L_p estimates of solutions to mixed boundary value problems for the Stokes system in polyhedral domains.

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Abstract

A mixed boundary value problem for the Stokes system in a polyhedral domain is considered. Here different boundary conditions (in particular, Dirichlet, Neumann, free surface conditions) are prescribed on the sides of the polyhedron. The authors prove the existence of solutions in (weighted and non-weighted) L_p Sobolev spaces and obtain regularity assertions for weak solutions. The results are based on point estimates of Green's matrix.

0 Introduction

Steady-state flows of incompressible viscous Newtonian fluids are modelled by the Navier-Stokes equations

$$-\nu\,\Delta u + (u\cdot\nabla)\,u + \nabla p = f, \qquad \nabla\cdot u = 0 \tag{0.1}$$

for the velocity u and the pressure p. To this system, one may add a variety of boundary conditions on different parts of the boundary (see e.g. [12]). For example, there is the Dirichlet condition $u = 0$ on solid walls. On other parts of the boundary (an artificial boundary such as the exit of a canal, or a free surface) a no-friction condition $2\nu\varepsilon(u)\,n - pn = 0$ may be useful. Here $\varepsilon(u)$ denotes the matrix with the components $\frac{1}{2}(\partial_{x_i}u_j + \partial_{x_j}u_i)$, and n is the outward normal. It is also of interest to consider boundary conditions containing components of the velocity and of the friction. Frequently used combinations are the normal component of the velocity and the tangential component of the friction (slip condition for uncovered fluid surfaces) or the tangential component of the velocity and the normal component of the friction (condition for in/out-stream surfaces).

In the present paper, we consider a mixed boundary value problem for the linear Stokes system

$$-\Delta u + \nabla p = f, \qquad -\nabla\cdot u = g \tag{0.2}$$

in a three-dimensional domain of polyhedral type, where components of the velocity and/or the friction are given on the boundary. To be more precise, we have one of the following boundary conditions on each side Γ_j:

(i) $u = h$,

(ii) $u_\tau = h, \quad -p + 2\varepsilon_{n,n}(u) = \phi$,

(iii) $u_n = h, \quad \varepsilon_{n,\tau}(u) = \phi$,

(iv) $-pn + 2\varepsilon_n(u) = \phi$,

where $u_n = u\cdot n$ denotes the normal and $u_\tau = u - u_n n$ the tangential component of u, $\varepsilon_n(u)$ is the vector $\varepsilon(u)\,n$, $\varepsilon_{n,n}(u)$ is the normal component and $\varepsilon_{n,\tau}(u)$ the tangential component of $\varepsilon_n(u)$.

In the last decades a number of mathematical papers appeared which treat elliptic boundary value problems in piecewise smooth domains. For a historical account of this development we refer to the books of Grisvard [5], Dauge [2], Nazarov and Plamenevskiĭ [26], Kozlov, Maz'ya and Rossmann [9]. Our main goal is to prove regularity assertions for weak solutions of the mixed problem to the Stokes system. For

the Dirichlet problem such results were obtained in papers by Maz'ya and Plamenevskiĭ [19] and Dauge [3]. Fabes, Kenig and Verchota [4] studied the Dirichlet problem for the Stokes system in Lipschitz domains. Spectral properties of operator pencils generated by the mixed boundary value problem in a cone were investigated by Kozlov, Maz'ya and Rossmann [10, Ch.6].

It is well-known that the singularities of solutions of elliptic problems near edges and corners have power (or power-logarithmic) form. For this reason, it is natural to use weighted Sobolev spaces, where the weights are powers of the distances to the edges and corners. Special boundary value problems (e.g., the Dirichlet problem) can be studied in weighted Sobolev spaces with "homogeneous" norms (see e.g. [19, 20]). However, the more general problem with boundary conditions (i)–(iv) requires the use of weighted spaces with "nonhomogenous" norms. This makes the consideration of the boundary value problem more difficult. On the other hand, in some cases (e.g. the Dirichlet problem in convex polyhedral domains), the results can be improved when considering solutions in weighted spaces with nonhomogeneous norms. We also note that the class of weighted Sobolev spaces with nonhomogeneous norms contains the nonweighted Sobolev spaces.

The largest part of the paper (Sections 3 and 4) concerns the boundary value problem for the Stokes system in a polyhedral cone $\mathcal{K}$ with sides $\Gamma_1, \dots, \Gamma_n$ and edges $M_1, \dots, M_n$. Section 3 deals with the existence of solutions $(u, p) \in W^{2,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{1,s}_{\beta,\delta}(\mathcal{K})$ of the boundary value problem if $f \in W^{0,s}_{\beta,\delta}(\mathcal{K})^3$, $g \in W^{1,s}_{\beta,\delta}(\mathcal{K})$, and the boundary data h, ϕ are from the corresponding trace spaces. Here, for integer $l \geq 0$, $\beta \in \mathbb{R}$, $\delta = (\delta_1, \dots, \delta_n) \in \mathbb{R}^n$, and $1 < s < \infty$, the space $W^{l,s}_{\beta,\delta}(\mathcal{K})$ is defined as the set all functions u such that

$$\rho^{\beta-l+|\alpha|} \prod_{k=1}^{n} \Big(r_k/\rho\Big)^{\delta_k} \partial_x^\alpha u \in L_s(\mathcal{K}) \quad \text{for } |\alpha| \leq l,$$

ρ is the distance to the vertex of the cone, and r_k denotes the distance to the edge $M_k j$. The estimates of the solutions in these spaces are essentially based on point estimates for Green's matrix obtained in our previous paper [24, 25]. It is shown that there is a uniquely determined solution if g and the boundary data satisfy certain compatibility conditions on the edges, the line $\operatorname{Re}\lambda = 2-\beta-3/s$ is free of eigenvalues of a certain operator pencil $\mathfrak{A}(\lambda)$, and $\max(2-\mu_k, 0) < \delta_k + 2/s < 2$ for $k = 1, \dots, n$, where μ_k are certain positive numbers depending on the angle θ_k at the edge M_k. For example, in the case of the Dirichlet problem, we have $\mu_k = \pi/\theta_k$ if $\theta_k < \pi$, while μ_k is the smallest positive solution of the equation $\sin(\mu\theta_k) + \mu \sin\theta_k = 0$ if $\theta_k > \pi$. Estimates for the eigenvalues of the pencil $\mathfrak{A}(\lambda)$ can be found e.g. in [3, 10, 11, 19].

In Section 4 we consider weak solutions of the boundary value problem, i.e. vector functions $(u, p) \in W^{1,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{0,s}_{\beta,\delta}(\mathcal{K})$ satisfying

$$2\int_{\mathcal{K}} \sum_{i,j=1}^{3} \varepsilon_{i,j}(u)\, \varepsilon_{i,j}(v)\, dx - \int_{\mathcal{K}} p\, \nabla\cdot v\, dx = F(v) \quad \text{for all } v \in W^{1,s'}_{-\beta,-\delta}(\mathcal{K})^3,\ S_j v = 0 \text{ on } \Gamma_j,$$
$$-\nabla\cdot u = g \ \text{ in } \mathcal{K}, \quad S_j u = h_j \ \text{ on } \Gamma_j,\ j = 1, \dots, n.$$

Here $S_j u = u$ in the case of the Dirichlet condition on Γ_j, $S_j u = u_\tau$ in the case of condition (ii), and $S_j u = u_n$ in the case of condition (iii). We prove that a unique weak solution exists if the boundary data h_j satisfy certain compatibility conditions on the edges, the line $\operatorname{Re}\lambda = 1-\beta-3/s$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$, and $\max(1-\mu_k, 0) < \delta_k + 2/s < 1$ for $k = 1, \dots, n$. In the case $s = 2$, the last condition can be replaced by $-\min(\mu_k, 1) < \delta_k \leq 0$.

Moreover, we obtain regularity assertions for the weak solution. For example, let $(u, p) \in W^{1,2}_{0,0}(\mathcal{K})^3 \times L_2(\mathcal{K})$ be the weak solution of the boundary value problem, where

$$F \in (W^{1,2}_{0,0}(\mathcal{K})^*)^3 \cap (W^{1,s'}_{-\beta,-\delta}(\mathcal{K})^*)^3, \quad g \in L_2(\mathcal{K}) \cap W^{0,s}_{\beta,\delta}(\mathcal{K}), \quad h_j \in W^{1/2,2}_{0,0}(\Gamma_j) \cap W^{1-1/s,s}_{\beta,\delta}(\Gamma_j),$$

$s' = s/(s-1)$. If $\max(1-\mu_k, 0) < \delta_k + 2/s < 1$ and there are no eigenvalues of the pencil $\mathfrak{A}(\lambda)$ in the strip $-1/2 < \operatorname{Re}\lambda \leq 1-\beta-3/s$, then

$$(u, p) \in W^{1,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{0,s}_{\beta,\delta}(\mathcal{K}).$$

Suppose the functional $F \in (W^{1,2}_{0,0}(\mathcal{K})^*)^3$ has the form

$$F(v) = \int_{\mathcal{G}} f \cdot v \, dx + \sum \int_{\Gamma_j} \phi_j \cdot v \, dx$$

where $f \in W^{l-2,s}_{\beta,\delta}(\mathcal{K})^3$, $\phi_j \in W^{l-1-1/s,s}_{\beta,\delta}(\Gamma_j)$, $l \geq 2$. If, moreover, $g \in W^{l-1,s}_{\beta,\delta}\mathcal{K})$, $h_j \in W^{l-1/s,s}_{\beta,\delta}(\Gamma_j)$, the data g, h_j and ϕ_j satisfy certain compatibility conditions on the edges of the cone, the components δ_k of δ satisfy the inequalities $\max(l-\mu_k, 0) < \delta_k + 2/s < l$, and the strip $-1/2 < \operatorname{Re}\lambda \leq l - \beta - 3/s$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$, then

$$(u,p) \in W^{l,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{l-1,s}_{\beta,\delta}(\mathcal{K}).$$

In Section 5 we consider the boundary value problem for the Stokes system (0.2) in a bounded domain $\mathcal{G}$ of polyhedral type. Under certain compatibility conditions, there exists a weak solution $(u,p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ which is unique up to a certain subspace of linear vector functions. Using the results of Section 4, we obtain regularity assertions for this solution.

As an example, we consider the weak solution of the Dirichlet problem to the Stokes system in a polyhedron with boundary data $h_j = 0$. From our results and from estimates for the eigenvalues of the pencil $\mathfrak{A}(\lambda)$ (see [3, 10, 11, 19]) it follows that

$$(u,p) \in W^{1,s}(\mathcal{G})^3 \times L_s(\mathcal{G}), \quad 2 < s \leq 3,$$

if $f \in W^{-1,s}(\mathcal{G})^3$, $g \in L_s(\mathcal{G})$. If the polyhedron $\mathcal{G}$ is convex, then this result is true for all $s > 2$. Furthermore, the following $W^{2,s}$-regularity result holds for the weak solution $(u,p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ of the Dirichlet problem:

$$(u,p) \in W^{2,s}(\mathcal{K})^3 \times W^{1,s}(\mathcal{K}), \quad 1 < s \leq 4/3,$$

if $f \in W^{-1,2}(\mathcal{G})^3 \cap L_s(\mathcal{G})^3$, $g \in L_2(\mathcal{G}) \cap W^{1,s}(\mathcal{G})$. If the edge angles are less than $3 \arccos\frac{1}{4} \approx 1.2587\pi$, then this result is true for $1 < s \leq 3/2$. In the case of a convex polyhedron, this result is true for $1 < s \leq 2$. If, moreover, the edge angles are less then $\frac{3}{4}\pi$, then the result holds even for $1 < s < 3$. However, in the case $s > 2$ the trace of the function g on the edges must be equal to zero, while in the case $s = 2$ the function g must be such that

$$\int_{\mathcal{G}} \rho_j^{-1} \prod_{k \in N_j} (r_k/\rho_j)^{-1} \, |g(x)|^2 \, dx < \infty, \tag{0.3}$$

for every j, where ρ_j denotes the distance to the vertex O_j, r_k denotes the distance to the edge M_k, and N_j is the set of all k such that $\bar{M}_k \ni O_j$. In the case $s = 2$ the $W^{2,s}$-regularity result for convex polyhedrons was also proved by Dauge [3].

Similar $W^{1,s}$ and $W^{2,s}$ regularity results can be obtained for Neumann and mixed problems. Let us consider, for example, the mixed boundary value problem with boundary conditions (i)–(iii). We assume that for every edge, the Dirichlet condition is given on at least one of the adjoining sides. Then the following $W^{1,s}$ regularity result holds. The weak solution $(u,p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ belongs to $W^{1,s}(\mathcal{G})^3 \times L_s(\mathcal{G})$ if $F \in (W^{1,s'}(\mathcal{G})^*)^3$, $g \in L_s(\mathcal{G})$, $h_j = 0$, $2 \leq s \leq 8/3$, $s' = s/(s-1)$. If at every edge with boundary condition (ii) or (iii) on one of the adjoining sides, the angle is less than $\frac{3}{2}\pi$, then this result is even true for $2 \leq s \leq 3$.

Lastly, we present a $W^{2,s}$ regularity result for the mixed problem with Dirichlet and Neumann boundary conditions. Suppose that $(u,p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ is a weak solution of this problem with data $f \in L_s(\mathcal{G})^3$, $g \in W^{1,s}(\mathcal{G})$, $h_j \in W^{2-1/s,s}(\Gamma_j)^3$, and $\phi_j \in W^{1-1/s,s}(\Gamma_j)^3$, $1 < s \leq 8/7$. Then $(u,p) \in W^{2,s}(\mathcal{G})^3 \times W^{1,s}(\mathcal{G})$.

Other examples are given at the end of Section 5. In a forthcoming paper, we extend the results to mixed problems for the nonlinear Navier-Stokes system.

1 Weighted Sobolev spaces

1.1 Weighted Sobolev spaces in a dihedron

Let $\mathcal{D}$ be the dihedron

$$\mathcal{D} = \{x = (x', x_3) : \ x' \in K, \ x_3 \in \mathbb{R}\}, \tag{1.1}$$

where K is an infinite angle which has the form $\{x' = (x_1, x_2) \in \mathbb{R}^2 :\ 0 < r < \infty,\ -\theta/2 < \varphi < \theta/2\}$ in polar coordinates r, φ. The boundary of $\mathcal{D}$ consists of the half-planes $\Gamma^{\pm} : \varphi = \pm\theta/2$ and the edge M. We denote by $V_\delta^{l,s}(\mathcal{D})$ and $W_\delta^{l,s}(\mathcal{D})$, $1 < s < \infty$, the weighted Sobolev spaces with the norms

$$\|u\|_{V_\delta^{l,s}(\mathcal{D})} = \Big(\int\limits_{\mathcal{D}} \sum_{|\alpha|\le l} |x'|^{s(\delta-l+|\alpha|)} \left|\partial_x^\alpha u\right|^s dx\Big)^{1/s}, \quad \|u\|_{W_\delta^{l,s}(\mathcal{D})} = \Big(\int\limits_{\mathcal{D}} \sum_{|\alpha|\le l} |x'|^{s\delta} \left|\partial_x^\alpha u\right|^s dx\Big)^{1/s}.$$

Analogously, the spaces $V_\delta^{l,s}(K)$ and $W_\delta^{l,s}(K)$ are defined (here in the above norms $\mathcal{D}$ has to be replaced by K and dx by dx'). By Hardy's inequality, every function $u \in C_0^\infty(\overline{\mathcal{D}})$ satisfies the inequality

$$\int_{\mathcal{D}} r^{s(\delta-1)} |u|^2\, dx \le c \int_{\mathcal{D}} r^{s\delta}\, |\nabla u|^s\, dx$$

for $\delta > 1 - 2/s$ with a constant c depending only on s and δ. Consequently, the space $W_\delta^{l,s}(\mathcal{D})$ is continuously imbedded into $W_{\delta-1}^{l-1,s}(\mathcal{D})$ if $\delta > 1 - 2/s$. If $\delta > l - 2/s$, then $W_\delta^{l,s}(\mathcal{D}) \subset V_\delta^{l,s}(\mathcal{D})$.

Let $V_\delta^{l-1/s,s}(\Gamma^{\pm})$ and $W_\delta^{l-1/s,s}(\Gamma^{\pm})$ be the trace spaces corresponding to $V_\delta^{l,s}(\mathcal{D})$ and $W_\delta^{l,s}(\mathcal{D})$, respectively. The trace spaces for $V_\delta^{l,s}(K)$ and $W_\delta^{l,s}(K)$ on the sides $\gamma^{\pm}$ of K are denoted by $V_\delta^{l-1/s,s}(\gamma^{\pm})$ and $W_\delta^{l-1/s,s}(\gamma^{\pm})$, respectively.

Note that the trace of a function $u \in W_\delta^{l,s}(\mathcal{D})$ or $u \in W_\delta^{l-1/s,s}(\Gamma^{\pm})$ on the edge M exists if $-2/s < \delta < l - 2/s$. It belongs to the Sobolev-Slobodetskiĭ space $W^{l-\delta-2/s,s}(M)$ if $l - \delta - 2/s$ is not integer. There is the following relation between the spaces $V_\delta^{l,s}$ and $W_\delta^{l,s}$ (see [21, 27]).

Lemma 1.1 1) *Let $u \in W_\delta^{l,s}(\mathcal{D})$, $-2/s < \delta \le l - 2/s$, If $\delta + 2/s$ is not integer, then*

$$u \in V_\delta^{l,s}(\mathcal{D}) \Leftrightarrow \partial_{x'}^\alpha u(x) = 0 \quad \text{on } M \text{ for } |\alpha| < l - \delta - 2/s. \tag{1.2}$$

If $\delta + 2/s$ is integer, then for the inclusion $u \in V_\delta^{l,s}(\mathcal{D})$ it is necessary and sufficient that the conditions (1.2) *and*

$$\int_{\mathcal{D}} r^{-2} \left|\partial_{x'}^\alpha u(x)\right|^s dx < \infty \quad \text{for } |\alpha| = l - \delta - 2/s$$

are satisfied.

2) *Let $u \in W_\delta^{l-1/s,s}(\Gamma^+)$, $-2/s < \delta \le l - 2/s$. If $\delta + 2/s$ is not integer, then*

$$u \in V_\delta^{l,s}(\Gamma^+) \Leftrightarrow \partial_r^j u(r, x_3) = 0 \quad \text{on } M \text{ for } j < l - \delta - 2/s. \tag{1.3}$$

If $\delta + 2/s$ is integer, then for the inclusion $u \in V_\delta^{l,s}(\mathcal{D})$ is it necessary and sufficient that the conditions (1.3) *and*

$$\int_{\mathbb{R}} \int_0^\infty r^{-1} \left|\partial_r^{l-\delta-2/s} u(r, x_3)\right|^s dr\, dx_3 < \infty$$

are satisfied.

We introduce the following extension operator E mapping $W^{l-\delta-2/s,s}(M)$ into $W_\delta^{l,s}(\mathcal{D})$ or $W_\delta^{l-1/p,p}(\Gamma^{\pm})$.

$$(Ef)(x) = \chi(r) \int_{\mathbb{R}} f(x_3 + tr)\, \psi(t)\, dt, \tag{1.4}$$

where $r = |x'|$, χ is a smooth function on $(0, \infty)$ with support in $[0, 1]$ equal to 1 in $(0, \frac{1}{2})$, and ψ is a smooth function on $\mathbb{R}$ with support in $[-1, +1]$ satisfying the condition

$$\int_{\mathbb{R}} \psi(t)\, dt = 1, \quad \int_{\mathbb{R}} t^j\, \psi(t)\, dt = 0 \ \text{ for } j = 1, 2, \ldots, l.$$

Since the function Ef depends only on r and x_3, it can be also considered as a function on the half-planes Γ^+ and Γ^-. For the following lemma we refer to [21].

Lemma 1.2 *Let $-2/s < \delta < l - 2/s$ and $\delta + 2/s$ be not integer. Then*

$$(\partial_{x_3}^j Ef)|_M = \partial_{x_3}^j f \quad \text{for } j < l - \delta - 2/s. \tag{1.5}$$

Moreover, if Ef is considered as a function on $\mathcal{D}$, then $\partial_{x'}^\alpha Ef \in V_\delta^{l-|\alpha|,s}(\mathcal{D})$ for $1 \le |\alpha| \le l$. In particular, the trace of $\partial_{x'}^\alpha Ef$ on M vanishes for $1 \le |\alpha| < l - \delta - 2/s$.

If Ef is considered as a function on $\Gamma^\pm$, then $\partial_r^j Ef \in V_\delta^{l-j-1/s,s}(\Gamma^\pm)$ for $j = 1, \dots, l-1$.

If $f \in W_\delta^{2-1/s,s}(\Gamma^+)$, $\delta < 1 - 2/2$, then the traces of f and $\partial_{x_3} f$ on M exist. Obviously, $(\partial_{x_3} f)|_M = \partial_{x_3}(f|_M)$. The following result for the limit case $\delta = 1 - 2/s$ follows from [27, Le.7, Rem.4].

Lemma 1.3 *If $f \in W_{1-2/s}^{2-1/s,s}(\Gamma^+)$ and $f|_M = 0$, then*

$$\int_{\mathbb{R}} \int_0^\varepsilon r^{-1} \left|\partial_{x_3} f(r, x_3)\right|^s dr\, dx_3 \le c \left\|f\right\|^s_{W_{1-2/s}^{2-1/s,s}(\Gamma^+)}$$

for arbitrary positive ε.

For the following lemma we refer to [23, Le.2.1].

Lemma 1.4 *If $\partial_{x_3}^j u \in V_\delta^{2,s}(\mathcal{D})$, $1 < s < 2$, for $j = 0, 1, 2$, then $u \in V_{\delta-3+2/s}^{0,2}(\mathcal{D})$.*

Corollary 1.1 *If $\partial_{x_3}^j u \in W_\delta^{3,s}(\mathcal{D})$ for $j = 0, 1, 2$, where $1 < s < 2$ and $\delta > 2 - 2/s$, then $u \in W_{\delta-3+2/s}^{1,2}(\mathcal{D})$.*

Proof: By Lemma 1.4, the inclusion $\partial_{x_3}^j u \in W_\delta^{2,s}(\mathcal{D}) = V_\delta^{2,p}(\mathcal{D})$ for $j \le 2$ implies $u \in V_{\delta-3+2/s}^{0,2}(\mathcal{D})$. Furthermore, by our assumptions, $\partial_{x_3}^j \nabla u \in V_\delta^{2,s}(\mathcal{D})^3$ for $j = 0, 1, 2$ and, therefore, $\nabla u \in V_{\delta-3+2/s}^{0,2}(\mathcal{D})^3$. The result follows. ∎

1.2 Weighted Sobolev spaces in a cone

Let $\mathcal{K}$ be the cone

$$\mathcal{K} = \{x \in \mathbb{R}^3 : \ x/|x| \in \Omega\} \tag{1.6}$$

where Ω is a domain on the unit sphere of polygonal type ...

We denote by $\mathcal{S}$ the set $M_1 \cup \dots \cup M_n \cup \{0\}$ of all singular boundary points. Furthermore, for an arbitrary point $x \in \mathcal{K}$ we denote by $\rho(x) = |x|$ the distance to the vertex of the cone, by $r_j(x)$ the distance to the edge M_j, and by $r(x)$ the regularized distance to $\mathcal{S}$, i.e., an infinitely differentiable function in $\mathcal{K}$ which satisfies the estimates

$$c_1 \operatorname{dist}(x, \mathcal{S}) < r(x) < c_2 \operatorname{dist}(x, \mathcal{S}) \quad \text{and} \quad |\partial_x^\alpha r(x)| \le c_\alpha \operatorname{dist}(x, \mathcal{S})^{1-|\alpha|}$$

for all $x \in \mathcal{K}$ and all multi-indices α. Here c_1, c_2, c_α are positive constants independent of x.

Let l be a nonnegative integer, $\beta \in \mathbb{R}$, $\delta = (\delta_1, \dots, \delta_n) \in \mathbb{R}^n$, $\delta_j > -2/s$ for $j = 1, \dots, n$, and $1 < s < \infty$. We define $V_{\beta,\delta}^{l,s}(\mathcal{K})$ and $W_{\beta,\delta}^{l,s}(\mathcal{K})$ as the weighted Sobolev spaces with the norms

$$\|u\|_{V_{\beta,\delta}^{l,s}(\mathcal{K})} = \Big(\int_{\mathcal{K}} \sum_{|\alpha| \le l} |x|^{s(\beta-l+|\alpha|)} \ |\partial_x^\alpha u|^s \prod_{j=1}^n \big(\frac{r_j(x)}{|x|}\big)^{s(\delta_j - l + |\alpha|)} dx \Big)^{1/s},$$

$$\|u\|_{W_{\beta,\delta}^{l,s}(\mathcal{K})} = \Big(\int_{\mathcal{K}} \sum_{|\alpha| \le l} |x|^{s(\beta-l+|\alpha|)} \ |\partial_x^\alpha u|^s \prod_{j=1}^n \big(\frac{r_j}{\rho}\big)^{s\delta_j} dx \Big)^{1/s},$$

respectively. Furthermore, we introduce the following notation. If d is real number, then $V_{\beta,d}^{l,s}(\mathcal{K})$ and $W_{\beta,d}^{l,s}(\mathcal{K})$ denote the above introduced spaces with $\delta = (d, \dots, d)$. If $\delta = (\delta_1, \dots, \delta_n)$ and d is a real number, then we define $W_{\beta,\delta+d}^{l,s}(\mathcal{K}) = W_{\beta,\delta'}^{l,s}(\mathcal{K})$, where $\delta' = (\delta_1 + d, \dots, \delta_n + d)$.

Passing to spherical coordinates $\rho = |x|, \omega = x/|x|$, one obtains the following equivalent norm in $W^{l,s}_{\beta,\delta}(\mathcal{K})$:

$$\|u\| = \Big(\int_0^\infty \rho^{s(\beta-l)+2} \sum_{k=0}^{l} \|(\rho\partial_\rho)^k u(\rho,\cdot)\|^s_{W^{l-k,s}_\delta(\Omega)} \, d\rho \Big)^{1/s},$$

where the norm in $W^{l,s}_\delta(\Omega)$ is given by

$$\|v\|_{W^{l,s}_\delta(\Omega)} = \Big(\int\limits_{\substack{\mathcal{K} \\ 1<|x|<2}} \sum_{|\alpha|\le l} \big|\partial_x^\alpha v(x)\big|^s \prod_{j=1}^{n} r_j^{s\delta_j} \, dx \Big)^{1/s}$$

(here the function v on Ω is extended by $v(x) = v\big(x/|x|\big)$ to the cone $\mathcal{K}$).

By Hardy's inequality, the space $W^{l+1,s}_{\beta+1,\delta'}(\Omega)$ is continuously imbedded into $W^{l,s}_{\beta,\delta}(\Omega)$ if $\delta = (\delta_1, \dots, \delta_n)$, $\delta' = (\delta'_1, \dots, \delta'_n)$ are such that $\delta_j, \delta'_j > -2/s$ and $\delta'_j - \delta_j \le 1$ for $j = 1, \dots, n$. This implies that, under the above assumptions on δ and δ', there is the imbedding $W^{l+1,s}_{\beta+1,\delta'}(\mathcal{K}) \subset W^{l,s}_{\beta,\delta}(\mathcal{K})$. In particular, we have $V^{l,s}_{\beta,\delta}(\mathcal{K}) = W^{l,s}_{\beta,\delta}(\mathcal{K})$ if $\delta_j > l - 2/s$ for $j = 1, \dots, n$.

Let ζ_k be smooth functions depending only on $\rho = |x|$ such that

$$\operatorname{supp}\zeta_k \subset (2^{k-1}, 2^{k+1}), \quad \sum_{k=-\infty}^{+\infty} \zeta_k = 1, \quad |(\rho\partial_\rho)^j \zeta_k(\rho)| \le c_j \tag{1.7}$$

with constants c_j independent of k and ρ. It can be easily shown (cf. [9, Le.6.1.1]) that the norms in $V^{l,s}_{\beta,\delta}(\mathcal{K})$ and $W^{l,s}_{\beta,\delta}(\mathcal{K})$ are equivalent to

$$\|u\| = \Big(\sum_{k=-\infty}^{+\infty} \|\zeta_k u\|^s_{V^{l,s}_{\beta,\delta}(\mathcal{K})} \Big)^{1/s} \quad \text{and} \quad \|u\| = \Big(\sum_{k=-\infty}^{+\infty} \|\zeta_k u\|^s_{W^{l,s}_{\beta,\delta}(\mathcal{K})} \Big)^{1/s}, \tag{1.8}$$

respectively. We denote the trace spaces for $V^{l,s}_{\beta,\delta}(\mathcal{K})$ and $W^{l,s}_{\beta,\delta}(\mathcal{K})$, $l \ge 1$, on Γ_j by $V^{l-1/s,s}_{\beta,\delta}(\Gamma_j)$, and $W^{l-1/s,s}_{\beta,\delta}(\Gamma_j)$, respectively. The norms in these spaces are also equivalent to

$$\|u\| = \Big(\sum_{k=-\infty}^{+\infty} \|\zeta_k u\|^s_{V^{l-1/s,s}_{\beta,\delta}(\Gamma_j)} \Big)^{1/s} \quad \text{and} \quad \|u\| = \Big(\sum_{k=-\infty}^{+\infty} \|\zeta_k u\|^s_{W^{l-1/s,s}_{\beta,\delta}(\Gamma_j)} \Big)^{1/s}, \tag{1.9}$$

respectively. The trace of a function $u \in W^{l,s}_{\beta,\delta}(\mathcal{K})$ (or $u \in W^{l-1/s,s}_{\beta,\delta}(\Gamma_j)$) on the edge M_k exists if $\delta_k < l - 2/s$. Using Lemma 1.3, we obtain the following result.

Lemma 1.5 *Let Γ be a side of the cone $\mathcal{K}$ adjacent to the edge M_k and let $f \in W^{2-1/s,s}_{\beta,\delta}(\Gamma)$, where $\delta_k = 1 - 2/s$. If $f|_{M_k} = 0$, then*

$$\int_0^\infty \int_0^{\varepsilon t} t^{s(\beta-1)+2} \, r^{-1} \, \big|\partial_t f(r,t)\big|^s \, dr \, dt < \infty$$

for sufficiently small $\varepsilon > 0$. Here $r = \operatorname{dist}(x, M_k)$ and t is the coordinate on M_k.

Proof: Let $\eta_k = \zeta_{k-1} + \zeta_k + \zeta_{k+1}$, $\tilde\eta_k(x) = \eta_k(2^k x)$, and $\tilde f(r,t) = f(2^k r, 2^k t)$. By our assumptions on the functions ζ_k, we have $\tilde\eta_k(x) = 1$ for $1/2 < |x| < 2$. From Lemma 1.3 it follows that

$$\int_{1/2}^2 \int_0^{\varepsilon t} r^{-1} \, \big|\partial_t \tilde f(r,t)\big|^s \, dr \, dt \le c \, \|\tilde\eta_k \tilde f\|^s_{W^{2-1/s,s}_{\beta,\delta}(\Gamma)}$$

Here,

$$\int_{1/2}^2 \int_0^{\varepsilon t} r^{-1} \, \big|\partial_t \tilde f(r,t)\big|^s \, dr \, dt = 2^{k(s-1)} \int_{2^{k-1}}^{2^{k+1}} \int_0^{\varepsilon t} r^{-1} \, \big|\partial_t f(r,t)\big|^s \, dr \, dt \quad \text{and}$$
$$\|\tilde\eta_k \tilde f\|^s_{W^{2-1/s,s}_{\beta,\delta}(\Gamma)} \le c \, 2^{-ks(\beta-2)-3k} \, \|\eta_k f\|^s_{W^{2-1/s,s}_{\beta,\delta}(\Gamma)}.$$

This implies

$$\int_{2^{k-1}}^{2^{k+1}} \int_0^{\varepsilon t} t^{s(\beta-1)+2}\, r^{-1} \left|\partial_t f(r,t)\right|^s dr\, dt \le c\, \|\eta_k f\|^s_{W^{2-1/s,s}_{\beta,\delta}(\Gamma)}.$$

Summing up over all integer k and using the equivalence of the norm in $W^{l-1/s,s}_{\beta,\delta}(\Gamma_j)$ with the the second norm in (1.9), we obtain the desired inequality. ∎

2 The boundary value problem in a dihedron

We consider a boundary value problem for the Stokes system, where on each of the sides $\Gamma^\pm$ one of the boundary conditions (i)–(iv) is given. Let $n^\pm = (n_1^\pm, n_2^\pm, 0)$ be the exterior normal to $\Gamma^\pm$, $\varepsilon_n^\pm(u) = \varepsilon(u)\, n^\pm$ and $\varepsilon_{nn}^\pm(u) = \varepsilon_n^\pm(u)\cdot n^\pm$. Furthermore, let $d^\pm \in \{0,1,2,3\}$ be integer numbers characterizing the boundary conditions on Γ^+ and Γ^-, respectively. We put

- $S^\pm u = u \quad$ for $d^\pm = 0$,
- $S^\pm u = u - (u\cdot n^\pm)n^\pm, \quad N^\pm(u,p) = -p + 2\varepsilon_{nn}^\pm(u) \quad$ for $d^\pm = 1$,
- $S^\pm u = u\cdot n^\pm, \quad N^\pm(u,p) = \varepsilon_n^\pm(u) - \varepsilon_{nn}^\pm(u)\, n^\pm \quad$ for $d^\pm = 2$
- $N^\pm(u,p) = -pn^\pm + 2\varepsilon_n^\pm(u) \quad$ for $d^\pm = 3$

and consider the boundary value problem

$$-\Delta u + \nabla p = f, \quad -\nabla\cdot u = g \quad \text{in } \mathcal{D}, \tag{2.1}$$

$$S^\pm u = h^\pm, \quad N^\pm(u,p) = \phi^\pm \quad \text{on } \Gamma^\pm. \tag{2.2}$$

Here the condition $N^\pm(u,p) = \phi^\pm$ is absent in the case $d^\pm = 0$, while the condition $S^\pm u = h^\pm$ is absent in the case $d^\pm = 3$. The Dirichlet problem for Stokes system and the mixed problem with Dirichlet condition (i) on Γ^+ and condition (ii) on Γ^- were studied by Maz'ya, Plamenevskiĭ and Stupelis [20] (see also the book of Stupelis [28]). In contrast to [20, 28], we will use here weighted Sobolev spaces $W^{l,s}_\delta(\mathcal{D})$ with nonhomogeneous norms.

2.1 Reduction to homogeneous boundary conditions

For the following lemma we refer to [24, 25].

Lemma 2.1 *Let $h^\pm \in V^{l-1/s,s}_\delta(\Gamma^\pm)^{3-d^\pm}$, $\phi^\pm \in V^{l-1-1/s,s}_\delta(\Gamma^\pm)^{d^\pm}$, $l \ge 2$. Then there exists a vector function $u \in V^{l,s}_\delta(\mathcal{D})^3$ such that $S^\pm u = h^\pm$ and $N^\pm(u,0) = \phi^\pm$ on $\Gamma^\pm$ satisfying the estimate*

$$\|u\|_{V^{l,s}_\delta(\mathcal{D})^3} \le c\left(\|h^\pm\|_{V^{l-1/s,s}_\delta(\Gamma^\pm)^{3-d^\pm}} + \|\phi^\pm\|_{V^{l-1-1/s,s}_\delta(\Gamma^\pm)^{d^\pm}}\right)$$

with a constant c independent of $h^\pm$ and $\phi^\pm$. If $h^\pm$ and $\phi^\pm$ vanish for $|x'| > 1$, then also u can be chosen such that $u(x) = 0$ for $|x'| > 1$.

Now let $h^\pm \in W^{2-1/s,s}_\delta(\Gamma^\pm)^{3-d^\pm}$, $\phi^\pm \in W^{1-1/s,s}_\delta(\Gamma^\pm)^{d^\pm}$ and $g \in W^{1,s}_\delta(\mathcal{D})$, $\delta < 2 - 2/s$, be given functions vanishing for $|x'| > 1$. We want to answer the question under which conditions there exist functions $u \in W^{2,s}_\delta(\mathcal{D})^3$ and $p \in W^{1,s}_\delta(\mathcal{D})$ such that

$$S^\pm u = h^\pm, \quad N^\pm(u,p) = \phi^\pm \ \text{ on } \Gamma^\pm \quad \text{and} \quad \nabla\cdot u + g \in V^{1,s}_\delta(\mathcal{D}) \tag{2.3}$$

For $\delta > 1 - 2/s$ the answer follows immediately from the following lemma and from Lemma 2.1.

Lemma 2.2 *Let $h^\pm \in W_\delta^{l-1/s,s}(\Gamma^\pm)^{3-d^\pm}$, $l-1-2/s < \delta < l-2/s$, $l \geq 1$, $h^\pm(x) = 0$ for $|x'| > 1$. Suppose that h^+ and h^- satisfy the compatibility condition*

$$\big(h^+|_M\,,\, h^-|_M\big) \in R(T), \tag{2.4}$$

where $R(T)$ denotes the range of the operator $T = (S^+, S^-)$ (here $S^\pm$ are considered as operators on $W^{l-\delta-2/s,s}(M)^3$). Then there exists a vector function $u \in W_\delta^{l,s}(\mathcal{D})^3$ such that $u(x) = 0$ for $|x'| > 1$ and $S^\pm u = h^\pm$.

Proof: By (2.4), there exists a vector function $\psi \in W^{l-\delta-2/s,s}(M)^3$ such that $S^\pm\psi = h^\pm|_M$. Let $v \in W_\delta^{l,s}(\mathcal{D})^3$ be an extension of ψ vanishing for $|x'| > 1$. Then the traces of $S^\pm v|_{\Gamma^\pm} - h^\pm$ are zero on M and, consequently, $S^\pm v|_{\Gamma^\pm} - h^\pm \in V_\delta^{l-1/s,s}(\Gamma^\pm)^{3-d^\pm}$ (see Lemma 1.1). Applying Lemma 2.1 (in the case $l = 1$ see [16, Le.3.1]), we obtain the assertion of the lemma. ∎

Note that condition (2.4) can be also written in the form

$$Ah^+|_M = Bh^-|_M\,, \tag{2.5}$$

where A and B are certain matrices. For example, $A = B = I$ in the case of the Dirichlet problem ($d^+ = d^- = 0$), $A = (n^-)^t = (n_1^-, n_2^-, 0)$, $B = 1$ if $d^+ = 0$ and $d^- = 2$.

If $h^\pm \in W_\delta^{2-1/s,s}(\Gamma^\pm)^{3-d^\pm}$, $\phi^\pm \in W_\delta^{l-1-1/s,s}(\Gamma^\pm)^{d^\pm}$ and $g \in W_\delta^{1,s}(\mathcal{D})$, $-2/s < \delta < 1-2/s$, then the traces of g, $h^\pm$, $\partial_r h^\pm$ and $\phi^\pm$ on M exist. Suppose that $(u,p) \in W_\delta^{2,s}(\mathcal{D})^3 \times W_\delta^{1,s}(\mathcal{D})$ satisfies (2.3). We put

$$b = u|_M\,, \quad c = (\partial_{x_1} u)|_M\,, \quad d = (\partial_{x_2} u)|_M \quad \text{and} \quad q = p|_M\,.$$

Then from the equations $S^\pm u = h^\pm$ on $\Gamma^\pm$ it follows that $S^\pm \partial_r u = \partial_r h^\pm$ on $\Gamma^\pm$, and therefore,

$$S^\pm b = h^\pm|_M\,, \tag{2.6}$$

$$S^\pm\big(c\cos\tfrac{\alpha}{2} \pm d\sin\tfrac{\alpha}{2}\big) = (\partial_r h^\pm)|_M\,. \tag{2.7}$$

Moreover $\nabla\cdot u + g \in V_\delta^{1,s}(\mathcal{D})$ if and only if the trace of $\nabla\cdot u + g$ on M vanishes, i.e.,

$$c_1 + d_2 + \partial_{x_3} b = -g|_M\,. \tag{2.8}$$

Obviously, the trace of $N^\pm(u,p)$ on M can be written as a linear form $M^\pm(c, d, \partial_{x_3} b, q)$. Thus, from $N^\pm(u,p) = \phi^\pm$ on $\Gamma^\pm$ it follows that

$$M^\pm(c, d, \partial_{x_3} b, q) = \phi^\pm|_M\,. \tag{2.9}$$

Lemma 2.3 *Suppose that there doesn't exist a pair $(u,p) \neq (0,0)$ of a linear vector function $u = cx_1 + dx_2$ and a constant p satisfying*

$$-\nabla\cdot u = 0 \quad \text{in } \mathcal{D}, \quad S^\pm u = 0, \quad N^\pm(u,p) = 0 \quad \text{on } \Gamma^\pm. \tag{2.10}$$

Then the linear system (2.7)–(2.9) has a unique solution (c,d,q) for arbitrary $h^\pm, \phi^\pm, g$, and b.

Proof: Inserting $u = cx_1 + dx_2$ and $p = q = const.$ into (2.10), we obtain

$$c_1 + d_2 = 0, \quad S^\pm\big(c\cos\tfrac{\alpha}{2} \pm d\sin\tfrac{\alpha}{2}\big) = 0, \quad \text{and} \quad M^\pm(c,d,0,q) = 0. \tag{2.11}$$

By the assumption of the lemma, the homogeneous system (2.11) of 7 linear equations with 7 unknowns has only the trivial solution $c = d = 0$, $q = 0$. Consequently, the inhomogeneous system (2.7)–(2.9) is uniquely solvable. ∎

The last lemma together with Lemmas 1.1 and 1.2 allows us to obtain the following result.

Lemma 2.4 *Let $h^\pm \in W_\delta^{2-1/s,s}(\Gamma^\pm)^{3-d^\pm}$, $\phi^\pm \in W_\delta^{1-1/s,s}(\Gamma^\pm)^{d^\pm}$ and $g \in W_\delta^{1,s}(\mathcal{D})$, $-2/s < \delta < 2-2/s$, be given functions vanishing for $|x'| > 1$, and let h^+ and h^- satisfy the compatibility condition (2.5) on M. If $\delta \le 1-2/s$ we assume additionally that the assumption of Lemma 2.3 is satisfied. Then there exist a vector function $u \in W_\delta^{2,s}(\mathcal{D})^3$ and a function $p \in W_\delta^{1,s}(\mathcal{D})$ vanishing for $|x'| > 1$ and satisfying (2.3) and the estimate*

$$\|u\|_{W_\delta^{2,s}(\mathcal{D})^3} + \|p\|_{W_\delta^{1,s}(\mathcal{D})} \le c\Big(\sum_\pm \|h^\pm\|_{W_\delta^{2-1/s,s}(\Gamma^\pm)^{3-d^\pm}} + \sum_\pm \|\phi^\pm\|_{W_\delta^{l-1-1/s,s}(\Gamma^\pm)^{d^\pm}} + \|g\|_{W_\delta^{1,s}(\mathcal{D})}\Big).$$

Proof: For $\delta > 1-2/s$ the assertion of the lemma follows immediately from Lemmas 2.1 and 2.2. Let $\delta < 1-2/s$. Then there exist $b \in W^{2-\delta-2/s,s}(M)^3$, $c,d \in W^{1-\delta-2/s,s}(M)^3$ and $q \in W^{1-\delta-2/s,s}(M)$ satisfying (2.6)–(2.9). We put

$$v = Eb + x_1\, Ec + x_2\, Ed, \quad p = Eq,$$

where E is the extension operator (1.4). Then, by Lemma 1.2,

$$S^\pm v|_M = h^\pm|_M\,, \quad (\partial_r S^\pm w)|_M = (\partial_r h^\pm)|_M\,, \quad -(\nabla\cdot w)|_M = g|_M\,.$$

and

$$N^\pm(v,p)|_M = M^\pm(c,d,\partial_{x_3}b,q) = \phi^\pm|_M\,.$$

Consequently, by Lemma 1.1, we have

$$S^\pm v - h^\pm \in V_\delta^{2-1/s,s}(\Gamma^\pm)^{3-d^\pm}, \quad \nabla\cdot v + g \in V_\delta^{1,s}(\mathcal{D}), \quad \text{and} \quad N^\pm(v,p) - \phi^\pm \in V_\delta^{1-1/s,s}(\Gamma^\pm)^{d^\pm}.$$

By Lemma 2.1, there exist a vector function $w \in V_\delta^{2,s}(\mathcal{D})^3$, $w(x) = 0$ for $|x'| > 1$ such that

$$S^\pm w = h^\pm - S^\pm v, \quad N^\pm(w,0) = \phi^\pm - N^\pm(v,p) \quad \text{on } \Gamma^\pm.$$

Then the pair $(u,p) = (v+w,p)$ has the desired properties. In the case $\delta = 1-2/s$ the lemma can be proved analogously using the relations between the spaces $V_\delta^{l,s}(\mathcal{D})$ and $W_\delta^{l,s}(\mathcal{D})$ given in [27]. ∎

Remark 2.1 The condition of Lemma 2.3 is satisfied for $d^+ + d^- = 3$, $\sin 2\theta \ne 0$ and for $d^+ + d^- \in \{1,5\}$, $\cos\theta\,\cos 2\theta \ne 0$. If $d^+ + d^-$ is an even number, then the condition of Lemma 2.3 fails for all θ. If d^+ and d^- are both even, then obviously $(u,p) = (0,1)$ satisfies (2.10), while in the case of odd d^+ and d^-, the vector $(u,p) = (x_1, -x_2, 0, 0)$ satisfies (2.10). In these cases the assertion of Lemma 2.4 holds only under additional compatibility conditions on the functions $h^\pm$, $\phi^\pm$ and g.

We give here the corresponding result for the Dirichlet boundary condition.

Lemma 2.5 *Let $h^\pm \in W_\delta^{2-1/s,s}(\Gamma^\pm)^3$ and $g \in W_\delta^{1,s}(\mathcal{D})$, $-2/s < \delta < 2-2/s$, be given functions vanishing for $|x'| > 1$ such that $h^+|_M = h^-|_M$. If $\delta < 1-2/s$, we assume additionally that $\theta \ne \pi$, $\theta \ne 2\pi$, and*

$$n^-\cdot\partial_r h^+|_M + n^+\cdot\partial_r h^-|_M = (g|_M + \partial_{x_3} h_3^+|_M)\,\sin\theta, \tag{2.12}$$

while for $\delta = 1-2/s$ the "generalized trace condition"

$$\int_0^\infty \int_{\mathbb{R}} r^{-1}\Big|n^-\cdot\partial_r h^+(r,x_3) + n^+\cdot\partial_r h^-(r,x_3) - \big(\mathring{g}\,(r,x_3) - \partial_{x_3} h_3^+(r,x_3)\big)\,\sin\theta\Big|^s\, dx_3\, dr < \infty$$

is assumed to be valid. Here

$$\mathring{g}\,(r,x_3) = \frac{1}{\theta}\int_{-\theta/2}^{\theta/2} g(r\cos\varphi, r\sin\varphi, x_3)\, d\varphi$$

denotes the average of g with respect to the variable φ. Then there exists a vector function $u \in W_\delta^{2,s}(\mathcal{D})^3$ vanishing for $|x'| > 1$ such that $u = h^\pm$ on $\Gamma^\pm$ and $\nabla\cdot u + g \in V_\delta^{2,s}(\mathcal{D})$.

Proof: If $\delta < 1 - 2/s$, then the traces of $h^\pm$, $\partial_r h^\pm$, and g on M exist and there are vector functions $c, d \in W^{1-\delta-2/s,s}(M)^3$ satisfying

$$c \cos\tfrac{\alpha}{2} \pm d \sin\tfrac{\alpha}{2} = (\partial_r h^\pm)|_M \quad \text{and} \quad c_1 + d_2 = -\partial_{x_3} h^+|_M - g|_M \,.$$

Analogously to the proof of Lemma 2.4, it can be shown that $v = Eh^+|_M + x_1\, Ec + x_2\, Ed$ satisfies the conditions $v|_{\Gamma^\pm} - h^\pm \in V_\delta^{2,s}(\Gamma^\pm)^3$ and $\nabla \cdot v + g \in V_\delta^{1,s}(\mathcal{D})$. Applying Lemma 2.1, we obtain the assertion of the lemma for $\delta < 1 - 2/s$. Analogously, it can be proved for $\delta = 1 - 2/s$. ∎

Remark 2.2 Analogous results are valid for even $d^+ + d^- \neq 0$. Then of course the conditions $h^+|_M = h^-|_M$ and (2.12) have to be replaced by another compatibility conditions on M. If $\delta < 1 - 2/s$, then the traces of $h^\pm, \partial_r h^\pm, \phi^\pm$ and g on M must be such that the system (2.6)–(2.9) with the unknowns b, c, d, q is solvable. For example, in the case of the Neumann problem ($d^+ = d^- = 3$), $\theta \neq \pi$, $\theta \neq 2\pi$, the boundary data ϕ^+ and ϕ^- must satisfy the condition

$$\phi^+ \cdot n^- = \phi^- \cdot n^+ \ \text{ on } M.$$

In the case $d^- = 0$, $d^+ = 2$, the data h^+, h^-, ϕ^+ and g must satisfy the compatibility conditions $h^- \cdot n^+ = h^+$ and

$$\partial_r h^+ \cos 2\theta - (2n^+ \cos\theta + n^-)\, \partial_r h^- + 2\sin^2\theta\, (\phi_1^+ \cos\theta/2 + \phi_2^+ \sin\theta/2) + \frac{1}{2}(g + \partial_{x_3} h_3^-) \sin 2\theta = 0$$

on the edge M.

2.2 Regularity results

The following two lemmas are proved in [23, Le.3.1,Le.3.4] for boundary value problems to elliptic systems of the form

$$-\sum_{i,j=1}^{3} A_{i,j}\, \partial_{x_i}\, \partial_{x_j} u = f$$

The proof for the Stokes system is essentially the same.

Lemma 2.6 *Let $(u,p) \in W_{loc}^{l,s}(\bar{\mathcal{D}}\backslash M)^3 \times W_{loc}^{l-1,s}(\bar{\mathcal{D}}\backslash M)$ be a solution of problem (2.1), (2.2). Furthermore, let ζ, η be infinitely differentiable functions with compact supports on $\bar{\mathcal{D}}$ such that $\eta = 1$ in a neighborhood of* $\operatorname{supp}\zeta$.

1) *If $\eta u \in V_{\delta-l}^{0,s}(\mathcal{D})^3$, $\eta p \in V_{\delta-l+1}^{0,s}(\mathcal{D})$, $\eta f \in V_\delta^{l-2,s}(\mathcal{D})^3$, $\eta g \in V_\delta^{l-1,s}(\mathcal{D})$, $\eta h^\pm \in V_\delta^{l-1/s,s}(\Gamma^\pm)^{3-d^\pm}$, and $\eta\phi^\pm \in V_\delta^{l-1-1/s,s}(\Gamma^\pm)^{d^\pm}$, $l \geq 2$, then $\zeta u \in V_\delta^{l,s}(\mathcal{D})^3$, $\zeta p \in V_\delta^{l-1,s}(\mathcal{D})$ and*

$$\begin{aligned}\|\zeta u\|_{V_\delta^{l,s}(\mathcal{D})^3} + \|\zeta p\|_{V_\delta^{l-1,s}(\mathcal{D})} \;\leq\; & c\Big(\|\eta u\|_{V_{\delta-l}^{0,s}(\mathcal{D})^3} + \|\eta p\|_{V_{\delta-l+1}^{0,s}(\mathcal{D})} + \|\eta f\|_{V_\delta^{l-2,s}(\mathcal{D})^3} + \|\eta g\|_{V_\delta^{l-1,s}(\mathcal{D})} \\ & + \sum_\pm \|\eta h^\pm\|_{V_\delta^{l-1/s,s}(\Gamma^\pm)^{3-d^\pm}} + \sum_\pm \|\eta\phi^\pm\|_{V_\delta^{l-1-1/s,s}(\Gamma^\pm)^{d^\pm}}\Big). \quad (2.13)\end{aligned}$$

2) *If $\eta u \in W_{\delta-l+k}^{k,s}(\mathcal{D})^3$, $\eta p \in W_{\delta-l+k}^{k-1,s}(\mathcal{D})$, $\eta f \in W_\delta^{l-2,s}(\mathcal{D})^3$, $\eta g \in W_\delta^{l-1,s}(\mathcal{D})$, $\eta h^\pm \in W_\delta^{l-1/s,s}(\Gamma^\pm)^{3-d^\pm}$, and $\eta\phi^\pm \in W_\delta^{l-1-1/s,s}(\Gamma^\pm)^{d^\pm}$, $l \geq k+1 \geq 2$, $\delta > l-k-2/s$, then $\zeta u \in W_\delta^{l,s}(\mathcal{D})^3$, $\zeta p \in W_\delta^{l-1,s}(\mathcal{D})$ and an estimate analogous to* (2.13) *holds.*

We define the operator $A(\lambda)$ as follows

$$A(\lambda)\,\big(U(\varphi), P(\varphi)\big) = \big(r^{2-\lambda}(-\Delta u + \nabla p)\,,\, -r^{1-\lambda}\nabla\cdot u\,,\, r^{-\lambda} S^\pm u|_{\varphi=\pm\theta/2}\,,\, r^{1-\lambda} N^\pm(u,p)|_{\varphi=\pm\theta/2}\big),$$

where $u = r^\lambda U(\varphi)$, $p = r^{\lambda-1}P(\varphi)$, $\lambda \in \mathbb{C}$, r, φ are the polar coordinates of the point $x' = (x_1, x_2)$. The operator $A(\lambda)$ depends quadratically on the parameter λ and realizes a continuous mapping

$$W^{2,s}((-\tfrac{\theta}{2}, +\tfrac{\theta}{2}))^3 \times W^{1,s}((-\tfrac{\theta}{2}, +\tfrac{\theta}{2})) \to W^{1,s}((-\tfrac{\theta}{2}, +\tfrac{\theta}{2}))^3 \times L^s((-\tfrac{\theta}{2}, \tfrac{\theta}{2})) \times \mathbb{C}^3 \times \mathbb{C}^3$$

for every $\lambda \in \mathbb{C}$. In [24, 25] a description of the spectrum of the pencil $A(\lambda)$ is given for different d^- and d^+. For example, in the cases of the Dirichlet problem ($d^+ = d^- = 0$) and Neumann problem ($d^+ = d^- = 3$), the spectrum of $A(\lambda)$ consists of the solutions of the equation

$$\sin(\lambda\theta)\left(\lambda^2 \sin^2\theta - \sin^2(\lambda\theta)\right) = 0,$$

$\lambda \neq 0$ for $d^+ = d^- = 0$. In the case $d^- = 0,\ d^+ = 1$, the eigenvalues of $A(\lambda)$ are the nonzero solutions of the equation

$$\sin(\lambda\theta)\left(\lambda \sin(2\theta) + \sin(2\lambda\theta)\right) = 0.$$

If $d^- = 0,\ d^+ = 2$, then the eigenvalues are the nonzero solutions of the equation

$$\sin(2\lambda\theta)\left(\lambda \sin(2\theta) - \sin(2\lambda\theta)\right) = 0,$$

while the nonzero solutions of the equation

$$\sin(2\lambda\theta)\left(\lambda^2 \sin^2\theta - \cos^2(\lambda\theta)\right) = 0$$

are eigenvalues of $A(\lambda)$ if $d^- = 0$ and $d^+ = 3$.

Lemma 2.7 *Let ζ, η be smooth functions on $\bar{\mathcal{D}}$ with compact supports such that $\eta = 1$ in a neighborhood of $\operatorname{supp}\zeta$, and let (u, p) be a solution of problem (2.1), (2.2) such that*

$$\eta u \in W^{l,s}_{\delta}(\mathcal{D})^3, \quad \eta p \in W^{l-1,s}_{\delta}(\mathcal{D}), \quad \eta\partial_{x_3} u \in W^{l,s}_{\delta'}(\mathcal{D})^3, \quad \eta\partial_{x_3} p \in W^{l-1,s}_{\delta'}(\mathcal{D})$$

where $l \geq 2$, $-2/s < \delta \leq \delta' \leq \delta + 1$. Furthermore, we assume that

$$\eta f \in W^{l-1,s}_{\delta'}(\mathcal{D})^3, \quad \eta g \in W^{l,s}_{\delta'}(\mathcal{D}), \quad \eta h^{\pm} \in W^{l+1-1/s,s}_{\delta'}(\Gamma^{\pm})^{3-d^{\pm}}, \quad \eta\phi^{\pm} \in W^{l-1/s,s}_{\delta'}(\Gamma^{\pm})^{d^{\pm}}.$$

If there are no eigenvalues of the pencil $A(\lambda)$ in the strip $l - \delta - 2/s \leq \operatorname{Re}\lambda \leq l + 1 - \delta' - 2/s$, then $\zeta u \in W^{l+1,s}_{\delta'}(\mathcal{D})^3$, $\zeta p \in W^{l,s}_{\delta'}(\mathcal{D})$.

3 Solvability of the boundary value problem in a polyhedral cone

Let $\mathcal{K}$ be the cone (1.6) introduced in Section 1.2. For every $j = 1, \dots, n$ let d_j be one of numbers $0, 1, 2, 3$. We consider the boundary value problem

$$-\Delta u + \nabla p = f, \quad -\nabla \cdot u = g \ \text{ in } \mathcal{K}, \tag{3.1}$$

$$S_j u = h_j\,, \quad N_j(u, p) = \phi_j \quad \text{on } \Gamma_j,\ j = 1, \dots, n. \tag{3.2}$$

Here S_j is defined as

$$S_j u = u \text{ if } d_j = 0, \quad S_j u = u_n = u \cdot n \text{ if } d_j = 2, \quad S_j u = u_\tau = u - u_n n \text{ if } d_j = 1,$$

while the operators N_j are defined as

$$N_j(u, p) = -p + 2\varepsilon_{n,n}(u) \text{ if } d_j = 1, \quad N_j(u, p) = \varepsilon_{n,\tau}(u) \text{ if } d_j = 2, \quad N_j(u, p) = -pn + 2\varepsilon_n(u) \text{ if } d_j = 3.$$

In the case $d_j = 0$ the condition $N_j(u, p) = \phi_j$ does not appear in (3.2), whereas the condition $S_j u = h_j$ does not appear if $d_j = 3$.

3.1 Reduction to homogeneous boundary conditions

Lemma 3.1 *Let $h_j \in V^{l-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, $\phi_j \in V^{l-1-1/s,s}_{\beta,\delta}(\Gamma_j)^{d_j}$, $l \geq 2$. Then there exists a vector function $u \in V^{l,s}_{\beta,\delta}(\mathcal{K})^3$ such that $S_j u = h_j$ and $N_j(u,0) = \phi_j$ on Γ_j and*

$$\|u\|_{V^{l,s}_{\beta,\delta}(\mathcal{K})^3} \leq c \sum_{j=1}^{n} \Big(\|h_j\|_{V^{l-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j}} + \|\phi_j\|_{V^{l-1-1/s,s}_{\beta,\delta}(\Gamma_j)^{d_j}} \Big) \tag{3.3}$$

with a constant c independent of h_j and ϕ_j.

Proof: Let ζ_k be smooth functions depending only on $\rho = |x|$ such that

$$\operatorname{supp} \zeta_k \subset (2^{k-1}, 2^{k+1}), \quad \sum_{k=-\infty}^{+\infty} \zeta_k = 1, \quad |(\rho\partial_\rho)^j \zeta_k(\rho)| \leq c_j \tag{3.4}$$

with constants c_j independent of k and ρ. We set $h_{k,j}(x) = \zeta_k(2^k x)\, h_j(2^k x)$, $\phi_{k,j}(x) = 2^k\, \zeta_k(2^k x)\, \phi_j(2^k x)$. These functions vanish for $|x| < \frac{1}{2}$ and $|x| > 2$. Consequently, by Lemma 2.1, there exist vector functions $v_k \in V^{l,s}_{\beta,\delta}(\mathcal{K})^3$ such that $S_j v_k = h_{k,j}$ and $N_j(v_k, 0) = \phi_{k,j}$ on Γ_j for $j = 1, \ldots, n$,

$$\|v_k\|_{V^{l,s}_{\beta,\delta}(\mathcal{K})^3} \leq c \sum_{j=1}^{n} \Big(\|h_{k,j}\|_{V^{l-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j}} + \|\phi_{k,j}\|_{V^{l-1-1/s,s}_{\beta,\delta}(\Gamma_j)^{d_j}} \Big), \tag{3.5}$$

and $v_k(x) = 0$ for $|x| < \frac{1}{4}$ and $|x| > 4$. Hence for the functions $u_k(x) = v_k(2^{-k}x)$ we obtain $S_j u_k = \zeta_k h_j$ and $N_j(u_k, 0) = \zeta_k \phi_j$ on Γ_j, $u_k(x) = 0$ for $|x| < 2^{k-2}$ and $|x| > 2^{k+2}$. Furthermore, u_k satisfies (3.5) with $\zeta_k h_j$ and $\zeta_k \phi_j$ instead of $h_{k,j}$ and $\phi_{k,j}$. Here the constant c is independent of k, h_j and ϕ_j. Consequently, for $u = \sum u_k$ we have $S_j u = h_j$ on and $N_j(u, 0) = \phi_j$ on Γ_j for $j = 1, \ldots, n$. Inequality (3.3) follows from the equivalence of the norms in $V^{l,s}_{\beta,\delta}(\mathcal{K})$ and $V^{l-1/s,s}_{\beta,\delta}(\Gamma_j)$ with the norms

$$\|u\| = \Big(\sum_{k=-\infty}^{+\infty} \|\zeta_k u\|^2_{V^{l,s}_{\beta,\delta}(\mathcal{K})} \Big)^{1/s} \quad \text{and} \quad \|h\| = \Big(\sum_{k=-\infty}^{+\infty} \|\zeta_k h\|^2_{V^{l-1/s,s}_{\beta,\delta}(\Gamma_j)} \Big)^{1/s}, \tag{3.6}$$

respectively (cf. [9, Sect.6.1]). ∎

An analogous result in $W^{2,s}_{\beta,\delta}(\mathcal{K})$ is only valid under additional compatibility conditions on the boundary data. Denote by Γ_{k_+} and Γ_{k_-} the sides of the cone $\mathcal{K}$ adjacent to the edge M_k and by θ_k the inner angle at M_k. If $u \in W^{2,s}_{\beta,\delta}(\mathcal{K})$ and $\delta_k < 2 - 2/s$, then the trace of u on M_k exists and from the equations $S_j u = h_j$ on Γ_j it follows that the pair $\big(h_{k_+}|_{M_j}, h_{k_-}|_{M_j}\big)$ belongs to the range of the matrix operator (S_{k_+}, S_{k_-}). This condition can be also written in the form

$$A_k h_{k_+}|_{M_k} = B_k h_{k_-}|_{M_k}\,, \tag{3.7}$$

where A_k, B_k are certain constant matrices (see Section 2.1).

Using Lemma 2.4 (see also Remark 2.1), one can prove the following result analogously to Lemma 3.1.

Lemma 3.2 *Let $h_j \in W^{2-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, $\phi_j \in W^{l-1-1/s,s}_{\beta,\delta}(\Gamma_j)^{d_j}$ and $g \in W^{1,s}_{\beta,\delta}(\mathcal{K})$, $-2/s < \delta_k < 2 - 2/s$ for $k = 1, \ldots, n$, be given functions such that the compatibility condition (3.7) is satisfied for $k = 1, \ldots, n$. In the case $\delta_k \leq 1 - 2/s$ we assume additionally that $d_{k_+} + d_{k_-}$ is odd and*

$$\sin 2\theta_k \neq 0 \quad \text{if } d_{k_+} + d_{k_-} = 3, \quad \cos\theta_k \cos 2\theta_k \neq 0 \quad \text{if } d_{k_+} + d_{k_-} \in \{1, 5\}.$$

Then there exist a vector function $u \in W^{2,s}_{\beta,\delta}(\mathcal{K})^3$ and a function $p \in W^{1,s}_{\beta,\delta}(\mathcal{K})$ satisfying

$$S_j u = h_j, \quad N_j(u,p) = \phi_j \text{ on } \Gamma_j,\ j = 1, \ldots, n, \quad \nabla \cdot u + g \in V^{1,s}_{\beta,\delta}(\mathcal{K}) \tag{3.8}$$

and the estimate

$$\|u\|_{W^{2,s}_{\beta,\delta}(\mathcal{K})^3} + \|p\|_{W^{1,s}_{\beta,\delta}(\mathcal{K})} \leq c \Big(\sum_{\pm} \|h_j\|_{W^{2-1/s,s}_{\beta,\delta}(\Gamma^{\pm})^{3-d_j}} + \sum_{\pm} \|\phi_j\|_{W^{l-1-1/s,s}_{\beta,\delta}(\Gamma_j)^{d_j}} + \|g\|_{W^{1,s}_{\beta,\delta}(\mathcal{K})} \Big).$$

If $\delta_k \le 1-2/s$ and $d_{k_+}+d_{k_-}$ is even for at least one k, then the assertion of Lemma 3.2 holds only under an additional compatibility condition on the edge M_k (cf. Lemma 2.5, Remark 2.2). We give here the corresponding result for the Dirichlet problem. An analogous result is valid in the general case.

Lemma 3.3 *Let $h_j \in W^{2-1/s,s}_{\beta,\delta}(\Gamma_j)^3$ and $g \in W^{1,s}_{\beta,\delta}(\mathcal{K})$, where $-2/s < \delta_k < 2-2/s$ for $k=1,\dots,n$, be given functions such that*

$$h_{k_+}|_{M_k} = h_{k_-}|_{M_k} \quad \text{for } k=1,\dots,n.$$

In the case $\delta_k < 1-2/s$ we assume additionally that $\theta_k \neq \pi$, $\theta_k \neq 2\pi$ and

$$n_{k_-}\cdot(\partial_r h_{k_+})|_{M_k} + n_{k_+}\cdot(\partial_r h_{k_-})|_{M_k} = \big(g|_{M_k} + \partial_t (h_{k_+}\cdot e_k)|_{M_k}\big)\,\sin\theta_k, \tag{3.9}$$

where e_k is the unit vector on M_k, $r=\operatorname{dist}(x,M_k)$, and t denotes the coordinate on M_k. For $\delta_k = 1-2/s$ instead of (3.9) the generalized trace condition

$$\int_0^\infty\int_0^{\varepsilon t} t^{s(\beta-1)+2}\, r^{-1}\left| n_{k_-}\cdot\partial_r h_{k_+}(r,t) + n_{k_+}\cdot\partial_r h_{k_-}(r,t) - \big(\overset{\circ}{g}(r,t) - \partial_t(h_{k_+}(r,t)\cdot e_k)\big)\,\sin\theta_k\right|^s dr\,dt < \infty$$

is assumed to be valid. Here ε is a small positive number, the functions $h_{k_\pm}$ are considered near M_k as functions in the variables r,t, and $\overset{\circ}{g}(r,t)$ denotes the average of g with respect to the angle φ in the plane perpendicular to M_k (cf. Lemma 2.5). Then there exists a vector function $u \in W^{2,s}_{\beta,\delta}(\mathcal{K})^3$ such that $u=h_j$ on Γ_j for $j=1,\dots,n$ and $\nabla\cdot u + g \in V^{1,s}_{\beta,\delta}(\mathcal{K})$.

3.2 Operator pencils generated by the boundary value problem

We introduce the following operator pencils $\mathfrak{A}$ and A_j.

1) Let $\Gamma_{k_\pm}$ be the sides of $\mathcal{K}$ adjacent to the edge M_k, and let θ_k be the angle at the edge M_k. We consider the Stokes system in the dihedron $\mathcal{D}_k$ bounded by the half-planes $\Gamma^\circ_{k_\pm} \supset \Gamma_{k_\pm}$ with the boundary conditions

$$S_{k_\pm}u = h^\pm, \quad N_{k_\pm}(u,p) = \phi^\pm \ \text{ on } \Gamma^\circ_{k_\pm}.$$

By $A_k(\lambda)$ we denote the operator pencil introduced before Lemma 2.7 for this problem. Furthermore, let $\lambda_1^{(k)}$ denote the eigenvalue with smallest positive real part of this pencil, while $\lambda_2^{(k)}$ is the eigenvalue with smallest real part greater than 1. Finally, we define

$$\mu_k = \begin{cases} \operatorname{Re}\lambda_1^{(k)} & \text{if } d_{k_+}+d_{k_-} \text{ is odd} \quad \text{or} \quad d_{k_+}+d_{k_-} \text{ is even and } \alpha_k \ge \pi/m_k, \\ \operatorname{Re}\lambda_2^{(k)} & \text{if } d_{k_+}+d_{k_-} \text{ is even and } \alpha_k < \pi/m_k, \end{cases} \tag{3.10}$$

where $m_k = 1$ if $d_{k_+} = d_{k_-}$, $m_k = 2$ if $d_{k_+} \neq d_{k_-}$.

2) Let $\rho=|x|$, $\omega = x/|x|$, $V_\Omega = \{u \in W^1(\Omega)^3 : S_j u = 0 \text{ on } \gamma_j \text{ for } j=1,\dots,n\}$, and

$$a\Big(\begin{pmatrix} u \\ p\end{pmatrix}, \begin{pmatrix} v \\ q\end{pmatrix}; \lambda\Big) = \frac{1}{\log 2}\int\limits_{\substack{\mathcal{K}\\ 1<|x|<2}} \Big(2\sum_{i,j=1}^3 \varepsilon_{i,j}(U)\cdot\varepsilon_{i,j}(V) - P\nabla\cdot V - (\nabla\cdot U)\,Q\Big)\,dx,$$

where $U=\rho^\lambda u(\omega)$, $V=\rho^{-1-\lambda}v(\omega)$, $P=\rho^{\lambda-1}p(\omega)$, $Q=\rho^{-2-\lambda}q(\omega)$, $u,v\in V_\Omega$, $p,q\in L_2(\Omega)$, and $\lambda\in\mathbb{C}$. The bilinear form $a(\cdot,\cdot;\lambda)$ generates the linear and continuous operator

$$\mathfrak{A}(\lambda):\ V_\Omega \times L_2(\Omega) \to V_\Omega^* \times L_2(\Omega)$$

by

$$\int_\Omega \mathfrak{A}(\lambda)\begin{pmatrix} u \\ p\end{pmatrix}\cdot\begin{pmatrix} v \\ q\end{pmatrix} d\omega = a\Big(\begin{pmatrix} u \\ p\end{pmatrix}, \begin{pmatrix} v \\ q\end{pmatrix}; \lambda\Big), \quad u,v\in V_\Omega,\ p,q\in L_2(\Omega).$$

3.3 Regularity results for the problem in the cone

The following results are based on Lemmas 2.6, 2.7

Lemma 3.4 *Let* $(u,p) \in W^{l,s}_{loc}(\bar{\mathcal{K}}\backslash\mathcal{S})^3 \times W^{l-1,s}_{loc}(\bar{\mathcal{K}}\backslash\mathcal{S})$ *be a solution of problem* (3.1), (3.2).

1) *If* $u \in V^{1,s}_{\beta-l+1,\delta-l+1}(\mathcal{K})^3$, $p \in V^{0,s}_{\beta-l+1,\delta-l+1}(\mathcal{K})$, $f \in V^{l-2,s}_{\beta,\delta}(\mathcal{K})^3$, $g \in V^{l-1,s}_{\beta,\delta}(\mathcal{K})$, $h_j \in V^{l-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, *and* $\phi_j \in V^{l-1-1/s,s}_{\delta}(\Gamma_j)^{d_j}$, $l \geq 2$, *then* $u \in V^{l,s}_{\beta,\delta}(\mathcal{K})^3$, $p \in V^{l-1,s}_{\beta,\delta}(\mathcal{K})$ *and*

$$\begin{aligned}\|u\|_{V^{l,s}_{\beta,\delta}(\mathcal{K})^3} + \|p\|_{V^{l-1,s}_{\beta,\delta}(\mathcal{K})} \;\leq\; & c\Big(\|u\|_{V^{1,s}_{\beta-l+1,\delta-l+1}(\mathcal{K})^3} + \|p\|_{V^{0,s}_{\beta-l+1,\delta-l+1}(\mathcal{K})} + \|f\|_{V^{l-2,s}_{\beta,\delta}(\mathcal{K})^3} + \|g\|_{V^{l-1,s}_{\beta,\delta}(\mathcal{K})} \\ & + \sum_{j=1}^{n} \|h_j\|_{V^{l-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j}} + \sum_{j=1}^{n} \|\phi_j\|_{V^{l-1-1/s,s}_{\beta,\delta}(\Gamma_j)^{d_j}}\Big). \qquad (3.11)\end{aligned}$$

2) *If* $u \in W^{k,s}_{\beta-l+k,\delta-l+k}(\mathcal{K})^3$, $p \in W^{k-1,s}_{\beta-l+k+1,\delta-l+k}(\mathcal{K})$, $f \in W^{l-2,s}_{\beta,\delta}(\mathcal{K})^3$, $g \in W^{l-1,s}_{\beta,\delta}(\mathcal{K})$, $h_j \in W^{l-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, *and* $\phi_j \in W^{l-1-1/s,s}_{\beta,\delta}(\Gamma_j)^{d_j}$, $l \geq k+1 \geq 2$, $\delta_j > l-k-2/s$, *then* $u \in W^{l,s}_{\beta,\delta}(\mathcal{K})^3$, $p \in W^{l-1,s}_{\beta,\delta}(\mathcal{K})$ *and an estimate analogous to* (3.11) *holds.*

Proof: 1) Due to Lemma 3.1, we may assume, without loss of generality, that $h_j = 0$ and $\phi_j = 0$ for $j = 1,\dots,n$. Let ζ_k be the same functions as in the proof of Lemma 3.1. Furthermore, let $\eta_k = \zeta_{k-1} + \zeta_k + \zeta_{k+1}$, $\tilde{\zeta}_k(x) = \zeta_k(2^k x)$, $\tilde{\eta}_k(x) = \eta_k(2^k x)$, $\tilde{u}(x) = u(2^k x)$, and $\tilde{p}(x) = 2^k p(2^k x)$. By (3.4), the support of $\tilde{\zeta}_k$ is contained in the set $\{x : 1/2 < |x| < 2\}$ and the derivatives of $\tilde{\zeta}_k$ are bounded by constants independent of k. Obviously,

$$\begin{aligned}&-\Delta\tilde{u} + \nabla\tilde{p} = \tilde{f}, \quad -\nabla\tilde{u} = \tilde{g} \;\text{ in } \mathcal{K} \\ &S_j\tilde{u} = 0,\; N_j(\tilde{u},\tilde{p}) = 0 \;\text{ on } \Gamma_j,\; j = 1,\dots,n,\end{aligned}$$

where $\tilde{f}(x) = 2^{2k} f(2^k x)$ and $\tilde{g}(x) = 2^k g(2^k x)$. Consequently, from Lemma 2.6 it follows that $\tilde{\zeta}_k\tilde{u} \in V^{l,s}_{\beta,\delta}(\mathcal{K})^3$, $\tilde{\zeta}_k\tilde{p} \in V^{l-1,s}_{\beta,\delta}(\mathcal{K})$, and

$$\begin{aligned}&\|\tilde{\zeta}_k\tilde{u}\|_{V^{l,s}_{\beta,\delta}(\mathcal{K})^3} + \|\tilde{\zeta}_k\tilde{p}\|_{V^{l-1,s}_{\beta,\delta}(\mathcal{K})} \\ &\leq c\Big(\|\tilde{\eta}_k\tilde{u}\|_{V^{0,s}_{\beta-l,\delta-l}(\mathcal{K})^3} + \|\tilde{\eta}_k\tilde{p}\|_{V^{0,s}_{\beta-l+1,\delta-l+1}(\mathcal{K})} + \|\tilde{\eta}_k\tilde{f}\|_{V^{l-2,s}_{\beta,\delta}(\mathcal{K})^3} + \|\tilde{\eta}_k\tilde{g}\|_{V^{l-1,s}_{\beta,\delta}(\mathcal{K})}\Big).\end{aligned}$$

where c is independent of u, p, and k. Using the coordinate change $2^k x = y$, we obtain the same estimate with $\zeta_k, \eta_k, u, p, f, g$ instead of $\tilde{\zeta}_k, \tilde{\eta}_k, \tilde{u}, \tilde{p}, \tilde{f}$ and $\tilde{g}$, respectively. Since the norm in $V^{l,s}_{\beta,\delta}(\mathcal{K})$ is equivalent to the first norm in (1.8), this implies (3.11) for $\zeta = \eta = 1$.

2) The second assertion can be proved analogously. ∎

Corollary 3.1 *Let* $(u,p) \in W^{l,s}_{loc}(\bar{\mathcal{K}}\backslash\mathcal{S})^3 \times W^{l-1,s}_{loc}(\bar{\mathcal{K}}\backslash\mathcal{S})$ *be a solution of problem* (3.1), (3.2), *and let* ζ, η *be infinitely differentiable functions on* $\bar{\mathcal{K}}$ *with compact supports such that* $\eta = 1$ *in a neighborhood of* $\operatorname{supp}\zeta$. *If* $\eta u \in V^{1,s}_{\beta-l+1,\delta-l+1}(\mathcal{K})^3$, $\eta p \in V^{0,s}_{\beta-l+1,\delta-l+1}(\mathcal{K})$, $\eta f \in V^{l-2,s}_{\beta,\delta}(\mathcal{K})^3$, $\eta g \in V^{l-1,s}_{\beta,\delta}(\mathcal{K})$, $\eta h_j \in V^{l-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, *and* $\eta\phi_j \in V^{l-1-1/s,s}_{\delta}(\Gamma_j)^{d_j}$, $l \geq 2$, *then* $\zeta u \in V^{l,s}_{\beta,\delta}(\mathcal{K})^3$, $\zeta p \in V^{l-1,s}_{\beta,\delta}(\mathcal{K})$ *and an estimate analogous to* (3.11) *holds.*

Proof: We apply Lemma 3.4 to the vector function $(\zeta u, \zeta p)$. Obviously,

$$-\Delta(\zeta u) + \nabla(\zeta p) = \zeta f - 2\sum_{j=1}^{3}(\partial_{x_j}\zeta)\,\partial_{x_j}u - u\,\Delta\zeta + p\,\nabla\zeta \quad\text{and}\quad -\nabla\cdot(\zeta u) = \zeta g - g\cdot\nabla\zeta,$$

Moreover, $(\zeta u, \zeta p)$ satisfies the boundary conditions (3.2) with the data $H_j = \zeta h_j$ and $\Phi_j = \zeta\phi_j + N_j'(\nabla\zeta)\,u$, where $N_j'(\nabla\zeta)$ are certain matrices depending on $\nabla\zeta$. Thus, in the case $l = 2$ the assertion of the corolllary follows immediately from Lemma 3.4. Using induction in l, we obtain the assertion for $l \geq 3$. ∎

Lemma 3.5 *Let $(u,p) \in W^{l,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{l-1,s}_{\beta,\delta}(\mathcal{K})$ be a solution of problem (3.1), (3.2) such that*

$$\partial_\rho u \in W^{l,s}_{\beta+1,\delta'}(\mathcal{K})^3, \quad \partial_\rho p \in W^{l-1,s}_{\beta+1,\delta'}(\mathcal{K}),$$

where $l \geq 2$, $-2/s < \delta_k \leq \delta'_k \leq \delta_k + 1$ for $k = 1, \ldots, n$. Furthermore, we assume that

$$f \in W^{l-1,s}_{\beta+1,\delta'}(\mathcal{K})^3, \quad g \in W^{l,s}_{\beta+1,\delta'}(\mathcal{K}), \quad h_j \in W^{l+1-1/s,s}_{\beta+1,\delta'}(\Gamma^\pm)^{3-d_j}, \quad \phi_j \in W^{l-1/s,s}_{\beta+1,\delta'}(\Gamma^\pm)^{d_j}, \quad j = 1, \ldots, n.$$

If there are no eigenvalues of the pencils $A_k(\lambda)$ in the strip $l - \delta_k - 2/s \leq \operatorname{Re}\lambda \leq l + 1 - \delta'_k - 2/s$, $k = 1, \ldots, n$, then $u \in W^{l+1,s}_{\beta+1,\delta'}(\mathcal{K})^3$, $\zeta p \in W^{l,s}_{\beta+1,\delta'}(\mathcal{K})$.

3.4 Representation of the solution by Green's matrix

Let $f \in W^{0,s}_{\beta,\delta}(\mathcal{K})^3$, $g \in W^{1,s}_{\beta,\delta}(\mathcal{K})$, $h_j \in W^{2-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, $\phi_j \in W^{1-1/s,s}_{\beta,\delta}(\Gamma_j)^{d_j}$. Our goal is to show that problem (3.1), (3.2) has a unique solution in $W^{2,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{1,s}_{\beta,\delta}(\mathcal{K})$ if the line $\operatorname{Re}\lambda = 2 - \beta - 3s$ does not contain eigenvalues of the pencil $\mathfrak{A}(\lambda)$ and $\max(2 - \mu_k, 0) < \delta_k + 2/s < 2$ for $k = 1, \ldots, n$.

Let κ be a fixed real number such that the closed strip between the lines $\operatorname{Re}\lambda = -\kappa - 1/2$ and $\operatorname{Re}\lambda = 2 - \beta - 3/s$ is free of eigenvalues of the pencil $\mathfrak{A}$. Then, according to [25, Th.4.5], there exists a unique solution $G(x,\xi) = \big(G_{i,j}(x,\xi)\big)_{i,j=1}^4$ of the problem

$$-\Delta_x \vec{G}_j(x,\xi) + \nabla_x G_{4,j}(x,\xi) = \delta(x-\xi)\,(\delta_{1,j}, \delta_{2,j}, \delta_{3,j})^t \quad \text{for } x, \xi \in \mathcal{K}, \tag{3.12}$$

$$-\nabla_x \cdot \vec{G}_j(x,\xi) = \delta_{4,j}\,\delta(x-\xi) \quad \text{for } x, \xi \in \mathcal{K}, \tag{3.13}$$

$$S_k \vec{G}_j(x,\xi) = 0, \quad N_k(\partial_x)\big(\vec{G}_j(x,\xi), G_{4,j}(x,\xi)\big) = 0 \quad \text{for } x \in \Gamma_k,\ \xi \in \mathcal{K}, \quad k = 1, \ldots, n, \tag{3.14}$$

(here $\vec{G}_j$ denotes the vector with the components $G_{1,j}, G_{2,j}, G_{3,j}$) such that the function $x \to \zeta(|x - \xi|/r(\xi))\,G_{i,j}(x,\xi)$ belongs to $W^1_{\kappa,0}(\mathcal{K})$ for $i = 1,2,3$ and to $W^0_{\kappa,0}(\mathcal{K})$ for $i = 4$, where ζ is an arbitrary smooth function on $(0,\infty)$ equal to one in $(1,\infty)$ and to zero in $(0, \frac{1}{2})$. We denote by $\Lambda_- < \operatorname{Re}\lambda < \Lambda_+$ the widest strip in the complex plane containing the line $\operatorname{Re}\lambda = -\kappa - 1/2$ which is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$.

If $h_j = 0$ and $\phi_j = 0$ for $j = 1, \ldots, n$, then the solution of problem (3.1), (3.2) has the form

$$u_i(x) = \sum_{j=1}^3 \int_{\mathcal{K}} \big(f_j(\xi) + \partial_{\xi_j} g(\xi)\big)\,G_{i,j}(x,\xi)\,d\xi + \int_{\mathcal{K}} g(\xi)\,G_{i,4}(x,\xi)\,d\xi, \quad i = 1,2,3, \tag{3.15}$$

$$p(x) = -g(x) + \sum_{j=1}^3 \int_{\mathcal{K}} \big(f_j(\xi) + \partial_{\xi_j} g(\xi)\big)\,G_{4,j}(x,\xi)\,d\xi + \int_{\mathcal{K}} g(\xi)\,G_{4,4}(x,\xi)\,d\xi \tag{3.16}$$

(see [25, Th.4.5]). In the following, we will show that (3.15) and (3.16) define a continuous mapping

$$W^{0,s}_{\beta,\delta}(\mathcal{K})^3 \times V^{1,s}_{\beta,\delta}(\mathcal{K}) \ni (f,g) \to (u,p) \in W^{2,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{1,s}_{\beta,\delta}(\mathcal{K})$$

if

$$\Lambda_- < 2 - \beta - 3/s < \Lambda_+ \quad \text{and} \quad \max(2 - \mu_k, 0) < \delta_k + 2/s < 2 \ \text{ for } k = 1, \ldots, n. \tag{3.17}$$

3.5 Estimates of Green's matrix

The following estimates of Green's matrix are proved in [24, 25].

1) For $|x| > 2|\xi|$ there is the estimate

$$\big|\partial_x^\alpha \partial_\xi^\gamma G_{i,j}(x,\xi)\big| \leq c\,|x|^{\Lambda_- - \delta_{i,4} - |\alpha| + \varepsilon}\,|\xi|^{-\Lambda_- - 1 - \delta_{j,4} - |\gamma| - \varepsilon} \times \prod_{k=1}^n \Big(\frac{r_k(x)}{|x|}\Big)^{\min(0, \mu_k - |\alpha| - \delta_{i,4} - \varepsilon)} \prod_{k=1}^n \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{\min(0, \mu_k - |\gamma| - \delta_{j,4} - \varepsilon)},$$

where ε is an arbitrarily small positive number. Analogously for $|\xi| > 2|x|$, there is the inequality

$$\begin{aligned}\big|\partial_x^\alpha \partial_\xi^\gamma G_{i,j}(x,\xi)\big| \;&\le\; c\,|x|^{\Lambda_+ - \delta_{i,4} - |\alpha| - \varepsilon}\;|\xi|^{-\Lambda_+ - 1 - \delta_{j,4} - |\gamma| + \varepsilon} \\ &\quad\times \prod_{k=1}^n \Big(\frac{r_k(x)}{|x|}\Big)^{\min(0,\mu_k - |\alpha| - \delta_{i,4} - \varepsilon)} \prod_{k=1}^n \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{\min(0,\mu_k - |\gamma| - \delta_{j,4} - \varepsilon)}.\end{aligned}$$

2) For $|x|/2 < |\xi| < 2|x|$, $|x-\xi| > \min(r(x), r(\xi))$, we have

$$\big|\partial_x^\alpha \partial_\xi^\gamma G_{i,j}(x,\xi)\big| \le c\,|x-\xi|^{-T-|\alpha|-|\gamma|} \Big(\frac{r(x)}{|x-\xi|}\Big)^{\min(0,\mu_x - |\alpha| - \delta_{i,4} - \varepsilon)} \Big(\frac{r(\xi)}{|x-\xi|}\Big)^{\min(0,\mu_\xi - |\gamma| - \delta_{j,4} - \varepsilon)},$$

where $T = 1 + \delta_{i,4} + \delta_{j,4}$.

3) Let $|x|/2 < |\xi| < 2|x|$ and $|x-\xi| < \min(r(x), r(\xi))$. Then

$$\big|\partial_x^\alpha \partial_\xi^\gamma G_{i,j}(x,\xi)\big| \le c\,|x-\xi|^{-T-|\alpha|-|\gamma|}.$$

Moreover for $i, j = 1, \dots, 4$, there are the representations

$$G_{4,j}(x,\xi) = -\nabla_x \cdot \vec{P}_j(x,\xi) + Q_j(x,\xi), \quad G_{i,4}(x,\xi) = -\nabla_\xi \cdot \vec{\mathcal{P}}_i(x,\xi) + \mathcal{Q}_i(x,\xi),$$

where $\vec{P}_j(x,\xi)\cdot n$ for $x \in \Gamma_k$, $k = 1, \dots, n$, $\xi \in \mathcal{D}$, $\vec{\mathcal{P}}_i(x,\xi)\cdot n$ for $\xi \in \Gamma_k$, $x \in \mathcal{D}$, and

$$\begin{aligned}&|\partial_x^\alpha \partial_\xi^\gamma \vec{P}_j(x,\xi)| \le c_{\alpha,\gamma}\,|x-\xi|^{-1-\delta_{j,4}-|\alpha|-|\gamma|}, \quad |\partial_x^\alpha \partial_\xi^\gamma Q_j(x,\xi)| \le c_{\alpha,\gamma}\,r(\xi)^{-2-\delta_{j,4}-|\alpha|-|\gamma|}, \\ &|\partial_x^\alpha \partial_\xi^\gamma \vec{\mathcal{P}}_i(x,\xi)| \le c_{\alpha,\gamma}\,|x-\xi|^{-1-\delta_{i,4}-|\alpha|-|\gamma|}, \quad |\partial_x^\alpha \partial_\xi^\gamma \mathcal{Q}_i(x,\xi)| \le c_{\alpha,\gamma}\,r(\xi)^{-2-\delta_{i,4}-|\alpha|-|\gamma|}\end{aligned}$$

for $|x|/2 < |\xi| < 2|x|$, $|x-\xi| < \min(r(x), r(\xi))$.

3.6 Auxiliary inequalities

In this subsection we prove estimates for an integral operator with kernel $K(x,\xi)$ which satisfies the same point estimates as the elements $G_{i,j}(x,\xi)$ of Green's matrix with $\sigma = \delta_{i,4}$ and $\tau = \delta_{j,4}$.

Lemma 3.6 *Let ζ_k be the same function as in the proof of Lemma* 2.1 *and let*

$$v(x) = \zeta_m(x) \int_{\mathcal{K}} \zeta_l(\xi)\, f(\xi)\, K(x,\xi)\, d\xi.$$

Suppose that $m \ge l+3$, $f \in W^{0,s}_{\beta-\tau,\delta-\tau}(\mathcal{K})$, and

$$\big|\partial_x^\alpha K(x,\xi)\big| \le c\,\frac{|x|^{\Lambda_- - \sigma - |\alpha| + \varepsilon}}{|\xi|^{\Lambda_- + 1 + \tau + \varepsilon}} \prod_{k=1}^n \Big(\frac{r_k(x)}{|x|}\Big)^{\min(0,\mu_k - \sigma - |\alpha| - \varepsilon)} \prod_{k=1}^n \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{\min(0,\mu_k - \tau - \varepsilon)} \tag{3.18}$$

for $|x| > 2|\xi|$, $|\alpha| \le 2 - \sigma$, where $\sigma, \tau \in \{0,1\}$ and ε is a sufficiently small positive real number. If β and δ satisfy condition (3.17), *then*

$$\|v\|_{W^{2-\sigma,s}_{\beta,\delta}(\mathcal{K})} \le c\,2^{-|m-l|\varsigma} \|\zeta_l f\|_{W^{0,s}_{\beta-\tau,\delta-\tau}(\mathcal{K})}$$

with positive constants c and ς independent of f. The same estimates holds if $l \ge m+3$ and

$$\big|\partial_x^\alpha K(x,\xi)\big| \le c\,\frac{|x|^{\Lambda_+ - \sigma - |\alpha| - \varepsilon}}{|\xi|^{\Lambda_+ + 1 + \tau - \varepsilon}} \prod_{k=1}^n \Big(\frac{r_k(x)}{|x|}\Big)^{\min(0,\mu_k - \sigma - |\alpha| - \varepsilon)} \prod_{k=1}^n \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{\min(0,\mu_k - \tau - \varepsilon)}$$

for $|\xi| > 2|x|$, $|\alpha| \le 2 - \sigma$.

Proof: For $x \in \operatorname{supp}\zeta_m$, $\xi \in \operatorname{supp}\zeta_l$, we have $2^{m-1} < |x| < 2^{m+1}$, $2^{l-1} < |\xi| < 2^{l+1}$. In particular, $|x| > 2|\xi|$ if $m \geq l+3$. Therefore, by (3.18) and by Hölder's inequality,

$$\begin{aligned}
&\int_{\mathcal{K}} |x|^{s(\beta-2+\sigma+|\alpha|)} \prod_{k=1}^{n} \Big(\frac{r_k(x)}{|x|}\Big)^{s\delta_k} \big|\partial_x^\alpha v(x)\big|^s \, dx \\
&\leq c\, 2^{sm(\Lambda_-+\beta-2+\varepsilon)} \int\limits_{\substack{\mathcal{K} \\ 2^{m-1}<|x|<2^{m+1}}} \prod_{k=1}^{n} \Big(\frac{r_k(x)}{|x|}\Big)^{s(\delta_k+\min(0,\mu_k-\sigma-|\alpha|-\varepsilon))} dx \\
&\quad \times \Big(\int_{\mathcal{K}} |\xi|^{-\Lambda_- -1-\tau-\varepsilon} \prod_{k=1}^{n} \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{\min(0,\mu_k-\tau-\varepsilon)} \big|\zeta_l(\xi) f(\xi)\big| \, d\xi \Big)^s \\
&\leq c\, 2^{sm(\Lambda_-+\beta-2+\varepsilon)} \, \|\zeta_l f\|^s_{W^{0,s}_{\beta-\tau,\delta-\tau}(\mathcal{K})} \int\limits_{\substack{\mathcal{K} \\ 2^{m-1}<|x|<2^{m+1}}} \prod_{k=1}^{n} \Big(\frac{r_k(x)}{|x|}\Big)^{s(\delta_k+\min(0,\mu_k-\sigma-|\alpha|-\varepsilon))} dx \\
&\quad \times \Big(\int\limits_{\substack{\mathcal{K} \\ 2^{l-1}<|\xi|<2^{l+1}}} |\xi|^{s'(-\Lambda_- -1-\beta-\varepsilon)} \prod_{k=1}^{n} \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{s'(\min(0,\mu_k-\tau-\varepsilon)-\delta_k+\tau)} d\xi \Big)^{s/s'}
\end{aligned}$$

for $|\alpha| \leq 2-\sigma$, where $s' = s/(s-1)$. Since $s(\delta_k+\min(0,\mu_k-\sigma-|\alpha|) > -2$ and $s'(\min(0,\mu_k-\tau)-\delta_k+\tau) > -2$, we obtain

$$\int_{\mathcal{K}} |x|^{s(\beta-2+\sigma+|\alpha|)} \prod_{k=1}^{n} \Big(\frac{r_k(x)}{|x|}\Big)^{s\delta_k} \big|\partial_x^\alpha v(x)\big|^s \, dx \leq c\, 2^{s(m-l)(\Lambda_-+\beta-2+\varepsilon+3/s)} \, \|\zeta_l f\|^s_{W^{0,s}_{\beta-\tau,\delta-\tau}(\mathcal{K})}$$

This proves the lemma for $m \geq l+3$. The proof for the case $l \geq m+3$ proceeds analogously. ∎

We will show an analogous result for the case $|l-m| \leq 2$. For this we need the following lemma.

Lemma 3.7 *Let $\mathcal{D}$ be the dihedron* (1.1)*, and let $r(x)$ denote the distance of x to the edge. If $\alpha+\beta > 3$ and $\beta < 2$, then*

$$\int\limits_{\substack{\mathcal{D} \\ |\xi-x|>r(x)/3}} |\xi-x|^{-\alpha} \, r(\xi)^{-\beta} \, d\xi \leq c\, r(x)^{3-\alpha-\beta}$$

with a constant c independent of x.

Proof: The substitution $y = x/r(x)$, $\eta = \xi/r(x)$ yields

$$\int\limits_{\substack{\mathcal{D} \\ |\xi-x|>r(x)/3}} |\xi-x|^{-\alpha} \, r(\xi)^{-\beta} \, d\xi = r(x)^{3-\alpha-\beta} \int\limits_{\substack{\mathcal{D} \\ |\eta-y|>1/3}} |\eta-y|^{-\alpha} \, r(\eta)^{-\beta} \, d\eta.$$

Since $r(y) = 1$, the integral on the right is majorized by a finite constant c. This proves the lemma. ∎

In the sequel, let $k(x)$ denote the smallest integer k such that $r_{k(x)} = r(x)$.

Corollary 3.2 *Let $c_1, c_2, \alpha, \beta_j, \gamma_j, \delta_j$ be real numbers such that $\gamma_j + \delta_j < 2$ and $3-\alpha+\beta_j-\delta_j < 0$ for $j = 1, \ldots, n$. Furthermore, let $\mathcal{K}_x = \big\{\xi \in \mathcal{K} : \, c_1|x| < |\xi| < c_2|x|, \ |\xi-x| > r(x)/3\big\}$. Then*

$$\int\limits_{\mathcal{K}_x} |x-\xi|^{-\alpha} \Big(\frac{r(x)}{|x-\xi|}\Big)^{-\beta_{k(x)}} \Big(\frac{r(\xi)}{|x-\xi|}\Big)^{-\gamma_{k(\xi)}} \prod_{j=1}^{n} \Big(\frac{r_j(\xi)}{|\xi|}\Big)^{-\delta_j} \, d\xi \leq c\, |x|^{3-\alpha} \prod_{j=1}^{n} \Big(\frac{r_j(x)}{|x|}\Big)^{3-\alpha-\delta_j} \tag{3.19}$$

with c independent of x.

Proof: Without loss of generality, we may assume that $r(x) = r_1(x)$, i.e. $k(x) = 1$. Then the left-hand side of (3.19) is equal to

$$|x|^{3-\alpha}\, r_1(y)^{-\beta_1} \int\limits_{\mathcal{K}_y} |y-\eta|^{-\alpha+\beta_1} \Big(\frac{r(\eta)}{|y-\eta|}\Big)^{-\gamma_{k(\eta)}} \prod_{j=1}^{n} \Big(\frac{r_j(\eta)}{|\eta|}\Big)^{-\delta_j} d\eta, \tag{3.20}$$

where $y = x/|x|$, $\eta = \xi/|x|$.

Suppose first that $r(y) = r_1(y) < r_i(y)$ for $i = 2, \dots, n$. We denote by $\mathcal{K}_y^{(1)}$ the set of all $\eta \in \mathcal{K}_y$ such that $r(\eta) = r_1(\eta) < r_j(\eta)$ for $j = 2, \dots, n$. Obviously, this set is contained in a dihedron $\mathcal{D}$ with edge $M \supset M_1$. Therefore, by Lemma 3.7,

$$\begin{aligned} \int\limits_{\mathcal{K}_y^{(1)}} |y-\eta|^{-\alpha+\beta_1} \Big(\frac{r(\eta)}{|y-\eta|}\Big)^{-\gamma_{k(\eta)}} \prod_{j=1}^{n} \Big(\frac{r_j(\eta)}{|\eta|}\Big)^{-\delta_j} d\eta &\le c \int\limits_{\mathcal{D}} |\eta-y|^{-\alpha+\beta_1+\gamma_1}\, r_1(\eta)^{-\gamma_1-\delta_1}\, d\eta \\ &\le c\, r_1(y)^{3-\alpha+\beta_1-\delta_1}. \end{aligned}$$

Let $\mathcal{K}_y^{(i)}$, $i = 2, \dots, n$, be the set of all $\eta \in \mathcal{K}_y$ such that $r(\eta) = r_i(\eta)$. Obviously, there exists a constant $c_0 > 0$ such that

$$c_0 < |y-\eta| < c_2 + 1 \quad \text{if } r_1(y) < r_i(y),\ \eta \in \mathcal{K}_y^{(i)},\ i \ge 2.$$

Consequently,

$$\int\limits_{\mathcal{K}_y^{(i)}} |y-\eta|^{-\alpha+\beta_1} \Big(\frac{r(\eta)}{|y-\eta|}\Big)^{-\gamma_{k(\eta)}} \prod_{j=1}^{n} \Big(\frac{r_j(\eta)}{|\eta|}\Big)^{-\delta_j} d\eta \le c \int\limits_{\mathcal{K}_y^{(i)}} r(\eta)^{-\gamma_{k(\eta)}-\delta_{k(\eta)}}\, d\eta \le c \le c\, r_1(y)^{3-\alpha+\beta_1-\delta_1}$$

for $i \ge 2$. This together with (3.20) proves (3.19) if $r(y) = r_1(y) < r_j(y)$ for $j = 2, \dots, n$.

Suppose now that $r(y) = r_1(y) = r_i(y)$ for a certain $i \ge 2$. Then, there are the inequalities

$$c_0 < r(y) \le |y| = 1 \quad \text{and} \quad c_0/3 < |y-\eta| < c_2 + 1 \ \text{ for } \eta \in \mathcal{K}_y$$

with a positive constant c_0. Therefore,

$$\int\limits_{\mathcal{K}_y} |y-\eta|^{-\alpha+\beta_1} \Big(\frac{r(\eta)}{|y-\eta|}\Big)^{-\gamma_{k(\eta)}} \prod_{j=1}^{n} \Big(\frac{r_j(\eta)}{|\eta|}\Big)^{-\gamma_j} d\eta \le c \int\limits_{\mathcal{K}_y} r(\eta)^{-\beta_{k(\eta)}-\delta_{k(\eta)}}\, d\eta \le c \le c\, r_1(y)^{3-\alpha+\beta_1-\delta_1}$$

what implies (3.19). The proof is complete. ∎

We introduce the functions

$$\chi^+(x,\xi) = \chi\Big(\frac{|x-\xi|}{r(x)}\Big), \quad \chi^-(x,\xi) = 1 - \chi^+(x,\xi), \tag{3.21}$$

where χ is an arbitrary smooth cut-off function on $[0,\infty)$, $\chi(t) = 1$ for $t < 1/4$, $\chi(t) = 0$ for $t > 1/2$. Furthermore, let $\mu_x = \mu_{k(x)}$, where $k(x)$ is the smallest integer k such that $r(x) = r_k(x)$.

Lemma 3.8 *Let ζ_k be the same function as in the proof of Lemma 2.1 and let*

$$v(x) = \zeta_m(x) \int_{\mathcal{K}} \zeta_l(\xi)\, f(\xi)\, \chi^-(x,\xi)\, K(x,\xi)\, d\xi,$$

where $|l-m| \le 2$ and $f \in W^{0,s}_{\beta-\tau,\delta-\tau}(\mathcal{K})$. Suppose that

$$\big|\partial_x^\alpha K(x,\xi)\big| \le c\, |x-\xi|^{-1-\sigma-\tau-|\alpha|} \Big(\frac{r(x)}{|x-\xi|}\Big)^{\min(0,\mu_x-\sigma-|\alpha|-\varepsilon)} \Big(\frac{r(\xi)}{|x-\xi|}\Big)^{\min(0,\mu_\xi-\tau-\varepsilon)} \tag{3.22}$$

for $|x|/32 < |\xi| < 32|x|$, $|x-\xi| > r(x)/4$, $|\alpha| \le 2-\sigma$, where $\sigma, \tau \in \{0,1\}$, ε is a sufficiently small positive real number. If $\max(0, 2-\mu_k) < \delta_k + 2/s < 2$ for $k = 1, \dots, n$, then

$$\|v\|_{W^{2-\sigma,s}_{\beta,\delta}(\mathcal{K})} \le c\, \|\zeta_l f\|_{W^{0,s}_{\beta-\tau,\delta-\tau}(\mathcal{K})}.$$

Proof: Let $|\alpha| \le 2-\sigma$. Obviously, $\big|\partial_x^\alpha v(x)\big| \le c \sum\limits_{j+|\gamma|=|\alpha|} |x|^{-j} A_\gamma(x)$, where

$$A_\gamma(x) = \int\limits_{\substack{\mathcal{K} \\ |x-\xi|>r(x)/4}} |x-\xi|^{-1-\sigma-\tau-|\gamma|} \Big(\frac{r(x)}{|x-\xi|}\Big)^{\min(0,\mu_x-\sigma-|\gamma|-\varepsilon)} \Big(\frac{r(\xi)}{|x-\xi|}\Big)^{\min(0,\mu_\xi-\tau-\varepsilon)} |\zeta_l(\xi) f(\xi)|\, d\xi$$

We have to prove that

$$\int_{\mathcal{K}} |x|^{s(\beta-2+\sigma+|\gamma|)} \prod_{k=1}^n \Big(\frac{r_k(x)}{|x|}\Big)^{s\delta_k} \big|A_\gamma(x)\big|^s\, dx \le c\, \|\zeta_l f\|^s_{V^{0,s}_{\beta-\tau,\delta-\tau}(\mathcal{K})} \tag{3.23}$$

for $|\gamma| \le 2-\sigma$. Let first $\sigma+\tau+|\gamma| \ne 0$, and let $s' = s/(s-1)$. Using Hölder's inequality and Corollary 3.2, we obtain

$$\begin{aligned}
|A_\gamma(x)|^s \le c \int\limits_{\substack{\mathcal{K} \\ |x-\xi|>r(x)/4}} & |x-\xi|^{-1-\sigma-\tau-|\gamma|} \Big(\frac{r(x)}{|x-\xi|}\Big)^{\min(0,\mu_x-\sigma-|\gamma|-\varepsilon)} \Big(\frac{r(\xi)}{|x-\xi|}\Big)^{\min(0,\mu_\xi-\tau-\varepsilon)} \\
& \times \prod_{k=1}^n \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{ss_k} |\zeta_l f|^s\, d\xi \Big(\int\limits_{\substack{|x|/32<|\xi|<32|x| \\ |x-\xi|>r(x)/4}} |x-\xi|^{-1-\sigma-\tau-|\gamma|} \Big(\frac{r(x)}{|x-\xi|}\Big)^{\min(0,\mu_x-\sigma-|\gamma|-\varepsilon)} \\
& \times \Big(\frac{r(\xi)}{|x-\xi|}\Big)^{\min(0,\mu_\xi-\tau-\varepsilon)} \prod_{k=1}^n \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{-s's_k} d\xi\Big)^{s-1} \\
\le\ |x|^{(s-1)(2-\sigma-\tau-|\gamma|)} \prod_{k=1}^n \Big(\frac{r_k(x)}{|x|}\Big)^{(s-1)(2-\sigma-\tau-|\gamma|)-ss_k} & \int\limits_{\substack{\mathcal{K} \\ |x-\xi|>r(x)/4}} |x-\xi|^{-1-\sigma-\tau-|\gamma|} \\
\times \Big(\frac{r(x)}{|x-\xi|}\Big)^{\min(0,\mu_x-\sigma-|\gamma|-\varepsilon)} & \Big(\frac{r(\xi)}{|x-\xi|}\Big)^{\min(0,\mu_\xi-\tau-\varepsilon)} \prod_{k=1}^n \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{ss_k} \big|\zeta_l(\xi)\, f(\xi)\big|^s\, d\xi
\end{aligned} \tag{3.24}$$

provided s_k satisfies the inequalities

$$\max(2-\sigma-\tau-|\gamma|, 2-\tau-\mu_k) < s's_k < \min(2, 2-\tau+\mu_k). \tag{3.25}$$

We put $t_k = -\tau$ if $\sigma+|\gamma| = 2$, $t_k = \max(1-\tau, 2-\tau-\mu_k) - \max(0, 2-\mu_k)$ if $\sigma+|\gamma| = 1$. In the case $\sigma = |\gamma| = 0$, $\tau = 1$ let t_k be arbitrary numbers in the intervals $1-\delta_k-2/s < t_k < 1-\delta_k-2/s+\min(1,\mu_k)$. Obviously, $-\tau \le t_k \le 2-\sigma-|\gamma|-\tau$. Due to condition (ii), the numbers s_k can be chosen such that, additionally to condition (3.25), the inequalities

$$\delta_k + t_k + \frac{1}{s}\max(0, \tau-\mu_k) < s_k < \delta_k + t_k + \frac{1}{s}\min(\sigma+\tau+|\gamma|, \tau+\mu_k).$$

are satisfied. Consequently, by Corollory 3.2,

$$
\begin{aligned}
&\int_{\mathcal{K}} |x|^{s(\beta-2+\sigma+|\gamma|)} \prod_{k=1}^{n} \Big(\frac{r_k(x)}{|x|}\Big)^{s\delta_k} \big|A_\gamma(x)\big|^s \, dx \\
&\le c \int\limits_{\substack{\mathcal{K} \\ 2^{m-1}<|x|<2^{m+1}}} |x|^{s(\beta-\tau)-2+\sigma+\tau+|\gamma|} \prod_{k=1}^{n} \Big(\frac{r_k(x)}{|x|}\Big)^{s(\delta_k-s_k)+(s-1)(2-\sigma-\tau-|\gamma|)} \int\limits_{\substack{\mathcal{K} \\ |x-\xi|>r(x)/4}} |x-\xi|^{-1-\sigma-\tau-|\gamma|} \\
&\qquad \times \Big(\frac{r(x)}{|x-\xi|}\Big)^{\min(0,\mu_x-\sigma-|\gamma|-\varepsilon)} \Big(\frac{r(\xi)}{|x-\xi|}\Big)^{\min(0,\mu_\xi-\tau-\varepsilon)} \prod_{k=1}^{n} \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{ss_k} |\zeta_l f|^s \, d\xi \, dx \\
&\le c \int_{\mathcal{K}} |\xi|^{s(\beta-\tau)-2+\sigma+\tau+|\gamma|} \prod_{k=1}^{n} \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{ss_k} \big|\zeta_l(\xi) f(\xi)\big|^s \int\limits_{\substack{2^{m-1}<|x|<2^{m+1} \\ |x-\xi|>r(\xi)/5}} |x-\xi|^{-1-\sigma-\tau-|\gamma|} \\
&\qquad \times \Big(\frac{r(x)}{|x-\xi|}\Big)^{\min(0,\mu_x-\sigma-|\gamma|-\varepsilon)} \Big(\frac{r(\xi)}{|x-\xi|}\Big)^{\min(0,\mu_\xi-\tau-\varepsilon)} \prod_{k=1}^{n} \Big(\frac{r_k(x)}{|x|}\Big)^{s(\delta_k-s_k+t_k)-2+\sigma+\tau+|\gamma|} dx \, d\xi \\
&\le c \int_{\mathcal{K}} |\xi|^{s(\beta-\tau)} \prod_{k=1}^{n} \Big(\frac{r_k(x)}{|x|}\Big)^{s(\delta_k+t_k)} \big|\zeta_l(\xi) f(\xi)\big|^s \, d\xi.
\end{aligned}
$$

This proves (3.23) for $\sigma+\tau+|\gamma| \neq 0$. If $\sigma=\tau=|\gamma|=0$, then Hölder's inequality yields

$$
\begin{aligned}
&\big|A_\gamma(x)\big|^s \le \int_{\mathcal{K}} |x-\xi|^{-1} \prod_{k=1}^{n} \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{s\delta_k} \big|\zeta_l(\xi) f(\xi)\big| \, d\xi \Big(\int\limits_{\substack{\mathcal{K} \\ 2^{l-1}<|\xi|<2^{l+1}}} |x-\xi|^{-1} \prod_{k=1}^{n} \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{-s'\delta_k} d\xi \Big)^{s-1} \\
&\le c\, 2^{2l(s-1)} \int_{\mathcal{K}} |x-\xi|^{-1} \prod_{k=1}^{n} \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{s\delta_k} \big|\zeta_l(\xi) f(\xi)\big| \, d\xi.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int_{\mathcal{K}} |x|^{s(\beta-2)} \prod_{k=1}^{n} \Big(\frac{r_k(x)}{|x|}\Big)^{s\delta_k} \big|A_\gamma(x)\big|^s \, dx \\
&\le c \int_{\mathcal{K}} |\xi|^{s\beta-2} \prod_{k=1}^{n} \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{s\delta_k} \big|\zeta_l(\xi) f(\xi)\big|^s \Big(\int\limits_{\substack{\mathcal{K} \\ 2^{m-1}<|x|<2^{m+1}}} |x-\xi|^{-1} \prod_{k=1}^{n} \Big(\frac{r_k(x)}{|x|}\Big)^{s\delta_k} dx \Big) \, d\xi \\
&\le c \int_{\mathcal{K}} |\xi|^{s\beta} \prod_{k=1}^{n} \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{s\delta_k} \big|\zeta_l(\xi) f(\xi)\big|^s \, d\xi.
\end{aligned}
$$

This proves the lemma. ∎

Lemma 3.9 *Let χ^+ be defined by (3.21), $\sigma \in \{0,1\}$, and*

$$
v(x) = \zeta_m(x) \int_{\mathcal{K}} \zeta_l(\xi)\, f(\xi)\, \chi^+(x,\xi)\, K(x,\xi)\, d\xi, \quad |l-m| \le 2.
$$

1) *If $f \in V^{0,s}_{\beta,\delta}(\mathcal{K})$, and $K(x,\xi)$ satisfies the estimate*

$$
|\partial_x^\alpha K(x,\xi)| \le c\, |x-\xi|^{-1-\sigma-|\alpha|} \quad \text{for } |x|/32 < |\xi| < 32|x|,\ |x-\xi| < r(x)/2,\ |\alpha| \le 1-\sigma, \tag{3.26}
$$

then

$$
\|v\|_{V^{1-\sigma,s}_{\beta-1,\delta-1}(\mathcal{K})} \le c\, \|\zeta_l f\|_{V^{0,s}_{\beta,\delta}(\mathcal{K})}.
$$

2) *If $K(x,\xi)$ has the representation $K(x,\xi)=\nabla_\xi P(x,\xi)+Q(x,\xi)$, where*

$$|\partial_x^\alpha P(x,\xi)|\le c\,|x-\xi|^{-1-\sigma-|\alpha|},\quad |\partial_x^\alpha Q(x,\xi)|\le c\,r(\xi)^{-2-\sigma-|\alpha|} \tag{3.27}$$

for $|x|/32<|\xi|<32|x|$, $|x-\xi|<r(x)/2$, $|\alpha|\le 1-\sigma$, and $P(x,\xi)\cdot n^{(j)}=0$ for $\xi\in\Gamma_j$, $j=1,\dots,n$ (here $n^{(j)}$ denotes the exterior normal to Γ_j), then

$$\|v\|_{V^{1-\sigma,s}_{\beta-1,\delta-1}(\mathcal{K})}\le c\,\|\zeta_l f\|_{V^{1,s}_{\beta,\delta}(\mathcal{K})}\,.$$

Proof: 1) By our definition, the function $\chi^+(x,\xi)$ vanishes for $|x-\xi|>r(x)/2$. Note that

$$\frac{1}{32}|x|<|\xi|<32|x|,\quad \frac{1}{2}r_k(x)\le r_k(\xi)\le\frac{3}{2}\,r_k(x),\quad\text{and}\quad \frac{1}{2}r(x)\le r(\xi)\le\frac{3}{2}\,r(x), \tag{3.28}$$

for $x\in\operatorname{supp}\zeta_m$, $\xi\in\operatorname{supp}\zeta_l$, and $|x-\xi|<r(x)/2$. Let $\mathcal{K}_x=\{\xi\in\mathcal{K}:\ |x|/32<|\xi|<32|x|,\ |x-\xi|<r(x)/2\}$. Then

$$\big|\partial_x^\alpha v(x)\big|\le c\int_{\mathcal{K}_x}|x-\xi|^{-1-\sigma-|\alpha|}\,\big|\zeta_l(\xi)\,f(\xi)\big|\,d\xi$$

and, consequently,

$$\begin{aligned}\big|\partial_x^\alpha v(x)\big|^s &\le c\int_{\mathcal{K}_x}|x-\xi|^{-1-\sigma-|\alpha|}\,\big|\zeta_l(\xi)\,f(\xi)\big|^s\,d\xi\,\Big(\int\limits_{|x-\xi|<r(x)/2}|x-\xi|^{-1-\sigma-|\alpha|}\,d\xi\Big)^{s-1}\\ &\le c\,r(x)^{(s-1)(2-\sigma-|\alpha|)}\int_{\mathcal{K}_x}|x-\xi|^{-1-\sigma-|\alpha|}\,\big|\zeta_l(\xi)\,f(\xi)\big|^s\,d\xi\end{aligned}$$

for $|\alpha|\le 1-\sigma$. Using (3.28), we obtain

$$\begin{aligned}&\int_{\mathcal{K}}|x|^{s(\beta-2+\sigma+|\alpha|)}\prod_{k=1}^n\Big(\frac{r_k(x)}{|x|}\Big)^{s(\delta_k-2+\sigma+|\alpha|)}\big|\partial_x^\alpha v(x)\big|^s\,dx\\ &\le c\int_{\mathcal{K}}|\xi|^{s(\beta-2+\sigma+|\alpha|)}\prod_{k=1}^n\Big(\frac{r_k(\xi)}{|\xi|}\Big)^{s(\delta_k-2+\sigma+|\alpha|)}r(\xi)^{(s-1)(2-\sigma-|\alpha|)}\,|\zeta_l f|^s\Big(\int\limits_{|x-\xi|<r(\xi)}|x-\xi|^{-1-\sigma-|\alpha|}\,dx\Big)\,d\xi\\ &\le c\int_{\mathcal{K}}|\xi|^{s\beta}\prod_{k=1}^n\Big(\frac{r_k(\xi)}{|\xi|}\Big)^{s\delta_k}\,\big|\zeta_l(\xi)f(\xi)\big|^s\,d\xi.\end{aligned}$$

This proves the first part.

2) The second part can be proved analogously using the estimate

$$\big|\partial_x^\alpha v(x)\big|\le c\int_{\mathcal{K}_x}\Big(|x-\xi|^{-1-\sigma-|\alpha|}\,\big|\nabla_\xi(\zeta_l f)\big|+r(\xi)^{-2-\sigma-|\alpha|}\,|\zeta_l f|\Big)\,d\xi$$

which follows from our assumptions on $K(x,\xi)$. ∎

3.7 Existence of solutions

Let $f\in W^{0,s}_{\beta,\delta}(\mathcal{K})^3$, $g\in W^{1,s}_{\beta,\delta}(\mathcal{K})$, $h_j\in W^{2-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, $\phi_j\in W^{1-1/s,s}_{\beta,\delta}(\Gamma_j)^{d_j}$. Our goal is to show that there exists a solution $(u,p)\in W^{2,s}_{\beta,\delta}(\mathcal{K})^3\times W^{1,s}_{\beta,\delta}(\mathcal{K})$ of problem (3.1), (3.2) if the following conditions are satisfied.

(i) there are no eigenvalues of the pencil $\mathfrak{A}(\lambda)$ on the line $\operatorname{Re}\lambda=2-\beta-3/s$,

(ii) $\max(2-\mu_k,0)<\delta_k+2/s<2$ for $k=1,\dots,n$.

(iii) h_j,ϕ_j and g are such that there exist $u\in W^{2,s}_{\beta,\delta}(\mathcal{K})^3$ and $p\in W^{1,s}_{\beta,\delta}(\mathcal{K})$ satisfying (3.8).

The last is a condition on the traces of g, ϕ_j, h_j and the derivatives of h_j on the edges of the cone $\mathcal{K}$ (see Section 3.1).

Lemma 3.10 *Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces of functions on $\mathcal{K}$ in each of them the multiplication with a scalar function from $C_0^\infty(\overline{\mathcal{K}}\setminus\{0\})$ is defined. We suppose that the inequalities*

$$\|f\|_{\mathcal{X}} \ge c\Big(\sum_{j=-\infty}^{+\infty} \|\zeta_j f\|_{\mathcal{X}}^s\Big)^{1/s}, \qquad \|u\|_{\mathcal{Y}} \le c\Big(\sum_{j=-\infty}^{+\infty} \|\zeta_j u\|_{\mathcal{Y}}^s\Big)^{1/s}$$

are satisfied for all $f \in \mathcal{X}$, $u \in \mathcal{Y}$. Furthermore, let $\mathcal{O}$ be a linear operator from $\mathcal{X}$ into $\mathcal{Y}$ defined on functions with compact support in $\overline{\mathcal{K}}\setminus\{0\}$ such that

$$\|\zeta_m \mathcal{O}\zeta_l f\|_{\mathcal{Y}} \le c\,2^{-\varsigma|l-m|}\,\|\zeta_l f\|_{\mathcal{X}}$$

with positive constants c, ς independent of l, m and f. Then $\|\mathcal{O}f\|_{\mathcal{Y}} \le c\,\|f\|_{\mathcal{X}}$ for all $f \in \mathcal{X}$ with compact support in $\overline{\mathcal{K}}\setminus\{0\}$.

The proof of this lemma can be found in [18].

Theorem 3.1 *Let $f \in W^{0,s}_{\beta,\delta}(\mathcal{K})^3$, $g \in W^{1,s}_{\beta,\delta}(\mathcal{K})$, $h_j \in W^{2-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, and $\phi_j \in W^{1-1/s,s}_{\beta,\delta}(\Gamma_j)^{d_j}$. Suppose that conditions* (i)–(iii) *are satisfied. Then there exists a solution $(u,p) \in W^{2,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{1,s}_{\beta,\delta}(\mathcal{K})$ of problem* (3.1), (3.2) *satisfying the estimate*

$$\|u\|_{W^{2,s}_{\beta,\delta}(\mathcal{K})^3} + \|p\|_{W^{1,s}_{\beta,\delta}(\mathcal{K})} \le c\,\Big(\|f|_{W^{0,s}_{\beta,\delta}(\mathcal{K})^3} + \|g\|_{W^{1,s}_{\beta,\delta}(\mathcal{K})} + \sum_{j=1}^{n}\Big(\|h_j\|_{W^{2-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j}} + \|\phi_j\|_{W^{1-1/s,s}_{\beta,\delta}(\Gamma_j)^{d_j}}\Big)\Big).$$

Proof: Without loss of generality we may assume that $h_j = 0$, $\phi_j = 0$, and $g \in V^{1,s}_{\beta,\delta}(\mathcal{K})$. We consider the operator

$$\mathcal{X} \stackrel{def}{=} W^{0,s}_{\beta,\delta}(\mathcal{K})^3 \times V^{1,s}_{\beta,\delta}(\mathcal{K}) \ni (f,g) \to \mathcal{O}(f,g) = (u,p),$$

where u and p are defined by (3.15), (3.16) and $G_{i,j}$ are the elements of Green's matrix introduced in Section 3.4. Then by Lemma 3.6,

$$\|\zeta_m \mathcal{O}\zeta_l(f,g)\|_{\mathcal{Y}} \le c\,2^{-|l-m|\varsigma}\|\zeta_l(f,g)\|_{\mathcal{X}} \tag{3.29}$$

for $|l-m| \ge 3$, where $\mathcal{Y} = W^{2,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{1,s}_{\beta,\delta}(\mathcal{K})$ and c, ς are positive constants independent of f, g, l, m. In order to prove the same inequality for $|l-m| \le 2$, we introduce the functions

$$u_i^{\pm}(x) = \sum_{j=1}^{3}\int_{\mathcal{K}} \zeta_l(\xi)\,f_j(\xi)\,\chi^{\pm}(x,\xi)\,G_{i,j}(x,\xi)\,d\xi + \int_{\mathcal{K}} \zeta_l(\xi) g(\xi)\,\chi^{\pm}(x,\xi)\,G_{i,4}(x,\xi)\,d\xi, \quad i = 1,2,3,$$

$$p^{\pm}(x) = -\zeta_l(x)\,g(x) + \sum_{j=1}^{3}\int_{\mathcal{K}} \zeta_l(\xi)\,f_j(\xi)\,\chi^{\pm}(x,\xi)\,G_{4,j}(x,\xi)\,d\xi + \int_{\mathcal{K}} \zeta_l(\xi)\,g(\xi)\,\chi^{\pm}(x,\xi) G_{4,4}(x,\xi)\,d\xi,$$

where χ^+ and χ^- are defined by (3.21). Then

$$-\Delta(u^+ + u^-) + \nabla(p^+ + p^-) = \zeta_l f, \quad -\nabla\cdot(u^+ + u^-) = \zeta_l g \ \text{ in } \mathcal{K}$$

and $S_j(u^+ + u^-) = 0$, $N_j(u^+ + u^-, p^+ + p^-) = 0$ on Γ_j. Furthermore, by Lemmas 3.8, 3.9, there are the inequalities

$$\|\zeta_m(u^-,p^-)\|_{\mathcal{Y}} \le c\,\|\zeta_l(f,g)\|_{\mathcal{X}}, \quad \|\zeta_m(u^+,p^+)\|_{V^{1,s}_{\beta-1,\delta-1}(\mathcal{K})^3 \times V^{0,s}_{\beta-1,\delta-1}(\mathcal{K})} \le c\,\|\zeta_l(f,g)\|_{\mathcal{X}} \tag{3.30}$$

if $|l-m| \le 3$, where c is independent of f, g, l, m. Let $\eta_m = \zeta_{m-1} + \zeta_m + \zeta_{m+1}$. Then, by Corollary 3.1,

$$\begin{aligned}
&\|\zeta_m u^+\|_{V^{2,s}_{\beta,\delta}(\mathcal{K})^3} + \|\zeta_m p^+\|_{V^{1,s}_{\beta,\delta}(\mathcal{K})} \\
&\le c\,\Big(\|\eta_m u^+\|_{V^{1,s}_{\beta-1,\delta-1}(\mathcal{K})^3} + \|\eta_m p^+\|_{V^{0,s}_{\beta-1,\delta-1}(\mathcal{K})} + \|\eta_m\zeta_l(f,g)\|_{\mathcal{X}} + \|\eta_m(u^-,p^-)\|_{\mathcal{Y}}\Big)
\end{aligned}$$

for $|l-m|\le 2$. Due to (3.30), the right hand side of the last inequality can be estimated by the norm of $\zeta_l(f,g)$ in $\mathcal{X}$. Consequently,

$$\|\zeta_m(u^+ + u^-, p^+ + p^-)\|_{\mathcal{Y}} \le c\,\|\zeta_l(f,g)\|_{\mathcal{X}} \quad \text{for } |l-m|\le 2.$$

Thus, estimate (3.29) is valid for arbitrary l and m. Now the assertion of the theorem follows immediately from Lemma 3.10. ∎

3.8 Uniqueness of the solution

First we prove the uniqueness of the solution in Theorem 3.1 in the case $s \ge 2$.

Lemma 3.11 *Let $s \ge 2$, and let the conditions* (i), (ii) *be satisfied. Then the homogeneous boundary value problem* (3.1), (3.2) *has only the trivial solution $(u,p) = (0,0)$ in $W^{2,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{1,s}_{\beta,\delta}(\mathcal{K})$.*

Proof: Let $(u,p) \in W^{2,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{1,s}_{\beta,\delta}(\mathcal{K})$ be a solution of the homogeneous problem (3.1), (3.2). By χ we denote a smooth cut-off function on $\overline{\mathcal{K}}$ equal to one for $|x|<1$ and to zero for $|x|>2$. Furthermore, we set $\beta' = \beta - \frac{3}{2} + \frac{3}{s}$ and $\delta_j' = \delta_j - 1 + \frac{2}{s}$ for $j = 1,\dots,n$. Then, by Hölder's inequality,

$$\begin{aligned}
&\int_{\mathcal{K}} \rho^{2(\beta'+\varepsilon-2+|\alpha|)} \prod \Big(\frac{r_j}{\rho}\Big)^{2(\delta_j'+\varepsilon)} |\partial_x^\alpha(\chi u)|^2\,dx \\
&\le \Big(\int_{\mathcal{K}} \rho^{s(\beta-2+|\alpha|)} \prod \Big(\frac{r_j}{\rho}\Big)^{s\delta_j} |\partial_x^\alpha(\chi u)|^s\,dx\Big)^{2/s} \Big(\int\limits_{\substack{\mathcal{K}\\ |x|\le 2}} \rho^{-3+s'\varepsilon} \prod \Big(\frac{r_j}{\rho}\Big)^{-2+q\varepsilon} dx\Big)^{2/s'},
\end{aligned}$$

where $s' = 2s/(s-2)$. The second integral on the right is finite if $\varepsilon > 0$. Consequently, $\chi u \in W^{2,2}_{\beta'+\varepsilon,\delta+\varepsilon}(\mathcal{K})^3$ and $\chi p \in W^{1,2}_{\beta'+\varepsilon,\delta+\varepsilon}(\mathcal{K})$. Analogously, we obtain $(1-\chi)u \in W^{2,2}_{\beta'-\varepsilon,\delta-\varepsilon}(\mathcal{K})^3$ and $(1-\chi)u \in W^{2,2}_{\beta'-\varepsilon,\delta-\varepsilon}(\mathcal{K})$. This implies

$$-\Delta(\chi u) + \nabla(\chi p) = \Delta\big((1-\phi)u\big) - \nabla\big((1-\chi)p\big) \in W^{0,2}_{\beta'-\varepsilon,\delta-\varepsilon}(\mathcal{K})^3,$$

and, analogously, $\nabla\cdot(\chi u) \in W^{0,2}_{\beta'-\varepsilon,\delta-\varepsilon}(\mathcal{K})$, $S_j(\chi u) \in W^{3/2,2}_{\beta'-\varepsilon,\delta-\varepsilon}(\Gamma_j)^{3-d_j}$ and $N_j(\chi u,\chi p) \in W^{3/2,2}_{\beta'-\varepsilon,\delta-\varepsilon}(\Gamma_j)^{d_j}$. From this and from [25, Th.4.1] it follows that $\chi u \in W^{2,2}_{\beta'-\varepsilon,\delta-\varepsilon}(\mathcal{K})^3$ and $\chi p \in W^{1,2}_{\beta'-\varepsilon,\delta-\varepsilon}(\mathcal{K})$. The same is then obviously true for u and p. Hence, by Theorem [25, Th.4.1], we have $u=0$ and $p=0$. ∎

It remains to prove the uniqueness of the solution in Theorem 3.1 in the case $s \le 2$. In this case we pass to the coordinates t,ω, where $t = \log\rho = \log|x|$ and $\omega = x/|x|$. We denote by $W^{l,s}_\delta(\mathbb{R}\times\Omega)$ the weighted Sobolev space with the norm

$$\|u\|_{W^{l,s}_\delta(\mathbb{R}\times\Omega)} = \Big(\int_{\mathbb{R}} \sum_{j=0}^{l} \|\partial_t^j u(t,\cdot)\|^s_{W^{l-j,s}_\delta(\Omega)}\,dt\Big)^{1/s}.$$

Note that $u \in W^{l,s}_{\beta,\delta}(\mathcal{K})$ if and only if $\rho^{\beta-l+3/s}u$ (as function of the variables t and ω) belongs to $W^{l,s}_\delta(\mathbb{R}\times\Omega)$.

For an arbitrary function $v \in W^{l,s}_\delta(\mathbb{R}\times\Omega)$ we define by v_ε the mollification with respect to the variable t of v, i.e.,

$$v_\varepsilon(t,\omega) = \int_{\mathbb{R}} v(\tau,\omega)\,h_\varepsilon(t-\tau)\,d\tau,$$

where $h_\varepsilon(t) = \varepsilon^{-1}h(t/\varepsilon)$ and h is a smooth function with compact support, $\int h(t)\,dt = 1$. Since

$$\partial_\omega^\alpha \partial_t^{j+k} v_\varepsilon(\omega,t) = \int_{\mathbb{R}} (\partial_\omega^\alpha \partial_t^k v)(\omega,\tau)\,h_\varepsilon^{(j)}(t-\tau)\,d\tau,$$

it follows that $\partial_t^j v_\varepsilon \in W^{l,s}_\delta(\mathbb{R}\times\Omega)$ for $v \in W^{3,s}_\delta(\mathbb{R}\times\Omega)$, $\varepsilon>0$, $j=0,1,\dots$.

Theorem 3.2 *Let* $\beta \in \mathbb{R}$, $\delta \in \mathbb{R}^n$, $f \in W^{0,s}_{\beta,\delta}(\mathcal{K})$, $g \in W^{1,s}_{\beta,\delta}(\mathcal{K})$, $h_j \in W^{2-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, *and* $\phi_j \in W^{1-1/s,s}_{\beta,\delta}(\Gamma_j)^{d_j}$ *be such that conditions* (i)–(iii) *are satisfied. Then problem* (3.1), (3.2) *has a unique solution* $(u,p) \in W^{2,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{1,s}_{\beta,\delta}(\mathcal{K})$.

Proof: The existence existence of the solution and the uniqueness for $s \geq 2$ are already proved. We show the uniqueness for the case $1 < s < 2$. Let $(u,p) \in W^{2,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{2,s}_{\beta,\delta}(\mathcal{K})$ be a solution of the homogeneous problem (3.1), (3.2). Since $W^{l,s}_{\beta,\delta}(\mathcal{K}) \subset W^{l,s}_{\beta,\delta'}(\mathcal{K})$ if $\delta_j \leq \delta'_j$ for $j = 1, \ldots, n$, we may assume, without loss of generality, that $\max(2-\mu_j, 1) < \delta_j + 2/s < 2$.

From Lemma 3.4 it follows that $u \in W^{3,s}_{\beta+1,\delta+1}(\mathcal{K})^\ell$ and $p \in W^{2,s}_{\beta+1,\delta+1}(\mathcal{K})$. We set $v = \rho^{\beta-2+3/s}u$ and $q = \rho^{\beta-1+3/s}p$. Then, in the coordinates $t = \log|x|$ and $\omega = x/|x|$, we have $v \in W^{3,s}_{\delta+1}(\mathbb{R}\times\Omega)^3$ and $q \in W^{2,s}_{\delta+1}(\mathbb{R}\times\Omega) = V^{2,s}_{\delta+1}(\mathbb{R}\times\Omega)$. Consequently, $\partial_t^j v_\varepsilon \in W^{3,s}_{\delta+1}(\mathbb{R}\times\Omega)^3$ for $j = 0, 1, 2, \ldots$. From Corollary 1.1 we conclude that $v_\varepsilon \in W^{1,2}_{\delta-2+2/s}(\mathbb{R}\times\Omega)^3 \subset W^{1,2}_0(\mathbb{R}\times\Omega)^3$. Thus, the function $u_\varepsilon = \rho^{-\beta+2-3/s}v_\varepsilon$ (as function in x) belongs to the spaces $W^{1,2}_{\beta+3/s-5/2,0}(\mathcal{K})^3$. Analogously, using Lemma 1.4, we obtain $p_\varepsilon = \rho^{-\beta+1-3/s}q_\varepsilon \in W^{0,2}_{\beta+3/s-5/2,0}(\mathcal{K})$. It can be easily seen that $(u_\varepsilon, p_\varepsilon)$ is also a solution of the homogeneous problem (3.1), (3.2). According to [25, Th.4.2], this problem has no nonzero solutions in $W^{1,2}_{\beta+3/s-5/2,0}(\mathcal{K})^3 \times W^{0,2}_{\beta+3/s-5/2,0}(\mathcal{K})$. Therefore, $u_\varepsilon = 0$, $p_\varepsilon = 0$ what implies $u = 0$, $v = 0$. The proof is complete. ∎

Theorem 3.3 *Let* $(u,p) \in W^{2,\sigma}_{\beta',\delta'}(\mathcal{K})^3 \times W^{1,\sigma}_{\beta',\delta'}(\mathcal{K})$ *be a solution of problem* (3.1), (3.2), *where*

$$f \in W^{0,s}_{\beta,\delta}(\mathcal{K})^3 \cap W^{0,\sigma}_{\beta',\delta'}(\mathcal{K})^3, \quad g \in W^{1,s}_{\beta,\delta}(\mathcal{K}) \cap W^{1,\sigma}_{\beta',\delta'}(\mathcal{K}),$$
$$h_j \in W^{2-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j} \cap W^{2-1/\sigma,\sigma}_{\beta',\delta'}(\Gamma_j)^{3-d_j}, \quad \phi_j \in W^{1-1/s,s}_{\beta,\delta}(\Gamma_j)^{d_j} \cap W^{1-1/\sigma,\sigma}_{\beta',\delta'}(\Gamma_j)^{d_j}.$$

Suppose that there are no eigenvalues of the pencil $\mathfrak{A}(\lambda)$ *in the closed strip between the lines* $\operatorname{Re}\lambda = 2-\beta-3/s$ *and* $\operatorname{Re}\lambda = 2-\beta'-3/\sigma$, δ *and* δ' *satisfy the inequalities* $\max(0, 2-\mu_k) > \delta_k + 2/s < 2$, $\max(0, 2-\mu_k) > \delta'_k + 2/\sigma < 2$ *and* g, h_j, ϕ_j *satisfy condition* (iii) *of Section* 3.6. *Then* $u \in W^{2,s}_{\beta,\delta}(\mathcal{K})^3$ *and* $p \in W^{1,s}_{\beta,\delta}(\mathcal{K})$.

Proof: By Theorem 3.2, there are unique solutions of problem (3.1), (3.2) in $W^{2,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{1,s}_{\beta,\delta}(\mathcal{K})$ and $W^{2,\sigma}_{\beta',\delta'}(\mathcal{K})^3 \times W^{1,\sigma}_{\beta,\delta}(\mathcal{K})$. These solutions coincide, since they are represented by the same Green's matrix. ∎

4 Weak solutions of the boundary value problem in a cone

4.1 Definition of weak solutions

Obviously, the bilinear form

$$b(u,v) = 2\int_{\mathcal{K}} \sum_{i,j=1}^{3} \varepsilon_{i,j}(u)\,\varepsilon_{i,j}(v)\,dx \tag{4.1}$$

is continuous on $W^{1,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{1,s'}_{-\beta,-\delta}(\mathcal{K})^3$, where $s' = s/(s-1)$. We suppose in this section that the line $\operatorname{Re}\lambda = 1-\beta-3/s$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$ and that

$$\max(1-\mu_k, 0) < \delta_k + 2/s < 1 \quad \text{for } k = 1, \ldots, n. \tag{4.2}$$

Then $-\delta_k > 1 - 2/s'$ and, therefore, $W^{1,s'}_{-\beta,-\delta}(\mathcal{K}) = V^{1,s'}_{-\beta,-\delta}(\mathcal{K})$. By $V^{-1,s}_{\beta,\delta}(\mathcal{K})$ we denote the dual space of $V^{1,s'}_{-\beta,-\delta}(\mathcal{K})^3$. It can be shown analogously to [1, Th.3.8] that every functional $F \in V^{-1,s}_{\beta,\delta}(\mathcal{K})^3$ has the form

$$F(v) = \int_{\mathcal{K}} f^{(0)} \cdot v\,dx + \sum_{k=1}^{3} \int_{\mathcal{K}} f^{(k)}\,\partial_{x_k} v \cdot dx \quad \text{for all } v \in V^{1,s'}_{-\beta,-\delta}(\mathcal{K})^3, \tag{4.3}$$

where $f^{(0)} \in V^{0,s}_{\beta+1,\delta+1}(\mathcal{K})^3$ and $f^{(k)} \in V^{0,s}_{\beta,\delta}(\mathcal{K})^3$, $k = 1, 2, 3$.

Let $F \in V^{-1,s}_{\beta,\delta}(\mathcal{K})^3$, $g \in W^{0,s}_{\beta,\delta}(\mathcal{K})$ and $h_j \in W^{1-1/s,s}_{\beta,\delta}(\Gamma_j)$, $j = 1, \dots, n$. By a weak solution of problem (3.1), (3.2) we mean a pair $(u, p) \in W^{1,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{1,s}_{\beta,\delta}(\mathcal{K})^3$ satisfying

$$b(u, v) - \int_{\mathcal{K}} p\, \nabla \cdot v = F(v) \ \text{ for all } v \in V^{1,s'}_{-\beta,-\delta}(\mathcal{K})^3,\ S_j v = 0 \text{ on } \Gamma_j,\ j = 1, \dots, n, \tag{4.4}$$

$$-\nabla \cdot u = g \ \text{ in } \mathcal{K}, \quad S_j u = h_j \ \text{ on } \Gamma_j,\ j = 1, \dots, n. \tag{4.5}$$

From Green's formula

$$b(u, v) - \int_{\mathcal{K}} p\, \nabla \cdot v\, dx = \int_{\mathcal{K}} (-\Delta u - \nabla\nabla \cdot u + \nabla p) \cdot v\, dx + \sum_{j=1}^{n} \int_{\Gamma_j} (-p n^{(j)} + 2\varepsilon(u) n^{(j)}) \cdot v\, dx$$

it follows that every solution $(u, p) \in W^{1,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{0,s}_{\beta,\delta}(\mathcal{K})^3$ of problem (4.4), (4.5) satisfies (3.1), (3.2) if $g \in W^{1,s}_{\beta+1,\delta+1}(\mathcal{K})$, $h_j \in W^{2-1/s}_{\beta+1,\delta+1}(\Gamma_j)$, and F has the form

$$F(v) = \int_{\mathcal{K}} (f + \nabla g) \cdot v\, dx + \sum_{j=1}^{n} \int_{\Gamma_j} \phi_j \cdot v\, dx \quad \text{for all } v \in V^{1,s'}_{-\beta,-\delta}(\mathcal{K})^3,\ S_j v = 0 \text{ on } \Gamma_j,\ j = 1, \dots, n,$$

where $f \in W^{0,s}_{\beta+1,\delta+1}(\mathcal{K})^3$, $\phi_j \in W^{1-1/s}_{\beta+1,\delta+1}(\Gamma_j)$.

Let κ be a fixed real number such that the closed strip between the lines $\operatorname{Re}\lambda = -\kappa - 1/2$ and $\operatorname{Re}\lambda = 1 - \beta - 3/s$ is free of eigenvalues of the pencil $\mathfrak{A}$. Then, according to [25, Th.4.5], there exists a unique solution $G(x, \xi) = \big(G_{i,j}(x, \xi)\big)_{i,j=1}^{4}$ of the problem (3.12)–(3.14) such that the function $x \to \zeta(|x-\xi|/r(\xi))\, G_{i,j}(x, \xi)$ belongs to $W^{1,2}_{\kappa,0}(\mathcal{K})$ for $i = 1, 2, 3$ and to $W^{0,2}_{\kappa,0}(\mathcal{K})$ for $i = 4$, where ζ is an arbitrary smooth function on $(0, \infty)$ equal to one in $(1, \infty)$ and to zero in $(0, \frac{1}{2})$. We denote by $\Lambda_- < \operatorname{Re}\lambda < \Lambda_+$ the widest strip in the complex plane containing the line $\operatorname{Re}\lambda = -\kappa - 1/2$ which is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$.

Suppose that $h_j = 0$ for $j = 1, \dots, n$ and $F \in V^{-1,s}_{\beta,\delta}(\mathcal{K})^3$ is given in the form (4.3). Then, analogously to (3.15), (3.16), the following representation for the solution of problem (4.4), (4.5) holds.

$$u_i(x) = \sum_{j=1}^{3} \int_{\mathcal{K}} \Big(f_j^{(0)}(\xi)\, G_{i,j}(x, \xi) + \sum_{k=1}^{3} f_j^{(k)}(\xi) \partial_{\xi_k} G_{i,j}(x, \xi) \Big)\, d\xi + \int_{\mathcal{K}} g(\xi)\, G_{i,4}(x, \xi)\, d\xi, \tag{4.6}$$

$$p(x) = -g(x) + \sum_{j=1}^{3} \int_{\mathcal{K}} \Big(f_j^{(0)}(\xi)\, G_{4,j}(x, \xi) + \sum_{k=1}^{3} f_j^{(k)}(\xi) \partial_{\xi_k} G_{4,j}(x, \xi) \Big)\, d\xi + \int_{\mathcal{K}} g(\xi)\, G_{4,4}(x, \xi)\, d\xi. \tag{4.7}$$

4.2 Auxiliary inequalities

Our goal is to prove that (4.6), (4.7) define a continuous mapping

$$V^{0,s}_{\beta+1,\delta+1}(\mathcal{K})^3 \times V^{0,s}_{\beta,\delta}(\mathcal{K})^9 \times V^{0,s}_{\beta,\delta}(\mathcal{K}) \ni \big(f^{(0)}, f^{(1)}, f^{(2)}, f^{(3)}, g\big) \to (u, p) \in W^{1,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{0,s}_{\beta,\delta}(\mathcal{K})$$

if δ satisfies condition (4.2) and β satisfies the inequalities

$$\Lambda_- < 1 - \beta - 3/s < \Lambda_+ . \tag{4.8}$$

The following lemmas allow us to estimate the integrals containing $f^{(k)}$, $k = 1, 2, 3$, and g in (4.6) and (4.7).

Lemma 4.1 *Let ζ_k be the same function as in the proof of Lemma* 2.1 *and let*

$$v(x) = \zeta_m(x) \int_{\mathcal{K}} \zeta_l(\xi)\, f(\xi)\, K(x, \xi)\, d\xi.$$

Suppose that $m \geq l+3$, $f \in W^{0,s}_{\beta,\delta}(\mathcal{K})$, *and*

$$\left|\partial_x^\alpha K(x,\xi)\right| \leq c\,\frac{|x|^{\Lambda_- -\sigma-|\alpha|+\varepsilon}}{|\xi|^{\Lambda_-+2+\varepsilon}} \prod_{k=1}^n \Big(\frac{r_k(x)}{|x|}\Big)^{\min(0,\mu_k-\sigma-|\alpha|-\varepsilon)} \prod_{k=1}^n \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{\min(0,\mu_k-1-\varepsilon)} \tag{4.9}$$

for $|x| > 2|\xi|$, $|\alpha| \leq 1-\sigma$, *where* $\sigma \in \{0,1\}$ *and* ε *is a sufficiently small positive real number. If* δ *and* β *satisfy conditions* (4.2) *and* (4.8), *then*

$$\|v\|_{W^{1-\sigma,s}_{\beta,\delta}(\mathcal{K})} \leq c\,2^{-|m-l|\varsigma}\|\zeta_l f\|_{W^{0,s}_{\beta,\delta}(\mathcal{K})}$$

with positive constants c *and* ς *independent of* f. *The same estimates holds if* $l \geq m+3$ *and*

$$\left|\partial_x^\alpha K(x,\xi)\right| \leq c\,\frac{|x|^{\Lambda_+ -\sigma-|\alpha|-\varepsilon}}{|\xi|^{\Lambda_++2-\varepsilon}} \prod_{k=1}^n \Big(\frac{r_k(x)}{|x|}\Big)^{\min(0,\mu_k-\sigma-|\alpha|-\varepsilon)} \prod_{k=1}^n \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{\min(0,\mu_k-1-\varepsilon)}$$

for $|\xi| > 2|x|$, $|\alpha| \leq 1-\sigma$.

The proof of Lemma 4.1 proceeds analogously to Lemma 3.6. Note that the elements $G_{i,4}(x,\xi)$ of Green's matrix and the derivatives $\partial_{\xi_k} G_{i,j}(x,\xi)$, $j=1,2,3$, satisfy the assumptions on the kernel $K(x,\xi)$ (with $\sigma = \delta_{i,4}$) of Lemma 4.1 and of the following lemma.

Lemma 4.2 *Let* ζ_k *be the same function as in the proof of Lemma* 2.1 *and let*

$$v(x) = \zeta_m(x) \int_{\mathcal{K}} \zeta_l(\xi)\, f(\xi)\, \chi^-(x,\xi)\, K(x,\xi)\, d\xi,$$

where $|l-m| \leq 2$, χ^- *is defined by* (3.21), *and* $f \in W^{0,s}_{\beta-\tau,\delta-\tau}(\mathcal{K})$. *Suppose that*

$$\left|\partial_x^\alpha K(x,\xi)\right| \leq c\,|x-\xi|^{-2-\sigma-|\alpha|}\Big(\frac{r(x)}{|x-\xi|}\Big)^{\min(0,\mu_x-\sigma-|\alpha|-\varepsilon)}\Big(\frac{r(\xi)}{|x-\xi|}\Big)^{\min(0,\mu_\xi-1-\varepsilon)} \tag{4.10}$$

for $|x|/32 < |\xi| < 32|x|$, $|x-\xi| > r(x)/4$, $|\alpha| \leq 1-\sigma$, *where* $\sigma \in \{0,1\}$, ε *is a sufficiently small positive real number. If* δ *satisfies condition* (4.2), *then*

$$\|v\|_{W^{1-\sigma,s}_{\beta,\delta}(\mathcal{K})} \leq c\,\|\zeta_l f\|_{W^{0,s}_{\beta,\delta}(\mathcal{K})}\,.$$

Proof: Let $|\alpha| \leq 1-\sigma$. Obviously, $\left|\partial_x^\alpha v(x)\right| \leq c \sum_{j+|\gamma|=|\alpha|} |x|^{-j} A_\gamma(x)$, where A_γ satisfies the inequality (3.24) with $\tau = 1$ provided

$$\max(1-\sigma-|\gamma|, 1-\mu_k) < s' s_k < \min(2, 1+\mu_k), \quad s' = s/(s-1). \tag{4.11}$$

If additionally

$$\delta_k + t_k + \frac{1}{s}\max(0,1-\mu_k) < s_k < \delta_k + t_k + \frac{1}{s}\min(1+\sigma+|\gamma|, 1+\mu_k), \tag{4.12}$$

where

$$t_k = \begin{cases} 1-\max(0,1-\mu_k) & \text{for } \sigma = |\gamma| = 0, \\ 0 & \text{for } \sigma + |\gamma| = 1, \end{cases}$$

then, using Corollory 3.2, we obtain

$$
\begin{aligned}
&\int_{\mathcal{K}} |x|^{s(\beta-1+\sigma+|\gamma|)} \prod_{k=1}^{n} \Big(\frac{r_k(x)}{|x|}\Big)^{s\delta_k} \big|A_\gamma(x)\big|^s \, dx \\
&\le c \int\limits_{\substack{\mathcal{K} \\ 2^{m-1}<|x|<2^{m+1}}} |x|^{s\beta-1+\sigma+|\gamma|} \prod_{k=1}^{n} \Big(\frac{r_k(x)}{|x|}\Big)^{s(\delta_k-s_k)+(s-1)(1-\sigma-|\gamma|)} \int\limits_{\substack{\mathcal{K} \\ |x-\xi|>r(x)/4}} |x-\xi|^{-2-\sigma-|\gamma|} \\
&\qquad \times \Big(\frac{r(x)}{|x-\xi|}\Big)^{\min(0,\mu_x-\sigma-|\gamma|-\varepsilon)} \Big(\frac{r(\xi)}{|x-\xi|}\Big)^{\min(0,\mu_\xi-1-\varepsilon)} \prod_{k=1}^{n} \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{ss_k} |\zeta_l f|^s \, d\xi \, dx \\
&\le c \int_{\mathcal{K}} |\xi|^{s\beta-1+\sigma+|\gamma|} \prod_{k=1}^{n} \Big(\frac{r_k(\xi)}{|\xi|}\Big)^{ss_k} \big|\zeta_l(\xi) f(\xi)\big|^s \int\limits_{\substack{2^{m-1}<|x|<2^{m+1} \\ |x-\xi|>r(\xi)/5}} |x-\xi|^{-2-\sigma-|\gamma|} \\
&\qquad \times \Big(\frac{r(x)}{|x-\xi|}\Big)^{\min(0,\mu_x-\sigma-|\gamma|-\varepsilon)} \Big(\frac{r(\xi)}{|x-\xi|}\Big)^{\min(0,\mu_\xi-1-\varepsilon)} \prod_{k=1}^{n} \Big(\frac{r_k(x)}{|x|}\Big)^{s(\delta_k-s_k+t_k)-1+\sigma+|\gamma|} dx \, d\xi \\
&\le c \int_{\mathcal{K}} |\xi|^{s\beta} \prod_{k=1}^{n} \Big(\frac{r_k(x)}{|x|}\Big)^{s(\delta_k+t_k)} \big|\zeta_l(\xi) f(\xi)\big|^s \, d\xi.
\end{aligned}
$$

It can be easily verified that for $\sigma + |\gamma| \le 1$ there exist real numbers s_k satisfying (4.11), (4.12). This proves the lemma. ∎

The assumptions on $K(x,\xi)$ in the following lemma are satisfied by $\partial_{\xi_k} G_{4,j}$, $j = 1,2,3$, and $G_{4,4}$.

Lemma 4.3 *Let*

$$
v(x) = \zeta_m(x) \int_{\mathcal{K}} \zeta_l(\xi)\, f(\xi)\, \chi^+(x,\xi)\, K(x,\xi)\, d\xi,
$$

where $|l-m| \le 2$, $f \in V^{0,s}_{\beta,\delta}(\mathcal{K})$, and χ^+ is defined by (3.21). *Suppose that $K(x,\xi)$ has the representation $K(x,\xi) = -\nabla_x \cdot P(x,\xi) + Q(x,\xi)$, where*

$$
|P(x,\xi)| \le c\,|x-\xi|^{-2}, \quad |Q(x,\xi)| \le c\, r(\xi)^{-3} \tag{4.13}
$$

for $|x|/32 < |\xi| < 32|x|$, $|x-\xi| < r(x)/2$, $|\alpha| \le 1-\sigma$, and $P(x,\xi)\cdot n^{(j)} = 0$ for $x \in \Gamma_j$, $j = 1,\dots,n$ (here $n^{(j)}$ denotes the exterior normal to Γ_j), then

$$
\|v\|_{V^{-1,s}_{\beta-1,\delta-1}(\mathcal{K})} \le c\, \|\zeta_l f\|_{V^{0,s}_{\beta,\delta}(\mathcal{K})}. \tag{4.14}
$$

with a constant c independent of l, m and f.

Proof: Let $w \in V^{1,s'}_{1-\beta,1-\delta}(\mathcal{K})$, $s' = s/(s-1)$. Then

$$
\int_{\mathcal{K}} v(x)\, w(x)\, dx = \int_{\mathcal{K}} A(x)\, w(x) + B(x)\cdot \nabla w(x)\, dx, \tag{4.15}
$$

where

$$
\begin{aligned}
A(x) &= \int_{\mathcal{K}} \Big(P(x,\xi)\, \nabla_x \big(\zeta_m(x)\, \chi^+(x,\xi)\big) + \zeta_m(x)\, \chi^+(x,\xi)\, Q(x,\xi) \Big)\, \zeta_l(\xi)\, f(\xi)\, d\xi, \\
B(x) &= \int_{\mathcal{K}} \zeta_m(x)\, \chi^+(x,\xi)\, P(x,\xi)\, \zeta_l(\xi)\, f(\xi)\, d\xi.
\end{aligned}
$$

We have to show that $A \in V^{0,s}_{\beta,\delta}(\mathcal{K})$, $B \in V^{0,s}_{\beta-1,\delta-1}(\mathcal{K})^3$, and

$$
\|A\|_{V^{0,s}_{\beta,\delta}(\mathcal{K})} + \|B\|_{V^{0,s}_{\beta-1,\delta-1}(\mathcal{K})^3} \le c\, \|\zeta_l f\|_{V^{0,s}_{\beta,\delta}(\mathcal{K})}. \tag{4.16}
$$

We introduce the set $\mathcal{K}_x = \{\xi \in \mathcal{K} : |x|/32 < |\xi| < 32|x|,\ |x-\xi| < r(x)/2\}$. Note that there are the inequalities (3.28) for $x \in \operatorname{supp}\zeta_m$, $\xi \in \operatorname{supp}\zeta_l$, and $|x-\xi| < r(x)/2$. Since

$$\big|P(x,\xi)\,\nabla_x\big(\zeta_m(x)\,\chi^+(x,\xi)\big|+\big|\zeta_m(x)\,\chi^+(x,\xi)\,Q(x,\xi)\big| \le c\,r(x)^{-1}\,|x-\xi|^{-2}$$

for $\xi \in \operatorname{supp}\zeta_l$, we obtain

$$\begin{aligned}
|A(x)|^s &\le c\,r(x)^{-s}\Big(\int_{\mathcal{K}_x}|x-\xi|^{-2}\,\big|\zeta_l(\xi)\,f(\xi)\big|\,d\xi\Big)^s\\
&\le c\,r(x)^{-s}\int_{\mathcal{K}_x}|x-\xi|^{-2}\,\big|\zeta_l(\xi)\,f(\xi)\big|^s\,d\xi\,\Big(\int\limits_{|x-\xi|<r(x)/2}|x-\xi|^{-2}\,d\xi\Big)^{s-1}\\
&\le c\,r(x)^{-1}\int_{\mathcal{K}_x}|x-\xi|^{-2}\,\big|\zeta_l(\xi)\,f(\xi)\big|^s\,d\xi.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\|A\|^s_{V^{0,s}_{\beta,\delta}(\mathcal{K})} \le c\int_{\mathcal{K}}|x|^{s\beta}\prod_{k=1}^n\Big(\frac{r_k(x)}{|x|}\Big)^{s\delta_k}r(x)^{-1}\Big(\int_{\mathcal{K}_x}|x-\xi|^{-2}\,\big|\zeta_l(\xi)\,f(\xi)\big|^s\,d\xi\Big)\,dx\\
&\le c\int_{\mathcal{K}}|\xi|^{s\beta}\prod_{k=1}^n\Big(\frac{r_k(\xi)}{|\xi|}\Big)^{s\delta_k}r(\xi)^{-1}\,\big|\zeta_l(\xi)\,f(\xi)\big|^s\Big(\int\limits_{|x-\xi|<r(\xi)}|x-\xi|^{-2}\,dx\Big)\,d\xi \le c\,\|\zeta_l f\|^s_{V^{0,s}_{\beta,\delta}(\mathcal{K})}.
\end{aligned}$$

Analogously, we obtain

$$|B(x)|^s \le c\,r(x)^{s-1}\int_{\mathcal{K}_x}|x-\xi|^{-2}\,\big|\zeta_l(\xi)\,f(\xi)\big|^s\,d\xi$$

what implies the desired estimate for the norm of B in $V^{0,s}_{\beta-1,\delta-1}(\mathcal{K})^3$. Estimate (4.14) is an immediate consequence of (4.15) and (4.16). ∎

4.3 Existence und uniqueness of weak solutions

Let $F \in V^{-1,s}_{\beta,\delta}(\mathcal{K})^3$, $g \in W^{0,s}_{\beta,\delta}(\mathcal{K})$ and $h_k \in W^{1-1/s,s}_{\beta,\delta}(\Gamma_k)^{3-d_k}$, $k = 1,\dots,n$. We suppose that the following conditions are satisfied.

(i) The line $\operatorname{Re}\lambda = 1-\beta-3/s$ does not contain eigenvalues of the pencil $\mathfrak{A}(\lambda)$.

(ii) $\max(0,1-\mu_k) < \delta_k + 2/s < 1$ for $k = 1,\dots,n$.

(iii) The vector functions $h_j \in W^{1-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j}$ are such that there exists a vector function $w \in W^{1,s}_{\beta,\delta}(\mathcal{K})^3$ satisfying the condition $S_j u = h_j$ on Γ_j, $j = 1,\dots,n$.

The last is a condition on the traces of the boundary data h_j on the edges of $\mathcal{K}$. These traces exist since $\delta_k + 2/s < 1$. Let Γ_{j_+}, Γ_{j_-} be the sides of $\mathcal{K}$ adjacent to the edge M_j. Then condition (iii) is equivalent to (3.7).

Theorem 4.1 *Let $F \in V^{-1,s}_{\beta,\delta}(\mathcal{K})^3$, $g \in W^{0,s}_{\beta,\delta}(\mathcal{K})$, and $h_j \in W^{1-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j}$. Suppose that conditions* (i)–(iii) *are satisfied. Then there exists a unique solution $(u,p) \in W^{1,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{0,s}_{\beta,\delta}\mathcal{K})$ of problem* (4.4), (4.5).

Proof: Without loss of generality, we may assume that $h_j = 0$. Suppose that the functional F is given in the form (4.3), where $f^{(0)} = 0$ and $f^{(k)} \in V^{0,s}_{\beta,\delta}(\mathcal{K})^3$ for $k = 1,2,3$. We consider the operator

$$\mathcal{X} \stackrel{def}{=} \Big(\prod_{k=1}^3 V^{0,s}_{\beta,\delta}(\mathcal{K})^3\Big)\times V^{0,s}_{\beta,\delta}(\mathcal{K}) \ni (f^{(1)},f^{(2)},f^{(3)},g) \to \mathcal{O}(f^{(1)},f^{(2)},f^{(3)},g) = (u,p),$$

where u and p are defined by (4.6), (4.7) (with $f^{(0)} = 0$) and $G_{i,j}$ are the elements of Green's matrix introduced in Section 3.4. Then by Lemma 4.1,

$$\|\zeta_m \mathcal{O} \zeta_l(f^{(1)}, f^{(2)}, f^{(3)}, g)\|_{\mathcal{Y}} \le c\, 2^{-|l-m|\varsigma}\, \|\zeta_l(f^{(1)}, f^{(2)}, f^{(3)}, g)\|_{\mathcal{X}} \tag{4.17}$$

for $|l-m| \ge 3$, where

$$\mathcal{Y} \stackrel{def}{=} W^{1,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{0,s}_{\beta,\delta}(\mathcal{K})$$

and c, ς are positive constants independent of $f^{(k)}, g, l, m$. In order to prove the same inequality for $|l-m| \le 2$, we introduce the functions

$$u_i^{\pm}(x) = \sum_{j=1}^{3}\sum_{k=1}^{3} \int_{\mathcal{K}} \zeta_l(\xi)\, f_j^{(k)}(\xi)\, \chi^{\pm}(x,\xi)\, \partial_{\xi_k} G_{i,j}(x,\xi)\, d\xi + \int_{\mathcal{K}} \zeta_l(\xi) g(\xi)\, \chi^{\pm}(x,\xi)\, G_{i,4}(x,\xi)\, d\xi,$$

$$p^{\pm}(x) = -\zeta_l(x)\, g(x) + \sum_{j=1}^{3}\sum_{k=1}^{3} \int_{\mathcal{K}} \zeta_l(\xi)\, f_j^{(k)}(\xi)\, \chi^{\pm}(x,\xi)\, \partial_{\xi_k} G_{4,j}(x,\xi)\, d\xi$$
$$+ \int_{\mathcal{K}} \zeta_l(\xi)\, g(\xi)\, \chi^{\pm}(x,\xi) G_{4,4}(x,\xi)\, d\xi,$$

where χ^+ and χ^- are defined by (3.21). Then

$$b(u^+ + u^-, v) - \int_{\mathcal{K}} (p^+ + p^-) \nabla \cdot v\, dx = F(\zeta_l v) \quad \text{for all } v \in V^{1,s'}_{-\beta,-\delta}(\mathcal{K})^3,\ S_k v = 0 \ \text{ on } \Gamma_k,$$

$-\nabla \cdot (u^+ + u^-) = \zeta_l g$ in $\mathcal{K}$, and $S_j(u^+ + u^-) = 0$ on Γ_j. Furthermore, by Lemma 4.2, we have $\zeta_m(u^-, p^-) \in \mathcal{Y}$ and

$$\|\zeta_m(u^-, p^-)\|_{\mathcal{Y}} \le c\, \|\zeta_l(f^{(1)}, f^{(2)}, f^{(3)}, g)\|_{\mathcal{X}} \tag{4.18}$$

if $|l-m| \le 3$, where c is independent of f, g, l, m. From Lemmas 3.9 and 4.3 it follows that

$$\|\zeta_m u^+\|_{V^{0,s}_{\beta-1,\delta-1}(\mathcal{K})^3} + \|\zeta_m p^+\|_{V^{-1,s}_{\beta-1,\delta-1}(\mathcal{K})} \le c\, \|\zeta_l(f^{(1)}, f^{(2)}, f^{(3)}, g)\|_{\mathcal{X}} \tag{4.19}$$

The vector function (u^+, p^+) is a solution of the problem

$$b(u^+, v) - \int_{\mathcal{K}} p^+ \nabla \cdot v\, dx = \tilde{F}(v) \quad \text{for all } v \in V^{1,s'}_{-\beta,-\delta}(\mathcal{K})^3,\ S_k v = 0 \ \text{ on } \Gamma_k,\ k = 1, \dots, n,$$
$$-\nabla \cdot u^+ = \zeta_l g + \nabla \cdot u^- \ \text{ in } \mathcal{K}, \quad S_j u^+ = -S_j u^- \ \text{ on } \Gamma_j,\ j = 1, \dots, n,$$

where

$$\tilde{F}(v) = F(\zeta_l v) - b(u^-, v) + \int_{\mathcal{K}} p^- \nabla \cdot v\, dx$$

Obviously, $\zeta_m \tilde{F} \in V^{-1,s}_{\beta,\delta}(\mathcal{K}$. Let $\eta_m = \zeta_{m-1} + \zeta_m + \zeta_{m+1}$. Then, by an estimate analogous to that in Corollary 3.1,

$$\|\zeta_m u^+\|_{V^{1,s}_{\beta,\delta}(\mathcal{K})^3} + \|\zeta_m p^+\|_{V^{0,s}_{\beta,\delta}(\mathcal{K})}$$
$$\le c \left(\|\eta_m u^+\|_{V^{0,s}_{\beta-1,\delta-1}(\mathcal{K})^3} + \|\eta_m p^+\|_{V^{-1,s}_{\beta-1,\delta-1}(\mathcal{K})} + \|\eta_m \zeta_l(f^{(1)}, f^{(2)}, f^{(3)}, g)\|_{\mathcal{X}} + \|\eta_m(u^-, p^-)\|_{\mathcal{Y}} \right)$$

for $|l-m| \le 2$. Due to (4.18) and (4.19), the right hand side of the last inequality can be estimated by the norm of $\zeta_l(f^{(1)}, f^{(2)}, f^{(3)}, g)$ in $\mathcal{X}$. Consequently,

$$\|\zeta_m(u^+ + u^-, p^+ + p^-)\|_{\mathcal{Y}} \le c\, \|\zeta_l(f^{(1)}, f^{(2)}, f^{(3)}, g)\|_{\mathcal{X}} \quad \text{for } |l-m| \le 2.$$

Thus, estimate (4.17) is valid for arbitrary l and m. Consequently, by Lemma 3.10, the operator $\mathcal{O}$ continuously maps $\mathcal{X}$ into $\mathcal{Y}$. This proves the existence of a solution of problem (4.4), (4.5) in the case when F has the form (4.3) with $f^{(0)} = 0$, $f^{(k)} \in V^{0,s}_{\beta,\delta}(\mathcal{K})^3$ for $k = 1, 2, 3$.

By what has been shown in the previous section, the mapping $f^{(0)} \to (v,q)$ defined by

$$v_i(x) = \sum_{j=1}^{3} \int_{\mathcal{K}} f_j^{(0)}(\xi)\, G_{i,j}(x,\xi)\, d\xi,\ i=1,2,3, \quad p(x) = -g(x) + \sum_{j=1}^{3} \int_{\mathcal{K}} f_j^{(0)}(\xi)\, G_{4,j}(x,\xi)\, d\xi \tag{4.20}$$

is continuous from $V^{0,s}_{\beta+1,\delta+1}(\mathcal{K})^3$ into the subspace $W^{2,s}_{\beta+1,\delta+1}(\mathcal{K})^3 \times W^{1,s}_{\beta+1,\delta+1}(\mathcal{K})$ of $\mathcal{Y}$. Thus, problem (4.4), (4.5) is solvable for arbitrary $F \in V^{-1,s}_{\beta,\delta}(\mathcal{K})$.

It remains to prove the uniqueness of the solution. Suppose $(u,p) \in \mathcal{Y}$ is a solution of the homogeneous problem (4.4), (4.5). Then (u,p) is also a solution of the homogeneous problem (3.1), (3.2), and from Lemma 3.4 it follows that $u \in W^{2,s}_{\beta+1,\delta+1}(\mathcal{K})^3$ and $p \in W^{1,s}_{\beta+1,\delta+1}(\mathcal{K})$. Consequently, by Theorem 3.2, we have $u = 0$ and $p = 0$. The proof is complete. ∎

Remark 4.3 In [25] the existence and uniqueness of weak solutions in $W^{1,2}_{\beta,0}(\mathcal{K})^3 \times W^{0,2}_{\beta,0}(\mathcal{K})$ was proved for arbitrary $F \in W^{-1,2}_{\beta,0}(\mathcal{K})^3 = (W^{1,2}_{-\beta,0}(\mathcal{K})^*)^3$, $g \in W^{0,2}_{\beta,0}(\mathcal{K})$ and $h_j \in W^{1/2,2}_{\beta,\delta}(\Gamma_j)$, $j = 1,\dots,n$. Note that every $F \in W^{-1,2}_{\beta,0}(\mathcal{K})$ has the form

$$F(v) = \int_{\mathcal{K}} f^{(0)} \cdot v\, dx + \sum_{k=1}^{3} \int_{\mathcal{K}} f^{(k)}\, \partial_{x_k} v \cdot dx \ \text{ for all } v \in W^{1,2}_{-\beta,0}(\mathcal{K})^3,$$

where $f \in W^{0,2}_{\beta+1,0}(\mathcal{K})$, $f^{(k)} \in W^{0,2}_{\beta,0}(\mathcal{K})$, $k = 1,2,3$. It can be easily shown that the assertions of Lemmas 4.1–4.3 are also valid if $s = 2$, $\delta = 0$, and β satisfies (4.8). Consequently, the weak solution $(u,p) \in W^{1,2}_{\beta,0}(\mathcal{K})^3 \times W^{0,2}_{\beta,0}(\mathcal{K})$ has also the form (4.6), (4.7) if $h_j = 0$.

4.4 Regularity assertions for weak solutions

Lemma 4.4 *Let $(u,p) \in W^{1,\sigma}_{\beta',\delta'}(\mathcal{K})^3 \times W^{0,\sigma}_{\beta',\delta'}(\mathcal{K})$ be a solution of problem*

$$b(u,v) - \int_{\mathcal{K}} p\, \nabla \cdot v = F(v) \ \text{ for all } v \in V^{1,\sigma'}_{-\beta',-\delta'}(\mathcal{K})^3,\ S_j v = 0 \text{ on } \Gamma_j,\ j = 1,\dots,n, \tag{4.21}$$

$$-\nabla \cdot u = g \ \text{ in } \mathcal{K}, \quad S_j u = h_j \ \text{ on } \Gamma_j,\ j = 1,\dots,n, \tag{4.22}$$

where $\sigma' = \sigma/(\sigma-1)$,

$$F \in V^{-1,s}_{\beta,\delta}(\mathcal{K})^3 \cap V^{-1,\sigma}_{\beta',\delta'}(\mathcal{K}), \quad g \in W^{0,s}_{\beta,\delta}(\mathcal{K}) \cap W^{0,\sigma}_{\beta',\delta'}(\mathcal{K}), \quad h_j \in W^{1-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j} \cap W^{1-1/\sigma,\sigma}_{\beta',\delta'}(\Gamma_j)^{3-d_j}.$$

If the closed strip between the lines $\operatorname{Re}\lambda = 1-\beta-3/s$ and $\operatorname{Re}\lambda = 1-\beta'-3/\sigma$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$ and δ, δ' satisfy the inequalities

$$\max(0, 1-\mu_k) < \delta_k + 2/s < 1, \quad \max(0, 1-\mu_k) < \delta'_k + 2/\sigma < 1 \ \text{ for } k = 1,\dots,n, \tag{4.23}$$

then $(u,p) \in W^{1,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{0,s}_{\beta,\delta}(\mathcal{K})$.

Proof: Under the assumptions on the lemma, the boundary data h_j satisfy the compatibility condition (3.7). Therefore, there exists a vector function $v \in W^{1,s}_{\beta,\delta}(\mathcal{K})^3 \cap W^{1,\sigma}_{\beta',\delta'}(\mathcal{K})^3$ satisfying $S_j v = h_j$ on Γ_j, $j = 1,\dots,n$. For this reason, we may restrict ourselves to the case $h_j = 0$. According to Theorem 4.1, there are unique solutions of problem (4.4), (4.5) in $W^{1,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{0,s}_{\beta,\delta}(\mathcal{K})$ and $W^{1,s'}_{\beta',\delta'}(\mathcal{K})^3 \times W^{0,s'}_{\beta',\delta'}(\mathcal{K})$. Both solutions are given by (4.6), (4.7) with the same Green matrix $G(x,\xi)$. This proves the lemma. ∎

The same result is true for weak solutions in $W^{1,2}_{\beta',0}(\mathcal{K})^3 \times W^{0,2}_{\beta',0}(\mathcal{K})$ (cf. Remark 4.3). Furthermore, the following generalization of Lemma 4.4 holds.

Theorem 4.2 *Let u, p, F, g, and h_j be as in Lemma 4.4. We assume that there are no eigenvalues of the pencil $\mathfrak{A}(\lambda)$ on the lines $\operatorname{Re}\lambda = 1-\beta-3/s$ and $\operatorname{Re}\lambda = 1-\beta'-3/\sigma$ and that δ, δ' satisfy the inequalities (4.23). Then (u,p) admits the decomposition*

$$(u,p) = \sum_{\nu=1}^{N} \sum_{j=1}^{I_\nu} \sum_{s=0}^{\kappa_{\nu,j}-1} c_{\nu,j,s} \sum_{\sigma=0}^{s} \frac{1}{\sigma!}\, (\log\rho)^\sigma \left(\rho^{\lambda_\nu} u^{(\nu,j,s-\sigma)}(\omega), \rho^{\lambda_\nu - 1} p^{(\nu,j,s-\sigma)}(\omega)\right) + (w,q) \tag{4.24}$$

where $(w,q) \in W^{1,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{0,s}_{\beta,\delta}(\mathcal{K})$ *is a weak solution of problem* (3.1)–(3.2), λ_ν *are the eigenvalues of the pencil* $\mathfrak{A}$ *between the lines* $\operatorname{Re}\lambda = 1-\beta-3/s$ *and* $\operatorname{Re}\lambda = 1-\beta'-3/\sigma$, *and* $\big(u^{(\nu,j,s)}, p^{(\nu,j,s)}\big)$ *are eigenvectors and generalized eigenvectors corresponding to the eigenvalue* λ_ν.

Proof: As in the proof of Lemma 4.4, we may restrict ourselves to the case $h_j = 0$. Let $\{F_i\} \subset C_0^\infty(\bar{\mathcal{K}}\backslash\{0\})^3$, $\{g_i\} \subset C_0^\infty(\bar{\mathcal{K}}\backslash\{0\})$ be sequences converging to F in $V^{-1,s}_{\beta,\delta}(\mathcal{K})^3 \cap V^{-1,\sigma}_{\beta',\delta'}(\mathcal{K})$ and g in $W^{0,s}_{\beta,\delta}(\mathcal{K}) \cap W^{0,\sigma}_{\beta',\delta'}(\mathcal{K})$, respectively. According to [25, Th.3.2], there exist unique solutions

$$(u^{(i)}, p^{(i)}) \in W^{1,2}_{\beta'-3/2+3/\sigma,0}(\mathcal{K})^3 \times W^{0,2}_{\beta'-3/2+3/\sigma,0}(\mathcal{K}) \text{ and } (w^{(i)}, q^{(i)}) \in W^{1,2}_{\beta-3/2+3/s,0}(\mathcal{K})^3 \times W^{0,2}_{\beta-3/2+3/s,0}(\mathcal{K})$$

of the problem

$$\begin{aligned} &b(u,v) - \int_{\mathcal{K}} p\,\nabla\cdot v = F_i(v) \ \text{ for all } v \in C_0^\infty(\bar{\mathcal{K}}\backslash\{0\})^3,\ S_j v = 0 \text{ on } \Gamma_j,\ j = 1,\dots,n,\\ &-\nabla\cdot u = g_i \ \text{ in } \mathcal{K}, \quad S_j u = 0 \ \text{ on } \Gamma_j,\ j = 1,\dots,n. \end{aligned}$$

By what has been shown above, the vector functions $(u^{(i)}, p^{(i)})$ and $(w^{(i)}, q^{(i)})$ belong also to $W^{1,\sigma}_{\beta',\delta'}(\mathcal{K})^3 \times W^{0,\sigma}_{\beta',\delta'}(\mathcal{K})$ and $W^{1,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{0,s}_{\beta,\delta}(\mathcal{K})$, respectively. Furthermore, from Theorem 4.1 it follows that the sequence $\{(u^{(i)}, p^{(i)})\}$ converges to (u,p), while $\{(w^{(i)}, q^{(i)})\}$ converges to a vector function $(w,q) \in W^{1,s}_{\beta,\delta}(\mathcal{K})^3 \times W^{0,s}_{\beta,\delta}(\mathcal{K})$. Let $\mathcal{X}$ denote the linear span of the vector functions

$$\sum_{\sigma=0}^{s} \frac{1}{\sigma!}\,(\log\rho)^\sigma \left(\rho^{\lambda_\nu} u^{(\nu,j,s-\sigma)}(\omega), \rho^{\lambda_\nu - 1} p^{(\nu,j,s-\sigma)}(\omega)\right)$$

appearing in (4.24). By [25, Th.4.4], we have $(u^{(i)} - w^{(i)}, p^{(i)} - q^{(i)}) \in \mathcal{X}$ and, consequently, also $(u-w, p-q) \in \mathcal{X}$. This proves the theorem. ∎

Theorem 4.3 *Let* $(u,p) \in W^{1,\sigma}_{\beta',\delta'}(\mathcal{K})^3 \times W^{0,\sigma}_{\beta',\delta'}(\mathcal{K})$ *be a solution of problem* (4.21), (4.22). *We suppose that* $F \in V^{-1,\sigma}_{\beta',\delta'}(\mathcal{K})^3$ *and*

$$F(v) = \int_{\mathcal{K}} f\cdot v\,dx + \sum_{j=1}^{n} \int_{\Gamma_j} \phi_j \cdot v\,dx \quad \text{for all } v \in C_0^\infty(\bar{\mathcal{K}}\backslash\{0\})^3,\ S_j v = 0 \ \text{ on } \Gamma_j,\ j = 1,\dots,n, \tag{4.25}$$

where $f \in W^{0,s}_{\beta,\delta}(\mathcal{K})^3$, $\phi_j \in W^{1-1/s,s}_{\beta,\delta}(\Gamma_j)^{d_j}$ *Furthermore, we assume that* $g \in W^{0,s}_{\beta,\delta}(\mathcal{K}) \cap W^{1,\sigma}_{\beta',\delta'}(\mathcal{K})$, $h_j \in W^{1-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j} \cap W^{2-1/\sigma,\sigma}_{\beta',\delta'}(\Gamma_j)^{3-d_j}$, *there are no eigenvalues of the pencil* $\mathfrak{A}(\lambda)$ *in the closed strip between the lines* $\operatorname{Re}\lambda = 1-\beta'-3/\sigma$ *and* $\operatorname{Re}\lambda = 2-\beta-3/s$, δ *and* δ' *satisfy the inequalities*

$$\max(0, 1-\mu_k) > \delta_k' + 2/\sigma < 1, \quad \max(0, 2-\mu_k) > \delta_k + 2/s < 2, \quad k = 1,\dots,n,$$

and g, h_j, ϕ_j *satisfy the compatibility condition* (iii) *of Section* 3.6. *Then* $u \in W^{2,s}_{\beta,\delta}(\mathcal{K})^3$ *and* $p \in W^{1,s}_{\beta,\delta}(\mathcal{K})$.

Proof: Suppose first that $\max(2-\mu_k, 1) < \delta_k + 2/s < 2$ for $k = 1,\dots,n$. Then $W^{1-1/s,s}_{\beta,\delta}(\Gamma_j) = V^{1-1/s,s}_{\beta,\delta}(\Gamma_j)$ and, therefore, the functional F defined by (4.25) belongs to $V^{-1,s}_{\beta-1,\delta-1}(\mathcal{K})^3$. Using Theorem 4.2, we obtain $u \in W^{1,s}_{\beta-1,\delta-1}(\mathcal{K})^3$ and $p \in W^{0,s}_{\beta-1,\delta-1}(\mathcal{K})$, and from the second part of Lemma 3.4 we conclude that $u \in W^{2,s}_{\beta,\delta}(\mathcal{K})^3$, $p \in W^{1,s}_{\beta,\delta}(\mathcal{K})$.

If $\delta_k + 2/s \le 1$ for at least one k, then in a first step we obtain $u \in W^{2,s}_{\beta,\delta''}(\mathcal{K})^3$ and $p \in W^{1,s}_{\beta,\delta''}(\mathcal{K})$, where δ_k'' are arbitrary numbers satisfying $\max(2-\mu_k, 1) < \delta_k'' + 2/s < 2$ and $\delta_k'' \ge \delta_k$. Then Theorem 3.3 implies the assertion of the theorem. ∎

Lemma 4.5 *Let* $g \in W^{l-1,s}_{\beta,\delta}(\mathcal{K})$, $h_j \in W^{l-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, $\phi_j \in W^{l-1-1/s,s}_{\beta,\delta}(\Gamma_j)^{d_j}$, $l \ge 3$, $-2/s < \delta_k \le 1 - 2/s$. *Suppose there exist* $u \in W^{2,s}_{\beta-l+2,\varepsilon-2/s}(\mathcal{K})^3$ *and* $p \in W^{1,s}_{\beta-l+2,\varepsilon-2/s}(\mathcal{K})$, $0 < \varepsilon < 1$, *such that*

$$\nabla\cdot u + g \in V^{1,s}_{\beta-l+2,\varepsilon-2/s}(\mathcal{K}), \quad S_j u = h_j, \quad N_j(u,p) = \phi_j \ \text{ on } \Gamma_j. \tag{4.26}$$

Then there exist $v \in W^{2,s}_{\beta-l+2,\delta'}(\mathcal{K})^3$, $q \in W^{1,s}_{\beta-l+2,\delta'}(\mathcal{K})$, *where* $\delta' = \delta$ *if* $l = 3$, $\delta'_k = \varepsilon - 2/s$ *if* $l \geq 4$, *such that*

$$\nabla \cdot v + (\rho\partial_\rho + 1)g \in V^{1,s}_{\beta-l+2,\delta'}(\mathcal{K}), \quad S_j v = \rho\partial_\rho h_j, \quad N_j(v,q) = (\rho\partial_\rho + 1)\phi_j \quad \text{on } \Gamma_j. \tag{4.27}$$

Proof: We prove the lemma for the Dirichlet problem. The proof for other boundary conditions proceeds analogously. The existence of u and p satisfying (4.26) is equivalent to the trace conditions in Lemma 3.3. We assume, without loss of generality, that M_k coincides with the x_3-axis. Then the trace conditions on M_k have the form

$$h_{k_+}|_{M_k} = h_{k_-}|_{M_k}, \quad n_{k_-} \cdot (\partial_r h_{k_+})|_{M_k} + n_{k_+} \cdot (\partial_r h_{k_-})|_{M_k} = \big(g|_{M_k} + \partial_{x_3} h_{3,k_+}|_{M_k}\big) \sin\theta_k. \tag{4.28}$$

Here Γ_{k_+} and Γ_{k_-} are the sides adjacent to the edge M_k, n_{k_+} and n_{k_-} denote the exterior normals to these sides, θ_k is the inner angle at M_k, and h_{3,k_+} denotes the third component of the vector h_{k_+}.

Suppose first that $l \geq 4$ or $l = 3$ and $\delta_k < 1 - 2/s$. Then the traces of $\partial_{x_3}\partial_r h_{k_\pm}$ and $\partial_{x_3} g$ on M_k exist, and from (4.28) it follows that $\partial_{x_3} h_{k_+}|_{M_k} = \partial_{x_3} h_{k_-}|_{M_k}$ and

$$n_{k_-} \cdot (x_3\partial_{x_3} + 1)\,(\partial_r h_{k_+})|_{M_k} + n_{k_+} \cdot (x_3\partial_{x_3} + 1)\,(\partial_r h_{k_-})|_{M_k} = (x_3\partial_{x_3} + 1)\,\big(g|_{M_k} + \partial_{x_3} h_{3,k_+}|_{M_k}\big) \sin\theta_k.$$

Since $\rho\partial_\rho = r\partial_r + x_3\partial_{x_3} = x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3}$ and $x_1 = x_2 = 0$ on M_k, from the last equalities it follows that $(\rho\partial_\rho h_{k_+})|_{M_k} = (\rho\partial_\rho h_{k_-})|_{M_k}$ and

$$n_{k_-} \cdot (\partial_r \rho\partial_\rho h_{k_+})|_{M_k} + n_{k_+} \cdot (\partial_r \rho\partial_\rho h_{k_-})|_{M_k} = \big((\rho\partial_\rho + 1)g|_{M_k} + \partial_{x_3}(\rho\partial_\rho h_{3,k_+})|_{M_k}\big) \sin\theta_k.$$

This is the trace condition on M_k for the existence of v and q satisfying (4.27). In the case $l = 3$, $\delta_k = 1 - 2/s$ the validity of the trace condition can be proved analogously by means of Lemma 1.5. ∎

Theorem 4.4 *Let* $(u,p) \in W^{1,\sigma}_{\beta',\delta'}(\mathcal{K})^3 \times W^{0,\sigma}_{\beta',\delta'}(\mathcal{K})$ *be a solution of problem* (4.21), (4.22). *We suppose that* $F \in V^{-1,\sigma}_{\beta',\delta'}(\mathcal{K})^3$ *has the representation* (4.25), *where* $f \in W^{l-2,s}_{\beta,\delta}(\mathcal{K})^3$, $\phi_j \in W^{l-1-1/s,s}_{\beta,\delta}(\Gamma_j)^{d_j}$. *Furthermore, we assume that* $g \in W^{l-1,s}_{\beta,\delta}(\mathcal{K}) \cap W^{0,\sigma}_{\beta',\delta'}(\mathcal{K})$, $h_j \in W^{l-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j} \cap W^{1-1/\sigma,\sigma}_{\beta',\delta'}(\Gamma_j)^{3-d_j}$, *there are no eigenvalues of the pencil* $\mathfrak{A}(\lambda)$ *in the closed strip between the lines* $\operatorname{Re}\lambda = 1 - \beta' - 3/\sigma$ *and* $\operatorname{Re}\lambda = l - \beta - 3/s$, δ *and* δ' *satisfy the inequalities*

$$\max(0, 1 - \mu_k) < \delta'_k + 2/\sigma < 1, \quad \max(0, l - \mu_k) < \delta_k + 2/s < l, \quad k = 1, \dots, n, \tag{4.29}$$

and g, h_j, ϕ_j *satisfy condition* (iii) *of Section* 3.6 *with* $\beta'' = \beta - l + 2$ *and* $\delta''_k = \max(\delta_k - l + 2, \frac{1}{2} - \frac{2}{s})$ *instead of* β *and* δ_k, *respectively. Then* $u \in W^{l,s}_{\beta,\delta}(\mathcal{K})^3$ *and* $p \in W^{l-1,s}_{\beta,\delta}(\mathcal{K})$.

Proof: By Theorem 4.3, the assertion of the theorem is true for $l = 2$. We suppose that the assertion is true for $l = m - 1 \geq 2$ and show that it is true for $l = m$. Let first $\delta_k + 2/s > 1$ for $k = 1, \dots, n$. Then $W^{j,s}_{\beta,\delta}(\mathcal{K}) \subset W^{j-1,s}_{\beta-1,\delta-1}(\mathcal{K})$, $W^{j+1-1/s,s}_{\beta,\delta}(\Gamma_k) \subset W^{j-1/s,s}_{\beta,\delta}(\Gamma_k)$ for $j \geq 1$ and from the induction hypothesis it follows that $u \in W^{l-1,s}_{\beta-1,\delta-1}(\mathcal{K})^3$, $p \in W^{l-2,s}_{\beta-1,\delta-1}(\mathcal{K})$. Applying Lemma 3.4, we obtain $u \in W^{l,s}_{\beta,\delta}(\mathcal{K})^3$ and $p \in W^{l-1,s}_{\beta,\delta}(\mathcal{K})$.

Suppose now that $\delta_k + 2/s \leq 1$ for all k. Then, in particular, $\mu_k > l - 1$ for all k. Since $W^{j,s}_{\beta,\delta}(\mathcal{K}) \subset W^{j-1,s}_{\beta-1,\varepsilon-2/s}(\mathcal{K})$ for $j \geq 1$, $\varepsilon > 0$, it follows from the induction hypothesis that $u \in W^{l-1,s}_{\beta-1,\varepsilon-2/s}(\mathcal{K})^3$ and $p \in W^{l-2,s}_{\beta-1,\varepsilon-2/s}(\mathcal{K})$. Using again Lemma 3.4, we conclude that $u \in W^{l,s}_{\beta,\varepsilon+1-2/s}(\mathcal{K})^3$ and $p \in W^{l-1,s}_{\beta,\varepsilon+1-2/s}(\mathcal{K})$. Consequently, $\rho\partial_\rho u \in W^{l-1,s}_{\beta-1,\varepsilon+1-2/s}(\mathcal{K})^3$ and $\rho\partial_\rho p \in W^{l-2,s}_{\beta-1,\varepsilon+1-2/s}(\mathcal{K})$. Since the vector function $(\rho\partial_\rho u, \rho\partial_\rho p + p)$ is a solution of the problem

$$-\Delta(\rho\partial_\rho u) + \nabla(\rho\partial_\rho p + p) = (\rho\partial_\rho + 2)f \in W^{l-3,s}_{\beta-1,\delta}(\mathcal{K})^3, \quad -\nabla\cdot(\rho\partial_\rho u) = (\rho\partial_\rho + 1)g \in W^{l-2,s}_{\beta-1,\delta}(\mathcal{K}),$$
$$S_j \rho\partial_\rho u = \rho\partial_\rho h_j \in W^{l-1-1/s,s}_{\beta-1,\delta}(\Gamma_j), \quad N_j(\rho\partial_\rho u, \rho\partial_\rho p + p) = (\rho\partial_\rho + 1)\phi_j \in W^{l-2-1/s,s}_{\beta-1,\delta}(\Gamma_j),$$

the induction hypothesis and Lemma 4.5 imply $\rho\partial_\rho u \in W^{l-1,s}_{\beta-1,\delta}(\mathcal{K})^3$ and $\rho\partial_\rho p \in W^{l-2,s}_{\beta-1,\delta}(\mathcal{K})$. This together with the inclusion $(u,p) \in W^{l-1,s}_{\beta-1,\varepsilon-2/s}(\mathcal{K})^3 \times W^{l-2,s}_{\beta-1,\varepsilon-2/s}(\mathcal{K})$ and Lemma 3.5 yields $u \in W^{l,s}_{\beta,\delta}(\mathcal{K})^3$ and $p \in W^{l-1,s}_{\beta,\delta}(\mathcal{K})$.

Finally, we assume that $\delta_k + 2/s \le 1$ for some but not all k. Then let $\psi_1, \dots, \psi_n$ be smooth functions on $\bar\Omega$ such that $\psi_k \ge 0$, $\psi_k = 1$ near $M_j \cap S^2$, and $\sum \psi_k = 1$. We extend ψ_k to $\mathcal{K}$ by the equality $\psi_k(x) = \psi_k(x/|x|)$. Then $\partial_x^\alpha \psi_k(x) \le c\,|x|^{-|\alpha|}$. From the results proved above it follows that $\psi_k u \in W^{l,s}_{\beta,\delta}(\mathcal{K})^3$ and $\psi_k p \in W^{l-1,s}_{\beta,\delta}(\mathcal{K})^3$ for $k = 1, \dots, n$. This completes the proof. ∎

Corollary 4.1 *Let u, p, F, g, h_j be as in Theorem 4.4. We assume that there are no eigenvalues of the pencil $\mathfrak{A}(\lambda)$ on the lines $\operatorname{Re}\lambda = 1 - \beta' - 3/\sigma$ and $\operatorname{Re}\lambda = l - \beta - 3/s$, and that δ, δ' satisfy the inequalities (4.29). Then (u, p) admits the decomposition (4.24), where $w \in W^{l,s}_{\beta,\delta}(\mathcal{K})^3$, $q \in W^{l-1,s}_{\beta,\delta}(\mathcal{K})$, and λ_ν are the eigenvalues of the pencil $\mathfrak{A}(\lambda)$ between the lines $\operatorname{Re}\lambda = 1 - \beta' - 3/\sigma$ and $\operatorname{Re}\lambda = l - \beta - 3/s$, and $\big(u^{(\nu,j,s)}, p^{(\nu,j,s)}\big)$ are eigenvectors and generalized eigenvectors corresponding to the eigenvalue λ_ν.*

Proof: Under our assumptions on δ', the functional F belongs to $V^{1,s}_{\beta-l+1,\delta''}(\mathcal{K})^3$ with arbitrary $\delta'' = (\delta_1'', \dots, \delta_n'')$, $\delta_k'' \ge \delta_k - l + 1$, $\max(0, 1 - \mu_k) < \delta_k'' + 2/s < 1$. Consequently, by Theorem 4.2, (u, p) has the representation (4.24), where $(w, q) \in W^{1,s}_{\beta-l+1,\delta''}(\mathcal{K})^3 \times W^{0,s}_{\beta-l+1,\delta''}(\mathcal{K})$ is a solution of problem (4.21), (4.22). Applying Theorem 4.4, we obtain $w \in W^{l,s}_{\beta,\delta}(\mathcal{K})^3$ and $q \in W^{l-1,s}_{\beta,\delta}(\mathcal{K})$. ∎

5 The problem in a bounded domain

Let $\mathcal{G}$ be a bounded domain of polyhedral type in $\mathbb{R}^3$. This means that

(i) the boundary $\partial\mathcal{G}$ consists of smooth (of class C^∞) open two-dimensional manifolds Γ_j (the faces of $\mathcal{G}$), $j = 1, \dots, n$, smooth curves M_k (the edges), $k = 1, \dots, m$, and corners $x^{(1)}, \dots, x^{(d)}$,

(ii) for every $\xi \in M_k$ there exist a neighborhood $\mathcal{U}_\xi$ and a diffeomorphism (a C^∞ mapping) κ_ξ which maps $\mathcal{G} \cap \mathcal{U}_\xi$ onto $\mathcal{D}_\xi \cap B_1$, where $\mathcal{D}_\xi$ is a dihedron of the form (1.1) and B_1 is the unit ball,

(iii) for every corner $x^{(j)}$ there exist a neighborhood $\mathcal{U}_j$ and a diffeomorphism κ_j mapping $\mathcal{G} \cap \mathcal{U}_j$ onto $\mathcal{K}_j \cap B_1$, where $\mathcal{K}_j$ is a cone with vertex at the origin.

We consider the problem

$$-\Delta u + \nabla p = f, \quad -\nabla \cdot u = g \ \text{ in } \mathcal{G}, \tag{5.1}$$

$$S_j u = h_j, \quad N_j(u, p) = \phi_j \ \text{ on } \Gamma_j, \ j = 1, \dots, n \tag{5.2}$$

where S_j and N_j as well as the numbers $d_j \in \{0, 1, 2, 3\}$ are defined as in Section 3.

5.1 Sobolev spaces in $\mathcal{G}$

We denote by $\rho_j(x)$ the distance of x to the corner $x^{(j)}$, by $\rho(x)$ the distance to the set $X = \{x^{(1)}, \dots, x^{(d)}\}$, and by $r_k(x)$ the distance to the edge M_k. Then $W^{l,s}_{\beta,\delta}(\mathcal{G})$ is defined as the weighted Sobolev space with the norm

$$\|u\|_{W^{l,s}_{\beta,\delta}(\mathcal{G})} = \Big(\int_{\mathcal{G}} \sum_{|\alpha| \le l} |\partial_x^\alpha u|^s \prod_{j=1}^{d} \rho_j^{s(\beta_j - l + |\alpha|)} \prod_{k=1}^{m} \big(\frac{r_k}{\rho}\big)^{s\delta_k} \, dx \Big)^{1/s}.$$

Here $1 < s < \infty$, $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{R}^d$, $\delta = (\delta_1, \dots, \delta_m) \in \mathbb{R}^m$, $\delta_k > -2/s$ for $k = 1, \dots, m$, and l is a nonnegative integer. Note that the space $W^{1,2}_{0,0}(\mathcal{G})$ (where both β and δ are zero) coincides with the nonweighted Sobolev space $W^{1,2}(\mathcal{G})$.

For arbitrary $\beta \in \mathbb{R}^d$, $\delta \in \mathbb{R}^m$, $1 < s < \infty$ and integer $l \ge 0$ let $V^{l,s}_{\beta,\delta}(\mathcal{G})$ be the weighted Sobolev space with the norm

$$\|u\|_{V^{l,s}_{\beta,\delta}(\mathcal{G})} = \Big(\int_{\mathcal{G}} \sum_{|\alpha| \le l} |\partial_x^\alpha u|^s \prod_{j=1}^{d} \rho_j^{s(\beta_j - l + |\alpha|)} \prod_{k=1}^{m} \big(\frac{r_k}{\rho}\big)^{s(\delta_k - l + |\alpha|)} \, dx \Big)^{1/s}.$$

The dual space of $V^{l,s}_{\beta,\delta}(\mathcal{G})$ is denoted by $V^{-l,s'}_{-\beta,-\delta}(\mathcal{G})$, where $s' = s/(s-1)$.

Finally, we denote the trace spaces on Γ_j for $V^{l,s}_{\beta,\delta}(\mathcal{G})$ and $W^{l,s}_{\beta,\delta}(\mathcal{G})$ by $V^{l-1/s,s}_{\beta,\delta}(\Gamma_j)$ and $W^{l-1/s,s}_{\beta,\delta}(\Gamma_j)$, respectively.

5.2 Model problems and corresponding operator pencils

We introduce the operator pencils generated by problem (5.1), (5.2) for the edge points and vertices of the domain $\mathcal{G}$.

1) Let ξ be a point on an edge M_k, and let $\Gamma_{k_+}, \Gamma_{k_-}$ be the faces of $\mathcal{G}$ adjacent to ξ. Then by $\mathcal{D}_\xi$ we denote the dihedron which is bounded by the half-planes $\Gamma^\circ_{k_\pm}$ tangent to $\Gamma_{k_\pm}$ at ξ. The angle between the half-planes $\Gamma^\circ_{k_\pm}$ is denoted by $\theta(\xi)$. We consider the model problem

$$\begin{aligned} &-\Delta u + \nabla p = f, \quad -\nabla \cdot u = g \quad \text{in } \mathcal{D}_\xi, \\ &S_{k_\pm} u = h_{k_\pm}, \quad N_{k_\pm}(u,p) = \phi_{k_\pm} \quad \text{on } \Gamma^\circ_{k_\pm}. \end{aligned}$$

The operator pencil corresponding to this model problem (see Section 2.2) is denoted by $A_\xi(\lambda)$. Furthermore, let $\lambda_1(\xi)$ be the eigenvalue with smallest positive real part of this pencil, while $\lambda_2(\xi)$ is the eigenvalue with smallest real part greater than 1. We define

$$\mu(\xi) = \begin{cases} \operatorname{Re}\lambda_1(\xi) & \text{if } d_{k_+} + d_{k_-} \text{ is odd} \quad \text{or} \quad d_{k_+} + d_{k_-} \text{ is even and } \theta(\xi) \geq \pi/m_k, \\ \operatorname{Re}\lambda_2(\xi) & \text{if } d_{k_+} + d_{k_-} \text{ is even and } \alpha_k < \pi/m_k, \end{cases}$$

where $m_k = 1$ if $d_{k_+} = d_{k_-}$, $m_k = 2$ if $d_{k_+} \neq d_{k_-}$. Finally, let

$$\mu_k = \inf_{\xi \in M_k} \mu(\xi). \tag{5.3}$$

2) Let $x^{(j)}$ be a corner of $\mathcal{G}$ and let I_j be the set of all indices k such that $x^{(j)} \in \overline{\Gamma}_k$. By our assumptions, there exist a neighborhood $\mathcal{U}$ of $x^{(j)}$ and a diffeomorphism κ mapping $\mathcal{G} \cap \mathcal{U}$ onto $\mathcal{K}_j \cap B_1$ and $\Gamma_k \cap \mathcal{U}$ onto $\Gamma^\circ_k \cap B_1$ for $k \in I_j$, where $\mathcal{K}_j$ is a polyhedral cone with vertex 0 and Γ°_k are the faces of this cone. Without loss of generality, we may assume that the Jacobian matrix $\kappa'(x)$ is equal to the identity matrix I at the point $x^{(j)}$. We consider the model problem

$$\begin{aligned} &-\Delta u + \nabla p = f, \quad -\nabla \cdot u = g \quad \text{in } \mathcal{K}_j, \\ &S_k u = h_k, \quad N_k(u,p) = \phi_k \quad \text{on } \Gamma^\circ_k \text{ for } k \in I_j. \end{aligned}$$

The operator pencil generated by this model problem (see Section 3.2) is denoted by $\mathfrak{A}_j(\lambda)$.

5.3 Existence of weak solutions

We introduce the spaces

$$V = \{u \in W^{1,2}(\mathcal{G})^3 : \ S_j u = 0 \text{ on } \Gamma_j, \ j = 1, \dots, n\}, \quad V_0 = \{u \in V : \ \nabla \cdot u = 0\}.$$

Furthermore, we denote by L_V the set of all $u \in V$ such that $\varepsilon_{i,j}(u) = 0$ for $i,j = 1,2,3$. It can be easily seen that L_V is contained in the span of all all constant vectors and of the vectors $(x_2, -x_1, 0)$, $(0, x_3, -x_2)$, $(-x_3, 0, x_1)$. In particular, we have $L_V \subset V_0$.

Let the bilinear form b be defined as

$$b(u,v) = 2 \int_{\mathcal{G}} \sum_{i,j=1}^{3} \varepsilon_{i,j}(u)\, \varepsilon_{i,j}(v)\, dx.$$

We consider the problem

$$b(u,v) - \int_{\mathcal{G}} p\, \nabla \cdot v\, dx = F(v) \quad \text{for all } v \in V \tag{5.4}$$

$$-\nabla \cdot u = g \ \text{ in } \mathcal{G}, \quad S_j u = h_j \ \text{ on } \Gamma_j, \ j = 1, \dots, n, \tag{5.5}$$

where F is a given linear an continuous functional on V, $g \in L_2(\mathcal{G})$, $h_j \in W^{1/2,2}(\Gamma_j)^{3-d_j}$. We assume that the vector functions h_j are such that there exists a vector function $v \in W^{1,2}(\mathcal{G})^3$ satisfying the boundary conditions $S_j v = h_j$ on Γ_j for $j = 1, \dots, n$. This means, the boundary data h_j must satisfy a certain trace condition on the edges of the domain (see Section 3.1).

Lemma 5.1 *Let $g \in L_2(\mathcal{G})$, and let $h_j \in W^{1/2,2}(\Gamma_j)^{3-d_j}$ are such that there exists a vector function $v \in W^{1,2}(\mathcal{G})^3$, $S_j v = h_j$ on Γ_j for $j = 1, \dots, n$. In the case when $d_j \in \{0, 2\}$ for all j, we assume additionally that*

$$\int_{\mathcal{G}} g\,dx + \sum_{j:\, d_j=0} \int_{\Gamma_j} h_j \cdot n\,dx + \sum_{j:\, d_j=2} \int_{\Gamma_j} h_j\,dx = 0. \tag{5.6}$$

Then there exists a vector function $u \in W^{1,2}(\mathcal{G})^3$ such that $\nabla \cdot u = -g$ and $S_j u = h_j$ on Γ_j for $j = 1, \dots, n$.

Proof: Let $v \in W^{1,2}(\mathcal{G})^3$, $S_j v = h_j$ on Γ_j for $j = 1, \dots, n$. We have to show that there exists a vector function $w \in V$ such that $\nabla \cdot w = -g - \nabla \cdot v$. Then $u = v + w$ is the desired vector function.

Let first $d_j \in \{0, 2\}$ for all j. By [6, Ch.1,Cor.2.4], there exists a vector function $w \in \overset{\circ}{W}{}^{1,2}(\mathcal{G})^3 \subset V$ satisfying $\nabla \cdot w = -g - \nabla \cdot v$ if

$$\int_{\mathcal{G}} (g + \nabla \cdot v)\,dx = 0.$$

The last condition is equivalent to (5.6).

We consider the case when $d_j \in \{1, 3\}$ for at least one $j = j_0$. Let $\phi \in C_0^\infty(\Gamma_{j_0})$ be a function the integral of which over Γ_{j_0} is equal to 1. Then there exists a vector function $\psi \in W^{1,2}(\mathcal{G})^3$ such that

$$\psi = 0 \ \text{ on } \Gamma_j \text{ for } j \neq 0, \quad \psi_n = \phi,\ \psi_\tau = 0 \ \text{ on } \Gamma_{j_0}$$

(see Lemma 3.1). Since $j_0 \in \{1, 3\}$, the vector function ψ belongs to V. We introduce the function

$$g' = g + \nabla \cdot v - c\,\nabla \cdot \psi, \quad \text{where } c = \int_{\mathcal{G}} (g + \nabla \cdot v)\,dx.$$

Since

$$\int_{\mathcal{G}} g'\,dx = c\left(1 - \int_{\mathcal{G}} \nabla \cdot \psi\,dx\right) = c\left(1 - \int_{\Gamma_{j_0}} \phi\,dx\right) = 0,$$

there exists a vector function $w' \in V$ such that $-\nabla \cdot w' = g'$. Consequently, $w = w' - c\psi$ satisfies the equation $\nabla \cdot w = -g - \nabla \cdot v$. The result follows. ∎

The necessity of condition (5.6) in Lemma 5.1 is obvious. Moreover, since $b(u, v) = 0$ and $\nabla \cdot v = 0$ for $v \in L_V$, for the solvability of problem (5.4), (5.5) it is necessary that

$$F(v) = 0 \quad \text{for all } v \in L_V. \tag{5.7}$$

Theorem 5.1 *Let g and h_j be as in Lemma 5.1, and let $F \in V^*$ be a functional satisfying the condition (5.7). Then there exists a solution $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ of problem (5.4), (5.5). Here p is uniquely determined if $d_j \in \{1, 3\}$ for at least one j and unique up to constants if $d_j \in \{0, 2\}$ for all j. The vector function u is unique up to elements from L_V.*

Proof: 1) Let first $g = 0$ and $h_j = 0$ for $j = 1, \dots, n$. We denote by $L_V^\perp$ the orthogonal complement of L_V in V_0. By Korn's inequality, we have

$$b(u, \bar{u}) \geq c\,\|u\|^2_{W^{1,2}(\mathcal{G})^3} \quad \text{for all } v \in L_V^\perp. \tag{5.8}$$

Consequently, there exists a unique vector function $u \in L_V^\perp$ such that $b(u, v) = F(v)$ for all $v \in L_V^\perp$. Since both $b(u, v)$ and $F(v)$ vanish for $v \in L_V$, it follows that

$$b(u, v) = F(v) \quad \text{for all } v \in V_0. \tag{5.9}$$

Let $V_0^\perp$ denote the orthogonal complement of V_0 in V. By Lemma 5.1, the operator $B = -\mathrm{div}$ is an isomorphism from $V_0^\perp$ onto $L_2(\mathcal{G})$ if $d_j \in \{1, 3\}$ for at least one j and onto the space

$$\overset{\circ}{L}{}_2(\mathcal{G}) = \{q \in L_2(\mathcal{G}) : \int_{\mathcal{G}} q(x)\,dx = 1\}$$

if $d_j \in \{0,2\}$ for all j. Suppose that $d_j \in \{0,2\}$ for all j. Then we consider the mapping

$$L_2(\mathcal{G}) \ni q \to \ell(q) = F(B^{-1} \overset{\circ}{q}) - b(u, B^{-1} \overset{\circ}{q}), \quad \text{where } \overset{\circ}{q} = q - \frac{1}{|\mathcal{G}|} \int_{\mathcal{G}} q(x)\, dx \in \overset{\circ}{L_2}(\mathcal{G}).$$

Obviously, ℓ defines a linear and continuous functional on $L_2(\mathcal{G})$. Consequently, there exists a function $p \in L_2(\mathcal{G})$ such that

$$\int_{\mathcal{G}} p\, q\, dx = \ell(q) \quad \text{for all } q \in L_2(\mathcal{G}).$$

Consequently,

$$-\int_{\mathcal{G}} p\, \nabla \cdot v\, dx = \ell(-\nabla \cdot v) = F(v) - b(u,v) \quad \text{for all } v \in V_0^{\perp}. \tag{5.10}$$

In the case when $d_j \in \{1,3\}$ for at least one j, the existence of $p \in L_2(\mathcal{G})$ satisfying (5.9) follows analogously from the continuity of the mapping

$$L_2(\mathcal{G}) \ni q \to \ell(q) = F(B^{-1}q) - b(u, B^{-1}q) \in \mathbb{C}.$$

Combining (5.9) and (5.10), we conclude that u and p satisfies (5.4). This proves the existence of a solution.

We prove the uniqueness. Let $u \in V_0$ and $p \in L_2(\mathcal{G})$ satisfy (5.4) with $F = 0$. Then, in particular, $b(u, \bar{u}) = 0$. Obviously, $b(u, \bar{u}) = b(u - w, \bar{u} - \bar{w})$, where w is the orthogonal projection of $u \in V_0$ onto L_V. Using (5.8), we obtain $u - w = 0$, i.e., $u \in L_V$. However, then $b(u,v) = 0$ for all $v \in V$ and, therefore,

$$\int_{\mathcal{G}} p\, \nabla \cdot v = 0 \quad \text{for all } v \in V.$$

If $d_j \in \{1,3\}$ for at least one j, then v can be chosen such that $\nabla v = \bar{p}$, and we obtain $p = 0$. If $d_j \in \{0,2\}$ for all j, then we obtain

$$\int_{\mathcal{G}} p\, q\, dx = 0 \quad \text{for all } q \in L_2(\mathcal{G}), \ \int_{\mathcal{G}} q\, dx = 0.$$

From this we conclude that p is constant. The proof is complete. ∎

5.4 Regularity assertions for weak solutions

Our goal is to show that the solution $(u,p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ of problem (5.4), (5.5) belongs to $W^{1,s}_{\beta,\delta}(\mathcal{G})^3 \times W^{0,s}_{\beta,\delta}(\mathcal{G})$ under certain conditions on F, g, h_j, β and δ. For this end, we consider the perturbed Stokes problem in the cone $\mathcal{K}_j$

$$b_1(u,v) + \int_{\mathcal{K}_j} p\, L_1 v\, dx = F(v) \ \text{ for all } v \in W^{1,2}(\mathcal{K}_j)^3, \ S_k v = 0 \text{ on } \Gamma_k^{\circ}, \ k \in I_j, \tag{5.11}$$

$$L_1 u = g \ \text{ in } \mathcal{K}_j, \quad S_k u = h_k \ \text{ on } \Gamma_k^{\circ}, \ k \in I_j, \tag{5.12}$$

where

$$b_1(u,v) = 2 \int_{\mathcal{K}_j} \sum_{k,l=1}^{3} \varepsilon_{k,l}(u)\, \varepsilon_{k,l}(v)\, dx + \sum_{\mu,\nu,k,l=1}^{3} \int_{\mathcal{K}_j} b_{\mu,\nu,k,l}(x)\, \frac{\partial u_k}{\partial x_\mu}\, \frac{\partial u_l}{\partial x_\nu}\, dx,$$

$$L_1 v = -\nabla \cdot v + \sum_{k,l=1}^{3} c_{k,l}(x) \frac{\partial v_k}{\partial x_l}\,.$$

We assume that

$$\sum_{\mu,\nu,k,l} |b_{\mu,\nu,k,l}(x)| + \sum_{k,l} |c_{k,l}(x)| < \varepsilon \tag{5.13}$$

with sufficiently small ε.

Lemma 5.2 *Let* $(u,p) \in W^{1,2}_{\beta',0}(\mathcal{K}_j)^3 \times W^{0,2}_{\beta',0}(\mathcal{K}_j)$ *be a solution of problem* (5.11), (5.12)*, where*

$$F \in W^{-1,s}_{\beta',0}(\mathcal{K}_j)^3 \cap V^{-1,s}_{\beta,\delta}(\mathcal{K}_j)^3, \quad g \in W^{0,2}_{\beta',0}(\mathcal{K}_j) \cap W^{0,s}_{\beta,\delta}(\mathcal{K}_j), \quad h_k \in W^{1/2,2}_{\beta',0}(\Gamma_k^\circ) \cap W^{1-1/s,s}_{\beta,\delta}(\Gamma_k^\circ) \quad (5.14)$$

for $k \in I_j$*. Suppose that there are no eigenvalues of the pencil* $\mathfrak{A}_j(\lambda)$ *in the closed strip between the lines* $\mathrm{Re}\lambda = -\beta' - 1/2$ *and* $\mathrm{Re}\lambda = 1 - \beta - 3/s$*, the components of* δ *satisfy the inequalities* $\max(1-\mu_k, 0) < \delta_k + 2/s < 1$*, and that the number* ε *in* (5.13) *is sufficiently small. Then* $u \in W^{1,s}_{\beta,\delta}(\mathcal{K}_j)^3$ *and* $p \in W^{0,s}_{\beta,\delta}(\mathcal{K}_j)$*.*

Proof: Let $\mathcal{W}_{s,\beta,\delta}$ be the space of all

$$\{h_k\}_{k\in I_j} \in \prod_{k\in I_j} W^{1-1/s,s}_{\beta,\delta}(\Gamma_k^\circ)^{3-d_k}$$

such that there exists a vector function $u \in W^{1,s}_{\beta,\delta}(\mathcal{K}_j)$ satisfying $S_k u = h_k$ on Γ_k° for $k \in I_j$. This is a subspace of vector-functions on Γ_k° satisfying certain compatibility conditions on the edges of the cone $\mathcal{K}_j$ (see Sections 2.1 and 3.1). We define A as the operator

$$W^{1,2}_{\beta',0}(\mathcal{K}_j)^3 \times W^{0,2}_{\beta',0}(\mathcal{K}_j) \ni (u,p) \to (F,g,h_j) \in W^{-1,s}_{\beta',0}(\mathcal{K}_j)^3 \times W^{0,2}_{\beta',0}(\mathcal{K}_j) \times \mathcal{W}_{2,\beta',0} \quad (5.15)$$

where F, g and h_k are given by (5.11), (5.12). Furthermore, let A_0 be the operator (5.15), where

$$F(v) = 2\int_{\mathcal{K}_j} \sum_{i,j=1}^{3} \varepsilon_{i,j}(u)\,\varepsilon_{i,j}(v)\,dx - \int_{\mathcal{K}_j} p\,\nabla\cdot v\,dx, \quad g = -\nabla\cdot u, \quad h_k = S_k u.$$

By [25, Th.4.2], the operator A_0 is an isomorphism. Furthermore, it follows from Theorem 4.1 and Lemma 4.4 that A_0 is an isomorphism

$$\begin{aligned}\Big(W^{1,2}_{\beta',0}(\mathcal{K}_j)^3 \times W^{0,2}_{\beta',0}(\mathcal{K}_j)\Big) \cap \Big(W^{1,s}_{\beta,\delta}(\mathcal{K}_j)^3 \times W^{0,s}_{\beta,\delta}(\mathcal{K}_j)\Big)\\ \to \Big(W^{-1,s}_{\beta',0}(\mathcal{K}_j)^3 \times W^{0,2}_{\beta',0}(\mathcal{K}_j) \times \mathcal{W}_{2,\beta',0}\Big) \cap \Big(V^{-1,s}_{\beta,\delta}(\mathcal{K}_j)^3 \times W^{0,s}_{\beta,\delta}(\mathcal{K}_j) \times \mathcal{W}_{s,\beta,\delta}\Big)\end{aligned} \quad (5.16)$$

Due to (5.13), the operator norm (5.16) of $A - A_0$ is less than $c\varepsilon$. Hence for sufficiently small ε, the operator $A - A_0$ is also an isomorphism (5.16). The result follows. ∎

Theorem 5.2 *Let* $(u,p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ *be a solution of problem* (5.4), (5.5)*, where*

$$F \in V^* \cap V^{-1,s}_{\beta,\delta}(\mathcal{K}_j)^3, \quad g \in L_2(\mathcal{G}) \cap W^{0,s}_{\beta,\delta}(\mathcal{G}), \quad h_k \in W^{1/2,2}(\Gamma_k) \cap W^{1-1/s,s}_{\beta,\delta}(\Gamma_k).$$

Suppose that there are no eigenvalues of the pencils $\mathfrak{A}_j(\lambda)$*,* $j = 1,\dots,d$ *in the closed strip between the lines* $\mathrm{Re}\lambda = -1/2$ *and* $\mathrm{Re}\lambda = 1 - \beta - 3/s$ *and that the components of* δ *satisfy the inequalities* $\max(1-\mu_k, 0) < \delta_k + 2/s < 1$*. Then* $u \in W^{1,s}_{\beta,\delta}(\mathcal{G})^3$*,* $p \in W^{0,s}_{\beta,\delta}(\mathcal{G})$*.*

Proof: It suffices to prove the theorem for vector functions (u,p) with small supports. For solutions with arbitrary support the assertion then can be easily proved by means of a partition of unity on $\mathcal{G}$. Let the support of (u,p) be contained in a sufficiently small neighborhood $\mathcal{U}$ of the vertex $x^{(j)}$, and let κ be a diffeomorphism mapping $\mathcal{G}\cap\mathcal{U}$ onto $\mathcal{K}_j \cap B$, where $\mathcal{K}_j$ is a cone with vertex at the origin and B is a ball centered about the origin. We assume that $\kappa'(x^{(j)}) = I$. Then the vector function $(w(x), q(x)) = \big(u\big(\kappa^{-1}(x)\big), p\big(\kappa^{-1}(x)\big)\big)$ is a solution of a perturbed Stokes problem (5.11), (5.12), where the coefficients $b_{\mu,\nu,k,l}$ and $c_{k,l}$ are zero at the origin and bounded by small constants on the support of (w,q). Applying Lemma 5.2, we obtain $(w,q) \in W^{1,s}_{\beta_j,\delta}(\mathcal{K}_j)^3 \times W^{0,s}_{\beta_j,\delta}(\mathcal{K}_j)$ and, therefore $(u,p) \in W^{1,s}_{\beta,\delta}(\mathcal{G})^3 \times W^{0,s}_{\beta,\delta}(\mathcal{G})$. For vector functions (u,p) with support in a neighborhood of an edge point, the assertion of the theorem can be proved analogously. ∎

Analogously, the following theorem can be proved (cf. Theorem 4.4).

Theorem 5.3 *Let $(u,p) \in W^{1,2}_{\beta',0}(\mathcal{G})^3 \times W^{0,2}_{\beta',0}(\mathcal{G})$ be a solution of problem (5.11), (5.12), where $g \in W^{l-1,s}_{\beta,\delta}(\mathcal{G})$, $h_j \in W^{l-1/s,s}_{\beta,\delta}(\Gamma_j)^{3-d_j}$, and $F \in V^*$ has the representation*

$$F(v) = \int_{\mathcal{G}} f \cdot v \, dx + \sum_{j=1}^{n} \int_{\Gamma_j} \phi_j \cdot v \, dx \quad \text{for all } v \in V$$

with $f \in W^{l-2,s}_{\beta,\delta}(\Gamma_j)$, $\phi_j \in W^{l-1-1/s}_{\beta,\delta}(\Gamma_j)^{d_j}$. We suppose that there are no eigenvalues of the pencils $\mathfrak{A}_j(\lambda)$, $j = 1,\dots,d$, in the closed strip between the lines $\operatorname{Re}\lambda = -1/2$ and $\operatorname{Re}\lambda = l - \beta - 3/s$ and that $\max(l - \mu_k, 0) < \delta_k + 2/s < l$ for $k = 1,\dots,m$. Furthermore, we assume that g, h_j and ϕ_j satisfy compatibility conditions on the edges M_k which guarantee that there exist $w \in W^{2,s}_{\beta-l+2,\delta'}(\mathcal{G})^3$ and $q \in W^{1,s}_{\beta-l+2,\delta'}(\mathcal{G})$, $\delta'_k = \max(\delta_k - l + 2, \frac{1}{2} - \frac{2}{s})$, such that

$$S_j w = h_j, \quad N_j(w,q) = \phi_j \quad \text{on } \Gamma_j, \ j = 1,\dots,n, \quad \nabla \cdot w + g \in V^{1,s}_{\beta-l+2,\delta'}(\mathcal{G}).$$

Then $u \in W^{l,s}_{\beta,\delta}(\mathcal{G})^3$ and $p \in W^{l-1,s}_{\beta,\delta}(\mathcal{G})$.

5.5 Examples

Here we establish some regularity assertions for weak solutions of special boundary value problems for the Stokes system in the class of the nonweighted spaces $W^{l,s}(\mathcal{G})$. Let $\mathcal{G}$ be a polyhedron with sides Γ_j, $j = 1,...,n$, and edges M_k, $k = 1,\dots,m$. We denote the angle at the edge M_k by θ_k. For the sake of simplicity, we restrict ourselves to homogeneous boundary conditions

$$S_j u = 0, \quad N_j(u,p) = 0 \quad \text{on } \Gamma_j, \ j = 1,\dots,n. \tag{5.17}$$

Analogous results are valid for inhomogeneous boundary conditions provided the boundary data satisfy certain compatibility conditions on the edges.

The Dirichlet problem for the Stokes system. Let $f \in W^{-1,2}(\mathcal{G})^3$ and $g \in L_2(\mathcal{G})$ satisfy the compatibility conditions of Theorem 5.1. Then there exists a solution $(u,p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ of the Dirichlet problem

$$-\Delta u + \nabla p = f, \quad -\nabla u = g \text{ in } \mathcal{G}, \quad u = 0 \text{ on } \Gamma_j, \ j = 1,\dots,n.$$

Here u is unique and p is unique up to a constant (see also [6, Th.5.1]). It is known that there are no eigenvalues of the pencils $\mathfrak{A}_j(\lambda)$ in the strip $-1 \le \operatorname{Re}\lambda \le 0$ (see [10, Th.5.5.6]). In the case, when $\mathcal{G}$ is convex, then even the strip $-2 < \operatorname{Re}\lambda < 1$ does not contain eigenvalues of the pencils $\mathfrak{A}_j(\lambda)$ (see [10, Th.5.5.5]). Moreover, it can be easily verified that $\mu_k > 1/2$, $\mu_k > 2/3$ if $\theta_k < 3\arccos\frac{1}{4} \approx 1.2587\pi$, $\mu_k > 1$ if $\theta_k < \pi$, and $\mu_k > 4/3$ if $\theta_k < \frac{3}{4}\pi$. Using these results together with Theorems 5.2 and 5.3, we obtain the following assertions.

- If $f \in (W^{1,s'}(\mathcal{G})^*)^3$ and $g \in L_s(\mathcal{G})$, $2 < s \le 3$, $s' = s/(s-1)$, then $(u,p) \in W^{1,s}(\mathcal{G})^3 \times L_s(\mathcal{G})$. If the polyhedron $\mathcal{G}$ is convex, then this assertion is true for all $s > 2$.
- If $f \in W^{-1,2}(\mathcal{G})^3 \cap L_s(\mathcal{G})^3$ and $g \in L_2(\mathcal{G}) \cap W^{1,s}(\mathcal{G})$, $1 < s \le 4/3$, then $(u,p) \in W^{2,s}(\mathcal{G})^3 \times W^{1,s}(\mathcal{G})$. If $\theta_k < 3\arccos\frac{1}{4} \approx 1.2587\pi$ for $k = 1,\dots,m$, then this result is true for $1 < s \le 3/2$. If $\mathcal{G}$ is convex, then this result is valid for $1 < s \le 2$ provided g satisfies (0.3) if $s = 2$. If, moreover, the angles at the edges are less than $\frac{3}{4}\pi$, then the result holds even for $1 < s < 3$ provided g satisfies (0.3) if $s = 2$ and $g = 0$ on M_k, $k = 1,\dots,m$, if $s > 2$.

Here we used also the facts that $W^{1,s'}(\mathcal{G}) = V^{1,s'}_{0,0}(\mathcal{G})$ for $s' < 2$ and $W^{1,s}(\mathcal{G}) = W^{1,s}_{0,0}(\mathcal{G})$ for $s < 3$. In the case $s = 2$ the $W^{2,s}$-regularity result for convex polyhedrons was also proved by Dauge [3], for convex two-dimensional polygonal domains we refer to Kellogg and Osborn [7].

The Neumann problem for the Stokes system. We consider the weak solution $u \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ of the Neumann problem

$$-\Delta u + \nabla p = f, \quad -\nabla u = g \text{ in } \mathcal{G}, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_j, \ j = 1,\dots,n.$$

For this problem it is known that the strip $-1 \le \operatorname{Re}\lambda \le 0$ contains only the eigenvalues $\lambda = 0$ and $\lambda = 1$ of the operator pencils $\mathfrak{A}_j(\lambda)$ (see [10, Th.6.3.2]) if $\mathcal{G}$ is a Lipschitz polyhedron. The numbers μ_k are the same as for the Dirichlet problem. Therefore, the following assertions are valid.

- If $f \in (W^{1,s'}(\mathcal{G})^*)^3$ and $g \in L_s(\mathcal{G})$, $2 < s < 3$, then $(u,p) \in W^{1,s}(\mathcal{G})^3 \times L_s(\mathcal{G})$.
- If $f \in (W^{1,2}(\mathcal{G})^*)^3 \cap L_s(\mathcal{G})^3$ and $g \in L_2(\mathcal{G}) \cap W^{1,s}(\mathcal{G})$, $1 < s \le 4/3$, then $(u,p) \in W^{2,s}(\mathcal{G})^3 \times W^{1,s}(\mathcal{G})$. If the angles θ_k are less than $3\arccos\frac{1}{4}$, then this result is true for $1 < s < 3/2$.

The mixed problem with Dirichlet and Neumann boundary conditions. We assume that on each side Γ_j either the Dirichlet condition $u = 0$ or the Neumann condition $\frac{\partial u}{\partial n} = 0$ is given. If on the adjoining sides of the edge M_k the same boundary conditions are given, then $\mu_k > 1/2$. If on one of the adjoining sides the Dirichlet condition and on the other side the Neumann condition is given, then $\mu_k > 1/4$. This implies the following result.

- If $f \in (W^{1,2}(\mathcal{G})^*)^3 \cap L_s(\mathcal{G})^3$ and $g \in L_2(\mathcal{G}) \cap W^{1,s}(\mathcal{G})$, $1 < s \le 8/7$, then the weak solution (u,p) belongs to $W^{2,s}(\mathcal{G})^3 \times W^{1,s}(\mathcal{G})$.

The mixed problem with boundary conditions (i)–(iii). Let $(u,p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ be a weak solution of problem (5.1), (5.17), where $d_k \le 2$ for all k (i.e., the Neumann condition does not appear in the boundary conditions). We assume that the Dirichlet condition is given on at least one of the adjoining sides of every edge. Then, by [10, Th.6.1.5], the strip $-1 \le \operatorname{Re}\lambda \le 0$ is free of eigenvalues of the pencils $\mathfrak{A}_j(\lambda)$. Furthermore, we have $\mu_k > 1/2$ if the Dirichlet condition is given on both adjoining sides of the edge M_k. For the other indices k, we have $\mu_k > 1/4$ and $\mu_k > 1/3$ if $\theta_k < \frac{3}{2}\pi$.

- If $f \in (W^{1,s'}(\mathcal{G})^*)^3$ and $g \in L_s(\mathcal{G})$, $2 < s \le 8/3$, then $(u,p) \in W^{1,s}(\mathcal{G})^3 \times L_s(\mathcal{G})$. Suppose that $\theta_k < \frac{3}{2}\pi$ if the boundary condition (ii) or (iii) is given on one of the adjoining sides of the edge M_k. Then this result is even true for $2 < s \le 3$.
- If $f \in (W^{1,2}(\mathcal{G})^*)^3 \cap L_s(\mathcal{G})^3$ and $g \in L_2(\mathcal{G}) \cap W^{1,s}(\mathcal{G})$, $1 < s \le 8/7$, then $(u,p) \in W^{2,s}(\mathcal{G})^3 \times W^{1,s}(\mathcal{G})$. Suppose that $\theta_k < 3\arccos\frac{1}{4}$ if the Dirichlet condition is given on both adjoining sides of M_k, $\theta_k < \frac{3}{2}\arccos\frac{1}{4}$ if the boundary condition (i) is given on one of the adjoining sides of M_k, and $\theta_k < \frac{3}{4}\pi$ if the boundary condition (ii) is given on one of the adjoining sides of M_k. Then the last result is true for $1 < s \le 3/2$.

Note that in the last case, we have $\mu_k > 2/3$ for $k = 1, \dots, m$.

Finally, we consider problem (5.1), (5.17) when the Dirichlet condition is given on the sides $\Gamma_1, \dots, \Gamma_{n-1}$, while the boundary condition (ii) is given on Γ_n. Let I be the set of all k such that $M_k \subset \bar{\Gamma}_n$ and $I' = \{1, \dots, n\} \setminus I$. We suppose that the polyhedron $\mathcal{G}$ is convex and $\theta_k < \pi/2$ for $k \in I$. Then $\mu_k > 1$ for all k, and the strip $-1/2 \le \operatorname{Re}\lambda < 1$ is free of eigenvalues of the pencils $\mathfrak{A}_j(\lambda)$ (see [10, Th.6.2.7]). If $\theta_k < \frac{3}{8}\pi$ for $k \in I$ and $\theta_k < \frac{3}{4}\pi$ for $k \in I'$, then even $\mu_k > 4/3$. This implies the following result.

- Let $f \in (W^{1,2}(\mathcal{G})^*)^3 \cap L_s(\mathcal{K})^3$ and $g \in L_2(\mathcal{G}) \cap W^{1,s}(\mathcal{G})$, $s > 1$. In the case $s > 2$, we suppose that $g|_{M_k} = 0$ for all k, while condition (0.3) is assumed to be valid for $s = 2$. Then the weak solution $(u,p) \in W^{1,2}(\mathcal{G})^3 \times L_2(\mathcal{G})$ of problem (5.1), (5.17) belongs to $W^{2,s}(\mathcal{G})^3 \times W^{1,s}(\mathcal{G})$ for $1 < s \le 2$. If $\theta_k < \frac{3}{8}\pi$ for $k \in I$ and $\theta_k < \frac{3}{4}\pi$ for $k \in I'$, then the result holds even for $1 < s < 3$.

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