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The spectrum of water waves produced by moving point sources, and a related inverse problem

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Abstract

Parameters of moving sources on the sea surface are recovered by spectral analysis of the induced surface waves. The method can be an alternative to the standard way of seeing the ship directly, in particular, when the direct observation is impossible.

1. Introduction

We consider the problem of recovering parameters of moving sources on the sea surface by spectral analysis of the surface waves. The main idea behind our approach is based on the fact that a moving point source produces surface waves whose spectral characteristics have singularities for some values of frequencies. These special frequencies form an unbounded curve L on the two-dimensional plane of all frequencies. One can imagine an instrument which analyses the surface waves and singles out the frequencies where the density of the spectral amplitude is most significant. As it was mentioned above, this density is infinite on the curve L. The unboundedness of the density of the spectral amplitude does not contradict the nature of the surface waves, since the waves in the spectral representation are given as a two-dimensional integral over the frequency plane. The next technical step would be to distinguish the singularities due to the motion of a sea source from singularities in the spectral representation of the surface waves caused by other reasons: wind, current, and so on. We assume that these more significant natural forces have smoother spectral characteristics and form such a background, that the unbounded frequencies of localised disturbances can be distinguished on that background. We do not want to speculate on the technical questions. Our goal here is purely theoretical: assuming that the curve L or some points of this curve are found, recover trajectories, velocities and intensities of the moving sea sources.

2. Spectrum of a wave source in the moving coordinate system

In the linear theory of surface waves, the Neumann–Kelvin problem on the uniform rectilinear motion of a point source is stated as follows. Let the water fill the half-space $\{(x, y, z) : z < 0\}$. Suppose that a point source is moving

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in the direction of the x-axis with constant velocity V in the free surface $z = 0$. We introduce a coordinate system moving with the source in such a way that the origin coincides with the position of the source and x-axis is parallel to the velocity vector. Then the velocity potential ϕ and the elevation of the free surface η satisfy

$$\Delta\phi = 0, \quad z < 0, \qquad V\frac{\partial\phi}{\partial x} - g\eta = \frac{p}{\rho}, \quad z = 0, \qquad \frac{\partial\phi}{\partial z} + V\frac{\partial\eta}{\partial x} = 0, \quad z = 0, \tag{1}$$

where g is the acceleration due to gravity, ρ is the density of water, $p(x, y) = p_0 + A\delta(x)\delta(y)$, and $p_0 =$ const. is the atmospheric pressure. By δ we denote the Dirac function. The positive constant A characterises the intensity of the source. Replacing η by $\eta - p_0/g\rho$ we can omit the term p_0/ρ in the right-hand side of the first boundary condition in (1). Then the elevation of the free surface is measured with respect to its elevation in the absence of the source.

The boundary value problem (1) may be written in the form

$$\Delta\phi = 0, \quad z < 0, \qquad \frac{\partial^2\phi}{\partial x^2} + \nu\frac{\partial\phi}{\partial z} = \frac{A}{V\rho}\delta'(x)\delta(y), \quad z = 0. \tag{2}$$

For the elevation of the free surface we have the formula

$$\eta(x, y) = \frac{V}{g}\frac{\partial\phi}{\partial x}(x, y, 0) - \frac{A}{g\rho}\delta(x)\delta(y). \tag{3}$$

After the Fourier transform with respect to x, y

$$\tilde{\phi}(k_1, k_2, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x, y, z)\,\mathrm{e}^{\mathrm{i}(k_1x+k_2y)}\,\mathrm{d}x\,\mathrm{d}y, \tag{4}$$

the boundary value problem (2) takes the form

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\tilde{\phi}(k, z) - |k|^2\tilde{\phi}(k, z) = 0, \quad z < 0, \qquad -k_1^2\tilde{\phi}(k, 0) + \nu\frac{\mathrm{d}}{\mathrm{d}z}\tilde{\phi}(k, 0) = -\mathrm{i}\frac{A}{V\rho}k_1,$$

where $k = (k_1, k_2)$ and $|k| = (k_1^2 + k_2^2)^{1/2}$.

The ordinary differential equation for $\tilde{\phi}$ on the half-axis $z < 0$ with the condition that $\tilde{\phi}$ vanishes as $z \to -\infty$, leads to the formula

$$\tilde{\phi}(k, z) = \tilde{\phi}(k, 0)\,\mathrm{e}^{|k|z}.$$

Using the boundary condition for $\tilde{\phi}$ at $z = 0$, we find

$$(-k_1^2 + \nu|k|)\tilde{\phi}(k, 0) = -\mathrm{i}\frac{A}{V\rho}k_1. \tag{5}$$

By (3)

$$\tilde{\eta} = -\mathrm{i}k_1\frac{V}{g}\tilde{\phi} - \frac{A}{g\rho}.$$

Multiplying this equality by $-k_1^2 + \nu|k|$ and using (5), we obtain

$$(-k_1^2 + \nu|k|)\tilde{\eta}(k) = -k_1^2\frac{A}{g\rho} - \frac{A}{g\rho}(-k_1^2 + \nu|k|).$$

Hence,

$$(k_1^2 - \nu|k|)\tilde{\eta}(k) = \frac{A\nu}{g\rho}|k|. \tag{6}$$

This is the main equation for $\tilde{\eta}$. The rest of this section is devoted to solving (6). Note, that one cannot find $\tilde{\eta}$ by simple division of both sides of (6) by $k_1^2 - \nu|k|$, since the last expression vanishes for some values of k. Hence,

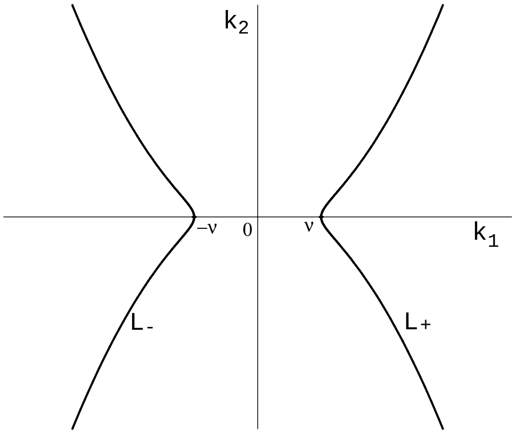

Fig. 1. Curves L_+ and L_- in plane (k_1, k_2) where Eq. (6) cannot be solved.

solution of (6) has to be understood in the sense of distributions. Solution of (6) is not unique, and we are looking for such a solution of (6) that the elevation $\eta(x, y)$ of the free surface has a character of a cylindrical wave downstream and decays rapidly upstream.

By L we denote the set on the plane (k_1, k_2) described by the equation

$$k_1^2 = \nu|k|. \tag{7}$$

It consists of two symmetric curves $L_\pm$ placed in the half-planes $\pm k_1 > 0$ (Fig. 1)

These curves $L_\pm$ may be described by the equations

$$k_1 = \pm\sqrt{\frac{1}{2}\left(\nu^2 + \sqrt{\nu^4 + 4k_2^2}\right)}.$$

In each quadrant of the plane (k_1, k_2), the set L can be expressed by a simpler equation

$$k_2 = \pm\frac{k_1}{\nu}(k_1^2 - \nu^2)^{1/2}, \quad |k_1| \geq \nu,$$

where the sign $\pm$ coincides with the sign of the product k_1k_2 in the quadrant. In polar coordinates (ρ, θ), $0 \leq \theta \leq 2\pi$, on the plane (k_1, k_2), the curves $L_\pm$ are determined by the equation

$$\rho = \pm\frac{\nu}{\cos^2\theta}.$$

Let $\mu_+\delta(L_+)$ be the δ-function of the curve L_+ with a continuous density $\mu_+ = \mu_+(k)$, i.e. $\mu_+\delta(L_+)$ is a distribution defined by the formula

$$\langle\mu_+\delta(L_+), h(k)\rangle = \int_{L_+} \mu_+(k)h(k)\,\mathrm{d}k_2,$$

where $h(k)$ is an arbitrary test function, which rapidly decreases at infinity. We have chosen k_2 as a parameter on L_+, since L_+ is uniquely projected on the k_2-axis. In a similar way one defines the distribution $\mu_-\delta(L_-)$. By definition

$$\mu\delta(L) = \mu_+\delta(L_+) + \mu_-\delta(L_-),$$

where $\mu = \mu_\pm$ on $L_\pm$

It is obvious that Eq. (6) has infinitely many solutions. In fact, the sum of any solution and the distribution $\mu\delta(L)$ with an arbitrary μ satisfies (6). One cannot ignore this property, since, for example, the backward motion of the source generates the same Eq. (6). The solutions $\tilde{\eta}_+$ and $\tilde{\eta}_-$, corresponding to opposite directions, differ by $\mu\delta(L)$ with a certain function μ.

In the exterior of any neighbourhood of L the solution of (6) is given by

$$\tilde{\eta}(k) = \frac{A\nu|k|}{g\rho(k_1^2 - \nu|k|)}. \tag{8}$$

The correct representation of $\tilde{\eta}$ in a neighbourhood of L should be such that the inverse Fourier transform of $\tilde{\eta}$ corresponds to a cylindrical wave downstream and decays rapidly upstream. We will show that these requirements define the solution $\tilde{\eta}$ of Eq. (6) uniquely, and the solution is given by the formula

$$\langle\tilde{\eta}(k), h(k)\rangle = \frac{A\nu}{g\rho}\left(\text{p.v.}\int_{R^2} -\frac{h(k)|k|\,\mathrm{d}k_1\,\mathrm{d}k_2}{k_1^2 - \nu|k|}\right) + \pi\mathrm{i}\langle\mu\delta(L), h(k)\rangle, \quad \mu = \frac{A\nu k_1|k|}{g\rho(2k_1^2 - \nu^2)}, \tag{9}$$

where h is an arbitrary test function which rapidly decreases at infinity. By p.v. $\int$ we denote the principal value of the integral, i.e.

$$\text{p.v.}\int_{R^2} \frac{h(k)|k|\,\mathrm{d}k_1\,\mathrm{d}k_2}{k_1^2 - \nu|k|} = \lim_{\varepsilon\to+0}\int_{|k_1^2-\nu|k||>\varepsilon} \frac{h(k)|k|\,\mathrm{d}k_1\,\mathrm{d}k_2}{k_1^2 - \nu|k|}. \tag{10}$$

The functional, defined by (10), will be denoted by

$$\text{p.v.}\frac{|k|}{k_1^2 - \nu|k|}.$$

Then formula (9) can be rewritten in the form

$$\tilde{\eta}(k) = \frac{A\nu}{g\rho}\left(\text{p.v.}\frac{|k|}{k_1^2 - \nu|k|}\right) + \frac{A\pi\mathrm{i}}{g\rho}\frac{k_1^3}{2k_1^2 - \nu^2}\delta(L). \tag{11}$$

One can justify formula (11) with the help of a well-known formula for Green's function of the Neumann–Kelvin problem. Green's function is a solution of the problem

$$G''_{xx} + G''_{yy} + G''_{zz} = 0, \quad z < 0, \qquad G''_{xx} + G'_z = 4\pi\delta(x, y), \quad z = 0 \tag{12}$$

with an appropriate behavior at infinity (it is bounded downstream and decays upstream). Conditions at infinity define Green's function uniquely. The velocity potential can be expressed through Green's function

$$\phi = \frac{A}{4\pi g\rho}\frac{\partial G}{\partial x}(\nu x, \nu y, \nu z),$$

and it is such a solution of (2), that the corresponding velocities decay rapidly upstream. So, according to (3) the function η is defined by the formula

$$\eta = \frac{A}{4\pi V\rho}\lim_{z\to-0}\frac{\partial^2}{\partial x^2}G(\nu x, \nu y, \nu z) - \frac{A}{g\rho}\delta(x, y). \tag{13}$$

By (13) we have

$$\tilde{\eta}(k) = \frac{-Ak_1^2}{4\pi g\rho\nu^2}\lim_{z\to-0}\tilde{G}\left(\frac{k_1}{\nu}, \frac{k_2}{\nu}, z\right) - \frac{A}{g\rho}. \tag{14}$$

Formula for Green's function can be found in [1] (formula (7)) or in [2] (formulas (6.6)–(6.8)). It can be shown that these formulae can be rewritten in the form

$$G = -\frac{1}{\pi}\text{p.v.}\int_{R^2}\frac{e^{z|k|-ixk_1-iyk_2}}{k_1^2-|k|}\,dk_1\,dk_2 - i\langle\mu\delta(L'), e^{-ixk_1-iyk_2}\rangle, \quad \mu = \frac{k_1}{2k_1^2-1},$$

where L' is the set L with $\nu = 1$. This formula gives a representation of Green's function as the inverse Fourier transform of the function $\tilde{G}$ such that for an arbitrary test function $h(k)$

$$\langle\tilde{G}(k,z), h(k)\rangle = \text{p.v.}\int_{R^2}\frac{-4\pi\,e^{z|k|}}{k_1^2-|k|}h(k)\,dk - 4\pi^2 i\langle\mu\delta(L'), h(k)\rangle,$$

where

$$\mu = \frac{k_1}{2k_1^2-1}.$$

Thus,

$$\tilde{G}(k,0) = \text{v.p.}\frac{-4\pi}{k_1^2-|k|} - 4\pi^2 i\frac{k_1}{2k_1^2-1}\delta(L').$$

This and (14) yield (11) if one takes into account that $\delta(L')$ is replaced by $\nu\delta(L)$ under the change $k \to k/\nu$ and that

$$\frac{k_1^3}{2k_1^2-\nu^2}\delta(L) = \frac{\nu k_1|k|}{2k_1^2-\nu^2}\delta(L).$$

3. Spectrum of a wave source in the immobile coordinate system

Let a point source move with a constant velocity V in the direction $l = (\cos\alpha, \sin\alpha)$ in the free surface. Suppose that this source is placed at the point (a, b) at the initial moment. The elevation of the free surface $H(x, y, t)$ at the point (x, y) at the moment t can be expressed by the formula

$$H(x,y,t) = \eta((x-a)\cos\alpha + (y-b)\sin\alpha - Vt, -(x-a)\sin\alpha + (y-b)\cos\alpha), \tag{15}$$

where η is the function defined in the previous section.

The Fourier transform of H with respect to variables (x, y) is given by

$$\tilde{H}(k,t) = \tilde{\eta}(k_1\cos\alpha + k_2\sin\alpha, -k_1\sin\alpha + k_2\cos\alpha)\,e^{i[k_1(a+Vt\cos\alpha)+k_2(b+Vt\sin\alpha)]} \tag{16}$$

with $\tilde{\eta}$ defined by (11).

The function $\tilde{H}$ has a singularity on the set

$$L(\alpha,\nu) = \{k : (k_1\cos\alpha + k_2\sin\alpha)^2 = \nu|k|\}. \tag{17}$$

It consists of two curves, which are obtained from $L_\pm$ (Fig. 1) by the counterclock rotation by the angle α.

4. Calculation of parameters of the motion of the source by the function $\tilde{H}(k, t)$ outside of some neighbourhood of the set $L(\alpha, \nu)$

We assume that the function $\tilde{H}(k, t)$ has been found experimentally. By (11) and (16) we obtain that the intensity coefficient A is connected with ν by the formula

$$A = \frac{g\rho}{\nu}|\tilde{H}(k,t)||\,|k|\cos^2(\theta-\alpha) - \nu|, \tag{18}$$

where k is any point of the plane which does not belong to $L(\alpha, \nu)$ and θ is the polar angle of the point k.

Writing this equality for two different points $k^{(1)}$ and $k^{(2)}$ with the same θ, we arrive at the following equation:

$$|\tilde{H}(k^{(1)},t)|\cdot||k^{(1)}|\cos^2(\theta-\alpha)-\nu|=|\tilde{H}(k^{(2)},t)|\cdot||k^{(2)}|\cos^2(\theta-\alpha)-\nu|, \tag{19}$$

which implies (in case $|\tilde{H}(k^{(1)},t)|\neq|\tilde{H}(k^{(2)},t)|$) that

$$\nu=C\cos^2(\theta-\alpha), \tag{20}$$

where constant C is given by one of the following expressions:

$$\frac{|k^{(1)}||\tilde{H}(k^{(1)},t)|-|k^{(2)}||\tilde{H}(k^{(2)},t)|}{|\tilde{H}(k^{(1)},t)|-|\tilde{H}(k^{(2)},t)|}\quad\text{or}\quad\frac{|k^{(1)}||\tilde{H}(k^{(1)},t)|+|k^{(2)}||\tilde{H}(k^{(2)},t)|}{|\tilde{H}(k^{(1)},t)|+|\tilde{H}(k^{(2)},t)|}.$$

One can extend Eq. (19) using one more point $k=k^{(3)}$ with the same polar angle θ. This will allow us to specify the constant C. Now α can be found by the equating the right-hand sides of (20) with different values of θ. One will need more than two different values of θ in order to find α uniquely. Then (20) and (18) define ν and A, respectively.

From (16) it also follows that

$$k_1(a+Vt\cos\alpha)+k_2(b+Vt\sin\alpha)=-\mathrm{i}\ln\left[\frac{g\rho}{A\nu}\tilde{H}(k,t)[|k|\cos^2(\theta-\alpha)-\nu]\right]. \tag{21}$$

From (21) written for two points $k^{(1)}$ and $k^{(2)}$ we get a system of two equations which enables us to find $a+Vt\cos\alpha$ and $b+Vt\sin\alpha$. Having these values for two moments $t^{(1)}$ and $t^{(2)}$ we find a, b and V (in fact, V could be found earlier, since $V=\sqrt{g/\nu}$).

One can suggest a different device to determine V and α. From (16) it follows that for any $k=k^{(0)}$ the following formula is valid:

$$\frac{\tilde{H}(k^{(0)},t)}{\tilde{H}(k^{(0)},t_1)}=\mathrm{e}^{\mathrm{i}[k_1^{(0)}V(t-t_1)\cos\alpha+k_2^{(0)}V(t-t_1)\sin\alpha]},$$

which gives the value

$$k_1^{(0)}V(t-t_1)\cos\alpha+k_2^{(0)}V(t-t_1)\sin\alpha.$$

Taking this value for two different points $k^{(0)}$ we obtain V and α.

5. Determination of the trajectory and of V by $L(\alpha,\nu)$

Suppose that the set $L(\alpha,\nu)$ is obtained experimentally. One may conduct such an experiment because the set $L(\alpha,\nu)$ does not depend on time and the function $\tilde{H}(k,t)$ tends to infinity as the point $k=(k_1,k_2)$ tends to this set. The set $L(\alpha,\nu)$ depends upon two parameters: α and $\nu=gV^{-2}$ only. After finding $L(\alpha,\nu)$ we can obtain the direction of the motion of the source $l=(\cos\alpha,\ \sin\alpha)$ and $V=(g/\nu)^{1/2}$.

We are going to show that the set $L(\alpha,\nu)$ (and the parameters α, ν) can be found, generally speaking, by three points of this set.

Introduce polar coordinates (ρ,θ), $0\le\theta\le 2\pi$, $\rho>0$, on the plane $k=(k_1,k_2)$. Eq. (17) can be rewritten in the form

$$\sqrt{\rho}\cos(\theta-\alpha)=\pm\sqrt{\nu}, \tag{22}$$

where signs $\pm$ correspond to different connected components of the set $L(\alpha,\nu)$.

Let the points $k^{(1)}=\rho_1\,\mathrm{e}^{\mathrm{i}\theta_1}$ and $k^{(2)}=\rho_2\,\mathrm{e}^{\mathrm{i}\theta_2}$ belong to $L(\alpha,\nu)$. Then

$$\sqrt{\rho_1}\cos(\theta_1-\alpha)=\pm\sqrt{\nu},\qquad\sqrt{\rho_2}\cos(\theta_2-\alpha)=\pm\sqrt{\nu}. \tag{23}$$

If the points $k^{(1)}, k^{(2)}$ are placed on the same connected component of $L(\alpha, \nu)$ then the signs in the right-hand side of (23) coincide and

$$\sqrt{\rho_1}\cos(\theta_1-\alpha)=\sqrt{\rho_2}\cos(\theta_2-\alpha).$$

In the opposite case

$$\sqrt{\rho_1}\cos(\theta_1-\alpha)=-\sqrt{\rho_2}\cos(\theta_2-\alpha).$$

By solving these equation we arrive at

$$\tan\alpha=\frac{\sqrt{\rho_1}\cos\theta_1-\sqrt{\rho_2}\cos\theta_2}{\sqrt{\rho_2}\sin\theta_2-\sqrt{\rho_1}\sin\theta_1} \tag{24}$$

in the first case and similarly

$$\tan\alpha=-\frac{\sqrt{\rho_1}\cos\theta_1+\sqrt{\rho_2}\cos\theta_2}{\sqrt{\rho_1}\sin\theta_1+\sqrt{\rho_2}\sin\theta_2} \tag{25}$$

in the second case.

Everywhere below we shall assume that the vector $l=(\cos\alpha, \sin\alpha)$ is directed into the right half-plane, i.e. $\cos\alpha \geq 0$. Then, if $V > 0$, then the source moves to the right. If $V < 0$ then it moves to the left.

Now by Eqs. (24) and (25) we obtain either

$$\cos\alpha=\frac{|\sqrt{\rho_2}\sin\theta_2-\sqrt{\rho_1}\sin\theta_2|}{|\rho_1+\rho_2-2\sqrt{\rho_1\rho_2}\cos(\theta_1-\theta_2)|^{1/2}}$$

or

$$\cos\alpha=\frac{|\sqrt{\rho_2}\sin\theta_2+\sqrt{\rho_1}\sin\theta_1|}{|\rho_1+\rho_2+2\sqrt{\rho_1\rho_2}\cos(\theta_1-\theta_2)|^{1/2}}$$

and, respectively, in the first case

$$\sin\alpha=\frac{\sqrt{\rho_1}\cos\theta_1-\sqrt{\rho_2}\cos\theta_2}{[\rho_1+\rho_2-2\sqrt{\rho_1\rho_2}\cos(\theta_1-\theta_2)]^{1/2}}\,\mathrm{sign}(\sqrt{\rho_2}\sin\theta_2-\sqrt{\rho_1}\sin\theta_1),$$

and in the second case

$$\sin\alpha=-\frac{\sqrt{\rho_1}\cos\theta_1+\sqrt{\rho_2}\cos\theta_2}{[\rho_1+\rho_2+2\sqrt{\rho_1\rho_2}\cos(\theta_1-\theta_2)]^{1/2}}\,\mathrm{sign}(\sqrt{\rho_2}\sin\theta_2+\sqrt{\rho_1}\sin\theta_1).$$

In the first case,

$$\nu=\frac{\rho_1\rho_2\sin^2(\theta_1-\theta_2)}{\rho_1+\rho_2-2\sqrt{\rho_1\rho_2}\cos(\theta_1-\theta_2)}.$$

In the second case,

$$\nu=\frac{\rho_1\rho_2\sin^2(\theta_1-\theta_2)}{\rho_1+\rho_2+2\sqrt{\rho_1\rho_2}\cos(\theta_1-\theta_2)}.$$

Hence, if points $k^{(1)}$ and $k^{(2)}$ belong to the set $L(\alpha, \nu)$, and if it is unknown whether they belong to the same connected component of $L(\alpha, \nu)$, then these points define two pairs of (α, ν) which can be calculated by the above formulae. Each of these pairs determines the set $L(\alpha, \nu)$ by formula (17). One of these sets corresponds to the situation when the points $k^{(1)}, k^{(2)}$ are located on the same connected component of $L(\alpha, \nu)$. The second set corresponds to the case, where $k^{(1)}, k^{(2)}$ belong to different components. One can specify the situation and pick up the appropriate set $L(\alpha, \nu)$ by taking one more point $k^{(3)}$ of $L(\alpha, \nu)$. The point $k^{(3)}$ has to be chosen not on the intersection of two sets $L(\alpha, \nu)$ which have been constructed. Thus, the trajectory of the source, i.e. the angle α, and the absolute value of the velocity can be found experimentally by a finite number of points on the curve $L(\alpha, \nu)$.

6. Calculation of the intensity of the source and its position by $\tilde{H}$ in a small neighbourhood of $L(\alpha, \nu)$

We assume that the parameters α and ν has been found and the set $L(\alpha, \nu)$ is known. Then we take the limit in formula (18) as $k \to k^{(0)}$, where $k^{(0)}$ is an arbitrary point of $L(\alpha, \nu)$. We obtain

$$A = \frac{g\rho}{\nu} \lim_{k \to k^{(0)}} [|\tilde{H}(k, t)[|k| \cos^2(\theta - \alpha) - \nu]|].$$

Similarly, by (21) we have

$$k_1^{(0)}(a + Vt\cos\alpha) + k_2^{(0)}(b + Vt\sin\alpha) = -\mathrm{i} \lim_{k \to k^{(0)}} \ln \left[\frac{g\rho}{A\nu} \tilde{H}(k, t)[|k| \cos^2(\theta - \alpha) - \nu]\right].$$

Taking two different points $k^{(0)}$ on $L(\alpha, \nu)$ we arrive at the algebraic system. By solving it we obtain $a + Vt\cos\alpha$ and $b + Vt\sin\alpha$. These values for $t = t^{(1)}$ and $t = t^{(2)}$ enable one to find a, b and α.

We give one more formula for the intensity of the source, which can be used when the remaining parameters of the source are known.

Let $\tilde{\eta}(k)$ be the Fourier transform of the elevation of the free surface in the moving coordinate system, found in Section 2. By p, q we denote two points of a connected component of the set L (both points belong to L_+ or to L_-). Let the function $\tilde{\eta}(k)$ be determined in the domain

$$D(\delta, p, q) = \{(k_1, k_2) : |k_1^2 - \nu|k|| < \delta, k_2(p) < k_2 < k_2(q)\},$$

where $k_2(p), k_2(q)$ are values of the coordinate k_2 at p, q and δ is a small positive number (see Fig. 2).

We integrate $\tilde{\eta}$ over $D(\delta, p, q)$, i.e. we put $h(k) = 1$ in $D(\delta, p, q)$ and $h(k) = 0$ outside of $D(\delta, p, q)$ in formula (9). It is easily seen that the first summand in the right-hand side of (9) tends to zero as $\delta \to 0$. The second term in

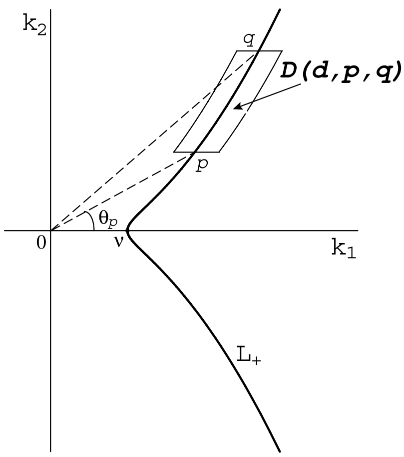

Fig. 2. $D(\delta, p, q)$ denotes domain where the intensity of the source and its position can be determined from the spectral density of the elevation of the free surface.

the right-hand side of (9) is equal to

$$\frac{A\pi \mathrm{i}}{g\rho}\int_{k_2(p)}^{k_2(q)} \frac{k_1^3}{2k_1^2-\nu^2}\,\mathrm{d}k_2. \tag{26}$$

We use the following change of variable in the latter integral: $k_2 = \nu\tau(1+\tau^2)^{1/2}$. Then by the equation of the set L we obtain that $k_1 = \pm\nu(1+\tau^2)^{1/2}$, where sign $\pm$ depends on the component $L_\pm$. Then $\tau = \pm k_2/k_1$ is equal to $\tan\theta$, where θ is an angle coordinate of the point k, $0 \le \theta \le 2\pi$. Integral (26) takes the form

$$\frac{A\pi \mathrm{i}}{g\rho}\int_{\tan\theta_p}^{\tan\theta_q} \frac{\nu^3(1+\tau^2)^{3/2}}{2\nu^2(1+\tau^2)-\nu^2}\frac{\nu(2\tau^2+1)}{\sqrt{1+\tau^2}}\,\mathrm{d}\tau = \frac{A\pi\nu^2 \mathrm{i}}{g\rho}\int_{\tan\theta_p}^{\tan\theta_q}(1+\tau^2)\,\mathrm{d}\tau$$
$$= \frac{A\pi\nu^2\mathrm{i}}{3g\rho}(3\tan\theta_q + \tan^3\theta_q - 3\tan\theta_p - \tan^3\theta_p).$$

Thus, we have the following approximate expression for the intensity A, which is valid for small δ:

$$A \approx \frac{(3g\rho\mathrm{i}/\pi\nu^2)\iint_{D(\delta,p,q)}\tilde{\eta}(k)\,\mathrm{d}k}{3\tan\theta_p + \tan^3\theta_p - 3\tan\theta_q - \tan^3\theta_q}. \tag{27}$$

Now let $\tilde{H}(k,t)$ be the Fourier transform of the elevation of the free surface in the immobile coordinate system. We denote the domain $D(\delta, p, q)$ rotated counterclock-wise by the angle α by D_α. Then from (16) it follows that

$$\tilde{\eta}(k_1\cos\alpha + k_2\sin\alpha, -k_1\sin\alpha + k_2\cos\alpha) = \tilde{H}(k,t)\,\mathrm{e}^{-\mathrm{i}[k_1(a+Vt\cos\alpha)+k_2(b+Vt\sin\alpha)]}. \tag{28}$$

Clearly, if we integrate the left-hand side in (28) over the domain D_α then the result will coincide with the integral of $\tilde{\eta}(k)$ over $D(\delta, p, q)$. Consequently, according to (28)

$$A \approx \frac{(3g\rho\mathrm{i}/\pi\nu^2)\iint_{D_\alpha}\tilde{H}(k,t)\,\mathrm{e}^{-\mathrm{i}[k_1(a+Vt\cos\alpha)+k_2(b+Vt\sin\alpha)]}\,\mathrm{d}k_1\,\mathrm{d}k_2}{3\tan\theta_p + \tan^3\theta_p - 3\tan\theta_q - \tan^3\theta_q}.$$

7. The case of several moving wave sources

Suppose that there are N point sources, which are moving in directions $l_j = (\cos\alpha_j, \sin\alpha_j)$ with the velocities V_j. The resulting elevation of the free surface is the sum of the elevations produced by each source:

$$H(x,y,t) = \sum_{1\le j\le N} H_j(x,y,t). \tag{29}$$

The function H_j is defined by formula (15) where the parameters $a, b, \alpha, V, \nu = gV^{-2}$ and the function η depend on j. Thus, after the Fourier transform, we have

$$\tilde{H}(k,t) = \sum_{1\le j\le N}\tilde{\eta}_j(k_1\cos\alpha_j + k_2\sin\alpha_j, -k_1\sin\alpha_j + k_2\cos\alpha_j)$$
$$\times\exp\mathrm{i}[k_1(\alpha_j + tV_j\cos\alpha_j) + k_2(b_j + tV_j\sin\alpha_j)], \tag{30}$$

where

$$\tilde{\eta}_j(k) = \frac{A_j\nu_j}{g\rho}\mathrm{v.p.}\frac{|k|}{k_1^2-\nu_j|k|} + \frac{A_j\pi\mathrm{i}}{g\rho}\frac{k_1^3}{2k_1^2-\nu_j^2}\delta(L_j). \tag{31}$$

Suppose that the sets $L_j = L(\alpha_j, \nu_j)$, $1 \le j \le N$, supporting singularities of the distribution $\tilde{H}(k,t)$, can be found experimentally. Then, as it was described in Section 5, from (30) we can obtain the direction α_j and the absolute value of the velocity, i.e. ν_j for each source. Assume, that afterwards we found $\tilde{H}$ in a small neighbourhood of $L(\alpha_j, \nu_j)$. Then we avoid the necessity to solve the system of equations which is related to more than one source.

8. The motion of a source on water waves

Here we will discuss very briefly a more general situation which is probably important in practice and which concerns the problem on the uniform motion of the source with account of harmonic oscillations of the water (see [3]).

Consider the coordinate system, moving with the point source, similarly to Section 2. The boundary conditions take the form

$$\left(V\frac{\partial}{\partial x} + \mathrm{i}\omega\right)\phi - g\eta = \frac{A\delta(x)\delta(y)}{\rho}, \qquad \frac{\partial\phi}{\partial z} + \left(V\frac{\partial}{\partial x} + \mathrm{i}\omega\right)\eta = 0,$$

where ω is the frequency of oscillations of the free surface.

The velocity potential satisfies the boundary value problem

$$\Delta\phi = 0, \qquad \left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\omega}{V}\right)^2\phi + \nu\frac{\partial\phi}{\partial z} = \frac{A}{V\rho}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\omega}{V}\right)\delta(x)\delta(y).$$

The elevation of the free surface η is expressed by the formula

$$\eta = \frac{1}{g}\left(V\frac{\partial}{\partial x} - \mathrm{i}\omega\right)\phi - \frac{A}{g\rho}\delta(x)\delta(y),$$

which leads to the following equation for $\tilde{\eta}(k)$:

$$\left[\left(k_1 + \frac{\omega}{V}\right)^2 - \nu|k|\right]\tilde{\eta}(k) = \frac{A}{g\rho}|k|$$

which can be investigated in the spirit of Sections 2–7.

In conclusion we remark that all our results can be extended to the case of a distributed pressure as well to the case of a submerged source.

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