Approximate Approximations on nonuniform grids

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We present an extension of approximate quasi-interpolation on uniformly distributed nodes, to functions given on a set of nodes close to an uniform, not necessarily cubic, grid.

1 Introduction

The method of approximate quasi-interpolation and its first related results were proposed in [5] and [6]. The method is characterized by a very accurate approximation in a certain range relevant for numerical computations, but in general the approximations do not converge in rigorous sense. For that reason such processes were called approximate approximations.

Suppose we want to approximate a smooth function $u(x), x \in \mathbb{R}^n$, when we prescribe the values of $u$ at the points of an uniform grid of mesh size $h$. We fix a positive parameter $\mathcal{D}$ and we choose a sufficiently smooth and rapidly decaying at infinity function $\eta$ - the generating function - such that the linear combination of dilated shifts of $\eta$ forms an approximate partition of the unity i.e.

$$\mathcal{D}^{-n/2} \sum_{m \in \mathbb{Z}^n} \eta \left( \frac{x - m}{\sqrt{\mathcal{D}}} \right) \approx 1.$$  

The method consists in approximating the function $u$ at the point $x$ by a linear combination of the form

$$M_{h,\mathcal{D}} u(x) = \mathcal{D}^{-n/2} \sum_{m \in \mathbb{Z}^n} u(hm) \eta \left( \frac{x - hm}{h \sqrt{\mathcal{D}}} \right), \quad x \in \mathbb{R}^n. \quad (1.1)$$

This type of formulas is known as quasi-interpolants and they have the property that $M_{h,\mathcal{D}} u(x)$ approximates $u(x)$, but $M_{h,\mathcal{D}} u(x)$ does not converge to $u(x)$ as the grid size $h$ tends to zero. However one can fix $\mathcal{D}$ such that the approximation error is as small as we wish so that the non-convergence is not perceptible in numerical computations (see [9], [10]). On the other hand, the simplicity of the generalizations to the multi-dimensional case together with a great flexibility in choosing the generating function $\eta$ compensate the lack of convergence.

The above mentioned flexibility is important in the applications because the generating function $\eta$ can be selected so that integral and pseudo-differential operators of mathematical physics applied to $\eta$ have analytically known expressions, obtaining semianalytic cubature formulas for these operators (see [8], [11], [13] and the review paper [14]). In some cases, e.g. for potentials, the cubature formulas converge even in a rigorous sense.

Another important application of the method is the possibility to develop explicit semi-analytic time marching algorithms for initial boundary value problems for linear and non linear evolution equations (see [7], [2]).

Quasi-interpolation formulas similar to (1.1) preserve the fundamental properties of approximate quasi-interpolation if the grid is a smooth image of the uniform one (see [12]) or if the grid is piecewise uniform (see [1]). The method of approximate quasi-interpolation has been
generalized to functions given on a set of nodes close to a uniform, not necessarily cubic, grid in [3]. More general scattered grids have been considered in [4].

To illustrate the unusual behavior of approximate approximations we assume \( \eta(x) = e^{-x^2}/\sqrt{\pi} \) as generating function and the following quasi-interpolant for a function \( u \) on \( \mathbb{R} \):

\[
M_{h,D} u(x) = \frac{1}{\sqrt{\pi D}} \sum_{m=-\infty}^{\infty} u(hm)e^{-\left(x-hm\right)^2/(Dh^2)}, \quad x \in \mathbb{R}.
\]

The application of Poisson’s summation formula to the function

\[
\Theta(\xi, D) = \frac{1}{\sqrt{\pi D}} \sum_{m=-\infty}^{\infty} e^{-\left(\xi-m\right)^2/D}
\]

yields to these equivalent representations for

\[
\Theta(\xi, D) = 1 + 2 \sum_{\nu=1}^{\infty} e^{-\pi^2 D \nu^2} \cos 2\pi \nu \xi
\]

and

\[
\Theta'(\xi, D) = -4\pi \sum_{\nu=1}^{\infty} \nu e^{-\pi^2 D \nu^2} \sin 2\pi \nu \xi.
\]

We deduce that

\[
|\Theta(\xi, D) - 1| \leq 2 \sum_{\nu=1}^{\infty} e^{-\pi^2 D \nu^2} < 2 \epsilon(D); \quad |\Theta'(\xi, D)| \leq 4\pi \sum_{\nu=1}^{\infty} \nu e^{-\pi^2 D \nu^2} < 4\pi \epsilon(D)
\]

with

\[
\epsilon(D) = e^{-\pi^2 D} + O(e^{-4\pi^2 D}).
\]

The rapid exponential decay ensures that we can choose \( D \) large enough such that \( \epsilon(D) \) can be made arbitrarily small, for example less that the needed accuracy or the machine precision. Therefore the integer shifts of the Gaussian \( \left\{ \frac{e^{-\left(\xi-m\right)^2/D}}{\sqrt{\pi D}}, m \in \mathbb{Z} \right\} \) form an approximate partition of unity for large \( D \).

If the approximated function \( u \) is smooth enough, the quasi-interpolant (1.2) can be represented in the form (see [6])

\[
M_{h,D} u(x) = u(x) + u(x) \left( \Theta\left(\frac{x}{h}, D\right) - 1 \right) + u'(x) \frac{hD}{2} \Theta'(\frac{x}{h}, D) + R_{h, D}(x)
\]

where the remainder term admits the estimate

\[
|R_{h, D}(x)| \leq c Dh^2 \max_{x \in \mathbb{R}} |u''(x)|
\]

with a constant \( c \) not depending on \( h, D, u \).

The difference between \( M_{h,D} u(x) \) and \( u(x) \) can be estimated by

\[
|M_{h,D} u(x) - u(x)| \leq c Dh^2 \max_{x \in \mathbb{R}} |u''(x)| + \epsilon(D) (2|u(x)| + \frac{hD}{2}|u'(x)|).
\]

(1.3)
This means that, above the tolerance \(1.3\), the quasi-interpolant (1.2) approximates \(u\) like usual second order approximations and, if \(\mathcal{D}\) is chosen appropriately, any prescribed accuracy can be reached. Then the non-convergent part - called saturation error because it does not converge to 0 - can be neglected and the approximation process behaves like a second order approximation process.

2 Quasi-interpolation on uniform grids

One of the advantages of the method is that quasi-interpolants in arbitrary space dimension \(n\) with approximation order larger than two, up to some prescribed accuracy, have the same simple form as second order quasi-interpolants. The quasi-interpolant in \(\mathbb{R}^n\) has the form

\[
M_{h,\mathcal{D}}u(x) = \mathcal{D}^{-n/2} \sum_{j \in \mathbb{Z}^n} u(hj) \eta \left( \frac{x - hj}{h \sqrt{\mathcal{D}}} \right)
\]

(2.1)

with the generating function \(\eta\) in the Schwartz space \(\mathcal{S}(\mathbb{R}^n)\) of smooth and rapidly decaying functions. Maz’ya and Schmidt have proved that formula (2.1) provides the following approximation result.

Theorem 2.1 ([12]) Suppose that

\[
\int_{\mathbb{R}^n} \eta(y)dy = 1, \quad \int_{\mathbb{R}^n} y^\alpha \eta(y)dy = 0, \quad \forall \alpha : 1 \leq |\alpha| < N
\]

and \(u \in W^N_\infty(\mathbb{R}^n)\). Then

\[
|M_{h,\mathcal{D}}u(x) - u(x)| \leq c_{\eta,N}(\sqrt{\mathcal{D}}h)^N \|
abla^N u\|_{L_\infty} + \sum_{k=0}^{N-1} \frac{(h \sqrt{\mathcal{D}})^k}{2\pi} \sum_{|\alpha| = k} \frac{\|
abla^\alpha u\|}{\alpha!} \sum_{\nu \in \mathbb{Z}^n \setminus 0} |\partial^\nu \mathcal{F}\eta(\sqrt{\mathcal{D}}\nu)|
\]

with the constant \(c_{\eta,N}\) not depending on \(u\), \(h\) and \(\mathcal{D}\).

Moreover for any \(\epsilon > 0\), there exists \(\mathcal{D} > 0\) such that for all \(\alpha, 0 \leq |\alpha| < N\),

\[
\sum_{\nu \in \mathbb{Z}^n \setminus 0} |\partial^\nu \mathcal{F}\eta(\sqrt{\mathcal{D}}\nu)| < \epsilon.
\]

\(\nabla_k u(x)\) denotes the vector of all partial derivatives \(\{\partial^\alpha u(x)\}_{|\alpha| = k}\) and \(\mathcal{F}\eta\) denotes the Fourier transform of \(\eta\). We deduce that for any \(\epsilon > 0\) there exists \(\mathcal{D} > 0\) such that \(M_{h,\mathcal{D}}u(x)\) approximates \(u(x)\) pointwise with the estimate (see [9],[10])

\[
|M_{h,\mathcal{D}}u(x) - u(x)| \leq c_{\eta,N}(\sqrt{\mathcal{D}}h)^N \|
abla^N u\|_{L_\infty} + \epsilon \sum_{k=0}^{N-1} (h \sqrt{\mathcal{D}})^k |\nabla_k u(x)|.
\]

Therefore \(M_{h,\mathcal{D}}u\) behaves like an approximation formula of order \(N\) up to the saturation term that can be ignored in numerical computations if \(\mathcal{D}\) is large enough. Similar estimates are also valid for integral norms (see [8]).

Several methods to construct generating functions satisfying the moment conditions (2.2) for arbitrarily large \(N\) have been developed (see [10],[12]). In fact any sufficiently smooth and rapidly decaying function \(\eta\) with \(\mathcal{F}\eta(0) \neq 0\) can be used to construct new generating functions \(\eta_N\) satisfying the moment conditions for arbitrary large \(N\) as shown in the next theorem.
Theorem 2.2 ([10]) Let $\eta \in S(\mathbb{R}^n)$ with $F\eta(0) \neq 0$. Then

$$\eta_N(x) = \sum_{|\alpha| = 0}^{N-1} \frac{\partial^\alpha (F\eta(\lambda)^{-1})|\lambda = 0}{\alpha!(2\pi i)^{|\alpha|}} \partial^\alpha \eta(x)$$

satisfies the moment conditions (2.2).

An interesting example is given by the Gaussian function $\eta(x) = e^{-|x|^2}$ where the application of Theorem 2.2 leads to the generating function

$$\eta_{2M}(x) = \pi^{-n/2} \sum_{j=0}^{M-1} (-1)^j \Delta^j e^{-|x|^2} = \pi^{-n/2} L_{M-1}^{(n/2)}(|x|^2)e^{-|x|^2}$$

with $N = 2M$ and the generalized Laguerre polynomial

$$L_k^{(\gamma)}(y) = \frac{e^y y^{-\gamma}}{k!} \left( \frac{d}{dy} \right)^k (e^{-y} y^{k+\gamma}), \gamma > -1.$$ 

Hence the quasi-interpolant

$$M_{h,D} u(x) = (\pi D)^{-n/2} \sum_{j \in \mathbb{Z}^n} u(hj) L_{M-1}^{(n/2)} \left( \frac{|x - hj|}{h \sqrt{D}} \right) e^{-\left|\frac{x-hj}{h \sqrt{D}}\right|^2}$$

is an approximation formula of order $N = 2M$ plus the saturation term.

The quasi-interpolation formula and the corresponding approximation results have been generalized in [1] and [3] to the case when the values of $u$ are given on uniform grids, not necessarily cubic, of this type

$$\Lambda_h := \{hA_j, j \in \mathbb{Z}^n\}$$

with a real nonsingular $n \times n$-matrix $A$.

Under the same assumptions on the generating function $\eta$, it is always possible to choose $D > 0$ such that the quasi-interpolant

$$M_{\Lambda_h} u(x) := \frac{\det A}{D^{n/2}} \sum_{j \in \mathbb{Z}^n} u(hA_j) \eta \left( \frac{x - hA_j}{\sqrt{D} h} \right)$$

(2.3)

satisfies an estimate similar to that obtained in Theorem 2.1 for uniform cubic grid i.e.

$$|M_{\Lambda_h} u(x) - u(x)| \leq c_{\eta,N}(\sqrt{D} h)^N \|\nabla u\|_{L^\infty} + \epsilon \sum_{k=0}^{N-1} (h \sqrt{D})^k \|\nabla_k u(x)\|$$

(2.4)

for any $\epsilon > 0$.

The first application of formula (2.3) is the construction of quasi-interpolants on a regular triangular grid in the plane, as indicated in Figure 1. The vertices $y^\Delta_j$ of a partition of the plane into equilateral triangles of side length 1 are given by

$$y^\Delta_j = A j; \quad A = \begin{pmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix}.$$ 

The application of formula (2.3) to the nodes of the regular triangular grid of size $h$

$$\Lambda_h = \{h y^\Delta_j\} = \{h A_j\}_{j \in \mathbb{Z}^2}$$

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gives the following quasi-interpolant

$$\mathcal{M}_h^\Delta u(x) := \frac{\sqrt{3}}{2D} \sum_{j \in \mathbb{Z}^2} u(hy_j^\Delta) \eta \left( \frac{x - hy_j^\Delta}{\sqrt{D}h} \right).$$

The system of functions \( \{ \frac{\sqrt{3}}{2D} \eta \left( \frac{x - y_j^\Delta}{\sqrt{D}} \right) \} \), centered at the points of the uniform triangular grid, forms an approximate partition of unity. Using Poisson’s summation formula one can bound the main term of the saturation error by

$$\left| 1 - \frac{\sqrt{3}}{2D} \sum_{j \in \mathbb{Z}^2} \eta \left( \frac{x - y_j^\Delta}{\sqrt{D}} \right) \right| \leq \sum_{\nu \in \mathbb{Z}^2 \backslash \{0\}} \left| \int_{\mathbb{R}^2} \eta(y) e^{-2\pi i \nu (x - y) \cdot d} dy \right|.$$ 

By assuming as generating function the Gaussian \( \eta(x) = \pi^{-1} e^{-|x|^2} \) we obtain

$$\left| 1 - \frac{\sqrt{3}}{2\pi D} \sum_{j \in \mathbb{Z}^2} e^{-|x - y_j^\Delta|^2 / D} \right| \leq \sum_{(\nu_1, \nu_2) \neq (0,0)} e^{-4\pi^2 D (\nu_1^2 + \nu_2^2)/3} = 6 e^{-4\pi^2 D/3} + O(e^{-4\pi^2 D}).$$

In Figure 2 the graph of the difference \( \frac{\sqrt{3}}{2\pi D} \sum_{j \in \mathbb{Z}^2} e^{-|x - y_j^\Delta|^2 / D} - 1 \) is plotted with two different values of \( D \).

As second example we construct quasi-interpolants with functions centered at the nodes of a regular hexagonal grid in the plane, as depicted in Figure 3. We obtain a hexagonal grid if, from the nodes of a regular triangular grid of side length 1, the nodes of another triangular grid of side length \( \sqrt{3} \) are removed (see Figure 4). Therefore the set of nodes \( X^\circ \) of the regular hexagonal grid are given by

$$X^\circ = \{ A j \}_{j \in \mathbb{Z}^2} \backslash \{ B j \}_{j \in \mathbb{Z}^2}$$

where

$$B = \begin{pmatrix} 3/2 & 0 \\ \sqrt{3}/2 & \sqrt{3} \end{pmatrix}$$

and \( B j, j \in \mathbb{Z}^2 \), denote the removed nodes.
Figure 2: The graph of $\frac{\sqrt{D}}{2\pi D} \sum_{j \in \mathbb{Z}^2} e^{-|x-y|^2/\sigma^2} - 1$ when $D = 2$ (on the left) and $D = 3$ (on the right).

Figure 3: Hexagonal grid

The quasi-interpolant on the $h$-scaled hexagonal grid

$$hX^o = \{hA_j\}_{j \in \mathbb{Z}^2} \setminus \{hB_j\}_{j \in \mathbb{Z}^2}$$

is defined as

$$\mathcal{M}_h^o u(x) := \frac{3\sqrt{3}}{4D} \sum_{y^o \in X^o} u(hy^o) \eta \left(\frac{x - hy^o}{\sqrt{D}h}\right).$$

For (2.5) the quasi-interpolant $\mathcal{M}_h^o u$ can be written in an equivalent way

$$\mathcal{M}_h^o u(x) = \frac{3\sqrt{3}}{4D} \left( \sum_{j \in \mathbb{Z}^2} u(hA_j) \eta \left(\frac{x - hA_j}{\sqrt{D}h}\right) - \sum_{j \in \mathbb{Z}^2} u(hB_j) \eta \left(\frac{x - hB_j}{\sqrt{D}h}\right) \right),$$

Therefore we derive that under the decay conditions and the moment conditions on $\eta$ the quasi-interpolant $\mathcal{M}_h^o u$ provides the estimate (2.4) for sufficiently large $D$.

From Poisson’s summation formula

$$\sum_{j \in \mathbb{Z}^2} \eta \left(\frac{x - A_j}{\sqrt{D}}\right) = \frac{D}{\text{det } A} \left(1 + \sum_{\nu \in \mathbb{Z}^2 \setminus 0} \mathcal{F}_\eta(\sqrt{D}(A^t)^{-1} \nu) e^{2\pi i (x.A^{-1} \nu)}\right),$$
we obtain an approximate partition of unity centered at the hexagonal grid:

\[
\frac{3\sqrt{3}}{4\pi D} \sum_{y^o \in X^o} \eta \left( \frac{x - y^o}{\sqrt{D}} \right) - 1 = \sum_{j \in \mathbb{Z}^2} \eta \left( \frac{x - Aj}{\sqrt{D}} \right) - \sum_{j \in \mathbb{Z}^2} \eta \left( \frac{x - Bj}{\sqrt{D}} \right) - 1 =
\]

\[
\frac{3}{2} \sum_{\nu \in \mathbb{Z}^2 \setminus \{0\}} \mathcal{F} \eta \left( \sqrt{D} (A^t)^{-1} \nu \right) e^{2\pi i (x, (A^t)^{-1} \nu)} - \frac{1}{2} \sum_{\nu \in \mathbb{Z}^2 \setminus \{0\}} \mathcal{F} \eta \left( \sqrt{D} (B^t)^{-1} \nu \right) e^{2\pi i (x, (B^t)^{-1} \nu)}.
\]

In the case of the exponential \( \eta(x) = \pi^{-1} e^{-|x|^2} \) we have estimated the main term of the saturation error by

\[
\left| 1 - \frac{3\sqrt{3}}{4\pi D} \sum_{y^o \in X^o} e^{-|x - y^o|^2 / D} \right| \leq \frac{1}{2} \sum_{(\nu_1, \nu_2) \neq (0,0)} (3e^{-4\pi^2 D (\nu_1^2 \nu_2^2 + \nu_1^2 \nu_2) / 3} + e^{-4\pi^2 D (\nu_1^2 - \nu_1^2 + \nu_2^2) / 3}) - 3e^{-4\pi^2 D / 3} + O(e^{-4\pi^2 D / 3}).
\]  

In Figure 5 the difference (2.6) is depicted for two different values of \( D \).

Figure 4: Nodes of a hexagonal grid. The eliminated triangular grid \( Bj \) is depicted with dashed lines.

Figure 5: The graph of \( \frac{3\sqrt{3}}{4\pi D} \sum_{y^o \in X^o} e^{-|x - y^o|^2 / D} - 1 \) when \( D = 4 \) (on the left) and \( D = 8 \) (on the right).
3 Results for nonuniform grids

Next we consider an extension of the approximate quasi-interpolation formulas on uniform grid to the case that the data are given on a set of scattered nodes \( X = \{ x_j \} \subset \mathbb{R}^n \) close to a uniform grid in the sense that we specify in Condition 1.

**Condition 1** There exists a uniform grid \( \Lambda \) such that the quasi-interpolants

\[
M_{h,\varnothing} u(x) = D^{-n/2} \sum_{y_j \in \Lambda} u(h y_j) \eta \left( \frac{x - h y_j}{h \sqrt{D}} \right)
\]

approximate sufficiently smooth functions \( u \) with the error

\[
|M_{h,\varnothing} u(x) - u(x)| \leq c_{N,\eta} (h \sqrt{D})^N \| \nabla^N u \|_{L_\infty(\mathbb{R}^n)} + \epsilon \sum_{k=0}^{N-1} (h \sqrt{D})^k |\nabla_k u(x)|
\]

(3.7)

for any \( \epsilon > 0 \).

Let \( X_h \) be a sequence of grids with the property that for \( \kappa_1 > 0 \) not depending on \( h \) and any \( y_j \in \Lambda \) the ball \( B(h y_j, \kappa_1) \) contains nodes of \( X_h \).

For example, if \( \eta \) satisfies the conditions of Theorem 2.1, we may assume as \( \Lambda \) the cubic grid \( \{ j \} \) or, in the plane, the triangular grid \( \{ y^a \} \) or the hexagonal grid \( \{ y^* \} \).

In order to construct an approximate quasi-interpolant which use the data at the nodes of \( X_h \) we introduce the following definition.

**Definition 3.1** Let \( x_j \in X_h \). A collection of \( m_N = \frac{(N - 1 + n)!}{n!(N - 1)!} - 1 \) nodes \( x_k \in X_h \) will be called star of \( x_j \) and denoted by \( \text{st}(x_j) \) if the Vandermonde matrix

\[
V_{j,h} = \left\{ \left( \frac{x_k - x_j}{h} \right)^\alpha \right\}, \quad |\alpha| = 1, ..., N - 1,
\]

is not singular.

**Condition 2** Denote by \( \tilde{x}_j \in X_h \) the node closest to \( h y_j \in h \Lambda \). There exists \( \kappa_2 > 0 \) such that for any \( y_j \in \Lambda \) the star \( \text{st}(\tilde{x}_j) \subset B(\tilde{x}_j, \kappa_2) \) with \( |\det V_{j,h}| \geq c > 0 \) uniformly in \( h \).

Let us denote by \( \{ b_{\alpha,k}^{(j)} \}, |\alpha| = 1, ..., N - 1, x_k \in \text{st}(\tilde{x}_j) \), the elements of the inverse matrix of \( V_{j,h} \), and consider the functional

\[
F_{j,h}(u) = u(\tilde{x}_j) \left( 1 - \sum_{|\alpha| = 1}^{N-1} \left( y_j - \frac{\tilde{x}_j}{h} \right)^\alpha \sum_{x_k \in \text{st}(\tilde{x}_j)} b_{\alpha,k}^{(j)} \right)
\]

\[
+ \sum_{x_k \in \text{st}(\tilde{x}_j)} u(x_k) \sum_{|\alpha| = 1}^{N-1} b_{\alpha,k}^{(j)} \left( y_j - \frac{\tilde{x}_j}{h} \right)^\alpha.
\]

The functional \( F_{j,h}(u) \) depends on the values of \( u \) at the nodes of \( \text{st}(\tilde{x}_j) \cup \tilde{x}_j \) i.e. \( m_N + 1 \) points close to \( h y_j \).

Let us define the following quasi-interpolant which uses the values of \( u \) on \( X_h \)

\[
M_{h,\varnothing} u(x) = D^{-n/2} \sum_{y_j \in \Lambda} F_{j,h}(u) \eta \left( \frac{x - h y_j}{h \sqrt{D}} \right).
\]

(3.8)

The following theorem states that, under the above mentioned conditions on the grid, \( M_{h,\varnothing} u \) has the same behavior as in the case of uniform grids.
Theorem 3.1 ([3]) Under the Conditions 1 and 2, for any \( \epsilon > 0 \) there exists \( D > 0 \) such that the quasi-interpolant (3.8) approximates any \( u \in W^N_{\infty}(\mathbb{R}^n) \) with

\[
|M_{h,D}u(x) - u(x)| \leq c_{N,\eta,D} h^{N}||\nabla N u||_{L_{\infty}(\mathbb{R}^n)} + \epsilon \sum_{k=0}^{N-1} (h \sqrt{D})^k |\nabla_k u(x)|,
\]

where \( c_{N,\eta,D} \) does not depend on \( u \) and \( h \).

One of the motivations of approximate approximations is the construction of cubature formulas for integral operators of convolution type

\[
K u(x) = \int_{\mathbb{R}^n} k(x - y) u(y) \, dy. \tag{3.9}
\]

A cubature formula of the multi-dimensional integral (3.9) can be obtained if the density \( u \) is replaced by the quasi-interpolant \( M_{h,D} u \). Then

\[
KM_{h,D} u(x) = D^{-n/2} \sum_{y_j \in \Lambda} F_{j,h}(u) \int_{\mathbb{R}^n} k(x - y) \eta(y/h) \, dy
\]

\[
= h^n \sum_{y_j \in \Lambda} F_{j,h}(u) \int_{\mathbb{R}^n} k(h \sqrt{D}(x - h y_j/h) - y) \eta(y) \, dy
\]

is a cubature formula for (3.9) with a generating function \( \eta \) chosen such that \( K \eta \) can be computed analytically or at least by some efficient quadrature method.

In (3.8) the generating function is centered at the nodes of the uniform grid \( h \Lambda \). This can be helpful to design fast methods for the approximation of (3.9). If we define

\[
a^{(k)}_{k-j} = \int_{\mathbb{R}^n} k(h y_k - y_j - \sqrt{D} y) \, \eta(y) \, dy.
\]

we reduce to the computation of the following sums

\[
KM_{h,D} u(h y_k) = h^n \sum_{y_j \in \Lambda} F_{j,h}(u) a^{(k)}_{k-j}
\]

which provide an approximation of (3.9) at the mesh points \( h y_k \).

A generalization of the method approximate approximations to functions with values given on a rather general grid was obtained in [4].

4 Numerical Experiments

The quasi-interpolant \( M_{h,D} u \) in (3.8) was tested by one- and two-dimensional experiments and the results of the numerical experiments confirm the predicted approximation orders. In all cases the grid \( X_h \) is chosen such that any ball \( B(h \mathbf{j}; h/2), \mathbf{j} \in \mathbb{Z}^n \), \( n = 1 \) or \( n = 2 \), contains one randomly chosen node, which we denote by \( x_j \).

The one-dimensional case. Figures 6 - 9 show the graphs of \( M_{h,D} u - u \) for different smooth functions \( u \) using the basis function \( \eta(x) = \pi^{-1/2} e^{-x^2} \) (Fig. 6 and 7) for which \( N = 2 \), and \( \eta(x) = \pi^{-1/2}(3/2 - x^2)e^{-x^2} \) (Fig. 8 and 9) for which \( N = 4 \), for different values of \( h \). We have chosen the parameter \( D = 4 \) in order to keep the saturation error less than \( 10^{-16} \).
Figure 6: The graphs of $\mathcal{M}_h \mathcal{D} u - u$ with $\eta(x) = \pi^{-1/2} e^{-x^2}$, $\mathcal{D} = 4$, $st(x_j) = \{x_{j+1}\}$, when $u(x) = \sin(x)$ (on the left) and $u(x) = \cos(x)$. Dashed and solid lines correspond to $h = 1/32$ and $h = 1/64$.

Figure 7: The graphs of $\mathcal{M}_h \mathcal{D} u - u$ with $\eta(x) = \pi^{-1/2} e^{-x^2}$, $\mathcal{D} = 4$, $st(x_j) = \{x_{j+1}\}$, when $u(x) = \sin(x)$ (on the left) and $u(x) = \cos(x)$. Dashed and solid lines correspond to $h = 1/128$ and $h = 1/256$.

The two-dimensional case. We depict in Figures 10 and 11 the quasi-interpolation error $\mathcal{M}_h \mathcal{D} u - u$ for the function $u(x) = (1 + |x|^2)^{-1}$ and different $h$ if generating functions of second (with $\mathcal{D} = 2$) and fourth (with $\mathcal{D} = 4$) order of approximation are used. The $h^2$- and respectively $h^4$-convergence of the corresponding two-dimensional quasi-interpolants are confirmed by the $L_\infty$- errors which are given in Table 1.

<table>
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<td>$2.95 \cdot 10^{-5}$</td>
<td>$6.61 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>$2^{-6}$</td>
<td>$5.51 \cdot 10^{-4}$</td>
<td>$1.01 \cdot 10^{-5}$</td>
<td>$2^{-6}$</td>
<td>$1.92 \cdot 10^{-6}$</td>
<td>$4.24 \cdot 10^{-6}$</td>
</tr>
<tr>
<td>$2^{-7}$</td>
<td>$1.42 \cdot 10^{-4}$</td>
<td>$2.52 \cdot 10^{-7}$</td>
<td>$2^{-7}$</td>
<td>$1.24 \cdot 10^{-7}$</td>
<td>$2.68 \cdot 10^{-7}$</td>
</tr>
<tr>
<td>$2^{-8}$</td>
<td>$3.56 \cdot 10^{-5}$</td>
<td>$6.50 \cdot 10^{-8}$</td>
<td>$2^{-8}$</td>
<td>$7.80 \cdot 10^{-9}$</td>
<td>$1.71 \cdot 10^{-8}$</td>
</tr>
</tbody>
</table>

Table 1: $L_\infty$ approximation error for the function $u(x) = (1 + |x|^2)^{-1}$ using $\mathcal{M}_h \mathcal{D} u$ with $\eta(x) = \pi^{-1} e^{-|x|^2}$, $N = 2$ (on the left), and $\eta(x) = \pi^{-1} (2 - |x|^2) e^{-|x|^2}$, $N = 4$ (on the right).
Figure 8: The graphs of $M_h D u - u$ with $\eta(x) = \pi^{-1/2}(3/2 - x^2)e^{-x^2}$, $D = 4$, st. $(x_j) = \{x_{j-2}, x_{j-1}, x_{j+1}\}$, when $u(x) = \sin(x)$ (on the left) and $u(x) = \cos(x)$. Dashed and solid lines correspond to $h = 1/32$ and $h = 1/64$.

Figure 9: The graphs of $M_h D u - u$ with $\eta(x) = \pi^{-1/2}(3/2 - x^2)e^{-x^2}$, $D = 4$, st. $(x_j) = \{x_{j-2}, x_{j-1}, x_{j+1}\}$, when $u(x) = \sin(x)$ (on the left) and $u(x) = \cos(x)$. Dashed and solid lines correspond to $h = 1/128$ and $h = 1/256$.

Figure 10: The graph of $M_h D u - u$ with $D = 2$, $\eta(x) = \pi^{-1/2}e^{-|x|^2}$, $N = 2$, $u(x) = (1 + |x|^2)^{-1}$, $h = 2^{-6}$ (on the left) and $h = 2^{-7}$ (on the right).
Figure 11: The graph of $\mathcal{M}_h, \mathcal{P}u - u$ with $\mathcal{D} = 4$, $\eta(x) = \pi^{-1} (2 - |x|^2)e^{-|x|^2}$, $N = 4$, $u(x) = (1 + |x|^2)^{-1}$, $h = 2^{-6}$ (on the left) and $h = 2^{-7}$ (on the right).
References


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