Approximate Approximations on nonuniform grids

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We present an extension of approximate quasi-interpolation on uniformly distributed nodes, to functions given on a set of nodes close to an uniform, not necessarily cubic, grid.

1 Introduction

The method of approximate quasi-interpolation and its first related results were proposed in [5] and [6]. The method is characterized by a very accurate approximation in a certain range relevant for numerical computations, but in general the approximations do not converge in rigorous sense. For that reason such processes were called *approximate approximations*.

Suppose we want to approximate a smooth function $u(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, when we prescribe the values of u at the points of an uniform grid of mesh size h. We fix a positive parameter $\mathcal{D}$ and we choose a sufficiently smooth and rapidly decaying at infinity function η - the generating function - such that the linear combination of dilated shifts of η forms an approximate partition of the unity *i.e.*

$$\mathcal{D}^{-n/2} \sum_{\mathbf{m}\in\mathbb{Z}^n} \eta\left(\frac{\xi - \mathbf{m}}{\sqrt{\mathcal{D}}}\right) \approx 1.$$

The method consists in approximating the function u at the point $\mathbf{x}$ by a linear combination of the form

$$M_{h,\mathcal{D}} u(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{\mathbf{m}\in\mathbb{Z}^n} u(h\mathbf{m}) \eta\left(\frac{\mathbf{x} - h\mathbf{m}}{h\sqrt{\mathcal{D}}}\right), \qquad \mathbf{x} \in \mathbb{R}^n. \tag{1.1}$$

This type of formulas is known as quasi-interpolants and they have the property that $M_{h,\mathcal{D}} u(\mathbf{x})$ approximates $u(\mathbf{x})$, but $M_{h,\mathcal{D}} u(\mathbf{x})$ does not converge to $u(\mathbf{x})$ as the grid size h tends to zero. However one can fix $\mathcal{D}$ such that the approximation error is as small as we wish so that the non-convergence is not perceptible in numerical computations (see [9], [10]). On the other hand, the simplicity of the generalizations to the multi-dimensional case together with a great flexibility in choosing the generating function η compensate the lack of convergence.

The above mentioned flexibility is important in the applications because the generating function η can be selected so that integral and pseudo-differential operators of mathematical physics applied to η have analitically known expressions, obtaining semianalytic cubature formulas for these operators (see [8], [11], [13] and the review paper [14]). In some cases, *e.g.* for potentials, the cubature formulas converge even in a rigorous sense.

Another important application of the method is the possibility to develop explicit semi-analytic time marching algorithms for initial boundary value problems for linear and non linear evolution equations (see [7], [2]).

Quasi-interpolation formulas similar to (1.1) preserve the fundamental properties of approximate quasi-interpolation if the grid is a smooth image of the uniform one (see [12]) or if the grid is piecewise uniform (see [1]). The method of approximate quasi-interpolation has been generalized to functions given on a set of nodes close to a uniform, not necessarly cubic, grid in [3]. More general scattered grids have been considered in [4].

To illustrate the unusual behavior of approximate approximations we assume $\eta(x) = \mathrm{e}^{-x^2}/\sqrt{\pi}$ as generating function and the following quasi-interpolant for a function u on $\mathbb{R}$:

$$M_{h,\mathcal{D}}u(x) = \frac{1}{\sqrt{\pi D}} \sum_{m=-\infty}^{\infty} u(hm)\mathrm{e}^{-(x-hm)^2/(\mathcal{D}h^2)}, \quad x \in \mathbb{R}. \tag{1.2}$$

The application of Poisson's summation formula to the function

$$\Theta(\xi, \mathcal{D}) = \frac{1}{\sqrt{\pi\mathcal{D}}} \sum_{m=-\infty}^{\infty} \mathrm{e}^{-(\xi-m)^2/\mathcal{D}}$$

yields to these equivalent representations for

$$\Theta(\xi, \mathcal{D}) = 1 + 2\sum_{\nu=1}^{\infty} \mathrm{e}^{-\pi^2\mathcal{D}\nu^2} \cos 2\pi\nu\xi$$

and

$$\Theta'(\xi, \mathcal{D}) = -4\pi \sum_{\nu=1}^{\infty} \nu\mathrm{e}^{-\pi^2\mathcal{D}\nu^2} \sin 2\pi\nu\xi.$$

We deduce that

$$|\Theta(\xi, \mathcal{D}) - 1| \leq 2\sum_{\nu=1}^{\infty} \mathrm{e}^{-\pi^2\mathcal{D}\nu^2} < 2\,\epsilon(\mathcal{D}); \qquad |\Theta'(\xi, \mathcal{D})| \leq 4\pi \sum_{\nu=1}^{\infty} \nu\mathrm{e}^{-\pi^2\mathcal{D}\nu^2} < 4\,\pi\,\epsilon(\mathcal{D})$$

with

$$\epsilon(D) = \mathrm{e}^{-\pi^2\mathcal{D}} + \mathcal{O}(\mathrm{e}^{-4\pi^2\mathcal{D}}).$$

The rapid exponential decay ensures that we can choose $\mathcal{D}$ large enough such that $\epsilon(\mathcal{D})$ can be made arbitrarly small, for example less that the needed accuracy or the machine precision. Therefore the integer shifts of the Gaussian $\{\frac{\mathrm{e}^{-(\xi-m)^2/\mathcal{D}}}{\sqrt{\pi\mathcal{D}}}, m \in \mathbb{Z}\}$ form an approximate partition of unity for large $\mathcal{D}$.

If the approximated function u is smooth enough, the quasi-interpolant (1.2) can be represented in the form (see [6])

$$M_{h,\mathcal{D}}u(x) = u(x) +$$
$$u(x)\left(\Theta(\frac{x}{h}, \mathcal{D}) - 1\right) + u'(x)\frac{h\mathcal{D}}{2}\Theta'(\frac{x}{h}, \mathcal{D}) + \mathcal{R}_{h,\mathcal{D}}(x)$$

where the remainder term admits the estimate

$$|\mathcal{R}_{h,\mathcal{D}}(x)| \leq c\,\mathcal{D}h^2 \max_{x\in\mathbb{R}} |u''(x)|$$

with a contant c not depending on $h, \mathcal{D}, u$.

The difference between $M_{h,\mathcal{D}}u(x)$ and $u(x)$ can be estimated by

$$|M_{h,\mathcal{D}}u(x) - u(x)| \leq c\mathcal{D}h^2 \max_{x\in\mathbb{R}} |u''(x)| +$$

$$\epsilon(\mathcal{D})(2|u(x)| + \frac{h\mathcal{D}}{2}|u'(x)|). \tag{1.3}$$

This means that, above the tolerance (1.3), the quasi-interpolant (1.2) approximates u like usual second order approximations and, if $\mathcal{D}$ is chosen appropriately, any prescribed accuracy can be reached. Then the non-convergent part - called *saturation error* because it does not converge to 0 - can be neglected and the approximation process behaves like a second order approximation process.

2 Quasi-interpolation on uniform grids

One of the advantages of the method is that quasi-interpolants in arbitrary space dimension n with approximation order larger than two, up to some prescribed accuracy, have the same simple form as second order quasi-interpolants. The quasi-interpolant in $\mathbb{R}^n$ has the form

$$M_{h,\mathcal{D}}u(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{\mathbf{j}\in\mathbb{Z}^n} u(h\mathbf{j})\ \eta\left(\frac{\mathbf{x}-h\mathbf{j}}{h\sqrt{\mathcal{D}}}\right) \tag{2.1}$$

with the generating function η in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of smooth and rapidly decaying functions. Maz'ya and Schmidt have proved that formula (2.1) provides the following approximation result.

Theorem 2.1 ([12]) *Suppose that*

$$\int_{\mathbb{R}^n} \eta(\mathbf{y})d\mathbf{y} = 1, \quad \int_{\mathbb{R}^n} \mathbf{y}^\alpha \eta(\mathbf{y})d\mathbf{y} = 0,\ \forall\alpha : 1 \le |\alpha| < N \tag{2.2}$$

and $u \in W_\infty^N(\mathbb{R}^n)$. *Then*

$$|M_{h,\mathcal{D}}u(\mathbf{x}) - u(\mathbf{x})| \le c_{\eta,N}(\sqrt{\mathcal{D}}h)^N \|\nabla_N u\|_{L_\infty} +$$

$$\sum_{k=0}^{N-1}\left(\frac{h\sqrt{\mathcal{D}}}{2\pi}\right)^k \sum_{|\alpha|=k}\frac{|\nabla_k u(\mathbf{x})|}{\alpha!}\sum_{\nu\in\mathbb{Z}^n\setminus 0}|\partial^\alpha\mathcal{F}\eta(\sqrt{\mathcal{D}}\nu)|$$

with the constant $c_{\eta,N}$ *not depending on* u, h *and* $\mathcal{D}$.

Moreover for any $\epsilon > 0$, *there exists* $\mathcal{D} > 0$ *such that for all* α, $0 \le |\alpha| < N$,

$$\sum_{\nu\in\mathbb{Z}^n\setminus 0}|\partial^\alpha\mathcal{F}\eta(\sqrt{\mathcal{D}}\nu)| < \epsilon\,.$$

$\nabla_k u(x)$ denotes the vector of all partial derivatives $\{\partial^\alpha u(x)\}_{|\alpha|=k}$ and $\mathcal{F}\eta$ denotes the Fourier transform of η. We deduce that for any $\epsilon > 0$ there exists $\mathcal{D} > 0$ such that $M_{h,\mathcal{D}}u(\mathbf{x})$ approximates $u(\mathbf{x})$ pointwise with the estimate (see [9],[10])

$$|M_{h,\mathcal{D}}u(\mathbf{x}) - u(\mathbf{x})| \le c_{\eta,N}(\sqrt{\mathcal{D}}h)^N\|\nabla_N u\|_{L_\infty} + \epsilon\sum_{k=0}^{N-1}(h\sqrt{\mathcal{D}})^k|\nabla_k u(\mathbf{x})|.$$

Therefore $M_{h,\mathcal{D}}u$ behaves like an approximation formula of order N up to the saturation term that can be ignored in numerical computations if $\mathcal{D}$ is large enough. Similar estimates are also valid for integral norms (see [8]).

Several methods to construct generating functions satisfying the moment conditions (2.2) for arbitrarily large N have been developed (see [10], [12]). In fact any sufficiently smooth and rapidly decaying function η with $\mathcal{F}\eta(0) \neq 0$ can be used to construct new generating functions η_N satisfying the moment conditions for arbitrary large N as shown in the next theorem.

Theorem 2.2 ([10]) *Let $\eta \in \mathcal{S}(\mathbb{R}^n)$ with $\mathcal{F}\eta(0) \neq 0$. Then*

$$\eta_N(\mathbf{x}) = \sum_{|\alpha|=0}^{N-1} \frac{\partial^\alpha (\mathcal{F}\eta(\lambda)^{-1})|_{\lambda=0}}{\alpha!(2\pi i)^{|\alpha|}} \partial^\alpha \eta(\mathbf{x})$$

satisfies the moment conditions (2.2).

An interesting example is given by the Gaussian function $\eta(\mathbf{x}) = \mathrm{e}^{-|\mathbf{x}|^2}$ where the application of Theorem 2.2 leads to the generating function

$$\eta_{2M}(\mathbf{x}) = \pi^{-n/2} \sum_{j=0}^{M-1} \frac{(-1)^j}{j!4^j} \Delta^j \mathrm{e}^{-|\mathbf{x}|^2} = \pi^{-n/2} L_{M-1}^{(n/2)}(|\mathbf{x}|^2)\mathrm{e}^{-|\mathbf{x}|^2}$$

with $N = 2M$ and the generalized Laguerre polynomial

$$L_k^{(\gamma)}(y) = \frac{\mathrm{e}^y y^{-\gamma}}{k!} \left(\frac{d}{dy}\right)^k (\mathrm{e}^{-y} y^{k+\gamma}),\ \gamma > -1\,.$$

Hence the quasi-interpolant

$$M_{h,\mathcal{D}} u(\mathbf{x}) = (\pi\mathcal{D})^{-n/2} \sum_{\mathbf{j}\in\mathbb{Z}^n} u(h\mathbf{j})\ L_{M-1}^{(n/2)}\Big(\left|\frac{\mathbf{x}-h\mathbf{j}}{h\sqrt{\mathcal{D}}}\right|^2\Big)\mathrm{e}^{-\left|\frac{\mathbf{x}-h\mathbf{j}}{h\sqrt{\mathcal{D}}}\right|^2}$$

is an approximation formula of order $N = 2M$ plus the saturation term.

The quasi-interpolation formula and the corresponding approximation results have been generalized in [1] and [3] to the case when the values of u are given on uniform grids, not necessarily cubic, of this type

$$\Lambda_h := \{hA\mathbf{j}\,,\ \mathbf{j} \in \mathbb{Z}^n\}$$

with a real nonsingular $n \times n$-matrix A.

Under the same assumptions on the generating function η, it is always possible to choose $\mathcal{D} > 0$ such that the quasi-interpolant

$$\mathcal{M}_{\Lambda_h} u(\mathbf{x}) := \frac{\det A}{\mathcal{D}^{n/2}} \sum_{\mathbf{j}\in\mathbb{Z}^n} u(hA\mathbf{j})\, \eta\Big(\frac{\mathbf{x}-hA\mathbf{j}}{\sqrt{\mathcal{D}}h}\Big) \tag{2.3}$$

satisfies an estimate similar to that obtained in Theorem 2.1 for uniform cubic grid *i.e.*

$$|\mathcal{M}_{\Lambda_h} u(\mathbf{x}) - u(\mathbf{x})| \leq c_{\eta,N}(\sqrt{\mathcal{D}}h)^N \|\nabla_N u\|_{L_\infty} + \epsilon \sum_{k=0}^{N-1} (h\sqrt{\mathcal{D}})^k |\nabla_k u(\mathbf{x})| \tag{2.4}$$

for any $\epsilon > 0$.

The first application of formula (2.3) is the construction of quasi-interpolants on a regular triangular grid in the plane, as indicated in Figure 1. The vertices $\mathbf{y}_\mathbf{j}^\triangle$ of a partition of the plane into equilateral triangles of side length 1 are given by

$$\mathbf{y}_\mathbf{j}^\triangle = A\mathbf{j}; \qquad A = \begin{pmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix}.$$

The application of formula (2.3) to the nodes of the regular triangular grid of size h

$$\Lambda_h = \{h\mathbf{y}_\mathbf{j}^\triangle\} = \{hA\mathbf{j}\}_{\mathbf{j}\in\mathbb{Z}^2}$$

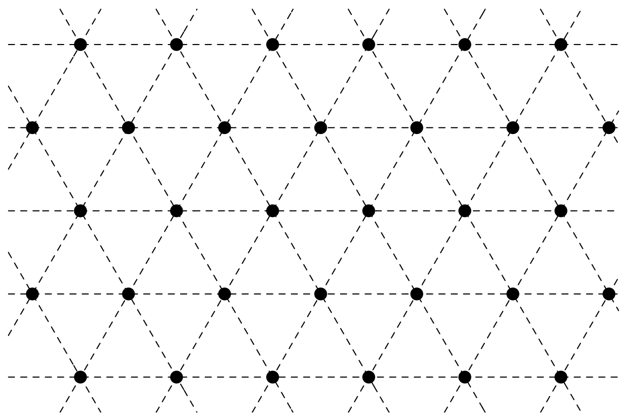

Figure 1: Tridiagonal grid

gives the following quasi-interpolant

$$\mathcal{M}_h^\triangle u(\mathbf{x}) := \frac{\sqrt{3}}{2\mathcal{D}} \sum_{\mathbf{j}\in\mathbb{Z}^2} u(h\mathbf{y}_\mathbf{j}^\triangle)\, \eta\Big(\frac{\mathbf{x}-h\mathbf{y}_\mathbf{j}^\triangle}{\sqrt{\mathcal{D}}h}\Big)\,.$$

The system of functions $\{\frac{\sqrt{3}}{2\mathcal{D}}\eta\Big(\frac{\mathbf{x}-\mathbf{y}_\mathbf{j}^\triangle}{\sqrt{\mathcal{D}}}\Big)\}$, centered at the points of the uniform triangular grid, forms an approximate partition of unity. Using Poisson's summation formula one can bound the main term of the saturation error by

$$\Big|1-\frac{\sqrt{3}}{2\mathcal{D}}\sum_{\mathbf{j}\in\mathbb{Z}^2}\eta\Big(\frac{\mathbf{x}-\mathbf{y}_\mathbf{j}^\triangle}{\sqrt{\mathcal{D}}}\Big)\Big| \le \sum_{\boldsymbol{\nu}\in\mathbb{Z}^2\setminus\mathbf{0}} \Big|\int_{\mathbb{R}^2}\eta(\mathbf{y})\,\mathrm{e}^{-2\pi i\sqrt{\mathcal{D}}(\mathrm{A}^{-1}\mathbf{y},\boldsymbol{\nu})}d\mathbf{y}\Big|\,.$$

By assuming as generating function the Gaussian $\eta(\mathbf{x}) = \pi^{-1}\mathrm{e}^{-|\mathbf{x}|^2}$ we obtain

$$\begin{aligned}&\Big|1-\frac{\sqrt{3}}{2\pi\mathcal{D}}\sum_{\mathbf{j}\in\mathbb{Z}^2}\mathrm{e}^{-|\mathbf{x}-\mathbf{y}_\mathbf{j}^\triangle|^2/\mathcal{D}}\Big|\\ &\le \sum_{(\nu_1,\nu_2)\ne(0,0)} \mathrm{e}^{-4\pi^2\mathcal{D}(\nu_1^2-\nu_1\nu_2+\nu_2^2)/3} = 6\,\mathrm{e}^{-4\pi^2\mathcal{D}/3} + \mathcal{O}(\mathrm{e}^{-4\pi^2\mathcal{D}})\,.\end{aligned}$$

In Figure 2 the graph of the difference $\frac{\sqrt{3}}{2\pi\mathcal{D}}\sum_{\mathbf{j}\in\mathbb{Z}^2}\mathrm{e}^{-|\mathbf{x}-\mathbf{y}_\mathbf{j}^\triangle|^2/\mathcal{D}}-1$ is plotted with two different values of D.

As second example we construct quasi-interpolants with functions centered at the nodes of a regular hexagonal grid in the plane, as depicted in Figure 3. We obtain a hexagonal grid if, from the nodes of a regular triangular grid of side length 1, the nodes of another triangular grid of side length $\sqrt{3}$ are removed (see Figure 4). Therefore the set of nodes $\mathbf{X}^\diamond$ of the regular hexagonal grid are given by

$$\mathbf{X}^\diamond = \{A\mathbf{j}\}_{\mathbf{j}\in\mathbb{Z}^2}\setminus\{B\mathbf{j}\}_{\mathbf{j}\in\mathbb{Z}^2}$$

where

$$B = \begin{pmatrix} 3/2 & 0 \\ \sqrt{3}/2 & \sqrt{3}\end{pmatrix}$$

and $B\mathbf{j}$, $\mathbf{j}\in\mathbb{Z}^2$, denote the removed nodes.

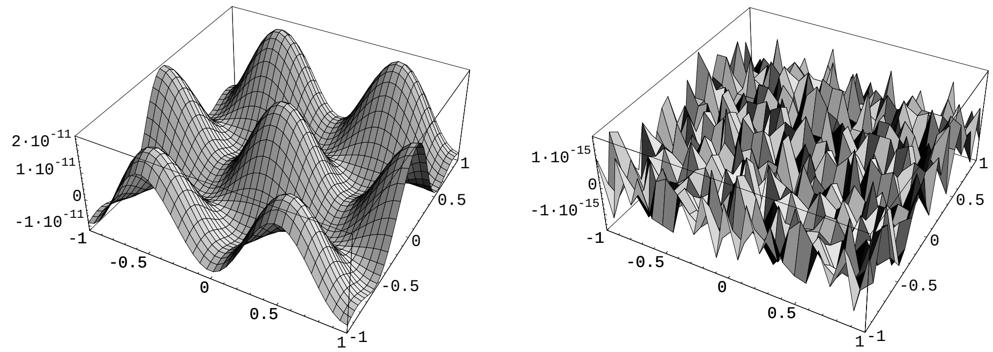

Figure 2: The graph of $\frac{\sqrt{3}}{2\pi\mathcal{D}}\sum_{\mathbf{j}\in\mathbb{Z}^2} \mathrm{e}^{-|\mathbf{x}-\mathbf{y}_\mathbf{j}^{\triangle}|^2/\mathcal{D}} - 1$ when $D = 2$ (on the left) and $D = 3$ (on the right).

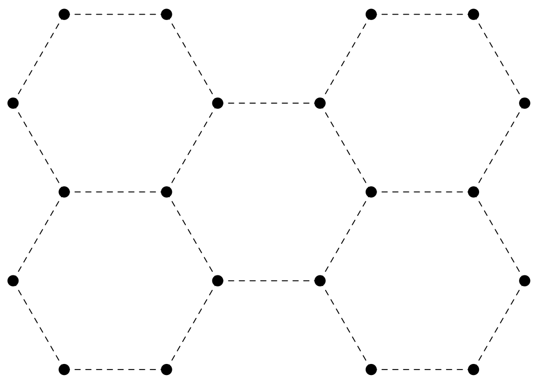

Figure 3: Hexagonal grid

The quasi-interpolant on the h-scaled hexagonal grid

$$h\mathbf{X}^\diamond = \{hA\mathbf{j}\}_{\mathbf{j}\in\mathbb{Z}^2} \setminus \{hB\mathbf{j}\}_{\mathbf{j}\in\mathbb{Z}^2} \tag{2.5}$$

is defined as

$$\mathcal{M}_h^\diamond u(\mathbf{x}) := \frac{3\sqrt{3}}{4\mathcal{D}} \sum_{\mathbf{y}^\diamond\in\mathbf{X}^\diamond} u(h\mathbf{y}^\diamond)\,\eta\Big(\frac{\mathbf{x}-h\mathbf{y}^\diamond}{\sqrt{\mathcal{D}}h}\Big)\,.$$

For (2.5) the quasi-interpolant $\mathcal{M}_h^\diamond u$ can be written in an equivalent way

$$\mathcal{M}_h^\diamond u(\mathbf{x}) = \frac{3\sqrt{3}}{4\mathcal{D}}\Big(\sum_{\mathbf{j}\in\mathbb{Z}^2} u(hA\mathbf{j})\eta\Big(\frac{\mathbf{x}-hA\mathbf{j}}{\sqrt{\mathcal{D}}h}\Big) - \sum_{\mathbf{j}\in\mathbb{Z}^2} u(hB\mathbf{j})\eta\Big(\frac{\mathbf{x}-hB\mathbf{j}}{\sqrt{\mathcal{D}}h}\Big)\Big)\,,$$

Therefore we derive that under the decay conditions and the moment conditions on η the quasi-interpolant $\mathcal{M}_h^\diamond u$ provides the estimate (2.4) for sufficiently large $\mathcal{D}$.

From Poisson's summation formula

$$\sum_{\mathbf{j}\in\mathbb{Z}^2} \eta\Big(\frac{\mathbf{x}-A\mathbf{j}}{\sqrt{\mathcal{D}}}\Big) = \frac{\mathcal{D}}{\det A}\Big(1 + \sum_{\boldsymbol{\nu}\in\mathbb{Z}^2\setminus\mathbf{0}} \mathcal{F}\eta(\sqrt{\mathcal{D}}(A^t)^{-1}\boldsymbol{\nu})\,\mathrm{e}^{2\pi\mathrm{i}(\mathbf{x},(A^t)^{-1}\boldsymbol{\nu})}\Big)\,,$$

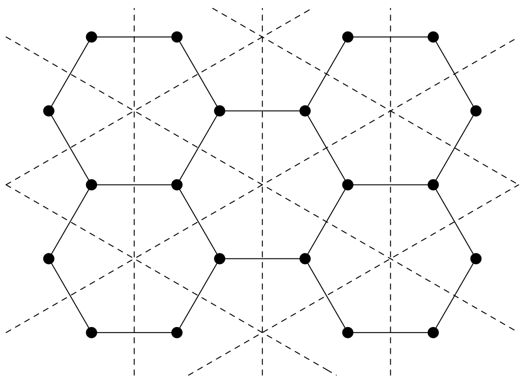

Figure 4: Nodes of a hexagonal grid. The eliminated triangular grid $B\mathbf{j}$ is depicted with dashed lines.

we obtain an approximate partition of unity centered at the hexagonal grid:

$$\frac{3\sqrt{3}}{4\mathcal{D}} \sum_{\mathbf{y}^\diamond \in \mathbf{X}^\diamond} \eta\Big(\frac{\mathbf{x}-\mathbf{y}^\diamond}{\sqrt{\mathcal{D}}}\Big) - 1 = \sum_{\mathbf{j}\in\mathbb{Z}^2} \eta\Big(\frac{\mathbf{x}-A\mathbf{j}}{\sqrt{\mathcal{D}}}\Big) - \sum_{\mathbf{j}\in\mathbb{Z}^2} \eta\Big(\frac{\mathbf{x}-B\mathbf{j}}{\sqrt{\mathcal{D}}}\Big) - 1 =$$

$$\frac{3}{2} \sum_{\boldsymbol{\nu}\in\mathbb{Z}^2\setminus\mathbf{0}} \mathcal{F}\eta(\sqrt{\mathcal{D}}(A^t)^{-1}\boldsymbol{\nu})\, \mathrm{e}^{\,2\pi \mathrm{i}(\mathbf{x},(\mathrm{A}^t)^{-1}\boldsymbol{\nu})} - \frac{1}{2} \sum_{\boldsymbol{\nu}\in\mathbb{Z}^2\setminus\mathbf{0}} \mathcal{F}\eta(\sqrt{\mathcal{D}}(B^t)^{-1}\boldsymbol{\nu})\, \mathrm{e}^{\,2\pi \mathrm{i}(\mathbf{x},(\mathrm{B}^t)^{-1}\boldsymbol{\nu})}.$$

In the case of the exponential $\eta(\mathbf{x}) = \pi^{-1}\mathrm{e}^{-|\mathbf{x}|^2}$ we have estimated the main term of the saturation error by

$$\Big|1 - \frac{3\sqrt{3}}{4\pi\mathcal{D}} \sum_{\mathbf{y}^\diamond\in\mathbf{X}^\diamond} \mathrm{e}^{\,-|\mathbf{x}-\mathbf{y}^\diamond|^2/\mathcal{D}}\Big| \tag{2.6}$$

$$\leq \frac{1}{2} \sum_{(\nu_1,\nu_2)\neq(0,0)} (3\mathrm{e}^{\,-4\pi^2\mathcal{D}(\nu_1^2-\nu_1\nu_2+\nu_2^2)/3} + \mathrm{e}^{\,-4\pi^2\mathcal{D}(\nu_1^2-\nu_1\nu_2+\nu_2^2)/9})$$

$$= 3\,\mathrm{e}^{-4\pi^2\mathcal{D}/9} + \mathcal{O}(\mathrm{e}^{-4\pi^2\mathcal{D}/3})\,.$$

In Figure 5 the difference (2.6) is depicted for two different values of $\mathcal{D}$.

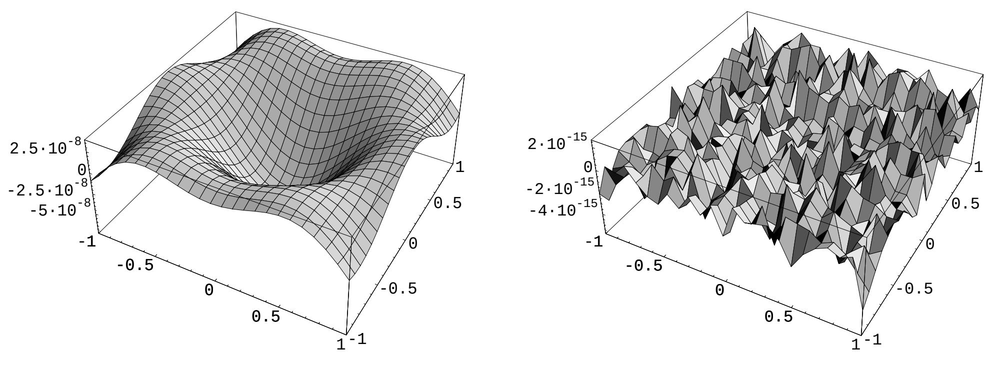

Figure 5: The graph of $\frac{3\sqrt{3}}{4\pi\mathcal{D}} \sum_{\mathbf{y}^\diamond\in\mathbf{X}^\diamond} \mathrm{e}^{\,-|\mathbf{x}-\mathbf{y}^\diamond|^2/\mathcal{D}} - 1$ when $D = 4$ (on the left) and $D = 8$ (on the right).

3 Results for nonuniform grids

Next we consider an extension of the approximate quasi-interpolation formulas on uniform grid to the case that the data are given on a set of scattered nodes $\mathbf{X} = \{\mathbf{x}_j\} \subset \mathbb{R}^n$ close to a uniform grid in the sense that we specify in Condition 1.

Condition 1 *There exists a uniform grid Λ such that the quasi-interpolants*

$$\mathcal{M}_{h,\mathcal{D}}\, u(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{\mathbf{y}_j \in \Lambda} u(h\mathbf{y}_j)\, \eta\Big(\frac{\mathbf{x} - h\mathbf{y}_j}{h\sqrt{\mathcal{D}}}\Big)$$

approximate sufficiently smooth functions u with the error

$$|\mathcal{M}_{h,\mathcal{D}} u(\mathbf{x}) - u(\mathbf{x})| \le c_{N,\eta}\, (h\sqrt{\mathcal{D}})^N \|\nabla_N u\|_{L_\infty(\mathbb{R}^n)} + \epsilon \sum_{k=0}^{N-1} (h\sqrt{\mathcal{D}})^k |\nabla_k u(\mathbf{x})| \qquad (3.7)$$

for any $\epsilon > 0$.

Let $\mathbf{X}_h$ be a sequence of grids with the property that for $\kappa_1 > 0$ not depending on h and any $\mathbf{y}_j \in \Lambda$ the ball $B(h\mathbf{y}_j, h\kappa_1)$ contains nodes of $\mathbf{X}_h$.

For example, if η satisfies the conditions of Theorem 2.1, we may assume as Λ the cubic grid $\{\mathbf{j}\}$ or, in the plane, the triangular grid $\{\mathbf{y}^\Delta\}$ or the hexagonal grid $\{\mathbf{y}^\diamond\}$.

In order to construct an approximate quasi-interpolant which use the data at the nodes of $\mathbf{X}_h$ we introduce the following definition.

Definition 3.1 *Let $\mathbf{x}_j \in \mathbf{X}_h$. A collection of $m_N = \dfrac{(N-1+n)!}{n!(N-1)!} - 1$ nodes $\mathbf{x}_k \in \mathbf{X}_h$ will be called star of $\mathbf{x}_j$ and denoted by $\mathrm{st}\,(\mathbf{x}_j)$ if the Vandermonde matrix*

$$V_{j,h} = \Big\{ \Big(\frac{\mathbf{x}_k - \mathbf{x}_j}{h}\Big)^\alpha \Big\}, \; |\alpha| = 1, ..., N-1,$$

is not singular.

Condition 2 *Denote by $\widetilde{\mathbf{x}}_j \in \mathbf{X}_h$ the node closest to $h\mathbf{y}_j \in h\Lambda$. There exists $\kappa_2 > 0$ such that for any $\mathbf{y}_j \in \Lambda$ the star $\mathrm{st}\,(\widetilde{\mathbf{x}}_j) \subset B(\widetilde{\mathbf{x}}_j, h\kappa_2)$ with $|\det V_{j,h}| \ge c > 0$ uniformly in h.*

Let us denote by $\{b^{(j)}_{\alpha,k}\}$, $|\alpha| = 1, \dots, N-1$, $\mathbf{x}_k \in \mathrm{st}\,(\widetilde{\mathbf{x}}_j)$, the elements of the inverse matrix of $V_{j,h}$, and consider the functional

$$F_{j,h}(u) = u(\widetilde{\mathbf{x}}_j)\Big(1 - \sum_{|\alpha|=1}^{N-1} \Big(\mathbf{y}_j - \frac{\widetilde{\mathbf{x}}_j}{h}\Big)^\alpha \sum_{\mathbf{x}_k \in \mathrm{st}\,(\widetilde{\mathbf{x}}_j)} b^{(j)}_{\alpha,k}\Big) + \sum_{\mathbf{x}_k \in \mathrm{st}\,(\widetilde{\mathbf{x}}_j)} u(\mathbf{x}_k) \sum_{|\alpha|=1}^{N-1} b^{(j)}_{\alpha,k}\Big(\mathbf{y}_j - \frac{\widetilde{\mathbf{x}}_j}{h}\Big)^\alpha .$$

The functional $F_{j,h}(u)$ depends on the values of u at the nodes of $\mathrm{st}\,(\widetilde{\mathbf{x}}_j) \cup \widetilde{\mathbf{x}}_j$ *i.e.* $m_N + 1$ points close to $h\,\mathbf{y_j}$.

Let us define the following quasi-interpolant which uses the values of u on $\mathbf{X}_h$

$$\mathbb{M}_{h,\mathcal{D}} u(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{\mathbf{y}_j \in \Lambda} F_{j,h}(u) \eta\Big(\frac{\mathbf{x} - h\,\mathbf{y}_j}{h\sqrt{\mathcal{D}}}\Big) . \qquad (3.8)$$

The following theorem states that, under the above mentioned conditions on the grid, $\mathbb{M}_{h,\mathcal{D}} u$ has the same behavior as in the case of uniform grids.

Theorem 3.1 ([3]) *Under the Conditions* 1 *and* 2, *for any* $\epsilon > 0$ *there exists* $\mathcal{D} > 0$ *such that the quasi-interpolant* (3.8) *approximates any* $u \in W_\infty^N(\mathbb{R}^n)$ *with*

$$|\mathbb{M}_{h,\mathcal{D}} u(\mathbf{x}) - u(\mathbf{x})| \le c_{N,\eta,\mathcal{D}}\, h^N \|\nabla_N u\|_{L_\infty(\mathbb{R}^n)} + \epsilon \sum_{k=0}^{N-1} (h\sqrt{\mathcal{D}})^k |\nabla_k u(\mathbf{x})|\,,$$

where $c_{N,\eta,\mathcal{D}}$ *does not depend on* u *and* h.

One of the motivations of approximate approximations is the construction of cubature formulas for integral operators of convolution type

$$\mathcal{K}u(\mathbf{x}) = \int_{\mathbb{R}^n} k(\mathbf{x}-\mathbf{y})u(\mathbf{y})\,d\mathbf{y}\,. \tag{3.9}$$

A cubature formula of the multi-dimensional integral (3.9) can be obtained if the density u is replaced by the quasi-interpolant $\mathbb{M}_{h,\mathcal{D}}u$. Then

$$\begin{aligned}\mathcal{K}\,\mathbb{M}_{h,\mathcal{D}} u(\mathbf{x}) &= \mathcal{D}^{-n/2} \sum_{\mathbf{y}_j\in\Lambda} F_{j,h}(u) \int_{\mathbb{R}^n} k(\mathbf{x}-\mathbf{y})\eta\Big(\frac{\mathbf{y}-h\,\mathbf{y}_j}{h\sqrt{\mathcal{D}}}\Big)d\mathbf{y}\\ &= h^n \sum_{\mathbf{y}_j\in\Lambda} F_{j,h}(u) \int_{\mathbb{R}^n} k\Big(h\sqrt{\mathcal{D}}\Big(\frac{\mathbf{x}-h\,\mathbf{y}_j}{h\sqrt{\mathcal{D}}} - \mathbf{y}\Big)\Big)\eta(\mathbf{y})d\mathbf{y}\end{aligned}$$

is a cubature formula for (3.9) with a generating function η chosen such that $\mathcal{K}\eta$ can be computed analytically or at least by some efficient quadrature method.

In (3.8) the generating function is centered at the nodes of the uniform grid $h\Lambda$. This can be helpful to design fast methods for the approximation of (3.9). If we define

$$a_{k-j}^{(h)} = \int_{\mathbb{R}^n} k\big(h(\mathbf{y}_k - \mathbf{y}_j - \sqrt{\mathcal{D}}\,\mathbf{y})\big)\,\eta(\mathbf{y})d\mathbf{y}\,.$$

we reduce to the computation of the following sums

$$\mathcal{K}\,\mathbb{M}_{h,\mathcal{D}} u(h\mathbf{y}_k) = h^n \sum_{\mathbf{y}_j\in\Lambda} F_{j,h}(u)\, a_{k-j}^{(h)}$$

which provide an approximation of (3.9) at the mesh points $h\mathbf{y}_k$.

A generalization of the method approximate approximations to functions with values given on a rather general grid was obtained in [4].

4 Numerical Experiments

The quasi-interpolant $\mathbb{M}_{h,\mathcal{D}}u$ in (3.8) was tested by one- and two-dimensional experiments and the results of the numerical experiments confirm the predicted approximation orders. In all cases the grid $\mathbf{X}_h$ is chosen such that any ball $B(h\mathbf{j}, h/2)$, $\mathbf{j}\in\mathbb{Z}^n$, $n=1$ or $n=2$, contains one randomly chosen node, which we denote by $\mathbf{x_j}$.

The one-dimensional case. Figures 6 – 9 show the graphs of $\mathbb{M}_{h,\mathcal{D}}u - u$ for different smooth functions u using the basis function $\eta(x) = \pi^{-1/2}\mathrm{e}^{-x^2}$ (Fig. 6 and 7) for which $N=2$, and $\eta(x) = \pi^{-1/2}(3/2 - x^2)\mathrm{e}^{-x^2}$ (Fig. 8 and 9) for which $N=4$, for different values of h. We have chosen the parameter $\mathcal{D} = 4$ in order to keep the saturation error less than 10^{-16}.

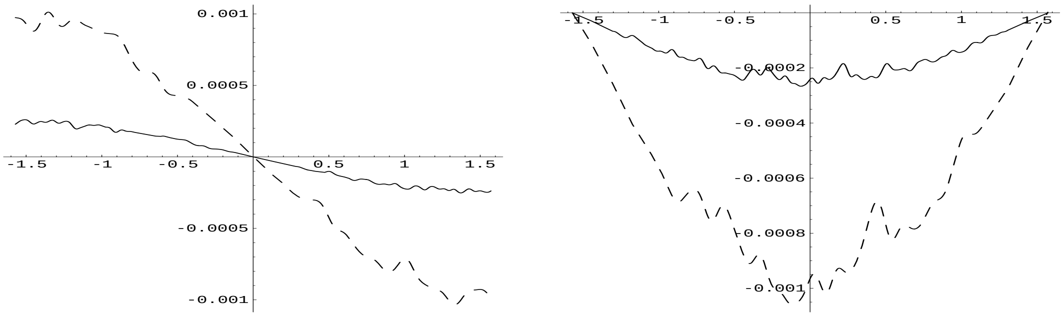

Figure 6: The graphs of $\mathbb{M}_{h,\mathcal{D}}u - u$ with $\eta(x) = \pi^{-1/2}\mathrm{e}^{-x^2}$, $\mathcal{D} = 4$, st $(x_j) = \{x_{j+1}\}$, when $u(x) = \sin(x)$ (on the left) and $u(x) = \cos(x)$. Dashed and solid lines correspond to $h = 1/32$ and $h = 1/64$.

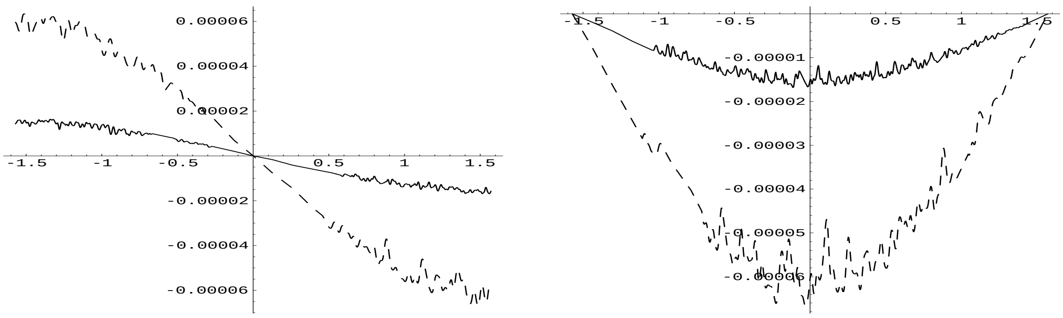

Figure 7: The graphs of $\mathbb{M}_{h,\mathcal{D}}u - u$ with $\eta(x) = \pi^{-1/2}\mathrm{e}^{-x^2}$, $\mathcal{D} = 4$, st $(x_j) = \{x_{j+1}\}$, when $u(x) = \sin(x)$ (on the left) and $u(x) = \cos(x)$. Dashed and solid lines correspond to $h = 1/128$ and $h = 1/256$.

The two-dimensional case. We depict in Figures 10 and 11 the quasi-interpolation error $\mathbb{M}_{h,\mathcal{D}}u - u$ for the function $u(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{-1}$ and different h if generating functions of second (with $\mathcal{D} = 2$) and fourth (with $\mathcal{D} = 4$) order of approximation are used. The h^2- and respectively h^4-convergence of the corresponding two-dimensional quasi-interpolants are confirmed by the $L_\infty-$ errors which are given in Table 1.

h	$\mathcal{D} = 2$	$\mathcal{D} = 4$
2^{-4}	$8.75 \cdot 10^{-3}$	$1.57 \cdot 10^{-2}$
2^{-5}	$2.21 \cdot 10^{-3}$	$4.00 \cdot 10^{-3}$
2^{-6}	$5.51 \cdot 10^{-4}$	$1.01 \cdot 10^{-3}$
2^{-7}	$1.42 \cdot 10^{-4}$	$2.52 \cdot 10^{-4}$
2^{-8}	$3.56 \cdot 10^{-5}$	$6.50 \cdot 10^{-5}$

h	$\mathcal{D} = 4$	$\mathcal{D} = 6$
2^{-4}	$4.42 \cdot 10^{-4}$	$9.59 \cdot 10^{-4}$
2^{-5}	$2.95 \cdot 10^{-5}$	$6.61 \cdot 10^{-5}$
2^{-6}	$1.92 \cdot 10^{-6}$	$4.24 \cdot 10^{-6}$
2^{-7}	$1.24 \cdot 10^{-7}$	$2.68 \cdot 10^{-7}$
2^{-8}	$7.80 \cdot 10^{-9}$	$1.71 \cdot 10^{-8}$

Table 1: L_∞ approximation error for the function $u(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{-1}$ using $\mathbb{M}_{h,\mathcal{D}}u$ with $\eta(\mathbf{x}) = \pi^{-1}\mathrm{e}^{-|\mathbf{x}|^2}$, $N = 2$ (on the left), and $\eta(\mathbf{x}) = \pi^{-1}(2 - |\mathbf{x}|^2)\mathrm{e}^{-|\mathbf{x}|^2}$, $N = 4$ (on the right).

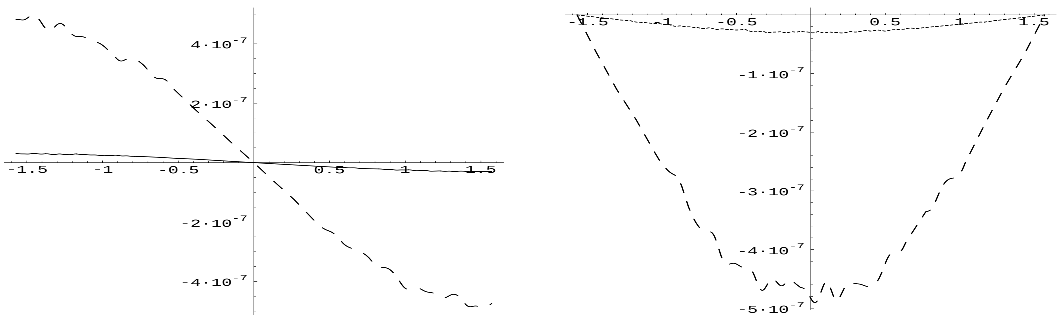

Figure 8: The graphs of $\mathbb{M}_{h,\mathcal{D}}u-u$ with $\eta(x)=\pi^{-1/2}(3/2-x^2)\mathrm{e}^{-x^2}$, $\mathcal{D}=4$, $\mathrm{st}\,(x_j)=\{x_{j-2},x_{j-1},x_{j+1}\}$, when $u(x)=\sin(x)$ (on the left) and $u(x)=\cos(x)$. Dashed and solid lines correspond to $h=1/32$ and $h=1/64$.

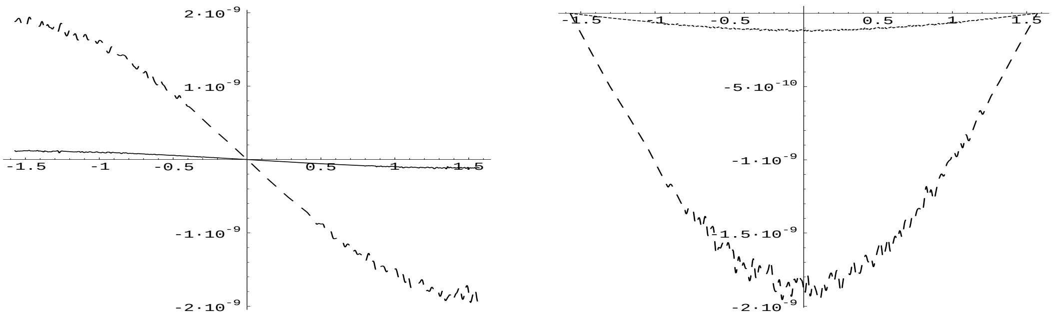

Figure 9: The graphs of $\mathbb{M}_{h,\mathcal{D}}u-u$ with $\eta(x)=\pi^{-1/2}(3/2-x^2)\mathrm{e}^{-x^2}$, $\mathcal{D}=4$, $\mathrm{st}\,(x_j)=\{x_{j-2},x_{j-1},x_{j+1}\}$, when $u(x)=\sin(x)$ (on the left) and $u(x)=\cos(x)$. Dashed and solid lines correspond to $h=1/128$ and $h=1/256$.

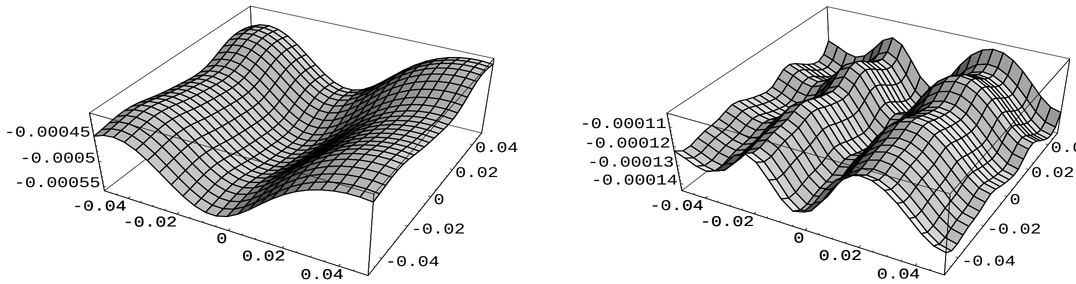

Figure 10: The graph of $\mathbb{M}_{h,\mathcal{D}}u-u$ with $\mathcal{D}=2$, $\eta(\mathbf{x})=\pi^{-1}\mathrm{e}^{-|\mathbf{x}|^2}$, $N=2$, $u(\mathbf{x})=(1+|\mathbf{x}|^2)^{-1}$, $h=2^{-6}$ (on the left) and $h=2^{-7}$ (on the right).

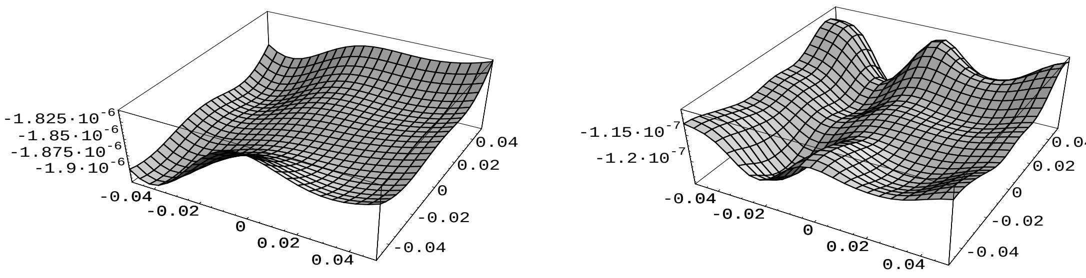

Figure 11: The graph of $\mathbb{M}_{h,\mathcal{D}}u - u$ with $\mathcal{D} = 4$, $\eta(\mathbf{x}) = \pi^{-1}(2 - |\mathbf{x}|^2)\mathrm{e}^{-|\mathbf{x}|^2}$, $N = 4$, $u(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{-1}$, $h = 2^{-6}$ (on the left) and $h = 2^{-7}$ (on the right).

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