Boundedness of solutions to the Schrödinger equation under Neumann boundary conditions

Andrea Cianchi
Dipartimento di Matematica "U.Dini", Università di Firenze
Piazza Ghiberti 27, 50122 Firenze, Italy

Vladimir Maz'ya
Department of Mathematical Sciences, M&O Building
University of Liverpool, Liverpool L69 3BX, UK
and
Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden

Abstract

We deal with Neumann problems for Schrödinger type equations, with non-necessarily bounded potentials, in possibly irregular domains in $\mathbb{R}^n$. Sharp balance conditions between the regularity of the domain and the integrability of the potential for any solution to be bounded are established. The regularity of the domain is described either through its isoperimetric function or its isocapacitary function. The integrability of the sole negative part of the potential plays a role, and is prescribed via its distribution function. The relevant conditions amount to the membership of the negative part of the potential to a Lorentz type space defined either in terms of the isoperimetric function, or of the isocapacitary function of the domain.

1 Introduction

It is our aim to exhibit minimal conditions on the domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and on the potential $V : \Omega \to \mathbb{R}$ ensuring the boundedness of any weak solution u to the Neumann problem for the Schrödinger type equation:

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) + V(x)u = 0 & \text{in } \Omega \\ A(x)\nabla u \cdot \mathbf{n} = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1.1}$$

Here, $\mathbf{n}$ denotes the outward unit normal to $\partial\Omega$, and A is a matrix-valued functions, with essentially bounded coefficients, satisfying for a.e. $x \in \Omega$, the ellipticity condition

$$A(x)\xi \cdot \xi \geq |\xi|^2 \quad \text{for } \xi \in \mathbb{R}^n. \tag{1.2}$$

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The boundedness of solutions is a classical issue in the theory of elliptic PDE's. Fundamental results on this problem can be traced back to [Sta], and to the reference monographs [LU] and [Mo]. Further contributions to this and closely related topics include [ALT, BFM, BMS, BST, Cha, Ci2, Li2, MS, Ma2, Ma6, Mi, Str, Ta1, Ta2, We]. Most of these references, which of course do not exhaust the rich literature on this matter, deal either with local problems, or with Dirichlet boundary value problems; results for Neumann problems in regular domains are also considered, and take a form very similar to those under Dirichlet boundary conditions. The situation is substantially different, both in the results and in the techniques, when Neumann problems in irregular domains are taken into account. Investigations in this generalized framework concerning existence and estimates for solutions are the subject of [ACMM, CM1, CM2, Li1, Ma4, Ma5, MP2]. Related spectral problems for the Neumann Laplacian in irregular domains are analyzed in [BD, DS, HSS, JMS, Si].

In the present paper, neither regularity on Ω, nor on V is a priori assumed. Sharp criteria for the boundedness of the solutions to (1.1) are formulated in terms of a balance between the (ir)regularity of Ω and the degree of integrability of V. The description of the regularity of Ω has a geometric-functional nature, and involves either the perimeter of sets relative to Ω, via its isoperimetric function λ_Ω, or the capacity of sets relative to Ω, via its isocapacitary function ν_Ω. As far as the potential V is concerned, only its negative part $V_- = \frac{1}{2}(|V| - V)$ plays a role. Note that V_- cannot vanish identically for a non-constant solution u to (1.1) to exist. Information on V_- is retained through its distribution function (namely the function measuring its level sets), or, equivalently, through its decreasing rearrangement V_-^*. The interplay to be required between Ω and V_- amounts to the membership of V_- to a classical Lorentz space depending on Ω.

Both the use of the isocapacitary function ν_Ω, and that of the isoperimetric function λ_Ω, lead to optimal criteria for the boundedness of solutions to Neumann problems of the form (1.1) in classes of domains Ω with prescribed behavior of V_-, and either of ν_Ω, or of λ_Ω, respectively. The situation is different when results for specific single domains are in question. For each domain satisfying customary conditions, including Lipschitz and Hölder domains, cusp-shaped domains, funnel-shaped unbounded domains, and John domains, the two approaches yield the same results (Examples 1-5, Section 6). However, there exist highly irregular domains Ω, such as γ-John domains, and certain domains with rooms and passages, to which our criteria involving ν_Ω apply, whereas those formulated in terms of λ_Ω do not, or require a stronger integrability on V_- (Examples 5-6, Section 6). These examples demonstrate how the approach to the regularity of solutions to problem (1.1) relying upon capacity turns out to be well fit to deal with specific domains with complicated geometric configurations, and provides us with precise information that cannot be derived via more standard methods exploiting isoperimetric inequalities.

2 Main results

We hereafter assume that

$$V \in L^1(\Omega),$$

and denote by $W^{1,2}_V(\Omega)$ the weighted Sobolev space

$$W^{1,2}_V(\Omega) = \left\{ u : u \text{ is weakly differentiable in } \Omega, \text{ and } \int_\Omega |\nabla u|^2 + |V(x)|u^2\,dx < \infty \right\}.$$

A function $u \in W^{1,2}_V(\Omega)$ is said to be a weak solution to (1.1) if

$$\int_\Omega A(x)\nabla u \cdot \nabla\phi\,dx + \int_\Omega V(x)u\phi\,dx = 0 \tag{2.1}$$

for every test function $\phi \in W^{1,2}_V(\Omega)$.

Existence of solutions to (1.1) will not be discussed. Let us just recall that, as noted above, a necessary condition for a non trivial (i.e. non-constant) weak solution to (1.1) to exist is that V be negative in a subset of Ω of positive measure. This is easily verified on choosing $\phi = u$ in (2.1), owing to the ellipticity condition (1.2). We thus henceforth assume that V_- does not vanish identically.

Uniqueness of solutions to (1.1) is not requested in what follows. For instance, (1.1) reduces to an eigenvalue problem for the Neumann Laplacian when $A(x)$ equals the identity matrix and $V(x)$ agrees with (a negative) constant in Ω. In this case, it is well known that, at least when Ω is regular enough, the Neumann Laplacian has a discrete spectrum. Hence, if $V(x)$ is constant and equals any of the eigenvalues, there exists a whole eigenspace of associated eigenfunctions u solving (1.1).

Domains, namely connected open sets, Ω such that

$$|\Omega| < \infty$$

are considered throughout. Here, $|\Omega|$ denotes the Lebesgue measure of Ω. The notion of isocapacitary function of Ω coming into play in our results is related to that of condenser capacity, and is a variant of that introduced in [Ma3]. Given sets $E \subset G \subset \Omega$, the capacity $C(E, G)$ of the condenser (E, G) relative to Ω is defined as

$$C(E,G) = \inf \left\{ \int_\Omega |\nabla u|^2\, dx : u \in W^{1,2}(\Omega), u \geq 1 \text{ in } E \text{ and } u \leq 0 \text{ in } \Omega \setminus G \right. \\ \left. \text{(up to sets of standard capacity zero)} \right\}, \tag{2.2}$$

where $W^{1,2}(\Omega)$ denotes the classical Sobolev space.
The isocapacitary function $\nu_\Omega : [0, |\Omega|/2] \to [0, \infty]$ of Ω is given by

$$\nu_\Omega(s) = \inf \{C(E,G) : E \text{ and } G \text{ are measurable subsets of } \Omega \text{ such that} \\ E \subset G \subset \Omega, \text{ and } s \leq |E| \leq |G| \leq |\Omega|/2\} \qquad \text{for } s \in [0, |\Omega|/2]. \tag{2.3}$$

The function ν_Ω is clearly non-decreasing. In what follows, we shall always deal with the left-continuous representative of ν_Ω, which, owing to the monotonicity of ν_Ω, is pointwise dominated by the right-hand side of (2.3).
The very definition of ν_Ω entails that the isocapacitary inequality

$$\nu_\Omega(|E|) \leq C(E, G) \tag{2.4}$$

holds for every measurable subsets of Ω such that $E \subset G \subset \Omega$ and $s \leq |E| \leq |G| \leq |\Omega|/2$.
Let us mention that, in the standard situation when Ω is a Lipschitz domain, one has that

$$\nu_\Omega(s) \approx \begin{cases} s^{\frac{n-2}{n}} & \text{if } n > 2, \\ \left(\log \frac{1}{s}\right)^{-1} & \text{if } n = 2, \end{cases} \tag{2.5}$$

near 0. Here, the notation $f \approx g$ for functions $f, g : (0, \infty) \to [0, \infty)$ stands for $f \preccurlyeq g \preccurlyeq f$, where, in turn, $f \preccurlyeq g$, means that there exists a positive c such that $f(s) \leq cg(cs)$ for $s > 0$. The condition $f \preccurlyeq g$ (and similarly $f \approx g$) is said to hold near 0, or near infinity, if there exist positive constants c and s_0 such that $f(s) \leq cg(cs)$ for $0 < s \leq s_0$ or for $s \geq s_0$, respectively.

For arbitrary domains Ω, the isocapacitary function ν_Ω may have a behavior at 0 different from the right-hand side of (2.5). Such behavior for some classes of domains can be determined - see e.g. [Ma7, Ma8, MP1], and Section 6 below. Heuristically speaking, the more irregular the domain Ω is, the faster $\nu_\Omega(s)$ decays to 0 as $s \to 0^+$.

Our first result provides an essentially weakest possible condition on the isocapacitary function ν_Ω of Ω, and on the decreasing rearrangement V_-^* of the negative part of the potential V for any solution u to (1.1) to be bounded. Such a condition amounts to requiring that V_- belongs to a Lorentz space. Recall that, given an integrable, non-increasing function $\varrho : (0, |\Omega|) \to [0, \infty)$, the classical Lorentz space $\Lambda(\varrho)(\Omega)$ is the Banach function space of those measurable functions $w : \Omega \to \mathbb{R}$ whose norm

$$\|w\|_{\Lambda(\varrho)(\Omega)} = \int_0^{|\Omega|} w^*(s)\varrho(s)\, ds$$

is finite. When $\varrho(s) = s^{-1+\frac{1}{p}}$ for some $p \in [1, \infty)$, the space $\Lambda(\varrho)(\Omega)$ is usually denoted by $L^{p,1}(\Omega)$.

Theorem 2.1 *Assume that $V_- \in \Lambda\big(\frac{1}{\nu_\Omega}\big)(\Omega)$, namely that*

$$\int_0 \frac{V_-^*(s)}{\nu_\Omega(s)}\, ds < \infty. \tag{2.6}$$

Then any weak solution u to (1.1) is essentially bounded, and there exists a constant $C = C(V_-^*, \nu_\Omega)$ such that

$$\|u\|_{L^\infty(\Omega)} \le C\|u\|_{L^2(\Omega)}\, . \tag{2.7}$$

In particular, inequality (2.7) holds with

$$C = \inf_{s \in (0, |\Omega|/2)} \frac{1}{s^{\frac{1}{2}}\Big(1 - \int_0^s \frac{V_-^*(r)}{\nu_\Omega(r)} dr\Big)_+}\, .$$

Note that, since $V_- \neq 0$, condition (2.6) implies that $\sup_{s \in (0, |\Omega|/2)} \frac{s}{\nu_\Omega(s)} < \infty$. Thus, the embedding $W_V^{1,2}(\Omega) \to L^2(\Omega)$ holds as a consequence [Ma7, Theorem 6.6]. Therefore, any weak solution to (1.1) actually belongs to $L^2(\Omega)$ under (2.6), and hence the norm on the right-hand side of (2.7) is finite.

Condition (2.6) is satisfied, in particular, whenever ν_Ω and V_- belong to a pair of mutually associated rearrangement invariant spaces – we refer to [BS, Chapter 2] for a comprehensive treatment of the theory of these spaces. For instance, Theorem 2.1 has the following corollary.

Corollary 2.2 *Let $p \in (1, \infty]$ and let $p' = \frac{p}{p-1}$ (with the usual modification if $p = \infty$). Assume that*

$$\int_0 \frac{ds}{\nu_\Omega(s)^{p'}} < \infty.$$

If $V_- \in L^p(\Omega)$, then the conclusion of Theorem 2.1 holds.

The sharpness of condition (2.6) in classes of domains Ω with prescribed asymptotic behavior of ν_Ω, and in classes of functions V with a prescribed upper bound for V_-^* near 0 is demonstrated by Theorem 2.3 below. The latter relies upon a careful analysis of problem (1.1) in suitable model domains Ω, and tells us that the conclusion of Theorem 2.1 may fail as soon as condition (2.6) is replaced by the slightly weaker assumption that

$$\lim_{s\to 0}\frac{1}{\nu_\Omega(s)}\int_0^s V_-^*(r)\,dr=0. \tag{2.8}$$

The notion of functions from the class Δ_2 is employed in the statement of Theorem 2.3. Recall that a non-decreasing function $f:(0,\infty)\to[0,\infty)$ is said to belong to the class Δ_2 near 0 if there exist constants c and s_0 such that

$$f(2s)\le cf(s) \qquad \text{if } 0<s\le s_0. \tag{2.9}$$

Theorem 2.3 *Let ν be a non-decreasing function of class Δ_2 such that there exists $\alpha>0$ for which*

$$\frac{\nu(s)}{s^\alpha} \quad \approx \text{ a non-decreasing function near } 0. \tag{2.10}$$

Let $h:(0,|\Omega|)\to[0,\infty)$ be a measurable function such that

$$\lim_{s\to 0}\frac{1}{\nu(s)}\int_0^s h^*(r)\,dr=0\,, \tag{2.11}$$

but

$$\int_0 \frac{h^*(s)}{\nu(s)}\,ds=\infty\,. \tag{2.12}$$

Then there exists a an open set $\Omega\subset\mathbb{R}^2$ such that $|\Omega|<\infty$ and

$$\nu_\Omega\approx\nu\,, \tag{2.13}$$

and a measurable function $V:\Omega\to(-\infty,0]$ such that

$$\int_0^s V^*(r)dr \preccurlyeq \int_0^s h^*(r)dr \tag{2.14}$$

(and hence such that (2.11) is fulfilled with h replaced with V) for which problem (1.1) has an unbounded solution u.

Remark 2.4 Assumption (2.10) is consistent with the fact that, by (2.5), the isocapacitary function $\nu_\Omega(s)$ cannot decay more slowly than $(\log\frac{1}{s})^{-1}$ as $s\to 0$, whatever the domain Ω is.

Remark 2.5 Conditions (2.11) and (2.12) are invariant under replacements of ν and h by equivalent (in the sense of $\approx$) functions. Indeed, condition (2.12) is equivalent to the membership of h to the Lorentz space $\Lambda(1/\nu)$; hence, the assertion follows from the boundedness of the dilation operator in r.i. spaces [BS, Chapter 3, Prop. 5.1]. On the other hand, it is easily seen that the assertion concerning (2.11) will follow if we show that (2.11) implies that

$$\lim_{s\to 0}\frac{1}{\nu(s)}\int_0^{cs} h^*(r)\,dr=0$$

for every $c>0$. This implication is a consequence of the fact that, since h is non-increasing,

$$\int_0^{cs} h^*(r)\,dr\le\max\{1,c\}\int_0^s h^*(r)\,dr \qquad \text{for } s\ge 0.$$

The relative isocapacitary inequality (2.4) can be regarded as a variant of the more classical relative isoperimetric inequality, where $P(E,\Omega)$, the perimeter of E relative to Ω, replaces the condenser capacity $C(E,G)$ on the right-hand side. Recall that $P(E;\Omega)$ agrees with $\mathcal{H}^{n-1}(\partial^M E\cap \Omega)$, where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure, and $\partial^M E$ is the essential boundary of E (see e.g. [AFP, Ma7]). According to [Ma1], the isoperimetric function $\lambda_\Omega : [0,|\Omega|/2] \to [0,\infty)$ of Ω is defined as

$$\lambda_\Omega(s) = \inf\{P(E,\Omega) : s \leq |E| \leq |\Omega|/2\} \qquad \text{for } s \in [0,|\Omega|/2]\,. \tag{2.15}$$

Then the relative isoperimetric inequality in Ω takes the form

$$\lambda_\Omega(|E|) \leq P(E;\Omega) \qquad \text{for every measurable set } E \subset \Omega \text{ with } |E| \leq |\Omega|/2.$$

Likewise that of ν_Ω, the asymptotic behavior of λ_Ω at 0 depends on the regularity of Ω. For instance, if Ω is a Lipschitz domain, then

$$\lambda(s) \approx s^{1/n'} \qquad \text{as } s \to 0^+ \tag{2.16}$$

[Ma7, Corollary 3.2.1/3]. Various qualitative and quantitative properties of λ_Ω for domains Ω in different classes are well-known – see e.g. [BuZa, Ci1, HK, KM, La, Ma7], and Section 6 below.

The next result contains a counterpart of Theorem 2.1, where an analogue of condition (2.6), involving λ_Ω instead of ν_Ω, plays a role. In the statement, $\Psi_\Omega : (0,|\Omega|/2) \to [0,\infty)$ denotes the function defined as

$$\Psi_\Omega(s) = \int_s^{|\Omega|/2} \frac{dr}{\lambda_\Omega(r)^2} \qquad \text{for } s \in (0,|\Omega|/2).$$

Theorem 2.6 *Assume that $V_- \in \Lambda(\Psi_\Omega)(\Omega)$, namely that*

$$\int_0 \frac{1}{\lambda_\Omega(s)^2} \int_0^s V_-^*(r)dr\; ds < \infty. \tag{2.17}$$

Then any weak solution u to (1.1) *is essentially bounded, and there exists a constant $C = C(V_-^*, \lambda_\Omega)$ such that*

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{L^2(\Omega)}\,. \tag{2.18}$$

Theorem 2.6 is a straightforward consequence of Theorem 2.1, owing to the inequality

$$\frac{1}{\nu_\Omega(s)} \leq \Psi_\Omega(s) \qquad \text{for } s \in (0,|\Omega|/2), \tag{2.19}$$

which holds for any domain Ω in $\mathbb{R}^n$ of finite measure (see the proof of [Ma7, Proposition 4.3.4/1]).

Assumption (2.17) in Theorem 2.6 is sharp in the same sense as (2.6) is sharp in Theorem 2.1. This is the content of the following result.

Theorem 2.7 *Let λ be a non-decreasing function such that*

$$\frac{\lambda(s)}{s^{1/2}} \approx \textit{a non-decreasing function near } 0. \tag{2.20}$$

Let $h : (0,|\Omega|) \to [0,\infty)$ be a measurable function such that

$$\lim_{s\to 0}\left(\int_s^{|\Omega|/2} \frac{dr}{\lambda(r)^2}\right)\left(\int_0^s h^*(r)dr\right) = 0\,, \tag{2.21}$$

but

$$\int_0 \frac{1}{\lambda(s)^2} \int_0^s h^*(r)dr\, ds = \infty\,. \tag{2.22}$$

Then there exists a an open set $\Omega \subset \mathbb{R}^2$ such that $|\Omega| < \infty$, and

$$\lambda_\Omega \approx \lambda\,,$$

and a measurable function $V : \Omega \to (-\infty, 0]$ such that

$$\int_0^s V^*(r)dr \preccurlyeq \int_0^s h^*(r)dr\,, \tag{2.23}$$

(and hence such that (2.21) *is fulfilled with h replaced with V) for which problem* (1.1) *has an unbounded solution u.*

A remark analogous to Remark 2.4 holds about assumption (2.20) in connection with (2.16). Also, a remark parallel to Remark 2.5 holds concerning the invariance of conditions (2.21) and (2.22) under replacements of λ and h with equivalent functions.

The remaining part of the paper is organized as follows. After recalling a few definitions on rearrangements, stating some properties related to the isoperimetric and the isocapacitary functions, and proving some preliminaries in Section 3, we establish Theorem 2.1 in Section 4. The question of the sharpness of our results is addressed in Section 5. In fact, we limit ourselves to proving Theorem 2.3; a close inspection of its proof will reveal that Theorem 2.7 is in fact established there in passing. The final Section 6 is devoted to a number of applications, as anticipated in Section 1.

3 Background and preliminaries

Given a measurable function $u : \Omega \to \mathbb{R}$, we denote by $\mu_u : \mathbb{R} \to [0, \infty)$ its distribution function defined as

$$\mu_u(t) = |\{x \in \Omega : u(x) \geq t\}| \qquad \text{for } t \in \mathbb{R}.$$

The decreasing rearrangement $u^* : (0, |\Omega|) \to [0, \infty)$ of u is then defined as

$$u^*(s) = \sup\{t : \mu_{|u|}(t) \geq s\} \qquad \text{for } s \in (0, |\Omega|),$$

and the signed decreasing rearrangement $u^\circ : (0, |\Omega|) \to \mathbb{R}$ as

$$u^\circ(s) = \sup\{t : \mu_u(t) \geq s\} \qquad \text{for } s \in (0, |\Omega|).$$

The function $u^{**} : (0, |\Omega|) \to [0, \infty)$ is given by

$$u^{**}(s) = \frac{1}{s} \int_0^s u^*(r)dr \qquad \text{for } s \in (0, |\Omega|). \tag{3.1}$$

One has that

$$u^{**}(s) = \frac{1}{s} \sup_{|E|=s} \int_E |u(x)|dx \qquad \text{for } s \in (0, |\Omega|). \tag{3.2}$$

The Hardy-Littlewood inequality is a basic property of rearrangements, which tells us that

$$\int_\Omega |u(x)v(x)|\,dx \le \int_0^{|\Omega|} u^*(s)v^*(s)\,ds \tag{3.3}$$

for any measurable functions $u\,,v : \Omega \to \mathbb{R}$.

For $p \in [1,\infty]$, we denote by $V^{1,p}(\Omega)$ the Sobolev type space of those weakly differentiable functions $u : \Omega \to \mathbb{R}$ such that $|\nabla u| \in L^p(\Omega)$. Given $u \in V^{1,2}(\Omega)$, let us define the functions ψ_u, $\psi_{u,A} : \mathbb{R} \to \mathbb{R}$ as

$$\psi_u(t) = \int_0^t \frac{d\tau}{\int_{\{u=\tau\}} |\nabla u|\,d\mathcal{H}^{n-1}(x)} \qquad \text{for } t \in \mathbb{R}\,, \tag{3.4}$$

and

$$\psi_{u,A}(t) = \int_0^t \frac{d\tau}{\int_{\{u=\tau\}} \frac{A(x)\nabla u\cdot\nabla u}{|\nabla u|}\,d\mathcal{H}^{n-1}(x)} \qquad \text{for } t \in \mathbb{R}\,, \tag{3.5}$$

respectively. In (3.5), and in what follows, $A(x)$ denotes a Borel representative of $A(x)$. Such a representative exists by a standard result in measure theory (see e.g. [AFP, Exercise 1.3]). By condition (1.2), one has that

$$\psi_{u,A}(t) \le \psi_u(t) \qquad \text{for } t \ge 0. \tag{3.6}$$

Set

$$\mathrm{med}(u) = u^\circ(|\Omega|/2),$$

the median of u. From an easy variant of [Ma7, Lemma 2.2.2/1], one can show that, if

$$\mathrm{med}(u) = 0\,, \tag{3.7}$$

then

$$\nu_\Omega(\mu_u(t)) \le \frac{1}{\psi_u(t)} \quad \text{for } t > 0, \tag{3.8}$$

and hence

$$\nu_\Omega(s) \le \frac{1}{\psi_u(u^\circ(s))} \qquad \text{for } s \in (0, |\Omega|/2). \tag{3.9}$$

Thus, owing to (3.6),

$$\nu_\Omega(s) \le \frac{1}{\psi_{u,A}(u^\circ(s))} \qquad \text{for } s \in (0, |\Omega|/2). \tag{3.10}$$

Note that $u^\circ(s) > 0$ if $s \in (0, |\Omega|/2)$, since (3.7) is in force.

Lemma 3.1 *Let $V \in L^1(\Omega)$, and let $u \in V^{1,1}(\Omega)$. Then the function*

$$(0,|\Omega|) \ni s \mapsto -\int_{\{u>u^\circ(s)\}} V(x)dx$$

is absolutely continuous. Moreover, if $\Phi : (0, |\Omega|) \to \mathbb{R}$ *is the function obeying*

$$\Phi(s) = -\frac{d}{ds} \int_{\{u > u^\circ(s)\}} V(x)dx \qquad \text{for a.e. } s \in (0, |\Omega|), \tag{3.11}$$

then

$$\Phi^{**}(s) \leq V_-^{**}(s) \quad \text{for } s \in (0, |\Omega|). \tag{3.12}$$

Assume, in addition, that $g : \mathbb{R} \to \mathbb{R}$ *is a non-increasing, locally absolutely continuous function such that* $Vg(u) \in L^1(\Omega)$*. Then*

$$-\int_{\{u > u^\circ(s)\}} g(u(x))V(x)dx = \int_0^s g(u^\circ(r))\Phi(r)\, dr \quad \text{for } s \in (0, |\Omega|). \tag{3.13}$$

Proof. Consider any family of pairwise disjoint intervals $(r_k, s_k) \subset (0, |\Omega|)$, $k \in \mathcal{K}$, where $\mathcal{K} \subset \mathbb{N}$. One has that

$$\begin{aligned} \sum_{k \in \mathcal{K}} \left| -\int_{\{u > u^\circ(s_k)\}} V(x)dx + \int_{\{u > u^\circ(r_k)\}} V(x)dx \right| &= \sum_{k \in \mathcal{K}} \left| \int_{\{u^\circ(s_k) < u \leq u^\circ(r_k)\}} V(x)dx \right| \\ &\leq \sum_{k \in \mathcal{K}} \int_{\{u^\circ(s_k) < u \leq u^\circ(r_k)\}} |V(x)|dx. \end{aligned} \tag{3.14}$$

The function $s \mapsto \int_{\{u > u^\circ(s)\}} |V(x)|dx$ is constant in any interval where u° is constant. Thus, if r_k belongs to such an interval, the rightmost side of (3.14) does not change if r_k is replaced by its left endpoint. After such a replacement (if necessary), $|\{u^\circ(s_k) < u \leq u^\circ(r_k)\}| \leq (s_k - r_k)$, whence $|\cup_{k \in \mathcal{K}} \{u^\circ(s_k) < u \leq u^\circ(r_k)\}| \leq \sum_{k \in \mathcal{K}} (s_k - r_k)$. Thus, by the Hardy-Littlewood inequality (3.3),

$$\sum_{k \in \mathcal{K}} \int_{\{u^\circ(s_k) < u \leq u^\circ(r_k)\}} |V(x)|dx \leq \int_0^{\sum_{k \in \mathcal{K}} (s_k - r_k)} V^*(s)ds. \tag{3.15}$$

The absolute continuity of the function $\int_{\{u > u^\circ(s)\}} V(x)dx$ follows from (3.14)–(3.15), since $V^* \in L^1(0, |\Omega|)$, inasmuch as $V \in L^1(\Omega)$.
Note that the same argument applies if V is replaced by any function from $L^1(\Omega)$, and hence the function $s \mapsto \int_{\{u > u^\circ(s)\}} g(u(x))V(x)dx$ is absolutely continuous as well, provided that $Vg(u) \in L^1(\Omega)$. Furhermore, we claim that

$$-\frac{d}{ds} \int_{\{u > u^\circ(s)\}} g(u(x))V(x)dx = g(u^\circ(s))\Phi(s) \quad \text{for a.e. } s \in (0, |\Omega|). \tag{3.16}$$

Indeed, given $s \in (0, |\Omega|)$ and $h > 0$, one has that

$$\begin{aligned} \frac{1}{h}\Big(-\int_{\{u > u^\circ(s+h)\}} g(u(x))V(x)dx &+ \int_{\{u > u^\circ(s)\}} g(u(x))V(x)dx \Big) \\ &= -\frac{1}{h} \int_{\{u^\circ(s+h) < u \leq u^\circ(s)\}} (g(u(x)) - g(u^\circ(s+h)))V(x)dx \\ &\quad - \frac{1}{h} \int_{\{u^\circ(s+h) < u \leq u^\circ(s)\}} g(u^\circ(s+h))V(x)dx. \end{aligned}$$

Since $u \in V^{1,1}(\Omega)$, the function u° is locally absolutely continuous (a.c. for short) in $(0, |\Omega|)$ (see e.g. [CEG, Lemma 6.6]), and hence $g(u^\circ)$ is locally a.c. as well, being the composition of monotone locally a.c. functions. Thus,

$$\begin{aligned}\lim_{h\to 0^+} -\frac{1}{h}\int_{\{u^\circ(s+h)<u\le u^\circ(s)\}} g(u^\circ(s+h))V(x)dx &= -g(u^\circ(s))\frac{d}{ds}\int_{\{u>u^\circ(s)\}} V(x)dx \\ &= g(u^\circ(s))\Phi(s) \qquad \text{for a.e. } s\in(0,|\Omega|);\end{aligned}$$

moreover,

$$\begin{aligned}&\frac{1}{h}\Big|\int_{\{u^\circ(s+h)<u\le u^\circ(s)\}} (g(u(x)) - g(u^\circ(s+h)))V(x)dx\Big| \\ &\qquad\qquad \le \frac{|g(u^\circ(s)) - g(u^\circ(s+h))|}{h}\int_{\{u^\circ(s+h)<u\le u^\circ(s)\}} |V(x)|dx,\end{aligned}$$

whence

$$\lim_{h\to 0^+} \frac{1}{h}\int_{\{u^\circ(s+h)<u\le u^\circ(s)\}} (g(u(x)) - g(u^\circ(s+h)))V(x)dx = 0 \quad \text{for a.e. } s\in(0,|\Omega|).$$

Altogether, we obtain equation (3.16), and hence, via integration, (3.13).
Let us now focus on (3.12). By property (3.2), it suffices to show that

$$\int_E |\Phi(r)|dr \le \int_0^{|E|} V_-^*(r)dr \qquad \text{for every measurable set } E\subset(0,|\Omega|). \tag{3.17}$$

First, it is easily verified via the very definition of derivative that

$$\Big| -\frac{d}{ds}\int_{\{u>u^\circ(s)\}} V(x)dx\Big| \le \frac{d}{ds}\int_{\{u>u^\circ(s)\}} V_-(x)dx \quad \text{for a.e. } s\in(0,|\Omega|). \tag{3.18}$$

Next observe that we may limit ourselves to proving (3.17) in the case when E is an open set, since any measurable set can be approximated from outside by open sets. Thus, we may assume that $E = \cup_{k\in\mathcal{K}}(r_k, s_k)$, where $\mathcal{K}\subset\mathbb{N}$ and the intervals (r_k, s_k) are pairwise disjoint. One has that

$$\begin{aligned}\int_E |\Phi(r)|dr &= \sum_{k\in\mathcal{K}}\int_{r_k}^{s_k}|\Phi(r)|dr \le \sum_{k\in\mathcal{K}}\int_{r_k}^{s_k}\Big(\frac{d}{dr}\int_{\{u>u^\circ(r)\}} V_-(x)dx\Big)dr \\ &= \sum_{k\in\mathcal{K}}\int_{\{u^\circ(s_k)<u<u^\circ(r_k)\}} V_-(x)dx = \int_{\cup_{k\in\mathcal{K}}\{u^\circ(s_k)<u<u^\circ(r_k)\}} V_-(x)dx\,,\end{aligned} \tag{3.19}$$

where the inequality holds by (3.18). Note that integration in the last two integrals is extended just over $\{u^\circ(s_k) < u < u^\circ(r_k)\}$, instead of $\{u^\circ(s_k) < u \le u^\circ(r_k)\}$, inasmuch as $\frac{d}{dr}\int_{\{u>u^\circ(r)\}} V_-(x)dx = 0$ in any interval where u° is constant. Since $|\{u^\circ(s_k) < u < u^\circ(r_k)\}| \le (s_k - r_k)$,

$$|\cup_{k\in\mathcal{K}}\{u^\circ(s_k) < u < u^\circ(r_k)\}| \le \sum_{k\in\mathcal{K}}(s_k - r_k) = |E|.$$

Thus, by the Hardy-Littlewood inequality (3.3) again,

$$\int_{\cup_{k\in\mathcal{K}}\{u^\circ(s_k)<u<u^\circ(r_k)\}} V_-(x)dx \le \int_0^{|E|} V_-^*(r)dr. \tag{3.20}$$

Inequality (3.17) follows from (3.19) and (3.20). □

4 Boundedness of solutions

Here, we establish Theorem 2.1.

Proof of Theorem 2.1. Given $s \in (0, |\Omega|)$ and $h > 0$, choose the test function ϕ defined as

$$\phi(x) = \begin{cases} 0 & \text{if } \ u(x) < u^\circ(s+h) \\ u(x) - u^\circ(s+h) & \text{if } \ u^\circ(s+h) \le u(x) \le u^\circ(s) \\ u^\circ(s) - u^\circ(s+h) & \text{if } \ u^\circ(s) < u(x)\,, \end{cases} \tag{4.1}$$

in equation (2.1). Notice that $\phi \in W^{1,2}_V(\Omega)$, since $\phi \in V^{1,2}(\Omega)$ by standard results on the truncation of Sobolev functions, and $\int_\Omega |V(x)|\phi^2 dx \le \int_\Omega |V(x)|u^2 dx < \infty$. We thus obtain that

$$\begin{aligned} &\int_{\{u^\circ(s+h)<u<u^\circ(s)\}} A(x)\nabla u \cdot \nabla u\, dx \\ &= - \int_{\{u^\circ(s+h)<u\le u^\circ(s)\}} u(x)\big(u(x){-}u^\circ(s{+}h)\big)\, V(x) dx - \big(u^\circ(s){-}u^\circ(s{+}h)\big) \int_{\{u>u^\circ(s)\}} u(x)\, V(x) dx\,. \end{aligned} \tag{4.2}$$

Consider the function $U : (0, |\Omega|) \to [0, \infty)$ given by

$$U(s) = \int_{\{u\le u^\circ(s)\}} A(x)\nabla u \cdot \nabla u\, dx \qquad \text{for } s \in (0, |\Omega|). \tag{4.3}$$

The function u° is locally a.c. in $(0, |\Omega|)$, owing to [CEG, Lemma 6.6]. The function

$$(0, \infty) \ni t \mapsto \int_{\{u\le t\}} A(x)\nabla u \cdot \nabla u\, dx$$

is also locally a.c., inasmuch as, by the coarea formula,

$$\int_{\{u\le t\}} A(x)\nabla u \cdot \nabla u\, dx = \int_{-\infty}^{t} \int_{\{u=\tau\}} \frac{A(x)\nabla u \cdot \nabla u}{|\nabla u|}\, d\mathcal{H}^{n-1}(x) d\tau \qquad \text{for } t \in \mathbb{R}. \tag{4.4}$$

Consequently, U is locally a.c., for it is the composition of monotone locally a.c. functions, and

$$U'(s) = u^{\circ\prime}(s) \int_{\{u=u^\circ(s)\}} \frac{A(x)\nabla u \cdot \nabla u}{|\nabla u|}\, d\mathcal{H}^{n-1}(x) \qquad \text{for a.e. } s \in (0, |\Omega|). \tag{4.5}$$

Thus, dividing through by h in (4.2), and passing to the limit as $h \to 0^+$ yield

$$\begin{aligned} &- u^{\circ\prime}(s) \int_{\{u=u^\circ(s)\}} \frac{A(x)\nabla u \cdot \nabla u}{|\nabla u|} d\mathcal{H}^{n-1}(x) \\ &\qquad = \ -u^{\circ\prime}(s)\Big(- \int_{\{u>u^\circ(s)\}} u(x)V(x)dx\Big) \quad \text{for a.e. } s \in (0, |\Omega|). \end{aligned} \tag{4.6}$$

Owing to (3.5), and (3.13) with $g(t) = t$, inequality (4.6) takes the form

$$- u^{\circ\prime}(s) = \big(- \psi_{u,A}(u^\circ(s))\big)' \int_0^s u^\circ(\varrho)\Phi(\varrho)\, d\varrho \qquad \text{for a.e. } s \in (0, |\Omega|). \tag{4.7}$$

Let $0 < s \le \varepsilon \le |\Omega|/2$. Via integration in (4.7), one obtains that

$$u^\circ(s) - \int_s^\varepsilon \Big(\int_0^r u^\circ(\varrho)\Phi(\varrho) d\varrho\Big) \big(- \psi_{u,A}(u^\circ(r))\big)' dr = u^\circ(\varepsilon) \qquad \text{for } s \in (0, \varepsilon). \tag{4.8}$$

Define the operator T as

$$Tf(s) = \int_s^{\varepsilon} \left(\int_0^r f(\varrho)\Phi(\varrho)d\varrho \right) \big(-\psi_{u,A}(u^\circ(r))\big)' \, dr \qquad \text{for } s \in (0,\varepsilon), \tag{4.9}$$

for an integrable function f in $(0,\varepsilon)$. Then, equation (4.8) reads

$$(I-T)(u^\circ) = u^\circ(\varepsilon)\,. \tag{4.10}$$

Set

$$v = u - \operatorname{med}(u),$$

and observe that $\operatorname{med}(v)=0$, and

$$v^\circ = u^\circ - \operatorname{med}(u).$$

In particular,

$$v^\circ(s) \geq 0 \qquad \text{if } s \in (0, |\Omega|/2). \tag{4.11}$$

Given $s\in(0,\varepsilon)$, one has that

$$\begin{aligned}
|Tf(s)| &= \left| \int_s^{\varepsilon} \left(\int_0^r f(\varrho)\Phi(\varrho)d\varrho \right) \big(-\psi_{u,A}(u^\circ(r))\big)' \, dr \right| \\
&= \left| \int_s^{\varepsilon} \left(\int_0^r f(\varrho)\Phi(\varrho)d\varrho \right) \big(-\psi_{v,A}(v^\circ(r))\big)' \, dr \right| \\
&\qquad\qquad\qquad \text{(since } \big(\psi_{u,A}(u^\circ)\big)' = \big(\psi_{v,A}(v^\circ)\big)' \text{)} \\
&\leq \int_s^{\varepsilon} \left(\int_0^r |f(\varrho)||\Phi(\varrho)|d\varrho \right) \big(-\psi_{v,A}(v^\circ(r))\big)' \, dr \\
&= \left(\int_s^{\varepsilon} \big(-\psi_{v,A}(v^\circ(r))\big)' \, dr \right) \int_0^s |f(\varrho)||\Phi(\varrho)|d\varrho \\
&\quad + \int_s^{\varepsilon} \left(\int_\varrho^{\varepsilon} \big(-\psi_{v,A}(v^\circ(r))\big)' dr \right) |f(\varrho)||\Phi(\varrho)|d\varrho \\
&\qquad\qquad\qquad \text{(by Fubini's theorem)} \\
&= \big(\psi_{v,A}(v^\circ(s)) - \psi_{v,A}(v^\circ(\varepsilon))\big) \int_0^s |f(\varrho)|\Phi(\varrho)||d\varrho \\
&\quad + \int_s^{\varepsilon} \big(\psi_{v,A}(v^\circ(\varrho)) - \psi_{v,A}(v^\circ(\varepsilon))\big)|f(\varrho)||\Phi(\varrho)|d\varrho \\
&\leq \psi_{v,A}(v^\circ(s)) \int_0^s |f(\varrho)|\Phi(\varrho)||d\varrho + \int_s^{\varepsilon} \psi_{v,A}(v^\circ(\varrho))|f(\varrho)||\Phi(\varrho)|d\varrho \\
&\qquad\qquad\qquad (\psi_{v,A}(v^\circ(\varepsilon)) \geq 0 \text{ by (4.11)})
\end{aligned} \tag{4.12}$$

$$\leq \frac{1}{\nu_\Omega(s)} \int_0^s |f(\varrho)||\Phi(\varrho)|d\varrho + \int_s^\varepsilon |f(\varrho)|\frac{|\Phi(\varrho)|}{\nu_\Omega(\varrho)}d\varrho$$

(by (3.10) with u replaced with v).

Thus,

$$\begin{aligned} \|Tf\|_{L^\infty(0,\varepsilon)} &\leq \|f\|_{L^\infty(0,\varepsilon)} \sup_{s\in(0,\varepsilon)} \left(\frac{1}{\nu_\Omega(s)} \int_0^s |\Phi(\varrho)|d\varrho + \int_s^\varepsilon \frac{|\Phi(\varrho)|}{\nu_\Omega(\varrho)}d\varrho\right) \\ &\leq \|f\|_{L^\infty(0,\varepsilon)} \sup_{s\in(0,\varepsilon)} \left(\int_0^s \frac{|\Phi(\varrho)|}{\nu_\Omega(\varrho)}d\varrho + \int_s^\varepsilon \frac{|\Phi(\varrho)|}{\nu_\Omega(\varrho)}d\varrho\right) \\ &= \|f\|_{L^\infty(0,\varepsilon)} \int_0^\varepsilon \frac{|\Phi(\varrho)|}{\nu_\Omega(\varrho)}d\varrho \leq \|f\|_{L^\infty(0,\varepsilon)} \int_0^\varepsilon \frac{V_-^*(\varrho)}{\nu_\Omega(\varrho)}d\varrho\,, \end{aligned} \tag{4.13}$$

where the second inequality holds since the function $1/\nu_\Omega$ is non increasing, and the last one by (3.12), owing to Hardy's Lemma [BS, Proposition 3.6, Chapter 2]. Hence, by (2.6), the operator $I - T$ is bounded from $L^\infty(0,\varepsilon)$ into $L^\infty(0,\varepsilon)$. Moreover,

$$\int_0^\varepsilon \frac{V_-^*(\varrho)}{\nu_\Omega(\varrho)}d\varrho < 1, \tag{4.14}$$

provided that ε is sufficiently small. For such a choice of ε, by a classical result of functional analysis, the restriction $(I-T)_\infty$ of $I-T$ to $L^\infty(0,\varepsilon)$,

$$(I-T)_\infty : L^\infty(0,\varepsilon) \to L^\infty(0,\varepsilon), \tag{4.15}$$

is invertible, with a bounded inverse, and

$$\|(I-T)_\infty^{-1}\| \leq \frac{1}{1 - \int_0^\varepsilon \frac{V_-^*(\varrho)}{\nu_\Omega(\varrho)}d\varrho}\,. \tag{4.16}$$

Let us now show that an analogous conclusion holds for the restriction $(I-T)_2$ of $I-T$ to $L^2((0,\varepsilon), |\Phi(s)|ds)$, where $L^2((0,\varepsilon), |\Phi(s)|ds)$ denotes the weighted Lebesgue space of those measurable functions in $(0,\Omega)$ with are square summable with respect to the measure $|\Phi(s)|ds$. Owing to (4.12), this will follow if we show that there exists a constant $C(\varepsilon)$ such that

$$\left(\int_0^\varepsilon \left(\frac{1}{\nu_\Omega(s)} \int_0^s |f(r)||\Phi(r)|dr\right)^2 |\Phi(s)|ds\right)^{1/2} \leq C(\varepsilon)\left(\int_0^\varepsilon |f(s)|^2|\Phi(s)|ds\right)^{1/2} \tag{4.17}$$

and

$$\left(\int_0^\varepsilon \left(\int_s^\varepsilon |f(r)|\frac{|\Phi(r)|}{\nu_\Omega(r)}\,dr\right)^2 |\Phi(s)|ds\right)^{1/q} \leq C(\varepsilon)\left(\int_0^\varepsilon |f(s)|^2|\Phi(s)|ds\right)^{1/q} \tag{4.18}$$

for every $f \in L^2((0,\varepsilon), |\Phi(s)|ds)$, and that $C(\varepsilon)$ can be made arbitrarily small, provided that ε is sufficiently small. A characterization of one-dimensional weighted Hardy type inequalities (see e.g. [Ma8, OK]) tells us that (4.17) and (4.18) hold provided that

$$\sup_{s\in(0,\varepsilon)} \left(\int_0^s |\Phi(r)|dr\right)^{\frac{1}{2}} \left(\int_s^\varepsilon \frac{|\Phi(r)|}{\nu_\Omega(r)^2}dr\right)^{\frac{1}{2}} < \infty\,, \tag{4.19}$$

and that the constant $C(\varepsilon)$ in (4.17) and (4.18) is equivalent (up to absolute multiplicative constants) to the left-hand side of (4.19). Since the function $1/\nu_\Omega$ is non increasing,

$$\begin{aligned}\left(\int_0^s |\Phi(r)|\,dr\right)^{\frac{1}{2}} \left(\int_s^\varepsilon \frac{|\Phi(r)|}{\nu_\Omega(r)^2}\,dr\right)^{\frac{1}{2}} &\leq \left(\int_0^s \frac{|\Phi(r)|}{\nu_\Omega(r)}\,dr\right)^{\frac{1}{2}} \left(\int_s^\varepsilon \frac{|\Phi(r)|}{\nu_\Omega(r)}\,dr\right)^{\frac{1}{2}} \\ &\leq \int_0^\varepsilon \frac{|\Phi(r)|}{\nu_\Omega(r)}\,dr \leq \int_0^\varepsilon \frac{V_-^*(r)}{\nu_\Omega(r)}\,dr \quad \text{if } s\in(0,\varepsilon).\end{aligned} \tag{4.20}$$

Thus, there exists an absolute constant C such that, if ε is so small that

$$C\int_0^\varepsilon \frac{V^*(r)}{\nu_\Omega(r)}dr < 1, \tag{4.21}$$

then

$$(I-T)_2 : L^2((0,\varepsilon), |\Phi(s)|ds) \to L^2((0,\varepsilon), |\Phi(s)|ds) \tag{4.22}$$

is invertible, with a bounded inverse, and

$$\|(I-T)_2^{-1}\| \leq \frac{1}{1 - C\int_0^\varepsilon \frac{V_-^*(r)}{\nu_\Omega(r)}dr}\,. \tag{4.23}$$

Since $u \in W_V^{1,2}(\Omega)$, we have that $\int_\Omega u(x)^2|V(x)|dx < \infty$. Thus, equation (3.13) with $g(t) = t^2$ entails that $u^\circ \in L^2((0,\varepsilon), |\Phi(s)|ds)$. Furthermore, $u^\circ(\varepsilon) \in L^\infty(0,\varepsilon) \subset L^2((0,\varepsilon), |\Phi(s)|ds)$. Thereby, from (4.10) we deduce that

$$u^\circ = (I-T)_2^{-1}(u^\circ(\varepsilon)) = (I-T)_\infty^{-1}(u^\circ(\varepsilon))\,. \tag{4.24}$$

Hence, $u^\circ \in L^\infty(0,\varepsilon)$, and

$$\|u^\circ\|_{L^\infty(0,\varepsilon)} \leq \frac{|u^\circ(\varepsilon)|}{1 - \int_0^\varepsilon \frac{V_-^*(r)}{\nu_\Omega(r)}dr}\,. \tag{4.25}$$

It is easily verified that

$$\|u\|_{L^2(\Omega)} \geq \|u^\circ\|_{L^2(0,\varepsilon)} \geq \varepsilon^{1/2}|u^\circ(\varepsilon)|\,. \tag{4.26}$$

Thus

$$u^\circ(0) \leq \|u^\circ\|_{L^\infty(0,\varepsilon)} \leq \frac{\|u\|_{L^2(\Omega)}}{\varepsilon^{\frac{1}{2}}\left(1 - \int_0^\varepsilon \frac{V_-^*(r)}{\nu_\Omega(r)}dr\right)}\,. \tag{4.27}$$

The same argument, applied to $-u$ (which is also a solution to problem (1.1)), yields a parallel estimate for $-u^\circ(|\Omega|)$. Since

$$\|u\|_{L^\infty(M)} = \max\{u^\circ(0), -u^\circ(|\Omega|)\},$$

inequality (2.7) easily follows. □

5 Sharpness of conditions

This section is concerned with the proof of Theorem 2.3. The construction of the set Ω in Theorem 2.3 relies upon the next result.

Theorem 5.1 *Let Ω be a domain in $\mathbb{R}^2$ of the form:*

$$\Omega = \{(x,y) \in \mathbb{R}^2 : x \in \mathbb{R}, |y| < \varphi(x)\},$$

where $\varphi : \mathbb{R} \to (0,\infty)$ is an even, twice continuously differentiable function, such that φ'' is bounded, φ is decreasing and convex near ∞, and $\int_{\mathbb{R}} \varphi(\varrho)d\varrho < \infty$. Define $B : (0,\infty) \to [0,\infty)$ as

$$B(r) = 2\int_r^\infty \varphi(\varrho)d\varrho \quad \textit{for } r > 0. \tag{5.1}$$

Then

$$\lambda_\Omega(s) \approx \varphi(B^{-1}(s)) \quad \textit{near } 0, \tag{5.2}$$

and

$$\nu_\Omega(s) \approx \frac{1}{\int_{B^{-1}(|\Omega|/2)}^{B^{-1}(s)} \frac{dr}{\varphi(r)}} \quad \textit{near } 0. \tag{5.3}$$

Moreover, if $h : (0,|\Omega|) \to [0,\infty)$ is a measurable function such that

$$\lim_{s\to 0} \frac{1}{\nu_\Omega(s)} \int_0^s h^*(r)\, dr = 0\,, \tag{5.4}$$

but

$$\int_0 \frac{h^*(s)}{\nu_\Omega(s)}\, ds = \infty\,, \tag{5.5}$$

then there exists a measurable function $V : \Omega \to (-\infty,0]$ such that

$$\int_0^s V^*(r)\, dr \precsim \int_0^s h^*(r)\, dr \qquad \textit{for } s \in (0,|\Omega|), \tag{5.6}$$

and problem (1.1) has an unbounded solution u.

Proof . Equations (5.2) and (5.3) follow from a variant of the arguments of [Ma7, 4.3.5/1] as, for instance, in the proof of [CM2, Theorem 4.1].
Assume now that (5.4) and (5.5) are in force. Define the function $\vartheta : [0,\infty) \to [0,\infty)$ as

$$\vartheta(r) = \int_0^r \frac{d\varrho}{\varphi(\varrho)} \qquad \text{for } r \geq 0, \tag{5.7}$$

and $q : [0,\infty) \to [0,\infty)$ as

$$q(s) = h^*(B(\vartheta^{-1}(s)))\varphi(\vartheta^{-1}(s))^2 \quad \text{for } s \geq 0. \tag{5.8}$$

Owing to (5.3), equations (5.4) and (5.5) are equivalent to

$$\lim_{s\to\infty} s\int_s^\infty q(r)dr = 0\,, \tag{5.9}$$

and

$$\int^\infty sq(s)\,ds = \infty\,, \tag{5.10}$$

respectively. Assumption (5.9) implies the compactness of the embedding of the one-dimensional weighted Sobolev space

$$W^{1,2}_{q,0}(0,\infty) = \big\{\omega \in W^{1,2}_q(0,\infty) : \omega(0) = 0\big\}$$

into $L^2((0,\infty), q(s)ds)$ – see e.g. [OK, Theorem 7.3]. Hence, by a standard variational argument making use of a weighted Reileigh quotient, there exists $\gamma > 0$ such that the problem

$$\begin{cases} \omega'' + \gamma q(s)\omega = 0 & \text{for } s \in (0,\infty), \\ \omega(0) = 0 & \end{cases} \tag{5.11}$$

has a (weak) solution in ω in $W^{1,2}_{q,0}(0,\infty)$. Moreover, owing to (5.10), any solution to (5.11) can be shown to be unbounded [CM2, Proof of Theorem 2.4]. Let us continue ω and q to the whole of $\mathbb{R}$ on setting $\omega(s) = -\omega(-s)$ and $q(s) = -q(-s)$ if $s < 0$.
Now, let

$$G = \{(s,t) : s \in \mathbb{R}, |t| < \pi/2\}.$$

By [Wa], there exists a conformal mapping $\zeta : \Omega \to G$ of the form

$$\zeta(x,y) = (\xi(x,y), \eta(x,y)) \quad \text{for } (x,y) \in \Omega,$$

where the functions $\xi, \eta : \Omega \to \mathbb{R}$ satisfy the Cauchy-Riemann equations

$$\xi_x = \eta_y, \quad \xi_y = -\eta_x \quad \text{in } \Omega. \tag{5.12}$$

Owing to the symmetry of Ω under reflections about the x-axis and the y-axis, we have that $\xi(x,y) = -\xi(-x,y)$ and $\eta(x,y) = -\eta(x,-y)$.
Consider the function $u : \Omega \to \mathbb{R}$ defined as

$$u(x,y) = \omega(\xi(x,y)) \quad \text{for } (x,y) \in \Omega,$$

and the function $V : \Omega \to (-\infty, 0]$ given by

$$V(x,y) = -\gamma|\nabla\xi(x,y)|^2 q(\xi(x,y)) \quad \text{for } (x,y) \in \Omega.$$

On making use of (5.11) and (5.12), one can easily verify that u is an unbounded weak solution to the Neumann problem

$$\begin{cases} -\Delta u + V(x,y)u = 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{5.13}$$

In particular, note that $u \in W^{1,2}_V(\Omega)$, as a consequence of the fact that

$$\begin{aligned}\int_\Omega |\nabla u|^2 + |V(x,y)|u^2\,dxdy &= \int_\Omega \big(\omega'(\xi(x,y))^2 + \gamma\omega(\xi(x,y))^2 q(\xi(x,y))\big)|\nabla\xi(x,y)|^2\,dxdy \\ &= 2\pi\int_0^\infty \omega'(s)^2 + \gamma\omega(s)^2 q(s)\,ds < \infty\,.\end{aligned}$$

It remains to prove (5.6). To this purpose, recall that the function η fulfills

$$\begin{cases}\Delta\eta = 0 \quad \text{in } \Omega,\\ \eta = \pi/2 \quad \text{on } \{y = \varphi(x)\},\\ \eta = -\pi/2 \quad \text{on } \{y = -\varphi(x)\}.\end{cases} \tag{5.14}$$

Given $x_0 > 0$, define the function $\rho_{x_0} : \mathbb{R} \to (0,\infty)$ as

$$\rho_{x_0}(x') = \frac{\varphi(\varphi(x_0)x' + x_0)}{\varphi(x_0)} \qquad \text{for } x' \in \mathbb{R},$$

and set

$$Q_{x_0} = \{(x,y) : |x - x_0| < \varphi(x_0), |y| < \varphi(x)\},$$
$$Q'_{x_0} = \Big\{(x',y') : |x'| < \frac{1}{2}, |y'| < \rho_{x_0}(x')\Big\},$$

and

$$Q''_{x_0} = \Big\{(x',y') : |x'| < 1, |y'| < \rho_{x_0}(x')\Big\}.$$

Note that Q_{x_0} is mapped into Q''_{x_0} under the affine change of variables $x = \varphi(x_0)x' + x_0$, $y = \varphi(x_0)y'$.
Let $\widehat{x}$ be such that φ is convex and decreasing in $(\widehat{x},\infty)$. We claim that there exist constants $\overline{x} \geq \widehat{x}$, $a > 0$ and $b > 0$ such that

$$a \leq \rho_{x_0}(x') \leq b \quad \text{if } x_0 \geq \overline{x} \text{ and } |x'| < 1. \tag{5.15}$$

Since φ is decreasing in $(\widehat{x},\infty)$, in order to prove (5.15) it suffices to show that

$$a \leq \frac{\varphi(x_0 + \varphi(x_0))}{\varphi(x_0)} \qquad \text{if } x_0 \geq \overline{x}, \tag{5.16}$$

and

$$\frac{\varphi(x_0 - \varphi(x_0))}{\varphi(x_0)} \leq b \qquad \text{if } x_0 \geq \overline{x}. \tag{5.17}$$

To verify (5.16), note that, if $x \geq x_0 \geq \widehat{x}$, then

$$\varphi(x) \geq \varphi(x_0) + \varphi'(x_0)(x - x_0),$$

inasmuch as φ is convex in $(\widehat{x},\infty)$. Hence

$$\varphi(x_0 + \varphi(x_0)) \geq \varphi(x_0) + \varphi'(x_0)\varphi(x_0) = \varphi(x_0)(1 + \varphi'(x_0)) \geq \frac{1}{2}\varphi(x_0)$$

if x_0 is sufficiently large, since $\lim_{x\to\infty}\varphi'(x) = 0$ by our assumptions on φ.

As for (5.17), for each $x < x_0$, there exists $c \in (x, x_0)$ such that

$$\varphi(x) = \varphi(x_0) + \varphi'(c)(x - x_0).$$

If $x \geq \widehat{x}$, then φ' is increasing in (x, ∞). Thus

$$\varphi(x) \leq \varphi(x_0) + \varphi'(x)(x - x_0) \tag{5.18}$$

if $\widehat{x} \leq x < x_0$. Since $\lim_{x\to\infty} \varphi(x) = 0$, we have that $x_0 - \varphi(x_0) \geq \widehat{x}$ provided that x_0 is sufficiently large. An application of (5.18) with $x = x_0 - \varphi(x_0)$ then yields

$$\varphi(x_0 - \varphi(x_0)) \leq \varphi(x_0) - \varphi'(x_0 - \varphi(x_0))\varphi(x_0) = \varphi(x_0)(1 - \varphi'(x_0 - \varphi(x_0))) \leq \varphi(x_0)(1 + \max|\varphi'|),$$

if x_0 is sufficiently large, whence (5.17) follows.
Let us next note that, since

$$\rho'_{x_0}(x') = \varphi'(\varphi(x_0)x' + x_0) \qquad \text{for } x' \in \mathbb{R}^n,$$

and

$$\rho''_{x_0}(x') = \varphi(x_0)\varphi''(\varphi(x_0)x' + x_0) \qquad \text{for } x' \in \mathbb{R}^n,$$

we have that

$$\text{the functions } \rho'_{x_0} \text{ and } \rho''_{x_0} \text{ are bounded, uniformly in } x_0, \tag{5.19}$$

owing to our assumptions on φ.
Now, let $T : Q''_{x_0} \to \mathbb{R}$ be the function defined by

$$T(x', y') = \eta(\varphi(x_0)x' + x_0, \varphi(x_0)y') \quad \text{for } (x', y') \in Q''_{x_0}.$$

Then

$$\begin{cases} \Delta T = 0 \quad \text{in } Q''_{x_0}, \\ T = \pi/2 \quad \text{on } \{y' = \frac{\varphi(\varphi(x_0)x'+x_0)}{\varphi(x_0)}\}, \\ T = -\pi/2 \quad \text{on } \{y' = -\frac{\varphi(\varphi(x_0)x'+x_0)}{\varphi(x_0)}\}. \end{cases} \tag{5.20}$$

By standard gradient estimates for harmonic functions,

$$\|\nabla T\|_{L^\infty(Q'_{x_0})} \leq C\|T\|_{L^\infty(Q''_{x_0})} \leq C\|\eta\|_{L^\infty(\Omega)} = C', \tag{5.21}$$

where the constants C and C' are independent of $x_0 > \overline{x}$, owing to (5.15) and (5.19). By (5.21),

$$|\nabla\xi(x_0, y_0)| = |\nabla\eta(x_0, y_0)| \leq \frac{1}{\varphi(x_0)}\|\nabla T\|_{L^\infty(Q'_{x_0})} \leq \frac{C'}{\varphi(x_0)}, \tag{5.22}$$

if $x_0 \geq \overline{x}$ and $|y| < \varphi(x_0)$, and hence, in particular,

$$|\nabla\xi(x, y)|^2 \leq \frac{C'^2}{\varphi(x)^2} \tag{5.23}$$

if $(x, y) \in \Omega$ and $x \geq \overline{x}$. We now show that there exists $\widetilde{x}$ such that, if $(x, y) \in \Omega$ and $x \geq \widetilde{x}$, then

$$\xi(x, y) \geq \vartheta(x), \tag{5.24}$$

namely

$$\xi(x,y) \geq \int_0^x \frac{dr}{\varphi(r)}. \tag{5.25}$$

By [Wa, Statement II], there exists a function $w(x_1, x_2, y_1, y_2)$ such that
(5.26)

$$\xi(x_2,y_2) \geq \xi(x_1,y_1) + \frac{\pi}{2}\int_{x_1}^{x_2} \frac{1-\varphi'(x)^2}{\varphi(x)}\, dx + w(x_1,x_2,y_1,y_2) \quad \text{if } (x_1,y_1),(x_2,y_2) \in \Omega,\ x_1 < x_2,$$

and $\lim_{x_1\to+\infty} w(x_1, x_2, y_1, y_2) = 0$, uniformly for $x_2 > x_1$, $|y_1| < \varphi(x_1)$, and $|y_2| < \varphi(x_2)$. Since, by our assumptions, $\lim_{x\to+\infty} \varphi'(x) = 0$, there exists x_1 such that $\frac{\pi}{2}(1 - \varphi'(x)^2) > 1$ if $x > x_1$. Thus,

$$\lim_{x\to+\infty} \frac{\xi(x,y)}{\int_0^x \frac{dr}{\varphi(r)}} > 1,$$

uniformly in y for $|y| < \varphi(x)$, whence (5.25) follows.
Inasmuch as φ is decreasing in $(\widehat{x}, \infty)$, we have, by (5.24), that

$$\varphi(\vartheta^{-1}(\xi(x,y))) \leq \varphi(x) \tag{5.27}$$

if $(x, y) \in \Omega$, and $x \geq \max\{\widehat{x}, \widetilde{x}\}$.
Finally, define the function $\Xi : G \to G$ as

$$\Xi(s,t) = (B(\vartheta^{-1}(s)), t) \qquad \text{for } (s,t) \in G.$$

Given any measurable set $E \subset \Omega \cap \{x > 0\}$, one has that

$$\begin{aligned} \int_E |V(x,y)|dxdy &= \int_E h^*(B(\vartheta^{-1}(\xi(x,y))))\varphi(\vartheta^{-1}(\xi(x,y)))^2|\nabla\xi(x,y)|^2\, dxdy \\ &= \int_{\zeta(E)} h^*(B(\vartheta^{-1}(s)))\varphi(\vartheta^{-1}(s))^2\, dsdt \\ &= \frac{1}{2}\int_{\Xi(\zeta(E))} h^*(r)\, drdt \\ &\leq \frac{1}{2}\int_0^{|\Xi(\zeta(E))|} h^*(\sigma/\pi)d\sigma\,, \end{aligned} \tag{5.28}$$

where the last inequality holds owing to the Hardy-Littlewood inequality (3.3). Thus, by the symmetry of $V(x,y)$ and of Ω about the y-axis, there exist constants c and C such that

$$\int_E |V(x,y)|dxdy \leq C\int_0^{c|\Xi(\zeta(E))|} h^*(\sigma)d\sigma \tag{5.29}$$

for every measurable set $E \subset \Omega$. Next, if $E \subset \Omega \cap \{x > \max\{\widetilde{x}, \overline{x}\}\}$,

$$\begin{aligned} |\Xi(\zeta(E))| &= \int_{\Xi(\zeta(E))} drdt = 2\int_{\zeta(E)} \varphi(\vartheta^{-1}(s))^2\, dsdt \\ &= 2\int_E \varphi(\vartheta^{-1}(\xi(x,y)))^2|\nabla\xi(x,y)|^2\, dxdy \\ &\leq 2C\int_E \varphi(\vartheta^{-1}(\xi(x,y)))^2\frac{1}{\varphi(x)^2}dxdy \\ &\leq 2C\int_E dxdy = 2C^2|E|, \end{aligned} \tag{5.30}$$

for some constant C, where the first inequality follows from (5.23), and the last one holds owing to (5.27). Since the functions $\varphi(\vartheta^{-1}(\xi(x,y)))^2$ and $\frac{1}{\varphi(x)^2}$ are bounded in $\Omega\cap\{0<x\le\max\{\widetilde{x},\overline{x}\}\}$, we can conclude that there exists a constant c such that

$$|\Xi(\zeta(E))|\le c|E| \tag{5.31}$$

for every measurable set $E\subset\Omega\cap\{x>0\}$, and hence, by the symmetry of Ω about the y-axis, for every $E\subset\Omega$. From (5.29) and (5.31) we conclude that there exist positive constants c and C such that

$$\int_E |V(x,y)|dxdy\le C\int_0^{c|E|} h^*(\sigma)d\sigma \tag{5.32}$$

for every measurable set $E\subset\Omega$. By property (3.2), equation (5.6) follows □

We are now in a position to prove Theorem 2.3.

Proof of Theorem 2.3. Let M be a positive number such that $\nu\in\Delta_2$ in $(0,M)$, and that (2.10) holds in $(0,M)$. Set $\nu_0=\nu$, and let ν_i, $i=1,2,3$ be the functions iteratively defined by

$$\nu_i(s)=\left(\int_0^s\frac{\nu_{i-1}(r)^{1/\alpha}}{r}dr\right)^\alpha\quad\text{ for } s\in(0,M).$$

It is easily seen that $\nu_3\in C^2(0,M)$, $\nu_3^{1/\alpha}$ is convex in $(0,M)$, and $\nu_3\approx\nu_2\approx\nu_1\approx\nu$ in $(0,M)$. In particular, the latter property holds owing to (2.10). Moreover,

$$\frac{1}{\alpha}\nu_3^{\frac{1}{\alpha}-1}\nu_3{}'(s)=(\nu_3(s)^{1/\alpha})'=\frac{\nu_2(s)^{1/\alpha}}{s}\approx\frac{\nu_3(s)^{1/\alpha}}{s}\quad\text{ for } s\in(0,M),$$

whence

$$s\nu_3{}'(s)\approx\nu_3(s)\qquad\text{for } s\in(0,M),$$

since $\nu_3\in\Delta_2$, inasmuch as $\nu\in\Delta_2$.
Thus, on replacing, if necessary, ν with ν_3, we may assume, without loss of generality, that

$$\nu\text{ is twice continuosly differentiable in }(0,M),\qquad \nu^{1/\alpha}\text{ is convex,} \tag{5.33}$$

and

$$s\nu'(s)\approx\nu(s)\qquad\text{for } s\in(0,M). \tag{5.34}$$

Now, define $\lambda:(0,M)\to(0,\infty)$ as

$$\lambda(s)=\frac{\nu(s)}{\sqrt{\nu'(s)}}\qquad\text{for } s\in(0,M). \tag{5.35}$$

Hence, given any $a\in(0,M)$,

$$\frac{1}{\nu(s)}-\frac{1}{\nu(a)}=\int_s^a\frac{dr}{\lambda(r)^2}\qquad\text{for } s\in(0,M). \tag{5.36}$$

Thus, there exists $\overline{s}\in(0,M)$ such that

$$\frac{1}{2\nu(s)}\le\int_s^a\frac{dr}{\lambda(r)^2}\le\frac{1}{\nu(s)}\qquad\text{if } 0<s<\overline{s}. \tag{5.37}$$

We claim that

$$\frac{\lambda(s)}{\sqrt{s}} \quad \text{is non-decreasing in } (0,M). \tag{5.38}$$

Indeed, owing to (5.34) and to the fact that $\nu \in \Delta_2$ in $(0,M)$,

$$\frac{\lambda(s)^2}{s} = \frac{\nu(s)^2}{\nu'(s)s} \approx \nu(s) \qquad \text{for } s \in (0,M)\,. \tag{5.39}$$

By (5.38), via an analogous argument as above, λ can be replaced, if necessary, by an equivalent function, still denoted by λ, satisfying (5.37) and such that $\lambda \in C^2(0,M)$ and λ^2 is convex in $(0,M)$.
From (2.12) and (5.37), via Fubini's theorem, we infer that

$$\int_0 \frac{r}{\lambda(r)^2} dr = \infty\,. \tag{5.40}$$

As a consequence,

$$\int_0 \frac{dr}{\lambda(r)} = \infty\,. \tag{5.41}$$

Indeed, if (5.41) were not true, namely if

$$\int_0 \frac{dr}{\lambda(r)} < \infty\,, \tag{5.42}$$

then $\lim_{s\to 0} \frac{s}{\lambda(s)} = 0$, and this limit, combined with (5.42), would imply the convergence of the integral in (5.40).
Let $N : [1,\infty) \to (0, M/2]$ be the function implicitly defined by

$$\int_{N(r)}^{M/2} \frac{dr}{\lambda(r)} = r - 1 \qquad \text{for } r \in [1,\infty)\,. \tag{5.43}$$

Clearly, $N \in C^2(1,\infty)$ and N decreases monotonically from $M/2$ to 0. Define $\varphi : [1,\infty) \to [0,\infty)$ as

$$\varphi(r) = \frac{1}{2}\lambda(N(r)) \qquad \text{for } r \in [1,\infty)\,, \tag{5.44}$$

and observe that $\varphi \in C^2(1,\infty)$. Since

$$\lambda(N(r)) = -N'(r) \qquad \text{for } r \in [1,\infty)\,, \tag{5.45}$$

and $\lim_{r\to\infty} N(r) = 0$, one has that

$$\int_r^\infty \lambda(N(\rho))d\rho = N(r) \qquad \text{for } r \in [1,\infty)\,, \tag{5.46}$$

whence

$$\lambda\bigg(\int_r^\infty \lambda(N(\rho))d\rho\bigg) = \lambda(N(r)) \qquad \text{for } r \in [1,\infty)\,. \tag{5.47}$$

By (5.44) and (5.47),

$$\lambda\Big(2\int_r^\infty \varphi(\rho)d\rho\Big) = 2\varphi(r) \qquad \text{for } r\in[1,\infty)\,, \tag{5.48}$$

and hence

$$\lambda(s) = 2\varphi(B^{-1}(s)) \qquad \text{for } s\in(0,M/2). \tag{5.49}$$

Observe that the function φ is decreasing in $(1,\infty)$ and $\lim_{r\to\infty}\varphi(r)=0$. Furthermore,

$$2\varphi'(r) = \lambda(N(r))' = \lambda'(N(r))N'(r) = -\lambda'(N(r))\lambda(N(r)) \quad \text{for } r\in(1,\infty), \tag{5.50}$$

and

$$\begin{aligned}2\varphi''(r) &= -[\lambda''(N(r))\lambda(N(r)) + \lambda'(N(r))^2]N'(r)\\ &= [\lambda''(N(r))\lambda(N(r)) + \lambda'(N(r))^2]\lambda(N(r)) = \frac{1}{2}(\lambda^2)''(N(r))\lambda(N(r)) \quad \text{for } r\in(1,\infty).\end{aligned}$$

Thus, φ is convex in $(1,\infty)$, since λ^2 is convex in $(0,M)$; moreover, φ'' is bounded in $(1,\infty)$. Let us continue φ to $\mathbb{R}$ in such a way that the assumptions of Theorem 5.1 are fulfilled, and

$$\int_{\mathbb{R}}\varphi(r)\,dr = \frac{M}{2}. \tag{5.51}$$

Now, let Ω be the set associated with φ as in the statement of Theorem 5.1. Owing to (5.51), we have that $|\Omega|=M$. By (5.3), (5.49) and a change of variable,

$$\nu_\Omega(s) \approx \frac{1}{\int_{B^{-1}(|\Omega|/2)}^{B^{-1}(s)}\frac{dr}{\varphi(r)}} = \frac{1}{2\int_s^{|\Omega|/2}\frac{dr}{\lambda(r)^2}} \qquad \text{near } 0. \tag{5.52}$$

Coupling (5.52) with (5.37) yields (2.13).
The existence of a function $V\in L^\infty(\Omega)$ such that problem (1.1) has an unbounded solution now follows from Theorem 2.3.
In view of a proof of Theorem 2.7, note also incidentally that, by (5.2),

$$\lambda_\Omega(s) \approx \varphi(H^{-1}(s)) \quad \text{near } 0, \tag{5.53}$$

and hence, by (5.49),

$$\lambda_\Omega(s) \approx \lambda(s) \quad \text{near } 0. \tag{5.54}$$

□

6 Applications and examples

Lipschitz domains.
Assume that Ω is a Lipschitz domain in $\mathbb{R}^n$. Owing to (2.5), condition (2.6) amounts to requiring that

$$V_-\in L^{\frac{n}{2},1}(\Omega) \text{ if } n>2, \;\text{ and } V_-\in L\log L(\Omega) \text{ if } n=2. \tag{6.1}$$

Recall that membership of V_- to the Zygmund space $L\log L(\Omega)$ entails that

$$\int_0^{|\Omega|} V_-^*(s)\Big(1+\log\frac{|\Omega|}{s}\Big)\,ds<\infty.$$

Thus, under (6.1), any solution u to (1.1) is essentially bounded, by Theorem 2.1. The same conclusion follows from Theorem 2.6, via (2.16). In particular, by Corollary 2.2, any solution u to (1.1) is essentially bounded if $V_-\in L^p(\Omega)$, where either $n>2$ and $p>\frac{n}{2}$, or $n=2$ and $p>1$.

Hölder domains.
Let Ω be a Hölder domain in $\mathbb{R}^n$ with exponent $\alpha\in(0,1)$. We have that

$$\nu_\Omega(s)\succapprox s^{1-\frac{2\alpha}{n-1+\alpha}}\quad\text{near }0, \tag{6.2}$$

and

$$\lambda(s)\succapprox s^{\frac{n-1}{n-1+\alpha}}\quad\text{near }0. \tag{6.3}$$

Inequalities (6.2) and (6.3) follow from a Sobolev embedding of [La]; inequality (6.3) for $n=2$ was earlier established in [Ci1].
From either Theorem 2.1 and (6.2), or Theorem 2.6 and (6.3) we deduce that any solution u to (1.1) is bounded, provided that

$$V_-\in L^{\frac{\alpha+1}{2\alpha},1}(\Omega),$$

and hence, in particular, if $V_-\in L^p(\Omega)$ for some $p>\frac{\alpha+1}{2\alpha}$.

Cusp-shaped domains.
Let $L>0$ and let $\vartheta:[0,L]\to[0,\infty)$ be a differentiable convex function such that $\vartheta(0)=0$. Consider the domain

$$\Omega=\{x\in\mathbb{R}^n:|x'|<\vartheta(x_n),0<x_n<L\}$$

(see Figure 1), where $x=(x',x_n)$ and $x'=(x_1,\dots,x_{n-1})\in\mathbb{R}^{n-1}$. Let $\Theta:[0,L]\to[0,\infty)$ be the function given by

$$\Theta(\rho)=n\omega_n\int_0^\rho\vartheta(r)^{n-1}dr\qquad\text{for }\rho\in[0,L],$$

where ω_n denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Then, [Ma7, Example 4.3.5/1] tells us that

$$\nu_\Omega(s)\approx\bigg(\int_{\Theta^{-1}(s)}^{\Theta^{-1}(|\Omega|/2)}\vartheta(r)^{1-n}dr\bigg)^{-1}\quad\text{near }0. \tag{6.4}$$

Thus, (2.6) is equivalent to

$$\int_0 V_-^*(s)\int_{\Theta^{-1}(s)}^{\Theta^{-1}(|\Omega|/2)}\vartheta(r)^{1-n}dr ds<\infty\,, \tag{6.5}$$

namely to

$$\int_0\vartheta(r)^{1-n}\int_0^{\Theta(r)}V_-^*(s)ds\,dr<\infty\,, \tag{6.6}$$

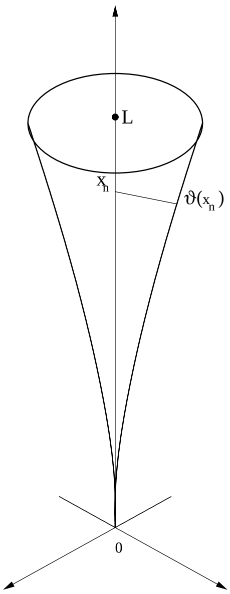

Figure 1: a cusp-shaped domain

and to

$$V_- \in \Lambda(\varrho)(\Omega), \tag{6.7}$$

where

$$\varrho(s) = \int_{\Theta^{-1}(s)}^{\Theta^{-1}(|\Omega|/2)} \vartheta(r)^{1-n} dr \quad \text{for } s \in (0, |\Omega|).$$

Owing to Theorem 2.1, any solution to (1.1) is essentially bounded provided that (6.7) is in force. The same result can be easily derived via Theorem 2.6, owing to the fact that

$$\lambda(s) \approx \vartheta(\Theta^{-1}(s))^{n-1} \qquad \text{near } 0,$$

by [Ma7, Example 3.3.3/1].

Unbounded funnel-shaped domains.
Let $\zeta : [0, \infty) \to (0, \infty)$ be a differentiable convex function such that $\lim_{\rho\to 0^+} \zeta'(\rho) > -\infty$ and $\lim_{\rho\to\infty} \zeta(\rho) = 0$. Consider the unbounded domain

$$\Omega = \{x \in \mathbb{R}^n : x_n > 0, |x'| < \zeta(x_n)\}$$

(see Figure 2), where $x = (x', x_n)$ and $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. Assume that

$$\int_0^\infty \zeta(r)^{n-1} dr < \infty\,, \tag{6.8}$$

whence $|\Omega| < \infty$. Let $\Upsilon : [0, \infty) \to [0, \infty)$ be the function given by

$$\Upsilon(\rho) = n\omega_n \int_\rho^\infty \zeta(r)^{n-1} dr \qquad \text{for } \rho > 0.$$

By [Ma7, Example 4.3.5/2],

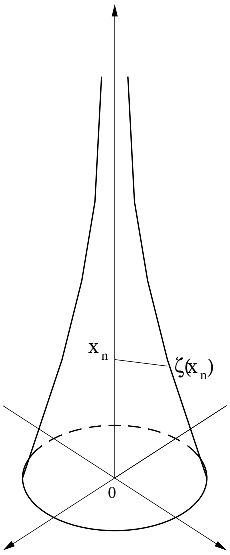

Figure 2: an unbounded domain

$$\nu_\Omega(s) \approx \left(\int_{\Upsilon^{-1}(|\Omega|/2)}^{\Upsilon^{-1}(s)} \zeta(r)^{1-n} dr \right)^{-1} \quad \text{near } 0.$$

Thus (2.6) is fulfilled, and hence any solution to (1.1) is essentially bounded by Theorem 2.1, provided that

$$\int_0 V_-^*(s) \int_{\Upsilon^{-1}(|\Omega|/2)}^{\Upsilon^{-1}(s)} \zeta(r)^{1-n} dr\, ds < \infty\,, \tag{6.9}$$

namely if

$$\int^{\infty} \zeta(r)^{1-n} \int_0^{\Upsilon(r)} V_-^*(s) ds\, dr < \infty\,, \tag{6.10}$$

or, equivalently, if

$$V_- \in \Lambda(\varrho)(\Omega),$$

where

$$\varrho(s) = \int_{\Upsilon^{-1}(|\Omega|/2)}^{\Upsilon^{-1}(s)} \zeta(r)^{1-n} dr \quad \text{for } s \in (0, |\Omega|).$$

Note that, by [Ma7, Example 3.3.3/2],

$$\lambda_\Omega(s) \approx \zeta(\Upsilon^{-1}(s))^{n-1} \qquad \text{near } 0,$$

and hence it is easily seen that condition (2.17) is equivalent to (6.9) and (6.10) in this case. Thus, Theorem 2.6 leads to the same conclusions as Theorem 2.1 also for this kind of domains.

John and γ-John domains.
Let $\gamma \geq 1$. Recall that a domain Ω in $\mathbb{R}^n$ is called a γ-John domain if there exist a positive constant c and a point $x_0 \in \Omega$ such that for every $x \in \Omega$ there exists a rectifiable curve $\varpi : [0, l] \to \Omega$, parametrized by arclenght, such that $\varpi(0) = x$, $\varpi(l) = x_0$, and

$$\operatorname{dist}(\varpi(r), \partial\Omega) \geq c r^\gamma \qquad \text{for } r \in [0, l].$$

The γ-John domains generalize the standard John domains, which correspond to the case when $\gamma = 1$ and arise in connection with the study of holomorphic dynamical systems and quasi-conformal mappings. The notion of John and γ-John domain has been used in recent years in the study of Sobolev inequalities (see e.g. [HK, KM]). Assume, for simplicity, that $n \geq 3$. As a consequence of [KM, Theorem 2.3] , the isocapacitary function of a γ-John domain Ω, with $1 \leq \gamma \leq \frac{n+1}{n-1}$, satisfies

$$\nu_\Omega(s) \succcurlyeq s^{\frac{(n-1)\gamma-1}{n}} \qquad \text{near } 0.$$

An application of Theorem 2.1 ensures that, if $1 \leq \gamma < \frac{n+1}{n-1}$, and

$$V_- \in L^{\frac{n}{n+1-\gamma(n-1)},1}(\Omega), \tag{6.11}$$

then any solution u to (1.1) is essentially bounded.
On the other hand, by [KM, Theorem 2.3],

$$\lambda_\Omega(s) \succcurlyeq s^{\frac{\gamma(n-1)}{n}} \qquad \text{near } 0.$$

From Theorem 2.6 one thus infers that any solution u to (1.1) is essentially bounded if

$$V_- \in L^{\frac{n}{2n-2\gamma(n-1)},1}(\Omega). \tag{6.12}$$

Since $\frac{n}{2n-2\gamma(n-1)} \geq \frac{n}{n+1-\gamma(n-1)}$ for $\gamma \geq 1$, with equality only if $\gamma = 1$, one has that

$$L^{\frac{n}{2n-2\gamma(n-1)},1}(\Omega) \subsetneqq L^{\frac{n}{n+1-\gamma(n-1)},1}(\Omega)$$

for every $\gamma > 1$. Thus, when Ω is a γ-John domain with $\gamma > 1$, Theorem 2.1 ensures the boundedness of the solutions to (1.1) under weaker assumptions on V_- than Theorem 2.6.
This shows that the characterization of a γ-John domain, with $\gamma > 1$, in terms of its isocapacitary function can be more effective in the analysis of the boundedness of solutions to (1.1) than its description via the isoperimetric function.

A family of domains with rooms and passages.
Let us consider problem (1.1) in the domain $\Omega \subset \mathbb{R}^2$ displayed in Figure 3 and inspired by an example from [CH]. In the figure, $L = 2^{-k}$ and $l = \sigma(2^{-k})$, where $k \in \mathbb{N}$ and $\sigma : [0, \infty) \to [0, \infty)$ is an increasing function such that $\lim_{s\to 0^+} \frac{\sigma(s)}{s} = 0$.
The asymptotic behaviors of the functions ν_Ω and λ_Ω, under some additional assumption on the function σ, are described in the following proposition.

Proposition 6.1 *Let Ω be the domain in Figure 3. Let $\sigma : [0, \infty) \to [0, \infty)$ be an increasing function of class Δ_2.*
(i) One has that

$$\lambda_\Omega(s) \preccurlyeq \sigma(s^{1/2}) \qquad \textit{near } 0. \tag{6.13}$$

If, in addition,

(6.14) *there exists $\beta > 0$ such that $\frac{s^{\beta+1}}{\sigma(s)}$ is non-increasing, and $\frac{s^2}{\sigma(s)}$ is non-decreasing,*

then

$$\lambda_\Omega(s) \approx \sigma(s^{1/2}) \qquad \textit{near } 0. \tag{6.15}$$

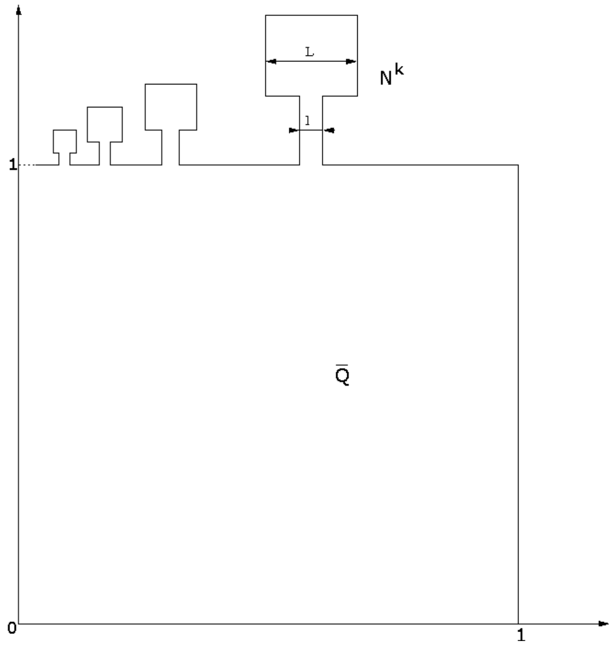

Figure 3: an example from [CH]

(ii) One has that

$$\nu_\Omega(s) \precsim \sigma(s^{1/2})s^{-\frac{1}{2}} \qquad \textit{near } 0. \tag{6.16}$$

If, in addition,

(6.17) *there exists* $\beta > 0$ *such that* $\frac{s^{\beta+1}}{\sigma(s)}$ *is non-increasing and* $\frac{s^3}{\sigma(s)}$ *is non-decreasing,*

then

$$\nu_\Omega(s) \approx \sigma(s^{1/2})s^{-\frac{1}{2}} \qquad \textit{near } 0. \tag{6.18}$$

By Theorem 2.1 and Proposition 6.1, any solution to (1.1) in Ω is bounded provided that σ fulfills (6.17), and

$$V_- \in \Lambda\Big(\frac{s^{1/2}}{\sigma(s^{1/2})}\Big),$$

namely

$$\int_0 V_-^*(s)\frac{s^{1/2}}{\sigma(s^{1/2})}ds < \infty\,. \tag{6.19}$$

For instance, when $\sigma(s) = s^\alpha$ for some $\alpha \in (1,3)$, condition (6.19) amounts to

$$V_- \in L^{\frac{2}{3-\alpha},1}(\Omega). \tag{6.20}$$

Theorem 2.6 applies to a more restricted family of domains from the class considered in the present example, and under stronger integrability assumptions on V_- . Indeed, by Proposition 6.1, condition (2.17) can only hold if

$$\int_0 \frac{1}{\sigma(s^{1/2})^2}\int_0^s V_-^*(r)dr\,ds < \infty. \tag{6.21}$$

Assumption (6.21) is stronger than (6.19) in general, as it is easily seen in the case when $\sigma(s) = s^\alpha$. In this case, (6.21) holds if $\alpha \in (1,2)$ and

$$V_- \in L^{\frac{1}{2-\alpha},1}(\Omega),$$

a more stringent assumption than (6.20).
This family of domains provides other examples in which the approach to the boundedness of solutions to (1.1) relying upon isocapacitary inequalities applies, whereas techniques exploiting isoperimetric inequalities fail, or yield the result under stronger assumptions on V_-.

Proposition 6.1 is a special case of a more general result contained in Proposition 6.2 below, which provides us with the asymptotic behavior of an isocapacitary function $\nu_{\Omega,p}$ associated with any exponent $p \in [1,2]$. In analogy with (2.2)–(2.3), the isocapacitary function $\nu_{\Omega,p} : [0, |\Omega|/2] \to [0,\infty]$ of Ω is given by

$$\begin{aligned}\nu_{\Omega,p}(s) = \inf \{C_p(E,G) : {}& E \text{ and } G \text{ are measurable subsets of } \Omega \text{ such that} \\ & E \subset G \subset \Omega \text{ and } s \le |E| \le |G| \le |\Omega|/2\} \qquad \text{for } s \in [0, |\Omega|/2],\end{aligned}$$

where, for any sets $E \subset G \subset \Omega$, the p-capacity $C_p(E,G)$ of the condenser (E,G) relative to Ω is defined as

$$C_p(E,G) = \inf \Big\{ \int_\Omega |\nabla u|^p \, dx : u \in W^{1,p}(\Omega), u \ge 1 \text{ in } E \text{ and } u \le 0 \text{ in } \Omega \setminus G \text{ (up to sets of standard capacity zero)} \Big\}. \tag{6.22}$$

Proposition 6.2 *Let Ω be the domain in Figure 3. Let $1 \le p \le 2$, and let $\sigma : [0,\infty) \to [0,\infty)$ be an increasing function of class Δ_2. Then,*

$$\nu_{\Omega,p}(s) \preccurlyeq \sigma(s^{1/2}) s^{-\frac{p-1}{2}} \qquad \textit{near } 0. \tag{6.23}$$

Assume, in addition, that

$$\frac{s^{\beta+1}}{\sigma(s)} \quad \textit{is non-increasing} \tag{6.24}$$

for some $\beta > 0$, and

$$\frac{s^{p+1}}{\sigma(s)} \quad \textit{is non-decreasing.} \tag{6.25}$$

Then

$$\nu_{\Omega,p}(s) \approx \sigma(s^{1/2}) s^{-\frac{p-1}{2}} \qquad \textit{near } 0. \tag{6.26}$$

Note that equation (6.18) of Proposition 6.1 agrees with (6.26) for $p = 2$, whereas equation (6.15) follows from (6.26) with $p = 1$, since

$$\nu_{\Omega,1}(s) \approx \lambda_\Omega(s) \quad \text{near } 0, \tag{6.27}$$

as shown by an analogous argument as in [Ma7, Lemma 2.2.5].

One step in the proof of Proposition 6.2 makes use of Orlicz spaces. Recall that given a Young function A, namely a convex function from $[0,\infty)$ into $[0,\infty)$ vanishing at 0, the Orlicz space $L^A(\Omega)$ is the Banach space of those measurable functions $w:\Omega\to\mathbb{R}$ whose Luxemburg norm

$$\|w\|_{L^A(\Omega)}=\inf\left\{k>0:\int_\Omega A\left(\frac{|w(x)|}{k}\right)dx\le 1\right\}$$

is finite. A generalized Hölder type inequality in Orlicz spaces (see [On]) tells us that if A_i, $i=1,2,3$, are Young functions such that $A_1^{-1}(r)A_2^{-1}(r)\le CA_3^{-1}(r)$ for some constant C, then there exists a constant C' such that

$$\|w_1w_2\|_{L^{A_3}(\Omega)}\le C'\|w_1\|_{L^{A_1}(\Omega)}\|w_2\|_{L^{A_2}(\Omega)} \tag{6.28}$$

for every $w_1\in L^{A_1}(\Omega)$ and $w_2\in L^{A_2}(\Omega)$.

Proof of Proposition 6.2.
Part I. Here we show that, if (6.24) and (6.25) are in force, then there exists a constant C such that

$$\nu_{\Omega,p}(s)\ge C\sigma(s^{1/2})s^{-\frac{p-1}{2}}\qquad\text{for } s\in(0,|\Omega|/2). \tag{6.29}$$

We split the proof of (6.29) is steps.
Step 1. Fix $\varepsilon\in(0,1)$, and let $Q=(-1/2,1/2)\times(0,1)$, $\Sigma_\varepsilon=(-\varepsilon/2,\varepsilon/2)$, $R_\varepsilon=\Sigma_\varepsilon\times(-1,0]$ and $N_\varepsilon=Q\cup R_\varepsilon$.
Let $q=\frac{2p}{2-p}$ if $p<2$, and let q be a sufficiently large number, to be chosen later, if $p=2$. We shall show that

$$\left(\int_Q|u|^q dxdy\right)^{\frac{p}{q}}\le\frac{C}{\varepsilon}\left(\int_{N_\varepsilon}|\nabla u|^p dxdy+\int_{\Sigma_\varepsilon}|u(x,-1)|^p dx\right) \tag{6.30}$$

for every $u\in W^{1,p}(N_\varepsilon)$, and for some constant C independent of ε and u. With abuse of notation, here, and in analogous occurrences below, $u(\cdot,-1)$ denotes the trace of u on $(\partial N_\varepsilon)\cap\{y=-1\}$.
Define

$$\overline{u}(x)=\int_0^1 u(x,y)dy\qquad\text{for a.e. } x\in(-1/2,1/2)\,. \tag{6.31}$$

One has that

$$\left(\int_Q|u|^q dxdy\right)^{\frac{p}{q}}\le 2^{p-1}\left[\left(\int_Q|u-\overline{u}|^q dxdy\right)^{\frac{p}{q}}+\left(\int_Q|\overline{u}|^q dxdy\right)^{\frac{p}{q}}\right], \tag{6.32}$$

where $\overline{u}$ is regarded as a function of (x,y), defined on N_ε. It is easily verified that the function $u-\overline{u}$ has mean value 0 on Q. Thus, a standard Poincaré inequality easily implies that

$$\left(\int_Q|u-\overline{u}|^q dxdy\right)^{\frac{p}{q}}\le C\int_Q|\nabla u|^p dxdy\,, \tag{6.33}$$

for some constant C independent of u. On the other hand, we have that

$$
\begin{aligned}
(6.34)\qquad \left(\int_Q |\overline{u}|^q dxdy\right)^{\frac{p}{q}} &= \left(\int_{-\frac12}^{\frac12}\left|\int_0^1 u(x,y)dy\right|^q dx\right)^{\frac{p}{q}} \\
&\le \max_{x\in[-1/2,1/2]}\left|\int_0^1 u(x,y)dy\right|^p \\
&\le \left(\int_{-\frac12}^{\frac12}\int_0^1 |u_x(x,y)|dy\,dx + \frac{1}{\varepsilon}\int_{\Sigma_\varepsilon}\left|\int_0^1 u(x,y)dy\right|dx\right)^p \\
&\le 2^{p-1}\left(\int_Q |\nabla u|^p dxdy + \frac{1}{\varepsilon}\int_{\Sigma_\varepsilon}\int_0^1 |u(x,y)|^p dydx\right).
\end{aligned}
$$

Note that a rigorous derivation of (6.34) involves an approximation argument for u by smooth functions in $\overline{Q}$. Since

$$
u(x,y) = \int_{-1}^{y} u_y(x,z)dz + u(x,-1) \quad \text{for a.e. } x\in\Sigma_\varepsilon,
$$

there exists a constant C such that

$$
|u(x,y)|^p \le C\int_{-1}^{y}|u_y(x,z)|^p dz + C|u(x,-1)|^p \quad \text{for a.e. } x\in\Sigma_\varepsilon.
$$

Thus,

(6.35)
$$
\begin{aligned}
\frac{1}{\varepsilon}\int_{\Sigma_\varepsilon}\int_0^1 |u(x,y)|^p dydx &\le \frac{C}{\varepsilon}\int_{-1}^1\int_{\Sigma_\varepsilon}\int_{-1}^{y}|u_y(x,z)|^p dzdxdy + \frac{C}{\varepsilon}\int_{-1}^1\int_{\Sigma_\varepsilon}|u(x,-1)|^p dxdy \\
&\le \frac{C}{\varepsilon}\int_{N_\varepsilon}|\nabla u|^p dxdy + \frac{2C}{\varepsilon}\int_{\Sigma_\varepsilon}|u(x,-1)|^p dx\,.
\end{aligned}
$$

Inequality (6.30) follows from (6.32)–(6.35).

Step 2. Let $N_{\varepsilon,\delta}$ be the set obtained on dilating N_ε by a factor δ, namely

$$
N_{\varepsilon,\delta} = \{(x,y) : (x/\delta, y/\delta)\in N_\varepsilon\}.
$$

Let $u\in W^{1,p}(N_{\varepsilon,\delta})$. From (6.30) we obtain that

$$
(6.36)\qquad \delta^{-\frac{2p}{q}}\left(\int_{Q_\delta}|u|^q dxdy\right)^{\frac{p}{q}} \le \frac{C}{\varepsilon}\left(\delta^{p-2}\int_{N_{\varepsilon,\delta}}|\nabla u|^p dxdy + \delta^{-1}\int_{\Sigma_{\varepsilon,\delta}}|u(x,-\delta)|^p dx\right),
$$

where Q_δ and $\Sigma_{\varepsilon,\delta}$ denote the sets obtained on dilating Q and Σ_ε, respectively, by a factor δ. Now, let A be a Young function whose inverse satisfies

$$
(6.37)\qquad A^{-1}(\delta^{-2}) \approx \frac{\delta^{p-1}}{\sigma(\delta)} \qquad \text{for } \delta>0.
$$

Notice that such a function A does exist. Indeed, the function $H:(0,\infty)\to[0,\infty)$ given by $H(t)=\frac{t^{-\frac{p-1}{2}}}{\sigma(t^{-\frac12})}$ for $t>0$ is increasing by (6.24), and the function $\frac{H(t)}{t}$ is non-increasing by (6.25). Thus, $\frac{H^{-1}(\tau)}{\tau}$ is a non-decreasing function, and, on choosing

$$
A(t) = \int_0^t \frac{H^{-1}(\tau)}{\tau}\,d\tau \qquad \text{for } t\ge 0,
$$

equation (6.37) holds, inasmuch as $A(t) \approx H^{-1}(t)$ for $t \geq 0$. Next, we claim that a Young function E exists whose inverse fulfils

$$E^{-1}(\tau) \approx \frac{A^{-1}(\tau)}{\tau^{p/q}} \qquad \text{for } \tau > 0. \tag{6.38}$$

To see this, note that the function $J(\tau) = \frac{A^{-1}(\tau)}{\tau^{p/q}}$ is equivalent to an increasing function $F(\tau)$ (for sufficiently large q, depending on β, if $p = 2$) by (6.24), and that the function $\frac{J(\tau)}{\tau} = \frac{A^{-1}(\tau)}{\tau^{1+p/q}}$ is trivially decreasing. Set $J_1(\tau) = \frac{J(\tau)}{\tau}$. Thus, $\frac{F(\tau)}{\tau} \approx J_1(\tau)$ for $\tau > 0$. As a consequence, one can show that $\frac{F^{-1}(t)}{t} \approx \frac{1}{J_1(F^{-1}(t))}$, and the latter is an increasing function. Thus the function E given by

$$E(t) = \int_0^t \frac{d\tau}{J_1(F^{-1}(\tau))} \qquad \text{for } t \geq 0,$$

is a Young function, and since $E(t) \approx \frac{t}{J_1(F^{-1}(t))} \approx F^{-1}(t)$, one has that $E^{-1}(\tau) \approx F(\tau) \approx J(\tau) = \frac{A^{-1}(\tau)}{t^{p/q}}$, whence (6.38) follows.
Owing to (6.38), inequality (6.28) ensures that

$$\begin{aligned} \| |u|^p \|_{L^A(Q_\delta)} &\leq C \| |u|^p \|_{L^{q/p}(Q_\delta)} \|1\|_{L^E(Q_\delta)} \\ &= C \|u\|^p_{L^q(Q_\delta)} \frac{1}{E^{-1}(1/|Q_\delta|)} \leq C \|u\|^p_{L^q(Q_\delta)} \frac{1}{E^{-1}(C'/\delta^2)}\,, \end{aligned} \tag{6.39}$$

for some constants C and C' independent of ε, δ and u. Combining (6.36)–(6.39) yields

$$\| |u|^p \|_{L^A(Q_\delta)} \leq \frac{C\sigma(\delta)}{\varepsilon\delta} \int_{N_{\varepsilon,\delta}} |\nabla u|^p dxdy + \frac{C\sigma(\delta)}{\varepsilon\delta^p} \int_{\Sigma_{\varepsilon,\delta}} |u(x,-\delta)|^p dx\,. \tag{6.40}$$

Now, choose

$$\varepsilon = \frac{\sigma(\delta)}{\delta}\,,$$

and obtain from (6.40)

$$\| |u|^p \|_{L^A(Q_\delta)} \leq C \int_{N_{\frac{\sigma(\delta)}{\delta},\delta}} |\nabla u|^p dxdy + C\delta^{1-p} \int_{\Sigma_{\frac{\sigma(\delta)}{\delta},\delta}} |u(x,-\delta)|^p dx \tag{6.41}$$

for every $u \in W^{1,p}(N_{\frac{\sigma(\delta)}{\delta},\delta})$.
Step 3. Choose $\delta_k = 2^{-k}$ for $k \in \mathbb{N}$, and set

$$Q^k = Q_{\delta_k}, \quad \Sigma^k = \Sigma_{\frac{\sigma(\delta_k)}{\delta_k},\delta_k}, \quad N^k = N_{\frac{\sigma(\delta_k)}{\delta_k},\delta_k}.$$

Note that $|\Sigma^k| = \sigma(\delta_k)$. Let us translate the sets N^k and Q^k – the translated sets being still denoted by N^k and Q^k – to construct the set Ω as in Figure 3. Moreover, define $\overline{Q} = (0,1)\times(0,1)$, the large square in Figure 3. Given $u \in W^{1,p}(\Omega)$, one has that

$$\| |u|^p \|_{L^A(\cup_k Q^k)} \leq \sum_{k\in\mathbb{N}} \| |u|^p \|_{L^A(Q^k)}\,, \tag{6.42}$$

and

$$\int_{\cup_k N^k} |\nabla u|^p dxdy = \sum_{k\in\mathbb{N}} \int_{N^k} |\nabla u|^p dxdy\,. \tag{6.43}$$

Note also that, if $1 \le p < 2$, then, since the intervals Σ^k are disjoint, there exist constants C and C' such that

$$
\begin{aligned}
(6.44)\qquad \sum_k \delta_k^{1-p} \int_{\Sigma^k} |u(x,1)|^p dx &= \int_0^1 |u(x,1)|^p \Big(\sum_k \chi_{\Sigma^k}(x)\delta_k^{1-p} \Big) dx \\
&\le \Big(\int_0^1 |u(x,1)|^{\frac{p}{2-p}} dx \Big)^{2-p} \Big(\int_0^1 \chi_{\Sigma^k}(x)\delta_k^{-1} dx \Big)^{p-1} \\
&= \Big(\int_0^1 |u(x,1)|^{\frac{p}{2-p}} dx \Big)^{2-p} \Big(\sum_k \frac{\sigma(\delta_k)}{\delta_k} \Big)^{p-1} \\
&\le C \Big(\int_{\overline{Q}} |\nabla u|^p dxdy + \int_{\overline{Q}} |u|^p dxdy \Big) \Big(\sum_k \frac{\sigma(\delta_k)}{\delta_k} \Big)^{p-1} \\
&\le C' \|u\|^p_{W^{1,p}(\overline{Q})} \Big(\int_0^1 \frac{\sigma(\delta)}{\delta^2} d\delta \Big)^{p-1},
\end{aligned}
$$

where the last but one inequality holds by a standard trace inequality on the square $\overline{Q}$.
If, instead, $p = 2$, then, for any $a > 1$

$$
\begin{aligned}
(6.45)\qquad \sum_k \delta_k^{-1} \int_{\Sigma^k} |u(x,1)|^2 dx &= \int_0^1 |u(x,1)|^2 \Big(\sum_k \chi_{\Sigma^k}(x)\delta_k^{-1} \Big) dx \\
&\le \Big(\int_0^1 |u(x,1)|^{2a} dx \Big)^{\frac{1}{a}} \Big(\int_0^1 \chi_{\Sigma^k}(x)\delta_k^{-a'} dx \Big)^{\frac{1}{a'}} \\
&= \Big(\int_0^1 |u(x,1)|^{2a} dx \Big)^{\frac{1}{a}} \Big(\sum_k \frac{\sigma(\delta_k)}{\delta_k^{a'}} \Big)^{\frac{1}{a'}} \\
&\le C \|u\|^2_{W^{1,2}(\overline{Q})} \Big(\int_0^1 \frac{\sigma(\delta)}{\delta^{a'+1}} d\delta \Big)^{\frac{1}{a'}},
\end{aligned}
$$

for some constant C. Owing to (6.24), if $1 \le p < 2$, then $\int_0^1 \frac{\sigma(\delta)}{\delta^2} d\delta < \infty$, and if $p = 2$, then $\int_0^1 \frac{\sigma(\delta)}{\delta^{a'+1}} d\delta < \infty$, provided that a is sufficiently large. By (6.41)–(6.45), there exists a constant C such that

$$
(6.46)\qquad \| |u|^p \|^{1/p}_{L^A(\cup_k Q^k)} \le C \|u\|_{W^{1,p}(\Omega)}
$$

for every $u \in W^{1,p}(\Omega)$.

Step 4. Denote by $R_{\frac{\sigma(\delta)}{\delta},\delta}$ the rectangle obtained on dilating $R_{\frac{\sigma(\delta)}{\delta}}$ by the factor δ. Hence, its sidelenghts are $\sigma(\delta)$ (along the x-axis) and δ (along the y-axis). Let y_i, $i = 1, \dots m$, be real numbers such that $y_1 = -\delta$, $y_m = 0$, $y_{i+1} - y_i$ is constant with respect to i, and

$$
(6.47)\qquad 1 \le \frac{y_{i+1} - y_i}{\sigma(\delta)} \le 2 \qquad \text{for } i = 1, \dots, m-1.
$$

Let

$$
R^i_\delta = \{(x,y) \in R_{\frac{\sigma(\delta)}{\delta},\delta} : y_i \le y \le y_{i+1}\} \qquad \text{for } i = 1, \dots, m-1.
$$

Given $u \in W^{1,p}(R_{\frac{\sigma(\delta)}{\delta},\delta})$, define

$$
\widehat{u}(y) = \frac{1}{\sigma(\delta)} \int_{\Sigma_{\frac{\sigma(\delta)}{\delta},\delta}} u(z,y) dz \qquad \text{for a.e. } y \in (-\delta, 0).
$$

We have that

$$\||u|^p\|_{L^A(R_{\frac{\sigma(\delta)}{\delta},\delta})} \leq 2^{p-1}\||u-\widehat{u}|^p\|_{L^A(R_{\frac{\sigma(\delta)}{\delta},\delta})} + 2^{p-1}\||\widehat{u}|^p\|_{L^A(R_{\frac{\sigma(\delta)}{\delta},\delta})}\,, \tag{6.48}$$

where $\widehat{u}$ is regarded as a function of (x,y), defined on $R_{\frac{\sigma(\delta)}{\delta},\delta}$, and A is the Young function introduced in Step 2. Furthermore,

$$\||u-\widehat{u}|^p\|_{L^A(R_{\frac{\sigma(\delta)}{\delta},\delta})} \leq \sum_{i=1}^{m-1} \||u-\widehat{u}|^p\|_{L^A(R^i_\delta)}\,. \tag{6.49}$$

The mean value of $u-\widehat{u}$ over each R^i_δ is 0. The rectangles $R^1_\delta, \cdots, R^{m-1}_\delta$ agree, up to translations. Moreover,

$$|R^i_\delta| \approx \sigma(\delta)^2\,,$$

owing to assumption (6.47). An analogous scaling argument as in the proof of Step 2 tells us that

$$\||u-\widehat{u}|^p\|_{L^A(R^i_\delta)} \leq C\frac{\sigma(\delta)^{p-2}}{A^{-1}(C'/\sigma(\delta)^2)}\|\nabla u\|^p_{L^p(R^i_\delta)}\,, \tag{6.50}$$

for some constants C and C'. Since

$$\frac{\sigma(\delta)}{\delta} \leq C \qquad \text{if } 0<\delta\leq 1, \tag{6.51}$$

for some constant C, we deduce from (6.49) and (6.50) that

$$\||u-\widehat{u}|^p\|_{L^A(R_{\frac{\sigma(\delta)}{\delta},\delta})} \leq C\frac{\sigma(\delta)^{p-1}}{\sigma(\delta)A^{-1}(C'/(\delta\sigma(\delta)))}\|\nabla u\|^p_{L^p(R_{\frac{\sigma(\delta)}{\delta},\delta})}\,, \tag{6.52}$$

for some positive constants C and C'.
Next, since

$$|\widehat{u}(y)|^p \leq 2^{p-1}|\widehat{u}(-\delta)|^p + 2^{p-1}\left(\int_{-\delta}^0 |\widehat{u}'(z)|dz\right)^p \qquad \text{for a.e. } y\in(-\delta,0),$$

one has that

(6.53)

$$\begin{aligned}
&\||\widehat{u}|^p\|_{L^A(R_{\frac{\sigma(\delta)}{\delta},\delta})} \\
&\quad\leq 2^{p-1}|\widehat{u}(-\delta)|^p\|1\|_{L^A(R_{\frac{\sigma(\delta)}{\delta},\delta})} + 2^{p-1}\left\|\left(\int_{-\delta}^0 \frac{1}{\sigma(\delta)}\int_{\Sigma_{\frac{\sigma(\delta)}{\delta},\delta}} |u_y(z,y)|dz\,dy\right)^p\right\|_{L^A(R_{\frac{\sigma(\delta)}{\delta},\delta})} \\
&\quad\leq \frac{2^{p-1}}{A^{-1}(1/|R_{\frac{\sigma(\delta)}{\delta},\delta}|)}|\widehat{u}(-\delta)|^p + 2^{p-1}\left\|\left(\int_{-\delta}^0 \frac{1}{\sigma(\delta)}\int_{\Sigma_{\frac{\sigma(\delta)}{\delta},\delta}} |\nabla u(z,y)|dz\,dy\right)^p\right\|_{L^A(R_{\frac{\sigma(\delta)}{\delta},\delta})} \\
&\quad\leq \frac{2^{p-1}}{A^{-1}(1/|R_{\frac{\sigma(\delta)}{\delta},\delta}|)}|\widehat{u}(-\delta)|^p + \frac{2^{p-1}}{\sigma(\delta)^p}\left\|\left(\int_{R_{\frac{\sigma(\delta)}{\delta},\delta}} |\nabla u|dxdy\right)^p\right\|_{L^A(R_{\frac{\sigma(\delta)}{\delta},\delta})} \\
&\quad\leq \frac{2^{p-1}}{A^{-1}(1/|R_{\frac{\sigma(\delta)}{\delta},\delta}|)}|\widehat{u}(-\delta)|^p + \frac{2^{p-1}}{\sigma(\delta)^p}\left\||R_{\frac{\sigma(\delta)}{\delta},\delta}|^{p-1}\|\nabla u\|^p_{L^p(R_{\frac{\sigma(\delta)}{\delta},\delta})}\right\|_{L^A(R_{\frac{\sigma(\delta)}{\delta},\delta})}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{2^{p-1}}{A^{-1}(1/|R_{\frac{\sigma(\delta)}{\delta},\delta}|)}|\widehat{u}(-\delta)|^p + \frac{2^{p-1}(\delta\sigma(\delta))^{p-1}}{\sigma(\delta)^p A^{-1}(1/|R_{\frac{\sigma(\delta)}{\delta},\delta}|)}\|\nabla u\|^p_{L^p(R_{\frac{\sigma(\delta)}{\delta},\delta})}\\
&\leq \frac{2^{p-1}}{A^{-1}(1/(\sigma(\delta)\delta))}|\widehat{u}(-\delta)|^p + \frac{2^{p-1}\delta^{p-1}}{\sigma(\delta)A^{-1}(1/(\sigma(\delta)\delta)}\|\nabla u\|^p_{L^p(R_{\frac{\sigma(\delta)}{\delta},\delta})}\\
&\leq \frac{2^{p-1}}{A^{-1}(1/(\sigma(\delta)\delta))}\bigg(\frac{1}{\sigma(\delta)}\int_{\Sigma_{\frac{\sigma(\delta)}{\delta},\delta}}|u(z,-\delta)|dz\bigg)^p + \frac{2^{p-1}\delta^{p-1}}{\sigma(\delta)A^{-1}(1/(\sigma(\delta)\delta)}\|\nabla u\|^p_{L^p(R_{\frac{\sigma(\delta)}{\delta},\delta})}.
\end{aligned}
$$

Again, the inequality between the leftmost side of (6.53) and its rightmost side is fully substantiated after an approximation argument for u. Combining (6.48), (6.52), and (6.53) yields

$$
\begin{aligned}
(6.54)\quad &\||u|^p\|_{L^A(R_{\frac{\sigma(\delta)}{\delta},\delta})}\\
&\leq \frac{C}{\sigma(\delta)A^{-1}(C'/(\sigma(\delta)\delta))}\int_{\Sigma_{\frac{\sigma(\delta)}{\delta},\delta}}|u(z,-\delta)|^p dz + \frac{C\delta^{p-1}}{\sigma(\delta)A^{-1}(C'/(\sigma(\delta)\delta))}\|\nabla u\|^p_{L^p(R_{\frac{\sigma(\delta)}{\delta},\delta})}
\end{aligned}
$$

for some positive constants C and C'. Denote by R^k the set obtained after translating $R_{\frac{\sigma(\delta_k)}{\delta_k},\delta_k}$ in the construction of Ω. Fix $u \in W^{1,p}(\Omega)$. By (6.54),

$$
\begin{aligned}
\||u|^p\|_{L^A(\cup_k R^k)} &\leq \sum_{k\in\mathbb{N}}\||u|^p\|_{L^A(R^k)}\\
&\leq C\sum_{k\in\mathbb{N}}\frac{1}{\sigma(\delta_k)A^{-1}(C'/(\sigma(\delta_k)\delta_k)}\int_{\Sigma^k}|u(z,1)|^p dz\\
&\quad + C\sum_{k\in\mathbb{N}}\frac{\delta_k^{p-1}}{\sigma(\delta_k)A^{-1}(C'/(\sigma(\delta_k)\delta_k)}\|\nabla u\|^p_{L^p(R^k)}.
\end{aligned}
\tag{6.55}
$$

By (6.51) and (6.37),

$$
\frac{\delta_k^{p-1}}{\sigma(\delta_k)A^{-1}(C'/(\delta_k\sigma(\delta_k)))} \leq \frac{\delta_k^{p-1}}{\sigma(\delta_k)A^{-1}(C''/\delta_k^2)} \leq C\,,
\tag{6.56}
$$

for some positive constants C and C''. Thus,

$$
\||u|^p\|_{L^A(\cup_k R^k)} \leq \sum_{k\in\mathbb{N}}\frac{C}{\delta_k^{p-1}}\int_{\Sigma^k}|u(z,1)|^p dz + C\|\nabla u\|^p_{L^p(\cup_k R^k)}\,,
\tag{6.57}
$$

for some constant C. Hence, we deduce from either (6.44) or (6.45) that

$$
\||u|^p\|^{1/p}_{L^A(\cup_k R^k)} \leq C\|u\|_{W^{1,p}(\Omega)}
\tag{6.58}
$$

for some constant C.

Step 5. A variant of [Ma7, Theorem 2.3.2], with analogous proof, tells us that given an open set $G \subset \mathbb{R}^2$ with $|G| < \infty$, and a Young function B, if the inequality

$$
\||u|^p\|^{1/p}_{L^B(G)} \leq C\big(\|\nabla u\|_{L^p(G)} + \|u\|_{L^p(G)}\big)
\tag{6.59}
$$

holds for some constant C and for every $u \in W^{1,p}(G)$, then

$$\frac{1}{B^{-1}(1/s)} \leq C'\nu_{G,p}(s) \qquad \text{for } s \in (0, |G|/2), \tag{6.60}$$

for some constant C'. The standard Sobolev inequality holds on the square $\overline{Q}$, and, consequently, (6.60) holds if $G = \overline{Q}$, and $B(t) = t^{\frac{2}{2-p}}$ if $1 \leq p < 2$, or $B(t) = t^a$ for any $a \geq 1$ if $p = 2$. Thus, since the right-hand side of (6.38) is equivalent to a non-decreasing function, inequality (6.60) also holds with $B = A$. Hence, there exists a constant C such that

$$\||u|^p\|_{L^A(\overline{Q})}^{1/p} \leq C\big(\|\nabla u\|_{L^p(\overline{Q})} + \|u\|_{L^p(\overline{Q})}\big) \tag{6.61}$$

for every $u \in W^{1,p}(\Omega)$. Combining (6.46), (6.58) and (6.61) tells us that

$$\||u|^p\|_{L^A(\Omega)}^{1/p} \leq C\big(\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\big) \tag{6.62}$$

for some constant C and for every $u \in W^{1,p}(\Omega)$. Hence,

$$\frac{1}{A^{-1}(1/s)} \leq C\nu_{\Omega,p}(s) \qquad \text{for } s \in (0, |\Omega|/2), \tag{6.63}$$

and (6.29) follows, owing to (6.37).
Part II. Here we show that, if $p \geq 1$, and σ is non-decreasing and of class Δ_2 near 0, then inequality (6.23) holds. Consider the sequence of condensers (Q^k, N^k). Let $\{u_k\}$ be the sequence of Lipschitz continuous functions given by $u_k = 1$ in Q^k, $u_k = 0$ in $\Omega \setminus N^k$ and such that u_k depends only on y and is a linear function of y in R^k. We have that

$$|Q^k| \approx \delta_k^2\,, \tag{6.64}$$

and

$$\int_\Omega |\nabla u_k|^p dxdy \approx \frac{|R^k|}{\delta_k^p} \approx \frac{\sigma(\delta_k)}{\delta_k^{p-1}}\,, \tag{6.65}$$

up to multiplicative constants independent of k. Thus, there exist constants C and C', independent of k, such that

$$\nu_{\Omega,p}(C\delta_k^2) \leq C_p(Q^k, N^k) \leq \frac{C'\sigma(\delta_k)}{\delta_k^{p-1}}\,. \tag{6.66}$$

It is easily seen that (6.66) continues to hold with δ_k replaced with any $s \in (0, |\Omega|/2)$. Hence (6.23) follows.
The proof is complete. □

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