ISSN 1063-4541, Vestnik St. Petersburg University. Mathematics, 2008, Vol. 41, No. 2, pp. 145–160. © Allerton Press, Inc., 2008.
Original Russian Text © V.G. Maz'ya, S.V. Poborchii, 2008, published in Vestnik Sankt-Peterburgskogo Universiteta. Seriya 1. Matematika, Mekhanika, Astronomiya, 2008, No. 2, pp. 69–85.

TO THE 100th ANNIVERSARY OF BIRTHDAY OF SOLOMON GRIGOR'EVICH MIKHLIN

Dedicated to the memory of S. G. Mikhlin

On the Solvability of the Neumann Problem for a Planar Domain with a Peak

V. G. Maz'ya and S. V. Poborchii

Received November 11, 2007

Abstract—The Neumann problem for second-order elliptic quasi-linear equations on a planar domain whose boundary contains the vertex of an outward or inward peak. Under certain conditions, the solvability problem for the Neumann problem is reduced to a description of the space dual to the boundary trace space $TW_p^1(\Omega)$ for functions from the Sobolev class $W_p^1(\Omega)$, where $1 < p < \infty$. This dual space is characterized in terms of Sobolev classes on Lipschitz curves with negative smoothness exponents and in terms of function spaces on the interval (0, 1) of the real line. The proofs of the main results are essentially based on an explicit description of the space $TW_p^1(\Omega)$ for a planar domain with a peak due to the author. Necessary and sufficient conditions for q to be such that the Neumann problem is solvable provided that the boundary function belongs to $L_q(\partial\Omega)$ are given.

DOI: 10.3103/S1063454108020088

INTRODUCTION

In a domain $\Omega \subset \mathbf{R}^n$ whose boundary contains the vertex of an outward or inward peak, consider the Neumann problem for a second-order quasilinear elliptic equation. Under certain conditions, studying the solvability of this problem reduces to describing the space $TW_p^1(\Omega)^*$ ($p \in (1, \infty)$ dual to the space $TW_p^1(\Omega)$ of boundary traces of functions from the Sobolev class $W_p^1(\Omega)$. For $n > 2$, the space $TW_2^1(\Omega)^*$ was characterized in [1]. In this paper, we continue its study. We explicitly describe the space $TW_p^1(\Omega)^*$, where $p \in (1, \infty)$, for a planar domain whose boundary contains the vertex of a peak.

This paper includes four sections. The first is essentially a detailed introduction, in which the Neumann problem is stated and the relationship between its solvability and a characterization of the dual space mentioned above is specified. In Section 2, we describe the class of domains under consideration and prove auxiliary assertions used in what follows. Section 3 is devoted to the proof of Theorem 1, which characterizes the space $TW_p^1(\Omega)^*$ for a planar domain with an outer peak in terms of the spaces $W_{p'}^{-1/p'}$ for Lipschitz curves, and to applications of Theorem 1 to the Neumann problem. Section 4 considers planar domains with inward peaks. It contains Theorem 2, which describes the space $TW_p^1(\Omega)^*$ for a planar domain with an inward peak.

Theorems 1 and 2 are the main results of this paper.

1. PRELIMINARIES

Let Ω be a domain in $\mathbf{R}^2$, and let $p \in (1, \infty)$. By $W_p^1(\Omega)$ we denote the space of functions on Ω for which the norm

$$\|u\|_{W_p^1(\Omega)} = (\|u\|_{L_p(\Omega)}^p + \|\nabla u\|_{L_p(\Omega)}^p)^{1/p}$$

is finite. Suppose also that $TW_p^1(\Omega)$ is the space of traces $u|_{\partial\Omega}$ of functions from $W_p^1(\Omega)$ endowed with the norm

$$\|v\|_{TW_p^1(\Omega)} = \inf\{\|u\|_{W_p^1(\Omega)} : u|_{\partial\Omega} = v\}.$$

For domains with "sufficiently good" boundary, the space $TW_p^1(\Omega)$ admits an explicit description. Thus, according to a theorem of Gagliardo [2], for planar domains of class $C^{0,1}$ (i.e., for domains with compact closure whose boundaries locally coincide with graphs of Lipschitz functions), the space $TW_p^1(\Omega)$ coincides with the space $W_p^{1-1/p}(S)$, where $S = \partial\Omega$, of functions on the curve S with finite norm

$$\|v\|_{W_p^{1-1/p}(S)} = |S|^{-1+1/p}\|v\|_{L_p(S)} + [v]_{p,S},$$

where

$$[v]_{p,S} = \left(\iint\limits_{S\times S}|v(Q)-v(M)|^p\frac{ds_Q ds_M}{|Q-M|^p}\right)^{1/p}, \tag{1}$$

$|S|$ is the length of S, and ds_Q and ds_M are length elements on S. In the general case, for $TW_p^1(\Omega)$ we can take the quotient space $W_p^1(\Omega)/\mathring{W}_p^1(\Omega)$, where $\mathring{W}_p^1(\Omega)$ denotes the closure of the set of $C_0^\infty(\Omega)$-smooth compactly supported functions on Ω in the norm of $W_p^1(\Omega)$.

Suppose that Ω has compact boundary and a normal to $\partial\Omega$ exists almost everywhere on $\partial\Omega$ (with respect to length). Let $\nu = \nu(x)$, where $x \in \partial\Omega$, denote the unit outer normal vector. Consider the Neumann problem[1]

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + a|u|^{p-2}u = 0 \text{ on } \Omega, \tag{2}$$

$$|\nabla u|^{p-2}\frac{\partial u}{\partial\nu}\bigg|_{\partial\Omega} = f, \tag{3}$$

where $p \in (1, \infty)$, $a \in L_\infty(\Omega)$, $a(x) \ge \text{const} > 0$ almost everywhere on Ω, and f is a homogeneous additive functional on $\mathcal{V} = W_p^1(\Omega) \cap L_\infty(\Omega) \cap C^\infty(\Omega)$ vanishing on the set of functions from $C_0^\infty(\Omega)$.

By a solution of problem (2), (3) we mean a function $u \in W_p^1(\Omega)$ such that, for all $v \in \mathcal{V}$, it satisfies the identity

$$L(u, v) = \langle f, v\rangle, \tag{4}$$

where $\langle\cdot, v\rangle$ is the functional value at v and

$$L(u, v) = \int\limits_\Omega(|\nabla u|^{p-2}\nabla u\cdot\nabla v + a|u|^{p-2}uv)dx.$$

For fixed $u \in W_p^1(\Omega)$, the mapping $W_p^1(\Omega) \ni v \mapsto L(u, v)$ is a continuous linear functional, and the set $\mathcal{V}$ is dense in $W_p^1(\Omega)$ [4, 3.1.2]; therefore, a necessary condition for the solvability of problem (2), (3) is that the functional on the right-hand side of (3) must have a (unique) extension to a functional from $W_p^1(\Omega)^*$ vanishing at the elements of $\mathring{W}_p^1(\Omega)$. According to the following lemma, this functional belongs to the space $TW_p^1(\Omega)^*$ dual to $TW_p^1(\Omega)$.

Lemma 1. *Suppose that X is a Banach space and X_0 is its subspace. Let $\dot{X}$ denote the quotient space X/X_0 with norm*

$$\|\dot{x}\| = \inf\{\|y\| : y \in \dot{x}\}.$$

[1] We consider the model problem (2), (3) for simplicity. In fact, what is said below is valid for more general elliptic equations [3].

(i) *If* $f \in X^*$, i.e., *f is a continuous linear functional on X, and* $f|_{X_0} = 0$, *then the functional* $\dot{X} \ni \dot{x} \mapsto \dot{f}(\dot{x}) = f(x)$ *belongs to the space* $(\dot{X})^*$, *and* $\|\dot{f}\| = \|f\|$.

(ii) *If* $\dot{f} \in (\dot{X})^*$, *then the functional* $X \ni x \mapsto f(x) = \dot{f}(\dot{x})$ *belongs to* X^* *and* $f|_{X_0} = 0$.

The proof of this assertion follows easily from the definition of a quotient space.

Now, let us show that if $F \in W_p^1(\Omega)^*$ (the inclusion $\operatorname{Ker} F \supset C_0^\infty(\Omega)$ is not required), then there exists a unique function $u \in W_p^1(\Omega)$ satisfying the equality

$$L(u, v) = \langle F, v\rangle \tag{5}$$

for all $v \in W_p^1(\Omega)$. Calculating the variation of the functional

$$W_p^1(\Omega) \ni v \mapsto G(v) = \int_\Omega (|\nabla v|^p + a|v|^p)dx - p\langle F, v\rangle, \tag{6}$$

we see that identity (5) is equivalent to the problem of minimizing functional (6) on $W_p^1(\Omega)$. Consider a minimizing sequence $\{v_k\}$ for $G(v)$. Since the functional $W_p^1(\Omega) \ni v \mapsto \langle F, v\rangle$ is continuous, it follows that, for some constants $c_1, c_2 > 0$, we have

$$G(v) \ge c_1\|v\|^p_{W_p^1(\Omega)} - c_2\|v\|_{W_p^1(\Omega)},$$

and since $p > 1$, it follows that $G(v) \longrightarrow +\infty$ as $\|v\|_{W_p^1(\Omega)} \longrightarrow \infty$. This implies the boundedness of the minimizing sequence; hence, this sequence has a subsequence (which we denote by the same symbol $\{v_k\}$) weakly convergent in $W_p^1(\Omega)$. Suppose that u is its limit. Let us show that

$$G(u) = \min\{G(v) \colon v \in W_p^1(\Omega)\}. \tag{7}$$

Consider the norm

$$[v] = \left(\int_\Omega (|\nabla v|^p + a|v|^p)dx\right)^{1/p} \tag{8}$$

on $W_p^1(\Omega)$, which is equivalent to $\|\cdot\|_{W_p^1(\Omega)}$. For any $\varepsilon > 0$, there exists a functional $f_0 \in W_p^1(\Omega)^*$ such that its norm (8) equals 1 and

$$[u]^p - \varepsilon < |\langle f_0, u\rangle|^p = \lim_{k\to\infty}|\langle f_0, v_k\rangle|^p.$$

Thus, $[v_k]^p > [u]^p$ for all sufficiently large k. For the same k, we have

$$G(v_k) > [u]^p - \varepsilon - p\langle F, v_k\rangle,$$

and the convergence $\langle F, v_k\rangle \longrightarrow \langle F, u\rangle$ implies

$$\inf\{G(v) \colon v \in W_p^1(\Omega)\} \ge [u]^p - \varepsilon - p\langle F, u\rangle.$$

The arbitrariness in the choice of ε implies (7).

Thus, if the functional on the right-hand side of (5) is continuous on $W_p^1(\Omega)$ (in particular, $F \in (TW_p^1(\Omega))^*$), then problem (2), (3) is solvable. Note also that the uniqueness of its solution follows from the strict convexity of functional (6). Indeed, assuming the existence of two different functions $u_1, u_2 \in W_p^1(\Omega)$ minimizing functional (6) and setting $u = (u_1 + u_2)/2$, we obtain the contradictory inequality

$$G(u) < (G(u_1) + G(u_2))/2 = \min\{G(v) \colon v \in W_p^1(\Omega)\}.$$

Thus, studying the solvability of problem (2), (3) (or, equivalently, of problem (4) in which the kernel of the functional f contains $C_0^\infty(\Omega)$) reduces to describing the space $TW_p^1(\Omega)^*$.

In what follows, for a Lipschitz curve S, we write $W_p^{-1/p'}(S)$ (where $p' = p/(p-1)$) instead of $(W_{p'}^{-1/p'}(S))^*$ and set

$$\|f\|_{W_{p'}^{-1/p'}} = \sup\{|\langle f, v\rangle|: v \in W_p^{1-1/p}(S), \|v\| = 1\}.$$

Below, we characterize the space $TW_p^1(\Omega)^*$ for a planar domain with a peak in terms of the spaces $W_{p'}^{-1/p'}$ on Lipschitz curves and some function spaces on the interval (0, 1) of the real line.

2. A CLASS OF DOMAINS AND AUXILIARY ASSERTIONS

In this section, we describe a planar domain with boundary containing the vertex of a peak. Let Ω be a domain with compact boundary in $\mathbf{R}^2$. Suppose that $O \in \partial\Omega$ and the curve $\partial\Omega\backslash\{O\}$ can locally be represented as the graph of a Lipschitz function. Consider Cartesian coordinates (x, y) with origin at O. Let φ_- and φ_+ be functions from $C^{0,1}[0, 1]$ such that $\varphi_\pm(0) = 0$, $\varphi'_\pm(t) \longrightarrow 0$ as $t \longrightarrow +0$, and the function $\varphi = \varphi_- + \varphi_+$ increases on the interval [0, 1].

Definition. A point O is said to be the vertex of an outward peak with respect to Ω if this point has a neighborhood U for which

$$U \cap \Omega = \{(x, y): x \in (0, 1), \varphi_-(x) < y < \varphi_+(x)\}.$$

A point O is said to be the vertex of a peak inward with respect to Ω if, for some neighborhood U of O,

$$U\backslash\overline{\Omega} = \{(x, y): x \in (0, 1), \varphi_-(x) < y < \varphi_+(x)\}.$$

For simplicity, we assume in what follows that $\partial\Omega \cap U = \{O\} \cup \Gamma_- \cup \Gamma_+$, where $\Gamma_\pm = \{(x, \varphi_\pm(x)): x \in (0, 1)\}$.

Below, we use the following notation. Let $\Gamma = \Gamma_- \cup \Gamma_+$. For a function v defined on Γ, we set

$$v_-(x) = v(x, \varphi_-(x)) \text{ and } v_+(x) = v(x, \varphi_+(x)), \text{ where } x \in (0, 1).$$

We write $v = (v_-, v_+)$. By $\tilde{v}$ we denote the function defined on Γ by the condition

$$(\tilde{v})_- = (\tilde{v})_+ = (v_+ + v_-)/2.$$

For $f \in TW_p^1(\Omega)^*$ and a Lipschitz function λ on $\partial\Omega$, we set

$$\langle \lambda f, v\rangle = \langle f, \lambda v\rangle, \text{ where } v \in TW_p^1(\Omega).$$

We say that a functional $f \in TW_p^1(\Omega)$ is supported on a curve $\gamma \subset \partial\Omega$ (and write supp$f \subset \gamma$) if $v|_\gamma = 0$ implies $\langle f, v\rangle = 0$.

We say that positive quantities a and b are equivalent (and write $a \sim b$) if $c_1 \le a/b \le c_2$ for positive constants c_1 and c_2 not depending on a and b. By c we denote positive constants which do not depend on the factors which they multiply and may take different values in the same chain of inequalities.

Throughout the rest of this section, Ω denotes a planar domain with an outward peak. Below, we define a special partition of unity on $\partial\Omega\backslash\{O\}$, which plays an important role in what follows. Consider the sequence $\{x_k\}$ defined by

$$x_0 \in (0, 1) \text{ and } x_{k+1} + \varphi(x_{k+1}) = x_k \text{ for } k = 0, 1, \dots.$$

Clearly, $\{x_k\}$ decreases; moreover,

$$x_k \longrightarrow 0, \quad x_{k+1}^{-1} x_k \longrightarrow 1, \text{ and } \varphi(x_{k+1})^{-1}\varphi(x_k) \longrightarrow 1.$$

Let $\{\mu_k\}_{k \geq 1}$ be a smooth partition of unity on the interval $(0, x_1]$ subordinate to the cover by the intervals $\Delta_k = (x_{k+1}, x_{k-1})$, i.e., a set of functions $\mu_k \in C_0^\infty(\Delta_k)$ satisfying the conditions

$$0 \leq \mu_k \leq 1 \text{ and } \sum_{k \geq 1} \mu_k(x) = 1 \text{ for } x \in (0, x_1].$$

This partition can be constructed so that, first,

$$\operatorname{dist}(\operatorname{supp}\mu_k, \mathbf{R}^1 \backslash \Delta_k) \geq \operatorname{const}\varphi(x_k) \text{ and } |\mu_k'| \leq \operatorname{const}\varphi(x_k)^{-1}, \tag{9}$$

where the constants depend only on φ, and, secondly, the equality $\sum_{k \geq 1} \mu_k(x) = 1$ holds at $x \in (0, \delta]$ for some $\delta > x_1$.

We set $\mu_0(x) = 0$ for $x < x_1$ and $\mu_0(x) = 1 - \mu_1(x)$ for $x \geq x_1$. Obviously, $\sum_{k \geq 0} \mu_k(x) = 1$ for all $x \in (0, 1]$. The partition of unity thus defined depends only on x_0 and the function φ. In what follows, we assume it to be fixed. We also set

$$\Gamma_0 = \partial\Omega \backslash \{(x, y) \in \bar{\Gamma}: x \leq x_1\}.$$

Note that the partition of unity on (0, 1] induces a partition of unity on $\partial\Omega \backslash \{O\}$, whose elements defined by $\mu_0 = 1$ on $\Gamma_0 \backslash \Gamma$, $\mu_k(x, y) = \mu_k(x)$ for $(x, y) \in \Gamma$ and $x \in \Delta_k$, where $k \geq 0$; for $k \geq 1$, $\mu_k = 0$ on $\partial\Omega \backslash \Gamma$.

For a plane domain with an outward peak, the space $TW_p^1(\Omega)$ can be described explicitly [5; 6, 6.4]: it consists of all functions of class $L_{p,\,\mathrm{loc}}(\partial\Omega \backslash \{O\})$ with finite norm

$$\begin{aligned} &\|v\|_{W_p^{1-1/p}(\Gamma_0)} + \left(\int_0^1 (|v_+|^p + |v_-|^p)\varphi(x)dx\right)^{1/p} + |v_+|_p + |v_-|_p \\ &+ \left(\int_0^1 |v_+(x) - v_-(x)|^p \varphi(x)^{1-p} dx\right)^{1/p}, \end{aligned} \tag{10}$$

where

$$|u|_p = \left(\iint_{\{t,\, \tau \in (0,1)\}} \frac{|u(t) - u(\tau)|^p}{|t - \tau|^p} \chi\left(\frac{|t - \tau|}{M(t, \tau)}\right) dt d\tau\right)^{1/p}, \tag{11}$$

$M(t, \tau) = \max\{\varphi(t), \varphi(\tau)\}$, and χ is the characteristic function of the interval (0, 1). Note that norm (10) is equivalent to the norm $\|v\|_{TW_p^1(\Omega)}$.

In what follows, we need the following auxiliary assertions.

Lemma 2. *Any function* $v \in TW_p^1(\Omega)$ *can be represented as the sum*

$$v = \mu_0 v + (1 - \mu_0)\tilde{v} + (1 - \mu_0)(v - \tilde{v})$$

of three terms, each of which is a continuous linear function of v *belonging to* $TW_p^1(\Omega)$.

Proof. Let us show that, for the first term,

$$\|\mu_0 v\|_{TW_p^1(\Omega)} \leq c\|v\|_{W_p^{1-1/p}(\Gamma_0)}. \tag{12}$$

It is sufficient to show that

$$\|\mu_0 v\|^p_{W_p^{1-1/p}(\Gamma_0)} \leq c\|v\|^p_{W_p^{1-1/p}(\Gamma_0)} \tag{13}$$

and

$$|\mu_0 v_\pm|_p^p \leq c\|v\|^p_{W_p^{1-1/p}(\Gamma_0)}. \tag{14}$$

Inequality (13) follows from

$$[\mu_0 v]^p_{p,\Gamma_0} \le c\int\limits_{\Gamma_0} |v(Q)|^p ds_Q \int\limits_{\Gamma_0} \frac{|\mu_0(Q)-\mu_0(M)|^p}{|Q-M|^p} ds_M$$

$$+ c\int\limits_{\Gamma_0} \mu_0(M)^p ds_M \int\limits_{\Gamma_0} \frac{|v(M)-v(Q)|^p}{|M-Q|^p} ds_Q$$

and the Lipschitz continuity of the function μ_0 on $\partial\Omega$ (recall that $[\cdot]_{p,\Gamma_0}$ denotes the seminorm (1)). Let us prove (14) for the function v_-. Definition (11) implies

$$|\mu_0 v_-|_p \le c\int\limits_0^1 |v_-(t)|^p dt \int\limits_{t-\varphi(t)}^{t} \frac{|\mu_0(t)-\mu_0(\tau)|^p}{|t-\tau|^p} d\tau$$
$$+ c\int\limits_0^1 dt \int\limits_{t-\varphi(t)}^{t} \frac{|v_-(t)-v_-(\tau)|^p}{|t-\tau|^p} d\tau.$$

The first term on the right-hand side does not exceed $c\|v\|^p_{L_p(\Gamma_-\cap\Gamma_0)}$, because $\mu_0|_{(0,x_1)} = 0$. In the second term, integration is essentially over $t > \tau > x_1$; therefore, this term does not exceed $c[v]^p_{p,\Gamma_-\cap\Gamma_0}$, where $[\cdot]_{p,\Gamma_-\cap\Gamma_0}$ denotes seminorm (1). A similar argument proves (14) for the function v_+. Thus, (12) does hold.

Now, let us show that

$$\|(1-\mu_0)\tilde{v}\|_{TW^1_p(\Omega)} \le c\|v\|_{TW^1_p(\Omega)}. \tag{15}$$

Since the support of the function $1-\mu_0$ is at a positive distance from $\Gamma_0\backslash\Gamma$, it follows that

$$\|(1-\mu_0)\tilde{v}\|^p_{L_p(\Gamma_0)} \le \|\tilde{v}\|^p_{L_p(\Gamma_0\cap\Gamma)} \le c\int\limits_{x_1}^1 |v_+ + v_-|^p \varphi(x)dx. \tag{16}$$

For the function $w = (1-\mu_0)\tilde{v}$, we also have

$$\int\limits_{\Gamma_0\cap\Gamma} |w(Q)|^p ds_Q \int\limits_{\Gamma_0\backslash\Gamma} \frac{ds_M}{|M-Q|^p} \le c\|\tilde{v}\|^p_{L_p(\Gamma_0\cap\Gamma)};$$

combining this relation with (16), we obtain

$$c[w]^p_{p,\Gamma_0} \le [w]^p_{p,\Gamma_0\cap\Gamma} + \int\limits_{x_1}^1 |v_+ + v_-|^p \varphi(x)dx. \tag{17}$$

Next,

$$[w]^p_{p,\Gamma_0\cap\Gamma} \le c[\tilde{v}]^p_{p,\Gamma_0\cap\Gamma} + c\iint\limits_{\{M,Q\in\Gamma_0\cap\Gamma\}} \frac{|\mu_0(M)-\mu_0(Q)|^p}{|M-Q|^p}|\tilde{v}(M)|^p ds_Q ds_M.$$

This, together with (17), implies

$$c[w]_{p,\Gamma_0} \le [v_- + v_+]p, (x_1, 1) + \left(\int\limits_0^1 |v_+ + v_-|^p \varphi(x)dx\right). \tag{18}$$

Note that (16) and (18) yield, in particular,

$$\|(1-\mu_0)\tilde{v}\|^p_{W_p^{1-1/p}(\Gamma_0)} \le c\|v\|^p_{W_p^{1-1/p}(\Gamma_0)}. \tag{19}$$

To complete the verification of (15), it suffices to show that

$$|(1-\mu_0)\bar{v}|_p^p \le c\int_0^1 |\bar{v}(x)|^p dx + c|\bar{v}|_p^p, \tag{20}$$

where $\bar{v}(x) = v_-(x) + v_+(x)$. Indeed, it follows from definition (11) that the left-hand side of (20) does not exceed the sum

$$c\int_0^1 |v(t)|^p dt \int_{t-\varphi(t)}^{t} \frac{|\mu_0(t)-\mu_0(\tau)|^p}{|t-\tau|^p} d\tau + c\int_0^1 dt \int_{t-\varphi(x)}^{t} \frac{|\bar{v}(t)-\bar{v}(\tau)|^p}{|t-\tau|^p} dt d\tau;$$

note that the first term of this sum does not exceed the first term on the right-hand side of (20), and the second term does not exceed the second term on the right-hand side of (20). To complete the proof of the lemma, it remains to note that (12) and (15) imply the estimate

$$\|(1-\mu_0)(v-\tilde{v})\|_{TW_p^1(\Omega)} \le c\|v\|_{TW_p^1(\Omega)}.$$

The following lemma asserts that each functional $f \in TW_p^1(\Omega)^*$ generates a family of functionals from $W_{p'}^{-1/p'}(\Delta_k)$ for $k = 1, 2, \ldots$.

Lemma 3. *The mapping* $W_p^{1-1/p}(\Delta_k) \ni u \mapsto (-\mu_k u, \mu_k u) \in TW_p^1(\Omega)$ *is continuous for* $k = 1, 2, \ldots$.

Proof. Let $\Omega_k = \{(x, y) \in W: x \in \Delta_k\}$. Consider the function v on $\partial\Omega_k$ defined by $v(x, y) = 0$ for $x = x_{k\pm1}$, $v(x, \varphi_-(x)) = -\mu_k(x)u(x)$, and $v(x, \varphi_+(x)) = \mu_k(x)u(x)$. Since $\mu_k \in C_0^\infty(\Delta_k)$ and $\varphi_\pm \in C^{0,1}([0, 1])$, it follows that $v \in W_p^{1-1/p}(\partial\Omega_k)$, and

$$\|v\|_{W_p^{1-1/p}(\partial\Omega_k)} \le c_k\|u\|_{W_p^{1-1/p}(\Delta_k)},$$

where the constant does not depend on u. By a theorem of Gagliardo [2], there exists a continuous extension operator $W_p^{1-1/p}(\partial\Omega_k) \ni v \mapsto V \in W_p^1(\Omega_k)$. Setting $V|_{\Omega\setminus\Omega_k} = 0$, we obtain a continuous extension operator $W_p^{1-1/p}(\partial\Omega_k) \ni v \mapsto V \in W_p^1(\Omega)$; therefore,

$$\|V\|_{W_p^1(\Omega)} \le c_k\|u\|_{W_p^{1-1/p}(\Delta_k)}.$$

It remains to note that $V_- = -\mu_k u$ and $V_+ = \mu_k u$.

Lemma 4. *If* $v \in L_{p,\,\mathrm{loc}}(\partial\Omega\setminus\{O\})$ *and* $v(x, y) = 0$ *outside the set* $\{(x, y) \in \Gamma: x < x_0\}$, *then the norm* $\|v\|_{TW_p^1(\Omega)}$ *is equivalent to norm* (10) *from which the first term is omitted.*

Proof. It is sufficient to verify the estimate

$$\|v\|^p_{W_p^{1-1/p}(\Gamma_0)} \le c|v_-|_p^p + |v_+|_p^p + c\int_0^1 (|v_-|^p + |v_+|^p)\varphi(x)dx. \tag{21}$$

Since the support of v is at a positive distance from $\Gamma_0\setminus\Gamma$, it follows that

$$\|v\|^p_{L_p(\Gamma_0)} = \|v\|^p_{L_p(\Gamma_0\cap\Gamma)} \le c\int_0^1 (|v_-|^p + |v_+|^p)\varphi(x)dx, \tag{22}$$

and

$$[v]^p_{p,\Gamma_0} \le [v]^p_{p,\Gamma_0\cap\Gamma} + c\int_{\Gamma_0\cap\Gamma} |v(Q)|^p ds_Q \int_{\Gamma_0\backslash\Gamma} \frac{ds_M}{|Q-M|^p}$$

$$\le [v]^p_{p,\Gamma_0\cap\Gamma} + c\int_0^1 (|v_-|^p + |v_+|^p)\varphi(x)dx.$$

Moreover,

$$c[v]^p_{p,\Gamma_0\cap\Gamma} \le [v]^p_{p,\Gamma_0\cap\Gamma_-} + [v]^p_{p,\Gamma_0\cap\Gamma_+} + \int_{\Gamma_0\cap\Gamma_-} ds_Q \int_{\Gamma_0\cap\Gamma_+} \frac{|v(Q)-v(M)|^p}{|Q-M|^p} ds_M.$$

In the last integral, $|Q - M| \ge \text{const} > 0$; hence, this integral does not exceed $c[v]^p_{L_p(\Gamma_0\cap\Gamma)}$ and, therefore, the right-hand side of (22). It remains to estimate the quantities $[v]^p_{p,\Gamma_0\cap\Gamma_\pm}$. Clearly,

$$[v]^p_{p,\Gamma_0\cap\Gamma_-} \le c \iint_{\{t,\tau\in(x_1,1)\}} \frac{|v_-(t)-v_-(\tau)|^p}{|t-\tau)^p} dt d\tau.$$

Let us represent the last integral as the sum of two integrals, over the set $|t - \tau| < \max\{\varphi(t), \varphi(\tau)\}$ and over the complementary set. The former integral is at most $|v_-|^p_p$, and in the latter is at most $|t - \tau| \ge \text{const} > 0$; therefore,

$$[v]^p_{p,\Gamma_0\cap\Gamma_-} \le c|v_-|^p_p + c\int_{x_1}^1 |v_-|^p \varphi(x)dx.$$

The estimate obtained by replacing v_- by v_+ in this inequality is derived in a similar way. This proves (21).

To prove the main result, we need one more lemma.

Lemma 5. *Suppose that* $u \in L_{p,\text{loc}}(0, 1)$ *and* $u(x) = 0$ for $x > x_0$. *Then, for sufficiently small* x_0, *seminorm* (11) *is transformed into an equivalent norm when the factor* $\chi(|t-\tau|/M(t,\tau))$ *in the integrand is replaced by* $\chi(|t-\tau|/(2M(t,\tau)))$.

For $p = 2$, this assertion is known (see Lemma 8.3/2 in [6]). The proof for $p \ne 2$ is the same except in minor details, and we omit it.

3. THE SPACE $TW^1_p(\Omega)^*$ FOR A PLANAR DOMAIN WITH AN OUTWARD PEAK

Let $W_p(0, 1)$ denote the space of functions from $L_{p,\text{loc}}(0, 1)$ that have finite norm

$$\|u\|_{W_p(0,1)} = \left(\int_0^1 |u(x)|^p \varphi(x)dx\right)^{1/p} + |u|_p,$$

where $|\cdot|_p$ is the seminorm defined by (11).

The space $TW^1_p(\Omega)^*$ for a planar domain with an outward peak is described by the following theorem.

Theorem 1. *Let* $\Omega \subset \mathbf{R}^2$ *be a domain with an outward peak, and let* $\{\mu_k\}$ *be the partition of unity on* $\partial\Omega\backslash\{O\}$ *constructed above.*

(i) *If* $f \in TW^1_p(\Omega)^*$, *then* $f = f^{(1)} + f^{(2)} + f^{(3)}$, *where*

$$TW^1_p(\Omega) \ni v \mapsto \langle f^{(1)}, v\rangle = \langle f, \mu_0 v\rangle,$$

$$TW_p^1(\Omega) \ni v \mapsto \langle f^{(2)}, v\rangle = \langle f, (1-\mu_0)\tilde{v}\rangle,$$

$$TW_p^1(\Omega) \ni v \mapsto \langle f^{(3)}, v\rangle = \langle f, (1-\mu_0)(v-\tilde{v})\rangle.$$

The functionals $f^{(j)}$ have the following properties. First, $f^{(j)} \in TW_p^1(\Omega)^*$ for $j = 1, 2, 3$. Moreover, $f^{(1)} \in W_{p'}^{-1/p'}(\Gamma_0)$ and supp $f^{(1)} \subset \Gamma_0$. *The functionals $f^{(2)}$ and $f^{(3)}$ are supported on the set $\{(x, y) \in U \cap \partial\Omega: x < x_0\}$. The functional $f^{(2)}$ belongs to the class $W_p(0, 1)^*$ in the sense that*

$$\left|\langle f^{(2)}, v\rangle\right| \le c\|v_- + v_+\|_{W_p(0,1)}, \tag{23}$$

where the constant does not depend on v. The functional $f^{(3)}$ has the representation

$$\langle f^{(3)}, v\rangle = \sum_{k\ge 1}\langle f_k, v_+ - v_-\rangle, \tag{24}$$

where the functionals f_k are defined by

$$W_p^{1-1/p}(\Delta_k) \ni u \mapsto \langle f_k, u\rangle = \langle f, (-\mu_k u, \mu_k u)\rangle$$

and belong to the class $W_{p'}^{-1/p'}(\Delta_k)$. Moreover,

$$\left(\sum_{k\ge 1}\|f_k\|^{p'}_{W_{p'}^{-1/p'}(\Delta_k)}\right)^{1/p'} \le c\|f^{(3)}\|_{TW_p^1(\Omega)^*}, \tag{25}$$

where the constant depends only on p and Ω.

(ii) *Suppose that, for $k \ge 1$, the functionals $f_k \in W_{p'}^{-1/p'}(\Delta_k)$ are supported inside Δ_k and the sum on the left-hand side of* (25) *is finite. Then, the functional $TW_p^1(\Omega) \ni v \mapsto \langle f^{(3)}, v\rangle$ defined by* (24) *is continuous and supported on $\{(x, y) \in U \cap \partial\Omega: x < x_0\}$; moreover,*

$$\|f^{(3)}\|_{TW_p^1(\Omega)^*} \le c\left(\sum_{k\ge 1}\|f_k\|^{p'}_{W_{p'}^{-1/p'}(\Delta_k)}\right)^{1/p'}, \tag{26}$$

where $c = c(p, \Omega)$. Suppose that, in addition, $h \in W_{p'}^{-1/p'}(\Gamma_0)$ and $g \in W_p(0, 1)^*$. Let *$f^{(1)} = \mu_0 h$, and let*

$$\langle f^{(2)}, v\rangle = \langle g, (1-\mu_0)(v_- + v_+)/2\rangle \quad \text{for} \quad v \in TW_p^1(\Omega). \tag{27}$$

Then, $f^{(1)}, f^{(2)} \in TW_p^1(\Omega)^*$ and $f^{(1)} \in W_{p'}^{-1/p'}(\Gamma_0)$.

Proof. (i) The inclusions $f^{(j)} \in TW_p^1(\Omega)^*$ follow from Lemma 2. In the proof of the same lemma, the inclusion $f^{(1)} \in W_{p'}^{-1/p'}(\Gamma_0)$ is established (see inequality (13)). Estimate (23) follows from the proof of Lemma 2 as well (see (16), (18), (20)).

Let us prove (25). By Lemma 3, $f_k \in W_{p'}^{-1/p'}(\Delta_k)$. We set $\alpha_k = \|f_k\|_{W_{p'}^{-1/p'}(\Delta_k)}$ and choose a function $u_k \in W_p^{1-1/p}(\Delta_k)$ for which

$$\|u_k\|_{W_p^{1-1/p}(\Delta_k)} \le 1 \quad \text{and} \quad \alpha_k \le 2\langle f_k, u_k\rangle.$$

For any positive integer N, we have

$$\sum_{k=1}^{N}\alpha_k^{p'} \le 2\left\langle f, \sum_{k=1}^{N}\alpha_k^{p'-1}(-\mu_k u_k, \mu_k u_k)\right\rangle. \tag{28}$$

Obviously, $\mu_k = (1 - \mu_0)\mu_k$ for $k \ge 2$. We have $\mu_1 = (1 - \mu_0)\mu_1$ on the interval $[x_2, x_1]$ and $\mu_1 = 1 - \mu_0$ on the interval $[x_1, x_0]$. Consider the function $\nu_1 \in C_0^\infty(\Delta_1)$ defined by $\nu_1 = \mu_1$ on $[x_2, x_1]$, $\nu_1(x) = 1$ for $x \in [x_1, x_0]$ and $x \in \operatorname{supp}\mu_1$, and $\nu_1 = 0$ in a neighborhood of x_0. The function μ_1 can be represented as $(1 - \mu_0)\nu_1$. Let $\nu_k = \mu_k$ for $k \ge 2$. Then, $\mu_k = (1 - \mu_0)\nu_k$ for all $k \ge 1$, and inequality (28) can be rewritten as

$$\sum_{k=1}^{N} \alpha_k^{p'} \le 2\langle f, (1-\mu_0)v\rangle, \tag{29}$$

where

$$v = \sum_{k=1}^{N} \alpha_k^{p'-1} v_k \quad \text{for} \quad v_k = (-\nu_k u_k, \nu_k u_k) \tag{30}$$

(recall that the expression $v = (u, w)$ means that v is a function on Γ for which $v_- = u$ and $v_+ = w$). Setting $\nu_k u_k = 0$ outside Δ_k, we extend v_k to $\partial\Omega$. Thus, the function v in (29), (30) is defined on $\partial\Omega$, and by virtue of $\tilde{v} = 0$, (29) implies

$$\sum_{k=1}^{N} \alpha_k^{p'} \le 2\langle f^{(3)}, v\rangle. \tag{31}$$

Let us estimate $\|v\|_{TW_p^1(\Omega)}$. Since $v(x, \varphi_\pm(x)) = 0$ for $x > x_0$ and $v_+ = -v_-$, it follows from Lemma 4 that

$$\|v\|^p_{TW_p^1(\Omega)} \le c|v_+|_p^p + c\int_0^1 |v_+(x)|^p \varphi(x)^{1-p}dx. \tag{32}$$

Here, $|\cdot|_p$ denotes the seminorm defined by (11). Each point $x \in (0, 1)$ belongs to at most two supports of functions from the set $\{\nu_k\}_{k=1}^N$. Therefore,

$$|v_+(x)|^p = \left|\sum_{k=1}^{N} \alpha_k^{p'-1}\nu_k(x)u_k(x)\right|^p \le c\sum_{k=1}^{N} \alpha_k^{p'}|u_k(x)|^p \chi_{\Delta_k}(x),$$

where χ_{Δ_k} is the characteristic function of the interval Δ_k. Thus,

$$\int_0^1 |v_+(x)|^p \varphi(x)^{1-p}dx \le c\sum_{k=1}^{N} \alpha_k^{p'} \int_{\Delta_k} |u_k|^p \varphi(x)^{1-p}dx.$$

Since $\|u_k\|_{W_p^{1-1/p}(\Delta_k)} \le 1$, it follows that the integral over Δ_k is uniformly bounded with respect to k, and the second term on the right-hand side of (32) does not exceed $c\sum_{k=1}^N \alpha_k^{p'}$. To estimate $|v_+|_p$, note that

$$|v_+(t) - v_+(\tau)|^p \le c\sum_{k=1}^{N} \alpha_k^{p'}|\nu_k(t)u_k(t) - \nu_k(\tau)u_k(\tau)|^p,$$

which implies

$$|v_+|_p^p \le c\sum_{k=1}^{N} \alpha_k^{p'}|\nu_k u_k|_p^p.$$

Since $\operatorname{supp}\nu_k \subset \Delta_k$, it follows that

$$c|\nu_k u_k|_p^p \le [\nu_k u_k]^p_{p, \Delta_k} + \int_{\Delta_k} |\nu_k(t)u_k(t)|^p dt \int_{\tau \notin \Delta_k} \chi\left(\frac{|t-\tau|}{M(t, \tau)}\right)\frac{d\tau}{|t-\tau|^p}, \tag{33}$$

where $[\cdot]_{p,\Delta_k}$ is the seminorm defined by (1). By virtue of (9), in the last integral, we have $|t-\tau| \ge c\varphi(x_k)$. Moreover, $\varphi(t) \sim \varphi(\tau)$ provided that $|t-\tau| < M(t, \tau)$. It follows that the double integral in (33) is bounded above by the quantity

$$c\varphi(x_k)^{1-p}\int_{\Delta_k}|\nu_k u_k|^p dx, \tag{34}$$

which is at most $c\|u_k\|^p_{W_p^{1-1/p}(\Delta_k)}$, that is, does not exceed c. For the first term on the right-hand side of (33), we have

$$\begin{aligned}[\nu_k u_k]^p_{p,\Delta_k} \le c \iint_{\Delta_k\times\Delta_k} |\nu_k(t)|^p |u_k(t)-u_k(\tau)|^p \frac{dtd\tau}{|t-\tau|^p} \\ + c\iint_{\Delta_k\times\Delta_k} |u_k(\tau)|^p|\nu_k(t)-\nu_k(\tau)|^p\frac{dtd\tau}{|t-\tau|^p}.\end{aligned} \tag{35}$$

The first term on the right-hand side at most $c[u_k]^p_{p,\Delta_k} \le c$. To estimate the second term, note that $|\nu'_k| \le c\varphi^{-1}(x_k)$, and hence this term has majorant (34). Thus, we have shown that the right-hand side of (32) does not exceed $c\sum_{k=1}^N \alpha_k^{p'}$. This, together with (31) and (32), implies

$$\sum_{k=1}^N \alpha_k^{p'} \le 2\|f^{(3)}\|_{TW_p^1(\Omega)^*}\|v\|_{TW_p^1(\Omega)} \le c\|f^{(3)}\|\left(\sum_{k=1}^N \alpha_k^{p'}\right)^{1/p},$$

which gives (25).

(ii) Suppose that $v \in TW_p^1(\Omega)$ and $u = v_+ - v_-$. Since $\operatorname{supp} f_k \subset \Delta_k$, it follows that

$$|\langle f_k, u\rangle| \le \left|\left\langle f_k, \sum_{|i-k|\le 1}\mu_i u\right\rangle\right| \le \sum_{|i-k|\le 1}\|f_k\|_{W_{p'}^{-1/p'}(\Delta_k)}\|\mu_i u\|_{W_p^{1-1/p}(\Delta_k)}.$$

Applying Hölder's inequality, we obtain

$$\sum_{k\ge 1}|\langle f_k, u\rangle| \le \left(\sum_{k,i}\|f_k\|^{p'}_{W_{p'}^{-1/p'}(\Delta_k)}\right)^{1/p'}\left(\sum_{k,i}\|\mu_i u\|^p_{W_p^{1-1/p}(\Delta_k)}\right)^{1/p}.$$

Thus, to prove estimate (26), it suffices to show that the last factor does not exceed $c\|v\|_{TW_p^1(\Omega)}$. For this purpose, we first estimate the general term of the last sum over $\{k \ge 1, |i-k| \le 1\}$. We have

$$\|\mu_i u\|^p_{W_p^{1-1/p}(\Delta_k)} \le c|\Delta_k|^{1-p}\|\mu_i u\|^p_{L_p(\Delta_k)} + [\mu_i u]^p_{p,\Delta_k},$$

where $|\Delta_k|$ is the length of Δ_k and $[\cdot]_{p,\Delta_k}$ is the seminorm defined by (1). It follows that

$$\begin{aligned}\|\mu_i u\|^p_{W_p^{1-1/p}(\Delta_k)} \le c\iint_{\Delta_k\times\Delta_k}\frac{|u(t)-u(\tau)|^p}{|t-\tau|^p}\mu_i(t)^p dtd\tau \\ + c\iint_{\Delta_k\times\Delta_k}\frac{|\mu_i(t)-\mu_i(\tau)|^p}{|t-\tau|^p}|u(\tau)|^p dtd\tau \quad + c\int_{\Delta_k}|u(x)|^p\varphi(x)^{1-p}dx.\end{aligned} \tag{36}$$

Since $|i-k| \le 1$, we have $|\mu_i'| \le c\varphi(x_k)^{-1}$, and the next to last term on the right-hand side of (36) is bounded above by the last term. Summing (36) over k and i and noting that $\Delta_k \times \Delta_k \subset \{t, \tau \in (0, 1): |t-\tau| < 2M(t, \tau)\}$, we obtain the estimate

$$\sum_{k,i} \|\mu_i u\|^p_{W_p^{1-1/p}(\Delta_k)} \le c\int_0^1 |u(x)|^p \varphi(x)^{1-p} + cI, \tag{37}$$

in which

$$I = \iint_{\{t, \tau \in (0,1)\}} |u(t) - u(\tau)|^p \chi\left(\frac{|t-\tau|}{2M(t,\tau)}\right)\frac{dt d\tau}{|t-\tau|^p}$$

(we use the same notation as in (11)). We have $I \le cI_1 + cI_2$, where I_1 and I_2 are the integrals obtained by replacing the factor $|u(t) - u(\tau)|^p$ in the integrand I by

$$|(1-\mu_0(t)u(t) - (1-\mu_0(\tau))u(\tau)|^p \text{ and } |\mu_0(t)u(t) - \mu_0(\tau))u(\tau)|^p,$$

respectively. By Lemma 5, we have $I_1 \le c|(1-\mu_0)u|_p^p$, whence

$$I_1 \le c|u|_p^p + c\iint_{\{|t-\tau| < M(t,\tau)\}} \frac{|\mu_0(t) - \mu_0(\tau)|^p}{|t-\tau|^p}|u(t)|^p dt d\tau;$$

therefore, $I_1 \le c|u|_p^p + \|u\varphi^{-1+1/p}\|_{L_p(0,1)}$. For I_2, we have $I_2 \le cI_3 + cI_4$, where

$$I_3 = \int_0^1 |u(t)|^p dt \int_{t-2\varphi(t)}^{t} \frac{|\mu_0(t) - \mu_0(\tau)|^p d\tau}{|t-\tau|^p} \text{ and } I_4 = \int_0^1 dt \int_{t-2\varphi(t)}^{t} \frac{|u(t) - u(\tau)|^p}{|t-\tau|^p}\mu(\tau)^p d\tau.$$

Since $\mu_0(t) = 0$ for $t < x_1$, it follows that $I_3 \le c\|u\|_{L_p(x_1, 1)}$ and $I_4 \le c[u]^p_{p,(x_1,1)}$. Combining (37) with these inequalities, we obtain

$$c\sum_{k,i} \|\mu_i u\|^p_{W_p^{1-1/p}(\Delta_k)} \le |u|_p^p + \int_0^1 |u|^p \varphi(x)^{1-p} dx + \|u\|^p_{W_p^{1-1/p}(x_1,1)},$$

where $u = v_+ - v_-$. The right-hand side of this inequality does not exceed a constant multiplied by norm (10) to the power p. This proves estimate (26).

To complete the proof of the theorem, note that the inclusion $f^{(1)} = \mu_0 h \in W_{p'}^{-1/p'}(\Gamma_0)$ for $h \in W_{p'}^{-1/p'}(\Gamma_0)$ follows from Lemma 2. The continuity of functional (27) follows from the inclusion $g \in W_p(0, 1)^*$ and the continuity of the mapping $TW_p^1(\Omega) \ni v \mapsto (1-\mu_0)\tilde{v}$, which is also implied by the proof of Lemma 2. This proves the theorem.

Below, we state several corollaries of the theorem proved above. The first of them follows from the definition of a solution to the Neumann problem considered in Section 1.

Corollary 1. *If Neumann problem* (2), (3) *is solvable on a planar domain with an outward peak, then the functional f can be represented as the sum of three terms* $f^{(1)}$, $f^{(2)}$, *and* $f^{(3)}$ *with the properties specified in assertion* (i) *of the theorem, and if the functional f is the sum of three terms with the properties specified in assertion* (ii) *of the theorem, then problem Neumann* (2), (3) *is uniquely solvable.*

Suppose that f is a function on $\partial\Omega$ integrable to some power. Let $f^{(3)}$ be the functional constructed for f in Theorem 1. Then, the minimal integrability exponent of f ensuring the continuity of $f^{(3)}$ is continuous is characterized as follows.

Corollary 2. *Let* $\Omega \subset \mathbf{R}^2$ *be a domain with an outward peak. We set* $q = p/(2-p)$ for $p < 2$, $q \in [1, \infty)$ *for* $p = 2$, *and* $q = \infty$ *for* $p > 2$. *Let* $q^{-1} + q'^{-1} = 1$. *Then, for* $f \in L_{q'}(\partial\Omega)$, *the functional*

$$TW_p^1(\Omega) \ni v \mapsto \int_\Gamma f(Q)(1-\mu_0(Q))(v(Q)-\tilde{v}(Q))ds_Q \tag{38}$$

is continuous on $TW_p^1(\Omega)$, *and its norm is at most* $c\|f\|_{L_{q'}(\partial\Omega)}$. *Moreover, for all* $v \in TW_p^1(\Omega)$,

$$\|v-\tilde{v}\|_{L_q(\Gamma)} \le c\|v\|_{TW_p^1(\Omega)}. \tag{39}$$

Proof. According to the theorem, it suffices to show that the right-hand side of (26) does not exceed $c\|f\|_{L_{q'}(\partial\Omega)}$, where the functionals $f_k \in W_{p'}^{-1/p'}(\Delta_k)$ have the form

$$\langle f_k, u\rangle = \int_{\Gamma_+} f(x, \varphi_+(x))\mu_k(x)u(x)ds_x - \int_{\Gamma_-} f(x, \varphi_-(x))\mu_k(x)u(x)ds_x.$$

Using Hölder's inequality, we obtain

$$|\langle f_k, u\rangle| \le c\|f\|_{L_{q'}(\Gamma_k)}\|u\|_{L_q(\Delta_k)}, \tag{40}$$

where $\Gamma_k = \{(x, y) \in \Gamma: x \in \Delta_k\}$. Setting $\varphi_k = \varphi(x_k)$ for short, we write the Sobolev embedding $W_p^{1-1/p}(\Delta_k) \subset L_q(\Delta_k)$ in the form

$$\|u\|_{L_q(\Delta_k)} \le c\varphi_k^{1+\frac{1}{q}-\frac{2}{p}}\left(\varphi_k^{-\frac{1}{p'}}\|u\|_{L_p(\Delta_k)} + [u]_{p,\Delta_k}\right), \text{ where } p' = p/(p-1).$$

Since the quantity in parentheses is equivalent to $\|u\|_{W_p^{1-1/p}(\Delta_k)}$, it follows from this inequality and (40) that

$$\|f_k\|_{W_{p'}^{-1/p'}(\Delta_k)} \le c\|f\|_{L_{q'}(\Gamma_k)}.$$

Applying the algebraic inequality

$$\left(\sum_{k\ge 1} a_k^\gamma\right)^{1/\gamma} \le \left(\sum_{k\ge 1} a_k^\beta\right)^{1/\beta}, \text{ where } a_k \ge 0 \text{ and } \gamma > \beta > 0,$$

and taking into account the inequality $q' \le p'$, we obtain

$$\left(\sum_{k\ge 1}\|f_k\|^{p'}_{W_{p'}^{-1/p'}(\Delta_k)}\right)^{1/p'} \le c\left(\sum_{k\ge 1}\|f\|^{q'}_{L_{q'}(\Gamma_k)}\right)^{1/q'} \le c\|f\|_{L_{q'}(\Gamma)}.$$

This proves the first of the required assertions. Consider estimate (39). For $v \in TW_p^1(\Omega)$ and $f \in L_{q'}(\Gamma)$, let $F_v(f)$ denote the integral in (38). What is said above implies

$$|F_v(f)| \le c\|f\|_{L_{q'}(\Gamma)}\|v\|_{TW_p^1(\Omega)}.$$

It follows that the norm of the functional $L_{q'}(\Gamma) \ni f \mapsto F_v(f)$ is at most $c\|v\|_{TW_p^1(\Omega)}$ and, therefore,

$$\|(1-\mu_0)(v-\tilde{v})\|_{L_q(\Gamma)} \le c\|v\|_{TW_p^1(\Omega)}.$$

To complete the proof of estimate (39), it remains to verify that

$$\|v-\tilde{v}\|_{L_q(\Gamma\cap\Gamma_0)} \le c\|v\|_{TW_p^1(\Omega)}.$$

The definition of $\tilde{v}$ and the continuity of the Sobolev embedding $W_p^{1-1/p}(\Gamma_0) \subset L_q(\Gamma_0)$ imply

$$\|\tilde{v}\|_{L_q(\Gamma\cap\Gamma_0)} \le c\|v\|_{L_q(\Gamma\cap\Gamma_0)} \le c\|v\|_{W_p^{1-1/p}(\Gamma_0)} \le c\|v\|_{TW_p^1(\Omega)},$$

which proves the required estimate.

In conclusion, we state one more assertion concerning the solvability of the Neumann problem with boundary condition from $L_{q'}(\partial\Omega)$.

Proposition 1. Suppose that Ω is a planar domain with an outward peak, $1 \le q \le p/(2-p)$ for $p < 2$, $q \in [1, \infty)$ for $p = 2$, $1 \le q \le \infty$ for $p > 2$, and $q^{-1} + q'^{-1} = 1$. Then, the following assertions are equivalent.

(A) Neumann problem (2), (3) is solvable for all $f \in L_{q'}(\partial\Omega)$;

(B) The space $TW_p^1(\Omega)$ is continuously embedded in $L_q(\partial\Omega)$;

(C) For all $f \in L_{q'}(\partial\Omega)$, the functional $TW_p^1(\Omega) \ni v \mapsto \int_{\partial\Omega} f\, v ds_x$ is continuous;

(D) The mapping $TW_p^1(\Omega) \ni v \mapsto \tilde{v} \in L_q(\Gamma)$ is continuous;

(E) The space $W_p(0, 1)$ is continuously embedded in $L_q(0, 1)$.

Proof. (A) $\longrightarrow$ (B) Let V be the unit ball of $W_p^1(\Omega)$, and let $\mathcal{V} = W_p^1(\Omega) \cap L_\infty(\Omega) \cap C^\infty(\Omega)$. For all $v \in V \cap \mathcal{V}$ and $f \in L_{q'}(\partial\Omega)$, equality (4) holds, which implies the pointwise boundedness of the family of functionals $L_{q'}(\partial\Omega) \ni f \mapsto \langle F_v, f\rangle = \int_{\partial\Omega} f\, v ds$ for $v \in V \cap \mathcal{V}$. Therefore, the norms $\|F_v\|$ are uniformly bounded with respect to v. Thus, $\|v\|_{L_q(\partial\Omega)} \le \text{const}$ for $v \in V \cap \mathcal{V}$. This implies (B), because the set $\mathcal{V}$ is dense in $W_p^1(\Omega)$.

(B) $\longrightarrow$ (C) Take $f \in L_{q'}(\partial\Omega)$ and $v \in TW_p^1(\Omega)$. Since $TW_p^1(\Omega)$ is embedded in $L_q(\partial\Omega)$, we have

$$\left|\int_{\partial\Omega} f v ds_x\right| \le \|f\|_{L_{q'}(\partial\Omega)} \|v\|_{L_q(\partial\Omega)} \le c\|f\|_{L_{q'}(\partial\Omega)} \|v\|_{TW_p^1(\Omega)},$$

which implies the required assertion.

(C) $\longrightarrow$ (A) As shown in Section 1, the continuity of the functional $TW_p^1(\Omega) \ni v \mapsto \int_{\partial\Omega} f\, v ds_x$ implies the solvability of the Neumann problem. The equivalence of assertions (B) and (D) follows from Corollary 2.

(D) $\longrightarrow$ (E) Let $u \in W_p(0, 1)$, and let $v \in TW_p^1(\Omega)$ be such that $v_+ = v_- = (1 - \mu_0)u$. Then, the continuity of the mapping $TW_p^1(\Omega) \ni v \mapsto \tilde{v} \in L_q(\Gamma)$ means that

$$\|(1 - \mu_0)u\|_{L_q(0,1)} \le c\|u\|_{W_p(0,1)}.$$

Thus, the space $W_p(0, 1)$ is embedded in $L_q(0, 1)$.

(E) $\longrightarrow$ (D) Let $v \in TW_p^1(\Omega)$. Then,

$$\|v_+ + v_-\|_{L_q(0,1)} \le c\|v_+ + v_-\|_{W_p(0,1)},$$

whence

$$\|\tilde{v}\|_{L_q(\Gamma)} \le c\|v\|_{TW_p^1(\Omega)}.$$

Remark. Using results of [7] (see also [6, 5.4]), we can supplement Proposition 1 as follows. It turns out that each of assertions (A)–(E) is equivalent to the following assertion.

(F) For $q < p$,

$$\int_0^1 \left(\left(\int_x^1 \frac{dt}{\varphi(t)^{1/(p-1)}}\right)^{q-1}\right)^{p/(p-q)} \frac{x^{p/(p-q)}}{\varphi(x)^{1/(p-1)}} dx < \infty,$$

and for $q \ge p$,

$$\sup_{x \in (0,1)} \left\{ x^{1/q} \left(\int_x^1 \frac{dt}{\varphi(t)^{1/(p-1)}}\right)^{(p-1)/p} \right\} < \infty.$$

Example. Consider the peak $\Omega = \{(x, y) : x \in (0, 1), c_1 < y/x^\lambda < c_2\}$, where $\lambda > 1$ and $c_2 > c_1 > 0$. Neumann problem (2), (3) is solvable for all $f \in L_{q'}(\partial\Omega)$ in the following cases:

(1) $p \geq 1 + \lambda$ and $q \in [1, p)$;

(2) $p < 1 + \lambda$ and $1 \leq q < p\min\{1, (1 + \lambda - p)^{-1}\}$;

(3) $p > 1 + \lambda$ and $p \leq q \leq \infty$;

(4) $p = 1 + \lambda$ and $p \leq q < \infty$;

(5) $1 < p < 1 + \lambda$ and $p \leq q \leq p/(1 + \lambda - p)$.

4. A PLANAR DOMAIN WITH AN INWARD PEAK

In this section, Ω denotes a planar domain with an inward peak vertex on the boundary in the sense of the definition given at the beginning of Section 2. To describe the space $TW_p^1(\Omega)^*$, we need a partition of unity on $\partial\Omega\backslash\{O\}$ different from that constructed in Section 2.

Let $x_k = 2^{-k-1}$ for $k = 0, 1, \ldots$. We set $\Delta_k = (x_{k+1}, x_{k-1})$ and take a smooth partition of unity $\{\mu_k\}_{k \geq 1}$ on the interval $(0, x_1]$ subordinate to the cover by the intervals Δ_k. We require that

$$\operatorname{dist}(\operatorname{supp}\mu_k, \mathbf{R}^1\backslash\Delta_k) \geq c2^{-k} \quad \text{and} \quad |\mu_k'| \leq c2^k \quad \text{for} \quad k = 1, 2, \ldots.$$

We set $\mu_0(x) = 0$ for $x < x_1$, $\mu_0(x) = 1 - \mu_1(x)$ for $x \in [x_1, 1]$, and $\mu_0 = 1$ on the curve $\partial\Omega\backslash\bar{\Gamma}$. Suppose that

$$\Gamma_k = \{(x, y) \in \Gamma : x \in \Delta_k\}, \quad k \geq 1, \quad \Gamma_0 = \partial\Omega\backslash\{(x, y) \in \bar{\Gamma} : x \leq x_1\}.$$

Then, the set of functions $\{\mu_k\}_{k \geq 0}$ induces a partition of unity on $\partial\Omega\backslash\{O\}$ subordinate to the cover $\{\Gamma_k\}_{k \geq 0}$ in the sense that $\mu_k = 0$ on $\partial\Omega\backslash\Gamma_k$.

The space $TW_p^1(\Omega)$ with $p \in (1, \infty)$ can be described explicitly as follows [8; 6, 6.5]. It consists of functions $v \in L_p(\partial\Omega)$ with finite norm

$$\|v\|_{W_p^{1-1/p}(\Gamma_0)} + \|v_\pm\|_{W_p^{1-1/p}(0,1)} + \left(\int_0^1 |v_+(x) - v_-(x)|^p \frac{dx}{x^{p-1}}\right)^{1/p}. \tag{41}$$

Moreover, the norm $\|v\|_{TW_p^1(\Omega)}$ is equivalent to (41). Lemmas 2 and 3 carry over to domains with an inward peak without any changes, and Lemma 4 becomes valid for such a domain when norm (10) is changed for (41). An argument similar to (and even somewhat simpler than) that used in the proof of Theorem 1 gives the following description of the space $TW_p^1(\Omega)^*$.

Theorem 2. *Let $\Omega \subset \mathbf{R}^2$ be a domain with an inward peak, and let $\{\mu_k\}$ be the partition of unity on $\partial\Omega\backslash\{O\}$ constructed in this section.*

(i) *If $f \in TW_p^1(\Omega)^*$, then $f = f^{(1)} + f^{(2)} + f^{(3)}$, where the functionals $f^{(j)}$ are defined as in Theorem* 1. *The functional $f^{(2)}$ belongs to the class $W_{p'}^{-1/p'}(0, 1)$ in the sense that*

$$|\langle f^{(2)}, v\rangle| \leq c\|v_- + v_+\|_{W_p^{1+1/p}(0,1)},$$

where the constant does not depend on $v \in TW_p^1(\Omega)$, and $f^{(1)}$ and $f^{(3)}$ have the properties specified in assertion (i) *of Theorem* 1.

(ii) *Suppose that $f_k \in W_{p'}^{-1/p'}(\Delta_k)$ and* $\operatorname{supp} f_k \subset \Delta_k$ *for $k \geq 1$. If the sum on the left-hand side of* (25) *is finite, then the functional $f^{(3)}$ defined by* (24) *is continuous and supported on the set $\{(x, y) \in U \cap \partial\Omega : x < x_0\}$, and estimate* (26) *holds. Moreover, if $h \in W_{p'}^{-1/p'}(\Gamma_0)$, $g \in W_{p'}^{-1/p'}(0, 1)$, $f^{(1)} = \mu_0 h$, and $f^{(2)}$ is the functional defined by* (27), *then $f^{(1)}, f^{(2)} \in TW_p^1(\Omega)^*$, and, moreover, $f^{(1)} \in W_{p'}^{-1/p'}(\Gamma_0)$.*

REFERENCES

1. V. G. Maz'ya and S. V. Poborchi, Georgian Math. J. **14** (3), 499–528 (2007).
2. E. Gadliardo, Rend. Sem. Mat. Univ. Padova **27**, 284–305 (1957).
3. J. Leray and J.-L. Lions, Bull. Soc. Math. Fr. **93**, 97–107 (1965).
4. V. G. Maz'ya, *Spaces of S. L. Sobolev* (Leningrad, 1985) [in Russian].
5. G. N. Yakovlev, Differ. Uravn. **1** (8), 1085–1098 (1965).
6. V. Maz'ya and S. V. Poborchii, *Embedding and Extension Theorems for Functions on Non-Lipschitz Domains* (Izd. S.-Peterburg. Univ., 2006) [in Russian].
7. S. V. Poborchii, Vestn. S.-Peterburg. Univ., Mat. Mekh. Astron., No. 3, 51–60 (2005).
8. G. N. Yakovlev, Dokl. Akad. Nauk SSSR **140**, 73–76 (1961).