Sharp pointwise estimates for solutions of strongly elliptic second order systems with boundary data from L^p

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Abstract

The strongly elliptic system $\mathcal{A}_{ij}\partial^2 \boldsymbol{u}/\partial x_i \partial x_j = \boldsymbol{0}$ with constant $m \times m$ matrix valued coefficients $\mathcal{A}_{ij} = \mathcal{A}_{ji}$ for a vector valued function $\boldsymbol{u} = (u_1, \dots, u_m)$ in the half-space $\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) : x_n > 0\}$ as well as in a domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$ and compact closure $\overline{\Omega}$ is considered. A representation for the sharp constant $\mathcal{C}_p$ in the inequality

$$|\boldsymbol{u}(x)| \le \mathcal{C}_p \; x_n^{(1-n)/p} \; ||\boldsymbol{u}|_{x_n=0}||_p$$

is obtained, where $|\cdot|$ is the length of a vector in the m-dimensional Euclidean space, $x \in \mathbb{R}^n_+$, and $||\cdot||_p$ is the L^p-norm of the modulus of an m-component vector valued function, $1 \le p \le \infty$.

It is shown that

$$\lim_{x \to \mathcal{O}_x} \big|x - \mathcal{O}_x\big|^{(n-1)/p} \; \sup\big\{|\boldsymbol{u}(x)| : ||\boldsymbol{u}|_{\partial\Omega}||_p \le 1\big\} = \mathcal{C}_p(\mathcal{O}_x),$$

where $\mathcal{O}_x$ is a point at $\partial\Omega$ nearest to $x \in \Omega$, $\boldsymbol{u}$ is the solution of Dirichlet problem in Ω for the strongly elliptic system $\mathcal{A}_{ij}\partial^2 \boldsymbol{u}/\partial x_i \partial x_j = \boldsymbol{0}$ with boundary data from $[L^p(\partial\Omega)]^m$, and $\mathcal{C}_p(\mathcal{O}_x)$ is the sharp constant in the above mentioned inequality for $\boldsymbol{u}$ in the tangent space $\mathbb{R}^n_+(\mathcal{O}_x)$ to $\partial\Omega$ at $\mathcal{O}_x$. As examples, Lamé and Stokes systems are considered. For instance, in the case of the Stokes system, the explicit formula

$$\mathcal{C}_p = \frac{2\Gamma\left(\frac{n+2}{2}\right)}{\pi^{(n+p-1)/(2p)}} \left\{ \frac{\Gamma\left(\frac{2p+n-1}{2p-2}\right)}{\Gamma\left(\frac{n+1}{2p-2}p\right)} \right\}^{(p-1)/p}$$

is derived, where $1 < p < \infty$.

Keywords: Boundary L^p-data; Pointwise estimates; Strongly elliptic systems; Lamé and Stokes systems

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0. Introduction

In this paper we consider solutions $\boldsymbol{u} = (u_1, \dots, u_m)$ of the strongly elliptic second order system

$$\sum_{i,j=1}^{n} \mathcal{A}_{ij} \, \frac{\partial^2 \boldsymbol{u}}{\partial x_i \partial x_j} = \boldsymbol{0} \tag{0.1}$$

with constant real $m \times m$ matrix valued coefficients $\mathcal{A}_{ij} = \mathcal{A}_{ji}$ in the half-space $\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n : x_n > 0\}$ as well as in a domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$ and compact closure $\overline{\Omega}$.

We find a representation for the sharp constant $\mathcal{C}_p$ in the inequality

$$|\boldsymbol{u}(x)| \le \mathcal{C}_p \; x_n^{(1-n)/p} \; ||\boldsymbol{u}|_{x_n=0}||_p, \tag{0.2}$$

where x is an arbitrary point in $\mathbb{R}^n_+$, $\boldsymbol{u}$ is the solution of the system (0.1) with boundary data from $[L^p(\partial\mathbb{R}^n_+)]^m$ (see [20]), $|\cdot|$ is the length of a vector in m-dimensional Euclidean space, and $||\cdot||_p$ is the L^p-norm of the modulus of an m-component vector valued function, $1 \le p \le \infty$.

The value $\mathcal{C}_p$ is connected with the asymptotic behaviour of solutions to system (0.1) near the boundary $\partial\Omega$. In particular, we show that

$$\lim_{x\to\mathcal{O}_x} \left|x-\mathcal{O}_x\right|^{(n-1)/p} \sup\left\{|\boldsymbol{u}(x)| : ||\boldsymbol{u}|_{\partial\Omega}||_p \le 1\right\} = \mathcal{C}_p(\mathcal{O}_x), \tag{0.3}$$

where $\mathcal{O}_x$ is a point at $\partial\Omega$ nearest to $x\in\Omega$, $\boldsymbol{u}$ is solution of the Dirichlet problem in Ω for strongly elliptic system (0.1) with boundary data from $[L^p(\partial\Omega)]^m$, and $\mathcal{C}_p(\mathcal{O}_x)$ is the sharp constant in the inequality (0.2) for the tangent space $\mathbb{R}^n_+(\mathcal{O}_x)$ to $\partial\Omega$ at $\mathcal{O}_x$.

The derivation of the explicit formulas for the sharp constant in the inequality (0.2) for solutions of the Lamé and Stokes systems is reduced to an optimization problem on the unit sphere $\mathbb{S}^{n-1}$. This problem can be explicitly solved for any $p\in[1,\infty]$ in the case of the Stokes system. We obtain also the estimates with sharp constants for the modulus of solutions to the Stokes and Lamé systems in the center of a ball by the integral means of order p of the modulus of boundary values on the sphere.

The first section is auxiliary. We consider the operator

$$T(\boldsymbol{f}) = \int_{\mathcal{X}} G(x)\boldsymbol{f}(x)d\mu(x),$$

acting from the space $[L^p(\mathcal{X},\mathcal{A},\mu)]^n$ ($[\boldsymbol{L}^p(\mathcal{X},\mathcal{A},\mu)]^n$) of real (complex) vector-valued n-component functions into the m-dimensional Euclidean space $\mathbb{R}^m$ (the unitary space $\mathbb{C}^m$). Here $(\mathcal{X},\mathcal{A},\mu)$ is the space with a measure, $G=((g_{ij}))$ is the $m\times n$ matrix-valued function with real (complex) components $g_{ij}\in L^q(\mathcal{X},\mathcal{A},\mu)$ ($g_{ij}\in \boldsymbol{L}^q(\mathcal{X},\mathcal{A},\mu)$), $1\le p\le\infty$, $1/p+1/q=1$ and the norm in $[L^p(\mathcal{X},\mathcal{A},\mu)]^n$ ($[\boldsymbol{L}^p(\mathcal{X},\mathcal{A},\mu)]^n$) is defined by

$$||\boldsymbol{f}||_p = \left\{\int_{\mathcal{X}} |\boldsymbol{f}(x)|^p d\mu(x)\right\}^{1/p},$$

for $1\le p<\infty$ and $||\boldsymbol{f}||_\infty = \text{ess sup}\{|\boldsymbol{f}(x)| : x\in\mathcal{X}\}$, where $|\cdot|$ is the length of a vector in $\mathbb{R}^m$ ($\mathbb{C}^m$). We derive a representation for the norm

$$||T||_p = \sup_{|\boldsymbol{z}|=1} ||G^*\boldsymbol{z}||_q, \tag{0.4}$$

where G^* stands for the transposed (adjoint) matrix of G and $\boldsymbol{z}\in\mathbb{R}^m$ ($\boldsymbol{z}\in\mathbb{C}^m$).

In Section 2, applying (0.4), we obtain a representation for the constant $\mathcal{C}_p$ in the inequality (0.2) for solutions of Dirichlet problem for the system (0.1) in the half-space $\mathbb{R}^n_+$. A representation of the sharp constant in (0.2) for the system (0.1) in case $p=\infty$ was derived earlier in [13]. Sharp pointwise estimates of solutions to elliptic systems with boundary data subject to some algebraic conditions are obtained in [9].

In Section 3 it is shown, that for any solution $\boldsymbol{u}$ of the Dirichlet problem in Ω for the system (0.1) with boundary data from $[L_p(\partial\Omega)]^m$ and all $x\in\Omega$ the relation

$$\sup\left\{|\boldsymbol{u}(x)| : ||\boldsymbol{u}|_{\partial\Omega}||_p \le 1\right\} = \mathcal{C}_p(\mathcal{O}_x)|x-\mathcal{O}_x|^{-(n-1)/p} + O\Big(|x-\mathcal{O}_x|^{\varepsilon-(n-1)/p}\Big) \tag{0.5}$$

holds for some $\varepsilon>0$. Here $\mathcal{O}_x$ is a point at $\partial\Omega$ nearest to $x\in\Omega$, and $\mathcal{C}_p(\mathcal{O}_x)$ is the best constant in (0.2) for the half-space $\mathbb{R}^n_+(\mathcal{O}_x)$. Equality (0.3) is an immediate consequence of (0.5).

In Section 4 we consider the Stokes system

$$\nu\Delta\boldsymbol{u} - \text{grad } p = \boldsymbol{0}, \quad \text{div } \boldsymbol{u} = 0,$$

in the half-space $\mathbb{R}^n_+$, $n\ge 2$, with the boundary condition

$$\boldsymbol{u}\big|_{x_n=0} = \boldsymbol{f},$$

where ν is the kinematic coefficient of viscosity, $\boldsymbol{u} = (u_1, \dots, u_n)$ is the velocity vector of a fluid, p is the pressure in the fluid, and $\boldsymbol{f} \in [L^p(\partial\mathbb{R}^n_+)]^n$. Despite the fact that the Stokes system is not strongly elliptic, the representation for the velocity vector $\boldsymbol{u}$ in the half-space is of the same nature as in the case of strongly elliptic systems. Hence, the result obtained in Section 2 applies to the Stokes system. It is shown, that for any $x \in \mathbb{R}^n_+$ the sharp coefficient $\mathcal{C}_p$ in the inequality (0.2) for the velocity vector $\boldsymbol{u}$ defined by a solution $(\boldsymbol{u}, p)$ of the Stokes system is given by

$$\mathcal{C}_1 = \frac{2\Gamma\left(\frac{n+2}{2}\right)}{\pi^{n/2}}, \qquad \mathcal{C}_\infty = \frac{2\Gamma\left(\frac{n+2}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)},$$

and

$$\mathcal{C}_p = \frac{2\Gamma\left(\frac{n+2}{2}\right)}{\pi^{(n+p-1)/(2p)}} \left\{ \frac{\Gamma\left(\frac{2p+n-1}{2p-2}\right)}{\Gamma\left(\frac{n+1}{2p-2}p\right)} \right\}^{(p-1)/p}$$

for $1 < p < \infty$. In particular, $\mathcal{C}_\infty = 4/\pi$ for $n = 2$ and $\mathcal{C}_\infty = 3/2$ for $n = 3$.

In Section 5 we find a representation of the sharp constant in the inequality

$$|\boldsymbol{u}(0)| \le \mathcal{H}_p \, M_p(\boldsymbol{u}; \partial\mathbb{B}_r),$$

where $\boldsymbol{u}$ is the velocity vector, and $M_p(\boldsymbol{u}; \partial\mathbb{B}_r)$ is the integral mean of order p for the modulus of $\boldsymbol{u}$ on the sphere $\partial\mathbb{B}_r = \{x \in \mathbb{R}^n : |x| = r\}$. In particular, it is shown that

$$\mathcal{H}_1 = \frac{n(n+1)}{2}, \qquad \mathcal{H}_2 = \frac{n\sqrt{n+3}}{2}.$$

Section 6 concerns the Lamé system

$$\mu \Delta \boldsymbol{u} + (\lambda + \mu) \text{grad div } \boldsymbol{u} = \boldsymbol{0}$$

in the half-space $\mathbb{R}^n_+$ with the boundary condition

$$\boldsymbol{u}\big|_{x_n = 0} = \boldsymbol{f},$$

where λ and μ are the Lamé constants, $\boldsymbol{u} = (u_1, \dots, u_n)$ is the displacement vector of an elastic medium, and $\boldsymbol{f} \in [L^p(\partial\mathbb{R}^n_+)]^n$. We find a representation for the sharp constant $\mathcal{C}_p$ in (0.2). In particular, we show that

$$\mathcal{C}_{1,\varkappa} = \frac{[1 + \varkappa(n-1)]\Gamma(n/2)}{\pi^{n/2}}, \qquad \mathcal{C}_{2,\varkappa} = \left\{ \frac{\Gamma\left(\frac{n}{2}\right)}{2^n \pi^{n/2}} \left[1 + (n-1)\varkappa^2 + \big(1 + (n-1)\varkappa\big)^2 \right] \right\}^{1/2},$$

where $\varkappa = (\lambda + \mu)(\lambda + 3\mu)^{-1}$.

The sharp constants in (0.2) for the Lamé and Stokes systems in the case $p = \infty$ were found in [13].

The concluding Section 7 is dedicated to the sharp constant in

$$|\boldsymbol{u}(0)| \le \mathcal{B}_{p,\varkappa} \, M_p(\boldsymbol{u}; \partial\mathbb{B}_r)$$

where $\boldsymbol{u}$ is a solution of the three-dimensional Lamé system in a ball. We show, in particular, that

$$\mathcal{B}_{1,\varkappa} = \frac{3(1 + 3\varkappa)}{3 - \varkappa}, \qquad \mathcal{B}_{2,\varkappa} = \frac{3}{3 - \varkappa}\big(17\varkappa^2 - 2\varkappa + 3\big)^{1/2}.$$

1. The norm of a linear bounded operator defined on the L^p-space of m-component vector valued functions and acting into $\mathbb{R}^m$

By $|\cdot|$ and $(\cdot,\cdot)$ we denote the length of a vector and the inner product in the unitary finite dimensional space $\mathbb{C}^m$ and in the Euclidean space $\mathbb{R}^m$, i.e. for $\boldsymbol{z}=(z_1,\dots,z_m)$ and $\boldsymbol{\zeta}=(\zeta_1,\dots,\zeta_m)\in\mathbb{C}^m$ we put $(\boldsymbol{z},\boldsymbol{\zeta})=\overline{z}_1\zeta_1+\dots+\overline{z}_m\zeta_m$ and $|\boldsymbol{z}|=(\boldsymbol{z},\boldsymbol{z})^{1/2}$. Let $(\mathcal{X},\mathcal{A},\mu)$ be a measure space, let p satisfy $1\leq p\leq\infty$, and let q be defined by $1/p+1/q=1$.

It is known (see, f.e., [4], Proposition 3.5.2 and Ex. 2, p. 153), that each element $g\in L^q(\mathcal{X},\mathcal{A},\mu)$ induces a bounded linear functional Φ_g on $L^p(\mathcal{X},\mathcal{A},\mu)$ by means of the formula

$$\Phi_g(f)=\int_{\mathcal{X}} f(x)g(x)d\mu(x),$$

and that the operator Φ mapping g to Φ_g is an isometry of $L^q(\mathcal{X},\mathcal{A},\mu)$ into $(L^p(\mathcal{X},\mathcal{A},\mu))^*$. A similar assertion holds for functionals on $\boldsymbol{L}^p(\mathcal{X},\mathcal{A},\mu)$ (see Sect. 3.3, 3.5 v [4]).

Moreover, the following statement is known ([4], Theorem 4.5.1).

Theorem 1. *If $p=1$ and μ is σ-finite, or if $1<p<\infty$ and μ is arbitrary, then the operator $\Phi: L^q(\mathcal{X},\mathcal{A},\mu)\to(L^p(\mathcal{X},\mathcal{A},\mu))^*$ in the real case, and the operator $\Phi:\ \boldsymbol{L}^q(\mathcal{X},\mathcal{A},\mu)\to(\boldsymbol{L}^p(\mathcal{X},\mathcal{A},\mu))^*$ in the complex case, defined above is an isometric isomorphism.*

We introduce the spaces $[L^p(\mathcal{X}\mathcal{A},\mu)]^n$ and $[\boldsymbol{L}^p(\mathcal{X},\mathcal{A},\mu)]^n$ of real and complex vector-valued functions $\boldsymbol{f}=(f_1,\dots,f_n)$ with components in $L^p(\mathcal{X},\mathcal{A},\mu)$ and $\boldsymbol{L}^p(\mathcal{X},\mathcal{A},\mu)$, respectively, endowed with the norm

$$||\boldsymbol{f}||_p=\left\{\int_{\mathcal{X}}|\boldsymbol{f}(x)|^p d\mu(x)\right\}^{1/p}, \tag{1.1}$$

for $1\leq p<\infty$, and $||\boldsymbol{f}||_\infty=\text{ess sup}\{|\boldsymbol{f}(x)|:x\in\mathcal{X}\}$.

The next assertion follows directly from the above theorem on the representation of a linear functional on the spaces $L^p(\mathcal{X},\mathcal{A},\mu)$ and $\boldsymbol{L}^p(\mathcal{X},\mathcal{A},\mu)$.

Corollary 1. *If $p=1$ and μ is σ-finite, or if $1<p<\infty$ and μ is arbitrary, then any linear bounded operator $T:[L^p(\mathcal{X},\mathcal{A},\mu)]^n\to\mathbb{R}^m$ ($T:[\boldsymbol{L}^p(\mathcal{X},\mathcal{A},\mu)]^n\to\mathbb{C}^m$) admits the representation*

$$T(\boldsymbol{f})=\int_{\mathcal{X}} G(x)\boldsymbol{f}(x)d\mu(x),$$

where $G=((g_{ij}))$ is the $m\times n$ matrix valued function with elements $g_{ij}\in L^q(\mathcal{X},\mathcal{A},\mu)$ (respectively, $g_{ij}\in\boldsymbol{L}^q(\mathcal{X},\mathcal{A},\mu)$).

Proof. We consider the space of real valued functions. The case of complex valued functions is treated in the same way. Let T be an arbitrary linear bounded operator $[L^p(\mathcal{X},\mathcal{A},\mu)]^n\to\mathbb{R}^m$,

$$T(\boldsymbol{f})=\begin{pmatrix} T_1(\boldsymbol{f})\\ \cdots\cdots\\ \cdots\cdots\\ T_m(\boldsymbol{f})\end{pmatrix}, \tag{1.2}$$

where $T_1,\dots,T_m$ are functionals on $[L^p(\mathcal{X},\mathcal{A},\mu)]^n$. Clearly, the functionals $T_1,\dots,T_m$ are linear and, in view of

$$|T_i(\boldsymbol{f})|\leq\left[\sum_{i=1}^m|T_i(\boldsymbol{f})|^2\right]^{1/2}=|T(\boldsymbol{f})|\leq||T||_p\ ||\boldsymbol{f}||_p,$$

they are bounded, where $||T||_p$ is the norm of the operator $T:[L^p(\mathcal{X},\mathcal{A},\mu)]^n\to\mathbb{R}^m$.

Further, for any function $\boldsymbol{f} = (f_1, \dots, f_n) \in [L^p(\mathcal{X}, \mathcal{A}, \mu)]^n$ there holds

$$T_i(\boldsymbol{f}) = T_i(f_1, 0, \dots, 0) + T_i(0, f_2, \dots, 0) + \cdots + T_i(0, \dots, 0, f_n). \tag{1.3}$$

We introduce the notation

$$T_{i1}(f_1) = T_i(f_1, 0, \dots, 0), \dots\dots\dots, T_{in}(f_n) = T_i(0, \dots, 0, f_n). \tag{1.4}$$

Clearly, each of the functionals $T_{i1}, \dots, T_{in}$ is linear and bounded on $L^p(\mathcal{X}, \mathcal{A}, \mu)$. Hence, by Theorem 1, $T_{ij}(f_j)$ admits the representation

$$T_{ij}(f_j) = \int_{\mathcal{X}} f_j(x) g_{ij}(x) d\mu(x), \tag{1.5}$$

$1 \le i \le m,\ 1 \le j \le n$, where $g_{ij} \in L^q(\mathcal{X}, \mathcal{A}, \mu)$. Thus, (1.3)-(1.5) imply

$$T_i(\boldsymbol{f}) = \sum_{j=1}^{n} T_{ij}(f_j) = \sum_{j=1}^{n} \int_{\mathcal{X}} f_j(x) g_{ij}(x) d\mu(x). \tag{1.6}$$

Combining (1.2), (1.6) and the notation $G = ((g_{ij})),\ 1 \le i \le m,\ 1 \le j \le n$, we complete the proof. $\square$

The following assertion contains a representation of the norm $||T||_p$ of the integral operator T defined on $[L^p(\mathcal{X}, \mathcal{A}, \mu)]^n$ ($[\boldsymbol{L}^p(\mathcal{X}, \mathcal{A}, \mu)]^n$), $1 \le p \le \infty$, and acting into $\mathbb{R}^m$ ($\mathbb{C}^m$).

Proposition 1. *Let $(\mathcal{X}, \mathcal{A}, \mu)$ be a measure space, let p satisfy $1 \le p \le \infty$, and let q be defined by $1/p + 1/q = 1$. Let, further, $G = ((g_{ij}))$ be an $m \times n$ matrix valued function with the elements $g_{ij} \in L^q(\mathcal{X}, \mathcal{A}, \mu)$ ($g_{ij} \in \boldsymbol{L}^q(\mathcal{X}, \mathcal{A}, \mu)$) whose values g_{ij} ($|g_{ij}|$) are everywhere finite. The norm of the linear continuous operator $T : [L^p(\mathcal{X}, \mathcal{A}, \mu)]^n \to \mathbb{R}^m$ ($T : [\boldsymbol{L}_p(\mathcal{X}, \mathcal{A}, \mu)]^n \to \mathbb{C}^m$) defined by*

$$T(\boldsymbol{f}) = \int_{\mathcal{X}} G(x) \boldsymbol{f}(x) d\mu(x) \tag{1.7}$$

is equal to

$$||T||_p = \sup_{|\boldsymbol{z}|=1} ||G^* \boldsymbol{z}||_q, \tag{1.8}$$

where G^* stands for the transposed (adjoint) matrix of G and $\boldsymbol{z} \in \mathbb{R}^m$ ($\boldsymbol{z} \in \mathbb{C}^m$).

Proof. We deduce (1.8) for the operator $T : [\boldsymbol{L}^p(\mathcal{X}, \mathcal{A}, \mu)]^n \to \mathbb{C}^m$ defined by (1.7). The case of the operator $T : [L^p(\mathcal{X}, \mathcal{A}, \mu)]^n \to \mathbb{R}^m$ is treated in the same way.

1. Upper estimate for $||T||_p$. For any vector $\boldsymbol{z} \in \mathbb{C}^m$,

$$(T(\boldsymbol{f}), \boldsymbol{z}) = \int_{\mathcal{X}} (G(x)\boldsymbol{f}(x), \boldsymbol{z}) d\mu(x) = \int_{\mathcal{X}} (\boldsymbol{f}(x), G^*(x)\boldsymbol{z}) d\mu(x), \tag{1.9}$$

where we denote by G^* the adjoint matrix of G. Hence by Hölder's inequality

$$|(T(\boldsymbol{f}), \boldsymbol{z})| \le \int_{\mathcal{X}} |(\boldsymbol{f}(x), G^*(x)\boldsymbol{z})| d\mu(x) \le \int_{\mathcal{X}} |G^*(x)\boldsymbol{z}||\boldsymbol{f}(x)| d\mu(x) \le ||G^*\boldsymbol{z}||_q ||\boldsymbol{f}||_p.$$

Therefore, taking into account that $|T(\boldsymbol{f})| = \sup\{|(T(\boldsymbol{f}), \boldsymbol{z})| : |\boldsymbol{z}| = 1\}$ we arrive at the estimate

$$||T||_p \le \sup_{|\boldsymbol{z}|=1} ||G^* \boldsymbol{z}||_q. \tag{1.10}$$

2. Lower estimate for $||T||_p$. Let us fix $\boldsymbol{z} \in \mathbf{S}^{m-1} = \{\boldsymbol{z} \in \mathbb{C}^m : |\boldsymbol{z}| = 1\}$. We introduce the vector-valued function with n components

$$\boldsymbol{h}_{\boldsymbol{z}}(x) = \boldsymbol{g}_{\boldsymbol{z}}(x) h(x), \tag{1.11}$$

where $h \in L_p(\mathcal{X}, \mathcal{A}, \mu)$, $||h||_p \le 1$, and

$$
\boldsymbol{g_z}(x) = \begin{cases} G^*(x)\boldsymbol{z}|G^*(x)\boldsymbol{z}|^{-1} & \text{for} \quad |G^*(x)\boldsymbol{z}| \neq 0, \\ \\ \boldsymbol{0} & \text{for} \quad |G^*(x)\boldsymbol{z}| = 0. \end{cases}
$$

Note that $\boldsymbol{h_z} \in [\boldsymbol{L}^p(\mathcal{X}, \mathcal{A}, \mu)]^n$ and $||\boldsymbol{h_z}||_p \le 1$. Setting (1.11) as $\boldsymbol{f}$ in (1.9) we find

$$
(T(\boldsymbol{h_z}), \boldsymbol{z}) = (T(\boldsymbol{g_z} h), \boldsymbol{z}) = \int_{\mathcal{X}} (\boldsymbol{g_z}(x), G^*(x)\boldsymbol{z}) h(x) d\mu(x) = \int_{\mathcal{X}} |G^*(x)\boldsymbol{z}| h(x) d\mu(x).
$$

Hence

$$
\begin{aligned}
||T||_p &= \sup_{||\boldsymbol{f}||_p \le 1} |T(\boldsymbol{f})| \ge \sup_{||h||_p \le 1} |T(\boldsymbol{g_z} h)| \ge \sup_{||h||_p \le 1} |(T(\boldsymbol{g_z} h), \boldsymbol{z})| \\
&= \sup_{||h||_p \le 1} \left| \int_{\mathcal{X}} |G^*(x)\boldsymbol{z}| h(x) d\mu(x) \right| = ||G^*\boldsymbol{z}||_q.
\end{aligned}
$$

By the arbitrariness of $\boldsymbol{z} \in \mathbf{S}^{m-1}$,

$$
||T||_p \ge \sup_{|\boldsymbol{z}|=1} ||G^*\boldsymbol{z}||_q,
$$

which together with (1.10) leads to (1.8). □

2. Elliptic systems in a half-space

We introduce some notation used henceforth. Let $x \in \mathbb{R}^n_+ = \big\{x = (x', x_n) : x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}, x_n > 0\big\}$ and $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. By $[\mathrm{C}^{(2)}(\mathbb{R}^n_+)]^m$ we denote the space of m-component vector valued functions with continuous derivatives up to the second order in $\mathbb{R}^m_+$. Further, $[\mathrm{C}(\overline{\mathbb{R}}^n_+)]^m$ and $[\mathrm{C}(\partial\mathbb{R}^n_+)]^m$ will stand for the spaces of continuous and bounded m-component vector valued functions on $\overline{\mathbb{R}}^n_+$ and $\partial\mathbb{R}^n_+$, respectively.

We introduce the strongly elliptic operator

$$
\mathfrak{A}(D_x) = \sum_{i,j=1}^{n} \mathcal{A}_{ij} \, \partial^2/\partial x_i \partial x_j, \tag{2.1}
$$

where $D_x = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$ and $\mathcal{A}_{ij} = \mathcal{A}_{ji}$ are constant real $m \times m$ matrices. The strong ellipticity of $\mathfrak{A}(D_x)$ means that the inequality

$$
\left(\sum_{i,j=1}^{n} \mathcal{A}_{ij} \sigma_i \sigma_j \boldsymbol{\zeta}, \boldsymbol{\zeta} \right) > 0
$$

is valid for all $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_m) \in \mathbb{R}^m \backslash \{\boldsymbol{0}\}$ and $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n \backslash \{\boldsymbol{0}\}$.

According to [12, 20], there exists a bounded solution of the problem

$$
\mathfrak{A}(D_x)\boldsymbol{u} = \boldsymbol{0} \ \text{ in } \ \mathbb{R}^n_+, \quad \boldsymbol{u} = \boldsymbol{f} \ \text{ on } \ \partial\mathbb{R}^n_+, \tag{2.2}
$$

where $\boldsymbol{f} \in [\mathrm{C}(\partial\mathbb{R}^n_+)]^m$, which is continuous up to $\partial\mathbb{R}^n_+$, and this solution can be represented in the form

$$
\boldsymbol{u}(x) = \int_{\partial\mathbb{R}^n_+} F\left(\frac{y-x}{|y-x|}\right) \frac{x_n}{|y-x|^n} \boldsymbol{f}(y') dy'. \tag{2.3}
$$

Here $y = (y', 0), y' = (y_1, \ldots, y_{n-1})$, and F is a $m \times m$ matrix valued function with continuous components on the closure of the hemisphere $\mathbb{S}^{n-1}_- = \big\{x \in \mathbb{R}^n : \ |x| = 1, \ x_n < 0\big\}$.

The uniqueness of a solution to the Dirichlet problem (2.2) in the class $[\mathrm{C}^{(2)}(\mathbb{R}^n_+)]^m$ with boundary data from $[L^p(\partial\mathbb{R}^n_+)]^m$ can be derived by means of a standard argument, from (2.3) and from local estimates for derivatives of solutions to elliptic systems (see [3, 17]).

By $||\cdot||_p$ we denote the norm in the space $[L^p(\partial\mathbb{R}^n_+)]^m$, that is

$$||\boldsymbol{f}||_p = \left\{ \int_{\partial\mathbb{R}^n_+} |\boldsymbol{f}(x')|^p dx' \right\}^{1/p},$$

if $1 \leq p < \infty$, and $||\boldsymbol{f}||_\infty = \mathrm{ess\ sup}\{|\boldsymbol{f}(x')| : x' \in \partial\mathbb{R}^n_+\}$.

Proposition 2. *Let x be an arbitrary point in $\mathbb{R}^n_+$ and let $\boldsymbol{z} \in \mathbb{R}^m$. The sharp coefficient $\mathcal{K}_p(x)$ in the inequality*

$$|\boldsymbol{u}(x)| \leq \mathcal{K}_p(x) ||\boldsymbol{u}|_{x_n=0}||_p \tag{2.4}$$

is given by

$$\mathcal{K}_p(x) = \mathcal{C}_p\, x_n^{(1-n)/p}, \tag{2.5}$$

where

$$\mathcal{C}_1 = \sup_{|\boldsymbol{z}|=1} \sup_{\sigma\in\mathbb{S}^{n-1}_-} |F^*(\boldsymbol{e}_\sigma)\boldsymbol{z}|(\boldsymbol{e}_\sigma, -\boldsymbol{e}_n)^n, \tag{2.6}$$

$$\mathcal{C}_\infty = \sup_{|\boldsymbol{z}|=1} \int_{\mathbb{S}^{n-1}_-} |F^*(\boldsymbol{e}_\sigma)\boldsymbol{z}| d\sigma, \tag{2.7}$$

and

$$\mathcal{C}_p = \sup_{|\boldsymbol{z}|=1} \left\{ \int_{\mathbb{S}^{n-1}_-} |F^*(\boldsymbol{e}_\sigma)\boldsymbol{z}|^{p/(p-1)} (\boldsymbol{e}_\sigma, -\boldsymbol{e}_n)^{n/(p-1)} d\sigma \right\}^{(p-1)/p} \tag{2.8}$$

for $1 < p < \infty$. Here $\boldsymbol{e}_\sigma$ is the n-dimensional unit vector joining the origin to a point $\sigma \in \mathbb{S}^{n-1}_-$, and $*$ denotes the transposition of a matrix.

Proof. By Proposition 1 and (2.3), the sharp constant $\mathcal{K}_p(x)$ in (2.4) is given by

$$\mathcal{K}_p(x) = \sup_{|\boldsymbol{z}|=1} \left\{ \int_{\partial\mathbb{R}^n_+} \left| F^*\left(\frac{y-x}{|y-x|}\right)\boldsymbol{z}\right|^q \frac{x_n^q}{|y-x|^{nq}} dy' \right\}^{1/q},$$

where $1 < p \leq \infty$. Putting $\rho = |y'-x'|$, we write the last equality as follows

$$\mathcal{K}_p(x) = \sup_{|\boldsymbol{z}|=1} \left\{ \int_{\mathbb{S}^{n-2}(x')} d\sigma \int_0^\infty \left| F^*\left(\frac{\rho\boldsymbol{e}'_\sigma - x_n\boldsymbol{e}_n}{\left[\rho^2+x_n^2\right]^{1/2}}\right)\boldsymbol{z}\right|^q \frac{x_n^q}{\left[\rho^2+x_n^2\right]^{nq/2}} \rho^{n-2} d\rho \right\}^{1/q}, \tag{2.9}$$

where $\boldsymbol{e}'_\sigma = (y'-x')|y'-x'|^{-1}$, and $\mathbb{S}^{n-2}(x')$ is the $(n-2)$-dimensional unit sphere with the center at the point x'.

Now we make the change of variable $\rho = x_n \tan\varphi$ in (2.9) with φ standing for the angle between the vectors $\rho\boldsymbol{e}'_\sigma - x_n\boldsymbol{e}_n$ and $-\boldsymbol{e}_n$, and obtain

$$\mathcal{K}_p(x) = x_n^{(1-n)(q-1)/q} \sup_{|\boldsymbol{z}|=1} \left\{ \int_{\mathbb{S}^{n-2}(x')} d\sigma \int_0^{\pi/2} |F^*\left(\boldsymbol{e}'_\sigma \sin\varphi - \boldsymbol{e}_n\cos\varphi\right)\boldsymbol{z}|^q \cos^{n(q-1)}\varphi\ \sin^{n-2}\varphi\, d\varphi \right\}^{1/q}.$$

Using the independence of the integral on x' and the notation

$$\boldsymbol{e}_\sigma = \boldsymbol{e}'_\sigma \sin\varphi - \boldsymbol{e}_n\cos\varphi,$$

we arrive at (2.5) with $\mathcal{C}_p$ defined by (2.8). In particular, (2.8) becomes (2.7) for $p = \infty$.

Next note that by Proposition 1 and (2.3), the sharp constant in (2.4) with $p = 1$ can be written as

$$\mathcal{K}_1(x) = \sup_{|\boldsymbol{z}|=1} \sup_{y \in \partial\mathbb{R}^n_+} \left| F^* \left(\frac{y-x}{|y-x|} \right) \boldsymbol{z} \right| \frac{x_n}{|y-x|^n}.$$

Setting here $|y' - x'| = x_n \tan\varphi,\ 0 \le \varphi < \pi/2$, we find

$$\mathcal{K}_1(x) = x_n^{1-n} \sup_{|\boldsymbol{z}|=1} \sup_{0 \le \varphi < \pi/2} |F^*(\boldsymbol{e}'_\sigma \sin\varphi - \boldsymbol{e}_n \cos\varphi)\boldsymbol{z}| \cos^n\varphi = x_n^{1-p} \sup_{|\boldsymbol{z}|=1} \sup_{\mathbb{S}^{n-1}_-} |F^*(\boldsymbol{e}_\sigma)\boldsymbol{z}|\, (\boldsymbol{e}_\sigma, -\boldsymbol{e}_n)^n,$$

which implies (2.5) with $p = 1$ and with $\mathcal{C}_1$ defined by (2.6). □

3. Asymptotic formula involving the Poisson matrix in a domain

In what follows, by smoothness we mean the membership in C^∞. Suppose Ω is a domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and compact closure $\overline{\Omega}$. By $\boldsymbol{\nu}(y)$ we denote the unit interior normal to $\partial\Omega$ at a point $y \in \partial\Omega$. Let $\mathbb{R}^n_+(y) = \big\{ x \in \mathbb{R}^n : (x, \boldsymbol{\nu}(y)) > 0 \big\}$, $\mathbb{R}^n_-(y) = \big\{ x \in \mathbb{R}^n : (x, \boldsymbol{\nu}(y)) < 0 \big\}$ and $\mathbb{R}^{n-1}(y) = \partial\mathbb{R}^n_+(y)$.

We consider the Dirichlet problem

$$\mathfrak{A}(D_x)\boldsymbol{u}_{\boldsymbol{g}} = \boldsymbol{0} \ \text{ in } \ \Omega, \quad \boldsymbol{u}_{\boldsymbol{g}}\big|_{\partial\Omega} = \boldsymbol{g} \tag{3.1}$$

for the strongly elliptic operator $\mathfrak{A}(D_x)$, defined by (2.1), with $\boldsymbol{g} \in [L_p(\partial\Omega)]^m$.

Theorem 2. *For all* $x \in \Omega$

$$\sup\big\{ |\boldsymbol{u}_{\boldsymbol{g}}(x)| : ||\boldsymbol{g}||_p \le 1 \big\} = \mathcal{C}_p(\mathcal{O}_x)|x - \mathcal{O}_x|^{-(n-1)/p} + O\Big(|x - \mathcal{O}_x|^{\varepsilon-(n-1)/p} \Big)$$

with some $\varepsilon > 0$. *Here* $\mathcal{O}_x$ *is a point at* $\partial\Omega$ *nearest to* $x \in \Omega$, *and* $\mathcal{C}_p(\mathcal{O}_x)$ *is the best constant in* (2.5) *for the half-space* $\mathbb{R}^n_+(\mathcal{O}_x)$.

Before giving a proof of this theorem, we formulate its obvious corollary.

Corollary 2. *The equality*

$$\lim_{x \to \mathcal{O}_x} \big| x - \mathcal{O}_x \big|^{(n-1)/p} \sup\big\{ |\boldsymbol{u}_{\boldsymbol{g}}(x)| : ||\boldsymbol{g}||_p \le 1 \big\} = \mathcal{C}_p(\mathcal{O}_x)$$

holds.

Proof of the Theorem. The Poisson matrix of problem (3.1) with singularity at the point $y \in \partial\Omega$ will be denoted by $P_\Omega(x, y)$. In other words, P_Ω satisfies the problem

$$\mathfrak{A}(D_x)P_\Omega(x, y) = \boldsymbol{0} \ \text{ for } \ x \in \Omega, \qquad P_\Omega(x, y) = \delta(y - x)I \ \text{ for } \ x \in \partial\Omega,$$

where δ is the Dirac function and I is the $m \times m$ identity matrix. We put $P_\Omega(x, y) = 0$ for $x \in \mathbb{R}^n \backslash \overline{\Omega}$.

The notation $\Pi(x, y)$ will be used for the Poisson matrix with singularity at y of the Dirichlet problem for the operator $\mathfrak{A}(D_x)$ in the half-space $\mathbb{R}^n_+(y)$. The matrix-function $x \to \Pi(x, y)$ is extended by zero to the half-space $\mathbb{R}^n_-(y)$. Clearly,

$$\Pi(x, y) = \frac{(y - x, \boldsymbol{\nu}(y))}{|y - x|^n}\, \mathcal{F}\left(\frac{y-x}{|y-x|},\ \boldsymbol{\nu}(y) \right), \tag{3.2}$$

where

$$\left|\left| \mathcal{F}\left(\frac{y-x}{|y-x|},\ \boldsymbol{\nu}(y) \right) - \mathcal{F}\left(\frac{y-x}{|y-x|},\ \boldsymbol{\nu}(\mathcal{O}_x) \right) \right|\right| \le c\, |y - \mathcal{O}_x|, \tag{3.3}$$

and $||\cdot||$ is the matrix norm induced by the Euclidean norm in $\mathbb{R}^m$. It is well-known (see Krasovskiĭ [6, 7], Solonnikov [18, 19]) that for $x \in \Omega$ and $y \in \partial\Omega$

$$||P_\Omega(x,y) - \Pi(x,y)|| \leq c(\varepsilon_o)\ |x-y|^{2-n-\varepsilon_o}, \tag{3.4}$$

where ε_o is an arbitrary positive number. Therefore, for $x \in \Omega$ and any $\boldsymbol{z} \in \mathbb{S}^{n-1}$

$$||P^*_\Omega(x,\boldsymbol{\cdot})\boldsymbol{z} - \Pi^*(x,\boldsymbol{\cdot})\boldsymbol{z}||_{[L_q(\partial\Omega)]^m} \leq c(\varepsilon_o)\ ||\ |x-\boldsymbol{\cdot}|^{2-n-\varepsilon_o}||_{L_q(\partial\Omega)} \leq c_1(\varepsilon)|x-\mathcal{O}_x|^{\varepsilon-(n-1)/p} \tag{3.5}$$

with some $\varepsilon > 0$. Using (3.2) and (3.3), we arrive at the estimate

$$\left|\ \left|\left|\frac{(\boldsymbol{\cdot}-x,\boldsymbol{\nu}(\boldsymbol{\cdot}))}{|\boldsymbol{\cdot}-x|^n}\ \mathcal{F}^*\left(\frac{\boldsymbol{\cdot}-x}{|\boldsymbol{\cdot}-x|},\ \boldsymbol{\nu}(\boldsymbol{\cdot})\right)\boldsymbol{z}\right|\right|_{[L_q(\partial\Omega)]^m} - \left|\left|\frac{(\boldsymbol{\cdot}-x,\boldsymbol{\nu}(\mathcal{O}_x))}{|\boldsymbol{\cdot}-x|^n}\ \mathcal{F}^*\left(\frac{\boldsymbol{\cdot}-x}{|\boldsymbol{\cdot}-x|},\ \boldsymbol{\nu}(\mathcal{O}_x)\right)\boldsymbol{z}\right|\right|_{[L_q(\partial\Omega)]^m}\right|$$

$$\leq c_2(\varepsilon)|x-\mathcal{O}_x|^{\varepsilon-(n-1)/p}. \tag{3.6}$$

Since $\partial\mathbb{R}^n_+(\mathcal{O}_x)$ is tangent to $\partial\Omega$ at the point $\mathcal{O}_x$, one can see that

$$\left|\ \left|\left|\frac{(\boldsymbol{\cdot}-x,\boldsymbol{\nu}(\mathcal{O}_x))}{|\boldsymbol{\cdot}-x|^n}\ \mathcal{F}^*\left(\frac{\boldsymbol{\cdot}-x}{|\boldsymbol{\cdot}-x|},\ \boldsymbol{\nu}(\mathcal{O}_x)\right)\boldsymbol{z}\right|\right|_{[L_q(\partial\Omega)]^m} - \left|\left|\frac{(\boldsymbol{\cdot}-x,\boldsymbol{\nu}(\mathcal{O}_x))}{|\boldsymbol{\cdot}-x|^n}\ \mathcal{F}^*\left(\frac{\boldsymbol{\cdot}-x}{|\boldsymbol{\cdot}-x|},\ \boldsymbol{\nu}(\mathcal{O}_x)\right)\boldsymbol{z}\right|\right|_{[L_q(\mathbb{R}^{n-1}(\mathcal{O}_x))]^m}\right|$$

$$\leq c_3(\varepsilon)|x-\mathcal{O}_x|^{\varepsilon-(n-1)/p}. \tag{3.7}$$

Using (3.2) and combining (3.5), (3.6) and (3.7), we obtain for any $\boldsymbol{z} \in \mathbb{S}^{n-1}$

$$\left|\ ||P^*_\Omega(x,\boldsymbol{\cdot})\boldsymbol{z}||_{[L_q(\partial\Omega)]^m} - \left|\left|\frac{(\boldsymbol{\cdot}-x,\boldsymbol{\nu}(\mathcal{O}_x))}{|\boldsymbol{\cdot}-x|^n}\ \mathcal{F}^*\left(\frac{\boldsymbol{\cdot}-x}{|\boldsymbol{\cdot}-x|},\ \boldsymbol{\nu}(\mathcal{O}_x)\right)\boldsymbol{z}\right|\right|_{[L_q(\mathbb{R}^{n-1}(\mathcal{O}_x))]^m}\right|$$

$$\leq c_4(\varepsilon)|x-\mathcal{O}_x|^{\varepsilon-(n-1)/p}. \tag{3.8}$$

The second norm in (3.8) is equal to

$$|x-\mathcal{O}_x|^{-(n-1)/p}\left\{\int_{\mathbb{R}^{n-1}(\mathcal{O}_x)}\left|\mathcal{F}^*\left(\frac{\sigma}{(1+|\sigma|^2)^{1/2}},\ \frac{-1}{(1+|\sigma|^2)^{1/2}}\right)\boldsymbol{z}\right|^q\frac{d\sigma}{(1+|\sigma|^2)^{nq/2}}\right\}^{1/q},$$

which, after taking the supremum over $\boldsymbol{z} \in \mathbb{S}^{n-1}$, becomes

$$\mathcal{C}_p(\mathcal{O}_x)|x-\mathcal{O}_x|^{-(n-1)/p}$$

by Proposition 2. Combining this with (3.8), we complete the proof of Theorem 2. □

4. The Stokes system in the half-space

Consider the Stokes system

$$\nu\Delta\boldsymbol{u} - \text{grad}\ p = \boldsymbol{0}, \quad \text{div}\ \boldsymbol{u} = 0 \quad \text{in}\ \ \mathbb{R}^n_+,\ \ n \geq 2, \tag{4.1}$$

with the boundary condition

$$\boldsymbol{u}\big|_{x_n=0} = \boldsymbol{f}, \tag{4.2}$$

where ν is the kinematic coefficient of viscosity, $\boldsymbol{u} = (u_1, \dots, u_n)$ is the velocity vector of a fluid, p is the pressure in the fluid, and $\boldsymbol{f} = (f_1, \dots, f_n)$ is a continuous and bounded vector valued function on $\partial\mathbb{R}^n_+$.

The solution $\boldsymbol{u}$ of the Dirichlet problem for the Stokes system in the half-space $\mathbb{R}^n_+$ which is bounded and continuous up to $\partial\mathbb{R}^n_+$ admits the representation (see [11])

$$\boldsymbol{u}(x) = \int_{\partial\mathbb{R}^n_+} S\left(\frac{y-x}{|y-x|}\right) \frac{x_n}{|y-x|^n}\, \boldsymbol{f}(y')dy', \tag{4.3}$$

where $x \in \mathbb{R}^n_+,\ y = (y', 0),\ y' = (y_1, \dots, y_{n-1})$, and $S(\boldsymbol{e}_\sigma)$ is the $n \times n$ matrix valued function on the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$ with the elements

$$\frac{2n}{\omega_n}(\boldsymbol{e}_\sigma, \boldsymbol{e}_i)(\boldsymbol{e}_\sigma, \boldsymbol{e}_j), \tag{4.4}$$

and $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the area of $\mathbb{S}^{n-1}$.

The uniqueness of solutions of the Dirichlet problem (4.1), (4.2) in the class $[\mathrm{C}^{(2)}(\mathbb{R}^n_+)]^n$ with boundary data from $[L^p(\partial\mathbb{R}^n_+)]^n$ can be derived by means of a standard argument from (4.3) and local estimates for derivatives of solutions to elliptic systems (see [3, 17]).

Theorem 3. *Let x be an arbitrary point in $\mathbb{R}^n_+$. The sharp coefficient $\mathcal{K}_p(x)$ in the inequality*

$$|\boldsymbol{u}(x)| \le \mathcal{K}_p(x)||\boldsymbol{u}|_{x_n=0}||_p \tag{4.5}$$

for the velocity vector $\boldsymbol{u}$ defined by a solution $(\boldsymbol{u}, p)$ of the Stokes system is given by

$$\mathcal{K}_p(x) = \mathcal{C}_p\, x_n^{(1-n)/p}, \tag{4.6}$$

where

$$\mathcal{C}_1 = \frac{2\Gamma\left(\frac{n+2}{2}\right)}{\pi^{n/2}}, \qquad\qquad \mathcal{C}_\infty = \frac{2\Gamma\left(\frac{n+2}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n+1}{2}\right)}, \tag{4.7}$$

and

$$\mathcal{C}_p = \frac{2\Gamma\left(\frac{n+2}{2}\right)}{\pi^{(n+p-1)/(2p)}} \left\{ \frac{\Gamma\left(\frac{2p+n-1}{2p-2}\right)}{\Gamma\left(\frac{n+1}{2p-2}p\right)} \right\}^{(p-1)/p} \tag{4.8}$$

for $1 < p < \infty$. In particular,

$$\mathcal{C}_2 = \left\{ \frac{(n+1)\Gamma\left(\frac{n+2}{2}\right)}{2^{n-1}\pi^{n/2}} \right\}^{1/2}. \tag{4.9}$$

Proof. Since the solution of the Dirichlet problem (4.1), (4.2) is given by (4.3), it obeys Proposition 2. Since the elements of the matrix S are defined by (4.4), it follows that

$$|S^*(\boldsymbol{e}_\sigma)\boldsymbol{z}| = \frac{2n}{\omega_n}\big|(\boldsymbol{e}_\sigma, \boldsymbol{z})\big|, \tag{4.10}$$

which together with (2.4), (2.5), and (2.8) implies

$$\mathcal{C}_p = \frac{2n}{\omega_n} \sup_{|\boldsymbol{z}|=1} \left\{ \int_{\mathbb{S}^{n-1}_-} \big|(\boldsymbol{e}_\sigma, \boldsymbol{z})\big|^{p/(p-1)} (\boldsymbol{e}_\sigma, -\boldsymbol{e}_n)^{n/(p-1)} d\sigma \right\}^{(p-1)/p},$$

where $1 < p \le \infty$. Noting that the function $|(\boldsymbol{e}_\sigma, -\boldsymbol{e}_n)|$ is even on the sphere $\mathbb{S}^{n-1}$, we can write $\mathcal{C}_p$ as

$$\mathcal{C}_p = \frac{2^{1/p} n}{\omega_n} \sup_{|\boldsymbol{z}|=1} \left\{ \int_{\mathbb{S}^{n-1}} \big|(\boldsymbol{e}_\sigma, \boldsymbol{z})\big|^{p/(p-1)} \big|(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big|^{n/(p-1)} d\sigma \right\}^{(p-1)/p}. \tag{4.11}$$

This immediately implies the lower estimate

$$\mathcal{C}_p \ge \frac{2^{1/p}n}{\omega_n}\left\{\int_{\mathbb{S}^{n-1}} \big|(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big|^{(p+n)/(p-1)} d\sigma\right\}^{(p-1)/p}. \tag{4.12}$$

Next we derive the upper estimate for $\mathcal{C}_p$. Setting

$$l = \frac{p+n}{n}, \quad s = \frac{p+n}{p},$$

and noting that $1/s + 1/l = 1$, we have by Hölder's inequality

$$\int_{\mathbb{S}^{n-1}} \big|(\boldsymbol{e}_\sigma, \boldsymbol{z})\big|^{p/(p-1)}\big|(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big|^{n/(p-1)} d\sigma \le \left\{\int_{\mathbb{S}^{n-1}} \big|(\boldsymbol{e}_\sigma, \boldsymbol{z})\big|^{ps/(p-1)} d\sigma\right\}^{1/s} \left\{\int_{\mathbb{S}^{n-1}} \big|(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big|^{nl/(p-1)} d\sigma\right\}^{1/l}.$$

Taking into account that $ps = nl = p + n$ and that the first integral in the right-hand side is independent on z, we find

$$\int_{\mathbb{S}^{n-1}} \big|(\boldsymbol{e}_\sigma, \boldsymbol{z})\big|^{p/(p-1)}\big|(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big|^{n/(p-1)} d\sigma \le \int_{\mathbb{S}^{n-1}} \big|(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big|^{(p+n)/(p-1)} d\sigma,$$

which together with (4.11) leads to

$$\mathcal{C}_p \le \frac{2^{1/p}n}{\omega_n}\left\{\int_{\mathbb{S}^{n-1}} \big|(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big|^{(p+n)/(p-1)} d\sigma\right\}^{(p-1)/p}.$$

Combining this with (4.12), we arrive at the equality

$$\mathcal{C}_p = \frac{2^{1/p}n}{\omega_n}\left\{\int_{\mathbb{S}^{n-1}} \big|(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big|^{(p+n)/(p-1)} d\sigma\right\}^{(p-1)/p}, \tag{4.13}$$

where $1 < p < \infty$. The formula (4.8) follows from (4.13) and

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} \big|(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big|^{(p+n)/(p-1)} d\sigma &= 2\omega_{n-1}\int_0^{\pi/2} \cos^{\frac{p+n}{p-1}} \vartheta \sin^{n-2}\vartheta d\vartheta \\ &= \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} B\left(\frac{2p+n-1}{2p-2}, \frac{n-1}{2}\right) = 2\pi^{(n-1)/2}\,\frac{\Gamma\left(\frac{2p+n-1}{2p-2}\right)}{\Gamma\left(\frac{n+1}{2p-2}p\right)},\end{aligned}$$

where $B(u, v)$ is the Beta-function. Passing to the limit in (4.8) as $p \to \infty$, we arrive at the second equality in (4.7).

Combining (2.6) and (4.10), we obtain

$$\mathcal{C}_1 = \sup_{|\boldsymbol{z}|=1}\ \sup_{\sigma\in\mathbb{S}^{n-1}_-} |S^*(\boldsymbol{e}_\sigma)\boldsymbol{z}|(\boldsymbol{e}_\sigma, -\boldsymbol{e}_n)^n = \sup_{\sigma\in\mathbb{S}^{n-1}_-}\ \sup_{|\boldsymbol{z}|=1} \frac{2n}{\omega_n}\big|(\boldsymbol{e}_\sigma, \boldsymbol{z})\big|(\boldsymbol{e}_\sigma, -\boldsymbol{e}_n)^n = \frac{2n}{\omega_n} = \frac{n\Gamma(n/2)}{\pi^{n/2}},$$

which proves the first equality in (4.7). □

5. The Stokes function in a ball

The Stokes function in a ball $\mathbb{B}_r = \{x \in \mathbb{R}^n : |x| < r\}$ is a solution $\boldsymbol{u} \in [C^2(\mathbb{B}_r)]^n \cap [C(\overline{\mathbb{B}}_r)]^n$ of the system

$$\nu\Delta\boldsymbol{u} - \text{grad}\, p = \boldsymbol{0}, \quad \text{div}\, \boldsymbol{u} = c,$$

where c is a constant.

The value of the Stokes function $\boldsymbol{u} \in [C(\partial \mathbb{B}_r)]^n$ at the center of the ball $\mathbb{B}_r$ satisfies (see Kratz [8])

$$\boldsymbol{u}(0) = \frac{n}{2\omega_n r^{n-1}} \int_{\partial \mathbb{B}_r} \mathcal{M}(\sigma)\boldsymbol{u}(\sigma)d\sigma, \tag{5.1}$$

where $\mathcal{M}$ is $n \times n$ matrix valued function on $\partial \mathbb{B}_r$ with the elements

$$-\delta_{ij} + (n+2)(\boldsymbol{e}_\sigma, \boldsymbol{e}_i)(\boldsymbol{e}_\sigma, \boldsymbol{e}_j). \tag{5.2}$$

The estimate of $|\boldsymbol{u}(0)|$ given below contains the integral mean of order p

$$M_p(\boldsymbol{u}; \partial \mathbb{B}_r) = \left\{ \frac{1}{\omega_n r^{n-1}} \int_{\partial \mathbb{B}_r} |\boldsymbol{u}(\sigma)|^p d\sigma \right\}^{1/p}.$$

Proposition 3. *The sharp constant $\mathcal{H}_p$ in the inequality*

$$|\boldsymbol{u}(0)| \le \mathcal{H}_p \; M_p(\boldsymbol{u}; \partial \mathbb{B}_r)$$

is given by

$$\mathcal{H}_1 = \frac{n(n+1)}{2}, \tag{5.3}$$

and

$$\mathcal{H}_p = \frac{n}{2^{1/p}} \left\{ \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)} \int_0^{\pi/2} \Big[1 + n(n+2)\cos^2 \vartheta\Big]^{\frac{p}{2(p-1)}} \sin^{n-2} \vartheta \, d\vartheta \right\}^{\frac{p-1}{p}} \tag{5.4}$$

for $1 < p \le \infty$. In particular,

$$\mathcal{H}_2 = \frac{n\sqrt{n+3}}{2}.$$

Proof. By (5.1) and Proposition 1, the sharp constant in

$$|\boldsymbol{u}(0)| \le \mathcal{H}_p \; M_p(\boldsymbol{u}; \partial \mathbb{B}_r)$$

can be written as

$$\mathcal{H}_p = \frac{n\left(\omega_n r^{n-1}\right)^{1/p}}{2\omega_n r^{n-1}} \sup_{|\boldsymbol{z}|=1} ||\mathcal{M}^* \boldsymbol{z}||_{L_q(\partial \mathbb{B}_r)}. \tag{5.5}$$

Hence, taking into account that

$$|\mathcal{M}^*(\sigma)\boldsymbol{z}| = \Big[1 + n(n+2)(\boldsymbol{e}_\sigma, \boldsymbol{z})^2\Big]^{1/2} \tag{5.6}$$

for any $|\boldsymbol{z}| = 1$ and $p \in (1, \infty]$, we obtain

$$\begin{aligned} \mathcal{H}_p &= \frac{n\left(\omega_n r^{n-1}\right)^{1/p}}{2\omega_n r^{n-1}} \sup_{|\boldsymbol{z}|=1} \left\{ \int_{\partial \mathbb{B}_r} \Big[1 + n(n+2)(\boldsymbol{e}_\sigma, \boldsymbol{z})^2\Big]^{q/2} d\sigma \right\}^{1/q} \\ &= \frac{n\left(\omega_n r^{n-1}\right)^{1/p} r^{(n-1)/q}}{2\omega_n r^{n-1}} \sup_{|\boldsymbol{z}|=1} \left\{ \int_{\mathbb{S}^{n-1}} \Big[1 + n(n+2)(\boldsymbol{e}_\sigma, \boldsymbol{z})^2\Big]^{q/2} d\sigma \right\}^{1/q} \\ &= \frac{n}{2\omega_n^{(p-1)/p}} \left\{ 2\omega_{n-1} \int_0^{\pi/2} \Big[1 + n(n+2)\cos^2 \vartheta\Big]^{\frac{p}{2(p-1)}} \sin^{n-2} \vartheta \, d\vartheta \right\}^{\frac{p-1}{p}}, \end{aligned} \tag{5.7}$$

which implies (5.4).

To get (5.3) we combine (5.5) and (5.6)

$$\mathcal{H}_1 = \frac{\omega_n r^{n-1} n}{2\omega_n r^{n-1}} \sup_{|\boldsymbol{z}|=1} \left[1 + n(n+2)(\boldsymbol{e}_\sigma, \boldsymbol{z})^2\right]^{1/2} = \frac{n(n+1)}{2}. \tag{5.8}$$

$\square$

Remark 1. For instance,

$$\mathcal{H}_1 = 3, \qquad \mathcal{H}_2 = \sqrt{5}, \qquad \mathcal{H}_\infty = \frac{6}{\pi}\, E\left(\frac{2\sqrt{2}}{3} \right)$$

for $n = 2$, and

$$\mathcal{H}_1 = 6, \qquad \mathcal{H}_2 = \frac{3}{2}\sqrt{6}, \qquad \mathcal{H}_\infty = 3 + \frac{\sqrt{15}}{20} \log\left(4 + \sqrt{15}\,\right)$$

for $n = 3$, where E is the complete elliptic integral of the second kind.

Note that the inequality

$$|\boldsymbol{u}(0)| \leq \mathcal{H}_\infty \sup_{\zeta \in \partial\mathcal{B}_1} |\boldsymbol{u}(\zeta)|$$

with the sharp constant

$$\mathcal{H}_\infty = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \int_0^{\pi/2} \left[1 + n(n+2)\cos^2\vartheta\right]^{1/2} \sin^{n-2}\vartheta \, d\vartheta$$

was obtained by Kratz [8].

6. The Lamé system in a half-space

Consider the Lamé system

$$\mu\Delta\boldsymbol{u} + (\lambda+\mu)\mathrm{grad}\ \mathrm{div}\ \boldsymbol{u} = \boldsymbol{0} \ \text{ in } \ \mathbb{R}^n_+, \ \ n \geq 2, \tag{6.1}$$

with the boundary condition

$$\boldsymbol{u}\big|_{x_n=0} = \boldsymbol{f}, \tag{6.2}$$

where λ and μ are the Lamé constants, $\boldsymbol{u} = (u_1, \dots, u_n)$ is the displacement vector of an elastic medium, $\boldsymbol{f} = (f_1, \dots, f_n)$ is a continuous and bounded vector valued function on $\partial\mathbb{R}^n_+$.

The solution $\boldsymbol{u}$ of the Dirichlet problem for the Lamé system in the half-space $\mathbb{R}^n_+$ which is bounded and continuous up to $\partial\mathbb{R}^n_+$ admits the representation (see [10])

$$\boldsymbol{u}(x) = \int_{\partial\mathbb{R}^n_+} L\left(\frac{y-x}{|y-x|}\right) \frac{x_n}{|y-x|^n}\, \boldsymbol{f}(y')dy', \tag{6.3}$$

where $x \in \mathbb{R}^n_+,\ y = (y', 0),\ y' = (y_1, \dots, y_{n-1})$. Here $L(\boldsymbol{e}_\sigma)$ is the $n \times n$ matrix valued function on the sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$ with the elements

$$\frac{2}{\omega_n}\left[(1-\varkappa)\delta_{ij} + n\varkappa(\boldsymbol{e}_\sigma, \boldsymbol{e}_i)(\boldsymbol{e}_\sigma, \boldsymbol{e}_j)\right], \tag{6.4}$$

where $\varkappa = (\lambda+\mu)(\lambda+3\mu)^{-1}$, and ω_n being the area of $\mathbb{S}^{n-1}$. From usual assumptions $\mu > 0,\ 3\lambda + 2\mu > 0$ of elasticity theory, it follows that $0 < \varkappa < 1$.

The uniqueness of solutions of the Dirichlet problem (6.1), (6.2) in the class $[\mathrm{C}^{(2)}(\mathbb{R}^n_+)]^n$ with boundary data from $[L^p(\partial\mathbb{R}^n_+)]^n$ can be derived by means of a standard argument from (6.3) and local estimates for derivatives of solutions of elliptic systems (see [3, 17]).

Theorem 4. *Let x be an arbitrary point in $\mathbb{R}^n_+$ and let $\boldsymbol{z} \in \mathbb{R}^n$. The sharp coefficient $\mathcal{K}_{p,\varkappa}(x)$ in the inequality*

$$|\boldsymbol{u}(x)| \le \mathcal{K}_{p,\varkappa}(x)||\boldsymbol{u}|_{x_n=0}||_p \tag{6.5}$$

for a solution $\boldsymbol{u}$ of the Lamé system is given by

$$\mathcal{K}_{p,\varkappa}(x) = \mathcal{C}_{p,\varkappa}\, x_n^{(1-n)/p}, \tag{6.6}$$

where

$$\mathcal{C}_{1,\varkappa} = \frac{[1+\varkappa(n-1)]\Gamma\left(n/2\right)}{\pi^{n/2}}, \tag{6.7}$$

$$\mathcal{C}_{\infty,\varkappa} = \frac{2\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \int_0^{\pi/2} \Big[(1-\varkappa)^2 + n\varkappa(n\varkappa - 2\varkappa + 2)\cos^2\theta\Big]^{1/2} \sin^{n-2}\theta d\theta, \tag{6.8}$$

and

$$\mathcal{C}_{p,\varkappa} = \frac{2^{1/p}}{\omega_n} \sup_{|\boldsymbol{z}|=1} \left\{ \int_{\mathbb{S}^{n-1}} \Big[(1-\varkappa)^2 + n\varkappa(n\varkappa - 2\varkappa + 2)(\boldsymbol{e}_\sigma, \boldsymbol{z})^2\Big]^{\frac{p}{2(p-1)}} |(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)|^{n/(p-1)} d\sigma \right\}^{(p-1)/p} \tag{6.9}$$

for $1 < p < \infty$.

In particular,

$$\mathcal{C}_{\frac{2k}{2k-1},\varkappa} = \frac{2^{\frac{2k-1}{2k}}}{\omega_n} \left\{ \int_{\mathbb{S}^{n-1}} \Big[(1-\varkappa)^2 + n\varkappa(n\varkappa - 2\varkappa + 2)(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)^2\Big]^k |(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)|^{(2k-1)n} d\sigma \right\}^{1/(2k)}, \tag{6.10}$$

where k is a natural number.

As a particular case of (6.10) *one has*

$$\mathcal{C}_{2,\varkappa} = \left\{ \frac{\Gamma\left(\frac{n}{2}\right)}{2^n \pi^{n/2}} \Big[1 + (n-1)\varkappa^2 + \big(1 + (n-1)\varkappa\big)^2\Big] \right\}^{1/2}. \tag{6.11}$$

Proof. Since the solution of the Dirichlet problem (6.1), (6.2) is given by (6.3), it follows that it obeys Proposition 2. Taking into account that the elements of the matrix L are given by (6.4), we find

$$|L^*(\boldsymbol{e}_\sigma)\boldsymbol{z}| = \frac{2}{\omega_n} \Big[(1-\varkappa)^2 + n\varkappa(n\varkappa - 2\varkappa + 2)(\boldsymbol{e}_\sigma, \boldsymbol{z})^2\Big]^{1/2}, \tag{6.12}$$

which together with (2.4), (2.5) and 2.8) leads to

$$\mathcal{C}_{p,\varkappa} = \frac{2}{\omega_n} \sup_{|\boldsymbol{z}|=1} \left\{ \int_{\mathbb{S}^{n-1}_-} \Big[(1-\varkappa)^2 + n\varkappa(n\varkappa - 2\varkappa + 2)(\boldsymbol{e}_\sigma, \boldsymbol{z})^2\Big]^{\frac{p}{2(p-1)}} (\boldsymbol{e}_\sigma, -\boldsymbol{e}_n)^{n/(p-1)} d\sigma \right\}^{(p-1)/p}$$

for $1 < p \le \infty$. The function $|(\boldsymbol{e}_\sigma, -\boldsymbol{e}_n)|$ is even on the sphere $\mathbb{S}^{n-1}$, therefore the last equality can be written as (6.9).

Passing to the limit in (6.9) as $p \to \infty$, we find

$$\mathcal{C}_{\infty,\varkappa} = \frac{1}{\omega_n} \sup_{|\boldsymbol{z}|=1} \int_{\mathbb{S}^{n-1}} \Big[(1-\varkappa)^2 + n\varkappa(n\varkappa - 2\varkappa + 2)(\boldsymbol{e}_\sigma, \boldsymbol{z})^2\Big]^{1/2} d\sigma,$$

which implies by the independence of the last integral on $\boldsymbol{z}$

$$\mathcal{C}_{\infty,\varkappa} = \frac{2\omega_{n-1}}{\omega_n} \int_0^{\pi/2} \Big[(1-\varkappa)^2 + n\varkappa(n\varkappa - 2\varkappa + 2)\cos^2\vartheta\Big]^{1/2} \sin^{n-2} d\vartheta.$$

Thus (6.8) follows.

By (2.6) and (6.12) we have

$$\begin{aligned}
\mathcal{C}_{1,\varkappa} &= \sup_{|\boldsymbol{z}|=1} \sup_{\sigma\in\mathbb{S}^{n-1}_-} |L^*(\boldsymbol{e}_\sigma)\boldsymbol{z}|(\boldsymbol{e}_\sigma, -\boldsymbol{e}_n)^n \\
&= \sup_{\sigma\in\mathbb{S}^{n-1}_-} \sup_{|\boldsymbol{z}|=1} \frac{2}{\omega_n}\Big[(1-\varkappa)^2 + n\varkappa(n\varkappa-2\varkappa+2)(\boldsymbol{e}_\sigma,\boldsymbol{z})^2\Big]^{1/2}(\boldsymbol{e}_\sigma,-\boldsymbol{e}_n)^n \\
&= \frac{2}{\omega_n}\Big[(1-\varkappa)^2 + n\varkappa(n\varkappa-2\varkappa+2)\Big]^{1/2} = \frac{[1+\varkappa(n-1)]\Gamma(n/2)}{\pi^{n/2}},
\end{aligned}$$

which results in (6.7).

Consider a particular case of (6.9) for $p = 2k(2k-1)^{-1}$. The lower estimate

$$\mathcal{C}_{\frac{2k}{2k-1},\varkappa} \ \geq\ \frac{2^{\frac{2k-1}{2k}}}{\omega_n}\left\{\int_{\mathbb{S}^{n-1}}\Big[(1-\varkappa)^2 + n\varkappa(n\varkappa-2\varkappa+2)(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)^2\Big]^k|(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)|^{(2k-1)n}d\sigma\right\}^{1/(2k)} \tag{6.13}$$

is a direct corollary of (6.9) for $p = 2k(2k-1)^{-1}$. Now we derive an upper estimate for the constant $\mathcal{C}_{\frac{2k}{2k-1},\varkappa}$. By (6.9)

$$\mathcal{C}_{\frac{2k}{2k-1},\varkappa} \ \leq\ \frac{2^{\frac{2k-1}{2k}}}{\omega_n}\left\{\sum_{j=0}^{k}\mathcal{T}_{jk}(\varkappa)\int_{\mathbb{S}^{n-1}}(\boldsymbol{e}_\sigma,\boldsymbol{z})^{2j}|(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)|^{(2k-1)n}d\sigma\right\}^{1/(2k)}, \tag{6.14}$$

where

$$\mathcal{T}_{jk}(\varkappa) = \binom{k}{j}(1-\varkappa)^{2(k-j)}[n\varkappa(n\varkappa-2\varkappa+2)]^j. \tag{6.15}$$

Adopting the notation

$$P_{jk} = \frac{2j+(2k-1)n}{2j}, \qquad Q_{jk} = \frac{2j+(2k-1)n}{(2k-1)n},$$

where $j = 1, 2, \dots, k$, we see that

$$\frac{1}{P_{jk}} + \frac{1}{Q_{jk}} = 1, \tag{6.16}$$

and

$$2j\ P_{jk} = (2k-1)n\ Q_{jk} = 2j + (2k-1)n. \tag{6.17}$$

Taking into account (6.16), we obtain by Hölder's inequality

$$\int_{\mathbb{S}^{n-1}}(\boldsymbol{e}_\sigma,\boldsymbol{z})^{2j}\big|(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\big|^{(2k-1)n}d\sigma \leq \left\{\int_{\mathbb{S}^{n-1}}\big|(\boldsymbol{e}_\sigma,\boldsymbol{z})\big|^{2jP_{jk}}d\sigma\right\}^{1/P_{jk}}\left\{\int_{\mathbb{S}^{n-1}}\big|(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\big|^{(2k-1)nQ_{jk}}d\sigma\right\}^{1/Q_{jk}}.$$

By (6.16), (6.17) as well as by the independence of the first integral in the right-hand side on $\boldsymbol{z}$, we find

$$\int_{\mathbb{S}^{n-1}}(\boldsymbol{e}_\sigma,\boldsymbol{z})^{2j}\big|(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\big|^{(2k-1)n}d\sigma \leq \int_{\mathbb{S}^{n-1}}\big|(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\big|^{2j+(2k-1)n}d\sigma,$$

which together with (6.14) leads to

$$\mathcal{C}_{\frac{2k}{2k-1},\varkappa} \ \leq\ \frac{2^{\frac{2k-1}{2k}}}{\omega_n}\left\{\sum_{j=0}^{k}\mathcal{T}_{jk}(\varkappa)\int_{\mathbb{S}^{n-1}}\big|(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\big|^{2j+(2k-1)n}d\sigma\right\}^{1/(2k)}.$$

Combined with (6.15), this estimate can be written as

$$\mathcal{C}_{\frac{2k}{2k-1},\varkappa} \ \leq\ \frac{2^{\frac{2k-1}{2k}}}{\omega_n}\left\{\int_{\mathbb{S}^{n-1}}\Big[(1-\varkappa)^2 + n\varkappa(n\varkappa-2\varkappa+2)(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)^2\Big]^k|(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)|^{(2k-1)n}d\sigma\right\}^{1/(2k)},$$

which by (6.13) results in (6.10). □

Remark 2. In the case $p = \infty$ we have the sharp constant in the inequality

$$|\boldsymbol{u}(x)| \leq \mathcal{C}_{\infty,\varkappa}\ \sup\{|\boldsymbol{u}(x')| : x' \in \partial\mathbb{R}^n_+\}$$

(see Agmon, Douglis and Nirenberg [1], Agmon [2], Fichera [5], Miranda [14]).

For instance, (6.8) implies the formulas, obtained in [13]:

$$\mathcal{C}_{\infty,\varkappa} = \frac{2}{\pi}(1+\varkappa)E\Big(\frac{2\sqrt{\varkappa}}{1+\varkappa}\Big)$$

for $n = 2$, and

$$\mathcal{C}_{\infty,\varkappa} = \frac{1}{2}\left(1 + 2\varkappa + \frac{(1-\varkappa)^2}{\sqrt{3\varkappa(\varkappa+2)}} \log \frac{1+2\varkappa+\sqrt{3\varkappa(\varkappa+2)}}{1-\varkappa} \right)$$

for $n = 3$, where E is the complete elliptic integral of the second kind.

Remark 3. The constant $\mathcal{C}_{p,\varkappa}$ in the previous assertion is defined by solving an optimization problem on the unit sphere $\mathbb{S}^{n-1}$.

In particular, in the proof of the Theorem 4 is shown that for $p = 2k(2k-1)^{-1}$, where k is a natural number, the supremum of the integral in the representation for $\mathcal{C}_{p,\varkappa}$ is attained on the vectors $\boldsymbol{z} = \boldsymbol{e}_n$ and $\boldsymbol{z} = -\boldsymbol{e}_n$. Note also that (6.10) can be written in the form

$$\begin{aligned}
\mathcal{C}_{\frac{2k}{2k-1},\varkappa} &= \frac{2^{\frac{2k-1}{2k}}}{\omega_n} \left\{ 2\omega_{n-1} \int_0^{\pi/2} \Big[(1-\varkappa)^2 + n\varkappa(n\varkappa - 2\varkappa + 2)\cos^2\vartheta \Big]^k \cos^{(2k-1)n}\vartheta\ \sin^{n-2}\vartheta\ d\vartheta \right\}^{1/(2k)} \\
&= \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{n/2}} \left\{ \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^{\pi/2} \Big[(1-\varkappa)^2 + n\varkappa(n\varkappa - 2\varkappa + 2)\cos^2\vartheta \Big]^k \cos^{(2k-1)n}\vartheta\ \sin^{n-2}\vartheta\ d\vartheta \right\}^{1/(2k)}.
\end{aligned}$$

7. The Lamé system in a ball

The elastic displacement in the center of the ball $\mathbb{B}_r = \{x \in \mathbb{R}^3 : |x| < r\}$ is given by the formula due to Natroshvili [15].

$$\boldsymbol{u}(0) = \frac{3}{4\pi(3-\varkappa)r^2} \int_{\partial\mathbb{B}_r} \mathcal{N}(\sigma)\boldsymbol{u}(\sigma)d\sigma, \tag{7.1}$$

where $\mathcal{N}$ is 3×3 matrix valued function on $\partial\mathbb{B}_r$ with the elements

$$(1-2\varkappa)\delta_{ij} + 5\varkappa(\boldsymbol{e}_\sigma, \boldsymbol{e}_i)(\boldsymbol{e}_\sigma, \boldsymbol{e}_j), \tag{7.2}$$

and

$$\varkappa = (\lambda+\mu)(\lambda+3\mu)^{-1}, \ \text{ and } \ \boldsymbol{u} \in [C(\partial\mathbb{B}_r)]^3.$$

Proposition 4. *The sharp constant $\mathcal{B}_{p,\varkappa}$ in the inequality*

$$|\boldsymbol{u}(0)| \leq \mathcal{B}_{p,\varkappa}\ M_p(\boldsymbol{u}; \partial\mathbb{B}_r)$$

is given by

$$\mathcal{B}_{1,\varkappa} = \frac{3(1+3\varkappa)}{3-\varkappa}, \tag{7.3}$$

$$\mathcal{B}_{\infty,\varkappa} = \frac{3}{2(3-\varkappa)}\left[1+3\varkappa+\frac{(1-2\varkappa)^2}{\sqrt{5\varkappa(\varkappa+2)}}\log\frac{1+3\varkappa+\sqrt{5\varkappa(\varkappa+2)}}{|1-2\varkappa|}\right],$$

and

$$\mathcal{B}_{p,\varkappa} = \frac{3}{3-\varkappa}\left\{\int_0^1\left[(1-2\varkappa)^2+5\varkappa(\varkappa+2)u^2\right]^{\frac{p}{2(p-1)}}du\right\}^{\frac{p-1}{p}} \tag{7.4}$$

for $1<p<\infty$.

In particular,

$$\mathcal{B}_{2,\varkappa} = \frac{3}{3-\varkappa}\big(17\varkappa^2-2\varkappa+3\big)^{1/2}.$$

Proof. According to (7.1) and Proposition 1, the sharp constant in

$$|\boldsymbol{u}(0)| \le \mathcal{B}_{p,\varkappa}\, M_p(\boldsymbol{u};\partial\mathbb{B}_r)$$

can be written as

$$\mathcal{B}_{p,\varkappa} = \frac{3\big(4\pi r^2\big)^{1/p}}{4\pi(3-\varkappa)r^2}\sup_{|\boldsymbol{z}|=1}||\mathcal{N}^*\boldsymbol{z}||_{L_q(\partial\mathbb{B}_r)}. \tag{7.5}$$

Hence, taking into account that

$$|\mathcal{N}^*(\sigma)\boldsymbol{z}| = \Big[(1-2\varkappa)^2+5\varkappa(\varkappa+2)(\boldsymbol{e}_\sigma,\boldsymbol{z})^2\Big]^{1/2}, \tag{7.6}$$

for $|\boldsymbol{z}|=1$ and $p\in(1,\infty]$ we find

$$\begin{aligned}
\mathcal{B}_{p,\varkappa} &= \frac{3\big(4\pi r^2\big)^{1/p}}{4\pi(3-\varkappa)r^2}\sup_{|\boldsymbol{z}|=1}\left\{\int_{\partial\mathbb{B}_r}\Big[(1-2\varkappa)^2+5\varkappa(\varkappa+2)(\boldsymbol{e}_\sigma,\boldsymbol{z})^2\Big]^{q/2}d\sigma\right\}^{1/q}\\
&= \frac{3\big(4\pi r^2\big)^{1/p}r^{2/q}}{4\pi(3-\varkappa)r^2}\sup_{|\boldsymbol{z}|=1}\left\{\int_{\mathbb{S}^2}\Big[(1-2\varkappa)^2+5\varkappa(\varkappa+2)(\boldsymbol{e}_\sigma,\boldsymbol{z})^2\Big]^{q/2}d\sigma\right\}^{1/q}\\
&= \frac{3}{\big(4\pi\big)^{(p-1)/p}(3-\varkappa)}\left\{4\pi\int_0^{\pi/2}\Big[(1-2\varkappa)^2+5\varkappa(\varkappa+2)\cos^2\vartheta\Big]^{\frac{p}{2(p-1)}}\sin\vartheta\, d\vartheta\right\}^{\frac{p-1}{p}},
\end{aligned} \tag{7.7}$$

which implies (7.4).

By (7.5) and (7.6)

$$\mathcal{B}_{1,\varkappa} = \frac{3}{3-\varkappa}\sup_{|\boldsymbol{z}|=1}\Big[(1-2\varkappa)^2+5\varkappa(\varkappa+2)(\boldsymbol{e}_\sigma,\boldsymbol{z})^2\Big]^{1/2} = \frac{3(1+3\varkappa)}{3-\varkappa}, \tag{7.8}$$

which proves (7.3). □

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