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Solutions for Quasilinear Nonsmooth Evolution Systems in L^p

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Abstract

We prove that nonsmooth quasilinear parabolic systems admit a local solution in L^p strongly differentiable with respect to time over a bounded three-dimensional polyhedral space domain. The proof rests essentially on new elliptic regularity results for polyhedral Laplace interface problems for anisotropic materials. These results are based on sharp pointwise estimates for Green's function, which are also of independent interest. To treat the nonlinear problem, we then apply a classical theorem of Sobolevskii for abstract parabolic equations and recently obtained resolvent estimates for elliptic operators and interpolation results. As applications we have in mind primarily reaction-diffusion systems. The treatment of such equations in an L^p context seems to be new and allows (by Gauss' theorem) the proper definition of the normal component of currents across the boundary.

1. Introduction

Various phenomena in physics, chemistry and biology are described by systems of evolution equations like

$$u_k' - \nabla \cdot (\mu_k \mathcal{J}_k(\mathbf{u}) \nabla u_k) = \mathcal{R}_k(\mathbf{u}, \nabla \mathbf{u}) \ , \ \mathbf{u}(T_0) = \mathbf{u}_0 \ ; \ \mathbf{u} = (u_1, \dots, u_m) \quad (1.1)$$

(see [1] and the references therein). In many applications, the data describing the properties of the medium involve discontinuities. The aim of this work is to establish conditions on the piecewise constant coefficients μ_k under which (1.1) admits a unique solution from a space

$$C([T_0, T], L^p(\Omega; \mathbb{R}^m)) \cap C^1((T_0, T], L^p(\Omega; \mathbb{R}^m)).$$

Throughout this paper we impose Dirichlet boundary conditions which may depend suitably on time. The underlying three-dimensional domain Ω is a Lipschitz polyhedron, which means that Ω is a bounded Lipschitz domain with piecewise plane boundary. Further, we assume that Ω is partitioned into a finite set of Lipschitz polyhedra $\Omega_1, \ldots, \Omega_J$ such that the (3×3)-matrix functions μ_k are constant on these subdomains. The dependence of the functions $\mathcal{R}_k$ on $\nabla \mathbf{u}$ is not stronger than quadratic.

The theory of systems of the form (1.1) is well developed if Ω and the coefficient functions μ_k are smooth (see, e.g., [9] or [33]). Furthermore, existence and uniqueness are studied exhaustively in the weak context; e.g., Hölder estimates have long been known also in this case (see [21] or [31]).

Note that the original formulation of (1.1) in terms of balance laws takes the form (see [29, Chapter 21])

$$\frac{\partial}{\partial t} \int_{\Xi} u_k \, dx + \int_{\partial \Xi} \nu \cdot j_k \, d\sigma = \int_{\Xi} \mathcal{R}_k(\mathbf{u}, \nabla \mathbf{u}) \, dx \,, \tag{1.2}$$

where Ξ stands for any suitable subdomain of Ω. Within the variational theory of weak solutions, however, the characteristic functions χ_Ξ of the subdomains are not admissible test functions. Therefore the integral formulation (1.2) is equivalent to (1.1) only if the weak solutions have some additional regularity. Moreover, the additional regularity is also of importance for the numerical treatment of (1.1), as the integral formulation is the basis of finite-volume methods.

The main advantage of our work in comparison to the concept of weak solutions is the strong differentiability of the solution with respect to time and the fact that the divergence of each corresponding current $j_k = \mu_k \mathcal{J}_k(t, \mathbf{u}) \nabla u_k$ is a function, not only a distribution. In a strict sense, only this justifies the application of Gauss' theorem to calculate the normal components of the currents over boundaries of (suitable) subdomains.

We address a general class of possible applications involving reaction-diffusion systems and heat conduction in Section 5. Though at this point our results are restricted to Dirichlet boundary conditions, we feel that the approach can be extended to mixed boundary conditions, which occur, e.g., in modelling semiconductor devices [7].

Global existence results for (1.1) cannot be expected within this rather general approach (see, e.g., [6] or [2] and the references therein) and are thus outside the scope of this paper.

Our regularity result for (1.1), Theorem 6.10, rests upon the classical theorem of Sobolevskii on abstract quasilinear parabolic equations in Banach spaces and estimates for elliptic transmission problems. The problem is to find an adequate function space with respect to which the hypotheses of this theorem can be verified; see Sections 5 and 6. In the three-dimensional case, this question comes down to checking whether the linear operators

$$\nabla \cdot \mu \nabla : H_0^{1,q}(\Omega) \mapsto (H_0^{1,q'}(\Omega))' \tag{1.3}$$

are topological isomorphisms for some $q > 3$ and any piecewise constant matrix $\mu = \mu_k$ occurring in (1.1). The operator (1.3) corresponds to an interface (or transmission) problem for the Laplacian, with different anisotropic materials given on the polyhedral subdomains $\Omega_1, \dots, \Omega_J$ of Ω and with Dirichlet conditions given on $\partial\Omega$.

Unfortunately, in contrast to the pure Laplacian on a Lipschitz domain (see [15, Theorem 0.5]), the solutions to such transmission problems only belong to $L^{2+\varepsilon}$ near vertices and edges where $\varepsilon > 0$ might be arbitrary small in general. This is even true for polygonal Laplace interface problems with four isotropic materials; see [16]. Therefore, a large part of this article is devoted to the optimal L^q regularity for (1.3).

It is well known that the singularities of solutions to elliptic boundary-value problems near corners and edges can be characterized in terms of the eigenvalues of certain polynomial operator pencils on domains of the unit sphere or the unit circle. We refer to [20] in the case of the Dirichlet and Neumann problems and to [11] for the polyhedral Laplace interface problem with two isotropic materials. To our knowledge, the corresponding analysis for several anisotropic materials has not been done so far. This will be the topic of Sections 2–4.

To avoid the rather complicated discussion of the optimal regularity near vertices, we exploit the somewhat surprising fact that if the solution of the interface problem belongs to L^q for some $q > 3$ near each interior point of the interface and boundary edges, then the operator (1.3) is in fact an isomorphism; see Sections 2 and 4. Thus we are able to reduce the regularity result for (1.3) to that for an interface problem on dihedral angles having one common edge; see Theorem 4.1. The proof of this relies essentially on sharp pointwise estimates of Green's function, which will be presented in detail in Section 3.

The main result of our linear regularity theory, Theorem 2.3, says that the operator (1.3) is an isomorphism provided that

$$q \in [2, 2/(1 - \widehat{\lambda}_\Omega))$$

and that the spectral parameter $\widehat{\lambda}_\Omega$ (cf. Definition 2.1) satisfies the inequality

$$\widehat{\lambda}_\Omega > \tfrac{1}{3}\,. \tag{1.4}$$

Note that $\widehat{\lambda}_\Omega$ can be expressed in terms of the eigenvalues of certain transmission problems on the unit circle, which are obtained applying the partial Fourier transform along an edge and the Mellin transform with respect to radial direction (see Section 3.5).

This result is sufficient for the treatment of the quadratic gradient terms in (1.1) if the Banach space is a suitably chosen L^p space. However, the condition (1.4) imposes a rather strong assumption on the geometry of the subdomains $\Omega_1, \dots, \Omega_J$ and the coefficient μ_k, or equivalently, on the eigenvalues of certain pencils of ordinary differential operators. We refer to Section 3.6 for a discussion of this condition, which can certainly be checked for many heterostructures of practical interest.

Let us introduce some notation. The space of complex-valued, Lebesgue measurable, p-integrable functions on Ω, $p \in [1, \infty)$, is denoted by $L^p(\Omega)$, whereas $L^\infty(\Omega)$ denotes the space of essentially bounded functions on Ω.

We use $H^{s,q}(\Omega)$, $s \in [0, 1]$, to denote the space of Bessel potentials according to the differentiability index s and integrability index q on the set Ω (see [32]).

(Note that for $s = 1$ these spaces coincide with the Sobolev spaces $W^{1,q}(\Omega)$.) By $H_0^{s,q}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $H^{s,q}(\Omega)$.

If $s \in [-1, 0]$, then $H^{s,q'}(\Omega)$ denotes the dual to $H^{-s,q}(\Omega)$ and $H_0^{s,q'}(\Omega)$ denotes the dual to $H_0^{-s,q}(\Omega)$ when $1/q + 1/q' = 1$ holds.

For two Banach spaces X and Y we denote the space of linear, bounded operators from X into Y by $\mathcal{B}(X, Y)$. If $X = Y$, then we abbreviate $\mathcal{B}(X)$. The norm in a Banach space X will be always indicated by $\|\cdot\|_X$; only in obvious cases the subscript sometimes will be omitted.

2. A linear regularity result: Reduction to a wedge problem

In the first part of the paper we study L^q regularity of weak solutions of the Dirichlet problem

$$-\nabla \cdot \Lambda(x)\nabla u = \phi\,, \quad x \in \Omega\,, \quad u|_{\partial\Omega} = 0\,, \tag{2.1}$$

with a real, symmetric and strictly positive-definite matrix function $\Lambda(x)$. The domain Ω is a Lipschitz polyhedron partitioned into a finite set of polyhedra $\Omega_1, \ldots, \Omega_J$ such that Λ is constant on each subdomain Ω_j. We are interested in determining under which conditions on Ω and Λ the solution of (2.1) satisfies $u \in H_0^{1,q}(\Omega)$ if the right-hand side $\phi \in H_0^{-1,q}(\Omega)$ for some $q > 3$.

To formulate the result, we need a parameter $\widehat{\lambda}_\Omega$ which can be obtained from the geometry of Ω and the coefficient $\Lambda(x)$ in the following way:

The matrix function Λ is constant on polyhedral subdomains of Ω and has therefore jumps at plane interfaces which intersect at certain interior or boundary edges. Let $\mathcal{M}$ be one of these edges or one of the edges of the polyhedron Ω. Choose a new coordinate system (y_1, y_2, y_3) with origin at a point P in the interior of $\mathcal{M}$ such that the direction of $\mathcal{M}$ coincides with the y_3-axis. Denote by $\tilde{\Lambda}(y)$ the piecewise constant matrix function which coincides in a neighbourhood of P with $A^{-1}\Lambda(A^{-1}(y + P))A$, where A denotes the corresponding orthogonal transformation matrix, and satisfies $\tilde{\Lambda}(ty', y_3) = \tilde{\Lambda}(y', 0)$, $y' = (y_1, y_2)$, for all $y_3 \in \mathbb{R}$, $t > 0$.

We assign to $\mathcal{M}$ a positive real number by solving the following nonlinear eigenvalue problem:

Let $r = |y'|, \theta$ be polar coordinates in the y' plane and set $U = r^\lambda u(\theta)$, $V = \overline{r^\lambda} v(\theta)$, $\lambda \in \mathbb{C}$, where the functions u, v are given on the intersection σ of the unit sphere S^1 in the y'-plane with the support $\widetilde{\Omega}$ of $\tilde{\Lambda}(y)$, $\sigma = \widetilde{\Omega} \cap S^1$. If $\mathcal{M}$ is an interior edge of Ω, then $\sigma = S^1$ and we denote by $\mathcal{H} = H^1(S^1)$ the periodic Sobolev space on the unit circle. Otherwise we set $\mathcal{H} = H_0^1(\sigma)$. Let $\widetilde{\Lambda}'(y')$ be the upper left 2×2 block of $\tilde{\Lambda}(y)$ and define the operator $\Pi(\lambda)$ by

$$\langle \Pi(\lambda)u, v\rangle_\sigma \stackrel{\text{def}}{=} \frac{1}{\log 2} \int\limits_{\{1<|y'|<2\}\cap\widetilde{\Omega}} \widetilde{\Lambda}'(y')\, \nabla_{y'} U \cdot \nabla_{y'} \overline{V}\, dy'\,, \quad u, v \in \mathcal{H}\,, \tag{2.2}$$

where $\langle \cdot, \cdot \rangle_\sigma$ is the $L^2(\sigma)$ duality. In Section 3 we will show that the spectrum of the operator pencil $\Pi(\lambda) : \mathcal{H} \mapsto \mathcal{H}'$ consists of isolated eigenvalues only. Denote by $\lambda_{\mathcal{M}}$ the eigenvalue with the smallest positive real part and set $\widehat{\lambda}_{\mathcal{M}} = \operatorname{Re} \lambda_{\mathcal{M}}$.

Definition 2.1. Let

$$\widehat{\lambda}_\Omega \stackrel{\text{def}}{=} \min(1, \widehat{\lambda}_{\mathcal{M}}),$$

where the minimum is taken over all interior and boundary edges $\mathcal{M}$ of Ω.

Definition 2.2. We say that the matrix Λ generates an *admissible decomposition* of the Lipschitz polyhedron Ω into the polyhedral subdomains Ω_j (where Λ is constant) if $\widehat{\lambda}_\Omega > \frac{1}{3}$.

The regularity result which is needed for the nonlinear problem is

Theorem 2.3. *If the piecewise constant matrix Λ generates an admissible decomposition of Ω, then the operator $-\nabla \cdot \Lambda(x)\nabla : H_0^1(\Omega) \mapsto H_0^{-1}(\Omega)$ provides a topological isomorphism between $H_0^{1,q}(\Omega)$ and $H_0^{-1,q}(\Omega)$ for all $q \in [2, 2/(1-\widehat{\lambda}_\Omega))$.*

Since $\|\nabla \cdot \|_{L^{q'}}$ is an equivalent norm on $H_0^{1,q'}(\Omega)$, any $\phi \in H_0^{-1,q}(\Omega) = (H_0^{1,q'}(\Omega))'$ can be represented as $\nabla \cdot \boldsymbol{f}$ with $\boldsymbol{f} \in L^q(\Omega)^3$, where the divergence is understood in the distributional sense. Hence, Theorem 2.3 is proved if we show that the unique solution u of the variational equation

$$\int_\Omega \Lambda(x)\nabla u \cdot \nabla \overline{\varphi}\, dx = \int_\Omega \boldsymbol{f} \cdot \nabla \overline{\varphi}\, dx \quad \forall \varphi \in H_0^1(\Omega) \tag{2.3}$$

satisfies the estimate

$$\|\nabla u\|_{L^q(\Omega)} \leqq c\, \|\boldsymbol{f}\|_{L^q(\Omega)} \tag{2.4}$$

with a constant c not depending on $\boldsymbol{f}$.

The proof of (2.4) is based on local estimates for solutions of the Dirichlet problem which can be obtained from model problems in an infinite wedge. Here we use the integral representation by Green's functions which are studied in Section 3. First we prove a result for differential operators in $\mathbb{R}^n$ with measurable coefficients, which will be applied in Section 4 to establish Theorem 2.3.

2.1. A preliminary result

Let Ω be a bounded polyhedral domain in $\mathbb{R}^n$, $n \geqq 3$, and consider the Dirichlet problem

$$L(x, \partial)u \stackrel{\text{def}}{=} \nabla \cdot A(x)\nabla u = \nabla \cdot \boldsymbol{g}\,, \quad x \in \Omega\,, \quad u|_{\partial\Omega} = 0\,, \tag{2.5}$$

with $\boldsymbol{g} \in L^q(\Omega)^n$. Here $A(x)$ is an $n \times n$ symmetric matrix of real, measurable and bounded functions satisfying

$$a\,|\xi|^2 \leqq A(x)\,\xi \cdot \overline{\xi} \leqq b\,|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n$$

uniformly in $x \in \Omega$ with $0 < a \leqq b$. Using Green's function $G(x, y)$, which satisfies

$$\int_\Omega A(y)\nabla_y G(x, y) \cdot \nabla\varphi \, dy = \varphi(x)\,, \quad \varphi \in H_0^1(\Omega)\,,$$

for all $x \in \Omega$, we can write the solution of (2.5) as

$$u(x) = -\int_\Omega \nabla_y G(x, y) \cdot \boldsymbol{g}(y)\, dy\,. \tag{2.6}$$

For almost all $x \in \overline{\Omega}$ the Green function belongs to the set $H^1(\Omega \setminus B_\rho(x)) \cap W_0^{1,p}(\Omega)$ for all $\rho > 0$ and $1 \leqq p < n/(n-1)$; cf. for example [30]. Here and in the following, $B_\rho(x)$ denotes the open ball in $\mathbb{R}^n$ with radius ρ and centre x. Moreover, for almost all $y \in \Omega \setminus \{x\}$ the estimate

$$0 \leqq G(x, y) \leqq \frac{c}{|x - y|^{n-2}} \tag{2.7}$$

holds and, if $|x - y| < \mathrm{dist}(x, \partial\Omega)/2$, then additionally

$$G(x, y) \geqq \frac{c}{|x - y|^{n-2}}\,, \tag{2.8}$$

where the constants depend on the ellipticity constants of $A(x)$.

By the De Giorgi-Nash theorem the solution of (2.5) is Hölder continuous. More precisely, there exists $\alpha \in (0, 1)$ such that $v \in C^\alpha(\overline{\Omega})$, and for any $x \in \overline{\Omega}$ and $0 < \rho < R$,

$$\sup_{\Omega \cap B_\rho(x)} u - \inf_{\Omega \cap B_\rho(x)} u \leqq c\,\rho^\alpha \Big(R^{-\alpha} \sup_{\Omega \cap B_R(x)} |u| + \|\boldsymbol{g}\|_{L^q} \Big)\,, \tag{2.9}$$

where C and α depend on the ellipticity constants of A, n, Ω and R (cf. [10]).

Theorem 2.4. *Let $n < q < n/(1-\alpha)$. For any $x_0 \in \overline{\Omega}$ the solution of* (2.5) *satisfies*

$$\int_\Omega |u(x) - u(x_0)|^q \frac{dx}{|x - x_0|^q} \leqq c \int_\Omega |\boldsymbol{g}|^q dx\,.$$

The proof relies on several lemmas. In the following, let $x_0 = 0$ and set $B_\rho = B_\rho(0)$.

Lemma 2.5. *Let $r > n$, $r' = r/(r-1)$. If $x \in B_\rho$, then*

$$\int_{\Omega \cap B_\rho} |\nabla_y G(x, y)|^{r'} dy \leqq c\,\rho^{(r-n)/(r-1)}$$

with a constant not depending on x.

Proof. We establish the stronger (because of $B_\rho \subset B_{2\rho}(x)$) inequality

$$\int\limits_{\Omega\cap B_{2\rho}(x)} |\nabla_y G(x,y)|^{r'} dy \leqq c\,\rho^{(r-n)/(r-1)}$$

by proving

$$I \overset{\text{def}}{=} \int\limits_{\Omega\cap B_\rho} |\nabla_y G(0,y)|^{r'} dy \leqq c\,\rho^{(r-n)/(r-1)}.$$

We have

$$\begin{aligned} I &\leqq c \sum_{k=0}^{\infty} (\rho\, 2^{-k})^n \fint\limits_{C_{\rho 2^{-k}}} |\nabla_y G(0,y)|^{r'} dy \\ &\leqq c \sum_{k=0}^{\infty} (\rho\, 2^{-k})^n \Big(\fint\limits_{C_{\rho 2^{-k}}} |\nabla_y G(0,y)|^2 dy \Big)^{r'/2}, \end{aligned}$$

where $C_\delta \overset{\text{def}}{=} \Omega \cap (B_{2\delta} \setminus B_\delta)$ and $\fint\limits_C f\,dx$ stands for $|C|^{-1} \int\limits_C f\,dx$.

We use a Caccioppoli-type inequality for spherical layers: Let $v \in H^1(\Omega \cap (B_{5\rho/2} \setminus B_{\rho/2}))$ with $L(x,\partial)\,v = 0$. Then

$$\int\limits_{C_\rho} |\nabla v|^2 dx \leqq \frac{c}{\rho^2} \Big(\int\limits_{\Omega\cap(B_{5\rho/2}\setminus B_{2\rho})} |v|^2 dx + \int\limits_{\Omega\cap(B_\rho\setminus B_{\rho/2})} |v|^2 dx \Big). \tag{2.10}$$

Because $L(x,\partial)\,G(x,y) = 0$ in $\Omega \cap (B_{\rho 2^{-k+2}} \setminus B_{\rho 2^{-k-1}})$, after applying (2.10) and (2.7) in $C_{\rho\, 2^{-k}}$ we obtain

$$\begin{aligned} \fint\limits_{C_{\rho\, 2^{-k}}} |\nabla_y G(0,y)|^2 dy &\leqq \frac{c_1}{(\rho\, 2^{-k})^2} \fint\limits_{\Omega\cap(B_{\rho 2^{-k+2}}\setminus B_{\rho 2^{-k-1}})} |G(0,y)|^2 dy \\ &\leqq \frac{c_2}{(\rho\, 2^{-k})^{2(n-1)}}. \end{aligned}$$

Thus

$$I \leqq c_3 \sum_{k=0}^{\infty} \frac{(\rho\, 2^{-k})^n}{(\rho\, 2^{-k})^{r'(n-1)}} = c_3\, \rho^{n-r'(n-1)} \sum_{k=0}^{\infty} 2^{-k(n-r'(n-1))},$$

and the series converges because $n - r'(n-1) = (r-n)/(r-1) > 0$. $\square$

Lemma 2.6. *Let* $|x| \leqq \frac{1}{2}|y|$, $x, y \in \Omega$. *Then*

$$|G(x, y) - G(0, y)| \leqq \frac{c\,|x|^\alpha}{|y|^{n-2+\alpha}} .$$

Proof. Set $\rho = |y|$. By (2.9) any solution of $L(x, \partial)\, v = 0$ satisfies

$$|v(x) - v(0)| \leqq c\,\frac{|x|^\alpha}{\rho^\alpha} \sup_{\Omega \cap B_{\rho/2}} |v|$$

for all $x \in \Omega \cap B_{\rho/2}$, and by (2.7) we obtain $G(x, y) \leqq c\,|x - y|^{2-n} \leqq c\,|y|^{2-n-2}$. □

Lemma 2.7. *Let* $\frac{5}{2}\rho < \operatorname{diam}(\Omega)$ *and* $|x| < \frac{1}{4}\rho$. *Then*

$$\Big(\fint_{\rho<|y|<2\rho} |\nabla_y (G(0, y) - G(x, y))|^2 dy \Big)^{1/2} \leqq \frac{c\,|x|^\alpha}{\rho^{n-1+\alpha}} .$$

Proof. By (2.10)

$$\fint_{C_\rho} |\nabla_y (G(0, y) - G(x, y))|^2 dy \leqq \frac{c}{\rho^2} \fint_{\Omega \cap (B_{5\rho/2} \setminus B_{\rho/2})} |G(0, y) - G(x, y)|^2 dy ,$$

and applying Lemma 2.6 gives the result. □

Proof of Theorem 2.4. Using the representation (2.6), we split

$$\begin{aligned}\int_\Omega |u(x) - u(0)|^q \frac{dx}{|x|^q} &\leqq \int_\Omega \Big| \int_{B_{4|x|}} \nabla_y (G(x, y) - G(0, y)) \cdot \boldsymbol{g}(y)\, dy \Big|^q \frac{dx}{|x|^q} \\ &\quad + \int_\Omega \Big| \int_{\mathbb{R}^n \setminus B_{4|x|}} \nabla_y (G(x, y) - G(0, y)) \cdot \boldsymbol{g}(y)\, dy \Big|^q \frac{dx}{|x|^q} \\ &\stackrel{\text{def}}{=} K_1 + K_2 ,\end{aligned}$$

where $\boldsymbol{g}$ is extended by zero onto $\mathbb{R}^n$. Let $n < r < q$. Then from Lemma 2.5,

$$\Big(\int_{\Omega \cap B_{4|x|}} (|\nabla_y G(x, y)|^{r'} + |\nabla_y G(0, y)|^{r'}) dy \Big)^{q/r'} \leqq c\,|x|^{q(r-n)/r} ,$$

and Hölder's inequality leads to

$$K_1 \leqq c \int_{\Omega} \frac{dx}{|x|^{nq/r}} \Big(\int_{B_{4|x|}} |\boldsymbol{g}(y)|^r dy \Big)^{q/r}$$
$$\leqq c \int_0^{\infty} \rho^{n-1-nq/r} d\rho \Big(\int_0^{\rho} \tau^{n-1} d\tau \int_{S^{n-1}} |\boldsymbol{g}(y)|^r d\sigma \Big)^{q/r}$$
$$\leqq c \int_0^{\infty} \rho^{n-1-nq/r+q/r} d\rho \Big(\rho^{n-1} \int_{S^{n-1}} |\boldsymbol{g}(y)|^r d\sigma \Big)^{q/r} \leqq c \int_{\Omega} |\boldsymbol{g}(y)|^q dy \,.$$

Since $nq/r - n > 0$, the second-to-last estimate follows from Hardy's inequality.

We proceed with

$$K_2 \leqq \int_{\Omega} \frac{dx}{|x|^q} \Big(\int_{|y|>4|x|} |\nabla_y (G(x,y) - G(0,y))| \, |\boldsymbol{g}(y)| \, dy \Big)^q$$
$$\leqq c \int_{\Omega} \frac{dx}{|x|^q} \Big(\int_{4|x|}^{\infty} \tau^{n-1} d\tau \fint_{B_{3\tau} \setminus B_\tau} |\nabla_y (G(x,y) - G(0,y))| \, |\boldsymbol{g}(y)| \, dy \Big)^q .$$

Here we use the fact that, for $f \geqq 0$, $a \geqq 0$,

$$\int_{|y|>a} f(y) \, dy = \log 3 \int_a^{\infty} \tau^{n-1} d\tau \Big(\frac{1}{\tau^n} \int_{\tau<|y|<3\tau} f(y) \, dy \Big) . \tag{2.11}$$

Then from Lemma 2.7,

$$K_2 \leqq \int_{\Omega} \frac{dx}{|x|^q} \Big(\int_{4|x|}^{\infty} \tau^{n-1} d\tau \Big(\fint_{B_{3\tau} \setminus B_\tau} |\nabla_y (G(x,y) - G(0,y))|^2 dy \Big)^{1/2}$$
$$\times \Big(\fint_{B_{3\tau} \setminus B_\tau} |\boldsymbol{g}(y)|^2 dy \Big)^{1/2} \Big)^q$$
$$\leqq c \int_{\Omega} \frac{dx}{|x|^{(1-\alpha)q}} \Big(\int_{4|x|}^{\infty} \tau^{-\alpha} d\tau \Big(\fint_{B_{3\tau} \setminus B_\tau} |\boldsymbol{g}(y)|^2 dy \Big)^{1/2} \Big)^q$$
$$\leqq c \int_0^{\infty} \rho^{n-1-(1-\alpha)q} \, d\rho \, \Big(\int_{\rho}^{\infty} \tau^{-\alpha} d\tau \Big(\fint_{B_{3\tau} \setminus B_\tau} |\boldsymbol{g}(y)|^2 dy \Big)^{1/2} \Big)^q ,$$

and in view of $n > (1-\alpha)q$, Hardy's inequality leads to

$$K_2 \leqq c \int_0^\infty \rho^{n-1} d\rho \fint_{B_{3\rho}\setminus B_\rho} |\boldsymbol{g}(y)|^q dy \leqq c \int_\Omega |\boldsymbol{g}(y)|^q dy \,. \qquad \square$$

2.2. Reduction to a wedge problem

We return to the variational equation (2.3) in $\Omega \subset \mathbb{R}^3$ with piecewise constant Λ. Let us choose a partition of unity of $\overline{\Omega}$ which isolates the corners, let χ be one of these cut-off functions, and define $\Omega_\chi = \Omega \cap \operatorname{supp} \chi$. From

$$\int_\Omega \Lambda\nabla(\chi u)\cdot\nabla\overline{\varphi}\,dx = \int_\Omega (\chi \boldsymbol{f} + u\,\Lambda\nabla\chi)\cdot\nabla\overline{\varphi}\,dx + \int_\Omega \overline{\varphi}\,(\boldsymbol{f} - \Lambda\nabla u)\cdot\nabla\chi\,dx$$

with

$$\left|\int_\Omega (\chi \boldsymbol{f} + u\,\Lambda\nabla\chi)\cdot\nabla\overline{\varphi}\,dx\right| \leqq c(\|\boldsymbol{f}\|_{L^q(\Omega_\chi)} + \|u\|_{L^q(\Omega_\chi)})\|\nabla\varphi\|_{L^{q'}}$$

and

$$\left|\int_\Omega \overline{\varphi}\,(\boldsymbol{f} - \Lambda\nabla u)\cdot\nabla\chi\,dx\right| \leqq c(\|\boldsymbol{f}\|_{L^q(\Omega_\chi)}\|\varphi\|_{L^{q'}} + \|u\|_{L^q(\Omega_\chi)}\|\nabla\varphi\|_{L^{q'}})$$

it follows that the function $\chi u \in H_0^1(\Omega_\chi)$ satisfies an equation of the form

$$\nabla\cdot\Lambda(x)\nabla(\chi u) = \nabla\cdot\boldsymbol{g} \quad \text{with } \|\boldsymbol{g}\|_{L^q(\Omega_\chi)} \leqq c\,(\|\boldsymbol{f}\|_{L^q(\Omega_\chi)} + \|u\|_{L^q(\Omega_\chi)})$$

and the constant c is independent of $\boldsymbol{f}$ and u. Then estimate (2.4) and consequently the assertion of Theorem 2.3 follows from the imbedding $H_0^{1,q}(\Omega) \subset L^{3q/(3-q)}(\Omega)$, if we show that

$$\|\nabla(\chi u)\|_{L^q} \leqq c\|\boldsymbol{g}\|_{L^q} \,. \tag{2.12}$$

Since $\overline{\Omega_\chi}$ contains exactly one of the corners, we have to consider the two cases of an interior corner point and of a boundary vertex, where additional homogeneous Dirichlet conditions are imposed. The case of an interior point corresponds to the problem in the full space $\mathbb{R}^3$ with a matrix $\Lambda(x)$ constant on infinite polyhedral cones Ω_j with their vertices at the origin O. Hence their edges are rays originating from O. In the case of a boundary corner point we get the Dirichlet problem in some infinite polyhedral cone denoted by D with vertex at O, and Λ is constant on polyhedral subcones $\Omega_j \subset D$. To unify notation we set $D = \mathbb{R}^3$ for the case of an interior corner and study the problem

$$\nabla\cdot\Lambda(x)\nabla v = \nabla\cdot\boldsymbol{g} \quad \text{with } \boldsymbol{g} \in L^q(D)^3 \tag{2.13}$$

where $\Lambda(x)$ is piecewise constant, satisfies $\Lambda(tx) = \Lambda(tx)$, $t > 0$, and $v = 0$ on ∂D if $D \neq \mathbb{R}^3$.

Lemma 2.8. *Denote* $D_\rho = B_\rho \cap D$ *and suppose that in the spherical layer* $D_3 \setminus D_{1/2}$ *the solution of* (2.13) *satisfies*

$$\|\nabla v\|_{L^q(D_2\setminus D_1)} \leqq c\Big(\|\boldsymbol{g}\|_{L^q(D_3\setminus D_{1/2})} + \|v\|_{L^q(D_3\setminus D_{1/2})}\Big) \tag{2.14}$$

for some $q > 3$. *Then*

$$\|\nabla v\|_{L^q(D)} \leqq c\|\boldsymbol{g}\|_{L^q(D)} \,.$$

Proof. Since the function $v_\delta(x) \overset{\text{def}}{=} v(\delta x)$, $\delta > 0$, satisfies

$$\nabla\cdot\Lambda(x)\nabla v_\delta = \delta^{-1}\,\nabla\cdot\boldsymbol{g}_\delta \,,$$

by dilation we obtain, from (2.14),

$$\|\nabla v\|_{L^q(D_{2\delta}\setminus D_\delta)} \leqq c\left(\|\boldsymbol{g}\|_{L^q(D_{3\delta}\setminus D_{\delta/2})} + \frac{1}{\delta}\|v\|_{L^q(D_{3\delta}\setminus D_{\delta/2})}\right). \tag{2.15}$$

Thus

$$\int_0^\infty \frac{d\delta}{\delta}\|\nabla v\|^q_{L^q(D_{2\delta}\setminus D_\delta)} \leqq c\int_0^\infty \frac{d\delta}{\delta}\Big(\|\boldsymbol{g}\|^q_{L^q(D_{3\delta}\setminus D_{\delta/2})} + \frac{1}{\delta^q}\|v\|^q_{L^q(D_{3\delta}\setminus D_{\delta/2})}\Big)$$

and from the relation

$$\int_0^\infty \frac{d\delta}{\delta}\int_{D_{2\delta}\setminus D_\delta} |u|^q dx = \int_D |u|^q dx \int_{|x|/2}^{|x|}\frac{d\delta}{\delta} = \log 2\int_D |u|^q dx$$

we therefore obtain

$$\int_D |\nabla v|^q\,dx \leqq c\Big(\int_D |\boldsymbol{g}|^q\,dx + \int_D \frac{|v|^q}{|x|^q}\,dx\Big).$$

Since $q > 3$, Theorem 2.4 implies the desired estimate. □

Lemma 2.8 reduces the proof of Theorem 2.3 to the proof of estimate (2.14). In the spherical layer $D_3 \setminus D_{1/2}$ the coefficient matrix $\Lambda(x)$ jumps at plane interfaces which meet only at certain edges. Next we perform in $D_3 \setminus D_{1/2}$ a partition of unity to isolate these edges. Let η be a cut-off function which isolates one edge. Then, in the domain $D_\eta := \operatorname{supp}\eta \cap (D_3 \setminus D_{1/2})$, ηv satisfies

$$\nabla\cdot\Lambda(x)\nabla(\eta v) = \nabla\cdot\boldsymbol{h}$$

with another right-hand side $\boldsymbol{h} \in L^q(D_\eta)^3$, and again we have $\|\boldsymbol{h}\|_{L^q(D_\eta)} \leqq c\,(\|\boldsymbol{g}\|_{L^q(D_\eta)} + \|v\|_{L^q(D_\eta)})$. Thus it remains to consider the localized problem

$$\nabla\cdot\Lambda(x)\nabla w = \nabla\cdot\boldsymbol{h}\,, \quad \boldsymbol{h}\in L^q(\mathbb{W})^3\,, \tag{2.16}$$

where $\mathbb{W}$ either coincides with $\mathbb{R}^3$ or is a wedge in $\mathbb{R}^3$, and the coefficient Λ is constant on dihedral angles E_j forming $\mathbb{W}$ and having the common edge $\mathcal{M}$. In the case of an exterior edge, $\mathbb{W} \neq \mathbb{R}^3$, the function w satisfies additionally $w|_{\partial\mathbb{W}} = 0$.

The regularity of the solution of the wedge problem (2.16) can be studied using the integral representation by Green's function, which is the topic of the next section.

3. The Green function of the wedge problem

Consider a wedge $\mathbb{W}$ with the edge $\mathcal{M} = \{(0, 0, x_3) : x_3 \in \mathbb{R}\}$. In the case of an interior edge we assume that $\mathbb{R}^3 = \mathbb{W}$ is divided into dihedral angles $E_j = e_j \times \mathbb{R}$ $(j = 1, \dots, n)$, where e_j are open sectors in the x'-plane, $x' = (x_1, x_2)$, with vertex at the origin. In the following we set $e = e_1 \cup \cdots \cup e_n$ and $E = E_1 \cup \cdots \cup E_n$. Then $\gamma = \partial e$ consists of n rays $\gamma_1, \dots, \gamma_n$ originating at $x' = 0$, and $\Gamma = \partial E = \Gamma_1 \cup \cdots \cup \Gamma_n$ with $\Gamma_j = \gamma_j \times \mathbb{R}$. Given real symmetric positive-definite matrices Λ_j $(j = 1, \dots, n)$, we suppose that $\Lambda(x) = \Lambda_j$ for $x \in E_j$ and consider the transmission problem

$$\begin{aligned} Lu &= -\nabla \cdot \Lambda(x)\nabla u(x) = f(x)\,, \quad x \in E, \\ [u]_\Gamma &= 0\,, \quad [\partial_{\nu,\Lambda} u]_\Gamma = g(x)\,, \quad x \in \Gamma, \end{aligned} \tag{3.1}$$

where $\partial_{\nu,\Lambda} \stackrel{\text{def}}{=} \Lambda\nu \cdot \nabla$ (ν denotes the normal to the interfaces) and $[\cdot]_\Gamma$ is the jump across Γ.

In the case $\mathbb{W} \neq \mathbb{R}^3$, the wedge is divided by the dihedral angles E_j and has the boundary $\partial\mathbb{W} = \Gamma_0 \cup \Gamma_n$. Here Γ denotes the interfaces $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_{n-1}$. Further, we introduce $\omega = \{x' : x \in \mathbb{W}\}$ with $\partial\omega = \gamma_0 \cup \gamma_n$ and $\gamma = \gamma_1 \cup \cdots \cup \gamma_{n-1}$. We have to consider the transmission problem (3.1) completed with the boundary condition

$$u|_{\partial\mathbb{W}} = 0\,. \tag{3.2}$$

With (3.1) and possibly (3.2) we associate the sesquilinear form

$$B(u, v) \stackrel{\text{def}}{=} \int_E \Lambda(x)\, \nabla u \cdot \nabla \bar{v}\, dx$$

and the energy space $H(E)$ which is the completion of $C_0^\infty(\mathbb{W})$ in the norm

$$\|u\|_{H(E)} = \|\nabla u\|_{L^2(E)} \stackrel{\text{def}}{=} \Big(\int_E |\nabla u|^2\, dx \Big)^{1/2}.$$

By Hardy's inequality, for any fixed $x_0 \in \mathcal{M}$ we have

$$\int_E |x - x_0|^{-2} |u|^2\, dx \leqq \|\nabla u\|^2_{L^2(E)}\,,$$

so that each $u \in H(E)$ belongs to $L^2_{\text{loc}}(\mathbb{W})$. Consider the variational problem corresponding to (3.1) and possibly (3.2):

$$B(u, v) = \int_E f\bar{v}\, dx + \int_\Gamma g\bar{v}\, d\sigma\,, \ \ v \in H(E)\,, \tag{3.3}$$

where u is sought in the energy space $H(E)$. Since $B(u, u) \sim \|\nabla u\|_{L^2}$, the problem (3.3) generates a continuous linear operator from $H(E)$ into $H(E)'$. In particular, if $f \in L^2_{\text{comp}}(\mathbb{W})$ and $g \in L^2_{\text{comp}}(\Gamma)$, then (3.3) has a unique solution $u \in H(E)$.

Theorem 3.1. (i) *There exists a unique solution of the boundary value problem*

$$\begin{aligned} L(\partial_x)G(x,\xi) &= \delta(x-\xi)\,, & x,\xi &\in E, \\ [G(x,\xi)]_{x\in\Gamma} &= [\partial_{\nu,\Lambda}G(x,\xi)]_{x\in\Gamma} = 0\,, & \xi &\in E, \\ G(x,\xi)|_{x\in\partial\mathbb{W}} &= 0 \quad \textit{if } \mathbb{W} \neq \mathbb{R}^3\,, & \xi &\in E, \end{aligned} \tag{3.4}$$

such that the function

$$x \mapsto (1-\chi(|x-\xi|\varepsilon^{-1})G(x,\xi)$$

belongs to $H(E)$ *for arbitrary fixed* $\xi = (\xi', \xi_3) \in E$ *and* $\varepsilon > 0$. *Here* χ *is a smooth function on* $[0,\infty)$ *satisfying* $\chi(t) = 1$ *for* $t \leqq 1/2$ *and* $\chi(t) = 0$ *if* $t \geqq 1$.

(ii) *The function* G *is infinitely differentiable with respect to* $x, \xi \in E, x \neq \xi$, *and homogeneous, i.e.,* $G(tx, t\xi) = t^{-1}G(x,\xi)$ *for* $t > 0$. *For* $|x-\xi| \leqq min(|x'|, |\xi'|)$ *the estimate*

$$|\partial_x^\alpha \partial_\xi^\beta G(x,\xi)| \leqq c\,|x-\xi|^{-1-|\alpha|-|\beta|} \tag{3.5}$$

holds, where c *is independent of* x *and* ξ.

(iii) G *is also the unique solution of the problem*

$$\begin{aligned} L(\partial_\xi)G(x,\xi) &= \delta(x-\xi)\,, & x,\xi &\in E, \\ [G(x,\xi)]_{\xi\in\Gamma} &= [\partial_{\nu,\Lambda}G(x,\xi)]_{\xi\in\Gamma} = 0\,, & x &\in E, \\ G(x,\xi)|_{\xi\in\partial\mathbb{W}} &= 0 \quad \textit{if } \mathbb{W} \neq \mathbb{R}^3\,, & x &\in E, \end{aligned}$$

such that the function

$$\xi \mapsto (1-\chi(|x-\xi|\varepsilon^{-1})G(x,\xi)$$

belongs to $H(E)$ *for arbitrary fixed* $x \in E$ *and* $\varepsilon > 0$.

Proof. (i) If G_1, G_2 are two solutions of (3.4), then $\tilde{G} = G_1 - G_2$ is infinitely smooth in a neighbourhood of ξ, implying that $\tilde{G} \in H(E)$ and hence $\tilde{G} = 0$, which shows the uniqueness of G. To verify its existence, let $\xi \in E_1$, for example, and let $\mathcal{E}_1$ be either the fundamental solution (if $\mathbb{W} = \mathbb{R}^3$) or Green's function for the Dirichlet problem in the wedge $\mathbb{W} \neq \mathbb{R}^3$ of the operator $-\nabla \cdot \Lambda_1 \nabla$. Reducing this to $-\Delta$ by a suitable unitary transformation and afterwards by a dilation with respect to each axis, it can be checked that $\mathcal{E}_1$ satisfies the estimate (3.5). For $\mathbb{W} = \mathbb{R}^3$ this is obvious since

$$\mathcal{E}_1(x,\xi) = c\big(a_1|x_1-\xi_1|^2 + a_2|x_2-\xi_2|^2 + a_3|x_3-\xi_3|^2\big)^{-1/2}$$

with some constants $c, a_1, a_2, a_3 > 0$, whereas the estimate for Green's function in the wedge follows from [23, Theorem 8.4]. Making the ansatz

$$G(x,\xi) = \mathcal{E}_1(x,\xi) + v(x,\xi)$$

for fixed $\xi \in E_1$, we observe that v satisfies the problem

$$\begin{gathered} -\nabla_x \cdot \Lambda(x)\nabla_x v(x,\xi) = f(x,\xi)\,, \quad x \in E\,, \\ [v(x,\xi)]_{x\in\Gamma} = 0\,, \quad [\partial_{\nu,\Lambda} v(x,\xi)]_{x\in\Gamma} = g(x,\xi)\,, \\ v(x,\xi)|_{x\in\partial\mathbb{W}} = 0 \quad \text{if } \mathbb{W} \neq \mathbb{R}^3, \end{gathered} \tag{3.6}$$

with

$$f(x,\xi) = 0\,,\ x \in E_1\,, \quad f(x,\xi) = \nabla_x \cdot \Lambda(x)\nabla_x \mathcal{E}_1(x,\xi)\,,\ x \in E \setminus E_1\,,$$

$$g(x,\xi) = -[\partial_{\nu,\Lambda}\mathcal{E}_1(x,\xi)]_{x\in\Gamma}\,.$$

To obtain a (unique) variational solution of (3.6) for fixed $\xi \in E_1$, we have to check that the corresponding right-hand side of (3.3) generates a continuous linear functional on $H(E)$: Note that f and g are infinitely smooth on $\bar{E}_k \cap \{|x| \leqq 1\}$ and $\bar{\Gamma}_k \cap \{|x| \leqq 1\}$, respectively, for any k. Moreover,

$$|f(x,\xi)| \leqq c\,|x-\xi|^{-3}, \quad |g(x,\xi)| \leqq c\,|x-\xi|^{-2}$$

with c independent of x and ξ. Then, with $v \in H(E)$,

$$\begin{aligned} \int_{E\cap\{|x|\geqq 1\}} |fv|\,dx &\leqq c\Big(\int_E \frac{|v|^2}{|x|^2}\,dx\Big)^{1/2}\Big(\int_{E\cap\{|x|\geqq 1\}} |x|^{-4}dx\Big)^{1/2} \\ &\leqq c_1\|\nabla v\|_{L^2(E)} \end{aligned}$$

by Hardy's inequality, which shows that $v \mapsto \int_E f\bar{v}$ is continuous on $H(E)$. To verify this for $v \mapsto \int_\Gamma g\bar{v}$, we note that

$$\int_{\Gamma\cap\{|x|\geqq 1\}} |gv|\,d\sigma \leqq c\Big(\int_{\Gamma\cap\{|x|\geqq 1\}} \frac{|v|^2}{|x|}\,d\sigma\Big)^{1/2}\Big(\int_{\Gamma\cap\{|x|\geqq 1\}} |x|^{-3}d\sigma\Big)^{1/2},$$

where the last integral is finite and the first on the right-hand side can be estimated by $c\|\nabla v\|_{L^2(E)}$ again; see [22, Section 1]. The construction of $G(x,\xi)$ for fixed $\xi \in E_k$ $(k = 2,\ldots,n)$ is analogous; hence (i) is proved. Since $(x,\xi) \mapsto v(\xi,x)$ is a solution of (3.6) with x,ξ interchanged, we obtain assertion (iii).

(ii) The homogeneity of G follows from that of the boundary-value problem (3.4) and the uniqueness of G. To prove the estimates (3.5) we apply well-known local elliptic estimates for transmission problems; see [27] and [26]:

Set $U = \{x \in \mathbb{W} : 1 < |x'| < 2\}$, $V = \{x \in \mathbb{W} : \frac{1}{2} < |x'| < 4\}$, and let $u \in H(E)$ be the solution of the variational problem (3.3). Then for any integer $l \geqq 0$ we have

$$\|\nabla^{l+2}u\|_{L^2(E\cap U)} \leqq c_\ell\big(\|\nabla^l f\|_{L^2(E\cap V)} + \|g\|_{H^{l+1/2}(\Gamma\cap V)} + \|\nabla^l u\|_{L^2(E\cap V)}\big). \tag{3.7}$$

Let $1 = |x-\xi| \leqq \min(|x'|,|\xi'|)$. By the homogeneity of G it is then sufficient to verify the estimates $|\partial_x^\alpha \partial_\xi^\beta G(x,\xi)| \leqq c_{\alpha\beta}$. It is enough to prove this for $x' \in \bar{U}$;

otherwise a translation with respect to x and ξ in direction x' may be performed. Consider, for example, the relation

$$\partial_\xi^\beta G(x,\xi) = \partial_\xi^\beta \mathcal{E}_1(x,\xi) + \partial_\xi^\beta v(x,\xi), \quad \xi \in E_1,$$

where $w(x,\xi) = \partial_\xi^\beta v(x,\xi)$ satisfies the problem (3.6) with right-hand sides

$$f_1 \overset{\text{def}}{=} \partial_\xi^\beta f(x,\xi)\,,\ g_1 \overset{\text{def}}{=} \partial_\xi^\beta g(x,\xi)\,.$$

Applying the estimates (3.7) to this problem, we find that for $|x-\xi| = 1$ the quantity

$$\|\partial_x^\alpha w(x,\xi)\|_{L^2(E\cap U)}$$

is uniformly bounded for any multi-index α since

$$\|\nabla_x^l f_1\|_{L^2(E\cap V)} \quad \text{and} \quad \|g_1\|_{H^{l+1/2}(\Gamma\cap V)}$$

are so for any l. Together with Sobolev's imbedding theorem, this implies

$$\sup_{|x-\xi|=1,\,x'\in\bar U} |\partial_x^\alpha w(x,\xi)| \leqq c_\alpha$$

for any multi-index α. This finishes the proof of (3.5). □

3.1. Estimates near the edge

Here we briefly recall the definition of the operator pencil $\Pi(\lambda)$ associated with the edge $\mathcal{M} = (0,0,x_3)$: Let $r = |x'|$, θ be polar coordinates in the x' plane and set $\Lambda'_j(x) = \Lambda'_j$ for $x' \in e_j$, where $\Lambda'_j = \big(a_{kl}^{(j)}\big)_{k,l=1,2}$ and $a_{kl}^{(j)}$ are the entries of the matrix Λ_j. Consider the family of sesquilinear forms

$$a(u,v;\lambda) \overset{\text{def}}{=} \frac{1}{\log 2} \int\limits_{\{1<|x'|<2\}\cap\mathbb{W}} \Lambda'(x')\,\nabla_{x'} U \cdot \nabla_{x'} \overline{V}\,dx'\ , \tag{3.8}$$

where $U = r^\lambda u(\theta)$, $V = \overline{r^\lambda} v(\theta)$ and $u, v \in \mathcal{H}$. Here $\mathcal{H} = H^1(S^1)$ if $\mathbb{W} = \mathbb{R}^3$ and $\mathcal{H} = H_0^1(\sigma)$, $\sigma = S^1 \cap \mathbb{W}$, otherwise. The form (3.8) generates a continuous linear operator $\Pi(\lambda) : \mathcal{H} \mapsto \mathcal{H}'$ by

$$(\Pi(\lambda)u, v)_\sigma \overset{\text{def}}{=} a(u,v;\lambda)\,, \quad u, v \in \mathcal{H}, \tag{3.9}$$

where $(\cdot,\cdot)_\sigma$ denotes the (extended) $L^2(\sigma)$ duality. The spectrum of the operator pencil $\Pi(\lambda)$ consists of isolated eigenvalues only (see Section 3.5 for a detailed discussion). Let λ_1 be the eigenvalue with the smallest positive real part and set $\widehat{\lambda} = \operatorname{Re}\lambda_1$.

Theorem 3.2. *For* $|x-\xi| \geqq \min(|x'|, |\xi'|)$ *the estimate*

$$|\partial_{x'}^{\alpha}\partial_{x_3}^{j}\partial_{\xi'}^{\beta}\partial_{\xi_3}^{k}G(x,\xi)| \leqq c\,|x-\xi|^{-1-|\alpha|-|\beta|-j-k}\Big(\frac{|x'|}{|x-\xi|}\Big)^{\delta_\alpha}\Big(\frac{|\xi'|}{|x-\xi|}\Big)^{\delta_\beta} \tag{3.10}$$

holds, where c is independent of x and ξ, $\delta_0 = 0$, $\delta_\alpha = \min(1, \widehat{\lambda}) - |\alpha| - \varepsilon$ *for* $\alpha \neq 0$ *and* ε *is an arbitrary small positive number.*

Corollary 3.3. *For* $\delta < \min(1, \widehat{\lambda})$,

$$|\nabla_x\nabla_\xi G(x,\xi)| \leqq c \begin{cases} |x-\xi|^{-3} & \text{if } |x-\xi| \leqq \min(|x'|, |\xi'|), \\ |x'|^{\delta-1}\,|\xi'|^{\delta-1}\,|x-\xi|^{-1-2\delta} & \text{if } |x-\xi| \geqq \min(|x'|, |\xi'|). \end{cases}$$

Proof. The first estimate follows from Theorem 3.1 (ii), whereas Theorem 3.2 implies the inequalities

$$|\nabla_{x'}\nabla_{\xi'}G(x,\xi)| \leqq c\,|x-\xi|^{-1-2\delta}\,|x'|^{\delta-1}\,|\xi'|^{\delta-1}\,,$$

$$|\partial_{x_3}\partial_{\xi_3}G(x,\xi)| \leqq c\,|x-\xi|^{-1-2\delta}\,|x'|^{\delta-1}\,|\xi'|^{\delta-1}\,,$$

for example. □

To prove Theorem 3.2, we follow the approach used in [24, Section 2] to obtain a corresponding result for Neumann problems in a dihedron. We also refer to [11] where the transmission problem with two isotropic materials, i.e., problem (3.1) with $n = 2$ and scalar (but in general complex-valued) quantities $\Lambda_1,\ \Lambda_2$, has been treated. That in our case the sesquilinear form is coercive simplifies the arguments of [11] at several places.

The proof of Theorem 3.2 relies on local estimates which will be discussed in the next section.

3.2. Local estimates near the edge

Theorem 3.4. *Let* $\varphi, \psi \in C_0^\infty(\overline{\mathbb{W}})$ *be such that* $\psi\varphi = \varphi$. *If* $u \in H(E)$ *is a solution of problem* (3.3) (*with right-hand side from* $H(E)'$) *where* $\psi f = 0$ *and* $\psi g = 0$, *then for all integers* $k, l \geqq 0$ *and* $\delta > \max(1-\widehat{\lambda}, 0)$ *the estimate*

$$\| |x'|^{\delta+k}\partial_{x_3}^{l}\nabla^{k+2}\varphi u\|_{L^2(E)} \leqq c\ \|\psi u\|_{H(E)} \tag{3.11}$$

holds, where c does not depend on u.

The proof of (3.11) for $k = 0$ is based on the following

Theorem 3.5. *For fixed* $R > 0$ *let* $u \in H(E)$ *be a solution of problem* (3.3) *such that* $\operatorname{supp} u \subset B_R(0)$, $f \in L^2(E)$ *and* $g \in H^{1/2}(\Gamma)$. *Then*

$$\begin{aligned}\|\partial_{x_3}^2 u\|_{L^2(E)} + \|\partial_{x_3}\nabla_{x'}u\|_{L^2(E)} + \||x'|^\delta\nabla_{x'}^2 u\|_{L^2(E)} \\ \leqq c\,\big(\|f\|_{L^2(E)} + \|g\|_{H^{1/2}(\Gamma)}\big),\end{aligned} \tag{3.12}$$

where c does not depend on f and g.

We proceed with the proof of Theorem 3.4, starting with the case $k = 0$: Assume first that $l = 0$. Then

$$L(\varphi u) = \varphi f + \nabla\varphi \cdot \Lambda\nabla u + (\nabla \cdot \Lambda\nabla\varphi)u \stackrel{\text{def}}{=} f_1\,,$$
$$[\varphi u]_\Gamma = 0\,, [\partial_{\nu,\Lambda}(\varphi u)]_\Gamma = \varphi g - [(\Lambda\nu \cdot \nabla\varphi)u]_\Gamma \stackrel{\text{def}}{=} g_1\,,$$
$$\varphi u|_{\partial\mathbb{W}} = 0 \quad \text{if } \mathbb{W} \neq \mathbb{R}^3\,.$$

Since $\varphi f = 0$, $\varphi g = 0$, the estimates

$$\|f_1\|_{L^2(E)} \leqq c\,\|\psi u\|_{H(E)}\,, \quad \|g_1\|_{H^{1/2}(\Gamma)} \leqq c\,\|\psi u\|_{H(E)}$$

hold, and (3.12) then implies

$$\|\partial^2_{x_3}\varphi u\|_{L^2(E)} + \|\partial_{x_3}\nabla_{x'}\varphi u\|_{L^2(E)} + \||x'|^\delta\nabla^2_{x'}\varphi u\|_{L^2(E)} \leqq \|\psi u\|_{H(E)}\,; \tag{3.13}$$

hence (3.11) holds for $k = l = 0$.

Setting $w = \partial_{x_3}\varphi u$, we obtain $w \in H(E)$ by (3.13), and w satisfies the boundary-value problem

$$Lw = \partial_{x_3} f_1\,, \quad [w]_\Gamma = 0\,, \quad [\partial_{\nu,\Lambda} w]_\Gamma = \partial_{x_3} g_1 \tag{3.14}$$

with homogeneous Dirichlet conditions if $\mathbb{W} \neq \mathbb{R}^3$. From (3.13) follows the estimate

$$\begin{aligned}||\partial_{x_3} f_1\|_{L^2(E)} + \|\partial_{x_3} g_1\|_{H^{1/2}(\Gamma)} &\leqq c\left(\|\partial_{x_3}\nabla\varphi_1 u\|_{L^2(E)} + \|\partial_{x_3}\varphi_1 u\|_{L^2(E)}\right)\\ &\leqq c\,\|\psi u\|_{H(E)},\end{aligned}$$

where $\varphi_1 \in C_0^\infty(\overline{\mathbb{W}})$ is such that $\varphi_1\varphi = \varphi$, $\psi\varphi_1 = \varphi_1$. Applying Theorem 3.5 to problem (3.14), we obtain (3.11) for $k = 0$, $l = 1$. Iterating this procedure, we obtain Theorem 3.4 for $k = 0$ and any l. Now we treat the case $k > 0$: Setting $v = \partial^l_{x_3}\varphi u$, we have to deduce the estimate

$$\||x'|^{\delta+k}\nabla^{k+2} v\|_{L^2(E)} \leqq c\|\psi u\|_{H(E)} \tag{3.15}$$

from the already established bound

$$\||x'|^\delta\nabla^2 v\|_{L^2(E)} \leqq c\,\|\psi u\|_{H(E)}. \tag{3.16}$$

Let $U_j = \left\{x \in \mathbb{W} : c_1 2^{-j-1} < |x'| < c_1 2^{-j}\right\}$ and $V_j = \left\{x \in \mathbb{W} : c_2 2^{-j-1} < |x'| < c_2 2^{-j}\right\}$ be such that $\left\{U_j\right\}_0^\infty$ is an open covering of supp φ and $\left\{V_j\right\}_0^\infty$ is another open covering with $\overline{U_j} \subset V_j$ and $V = \cup_j V_j \subset$ supp ψ. Set $U = \cup_j U_j$ and $v_1 = \partial^l_{x_3} u$. Using the local elliptic estimates (3.7) (with $f = g = 0$) and a scaling argument, we obtain

$$\|\nabla^{k+2} v_1\|_{L^2(U_j\cap E)} \leqq c_k\, 2^{2j}\, \|\nabla^k v_1\|_{L^2(V_j\cap E)}$$

for any k, $j \geqq 0$. Multiplying this inequality by $2^{-j(\delta+k)}$ and summing over j gives

$$\||x'|^{\delta+k}\nabla^{k+2} v_1\|_{L^2(U\cap E)} \leqq c_k\,\||x'|^{\delta+k-2}\nabla^k v_1\|_{L^2(V\cap E)}. \tag{3.17}$$

Now we have from (3.16) and (3.17) (for $k = 1$)

$$\begin{aligned} \| |x'|^{\delta+1} \nabla^3 v \|_{L^2(E)} &\leqq c\big(\| |x'|^{\delta-1} \nabla v_1 \|_{L^2(V\cap E)} + \|\psi u\|_{H(E)}\big) \\ &\leqq c\big(\| |x'|^{\delta} \nabla^2 v_1 \|_{L^2(V\cap E)} + \|\psi u\|_{H(E)}\big) \\ &\leqq c\, \|\psi u\|_{H(E)}\,, \end{aligned}$$

where we have applied Hardy's inequality in the second-to-last estimate. Furthermore, from (3.16) and (3.17) (for $k = 2$) we obtain

$$\| |x'|^{\delta+2} \nabla^4 v \|_{L^2(E)} \leqq c\big(\| |x'|^{\delta} \nabla^2 v_1 \|_{L^2(V\cap E)} + \|\psi u\|_{H(E)}\big) \leqq c\, \|\psi u\|_{H(E)}.$$

Proceeding this way, we get (3.15) for $k > 2$. □

3.3. Proof of Theorem 3.2

To deduce Theorem 3.2 from Theorem 3.4 we proceed exactly as in [24, Section 2.5] and first establish

Lemma 3.6. *Let* $x_0 \in \mathbb{W}$ *be such that* $\operatorname{dist}(x_0, \mathcal{M}) \leqq 4$. *Moreover, let* φ, ψ *be infinitely differentiable functions with support in* $B_1(x_0)$ *such that* $\psi = 1$ *on* $\operatorname{supp} \varphi$. *If* $u \in H(E)$, $Lu = 0$ *in* $E \cap B_1(x_0)$ *and* $[\partial_{\nu,\Lambda} u]_\Gamma = 0$ *on* $\Gamma \cap B_1(x_0)$, *then*

$$\sup_{x\in E} |x'|^{-\delta_\alpha}\, |\partial_{x_3}^j \partial_{x'}^\alpha \varphi(x) u(x)| \leqq c\, \|\psi u\|_{H(E)}, \tag{3.18}$$

where δ_α *is defined as in Theorem* 3.4 *and* c *does not depend on* u *and* x_0.

Proof. Consider first the case $\alpha \neq 0$ and let δ be a real number with $\max(1-\widehat{\lambda}, 0) < \delta < 1$. Then the estimates (3.18) can be written as

$$\sup_{x\in E} |x'|^{\delta+k-1}\, |\partial_{x_3}^l \partial_{x'}^k \varphi(x) u(x)| \leqq c\, \|\psi u\|_{H(E)}, \quad k \geqq 1,\ l \geqq 0. \tag{3.19}$$

From Theorem 3.4 it follows that $|x'|^{\delta+k} \partial_{x_3}^l \nabla_{x'}^{k+2} (\varphi u)(\cdot, x_3) \in L^2(e)$ for all k, l and almost all x_3. Applying Sobolev's theorem and Hardy's inequality we can show that

$$\sup_{x\in E} |x'|^{\delta+k-1}\, |\partial_{x_3}^l \nabla_{x'}^k (\varphi u)(x)| \leqq c \sup_{x_3\in\mathbb{R}} \| |x'|^{\delta+k} \partial_{x_3}^l \nabla_{x'}^{k+2} (\varphi u)(\cdot, x_3)\|_{L^2(e)}.$$

Moreover, by the continuity of the imbedding $H^1(\mathcal{M}) \hookrightarrow C(\mathcal{M})$ and Theorem 3.4, the last expression can be bounded by

$$c\Big(\| |x'|^{\delta+k} \partial_{x_3}^l \nabla_{x'}^{k+2} \varphi u \|_{L^2(E)} + \| |x'|^{\delta+k} \partial_{x_3}^{l+1} \nabla_{x'}^{k+2} \varphi u \|_{L^2(E)}\Big) \leqq c \|\psi u\|_{H(E)},$$

which gives (3.19).

Now let $\alpha = 0$. By Theorem 3.4 we have $x' \mapsto |x'|^\delta \partial_{x_3}^j (\varphi u)(x', x_3) \in L^2(e)$ for almost all x_3, and together with the imbedding result of [19, Lemma 7.1.3] this implies that

$$\sup_{x\in E} |\partial_{x_3}^j (\varphi u)(x)| \leqq c \sup_{x_3\in\mathbb{R}} \| |\cdot|^\delta \partial_{x_3}^j \nabla_{x'}^2 (\varphi u)(\cdot, x_3)\|_{L^2(e)}.$$

Proceeding as in the case $\alpha \neq 0$, we obtain (3.18). □

Proof of Theorem 3.2. Because of the homogeneity of G, we may assume that $|x-\xi| = 2$, which implies that $\max(|x'|, |\xi'|) \leqq 4$. Let φ and ψ be infinitely differentiable functions with support $B_1(x)$ or $B_1(\xi)$. Applying Lemma 3.6 to the function $\partial_{x_3}^j \partial_{x'}^\alpha G(x,\cdot)$, we obtain

$$|\xi'|^{-\delta_\beta}\, |\partial_{x_3}^j \partial_{x'}^\alpha \partial_{\xi_3}^k \partial_{\xi'}^\beta G(x,\xi)| \leqq c\, \|\psi(\cdot)\, \partial_{x_3}^j \partial_{x'}^\alpha G(x,\cdot)\|_{H(E)}. \tag{3.20}$$

Consider the solution $u(x) = (\psi(\cdot) f(\cdot), G(x,\cdot))_E$ of the variational problem (cf. (3.3))

$$B(u,v) = (F,v)_E \quad \forall\, v \in H(E)$$

with $F \in H(E)'$. Since ψF vanishes in the ball $B_1(x)$, we conclude from Lemma 3.6 that

$$|x'|^{-\delta_\alpha}\, |\partial_{x_3}^j \partial_{x'}^\alpha u(x)| \leqq c\, \|\varphi u\|_{H(E)}.$$

Therefore, the mapping

$$H(E)' \ni F \mapsto |x'|^{-\delta_\alpha} \partial_{x_3}^j \partial_{x'}^\alpha u(x) = |x'|^{-\delta_\alpha}\, (F(\cdot), \psi(\cdot) \partial_{x_3}^j \partial_{x'}^\alpha G(x,\cdot))_E$$

represents a continuous linear functional on $H(E)'$ for arbitrary $x \in E$, with norm independent of x. This implies that

$$|x'|^{-\delta_\alpha}\, \|\psi(\cdot) \partial_{x_3}^j \partial_{x'}^\alpha G(x,\cdot)\|_{H(E)} \leqq c,$$

which, together with (3.20), yields the desired estimate and finishes the proof of Theorem 3.2. □

The following subsections are devoted to the proof of Theorem 3.5 (which implies Theorem 3.4) and to the spectral properties of the operator pencil $\Pi(\lambda)$.

3.4. Reduction of Theorem 3.5 *to a two-dimensional problem*

Following the standard approach for elliptic problems in domains with edges, we apply the partial Fourier transform $F_{x_3 \mapsto \eta}$ to the problem (3.1). We use the notation

$$\hat{u}(\eta) \stackrel{\text{def}}{=} F_{x_3 \mapsto \eta} u(x', x_3) = \hat{u}(x', \eta)\,.$$

Then (3.1) takes the form

$$\begin{aligned} L(\eta)\hat{u} \stackrel{\text{def}}{=}\ & -\nabla_{x'} \cdot \Lambda' \nabla_{x'} \hat{u} - 2i\eta(a_{13}\partial_{x_1} + a_{23}\partial_{x_2})\hat{u} + \eta^2 a_{33}\hat{u} = \hat{f} \quad \text{in } e\,, \\ [\hat{u}]_\gamma =\ & 0, \quad [\partial_{\nu,\Lambda'}\hat{u} + i\eta(a_{13}\nu_1 + a_{23}\nu_2)\hat{u}]_\gamma = \hat{g} \quad \text{on } \gamma\,, \quad \eta \in \mathbb{R}\,, \\ & u|_{\partial\omega} = 0 \quad \text{if } \omega \neq \mathbb{R}^2\,, \end{aligned} \tag{3.21}$$

where $\partial_{\nu,\Lambda'} \stackrel{\text{def}}{=} \Lambda'\nu \cdot \nabla_{x'}$, $\Lambda' = \big(a_{kl}\big)_{k,l=1,2}$. Recall that ω is the intersection of $\mathbb{W}$ with the x'-plane and a_{kl} $(k,l = 1,2,3)$ are the entries of Λ which are constant in each sector e_j. Since

$$F_{x_3 \mapsto \eta}\big(\Lambda \nabla u \cdot \nabla \overline{v}\big) = \Lambda \begin{pmatrix} \nabla_{x'}\hat{u} \\ i\eta\hat{u} \end{pmatrix} \cdot \begin{pmatrix} \nabla_{x'}\overline{\hat{v}} \\ -i\eta\overline{\hat{v}} \end{pmatrix},$$

the sesquilinear form B transforms to

$$B(\hat{u}, \hat{v}; \eta) \stackrel{\text{def}}{=} \int_e \big(\Lambda' \nabla_{x'} \hat{u} \cdot \nabla_{x'} \overline{\hat{v}} + i\eta \hat{u}\,(a_{13}\partial_{x_1} + a_{23}\partial_{x_2})\overline{\hat{v}} - i\eta \overline{\hat{v}}\,(a_{13}\partial_{x_1} + a_{23}\partial_{x_2})\hat{u} + \eta^2 a_{33} \hat{u}\overline{\hat{v}}\big)\, dx' .$$

Since Λ is positive-definite (uniformly in x), for any $\hat{u} \in C_0^\infty(\omega)$ we obtain the estimate

$$B(\hat{u}, \hat{u}; \eta) \geqq c \int_e (|\nabla_{x'} \hat{u}|^2 + \eta^2 |\hat{u}|^2)\, dx' , \qquad \eta \in \mathbb{R}. \tag{3.22}$$

With the variational problem (3.3) we can now associate the family of problems

$$B(\hat{u}, \hat{v}; \eta) = \int_e \hat{f}\,\overline{\hat{v}}\, dx' + \int_\gamma \hat{g}\,\overline{\hat{v}}\, d\sigma \quad \forall \hat{v} \in H(e;\eta)\,, \quad \eta \in \mathbb{R}, \tag{3.23}$$

where the solution $\hat{u}$ is sought in the energy space $H(e;\eta)$, which is the completion of $C_0^\infty(\omega)$ with respect to the norm

$$\|\hat{u}\|_{H(e;\eta)} = \|\nabla_{x'}\hat{u}\|_{L^2(e)} + |\eta|\ \|\hat{u}\|_{L^2(e)}.$$

Now let $u \in H(E)$ be a solution of (3.3) with support in the ball $B_R(0)$ such that $f \in L^2_{\text{comp}}(\mathbb{R}^3)$ and $g \in H^{1/2}_{\text{comp}}(\Gamma)$. For every $\eta \in \mathbb{R}$ the function $\hat{u}(\eta)$ then belongs to $H(e;\eta)$ and satisfies (3.21) or (3.23). Furthermore, inequality (3.12) in the case $g = 0$ is equivalent to

$$\begin{aligned} &\|\eta^2\, \hat{u}(\eta)\|_{L^2(e)} + \|\eta \nabla_{x'} \hat{u}(\eta)\|_{L^2(e)} + \|\, |x'|^\delta \nabla^2_{x'} \hat{u}(\eta)\|_{L^2(e)} \\ &\quad \leqq c\, \|\hat{f}(\eta)\|_{L^2(e)}\,, \ \ \eta \in \mathbb{R}. \end{aligned} \tag{3.24}$$

Remark 3.7. It is sufficient to prove (3.12) for $g = 0$. In the general case we may choose $u_0 \in H(E)$ with compact support such that $u_0|_{E_j} \in H^2(E_j)$ for all j, $[u_0]_\Gamma = 0$ and $[\partial_{\nu,\Lambda} u_0]_\Gamma = g$, according to the trace theorem. Then $u_1 \stackrel{\text{def}}{=} u - u_0$ satisfies the problem

$$Lu_1 = f - Lu_0 \in L^2(E)\,, \quad [u_1]_\Gamma = 0\,, \quad [\partial_{\nu,\Lambda} u_1]_\Gamma = 0$$

with homogeneous Dirichlet conditions if $\mathbb{W} \neq \mathbb{R}^3$, and the desired estimate of $u = u_1 + u_0$ follows from that of u_1.

To reduce the estimate (3.24) to the case $\eta = 0$, we need the following result which will be proved in the next section.

Theorem 3.8. *Let $u \in H^1(e)$ be a solution of the problem*

$$\begin{gathered} L(0)u = -\nabla_{x'} \Lambda' \nabla u = f \quad \text{in } e\,, \\ [u]_\gamma = 0\,, [\partial_{\nu,\Lambda'} u]_\gamma = g \quad \text{on } \gamma, \\ u|_{\partial\omega} = 0 \quad \text{if } \omega \neq \mathbb{R}^2\,, \end{gathered} \tag{3.25}$$

such that supp $u \subset B_R(0)$*, $f \in L^2(e)$ and $g \in H^{1/2}(\gamma)$. Then the estimate*

$$\|\, |x'|^\delta \nabla^2_{x'} u\|_{L^2(e)} \leqq c\,\big(\|f\|_{L^2(e)} + \|g\|_{H^{1/2}(\gamma)}\big) \tag{3.26}$$

holds, where c does not depend on f and g.

Proof of estimate (3.24) for $\eta \neq 0$. From

$$L(\eta)\hat{u}(\eta) = \hat{f}(\eta)\,,\ [\hat{u}(\eta)]_\gamma = 0\,,\ [\partial_{\nu,\Lambda'}\hat{u}(\eta) + i\eta(a_{13}\nu_1 + a_{23}\nu_2)\hat{u}(\eta)]_\gamma = 0$$

we obtain

$$\begin{aligned} L(0)\hat{u}(\eta) &= \hat{f}(\eta) + 2i\eta(a_{13}\partial_{x_1} + a_{23}\partial_{x_2})\hat{u}(\eta) - \eta^2 a_{33}\hat{u}(\eta) \overset{\text{def}}{=} f_1(\eta)\,, \\ [\partial_{\nu,\Lambda'}\hat{u}(\eta)]_\gamma &= -i\eta[(a_{13}\nu_1 + a_{23}\nu_2)\hat{u}(\eta)]_\gamma \overset{\text{def}}{=} g_1(\eta). \end{aligned} \tag{3.27}$$

Setting $\hat{v} = \hat{u}$ in (3.23) and using (3.22) gives

$$\begin{aligned} \|\hat{f}(\eta)\|_{L^2(e)}\,\|\hat{u}(\eta)\|_{L^2(e)} &\geqq B(\hat{u},\hat{u};\eta) \\ &\geqq c\big(\|\nabla_{x'}\hat{u}(\eta)\|^2_{L^2(e)} + \eta^2\|\hat{u}(\eta)\|^2_{L^2(e)}\big) \\ &\geqq c\eta\ \|\nabla_{x'}\hat{u}(\eta)\|_{L^2(e)}\,\|\hat{u}(\eta)\|_{L^2(e)}. \end{aligned} \tag{3.28}$$

On the other hand, it follows from (3.27) and (3.28) that

$$\begin{aligned} \|f_1(\eta)\|_{L^2(e)} &\leqq c\|\hat{f}(\eta)\|_{L^2(e)}\,, \\ \|g_1(\eta)\|_{H^{1/2}(\gamma)} &\leqq c\eta\|\nabla_{x'}\hat{u}(\eta)\|_{L^2(e)} \leqq c\,\|\hat{f}(\eta)\|_{L^2(e)}. \end{aligned}$$

Applying Theorem 3.8 to problem (3.27) then yields the bound

$$\|\,|x'|^\delta\nabla^2_{x'}\hat{u}(\eta)\|_{L^2(e)} \leqq c\,\|\hat{f}(\eta)\|_{L^2(e)},$$

which, together with (3.28), implies (3.24) for any $\eta \neq 0$. Thus we have reduced the proof of Theorem 3.5 to that of Theorem 3.8. □

Remark 3.9. As in Remark 3.7, we reduce the assertion of Theorem 3.8 to the case $g = 0$.

3.5. Proof of Theorem 3.8*: Reduction to a one-dimensional eigenvalue problem*

Consider the boundary-value problem

$$L(0)u = f\,,\quad [u]_\gamma = [\partial_{\nu,\Lambda'}u]_\gamma = 0\,,\quad u|_{\partial\omega} = 0\ \text{ if }\ \omega \neq \mathbb{R}^2 \tag{3.29}$$

and let $u \in H^1(e)$ be a solution of (3.29) such that $\operatorname{supp} u \subset B_R(0)$ and $f \in L^2(e)$. Passing to polar coordinates $x' = (x_1, x_2) = r(\cos\theta, \sin\theta)$ we have

$$\begin{aligned} \nabla = \nabla_{x'} &= (\cos\theta\,\partial_r - r^{-1}\sin\theta\,\partial_\theta\,,\ \sin\theta\,\partial_r + r^{-1}\cos\theta\,\partial_\theta)\,, \\ r^2\,L(0) &= -r\nabla\cdot\Lambda' r\nabla + (\nabla r)\cdot\Lambda' r\nabla \\ &= -\begin{pmatrix}\cos\theta\,r\partial_r - \sin\theta\,\partial_\theta \\ \sin\theta\,r\partial_r + \cos\theta\,\partial_\theta\end{pmatrix}\cdot\begin{pmatrix}a_{11} & a_{12} \\ a_{12} & a_{22}\end{pmatrix}\begin{pmatrix}\cos\theta\,r\partial_r - \sin\theta\,\partial_\theta \\ \sin\theta\,r\partial_r + \cos\theta\,\partial_\theta\end{pmatrix} \\ &\quad + \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}\cdot\begin{pmatrix}a_{11} & a_{12} \\ a_{12} & a_{22}\end{pmatrix}\begin{pmatrix}\cos\theta\,r\partial_r - \sin\theta\,\partial_\theta \\ \sin\theta\,r\partial_r + \cos\theta\,\partial_\theta\end{pmatrix}. \end{aligned} \tag{3.30}$$

The transmission condition of (3.29) can be written in the form

$$\begin{aligned} [u]_{\theta_j} &= 0\,, \\ [r\partial_{\nu,\Lambda'}u]_{\theta_j} &= [\nu\cdot\Lambda' r\nabla u]_{\theta_j} \end{aligned}$$

$$= \left[\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} \cos\theta\, r\partial_r - \sin\theta\,\partial_\theta \\ \sin\theta\, r\partial_r + \cos\theta\,\partial_\theta \end{pmatrix} u \right]_{\theta_j}$$
$$= 0\,, \tag{3.31}$$

where the angle θ_j corresponds to the ray γ_j, $j = 1, \ldots, n$ if $\omega = \mathbb{R}^2$, or $j = 1, \ldots, n-1$ otherwise.

Following KONDRATIEV's method [17], we now apply the Mellin transform with respect to the radial variable:

$$\tilde{u}(\lambda, \theta) = \int\limits_{\mathbb{R}^+} r^{-\lambda-1} u(r, \theta)\, dr\,, \quad \lambda \in \mathbb{C}.$$

Using (3.30) and the relation $\widetilde{r\partial_r u} = -\lambda\tilde{u}$, we obtain

$$\begin{aligned} r^2\widetilde{L(0)u} = &- \partial_\theta(a_{11}\sin^2\theta - 2a_{12}\sin\theta\cos\theta + a_{22}\cos^2\theta)\partial_\theta\tilde{u} \\ &+ \lambda\partial_\theta\big((a_{22} - a_{11})\sin\theta\cos\theta + a_{12}(\cos^2\theta - \sin^2\theta)\big)\tilde{u} \\ &+ \lambda\big((a_{22} - a_{11})\sin\theta\cos\theta + a_{12}(\cos^2\theta - \sin^2\theta)\big)\partial_\theta\tilde{u} \\ &- \lambda^2(a_{11}\cos^2\theta + 2a_{12}\sin\theta\cos\theta + a_{22}\sin^2\theta)\tilde{u}. \end{aligned}$$

Furthermore, taking the Mellin transform of (3.31) and using the notation $g \stackrel{\text{def}}{=} r^2 f$ and

$$\begin{aligned} b_0(\theta) &\stackrel{\text{def}}{=} a_{11}\cos^2\theta + 2a_{12}\sin\theta\cos\theta + a_{22}\sin^2\theta\,, \\ b_2(\theta) &\stackrel{\text{def}}{=} a_{11}\sin^2\theta - 2a_{12}\sin\theta\cos\theta + a_{22}\cos^2\theta\,, \\ b_1(\theta) &\stackrel{\text{def}}{=} (a_{22} - a_{11})\sin\theta\cos\theta + a_{12}(\cos^2\theta - \sin^2\theta)\,, \end{aligned}$$

we can write the boundary-value problem (3.29) as the following one-dimensional problem with parameter λ on $\sigma \cap e$:

$$\begin{gathered} \Pi(\lambda)\tilde{u} \stackrel{\text{def}}{=} -\partial_\theta b_2\partial_\theta\tilde{u} + \lambda\partial_\theta b_1\tilde{u} + \lambda b_1\partial_\theta\tilde{u} - \lambda^2 b_0\tilde{u} = \tilde{g}\,, \\ [\tilde{u}]_{\theta_j} = 0\,, \quad [b_2\partial_\theta\tilde{u} - \lambda b_1\tilde{u}]_{\theta_j} = 0\,, \quad j = 1, \ldots, n \quad \text{if} \quad \sigma = S^1, \\ \text{or } j = 1, \ldots n-1, \quad \text{and additionally} \quad \tilde{u}(\theta_0) = \tilde{u}(\theta_n) = 0\,. \end{gathered} \tag{3.32}$$

For fixed $\lambda \in \mathbb{C}$, problem (3.32) generates a continuous linear operator $\Pi(\lambda) : \mathcal{H} \mapsto \mathcal{H}'$ via the sesquilinear form

$$a(\tilde{u}, \tilde{v}; \lambda) \stackrel{\text{def}}{=} \int\limits_\sigma \big(b_2\partial_\theta\tilde{u}\partial_\theta\overline{\tilde{v}} - \lambda b_1\tilde{u}\partial_\theta\overline{\tilde{v}} + \lambda b_1\partial_\theta\tilde{u}\overline{\tilde{v}} - \lambda^2 b_0\tilde{u}\overline{\tilde{v}}\big)\, d\theta\,. \tag{3.33}$$

Note that $\Pi(\lambda)$ is just the operator pencil defined at the beginning of Section 3.1. We now investigate its spectral properties.

Lemma 3.10. *Let* $\lambda = \xi + i\eta$, $|\xi| \leqq C$. *Then there exist constants* $c > 0$ *and* $c_1 = c_1(C)$ *such that*

$$\text{Re } a(\tilde{u}, \tilde{u}; \lambda) \geqq c\left(\|\partial_\theta \tilde{u}\|^2_{L^2(\sigma)} + \eta^2 \|\tilde{u}\|^2_{L^2(\sigma)}\right)$$

for all $\tilde{u} \in \mathcal{H}$ *and* $|\eta| \geqq c_1$.

Proof. From (3.33) we have

$$\begin{aligned}
&\text{Re}\, a(\tilde{u}, \tilde{u}; \lambda) \\
&\quad \geqq \int_\sigma \left(b_2 |\partial_\theta \tilde{u}|^2 + (\eta^2 - \xi^2) b_0 |\tilde{u}|^2 - 2\eta b_1 \text{Im}\,(\partial_\theta \tilde{u} \overline{\tilde{u}})\right) d\theta \\
&\quad \geqq c \int_\sigma \left(b_2 |\partial_\theta \tilde{u}|^2 + \eta^2 b_0 |\tilde{u}|^2 - 2|\eta|\, |b_1|\, |\partial_\theta \tilde{u}|\, |\tilde{u}|\right) d\theta - c_2 \xi^2 \|\tilde{u}\|_{L^2(\sigma)} .
\end{aligned}$$

A straightforward calculation gives $b_2 b_0 - b_1^2 = a_{11} a_{22} - a_{12}^2$, and this quantity is bounded from below (on σ) by a positive constant since Λ' is (uniformly) positive-definite. Therefore the last integral can be estimated from below by

$$c\left(\|\partial_\theta \tilde{u}\|_{L^2(\sigma)} + \eta^2 \|\tilde{u}\|^2_{L^2(\sigma)}\right). \qquad \square$$

Corollary 3.11. *The operator* $\Pi(\lambda)$ *is an analytic Fredholm operator function which has only isolated eigenvalues of finite multiplicity. For* $|\xi| \leqq C$ *and sufficiently large* $|\eta|$ *the operator* $\Pi(\lambda)$ $(\lambda = \xi + i\eta)$ *is invertible with the uniform bound*

$$\|\Pi(\lambda)^{-1} \tilde{g}\|_{H^2(e \cap \sigma)} + \eta^2 \|\Pi(\lambda)^{-1} \tilde{g}\|_{L^2(\sigma)} \leqq c\, \|\tilde{g}\|_{L^2(\sigma)}. \tag{3.34}$$

Here the norm in $H^2(e \cap \sigma)$ *is defined by*

$$\|v\|_{H^2(e \cap \sigma)} = \|v\|_{\mathcal{H}} + \sum_j \|\partial^2_\theta v\|_{L^2(e_j \cap \sigma)}.$$

If λ_0 *is an eigenvalue of maximal rank* k, *then in a neighbourhood of* λ_0 *the representation*

$$\Pi(\lambda)^{-1} = B(\lambda) + \sum_{l=1}^{k} B_l (\lambda - \lambda_0)^{-l} \tag{3.35}$$

holds, where $B_1, \ldots, B_k$ *are finite-rank operators and*

$$B(\lambda) : H^2(e \cap \sigma) \longmapsto L^2(\sigma)$$

is an analytic operator function.

Proof. From $a(\tilde{u}, \tilde{u}; \lambda) = \int_\sigma \tilde{g} \overline{\tilde{u}}$ and Lemma 3.10 we obtain

$$|\eta| \|\partial_\theta \tilde{u}\|_{L^2(\sigma)} \|\tilde{u}\|_{L^2(\sigma)} \leqq c\left(\|\partial_\theta \tilde{u}\|^2_{L^2(\sigma))} + \eta^2 \|\tilde{u}\|_{L^2(\sigma)}\right) \leqq c \|\tilde{g}\|_{L^2(\sigma)} \|\tilde{u}\|_{L^2(\sigma)},$$

which implies that

$$\eta^2 \|\tilde{u}\|_{L^2(\sigma)} + |\eta| \|\partial_\theta \tilde{u}\|_{L^2(\sigma)} \leqq c\, \|\tilde{g}\|_{L^2(\sigma)}.$$

Moreover, from (3.32) and $b_2 \geqq c > 0$ on σ, we have

$$\sum_j \|\partial_\theta^2 \tilde{u}\|_{L^2(e_j \cap \sigma)} \leqq c\, \|\tilde{g}\|_{L^2(\sigma)},$$

which completes the proof of (3.34). The other assertions follow from standard results on Fredholm operator functions; see e.g. [18, Appendix A]. □

Lemma 3.12. *If* $\mathbb{W} \neq \mathbb{R}^3$, *then the line* $\operatorname{Re}\lambda = 0$ *contains no eigenvalues; otherwise it contains the single eigenvalue* $\lambda = 0$.

Proof. Let $\lambda = i\eta$ ($\eta \in \mathbb{R}$) be an eigenvalue and u_0 an eigenfunction. From (3.33) we have

$$\begin{aligned} 0 = a(u_0, u_0; i\eta) &= \int_\sigma \big(b_2 |\partial_\theta u_0|^2 + b_0 \eta^2 |u_0|^2 - 2\eta b_1 \operatorname{Im}(\partial_\theta u_0 \bar{u}_0)\big)\, d\theta \\ &\geqq c\,\big(\|\partial_\theta \tilde{u}\|_{L^2(\sigma)} + \eta^2 \|\tilde{u}\|^2_{L^2(\sigma)}\big)\,; \end{aligned}$$

cf. the proof of Lemma 3.10. This gives $\eta = 0$ and $u_0 = \text{const}$ if $\sigma = S^1$; otherwise the Dirichlet conditions imply $u_0 = 0$. □

Proof of Theorem 3.8. Let $u \in H^1(e)$ be a solution of problem (3.29) such that $\operatorname{supp} u \subset B_R(0)$ and $f \in L^2(e)$, where $R > 0$ is fixed. We have to show that for $0 < \delta < 1$, $\delta > 1 - \widehat{\lambda}$, the estimate

$$\|r^\delta \nabla^2 u\|_{L^2(e)} \leqq c\, \|f\|_{L^2(e)} \tag{3.36}$$

holds, where c is independent of f. Recall that $\widehat{\lambda} = \operatorname{Re}\lambda_1$, where λ_1 is the eigenvalue of $\Pi(\lambda)$ with smallest positive real part. By Corollary 3.11 and Lemma 3.12, for any sufficiently small $\varepsilon > 0$ the operator function $\Pi(\lambda)^{-1}$ is analytic in the strip $\mathcal{S} = \{-\varepsilon < \operatorname{Re}\lambda < \widehat{\lambda} - \varepsilon\}$ obeying the bound (3.34) with the possible exception of a neighbourhood of $\lambda = 0$. By Lemma 3.12 this is the case if $\mathbb{W} = \mathbb{R}^3$ and then the representation (3.35) holds. We set

$$\tilde{v}(\lambda, \theta) = \Pi(\lambda)^{-1} \tilde{g}(\lambda, \theta)\,, \quad \lambda \in \mathcal{S}\,,\ \theta \in \sigma.$$

Note that $g = r^2 f$ has a finite weighted L^2 norm

$$\|r^\beta g\|_{L^2(e)} \leqq c(\beta)\, \|f\|_{L^2(e)} \quad \text{for any } \beta \geqq -2\,.$$

Let β such that $-1 - \beta \in \mathcal{S}$ and $\beta \neq -1$. Denoting by v_β the inverse Mellin transform of $\tilde{v}$ on the line $\operatorname{Re}\lambda = -1 - \beta$, we find as in [17] that v_β satisfies the problem (3.29) and satisfies the estimate

$$\begin{aligned} \|r^{\beta+2} \nabla^2 v_\beta\|_{L^2(e)} + \|r^{\beta+1} \nabla v_\beta\|_{L^2(e)} + \|r^\beta v_\beta\|_{L^2(e)} &\leqq c\, \|r^\beta g\|_{L^2(e)} \\ &\leqq c\, \|f\|_{L^2(e)}. \end{aligned} \tag{3.37}$$

Note that the left-hand side of (3.37) can be estimated from above by

$$c\Big(\int_{\operatorname{Re}\lambda = -1-\beta} \big(|\lambda|^4 \|\tilde{v}(\lambda, \cdot)\|^2_{L^2(\sigma)} + \|\tilde{v}(\lambda, \cdot)\|^2_{H^2(\sigma \cap d)}\big) d\lambda \Big)^{1/2},$$

which can be bounded by

$$c\Big(\int\limits_{\operatorname{Re}\lambda=-1-\beta} \|\tilde{g}(\lambda,\cdot)\|^2_{L^2(\sigma)}\,d\lambda\Big)^{1/2} \leqq c\,\|r^\beta g\|_{L^2(e)}$$

by using the estimate (3.34). Furthermore, since

$$\|r^\beta u\|_{L^2(e)} \leqq c\|r^{\beta+1}\nabla u\|_{L^2(e)} \leqq c\,\|\nabla u\|_{L^2(e)}\,, \quad \beta > -1$$

by Hardy's inequality, we obtain $u = v_{\varepsilon-1}$ for some $\varepsilon > 0$ sufficiently small. Hence, if $\Pi(\lambda)^{-1}$ is analytic in the strip $\mathcal{S} = \{-\varepsilon < \operatorname{Re}\lambda < 1-\delta+\varepsilon\}$, then also $u = v_{\delta-2}$ and (3.37) gives (3.36).

Otherwise we use the Residue Theorem (see [17]) which gives the equation

$$u = \psi v_{\delta-2} + \sum_{0\leqq j\leqq k-1} c_j(g)\,\psi\,\log^j(r)\,u_j(\theta)\,, \tag{3.38}$$

where ψ is a smooth cut-off function with $\psi = 1$ on $B_R(0)$, $u_0 = 1$ is the eigenfunction and u_j $(j \geqq 1)$ are generalized eigenfunctions corresponding to the eigenvalue $\lambda = 0$ of $\Pi(\lambda)$. Moreover, $c_j(g)$ are continuous linear functionals in the sense that

$$|c_j(g)| \leqq c\left(\|r^{\varepsilon-1}g\|_{L^2(e)} + \|r^{\delta-2}g\|_{L^2(e)}\right).$$

Using (3.37) for $\beta = \delta - 2$ and the fact that $u \in H^1(e)$, we see that $c_j(g) = 0$ for $j \geqq 1$ in (3.38), which implies the desired estimate (3.36) for u. □

3.6. Verification of the condition (1.4)

We conclude this section with some remarks on the verification of the condition (1.4), which is needed for our main regularity result (Theorem 2.3). For any operator pencil $\Pi(\lambda)$ of the form (3.32) corresponding to problem (2.1) near an interface or boundary edge, we have to check that

$$\widehat{\lambda} = \operatorname{Re}\lambda_1 > \tfrac{1}{3}\,, \tag{3.39}$$

where λ_1 is the eigenvalue of $\Pi(\lambda)$ with smallest positive real part.

Following [5], we may determine the eigenvalues of $\Pi(\lambda)$ by using a solution basis of the form

$$\varphi^+(\lambda,\theta) \overset{\text{def}}{=} e^{-i\lambda\theta}(\alpha^+ e^{2i\theta}+1)^\lambda\,, \quad \varphi^-(\lambda,\theta) \overset{\text{def}}{=} e^{i\lambda\theta}(\alpha^- e^{-2i\theta}+1)^\lambda \tag{3.40}$$

on each arc $e_j \cap \sigma$, where

$$\alpha^+ = \frac{i+\beta}{i-\beta}\,, \quad \alpha^- = \overline{\alpha^+}$$

and β is the root of the quadratic equation

$$a_{22}\beta^2 + 2a_{12}\beta + a_{11} = 0$$

satisfying $\operatorname{Im}\beta < 0$, with the real constants $a_{11}, a_{12} = a_{21}, a_{22}$ from (3.30). Thus we have

$$\beta = -\frac{1}{a_{22}}(a_{12} + i\kappa)\,, \quad \alpha^+ = \frac{i(a_{22} - \kappa) - a_{12}}{i(a_{22} + \kappa) + a_{12}}\,, \quad \kappa \overset{\text{def}}{=} (a_{11}a_{22} - a_{12}^2)^{1/2}\,.$$

Moreover, we assume without loss of generality that all polar angles corresponding to rays γ_j satisfy $\theta_j \in (-\pi, \pi)$ and define the function z^λ occurring in (3.40) by

$$z^\lambda \overset{\text{def}}{=} \exp(\lambda \log|z| + i\lambda \arg z)\,, \quad \arg z \in (-\pi, \pi]\,.$$

If all materials are orthotropic, we may assume that

$$a_{11} = 1\,,\ a_{12} = 0\,,\ a_{22} = t^2$$

with some $t > 0$ on each sector e_j. Then (3.40) takes the form

$$\varphi^+(\lambda, \theta) = (t\cos\theta + i\sin\theta)^\lambda\,, \quad \varphi^-(\lambda, \theta) = (t\cos\theta - i\sin\theta)^\lambda\,.$$

Denoting the functions (3.40) on $e_j \cap \sigma$ by $\varphi_j^\pm$ and inserting the ansatz

$$\varphi(\lambda, \theta) \overset{\text{def}}{=} C_j^+\varphi_j^+(\lambda, \theta) + C_j^-\varphi_j^-(\lambda, \theta)\,, \quad \theta \in e_j \cap \sigma\,,\ j = 1, \dots, n$$

into the homogeneous equations (3.32), we obtain a linear system

$$C(\lambda)z = 0\,, \quad z = ((C_j^+, C_j^-) : j = 1, \dots, n)\,,$$

for the unknown vector $z \in \mathbb{C}^{2n}$. Then the eigenvalues of $\Pi(\lambda)$ are given by the roots of the transcendental equation

$$\det C(\lambda) = 0\,, \tag{3.41}$$

and by determining the location of its roots we may find lower bounds of the quantity $\widehat{\lambda}$. We refer to [5] for further discussion and algorithmic aspects. The explicit form of equation (3.41) in the case of a pure transmission problem with two anisotropic materials can be found in [14].

If all materials are isotropic, then (3.32) is a Sturm-Liouville problem having only real eigenvalues, and we refer to [25] for a detailed discussion of (3.41). Moreover, from [25, Theorem 6.2] we find that condition (3.39) is fulfilled if additionally

(i) (3.32) is a pure transmission problem with at most three materials (i.e., $n \leqq 3$) and $\hat{\theta} \leqq 3\pi/2$, or
(ii) (3.32) corresponds to an interface problem with two materials and Dirichlet conditions on the boundary, and $\hat{\theta} \leqq 3\pi/2$.

Here $\hat{\theta}$ denotes the maximal interior angle of the sectors e_j.

4. Proof of the linear regularity result

Now we are in the position to study the L^p regularity of the wedge problem (2.16). Its solution can be represented as

$$w(x) = -\int_{\mathbb{W}} \nabla_y G(x, y) \cdot \boldsymbol{h}(y) dy\,, \tag{4.1}$$

with the corresponding Green function $G(x, y)$. Recall the definition of the number $\widehat{\lambda}_{\mathcal{M}}$ given in Section 2. It is uniquely determined by the values of $\Lambda(x)$ near $\mathcal{M}$.

Theorem 4.1. *If $\boldsymbol{h} \in L^q(\mathbb{W})^3$ with $q \in [2, 2/(1-\widehat{\lambda}_{\mathcal{M}}))$, then the solution of* (2.16) *satisfies*

$$\|\nabla w\|_{L^q(\mathbb{W})} \leqq c\, \|\boldsymbol{h}\|_{L^q(\mathbb{W})}\,.$$

The proof of Theorem 4.1 follows from

Lemma 4.2. *Let $\delta \in (0, \widehat{\lambda}_{\mathcal{M}})$. There exists a constant c such that, for any $r > 0$,*

$$\begin{aligned}\Big(\frac{1}{r^2}\int_{C_r} |\nabla w|^q dx\Big)^{1/q} \leqq c\Big(\frac{1}{r^2}\int_{C_r} |h|^q dx\Big)^{1/q} + \frac{c}{r^{1-\delta}}\int_r^\infty \Big(\frac{1}{\rho^2}\int_{C_\rho} |\boldsymbol{h}|^q dx\Big)^{1/q} \frac{d\rho}{\rho^\delta} \\ + \frac{c}{r^{1+\delta}}\int_0^r \Big(\frac{1}{\rho^2}\int_{C_\rho} |\boldsymbol{h}|^q dx\Big)^{1/q} \rho^\delta d\rho\,,\end{aligned} \tag{4.2}$$

where C_ρ denotes the cylindrical layer $C_r \overset{\text{def}}{=} \{x : r < |x'| < 2r, x_3 \in \mathbb{R}\} \cap \mathbb{W}$.

Proof of Theorem 4.1. We simply integrate the q-th power of the terms in (4.2) over $r\,dr$. Then

$$\begin{aligned}&\int_0^\infty \frac{r\,dr}{r^2}\int_r^{2r} \tau\, d\tau \int_\sigma \int_{\mathbb{R}} |\nabla w(r, \theta, x_3)|^q\, d\theta\, dx_3 \\ &= \log 2 \int_0^\infty \int_\sigma \int_{\mathbb{R}} |\nabla w(r, \theta, x_3)|^q\, \tau\, d\tau\, d\theta\, dx_3\,.\end{aligned}$$

To the last two terms on the right-hand side of (4.2) we apply additionally Hardy's inequality which provides, for $q - q\delta < 2$,

$$\begin{aligned}&\int_0^\infty r\,dr\Big(r^{\delta-1}\int_r^\infty \Big(\rho^{-2}\int_{C_\rho} |\boldsymbol{h}|^q dx\Big)^{1/q} \frac{d\rho}{\rho^\delta}\Big)^q \\ &= \int_0^\infty r^{1-q+q\delta}\, dr\Big(\int_r^\infty \Big(\int_{C_\rho} |\boldsymbol{h}|^q dx\Big)^{1/q} \frac{d\rho}{\rho^{\delta+2/q}}\Big)^q \\ &\leqq c\int_0^\infty r^{1-q+q\delta+q-2-\delta q}\, dr \int_{C_r} |\boldsymbol{h}|^q dx = c\int_0^\infty \frac{dr}{r}\int_{C_r} |\boldsymbol{h}|^q\, dx\,,\end{aligned}$$

and for $q + q\delta > 2$ we obtain

$$\int_0^\infty r\,dr\Big(r^{-\delta-1}\int_0^r \Big(\rho^{-2}\int_{C_\rho} |\boldsymbol{h}|^q dx\Big)^{1/q}\rho^\delta d\rho\Big)^q$$

$$= \int_0^\infty r^{1-q-q\delta}\,dr\Big(\int_0^r \Big(\int_{C_\rho} |\boldsymbol{h}|^q dx\Big)^{1/q}\frac{d\rho}{\rho^{2/q-\delta}}\Big)^q$$

$$\leqq c\int_0^\infty r^{1-q-q\delta+q-2+\delta q}\,dr\int_{C_r}|\boldsymbol{h}|^q dx = c\int_0^\infty \frac{dr}{r}\int_{C_r}|\boldsymbol{h}|^q\,dx\,. \qquad \square$$

Proof of Lemma 4.2. Using (4.1) and the notation $\displaystyle\fint_{C_r} u\,dx \overset{\text{def}}{=} r^{-2}\int_{C_r} u\,dx$, we split the integral on the left-hand side into three parts:

$$\Big(\fint_{C_r}|\nabla w|^q dx\Big)^{1/q} \leqq \Big(\fint_{C_r}\Big(\int_{|y'|<r/4}|\nabla_x\nabla_y G(x,y)|\,|\boldsymbol{h}(y)|\,dy_3\,dy'\Big)^q dx\Big)^{1/q}$$

$$+\Big(\fint_{C_r}\Big(\int_{|y'|>4r}|\nabla_x\nabla_y G(x,y)|\,|\boldsymbol{h}(y)|\,dy_3\,dy'\Big)^q dx\Big)^{1/q}$$

$$+\Big(\fint_{C_r}\Big|\nabla_x\int_{r/4<|y'|<4r}\nabla_y G(x,y)\cdot\boldsymbol{h}(y)\,dy_3\,dy'\Big|^q dx\Big)^{1/q}$$

$$\overset{\text{def}}{=} I_1 + I_2 + I_3\,,$$

where $\boldsymbol{h}$ is extended by zero onto the whole space if $\mathbb{W} \neq \mathbb{R}^3$. For the first and second integral we have $|x - y| \geqq \min(|x'|, |y'|)$; hence by Corollary 3.3 and $|x'| > r$,

$$I_1 \leqq c\Big(\fint_{C_r}\Big(\int_{|y'|<r/4}\frac{|\boldsymbol{h}(y)|\,dy'\,dy_3}{(|x'-y'|+|x_3-y_3|)^{1+2\delta}|x'|^{1-\delta}|y'|^{1-\delta}}\Big)^q dx\Big)^{1/q}$$

$$\leqq \frac{c}{r^{1-\delta}}\Big(\fint_{K_r}dx'\int_{\mathbb{R}}dx_3\Big(\int_{\mathbb{R}}\frac{H(y_3,r)\,dy_3}{(|x'|+|x_3-y_3|)^{1+2\delta}}\Big)^q\Big)^{1/q},$$

where $K_r = \{x' : x \in C_r\}$ and the notation

$$H(y_3, r) \overset{\text{def}}{=} \int_{|y'|<r/4}\frac{|\boldsymbol{h}(y', y_3)|\,dy'}{|y'|^{1-\delta}}$$

is used. From Young's convolution theorem we have

$$\int_{\mathbb{R}} dx_3 \Big| \int_{\mathbb{R}} \frac{H(y_3, r)\, dy_3}{(|x'| + |x_3 - y_3|)^{1+2\delta}} \Big|^q \leqq \frac{c}{|x'|^{2\delta q}} \int_{\mathbb{R}} |H(y_3, r)|^q dy_3 \,,$$

which implies together with Minkowski's inequality that

$$\begin{aligned} I_1 &\leqq \frac{c}{r^{1-\delta}} \Big(\fint_{K_r} \frac{dx'}{|x'|^{2\delta q}} \Big)^{1/q} \Big(\int_{\mathbb{R}} |H(y_3, r)|^q dy_3 \Big)^{1/q} \\ &\leqq \frac{c}{r^{1+\delta}} \Big(\int_{\mathbb{R}} \Big(\int_0^{r/4} \frac{\rho\, d\rho}{\rho^{1-\delta}} \fint_{K_\rho} |\boldsymbol{h}(y', y_3)|\, dy' \Big)^q dy_3 \Big)^{1/q} \\ &\leqq \frac{c}{r^{1+\delta}} \int_0^{r/4} \rho^\delta d\rho \, \Big\| \fint_{K_\rho} |\boldsymbol{h}(y', \cdot)|\, dy' \Big\|_{L^q(\mathbb{R})} \\ &\leqq \frac{c}{r^{1+\delta}} \int_0^{r/4} \rho^\delta d\rho \, \Big(\fint_{C_\rho} |\boldsymbol{h}(y)|^q dy \Big)^{1/q}. \end{aligned}$$

The second integral can be estimated by

$$\begin{aligned} I_2 &\leqq c \Big(\fint_{K_r} dx' \int_{\mathbb{R}} dx_3 \Big(\int_{|y'|>4r} \frac{|\boldsymbol{h}(y', y_3)|\, dy}{(|x' - y'| + |x_3 - y_3|)^{1+2\delta} |x'|^{1-\delta} |y'|^{1-\delta}} \Big)^q \Big)^{1/q} \\ &\leqq \frac{c}{r^{1-\delta}} \Big(\int_{\mathbb{R}} dx_3 \Big(\int_{|y'|>4r} \frac{|\boldsymbol{h}(y', y_3)|\, dy}{(|y'| + |x_3 - y_3|)^{1+2\delta} |y'|^{1-\delta}} \Big)^q \Big)^{1/q} \\ &\leqq \frac{c}{r^{1-\delta}} \Big(\int_{\mathbb{R}} dx_3 \Big(\int_{4r}^{\infty} \frac{\rho\, d\rho}{\rho^{1-\delta}} \int_{\mathbb{R}} dy_3 \fint_{K_\rho} \frac{|\boldsymbol{h}(y', y_3)|\, dy'}{(\rho + |x_3 - y_3|)^{1+2\delta}} \Big)^q \Big)^{1/q}. \end{aligned}$$

Applying again Minkowski's inequality, we obtain, similar to I_1,

$$\begin{aligned} I_2 &\leqq \frac{c}{r^{1-\delta}} \int_{4r}^{\infty} \rho^\delta d\rho \Big(\int_{\mathbb{R}} dx_3 \Big(\int_{\mathbb{R}} dy_3 \fint_{K_\rho} \frac{|\boldsymbol{h}(y', y_3)|\, dy'}{(\rho + |x_3 - y_3|)^{1+2\delta}} \Big)^q dx_3 \Big)^{1/q} \\ &\leqq \frac{c}{r^{1-\delta}} \int_{4r}^{\infty} \rho^\delta d\rho \frac{1}{\rho^{2\delta}} \Big(\int_{\mathbb{R}} \Big(\fint_{K_\rho} |\boldsymbol{h}(y', y_3)|\, dy' \Big)^q dy_3 \Big)^{1/q} \\ &\leqq \frac{c}{r^{1-\delta}} \int_{4r}^{\infty} \frac{d\rho}{\rho^\delta} \Big(\fint_{C_\rho} |\boldsymbol{h}(y)|^q dy \Big)^{1/q}. \end{aligned}$$

To estimate the third integral I_3, note that

$$\fint_{C_r} |\nabla w_r(x)|^q dx = \frac{1}{2r} \int_{\mathbb{R}} dt \int_{t-r}^{t+r} dx_3 \fint_{K_r} |\nabla w_r(x', x_3)|^q \, dx'$$

$$= \int_{\mathbb{R}} dt \fint_{Q_{r,t}} |\nabla w_r(x)|^q \, dx \,,$$

where

$$w_r(x) \stackrel{\text{def}}{=} - \int_{r/4<|y'|<4r} \nabla_y G(x, y) \cdot \boldsymbol{h}(y) \, dy \,,$$

$$Q_{r,t} \stackrel{\text{def}}{=} \{x : x' \in K_r, t - r < x_3 < t + r\} \,.$$

Next we split

$$\fint_{Q_{r,t}} |\nabla_x w_r|^q dx \leqq \fint_{Q_{r,t}} |\nabla_x (w_r - w_{r,t})|^q dx + \fint_{Q_{r,t}} |\nabla_x w_{r,t}|^q dx \stackrel{\text{def}}{=} J_1 + J_2 \,,$$

where

$$w_{r,t}(x) \stackrel{\text{def}}{=} - \int_{Q'_{r,t}} \nabla_y G(x, y) \cdot \boldsymbol{h}(y) \, dy$$

with

$$Q'_{r,t} \stackrel{\text{def}}{=} \{x : r/4 < |x'| < 4r, t - 2r < x_3 < t + 2r\} \,.$$

If $x \in Q_{r,t}$ and $\frac{1}{4}r < |y'| < 4r$, $|y_3 - t| > 2r$, then again $|x - y| \geqq \min(|x'|, |y'|)$, and additionally $|x' - y'| + |x_3 - y_3| \geqq c(|x'| + |x_3 - y_3|)$. Hence

$$J_1 = \fint_{Q_{r,t}} \Big| \int_{\substack{r/4<|y'|<4r \\ |y_3-t|>2r}} \nabla_x \nabla_y G(x, y) \cdot \boldsymbol{h}(y) \, dy \Big|^q dx$$

$$\leqq \frac{c}{r^{(1-\delta)q}} \fint_{Q_{r,t}} \Big(\int_{\substack{r/4<|y'|<4r \\ |y_3-t|>2r}} \frac{|\boldsymbol{h}(y)| \, dy}{(|x'| + |x_3 - y_3|)^{1+2\delta} |y'|^{1-\delta}} \Big)^q dx \,.$$

Integration over t gives

$$\int_{\mathbb{R}} J_1\,dt \leqq \frac{c}{r^{(1-\delta)q}} \fint_{C_r} \Big(\int_{r/4<|y'|<4r} \frac{|\boldsymbol{h}(y)|\,dy}{(|x'|+|x_3-y_3|)^{1+2\delta}|y'|^{1-\delta}} \Big)^q dx$$

$$\leqq \frac{c}{r^{(1-\delta)q}} \fint_{K_r} dx' \int_{\mathbb{R}} dx_3 \times \Big(\int_{\mathbb{R}} \frac{dy_3}{(|x'|+|x_3-y_3|)^{1+2\delta}} \int_{r/4<|y'|<4r} \frac{|\boldsymbol{h}(y',y_3)|\,dy'}{|y'|^{1-\delta}} \Big)^q,$$

and proceeding as in the estimation of the integral I_1, we obtain

$$\Big(\int_{\mathbb{R}} dt \fint_{Q_{r,t}} |\nabla(w_r - w_{r,t})|^q dx \Big)^{1/q} = \Big(\int_{\mathbb{R}} J_1\,dt \Big)^{1/q}$$

$$\leqq \frac{c}{r^{1+\delta}} \int_{r/4}^{4r} \rho^\delta d\rho \Big(\fint_{C_\rho} |\boldsymbol{h}(y)|^q dy \Big)^{1/q}.$$

To estimate J_2, note that by the homogeneity of Green's function

$$\int_{Q_{r,t}} dx \Big| \int_{Q'_{r,t}} \nabla_x \nabla_y G(x,y) \cdot \boldsymbol{h}(y)\,dy \Big|^q$$

$$= r^3 \int_{Q_{1,0}} dx \Big| \int_{Q'_{1,0}} \nabla_x \nabla_y G(x,y) \cdot \boldsymbol{h}(r(y+(0,0,t)))\,dy \Big|^q.$$

We can choose C^∞ cut-off functions $\varphi(x)$ and $\psi(y)$ equal to 1 on $Q_{1,0}$ respectively $Q'_{1,0}$ such that

$$\mathcal{G}(x,y) \overset{\text{def}}{=} \varphi(x)\psi(y)\nabla_x\nabla_y G(x,y)$$

satisfies the estimate

$$|\partial_x^\alpha \partial_y^\beta \mathcal{G}(x,y)| \leqq c\,|x-y|^{-3-|\alpha|-|\beta|}$$

for all $x, y \in \mathbb{R}^3$. This follows easily from Theorem 3.1. Hence, $\mathcal{G}$ fulfils the requirements of a Calderon-Zygmund kernel (cf. [4]). To prove that the mapping

$$\mathcal{K}\boldsymbol{g}(x) \overset{\text{def}}{=} -\int_{\mathbb{R}^3} \mathcal{G}(x,y)\,\boldsymbol{g}(y)\,dy \tag{4.3}$$

is a Calderon-Zygmund operator, it remains to show that $\mathcal{K} : L^2(\mathbb{R}^3)^3 \mapsto L^2(\mathbb{R}^3)^3$ is bounded. For any $\boldsymbol{g} \in L^2(\mathbb{R}^3)^3$ the vector function $\mathcal{K}\boldsymbol{g}$ can be written as $\mathcal{K}\boldsymbol{g}(x) = \varphi(x)\nabla u(x)$, where u satisfies the partial differential equation

$$\nabla \cdot \Lambda(x)\nabla u = \nabla \cdot (\psi \boldsymbol{g}) ,$$

and obviously

$$\|\varphi \nabla u\|_{L_2} \leqq \|\nabla u\|_{L_2} \leqq c\|\psi \boldsymbol{g}\|_{L_2} \leqq \|\boldsymbol{g}\|_{L_2} .$$

Consequently, $\mathcal{K} \in \mathcal{B}(L^q(\mathbb{R}^3)^3)$ for any $1 < q < \infty$, and therefore

$$\begin{aligned} &\int_{Q_{1,0}} dx \Big| \int_{Q'_{1,0}} \nabla_x \nabla_y G(x, y) \cdot \boldsymbol{h}(r(y + (0, 0, t)))\, dy \Big|^q \\ &\quad = \| \mathcal{K}\boldsymbol{h}(r(\cdot + (0, 0, t)))\|^q_{L^q(Q'_{1,0})} \leqq c\, r^{-3} \|\boldsymbol{h}\|^q_{L^q(Q'_{r,t})} . \end{aligned}$$

Hence

$$\int_{\mathbb{R}} dt \fint_{Q_{r,t}} |\nabla w_{r,t}|^q dx \leqq c \int_{\mathbb{R}} dt \fint_{Q'_{r,t}} |\boldsymbol{h}(y)|^q dy$$

and therefore

$$\begin{aligned} \Big(\fint_{C_r} |\nabla w_r|^q dx\Big)^{1/q} &\leqq c\Big(\fint_{r/2<|y'|<4r} |\boldsymbol{h}(y)|^q dy\Big)^{1/q} \\ &\quad + \frac{c}{r^{1+\delta}} \int_{r/4}^{4r} \rho^\delta d\rho \Big(\fint_{C_\rho} |\boldsymbol{h}(y)|^q dy\Big)^{1/q} . \qquad \square \end{aligned}$$

5. The nonlinear system: Assumptions, exact formulation of the problem, functional analytic tools

Having Theorem 2.3 at hand, we now develop the tools for solving the nonlinear equation (1.1) during the subsequent sections. We start this section by formulating our assumptions on the coefficient functions $\mathcal{J}_k$, the right-hand sides $\mathcal{R}_k$ and the boundary values. Afterwards we give equation (1.1) a precise meaning between appropriate spaces.

To simplify notation, in the following we denote by L^p, $H^{s,q}$ and $H_0^{s,q}$ the corresponding function spaces over the given polyhedron Ω. For the sake of brevity, the cross products of m copies of these spaces are denoted by $\mathbf{L}^p$, $\mathbf{H}^{s,q}$ and $\mathbf{H}_0^{s,q}$, correspondingly.

Because we have to deal also with spaces of real-valued functions, we use the notation $Z_{\mathbb{R}}$ for the real analog of a complex space Z from above.

Definition 5.1. We define for $k \in \{1, \dots, m\}$ the operators

$$-\nabla \cdot \mu_k \nabla : H_0^{1,2} \longmapsto H_0^{-1,2}$$

as usual via the corresponding forms. The operator

$$\mathbf{H}_0^{1,2} \ni (\psi_1, \dots, \psi_m) \longmapsto (-\nabla \cdot \mu_1 \nabla \psi_1, \dots, -\nabla \cdot \mu_m \nabla \psi_m) \in \mathbf{H}_0^{-1,2}$$

will be denoted by $-\mathbf{div}\, \mu\, \mathbf{grad}$.

Remark 5.2. For the restriction of these operator to an L^p space, this definition incorporates homogeneous Dirichlet conditions in the usual way (see [8] or [3]).

Assumption 5.3. *We suppose that for each k the piecewise constant* (3×3)*-matrices* μ_k *generate an admissible decomposition of* Ω (*see Definition 2.2*).

Remark 5.4. Theorem 2.3 guarantees for any k the existence of a number $q_k > 3$ such that the corresponding operator $-\nabla \cdot \mu_k \nabla$ provides a topological isomorphism between H_0^{1,q_k} and H_0^{-1,q_k}. Furthermore, each of these operators is also an isomorphism between $H_0^{1,2}$ and $H_0^{-1,2}$. Interpolation (see Proposition 5.16 below) between H_0^{1,q_k} and $H_0^{1,2}$ (H_0^{-1,q_k} and $H_0^{-1,2}$, respectively) then shows that the operator also establishes an isomorphism between $H_0^{1,q}$ and $H_0^{-1,q}$ if $q \in [2, q_k]$.

Definition 5.5. Let $q \in (3, 4]$ be a number such that each of the operators $-\nabla \cdot \mu_1 \nabla, \dots, -\nabla \cdot \mu_m \nabla$ provides a topological isomorphism between $H_0^{1,q}$ and $H_0^{-1,q}$. We define p as the number $\frac{q}{2}$. Finally, we denote by $\mathcal{D}$ the domain of the operator $-\mathbf{div}\, \mu\, \mathbf{grad}$ when the range space is restricted to $\mathbf{L}^p$. The real part of $\mathcal{D}$ is denoted by D.

We now formulate our assumptions on the operators $\mathcal{J}_k$, $\mathcal{R}_k$ and the boundary values. The reader will notice that the assumptions on $\mathcal{R}_k$ also include nonlocal operators, which enlarges the class of possible applications considerably (see Example 5.8).

Assumption 5.6. (i) *For any* $k \in \{1, \dots, m\}$ *there is a twice continuously differentiable mapping* $\zeta_k : [T_0, T_1] \times \mathbb{R}^m \mapsto (0, \infty)$ *such that the operator*

$$\mathcal{J}_k : [T_0, T_1] \times \mathbf{H}_{\mathbb{R}}^{1,q} \mapsto H_{\mathbb{R}}^{1,q}$$

is given by

$$\mathcal{J}_k(t, \mathbf{u})(x) \stackrel{\text{def}}{=} \zeta_k(t, u_1(x), \dots, u_m(x)), \quad \mathbf{u} = (u_1, \dots, u_m), \ x \in \Omega .$$

(ii) *The operator* $\mathcal{R}_k$ *maps* $[T_0, T_1] \times \mathbf{H}_{\mathbb{R}}^{1,q}$ *into* $L_{\mathbb{R}}^p$. *Additionally, there is a constant* $\eta \in (0, 1)$ *and for any* $R > 0$ *a constant* $C(R)$ *such that*

$$\|\mathcal{R}_k(t_1, \psi_1) - \mathcal{R}_k(t_2, \psi_2)\|_{L_{\mathbb{R}}^p} \leqq C(R) \left(|t_1 - t_2|^\eta + \|\psi_1 - \psi_2\|_{\mathbf{H}_{\mathbb{R}}^{1,q}}\right)$$

for all $(t_1, \psi_1), (t_2, \psi_2) \in [T_0, T_1] \times \mathbf{H}_{\mathbb{R}}^{1,q}$, $\|\psi_1\|_{\mathbf{H}_{\mathbb{R}}^{1,q}}, \|\psi_2\|_{\mathbf{H}_{\mathbb{R}}^{1,q}} \leqq R$ *and* $k = 1, \dots, m$.

(iii) *We assume the existence of functions* $\Phi_1, \dots, \Phi_m$,

$$[T_0, T_1] \ni t \mapsto \Phi_k(t) \in H^{1,q}_{\mathbb{R}} ,$$

such that the corresponding distributional derivatives $-\nabla \cdot \mu_k \nabla \Phi_k$ *are from the space* $L^p_{\mathbb{R}}$ *and for* $t \in [T_0, T_1]$ *the mappings*

$$\begin{aligned} &t \mapsto \Phi_k(t) \in H^{1,q}_{\mathbb{R}}, \\ &t \mapsto -\nabla \cdot \mu_k \nabla \Phi_k(t) \in L^p_{\mathbb{R}}, \\ &t \mapsto \frac{\partial \Phi_k}{\partial t} \in L^p_{\mathbb{R}} \end{aligned}$$

are Hölder continuous with exponent η. *For any* $k \in \{1, \dots, m\}$ *the function* Φ_k *represents the boundary conditions for* u_k *in the sense of traces; we have*

$$u_k(t)|_{\partial\Omega} = \Phi_k(t)|_{\partial\Omega}. \tag{5.1}$$

In what follows, we will denote the function $t \mapsto (\Phi_1(t), \dots, \Phi_m(t))$ by Φ.

We now give two examples for mappings $\mathcal{R} = (\mathcal{R}_1, \dots, \mathcal{R}_m)$:

Example 5.7. Let

$$\mathcal{S}_k : [T_0, T_1] \times \mathbb{R}^m \times \mathbb{R}^{3m} \mapsto \mathbb{R}$$

be functions which satisfy the following condition: There is a positive constant η and for any compact set $\mathcal{K} \subset \mathbb{R}^m$ a constant Υ such that for any $t_1, t_2 \in [T_0, T_1]$, $\mathbf{a}, \mathbf{b} \in \mathcal{K}$, $\mathbf{d}, \mathbf{e} \in \mathbb{R}^{3m}$ and $k \in \{1, \dots, m\}$ the inequality

$$\begin{aligned} |\mathcal{S}_k(t_1, \mathbf{a}, \mathbf{d}) - \mathcal{S}_k(t_2, \mathbf{b}, \mathbf{e})| \leqq \Upsilon\big(|t_1 - t_2|^\eta + |\mathbf{a} - \mathbf{b}|_{\mathbb{R}^m} \left(|\mathbf{d}|^2_{\mathbb{R}^{3m}} + |\mathbf{e}|^2_{\mathbb{R}^{3m}}\right) \\ + \Upsilon |\mathbf{d} - \mathbf{e}|_{\mathbb{R}^{3m}} \left(|\mathbf{d}|_{\mathbb{R}^{3m}} + |\mathbf{e}|_{\mathbb{R}^{3m}}\right) \end{aligned}$$

holds. Then $\mathcal{S} = (\mathcal{S}_1, \dots, \mathcal{S}_m)$ defines a mapping $\mathcal{R}$ in the following way: For every $\mathbf{u} \in C^\infty(\overline{\Omega}; \mathbb{R}^m)$ we put

$$\mathcal{R}_k(t, \mathbf{u})(x) = \mathcal{S}_k(t, \mathbf{u}(x), (\nabla \mathbf{u})(x)) \quad \text{for } x \in \Omega$$

and afterwards extend $\mathcal{R}$ by continuity to the whole set $[T_0, T_1] \times \mathbf{H}^{1,q}_{\mathbb{R}}$.

Example 5.8. Assume $\upsilon : \mathbb{R} \mapsto (0, \infty)$ to be a positive, continuously differentiable function. Further, let $\mathcal{L} : H^{1,q}_{\mathbb{R}} \mapsto H^{1,q}_{\mathbb{R}}$ be the mapping which assigns to $u \in H^{1,q}_{\mathbb{R}}$ the solution φ of the (inhomogeneous) Dirichlet problem

$$-\nabla \cdot \upsilon(u) \nabla \varphi = 0.$$

If we define

$$\mathcal{R}(u) = |\nabla(\mathcal{L}(u))|^2$$

then, under a reasonable condition on the boundary value of φ, $\mathcal{R}$ satisfies Assumption 5.6 (ii) with $m = 1$.

This second example comes from a model which describes electrical heat conduction; see [2] and the references therein.

We now present a formulation of (1.1) and (5.1) which will later enable us to prove local existence and uniqueness for the system under our consideration:

Definition 5.9. Let $F_k : [T_0, T_1] \times \mathbf{H}^{1,q}_{0,\mathbb{R}} \mapsto H^{1,q}_{\mathbb{R}}$ be defined by

$$F_k(t, \mathbf{w}) = \mathcal{J}_k(t, \mathbf{w} + \Phi(t))$$

and the mapping $X_k : [T_0, T_1] \times \mathbf{H}^{1,2p}_{0,\mathbb{R}} \mapsto L^p_{\mathbb{R}}$ be given by

$$X_k(t, \mathbf{w}) = \mathcal{R}_k(t, \mathbf{w} + \Phi(t), \nabla \mathbf{w} + \nabla \Phi(t)) .$$

Then we say that $\mathbf{u}$ is a *local solution* to (1.1) including the boundary condition (5.1) if

$$\mathbf{v} = \mathbf{u} - \Phi \in C((T_0, T], D) \cap C^1((T_0, T], \mathbf{L}^p_{\mathbb{R}}) \cap C([T_0, T], \mathbf{L}^p_{\mathbb{R}})$$

satisfies

$$\begin{aligned} &\frac{\partial v_k}{\partial t} - F_k(t, \mathbf{v}) \nabla \cdot \mu_k \nabla v_k \\ &\quad = \nabla F_k(t, \mathbf{v}) \cdot \mu_k \nabla v_k + X_k(t, \mathbf{w}) - \frac{\partial \Phi_k}{\partial t} \\ &\qquad + F_k(t, \mathbf{v}) \nabla \cdot \mu_k \nabla \Phi_k + \nabla F_k(t, \mathbf{v}) \cdot \mu_k \nabla \Phi_k , \quad k = 1, \dots , m \end{aligned} \tag{5.2}$$

on an interval $(T_0, T]$ and $\mathbf{v}(T_0) = \mathbf{u}_0 - \Phi(T_0)$.

In this definition an initial-value problem for a system of operator differential equations in the real space $\mathbf{L}^p_{\mathbb{R}}$ has been formulated. However, the methods for its solution operate in complex Banach spaces; cf. Proposition 5.13. That is why we now pass over to a complex version of the problem. We start with

Definition 5.10. Let $\mathbf{P} : \mathbf{H}^{1,q} \to \mathbf{H}^{1,q}_{\mathbb{R}}$ denote the mapping onto the real part of $\mathbf{H}^{1,q}$ which takes componentwise the real part of the function, and let $Q : L^p_{\mathbb{R}} \to L^p$ denote the canonical imbedding of the real space into the complex one. Further, we define for $\mathbf{v} \in \mathbf{H}^{1,q}$,

$$\mathcal{F}_k(t, \mathbf{v}) \overset{\text{def}}{=} F_k(t, \mathbf{P}\mathbf{v}) \quad \text{and} \quad \mathcal{X}_k(t, \mathbf{v}) \overset{\text{def}}{=} Q X_k(t, \mathbf{P}\mathbf{v}).$$

For the sake of simplicity, we denote the complexified functions $Q\Phi_k$ and the vector $\left(Q\Phi_1, \dots , Q\Phi_m\right)$ again by Φ_k and Φ, respectively.

Remark 5.11. It is easy to see that the continuity properties of F_k and X_k carry over to $\mathcal{F}_k$ and $\mathcal{X}_k$.

Furthermore, in referring to the assumptions on F_k we also implicitly refer to Remark 5.11. Thus, the complexified version of (5.2) reads as follows:

Problem 5.12. Find a function

$$\mathbf{v} \in C((T_0, T], \mathcal{D}) \cap C^1((T_0, T], \mathbf{L}^p) \cap C([T_0, T], \mathbf{L}^p)$$

which satisfies

$$\begin{aligned} &\frac{\partial v_k}{\partial t} - \mathcal{F}_k(t, \mathbf{v})\nabla \cdot \mu_k \nabla v_k \\ &\quad = \nabla \mathcal{F}_k(t, \mathbf{v}) \cdot \mu_k \nabla v_k + \mathcal{X}_k(t, \mathbf{v}) - \frac{\partial \Phi_k}{\partial t} \\ &\qquad + \mathcal{F}_k(t, \mathbf{v})\nabla \cdot \mu_k \nabla \Phi_k + \nabla \mathcal{F}_k(t, \mathbf{v}) \cdot \mu_k \nabla \Phi_k\,, \quad k = 1, \dots, m \end{aligned} \tag{5.3}$$

on an interval $(T_0, T]$ and $\mathbf{v}(T_0) = \mathbf{u}_0 - \Phi(T_0)$.

For the convenience of the reader, we now establish the functional-analytic background we will use in the following. We start by quoting Sobolevskii's theorem, which will serve as the ultimate instrument for solving our quasilinear problem. Then we continue with a resolvent estimate for elliptic operators on L^p spaces and finish this section with two interpolation results which will be needed in the next section.

Proposition 5.13. [28] *Let A_0 be an operator on a (complex) Banach space X with dense domain $\mathcal{D}_0$. Assume that A_0 admits the resolvent estimate*

$$\sup_{\operatorname{Re} z \geqq 0} (1 + |z|)\|(A_0 + z)^{-1}\|_{\mathcal{B}(X)} < \infty. \tag{5.4}$$

Suppose $\beta > \alpha$ and $v_0 \in \operatorname{dom}(A_0^\beta)$. Additionally, let

$$[T_0, T_1] \times \operatorname{dom}(A_0^\alpha) \ni (t, v) \longmapsto A(t, v) \in \mathcal{B}(\mathcal{D}_0, X)$$

be a mapping satisfying $A(T_0, v_0) = A_0$ and

$$\begin{aligned} &\|(A(t_1, A_0^{-\alpha} v_1) - A(t_2, A_0^{-\alpha} v_2))A_0^{-1}\|_{\mathcal{B}(X)} \\ &\qquad \leqq c(R)\left(|t_1 - t_2|^\eta + \|v_1 - v_2\|_X\right) \end{aligned} \tag{5.5}$$

for $t_1, t_2 \in [T_0, T_1]$ and $\|v_1\|_X, \|v_2\|_X \leqq R$. Finally, let

$$[T_0, T_1] \times \operatorname{dom}(A_0^\alpha) \ni (t, v) \longmapsto f(t, v) \in X$$

be a mapping obeying the estimate

$$\|f(t_1, A_0^{-\alpha} v_1) - f(t_2, A_0^{-\alpha} v_2)\|_X \leqq c(R)\left(|t_1 - t_2|^\eta + \|v_1 - v_2\|_X\right) \tag{5.6}$$

for $t_1, t_2 \in [T_0, T_1]$ and $\|v_1\|_X, \|v_2\|_X \leqq R$.

If $\|A_0^\alpha v_0\|_X < R$, then there is a (nontrivial) interval $[T_0, T]$ such that the equation

$$\frac{\partial v}{\partial t} + A(t, v(t))v = f(t, v)\,, \quad v(T_0) = v_0$$

admits exactly one solution on $[T_0, T]$ which belongs to the space

$$C([T_0, T]; \operatorname{dom}(A_0^\alpha)) \cap C^1((T_0, T]; X) \cap C((T_0, T]; \mathcal{D}_0)\,.$$

The next result, which is proved in [12], says in essence that the operator A_0, specified in Definition 6.1, satisfies the required resolvent estimate (5.4).

Proposition 5.14. [12]. *Let Λ be a measurable function on Ω with values in the set of the real, symmetric (3×3)-matrices which is essentially bounded, and assume that*

$$\underline{\Lambda} \stackrel{\text{def}}{=} \operatorname*{ess\,inf}_{x\in\Omega} \inf_{\|y\|_{\mathbb{R}^3}=1} \Lambda(x)y \cdot y > 0\,.$$

Let Θ be an $L^\infty_{\mathbb{R}}(\Omega)$ function with positive upper and lower bounds $\hat{\Theta}$ and $\underline{\Theta}$, respectively. Assume that $r \in (1, \infty)$ and denote by A_Λ the restriction of the operator $-\nabla \cdot \Lambda\nabla$ (including homogeneous Dirichlet conditions) to L^r. Then the operator $-\Theta A_\Lambda$ generates an analytic semigroup on L^r and satisfies the following resolvent estimate for z with $\operatorname{Re} z \geqq 0$:

$$\|(\Theta A_\Lambda + z)^{-1}\|_{\mathcal{B}(L^r)} \leqq \frac{\hat{\Theta}}{\underline{\Theta}} M\Big(\frac{\|\Lambda\|_{L^\infty}}{\underline{\Lambda}}, r\Big) \frac{1}{1+|z|}\,,$$

where

$$M : (1,\infty) \times (1,\infty) \mapsto (0,\infty)$$

is locally bounded.

The subsequent proposition will allow us to substitute the domain of fractional powers (including the corresponding graph norm) by a suitable interpolation space between the domain and the Banach space (and vice versa).

Proposition 5.15. *Let Z be a Banach space and B a densely defined operator on X satisfying the resolvent estimate*

$$\sup_{t\in[0,\infty)} (1+t)\|(B+t)^{-1}\|_{\mathcal{B}(Z)} < \infty.$$

If $\vartheta, \theta \in (0, 1)$ and $\vartheta < \theta$, then

$$[Z, \operatorname{dom}(B)]_\theta \hookrightarrow \operatorname{dom}\big(B^\vartheta\big)\,, \quad \operatorname{dom}\big(B^\theta\big) \hookrightarrow [Z, \operatorname{dom}(B)]_\vartheta$$

(the domains being topologized by a norm equivalent to the graph norm of the corresponding operator).

Proof. The assertions are obtained from [32, 1.15.2, 1.10.3, 1.3.3]. □

Finally, we will exploit the following interpolation result which was proved in [13] for the more general case of Lipschitz domains and mixed boundary conditions.

Proposition 5.16. *Let $\gamma \in (0, 1)$, $1 < p_0, p_1 < \infty$. Furthermore, suppose that $\gamma \neq 1/p = (1-\gamma)/p_0 + \gamma/p_1$. Then*

$$[L^{p_0}, H^{1,p_1}_0]_\gamma = H^{\gamma,p}_0.$$

6. The nonlinear system: Existence and uniqueness of the solution

In this section we show that (1.1) has a (local) solution in the spirit of Definition 5.9, which is also unique. Having an application of Proposition 5.13 in mind, the outline of the section is as follows: First we define an operator-valued mapping $\mathcal{A}$ on $[T_0, T_1] \times \mathbf{H}^{1,q}$, the restriction of which later on becomes the operator-valued mapping A from Proposition 5.13.

Having fixed in particular the operator A_0 within this procedure, we then prove that $\mathrm{dom}(A_0^\alpha)$ continuously imbeds into $\mathbf{H}_0^{1,q}$ for suitably chosen α. Thus, the restriction of $\mathcal{A}$ to $[T_0, T_1] \times \mathrm{dom}(A_0^\alpha)$ makes sense. Denoting this restriction by A, we then prove that A satisfies the hypotheses of Proposition 5.13. Afterwards we show that the same is true for the right-hand side of (5.3), which then enables us to apply Proposition 5.13. Finally, we prove that the solution in fact belongs to the corresponding real space.

Let us start with the following

Definition 6.1. We define a mapping

$$\mathcal{A} : [T_0, T_1] \times \mathbf{H}^{1,q} \longmapsto \mathcal{B}(\mathcal{D}, \mathbf{L}^p)$$

by putting, for $\psi = (\psi_1, \ldots, \psi_m) \in \mathcal{D}$,

$$\mathcal{A}(t, \mathbf{w})(\psi_1, \ldots, \psi_l) \stackrel{\text{def}}{=} \big(-\mathcal{F}_1(t, \mathbf{w})\nabla \cdot \mu_1 \nabla\psi_1, \ldots, -\mathcal{F}_m(t, \mathbf{w})\nabla \cdot \mu_m \nabla\psi_l\big).$$

Moreover, we set

$$A_0 \stackrel{\text{def}}{=} \mathcal{A}(T_0, \mathbf{u}_0 - \Phi(T_0)) .$$

Remark 6.2. This definition is justified because for any $(t, \mathbf{w}) \in [T_0, T_1] \times \mathbf{H}^{1,q}$ the function $\mathcal{F}_k(t, \mathbf{w})$ is from $H^{1,q} \hookrightarrow L^\infty$, and, hence, a multiplier on L^p. Additionally, any function $\mathcal{F}_k(t, \mathbf{w})$ is bounded from below by a positive constant, cf. Definition 5.9 and Assumption 5.6.

As announced above, our first goal is to prove

Theorem 6.3. *For every $\alpha \in (\frac{1}{2} + \frac{3}{2q}, 1)$ the space $\mathrm{dom}\big(A_0^\alpha\big)$ (equipped with the norm $\|A_0^\alpha(\cdot)\|_{\mathbf{L}^p}$) continuously imbeds into $\mathbf{H}_0^{1,q}$.*

For the proof we need

Lemma 6.4. *Assume $s = \frac{3}{p} - \frac{3}{2}$ and $\tau = \frac{3}{q} - \frac{1}{2}$ and set $\varrho = \tau - s$. Then for any $k \in \{1, \ldots, m\}$ the operator $(-\nabla \cdot \mu_k \nabla)^{\varrho/2}$ maps $H_0^{-s,2}$ continuously onto $H_0^{-\tau,2}$.*

Proof. First we observe that ϱ is positive because $q > 3$ and s is nonnegative because $p \leqq 2$. Secondly, the operator $\nabla \cdot \mu_k \nabla$ generates analytic semigroups on both, $H_0^{1,2}$ and L^2. Thus, powers of $-\nabla \cdot \mu_k \nabla$ and $-\nabla \cdot \mu_k \nabla|_{L^2}$ are well defined and the usual rules for calculus hold. In this spirit, we consider the operators

$$B \stackrel{\text{def}}{=} (-\nabla \cdot \mu_k \nabla)^{1/2} : H_0^{1,2} \mapsto L^2 ,$$
$$C \stackrel{\text{def}}{=} (-\nabla \cdot \mu_k \nabla)^{1/2} : L^2 \mapsto H_0^{-1,2} .$$

Clearly, we have

$$(B^*)^{\varrho} = (B^{\varrho})^* = C^{\varrho}.$$

By a well-known theorem (see [32, Chapter 1.15.2]), B^{ϱ} maps $\mathrm{dom}(B^{\tau})$ isomorphically onto $\mathrm{dom}(B^{s})$. On the other hand, B is positive and selfadjoint, so that

$$\mathrm{dom}(B^{\gamma}) = [L^2, \mathrm{dom}(B)]_{\gamma} = [L^2, H_0^{1,2}]_{\gamma}\ ,\ \ \gamma = \tau, s$$

(see [32, Chapter 1.18.10]). Because these interpolation spaces are identical with $H_0^{\gamma,2}$ (see Proposition 5.16), B^{ϱ} provides a topological isomorphism between $H_0^{\tau,2}$ and $H_0^{s,2}$. Hence, by duality, $C^{\varrho} = (-\nabla\cdot\mu_k\nabla)^{\varrho/2}$ maps $H_0^{-s,2}$ isomorphically onto $H_0^{-\tau,2}$. □

Proof of Theorem 6.3. Obviously, it suffices to show for all $k \in \{1, \dots, m\}$ and $\alpha \in (\frac{1}{2} + \frac{3}{2q}, 1)$, the existence of an imbedding

$$\mathrm{dom}\big((-\Theta\nabla\cdot\mu_k\nabla)^{\alpha}\big) \hookrightarrow H_0^{1,q}$$

whenever Θ is a real L^{∞} function bounded from below by a positive constant. In order to do so, we first notice that the definition of s and τ yield the (continuous) imbeddings

$$L^p \hookrightarrow H_0^{-s,2} \quad \text{and} \quad H_0^{-\tau,2} \hookrightarrow H_0^{-1,q},$$

(see [32, Chapter 4.6.1]). Denoting by κ_1 and κ_2 the imbedding constants between the corresponding spaces, we may estimate

$$\begin{aligned}
&\|(-\nabla\cdot\mu_k\nabla)^{\varrho/2-1}\|_{\mathcal{B}(L^p,H_0^{1,q})}\\
&\quad\leqq \|(-\nabla\cdot\mu_k\nabla)^{-1}\|_{\mathcal{B}(H_0^{-\tau,2},H_0^{1,q})}\ \|(-\nabla\cdot\mu_k\nabla)^{\varrho/2}\|_{\mathcal{B}(L^p,H_0^{-\tau,2})}\\
&\quad\leqq \kappa_1\kappa_2\, \|(-\nabla\cdot\mu_k\nabla)^{-1}\|_{\mathcal{B}(H_0^{-1,q},H_0^{1,q})}\ \|(-\nabla\cdot\mu_k\nabla)^{\varrho/2}\|_{\mathcal{B}(H_0^{-s,2},H_0^{-\tau,2})}.
\end{aligned}$$

The third factor is finite by Definition 5.5 and the last factor is finite by Lemma 6.4. Thus,

$$\mathrm{dom}\big((-\nabla\cdot\mu_k\nabla|_{L^p})^{1-\varrho/2}\big) \hookrightarrow H_0^{1,q}.$$

Hence, if $\alpha > 1 - \frac{1}{2}\varrho$, then Proposition 5.15 implies that

$$[L^p, \mathrm{dom}\big(-\nabla\cdot\mu_k\nabla|_{L^p}\big)]_{\alpha} \hookrightarrow H_0^{1,q}. \tag{6.1}$$

Because the domains of $-\nabla\cdot\mu_k\nabla|_{L^p}$ and $-\Theta\nabla\cdot\mu_k\nabla|_{L^p}$ are identical including the equivalence of the corresponding graph norms, (6.1) gives

$$[L^p, \mathrm{dom}\big(-\Theta\nabla\cdot\mu_k\nabla|_{L^p}\big)]_{\alpha} \hookrightarrow H_0^{1,q}.$$

Another application of Proposition 5.15 then leads to the assertion of Theorem 6.3. □

Before we can prove one key result which afterwards enables us to apply Sobolevskii's theorem, we have to reinforce the above assumption on the initial value $\mathbf{u}_0$:

Assumption 6.5. *There exists a number* $\beta \in (\frac{1}{2} + \frac{3}{2q}, 1]$ *such that*

$$\mathbf{u}_0 - \Phi(T_0) \in [\mathbf{L}^p_{\mathbb{R}}, D]_\beta \subset [\mathbf{L}^p, \mathcal{D}]_\beta \,.$$

In what follows we fix a number $\alpha \in (\frac{1}{2} + \frac{3}{2q}, \beta)$ and denote the imbedding constant from $\operatorname{dom}\big(A_0^\alpha\big)$ into $\mathbf{H}^{1,q}$ by κ.

Definition 6.6. Let A be the restriction of $\mathcal{A}$ to $[T_0, T_1] \times \operatorname{dom}(A_0^\alpha)$.

The reader should notice that the definition of A is justified by Theorem 6.3.

Lemma 6.7. *Let* M *be a bounded set in* $\operatorname{dom}\big(A_0^\alpha\big)$. *Then there is a constant* $c_k(M)$ *such that, for any* $\mathbf{y_1}, \mathbf{y_2} \in M$,

$$\|\mathcal{F}_k(t_1, \mathbf{y}_1) - \mathcal{F}_k(t_2, \mathbf{y}_1)\|_{H^{1,q}} \leqq c_k(M) \left(|t_1 - t_2|^\eta + \|A_0^\alpha \mathbf{y}_1 - A_0^\alpha \mathbf{y}_2\|_{\mathbf{L}^p}\right).$$

Moreover,

$$\sup_{(t,\mathbf{y})\in[T_0,T_1]\times M} \|\mathcal{F}_k(t, \mathbf{y})\|_{H^{1,q}} < \infty.$$

Proof. By Theorem 6.3, M constitutes a bounded set in $\mathbf{H}^{1,q}$. Thus, applying Definition 5.9 we may estimate

$$\begin{aligned} &\|\mathcal{F}_k(t_1, \mathbf{y}_1) - \mathcal{F}_k(t_2, \mathbf{y}_2)\|_{H^{1,q}} \\ &\quad \leqq \|\mathcal{J}_k(t_1, \mathbf{P}\mathbf{y}_1 + \Phi(t_1)) - \mathcal{J}_k(t_2, \mathbf{P}\mathbf{y}_2 + \Phi(t_2))\|_{H^{1,q}}. \end{aligned} \tag{6.2}$$

If we bring Assumption 5.6 into play, we obtain a constant $c(M)$ such that the right-hand side of (6.2) is not greater than

$$c(M) \left(|t_1 - t_2|^\eta + \|\mathbf{P}\mathbf{y}_1 + \Phi(t_1) - \mathbf{P}\mathbf{y}_2 - \Phi(t_2)\|_{\mathbf{H}^{1,q}}\right). \tag{6.3}$$

Let Λ_Φ denote the Hölder constant of Φ (cf. Assumption 5.6 (iii)); then (6.3) is less than or equal to

$$\begin{aligned} &c(M) \left(|t_1 - t_2|^\eta + \Lambda_\Phi |t_1 - t_2|^\eta + \|\mathbf{y}_1 - \mathbf{y}_1\|_{\mathbf{H}^{1,q}}\right) \\ &\qquad \leqq c(M) \left((1 + \Lambda_\Phi)|t_1 - t_2|^\eta + \kappa \|A_0^\alpha \mathbf{y}_1 - A_0^\alpha \mathbf{y}_1\|_{\mathbf{L}^p}\right). \end{aligned}$$

The second assertion follows from the first. □

Theorem 6.8. *The domain* $\mathcal{D}$ *of* A_0 (*cf. Definition 6.1*) *is dense in* $\mathbf{L}^p$ *and* A_0 *satisfies the resolvent estimate* (5.4). *Moreover,* A *satisfies the estimate* (5.5) *from Proposition 5.13.*

Proof. The density of the domain and the resolvent estimate (5.4) for A_0 are implied by Remark 6.2 and Proposition 5.14.

Let $B_R \stackrel{\text{def}}{=} \{\|\mathbf{w}\|_{\mathbf{L}^p} \leqq R\}$ be the closed ball of radius R in $\mathbf{L}^p$. Clearly, the set $A_0^{-\alpha} B_R$ is then identical to the R-ball in $\operatorname{dom}(A_0^\alpha)$ and, consequently, a bounded subset of $\mathbf{H}^{1,q}$ (cf. Theorem 6.3). Assume now $t_1, t_2 \in [T_0, T_1]$ and $\mathbf{w}_1, \mathbf{w}_2 \in B_R$. If we denote $A_0^{-\alpha}\mathbf{w}_1$ by $\mathbf{y}_1$ and $A_0^{-\alpha}\mathbf{w}_2$ by $\mathbf{y}_2$, then

$$\begin{aligned}
&\|(\mathcal{F}_k(t_1, \mathbf{y}_1)\nabla \cdot \mu_k \nabla - \mathcal{F}_k(t_2, \mathbf{y}_2)\nabla \cdot \mu_k \nabla)(\mathcal{F}_k(T_0, \mathbf{u}_0 - \Phi(T_0))\nabla \cdot \mu_k \nabla)^{-1}\| \\
&\quad = \|\big(\mathcal{F}_k(t_1, \mathbf{y}_1) - \mathcal{F}_k(t_2, \mathbf{y}_2)\big)\nabla \cdot \mu_k \nabla)(\nabla \cdot \mu_k \nabla)^{-1}(\mathcal{F}_k(T_0, \mathbf{u}_0 - \Phi(T_0))^{-1}\| \\
&\quad \leqq \frac{1}{\inf \mathcal{F}_k(T_0, \mathbf{u}_0 - \Phi(T_0))} \|\mathcal{F}_k(t_1, \mathbf{y}_1) - \mathcal{F}_k(t_2, \mathbf{y}_2)\|_{L^\infty},
\end{aligned}$$

where the operator norm is taken in $\mathcal{B}(L^p)$. Applying Lemma 6.7 and inserting for $\mathbf{y}_1, \mathbf{y}_2$, we obtain the assertion. □

In order to apply Sobolevskii's result for our quasilinear system, we still have to prove that the right-hand side of (5.3) satisfies the estimate (5.6) of Proposition 5.13. This will be done now:

Theorem 6.9. *Define the mapping*

$$\begin{aligned}
\boldsymbol{f}_k(t, \mathbf{w}) = {} & \nabla \mathcal{F}_k(t, \mathbf{w}) \cdot \mu_k \nabla w_k + \mathcal{X}_k(t, \mathbf{w}) - \Phi_k'(t) \\
& + \mathcal{F}_k(t, \mathbf{w})\nabla \cdot \mu_k \nabla \Phi_k(t) + \nabla \mathcal{F}_k(t, \mathbf{w}) \cdot \mu_k \nabla \Phi_k(t)
\end{aligned}$$

for $(t, \mathbf{w}) \in [T_0, T_1] \times \mathbf{H}^{1,q}$, $\mathbf{w} = (w_1, \dots, w_m)$. *Then* f_k *maps* $[T_0, T_1] \times \mathbf{H}^{1,q}$ *into* L^p. *Moreover, there exists a constant* $C(R)$ *such that*

$$\|f_k(t_1, A_0^{-\alpha}\mathbf{w}_1) - f_k(t_2, A_0^{-\alpha}\mathbf{w}_2)\|_{L^p} \leqq C(R)\left(|t_1 - t_2|^\eta + \|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{L}^p}\right)$$

for any $t_1, t_2 \in [T, T_0]$ *and any* $\mathbf{w}_1, \mathbf{w}_2 \in B_R = \{\mathbf{w} : \|\mathbf{w}\|_{\mathbf{L}^p} \leqq R\}$.

Proof. The first assertion immediately follows from the estimates (6.2) and (6.3), Definition 5.9 and the assumptions on the mappings $\mathcal{R}_k$ and the functions Φ_k (see Section 5).

To prove the second assertion we put $\mathbf{y} = A_0^{-\alpha}\mathbf{w}_1$ and $\hat{\mathbf{y}} = A_0^{-\alpha}\mathbf{w}_2$ with $\mathbf{w}_1, \mathbf{w}_2 \in B_R$. Then

$$\begin{aligned}
&\|f_k(t_1, \mathbf{y}) - f_k(t_2, \hat{\mathbf{y}})\|_{L^p} \\
&\quad \leqq \|\Phi_k'(t_1) - \Phi_k'(t_2)\|_{L^p} \\
&\qquad + \|\nabla \mathcal{F}_k(t_1, \mathbf{y}) \cdot \mu_k \nabla y_k - \nabla \mathcal{F}_k(t_2, \hat{\mathbf{y}}) \cdot \mu_k \nabla \hat{y}_k\|_{L^p} \\
&\qquad + \|\mathcal{X}_k(t_1, \mathbf{y}) - \mathcal{X}`_k(t_2, \hat{\mathbf{y}})\|_{L^p} \\
&\qquad + \|\mathcal{F}_k(t_1, \mathbf{y})\nabla \cdot \mu_k \nabla \Phi_k(t_1) - \mathcal{F}_k(t_2, \hat{\mathbf{y}})\nabla \cdot \mu_k \nabla \Phi_k(t_2)\|_{L^p} \\
&\qquad + \|\nabla \mathcal{F}_k(t_1, \mathbf{y}) \cdot \mu_k \nabla \Phi_k(t_1) - \nabla \mathcal{F}_k(t_2, \hat{\mathbf{y}}) \cdot \mu_k \nabla \Phi_k(t_2)\|_{L^p}. \qquad (6.4)
\end{aligned}$$

We consider the terms on the right-hand side of (6.4) separately and show that each of them has an upper bound of the form

$$C\left(|t_1 - t_2|^\eta + \|A_0^\alpha \mathbf{y} - A_0^\alpha \hat{\mathbf{y}}\|_{\mathbf{L}^p}\right). \qquad (6.5)$$

For the first term this follows directly from Assumption 5.6 (iii), whereas the second term can be estimated as follows:

$$\begin{aligned}
&\leqq \|\nabla\big(\mathcal{F}_k(t_1,\mathbf{y}) - \mathcal{F}_k(t_2,\hat{\mathbf{y}})\big)\cdot\mu_k\nabla y_k\|_{L^p} + \|\nabla\mathcal{F}_k(t_2,\hat{\mathbf{y}})\cdot\mu_k\nabla(y_k-\hat{y}_k)\|_{L^p}\\
&\leqq \|\mathcal{F}_k(t_1,\mathbf{y}) - \mathcal{F}_k(t_2,\hat{\mathbf{y}})\|_{H^{1,q}} \sup_{x\in\Omega}\|\mu_k(x)\|_{\mathcal{B}(\mathbb{R}^3)} \|\mathbf{y}\|_{\mathbf{H}^{1,q}}\\
&\quad + \|\mathcal{F}_k(t_2,\hat{\mathbf{y}})\|_{H^{1,q}} \sup_{x\in\Omega}\|\mu_k(x)\|_{\mathcal{B}(\mathbb{R}^3)} \|\mathbf{y}-\hat{\mathbf{y}}\|_{\mathbf{H}^{1,q}}.
\end{aligned}$$

Taking into account the imbedding $\mathrm{dom}\big(A_0^\alpha\big) \hookrightarrow \mathbf{H}^{1,q}$ and again applying Lemma 6.7, we estimate this sum by (6.5).

By Assumption 5.6, the third term may be bounded by

$$\begin{aligned}
&C(A_0^{-\alpha}B_R)\big(|t_1-t_2|^\eta + \|\mathbf{y}-\hat{\mathbf{y}}\|_{\mathbf{H}^{1,q}}\big)\\
&\leqq C(A_0^{-\alpha}B_R)\,\max(1,\kappa)\,\big(|t_1-t_2|^\eta + \|A_0^\alpha\mathbf{y} - A_0^\alpha\hat{\mathbf{y}}\|_{\mathbf{L}^p}\big).
\end{aligned}$$

Moreover, the fourth term may be estimated by

$$\begin{aligned}
&\|\mathcal{F}_k(t_1,\mathbf{y}) - \mathcal{F}_k(t_2,\hat{\mathbf{y}})\|_{L^\infty}\ \|\nabla\cdot\mu_k\nabla\Phi_k(t_1)\|_{L^p}\\
&+ \|\mathcal{F}_k(t_2,\mathbf{y})\big(\nabla\cdot\mu_k\nabla\Phi_k(t_1) - \nabla\cdot\mu_k\nabla\Phi_k(t_2)\big)\|_{L^p}.
\end{aligned}$$

Then another application of Lemma 6.7 and Assumption 5.6 yield an estimate of the form (6.5).

Finally, the fifth term is not greater than

$$\begin{aligned}
&\|\nabla\big(\mathcal{F}_k(t_1,\mathbf{y}) - \mathcal{F}_k(t_2,\hat{\mathbf{y}})\big)\cdot\mu_k\nabla\Phi_k(t_1)\|_{L^p}\\
&\quad + \|\nabla\mathcal{F}_k(t_2,\hat{\mathbf{y}})\cdot\mu_k\nabla\big(\Phi_k(t_1)-\Phi_k(t_2)\big)\|_{L^p}\\
&\leqq \|\mathcal{F}_k(t_1,\mathbf{y}) - \mathcal{F}_k(t_2,\hat{\mathbf{y}})\|_{H^{1,q}} \sup_{x\in\Omega}\|\mu_k(x)\|_{\mathcal{B}(\mathbb{C}^3)} \sup_{t\in[T_0,T_1]}\|\Phi_k(t)\|_{H^{1,q}_{\mathbb{R}}}\\
&\quad + \sup_{x\in\Omega}\|\mu_k(x)\|_{\mathcal{B}(\mathbb{C}^3)} \sup\|\mathcal{F}_k(t,\mathbf{y})\|_{H^{1,q}}\ \|\Phi_k(t_1)-\Phi_k(t_2)\|_{H^{1,q}_{\mathbb{R}}},
\end{aligned}$$

where the last supremum is taken over $(t,\mathbf{y}) \in [T_0,T_1]\times A_0^{-\alpha}B_R$. Applying Lemma 6.7 together with Assumption 5.6 yields the desired estimate for the last term. If we insert for $\mathbf{y}$ and $\hat{\mathbf{y}}$, we obtain the assertion. □

After these preparations we can formulate our final result:

Theorem 6.10. *Problem 5.12 admits exactly one solution* $\mathbf{v}$ *in*

$$C([T_0,T],\mathrm{dom}(A_0^\alpha)) \cap C((T_0,T],\mathcal{D}) \cap C^1((T_0,T],\mathbf{L}^p)$$

with $T \in (T_0,T_1]$.

The function $\mathbf{u} \overset{\mathrm{def}}{=} \mathbf{v}+\Phi$ *is then a solution of* (1.1) *in the sense of Definition 5.9.*

Proof. Assumption 6.5 together with Proposition 5.15 gives

$$\mathbf{v}(T_0) = \mathbf{u}_0 - \Phi(T_0) \in \operatorname{dom}(A_0^\gamma)$$

whenever $\gamma < \beta$. Thus, the first assertion is implied by Proposition 5.13, Theorem 6.8 and Theorem 6.9.

Furthermore, it is easy to see that the complex conjugate $\overline{\mathbf{v}} = (\bar{v}_1, \ldots, \bar{v}_m)$ is also a solution of (5.3) and has the same initial value. Hence, $\overline{\mathbf{v}}$ and $\mathbf{v}$ must coincide. Thus, $\mathbf{v}$ takes its values in $\mathbb{R}^m$ and also satisfies (5.2), which proves the second statement. □

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