})\, \mathrm{e}^{2\pi i(\mathbf{x},(A^t)^{-1}\boldsymbol{\nu})}\Big)\,,$$

thus we obtain

$$\sum_{\mathbf{y}^\diamond\in\mathbf{X}^\diamond} \eta\Big(\frac{\mathbf{x}-\mathbf{y}^\diamond}{\sqrt{\mathcal{D}}}\Big) = \frac{2\mathcal{D}}{\sqrt{3}} + \frac{2\mathcal{D}}{\sqrt{3}} \sum_{\boldsymbol{\nu}\in\mathbb{Z}^2\setminus\mathbf{0}} \mathcal{F}\eta(\sqrt{\mathcal{D}}(A^t)^{-1}\boldsymbol{\nu})\, \mathrm{e}^{2\pi i(\mathbf{x},(A^t)^{-1}\boldsymbol{\nu})}$$
$$- \frac{2\mathcal{D}}{3\sqrt{3}} - \frac{2\mathcal{D}}{3\sqrt{3}} \sum_{\boldsymbol{\nu}\in\mathbb{Z}^2\setminus\mathbf{0}} \mathcal{F}\eta(\sqrt{\mathcal{D}}(B^t)^{-1}\boldsymbol{\nu})\, \mathrm{e}^{2\pi i(\mathbf{x},(B^t)^{-1}\boldsymbol{\nu})}.$$

Hence an approximate partition of unity centered at the hexagonal grid is given by

$$\frac{3\sqrt{3}}{4\mathcal{D}} \sum_{\mathbf{y}^\diamond\in\mathbf{X}^\diamond} \eta\Big(\frac{\mathbf{x}-\mathbf{y}^\diamond}{\sqrt{\mathcal{D}}}\Big) = 1 + \frac{3}{2} \sum_{\boldsymbol{\nu}\in\mathbb{Z}^2\setminus\mathbf{0}} \mathcal{F}\eta(\sqrt{\mathcal{D}}(A^t)^{-1}\boldsymbol{\nu})\, \mathrm{e}^{2\pi i(\mathbf{x},(A^t)^{-1}\boldsymbol{\nu})}$$
$$- \frac{1}{2} \sum_{\boldsymbol{\nu}\in\mathbb{Z}^2\setminus\mathbf{0}} \mathcal{F}\eta(\sqrt{\mathcal{D}}(B^t)^{-1}\boldsymbol{\nu})\, \mathrm{e}^{2\pi i(\mathbf{x},(B^t)^{-1}\boldsymbol{\nu})}.$$

Now we define the quasi-interpolant on the h-scaled hexagonal grid

$$h\mathbf{X}^\diamond = \{hA\mathbf{j}\}_{\mathbf{j}\in\mathbb{Z}^2} \setminus \{hB\mathbf{j}\}_{\mathbf{j}\in\mathbb{Z}^2}$$

as

$$\mathcal{M}_h^\diamond u(\mathbf{x}) := \frac{3\sqrt{3}}{4\mathcal{D}} \sum_{\mathbf{y}^\diamond \in \mathbf{X}^\diamond} u(h\mathbf{y}^\diamond)\, \eta\Big(\frac{\mathbf{x} - h\mathbf{y}^\diamond}{\sqrt{\mathcal{D}}h}\Big) . \tag{2.9}$$

Since it can be written in the form

$$\mathcal{M}_h^\diamond u(\mathbf{x}) = \frac{3\sqrt{3}}{4\mathcal{D}} \Big(\sum_{\mathbf{j}\in\mathbb{Z}^2} u(hA\mathbf{j})\eta\Big(\frac{\mathbf{x} - hA\mathbf{j}}{\sqrt{\mathcal{D}}h}\Big) - \sum_{\mathbf{j}\in\mathbb{Z}^2} u(hB\mathbf{j})\eta\Big(\frac{\mathbf{x} - hB\mathbf{j}}{\sqrt{\mathcal{D}}h}\Big)\Big) ,$$

we see that under the conditions (2.1) and (2.2) the quasi-interpolant $\mathcal{M}_h^\diamond u$ provides the estimates (1.3) and (2.4) for sufficiently large $\mathcal{D}$.

Because of

$$B^{-1} = \begin{pmatrix} 2/3 & 0 \\ -1/3 & \sqrt{3}/3 \end{pmatrix}$$

we obtain, by using (2.8), Poisson's summation formula for Gaussians on the hexagonal grid

$$\begin{aligned}
&\frac{3\sqrt{3}}{4\pi\mathcal{D}} \sum_{\mathbf{y}^\diamond \in \mathbf{X}^\diamond} \mathrm{e}^{-|\mathbf{x}-\mathbf{y}^\diamond|^2/\mathcal{D}} = \frac{3\sqrt{3}}{4\pi\mathcal{D}} \Big(\sum_{\mathbf{j}\in\mathbb{Z}^2} \mathrm{e}^{-|\mathbf{x}-A\mathbf{j}|^2/\mathcal{D}} - \sum_{\mathbf{j}\in\mathbb{Z}^2} \mathrm{e}^{-|\mathbf{x}-B\mathbf{j}|^2/\mathcal{D}} \Big) \\
&= \frac{3}{2} \sum_{\boldsymbol{\nu}\in\mathbb{Z}^2} \mathrm{e}^{-4\pi^2\mathcal{D}(\nu_1^2-\nu_1\nu_2+\nu_2^2)/3}\, \mathrm{e}^{2\pi i(x_1\nu_1+x_2(2\nu_2-\nu_1)/\sqrt{3})} \\
&\quad - \frac{1}{2} \sum_{\boldsymbol{\nu}\in\mathbb{Z}^2} \mathrm{e}^{-4\pi^2\mathcal{D}(\nu_1^2-\nu_1\nu_2+\nu_2^2)/9}\, \mathrm{e}^{2\pi i(x_1(2\nu_1-\nu_2)/3+x_2\nu_2/\sqrt{3})} .
\end{aligned}$$

Hence, for the generating function $\eta(\mathbf{x}) = \pi^{-1}\mathrm{e}^{-|\mathbf{x}|^2}$ the factor of the main saturation error is bounded by

$$\begin{aligned}
&\Big| 1 - \frac{3\sqrt{3}}{4\pi\mathcal{D}} \sum_{\mathbf{y}^\diamond \in \mathbf{X}^\diamond} \mathrm{e}^{-|\mathbf{x}-\mathbf{y}^\diamond|^2/\mathcal{D}} \Big| \\
&\qquad \leq \frac{1}{2} \sum_{(\nu_1,\nu_2)\neq(0,0)} 3\mathrm{e}^{-4\pi^2\mathcal{D}(\nu_1^2-\nu_1\nu_2+\nu_2^2)/3} + \mathrm{e}^{-4\pi^2\mathcal{D}(\nu_1^2-\nu_1\nu_2+\nu_2^2)/9} \\
&\qquad = 3\,\mathrm{e}^{-4\pi^2\mathcal{D}/9} + O(\mathrm{e}^{-4\pi^2\mathcal{D}/3}) .
\end{aligned}$$

3. Quasi-interpolants for data on perturbed grids

Here we give a simple extension of the quasi-interpolation operator on a uniform grid, considered in the previous section, to quasi-interpolants, which use the values $u(\mathbf{x}_j)$ on a set of scattered nodes $\mathbf{X} = \{\mathbf{x}_j\}_{j\in J} \subset \mathbb{R}^n$ close to a uniform grid. Precisely we suppose

Condition 3.1. There exists a uniform grid Λ such that the quasi-interpolants

$$\mathcal{M}_{h,\mathcal{D}}\, u(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{\mathbf{y}_j \in \Lambda} u(h\mathbf{y}_j)\, \eta\Big(\frac{\mathbf{x} - h\mathbf{y}_j}{h\sqrt{\mathcal{D}}}\Big) \tag{3.1}$$

approximate sufficiently smooth functions u with the error (1.3) for any $\varepsilon > 0$. Let $\mathbf{X}_h$ be a sequence of grids with the property that for $\kappa_1 > 0$ not depending on h and any $\mathbf{y}_j \in \Lambda$ the ball $B(h\mathbf{y}_j, h\kappa_1)$ contains nodes of $\mathbf{X}_h$.

3.1. Construction.

Definition 3.1. Let $\mathbf{x}_j \in \mathbf{X}_h$. A collection of $m_N = \dfrac{(N-1+n)!}{n!(N-1)!} - 1$ nodes $\mathbf{x}_k \in \mathbf{X}_h$ will be called *star* of $\mathbf{x}_j$ and denoted by $\operatorname{st}(\mathbf{x}_j)$ if the Vandermonde matrix

$$V_{j,h} = \Big\{\Big(\frac{\mathbf{x}_k - \mathbf{x}_j}{h}\Big)^{\boldsymbol{\alpha}}\Big\},\ |\boldsymbol{\alpha}| = 1, ..., N-1, \tag{3.2}$$

is not singular.

Condition 3.2. Denote by $\widetilde{\mathbf{x}}_j \in \mathbf{X}_h$ the node closest to $h\mathbf{y}_j \in h\Lambda$. There exists $\kappa_2 > 0$ such that for any $\mathbf{y}_j \in \Lambda$ the star $\operatorname{st}(\widetilde{\mathbf{x}}_j) \subset B(\widetilde{\mathbf{x}}_j, h\kappa_2)$ with $|\det V_{j,h}| \ge c > 0$ uniformly in h.

To describe the construction of the quasi-interpolants which use the data at $\mathbf{X}_h$ we denote the elements of the inverse matrix of $V_{j,h}$ by $\{b^{(j)}_{\boldsymbol{\alpha},k}\}$, $|\boldsymbol{\alpha}| = 1, \dots, N-1$, $\mathbf{x}_k \in \operatorname{st}(\widetilde{\mathbf{x}}_j)$, and define the functional

$$\begin{aligned} F_{j,h}(u) = u(\widetilde{\mathbf{x}}_j)\Big(1 - \sum_{|\boldsymbol{\alpha}|=1}^{N-1} \Big(\mathbf{y}_j - \frac{\widetilde{\mathbf{x}}_j}{h}\Big)^{\boldsymbol{\alpha}} \sum_{\mathbf{x}_k \in \operatorname{st}(\widetilde{\mathbf{x}}_j)} b^{(j)}_{\boldsymbol{\alpha},k}\Big) \\ + \sum_{\mathbf{x}_k \in \operatorname{st}(\widetilde{\mathbf{x}}_j)} u(\mathbf{x}_k) \sum_{|\boldsymbol{\alpha}|=1}^{N-1} b^{(j)}_{\boldsymbol{\alpha},k}\Big(\mathbf{y}_j - \frac{\widetilde{\mathbf{x}}_j}{h}\Big)^{\boldsymbol{\alpha}}. \end{aligned} \tag{3.3}$$

The quasi-interpolants is then defined as the sum

$$\mathbb{M}_{h,\mathcal{D}} u(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{\mathbf{y}_j \in \Lambda} F_{j,h}(u)\eta\Big(\frac{\mathbf{x} - h\,\mathbf{y}_j}{h\sqrt{\mathcal{D}}}\Big), \tag{3.4}$$

i.e., the generating functions are centered at the nodes of the uniform grid $h\Lambda$. This can be advantageous to design fast methods for the approximation of convolution integrals

$$\mathcal{K}u(\mathbf{x}) = \int_{\mathbb{R}^n} g(\mathbf{x} - \mathbf{y})u(\mathbf{y})\, d\mathbf{y}\,. \tag{3.5}$$

Here a cubature formula can be defined as

$$\begin{aligned}\mathcal{K}\,\mathbb{M}_{h,\mathcal{D}}u(\mathbf{x}) &= \mathcal{D}^{-n/2}\sum_{\mathbf{y}_j\in\Lambda} F_{j,h}(u)\int_{\mathbb{R}^n} g(\mathbf{x}-\mathbf{y})\eta\Big(\frac{\mathbf{y}-h\,\mathbf{y}_j}{h\sqrt{\mathcal{D}}}\Big)d\mathbf{y}\\ &= h^n\sum_{\mathbf{y}_j\in\Lambda} F_{j,h}(u)\int_{\mathbb{R}^n} g\Big(h\sqrt{\mathcal{D}}\Big(\frac{\mathbf{x}-h\,\mathbf{y}_j}{h\sqrt{\mathcal{D}}}-\mathbf{y}\Big)\Big)\eta(\mathbf{y})d\mathbf{y}\,.\end{aligned}$$

Then the computation of $\mathcal{K}\,\mathbb{M}_{h,\mathcal{D}}u(h\mathbf{y}_k)$ for $\mathbf{y}_k\in\Lambda$ leads to the discrete convolution

$$\mathcal{K}\,\mathbb{M}_{h,\mathcal{D}}u(h\mathbf{y}_k) = h^n\sum_{\mathbf{y}_j\in\Lambda} F_{j,h}(u)\,a_{k-j}^{(h)}$$

with the coefficients

$$a_{k-j}^{(h)} = \int_{\mathbb{R}^n} g\big(h(\mathbf{y}_k-\mathbf{y}_j-\sqrt{\mathcal{D}}\,\mathbf{y})\big)\,\eta(\mathbf{y})d\mathbf{y}\,.$$

3.2. Estimates.

Theorem 3.2. *Under the Conditions* 3.1 *and* 3.2*, for any* $\varepsilon>0$ *there exists* $\mathcal{D}$ *such that the quasi-interpolant* (3.4) *approximates any* $u\in W_\infty^N(\mathbb{R}^n)$ *with*

$$|\mathbb{M}_{h,\mathcal{D}}u(\mathbf{x})-u(\mathbf{x})| \le c_{N,\eta,\mathcal{D}}\; h^N\|\nabla_N u\|_{L_\infty(\mathbb{R}^n)} + \varepsilon\sum_{k=0}^{N-1}(h\sqrt{\mathcal{D}})^k|\nabla_k u(\mathbf{x})|\,, \quad (3.6)$$

where $c_{N,\eta,\mathcal{D}}$ *does not depend on* u *and* h.

Proof. For given $u\in W_\infty^N(\mathbb{R}^n)$ and the grid $\mathbf{X}_h$ we consider the quasi-interpolant (3.1) on the uniform grid $h\Lambda$

$$\mathcal{M}_{h,\mathcal{D}}\,u(\mathbf{x}) = \mathcal{D}^{-n/2}\sum_{\mathbf{y}_j\in\Lambda} u(h\mathbf{y}_j)\,\eta\Big(\frac{\mathbf{x}-h\mathbf{y}_j}{h\sqrt{\mathcal{D}}}\Big)\,.$$

According to Condition 3.1 we can find $\mathcal{D}$ such that $\mathcal{M}u$ satisfies the inequality

$$|\mathcal{M}_{h,\mathcal{D}}u(\mathbf{x})-u(\mathbf{x})| \le c_{N,\eta}\;(h\sqrt{\mathcal{D}})^N\|\nabla_N u\|_{L_\infty(\mathbb{R}^n)} + \varepsilon\sum_{k=0}^{N-1}|\nabla_k u(\mathbf{x})|(h\sqrt{\mathcal{D}})^k\,. \quad (3.7)$$

So it remains to estimate $|\mathcal{M}_{h,\mathcal{D}}u(\mathbf{x})-\mathbb{M}_{h,\mathcal{D}}u(\mathbf{x})|$. Recall the Taylor expansion of u around $\mathbf{t}\in\mathbb{R}^n$

$$u(\mathbf{x}) = \sum_{|\boldsymbol{\alpha}|=0}^{N-1}\frac{\partial^{\boldsymbol{\alpha}}u(\mathbf{t})}{\boldsymbol{\alpha}!}(\mathbf{x}-\mathbf{t})^{\boldsymbol{\alpha}} + R_N(\mathbf{x},\mathbf{t}) \quad (3.8)$$

with the remainder satisfying

$$|R_N(\mathbf{x},\mathbf{t})| \le c_N |\mathbf{x}-\mathbf{t}|^N \sup_{B(\mathbf{t},|\mathbf{x}-\mathbf{t}|)} |\nabla_N u| \,. \tag{3.9}$$

For $\mathbf{y}_j \in \Lambda$ we choose $\widetilde{\mathbf{x}}_j \in \mathbf{X}_h$ and use (3.8) with $\mathbf{t} = \widetilde{\mathbf{x}}_j$. We split

$$\mathcal{M}_{h,\mathcal{D}} u(\mathbf{x}) = M^{(1)} u(\mathbf{x}) + \mathcal{D}^{-n/2} \sum_{\mathbf{y}_j \in \Lambda} R_N(h\mathbf{y}_j, \widetilde{\mathbf{x}}_j)\, \eta\Big(\frac{\mathbf{x}-h\mathbf{y}_j}{h\sqrt{\mathcal{D}}}\Big)$$

with

$$M^{(1)} u(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{\mathbf{y}_j \in \Lambda} \sum_{|\boldsymbol{\alpha}|=0}^{N-1} \frac{\partial^{\boldsymbol{\alpha}} u(\widetilde{\mathbf{x}}_j)}{\boldsymbol{\alpha}!} (h\mathbf{y}_j - \widetilde{\mathbf{x}}_j)^{\boldsymbol{\alpha}}\, \eta\Big(\frac{\mathbf{x}-h\mathbf{y}_j}{h\sqrt{\mathcal{D}}}\Big) \,. \tag{3.10}$$

Because of $|h\mathbf{y}_j - \widetilde{\mathbf{x}}_j| \le \kappa_1 h$ for any $\mathbf{y}_j$ we derive from (3.9)

$$|M^{(1)} u(\mathbf{x}) - \mathcal{M}_{h,\mathcal{D}} u(\mathbf{x})| \le c_N\, (\kappa_1 h)^N \mathcal{D}^{-n/2} \sum_{\mathbf{y}_j \in \Lambda} \Big|\eta\Big(\frac{\mathbf{x}-h\mathbf{y}_j}{h\sqrt{\mathcal{D}}}\Big)\Big| \sup_{B(\mathbf{x},h\kappa_1)} |\nabla_N u| \,. \tag{3.11}$$

The next step is to approximate $\partial^{\boldsymbol{\alpha}} u(\widetilde{\mathbf{x}}_j)$, $1 \le |\boldsymbol{\alpha}| < N$, by a linear combination of $u(\mathbf{x}_k)$, $\mathbf{x}_k \in \mathrm{st}\,(\widetilde{\mathbf{x}}_j)$. Let $\{a_{\boldsymbol{\alpha}}^{(j)}\}_{1 \le |\boldsymbol{\alpha}| \le N-1}$ be the unique solution of the linear system with m_N unknowns

$$\sum_{|\boldsymbol{\alpha}|=1}^{N-1} \frac{a_{\boldsymbol{\alpha}}^{(j)}}{\boldsymbol{\alpha}!} (\mathbf{x}_k - \widetilde{\mathbf{x}}_j)^{\boldsymbol{\alpha}} = u(\mathbf{x}_k) - u(\widetilde{\mathbf{x}}_j)\,, \quad \mathbf{x}_k \in \mathrm{st}\,(\widetilde{\mathbf{x}}_j)\,. \tag{3.12}$$

It follows from (3.8) and (3.12) that

$$\sum_{|\boldsymbol{\alpha}|=1}^{N-1} \frac{h^{|\boldsymbol{\alpha}|}}{\boldsymbol{\alpha}!} (a_{\boldsymbol{\alpha}}^{(j)} - \partial^{\boldsymbol{\alpha}} u(\widetilde{\mathbf{x}}_j)) \Big(\frac{\mathbf{x}_k - \widetilde{\mathbf{x}}_j}{h}\Big)^{\boldsymbol{\alpha}} = R_N(\mathbf{x}_k, \widetilde{\mathbf{x}}_j)\,.$$

By Condition 3.2 the norms of $V_{j,h}^{-1}$ are bounded uniformly for all j and h, this leads together with (3.9) to the inequalities

$$\frac{|a_{\boldsymbol{\alpha}}^{(j)} - \partial^{\boldsymbol{\alpha}} u(\widetilde{\mathbf{x}}_j)|}{\boldsymbol{\alpha}!} \le C_2\, h^{N-|\boldsymbol{\alpha}|} \sup_{B(\widetilde{\mathbf{x}}_j, h\kappa_2)} |\nabla_N u|\,,\ 0 \le |\boldsymbol{\alpha}| < N\,. \tag{3.13}$$

Hence, if we replace the derivatives $\partial^{\boldsymbol{\alpha}} u(\widetilde{\mathbf{x}}_j)$ in (3.10) by $a_{\boldsymbol{\alpha}}^{(j)}$, then we get the sum

$$\mathcal{D}^{-n/2} \sum_{\mathbf{y}_j \in \Lambda} \Big(u(\widetilde{\mathbf{x}}_j) + \sum_{|\boldsymbol{\alpha}|=1}^{N-1} \frac{a_{\boldsymbol{\alpha}}^{(j)}}{\boldsymbol{\alpha}!} (h\mathbf{y}_j - \widetilde{\mathbf{x}}_j)^{\boldsymbol{\alpha}}\Big)\, \eta\Big(\frac{\mathbf{x} - h\,\mathbf{y}_j}{h\sqrt{\mathcal{D}}}\Big),$$

which in view of

$$a_{\boldsymbol{\alpha}}^{(j)} = \frac{\boldsymbol{\alpha}!}{h^{|\boldsymbol{\alpha}|}} \sum_{\mathbf{x}_k \in \mathrm{st}\,(\widetilde{\mathbf{x}}_j)} b_{\boldsymbol{\alpha},k}^{(j)} \left(u(\mathbf{x}_k) - u(\widetilde{\mathbf{x}}_j)\right)$$

coincides with the quasi-interpolant $\mathbb{M}_{h,\mathcal{D}}u$, defined by (3.4). Moreover, by (3.10) and (3.13)

$$\begin{aligned} &| \mathbb{M}_{h,\mathcal{D}}u(\mathbf{x}) - M^{(1)}u(\mathbf{x})| \\ &\le C_2 h^N \sum_{|\boldsymbol{\alpha}|=1}^{N-1} \kappa_1^{|\boldsymbol{\alpha}|} \, \mathcal{D}^{-n/2} \sum_{\mathbf{y}_j \in \Lambda} \Big|\eta\Big(\frac{\mathbf{x} - h\,\mathbf{y}_j}{h\sqrt{\mathcal{D}}}\Big)\Big| \sup_{B(\mathbf{x},h\kappa_2)} |\nabla_N u| \,. \end{aligned} \tag{3.14}$$

Now the inequality

$$\sup_{\mathbf{x}\in\mathbb{R}^n} \mathcal{D}^{-n/2} \sum_{\mathbf{y}_j \in \Lambda} \Big|\eta\Big(\frac{\mathbf{x} - \mathbf{y}_j}{\sqrt{\mathcal{D}}}\Big)\Big| \le C_3$$

for all $\mathcal{D} \ge \mathcal{D}_0 > 0$ implies that (3.11) and (3.14) lead to

$$|\mathcal{M}_{h,\mathcal{D}}u(\mathbf{x}) - \mathbb{M}_{h,\mathcal{D}}u(\mathbf{x})| \le C_4 h^N \sup_{B(\mathbf{x},h\kappa_2)} |\nabla_N u| \,,$$

which establishes together with (3.7) the estimate (3.6). □

4. Numerical Experiments with Quasi-interpolants

The behavior of the quasi-interpolant $\mathbb{M}_{h,\mathcal{D}}u$ was tested by one- and two-dimensional experiments. In all cases the scattered grid is chosen such that any ball $B(h\mathbf{j}, h/2)$, $\mathbf{j} \in \mathbb{Z}^n$, $n = 1$ or $n = 2$, contains one randomly chosen node, which we denote by $\mathbf{x}_\mathbf{j}$. All the computations were carried out with MATHEMATICA®.

In the one-dimensional case Figures 4 and 5 show the graphs of $\mathbb{M}_{h,\mathcal{D}}u - u$ for different smooth functions u using the basis function $\eta(x) = \pi^{-1/2}\mathrm{e}^{-x^2}$ (Fig. 4) for which $N = 2$, and $\eta(x) = \pi^{-1/2}(3/2 - x^2)\mathrm{e}^{-x^2}$ (Fig. 5) for which $N = 4$, with $h = 1/32$ (dashed line) and $h = 1/64$ (solid line).

As two-dimensional examples we depict in Figures 6 and 7 the quasi-interpolation error $\mathbb{M}_{h,\mathcal{D}}u - u$ for the function $u(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{-1}$ and different h if generating functions of second (with $\mathcal{D} = 2$) and fourth (with $\mathcal{D} = 4$) order of approximation are used. The h^2- and respectively h^4-convergence of the corresponding two-dimensional quasi-interpolants are confirmed by the L_∞-errors which are given in Table 1.

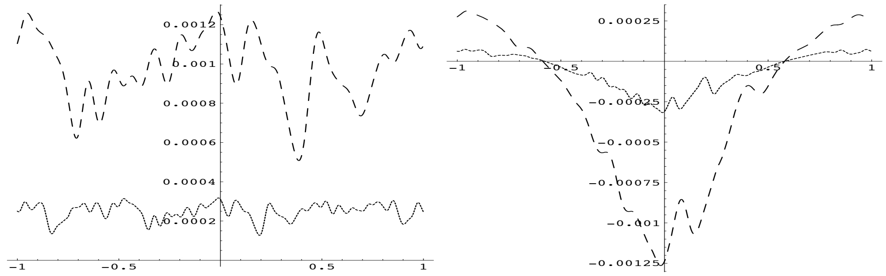

Figure 4: The graphs of $\mathbb{M}_{h,\mathcal{D}}u - u$ with $\eta(x) = \pi^{-1/2}\mathrm{e}^{-x^2}$, $\mathcal{D} = 2$, $\mathrm{st}\,(x_j) = \{x_{j+1}\}$, when $u(x) = x^2$ (on the left) and $u(x) = (1+x^2)^{-1}$. Dashed and solid lines correspond to $h = 1/32$ and $h = 1/64$.

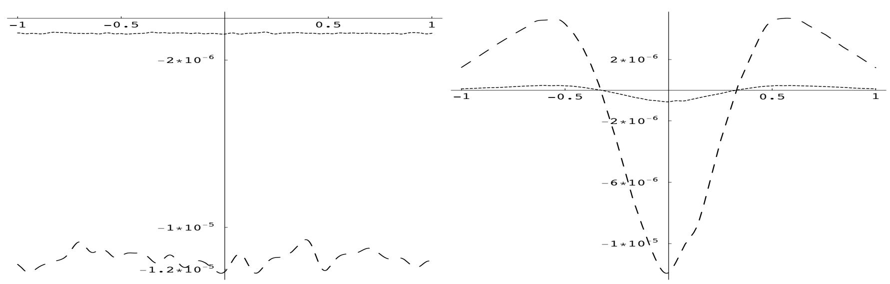

Figure 5: The graphs of $\mathbb{M}_{h,\mathcal{D}}u - u$ with $\eta(x) = \pi^{-1/2}(3/2 - x^2)\mathrm{e}^{-x^2}$, $\mathcal{D} = 4$, $\mathrm{st}\,(x_j) = \{x_{j-2}, x_{j-1}, x_{j+1}\}$, when $u(x) = x^4$ (on the left) and $u(x) = (1+x^2)^{-1}$. Dashed and solid lines correspond to $h = 1/32$ and $h = 1/64$.

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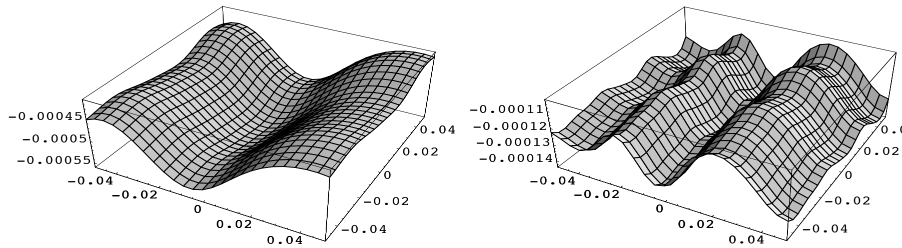

Figure 6: The graph of $\mathbb{M}_{h,\mathcal{D}}u - u$ with $\mathcal{D} = 2$, $\eta(\mathbf{x}) = \pi^{-1}\mathrm{e}^{-|\mathbf{x}|^2}$, $N = 2$, $u(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{-1}$, $h = 2^{-6}$ (on the left) and $h = 2^{-7}$ (on the right).

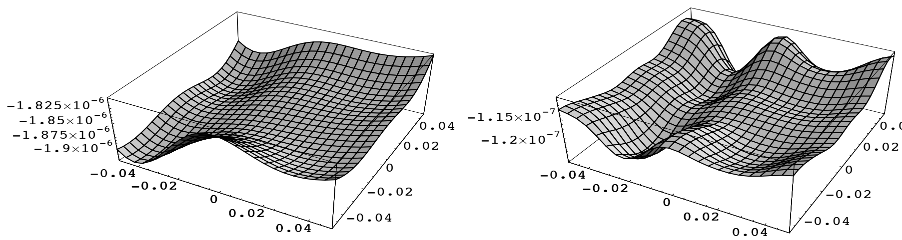

Figure 7: The graph of $\mathbb{M}_{h,\mathcal{D}}u - u$ with $\mathcal{D} = 4$, $\eta(\mathbf{x}) = \pi^{-1}(2 - |\mathbf{x}|^2)\mathrm{e}^{-|\mathbf{x}|^2}$, $N = 4$, $u(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{-1}$, $h = 2^{-6}$ (on the left) and $h = 2^{-7}$ (on the right).

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h	$\mathcal{D}=2$	$\mathcal{D}=4$
2^{-4}	$8.75 \cdot 10^{-3}$	$1.57 \cdot 10^{-2}$
2^{-5}	$2.21 \cdot 10^{-3}$	$4.00 \cdot 10^{-3}$
2^{-6}	$5.51 \cdot 10^{-4}$	$1.01 \cdot 10^{-3}$
2^{-7}	$1.42 \cdot 10^{-4}$	$2.52 \cdot 10^{-4}$
2^{-8}	$3.56 \cdot 10^{-5}$	$6.50 \cdot 10^{-5}$

h	$\mathcal{D}=4$	$\mathcal{D}=6$
2^{-4}	$4.42 \cdot 10^{-4}$	$9.59 \cdot 10^{-4}$
2^{-5}	$2.95 \cdot 10^{-5}$	$6.61 \cdot 10^{-5}$
2^{-6}	$1.92 \cdot 10^{-6}$	$4.24 \cdot 10^{-6}$
2^{-7}	$1.24 \cdot 10^{-7}$	$2.68 \cdot 10^{-7}$
2^{-8}	$7.80 \cdot 10^{-9}$	$1.71 \cdot 10^{-8}$

Table 1: L_∞ approximation error for the function $u(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{-1}$ using $\mathbb{M}_{h,\mathcal{D}}u$ with $\eta(\mathbf{x}) = \pi^{-1}\mathrm{e}^{-|\mathbf{x}|^2}$, $N = 2$ (on the left), and $\eta(\mathbf{x}) = \pi^{-1}(2-|\mathbf{x}|^2)\mathrm{e}^{-|\mathbf{x}|^2}$, $N = 4$ (on the right).

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